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Finite energy solutions for nonlinear elliptic equations with competing gradient, singular and L^1 terms

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Abstract

In this paper we deal with the following boundary value problem

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = h(u)f & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, where $1 \leq p < N$, g is a positive and continuous function on $[0, \infty)$, and h is a continuous function on $[0, \infty)$ (possibly blowing up at the origin). We show how the presence of regularizing terms h and g allows to prove existence of finite energy solutions for nonnegative data f only belonging to $L^1(\Omega)$.

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Keywords: 1-Laplacian; p-Laplacian; Natural growth gradient terms; Regularizing effects; L^1 data; Singular problems

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Contents

1. Introduction

The aim of this work is the study of the following boundary value problem

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = h(u)f & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 \leq p < N$. Here $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is an open and bounded set with Lipschitz boundary, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual $p-$laplacian, and g is a positive and continuous function on $[0, \infty)$. Finally h is a continuous function on $[0, \infty)$ that is allowed to blow-up at the origin and is bounded at infinity. In particular, the case of continuous, bounded and non-monotone functions g, h is covered by the above assumptions.

Our goal is the study of existence of finite energy solutions to (1.1); by a finite energy solution we mean a function lying in the natural space in which such problems are naturally built-in in case of smooth nonlinear terms and data, i.e. $u \in W_0^{1,p}(\Omega)$ if $p > 1$ and $u \in BV(\Omega)$ if $p = 1$.

We are interested to deeply explore the interplay between the first order absorption term involving g and the zero order and possibly singular nonlinearity h in presence of a merely integrable datum f. In particular we deal with the regularizing effect, in terms of Sobolev regularity, provided by the lower order terms to the solutions of problems as (1.1).

These kinds of regularizing effects given by the gradient terms with natural growth in elliptic problems with rough data are nowadays quite classical. If $h \equiv 1$ it is well known that, see for instance [8,28], quadratic gradient terms satisfying a sign condition (i.e. $g(s)s \geq 0$) in problems as (1.1) gives finite energy solutions if f is a merely integrable function (or even a measure). Let also stress that it is well known that solutions have, in general, infinite energy if $g \equiv 0$ (see [7]).

On the other hand problems as (1.1) with $g \equiv 0$ and in presence of possibly singular nonlinearities has reached great attention in the last decades starting from the pioneering papers [13,24,29]; if $p = 2$, $h(s) = s^{-\gamma}$ ($\gamma > 0$), and f is smooth, existence of classical solutions in $C^2(\Omega) \cap C^0(\overline{\Omega})$ follows by suitable approximation with desingularized problems. Only later, in [10], the authors prove existence of a distributional solution in case of a Lebesgue datum f and remarked the regularizing effect given by the right-hand of (1.1) when, once again, $p = 2$ and $h(s) = s^{-\gamma}$ ($\gamma > 0$): namely the solution, compared to the case $\gamma = 0$, always lies in a smaller Sobolev space when $0 < \gamma \le 1$. Moreover if $\gamma = 1$ the solution is always in $H_0^1(\Omega)$ even if f is just an L^1-function as one can formally deduce by taking u itself as test function in (1.1) while if $\gamma > 1$ the solution belongs only locally to $H^1(\Omega)$ and the boundary datum is meant as a suitable power of the solution having zero Sobolev trace. It is also worth to mention that, in general, one can not expect finite energy solutions for $\gamma \ge 3 - \frac{2}{m}$ if $f \in L^m(\Omega)$, $m > 1$ ([24,25]).

Similar results, again in the case $g \equiv 0$, were then extended to the case $p \ge 1$; i.e. let us consider nonnegative finite energy solutions for

$$\begin{cases} -\Delta_p u = \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It was shown in [15] that, if $p > 1$, then finite energy solutions exist either, for smooth datum f, up to $\gamma < 2 + \frac{1}{p-1}$ or if $\gamma \le 1$ with $f \in L^m(\Omega)$ for some $m \ge 1$. As observed in [16], the formal case of $p \to 1^+$ is also included as finite energy BV solutions are expected to exist either for any $\gamma > 0$ in case of smooth f, or for $\gamma \le 1$ in the general case of $f \in L^m(\Omega)$ for some $m \ge 1$.

Problems as

$$\begin{cases} -\Delta_1 u + g(u)|Du| = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

have also been recently considered as a model for the level set formulation proposed in [20] for the inverse mean curvature flow (see also [26]) in order to prove the well-known Penrose inequality in the case of a single black hole.

From the purely mathematical point of view, regularizing effect of the presence of gradient terms (also in case of a nonlinearity g with a generic sign) and $f \in L^N(\Omega)$ was recently investigated (see [5,26,22,18]). Among the others let only stress that solutions to these problems are, in general, BV functions with no jump part in its gradient.

The main goal of our study concerns problems as

$$\begin{cases} -\Delta_1 u + g(u)|Du| = h(u)f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where g is a positive and bounded continuous function on $[0,\infty)$, h is a continuous function on $[0,\infty)$ (possibly blowing up at the origin) and bounded at infinity, and f is a nonnegative function in $L^1(\Omega)$. In particular we will focus on the interplay between the data g, h and f in order to get nontrivial solutions in $BV(\Omega)$, the space of functions in $L^1(\Omega)$ whose derivatives have finite total bounded variation over Ω. We stress again that this goal is forbidden in presence of merely integrable data whenever $g \equiv 0$ (see [21]). It is also worth mentioning that, if this is the case, existence of solutions is expected for more regular data, but only under a suitable smallness condition on the norm ([17,16]), which here is not requested.

As a further feature of these types of problems with gradient type absorption terms one has that solutions u of (1.2) are in fact "zero" $\mathcal{H}^{N-1}$ a.e. on $\partial\Omega$ in contrast with the case of reaction terms ([22,18]) or no reaction at all ([16]) in which constant solutions are allowed.

We also stress that a standard approach to deal with 1-Laplace type problems as (1.2) consists in approximate them with p-Laplace ones where $p > 1$. Our method will follow this line but let us emphasize that the case $p > 1$ is interesting and new as well; so that, in Section 3 we set, as a preparatory tool but in full generality, the theory of existence and weak regularity of solutions for problems as in (1.1).

The plan of the paper is the following: in Section 2 we provide some preliminaries tools and notation. In Section 3 we study the Dirichlet problem in presence of a principal operator modeled by the p-Laplacian with $p > 1$; apart from being interesting itself, it is the preparatory study for the main Section 4, in which the limit case $p = 1$ is investigated. Finally in Section 5, we give some further insights on how to extend the result in various directions as the case of merely nonnegative data and the case of a nonlinearity g possibly blowing-up at infinity.

2. Notation and preparatory tools

For the entire paper Ω denotes an open bounded subset of $\mathbb{R}^N$, for $N \geq 2$, with Lipschitz boundary. We stress that for the results of Section 3 the Lipschitz regularity is not needed. By $\mathcal{H}^{N-1}(E)$ we mean the $(N-1)$-dimensional Hausdorff measure of a set E while $|E|$ stands for its N-dimensional Lebesgue measure. The space $\mathcal{M}(\Omega)$ is the usual one of Radon measures with finite total variation over Ω. Its local counterpart $\mathcal{M}_{\text{loc}}(\Omega)$ is the space of Radon measures which are locally finite in Ω.

For a fixed $k > 0$, we use the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ and $G_k : \mathbb{R} \to \mathbb{R}$ defined by

$$T_k(s) := \max(-k, \min(s, k)) \ \text{ and } \ G_k(s) := s - T_k(s). \tag{2.1}$$

Moreover we define

$$V_\delta(s) := \begin{cases} 1 & 0 \leq s \leq \delta, \\ \dfrac{2\delta - s}{\delta} & \delta < s < 2\delta, \\ 0 & s \geq 2\delta. \end{cases} \tag{2.2}$$

Finally, for a Banach space X we denote by $C_b^0(X)$ the space of bounded and continuous real functions on X.

If no otherwise specified, we denote by C several positive constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data but they will never depend on the indexes of the sequences we will gradually introduce. Let us explicitly mention that we will not relabel an extracted compact subsequence.

For simplicity's sake, and if there is no ambiguity, we will often use the following notation:

$$\int_\Omega f := \int_\Omega f(x)\, dx.$$

2.1. Basics on BV spaces

The Banach space of bounded variation functions on Ω is defined as:

$$BV(\Omega) := \{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega)^N\},$$

endowed with the norm

$$||u||_{BV(\Omega)} = \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} + \int_{\Omega} |Du|,$$

where $|Du|$ denotes the total variation of the measure Du. With L_u we mean the set of Lebesgue points of a function u, with $S_u := \Omega \setminus L_u$ and with J_u the jump set. Let recall that any function $u \in BV(\Omega)$ can be identified with its precise representative u^* which is the Lebesgue representative in L_u while $u^* = \frac{u^+ + u^-}{2}$ in J_u where u^+, u^- denote the approximate limits of u. Moreover it can be shown that $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ and that u^* is well defined $\mathcal{H}^{N-1}$-a.e.

Let us highlight that, once saying that $D^j u = 0$, we understand that $\mathcal{H}^{N-1}(J_u) = 0$, or, in other terms, that $Du = \tilde{D}u$ where $\tilde{D}u$ is the absolutely continuous part of Du with respect to the Lebesgue measure. In this case we will also denote by u instead of u^* the precise representative of u as no ambiguity is possible when integrating against a measure which is absolutely continuous with respect to $\mathcal{H}^{N-1}$.

2.2. The Anzellotti-Chen-Frid theory

In order to be self-contained we summarize the L^∞-divergence-measure vector fields theory due to [6] and [12]. We denote by

$$\mathcal{DM}^\infty(\Omega) := \{z \in L^\infty(\Omega)^N : \operatorname{div} z \in \mathcal{M}(\Omega)\},$$

and by $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ its local version, namely the space of bounded vector field z with $\operatorname{div} z \in \mathcal{M}_{\rm loc}(\Omega)$. We first recall that if $z \in \mathcal{DM}^\infty(\Omega)$ then $\operatorname{div} z$ is an absolutely continuous measure with respect to $\mathcal{H}^{N-1}$.
In [6] the following distribution $(z, Dv) : C^1_c(\Omega) \to \mathbb{R}$ is introduced:

$$\langle (z, Dv), \varphi \rangle := -\int_{\Omega} v^* \varphi \operatorname{div} z - \int_{\Omega} v z \cdot \nabla\varphi, \quad \varphi \in C^1_c(\Omega), \tag{2.3}$$

in order to define a generalized pairing between vector fields in $\mathcal{DM}^\infty(\Omega)$ and derivatives of BV functions. In [27] and [11], in fact, the authors prove that (z, Dv) is well defined if $z \in \mathcal{DM}^\infty(\Omega)$ and $v \in BV(\Omega) \cap L^\infty(\Omega)$ since one can show that $v^* \in L^\infty(\Omega, \operatorname{div} z)$. Moreover in [17] the authors show that (2.3) is well posed if $z \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $v \in BV_{\rm loc}(\Omega) \cap L^1_{\rm loc}(\Omega, \operatorname{div} z)$ and it holds that

$$|\langle (z, Dv), \varphi \rangle| \le ||\varphi||_{L^\infty(U)} ||z||_{L^\infty(U)^N} \int_U |Dv|,$$

for all open sets $U \subset\subset \Omega$ and for all $\varphi \in C_c^1(U)$. Moreover one has

$$\left| \int_B (z, Dv) \right| \le \int_B |(z, Dv)| \le ||z||_{L^\infty(U)^N} \int_B |Dv| , \tag{2.4}$$

for all Borel sets B and for all open sets U such that $B \subset U \subset \Omega$.

Observe that, if $z \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $w \in BV_{\rm loc}(\Omega) \cap L^\infty(\Omega)$, then

$$\operatorname{div}(wz) = (z, Dw) + w^* \operatorname{div} z , \tag{2.5}$$

so that $wz \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$.

We recall that in [6] it is proved that every $z \in \mathcal{DM}^\infty(\Omega)$ possesses a weak trace on $\partial\Omega$ of its normal component which is denoted by $[z, \nu]$, where $\nu(x)$ is the outward normal unit vector defined for $\mathcal{H}^{N-1}$-almost every $x \in \partial\Omega$. Moreover, it holds that

$$||[z, \nu]||_{L^\infty(\partial\Omega)} \le ||z||_{L^\infty(\Omega)^N} , \tag{2.6}$$

and also that, if $z \in \mathcal{DM}^\infty(\Omega)$ and $v \in BV(\Omega) \cap L^\infty(\Omega)$, then

$$v[z, \nu] = [vz, \nu],$$

(see [11]).
Finally the following Green formula holds (see [17]).

Lemma 2.1. *Let $z \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and set $\mu = \operatorname{div} z$. Let $v \in BV(\Omega) \cap L^\infty(\Omega)$ be such that $v^* \in L^1(\Omega, \mu)$. Then $vz \in \mathcal{DM}^\infty(\Omega)$ and the following holds:*

$$\int_\Omega v^* \, d\mu + \int_\Omega (z, Dv) = \int_{\partial\Omega} [vz, \nu] \, d\mathcal{H}^{N-1} . \tag{2.7}$$

Analogously to (2.6), it can be proved (see [17, Proposition 2.7]) that, for $z \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ such that the product $vz \in \mathcal{DM}^\infty(\Omega)$ for some $v \in BV(\Omega) \cap L^\infty(\Omega)$,

$$|[vz, \nu]| \le |v\llcorner_{\partial\Omega}| \, \|z\|_{L^\infty(\Omega)^N} \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{2.8}$$

When dealing with compositions with nonlinear functions it is sometimes useful to define a slightly "different" pairing measure as follows (see for instance [26]): let $\beta : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function and $v \in BV_{\rm loc}(\Omega)$, then we define

$$\beta(v)^\# := \begin{cases} \dfrac{1}{v^+ - v^-} \displaystyle\int_{v^-}^{v^+} \beta(s) \, ds & \text{if } x \in J_v, \\ \beta(v) & \text{otherwise.} \end{cases}$$

Observe that $\beta(v)^{\#}$ turns out to coincide with $\beta(v)^{*}$ on the jump set if and only if $\beta(s)=s$. As for (2.3), if $z\in\mathcal{DM}^{\infty}_{\rm loc}(\Omega)$ and $v\in BV_{\rm loc}(\Omega)$ is such that $\beta(v)\in BV_{\rm loc}(\Omega)\cap L^{\infty}_{\rm loc}(\Omega)$, it is possible to define the measure $(z,D\beta(v)^{\#})$ by

$$\langle (z,D\beta(v)^{\#}),\varphi\rangle := -\int_{\Omega}\beta(v)^{\#}\varphi\,\mathrm{div}\,z-\int_{\Omega}\beta(v)z\cdot\nabla\varphi,\quad \varphi\in C^1_c(\Omega). \tag{2.9}$$

By [26, Lemma 2.5], this new pairing $(z,D\beta(v)^{\#})$ is a well defined measure absolutely continuous with respect to $\mathcal{H}^{N-1}$, and moreover

$$\left|\int_B (z,D\beta(v)^{\#})\right|\le\int_B\left|(z,D\beta(v)^{\#})\right|\le ||z||_{L^\infty(U)^N}\int_B|D\beta(v)|, \tag{2.10}$$

for all Borel sets B and for all open sets U such that $B\subset U\subset\Omega$.

In what follows we will use the classical chain rule formula for functions in BV ([2, Theorem 3.99]).

Lemma 2.2. *Let $u\in BV(\Omega)$ and let $\Psi:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function. Then $v=\Psi(u)\in BV(\Omega)$ and*

$$Dv=\Psi'(u)^{\#}Du. \tag{2.11}$$

In particular, if $D^ju=0$, then $\tilde{D}v=\Psi'(u)\tilde{D}u$.

2.3. A general result on the jump part of a BV function

We show a general result showing that a function $w\in BV_{\rm loc}(\Omega)$ satisfying an inequality involving its gradient has no jump part. The proof is a suitable re-adaptation of an idea in [19].

Lemma 2.3. *Let α and β be two locally Lipschitz increasing functions on $\mathbb{R}$. Let $z\in\mathcal{DM}^{\infty}_{\rm loc}(\Omega)$ such that $\|z\|_{L^\infty(\Omega)^N}\le 1$, $w,\alpha(w)\in BV_{\rm loc}(\Omega)$, $\beta(w)\in BV_{\rm loc}(\Omega)\cap L^{\infty}_{\rm loc}(\Omega)$, and $\lambda\in L^1_{\rm loc}(\Omega)$. Moreover assume that*

$$-\mathrm{div}\,z+|D\alpha(w)|\le\lambda \tag{2.12}$$

and

$$(z,D\beta(w)^{\#})=|D\beta(w)|, \tag{2.13}$$

as measures. Then $D^jw=0$.

Proof. Since $\beta(w)\in BV_{\rm loc}(\Omega)$, by [2, Theorem 3.78], $S_{\beta(w)}$ is a (locally) countably $\mathcal{H}^{N-1}-$rectifiable set and then there exist regular hypersurfaces ξ_i such that $\mathcal{H}^{N-1}\left(S_{\beta(w)}\setminus\bigcup_{i=1}^{\infty}\xi_i\right)=0$.

By monotonicity of β, the proof follows once we prove that $|D\beta(w)|(\xi_i)=0$ for any $i\in\mathbb{N}$.

First observe that by [1, Proposition 3.4] one has that

$$\operatorname{div} z = [z,\nu]^+ - [z,\nu]^- \text{ on } \xi_i, \tag{2.14}$$

where $[z,\nu]^+$ and $[z,\nu]^-$ are the traces of the normal components of z over ξ_i defined as in [1, Definition 3.3].

Hence, using [14, Corollary 3.2], one has that

$$\begin{aligned}(z, D\beta(w)^\#) &= -\beta(w)^\# \operatorname{div} z + \operatorname{div}(\beta(w)z)\\ &= [z,\nu]^+(\beta(w)^+ - \beta(w)^\#) + [z,\nu]^-(\beta(w)^\# - \beta(w)^-)\\ &\le |\beta(w)^+ - \beta(w)^-| \text{ on } \xi_i,\end{aligned} \tag{2.15}$$

with the equality sign if and only if $[z,\nu]^+ = [z,\nu]^- = \operatorname{sgn}(\beta(w)^+ - \beta(w)^-)$.

Now, as

$$|D\beta(w)| = |\beta(w)^+ - \beta(w)^-| \text{ on } \xi_i,$$

from (2.13) we then get the equality sign in (2.15); so that

$$[z,\nu]^+ = [z,\nu]^- = \operatorname{sgn}(\beta(w)^+ - \beta(w)^-) \text{ on } \xi_i. \tag{2.16}$$

Observe that both $\alpha(w)$ and $\beta(w)$ share the same jump set so that we gather together (2.12), (2.14) and (2.16) to obtain that

$$|\alpha(w)^+ - \alpha(w)^-| = |D\alpha(w)| = 0 \text{ on } \xi_i,$$

which allows us to conclude that

$$|D\alpha(w)| = 0 \text{ on } \xi_i,$$

and, as α is strictly monotone, that finally $D^j w = 0$. $\square$

3. The case $p > 1$ for general monotone operators

As we mentioned, in this section we set, not only as a preparatory tool, the theory of existence and weak regularity of solutions for problems as in (1.1) for Leray-Lions nonlinear operators as leading terms.

Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be an open and bounded set (no further regularity is needed here), and let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying the classical Leray-Lions assumptions, i.e. there exist $\alpha, \beta > 0$ and $1 < p < N$ such that, for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and every $\xi, \xi' \in \mathbb{R}^N$:

$$a(x,s,\xi)\cdot\xi \ge \alpha|\xi|^p, \tag{3.1}$$

$$|a(x,s,\xi)| \le \beta\left(b(x) + |s|^{p-1} + |\xi|^{p-1}\right), \tag{3.2}$$

$$\big(a(x,s,\xi) - a(x,s,\xi')\big)\cdot(\xi - \xi') > 0; \tag{3.3}$$

let us recall that, by (3.1), one has $a(x,s,0)=0$, for any $s\in\mathbb{R}$ and a.e. $x\in\Omega$.

Consider the following boundary value problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u))+g(u)|\nabla u|^p=h(u)f & \text{in } \Omega,\\ u\geq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $g:[0,\infty)\to[0,\infty)$ is a continuous function such that

$$\liminf_{s\to\infty} g(s)>0. \tag{3.5}$$

Moreover $h:[0,\infty)\to[0,\infty]$ is a continuous function such that $h(0)>0$,

$$\exists\, 0\leq\gamma\leq 1, c_1, s_1>0: h(s)\leq\frac{c_1}{s^\gamma} \quad \text{for all } s\leq s_1, \tag{3.6}$$

and

$$h\in C_b^0([\delta,\infty)) \quad \forall\delta>0. \tag{3.7}$$

Let us stress that the classical case $g,h\equiv 1$ is covered by the above assumptions as γ could be zero and g does not necessarily satisfy $g(0)=0$.

As for the datum, we assume that $f\in L^1(\Omega)$ is nonnegative.

Let us introduce the notion of distributional solution for problem (3.4).

Definition 3.1. Let $1<p<N$ then a nonnegative $u\in W_0^{1,p}(\Omega)$ is a distributional solution to (3.4) if $a(x,u,\nabla u)\in L^1_{\mathrm{loc}}(\Omega)^N$, $g(u)|\nabla u|^p, h(u)f\in L^1_{\mathrm{loc}}(\Omega)$ and if it holds

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi+\int_\Omega g(u)|\nabla u|^p\varphi=\int_\Omega h(u)f\varphi, \quad \forall\varphi\in C_c^1(\Omega). \tag{3.8}$$

Remark 3.2. It is worth mentioning that all the terms appearing in the weak formulation of Definition 3.1 are well defined. In particular, as the vector field a satisfies (3.1)-(3.3), then $a(x,u,\nabla u)\in L^{p'}(\Omega)^N$ since $u\in W_0^{1,p}(\Omega)$, where $p'=\frac{p}{p-1}$ is the standard Hölder conjugate exponent of p.

Now we are ready to state the main result of this section.

Theorem 3.3. *Let a satisfy (3.1)-(3.3) with $1<p<N$. Let g satisfy (3.5) and let h satisfy (3.6)-(3.7). Finally let $f\in L^1(\Omega)$ be nonnegative. Then there exists a solution u to (3.4) in the sense of Definition 3.1 such that $g(u)|\nabla u|^p\in L^1(\Omega)$.*

3.1. Approximation scheme

In order to prove Theorem 3.3 we work by approximation. We will show the existence of a nonnegative solution for the following

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n)) + g_n(u_n)|\nabla u_n|^p = h_n(u_n) f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where

$$g_n(s) := \begin{cases} \min\{n, g(0)\} & \text{if } s \le 0, \\ T_n(g(s)) & \text{if } s > 0, \end{cases}$$

$f_n := T_n(f)$ and $h_n := T_n(h)$ with $n > 0$ ($T_n(s)$ is defined in (2.1)).

The proof is standard and it is based on an application of the Schauder fixed point. We will sketch it for the sake of completeness.

Lemma 3.4. *Let a satisfy (3.1)-(3.3) with $1 < p < N$, then there exists a nonnegative $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which satisfies*

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi + \int_\Omega g_n(u_n)|\nabla u_n|^p\varphi = \int_\Omega h_n(u_n) f_n\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Proof. We set

$$S : L^p(\Omega) \to L^p(\Omega),$$

as the map that, for any $v \in L^p(\Omega)$, gives the weak solution w to

$$\begin{cases} -\operatorname{div}(a(x,w,\nabla w)) + g_n(w)|\nabla w|^p = h_n(|v|) f_n & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

whose existence is guaranteed from [9, Theorem 1]. In particular $w \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ satisfies

$$\int_\Omega a(x,w,\nabla w)\cdot\nabla\varphi + \int_\Omega g_n(w)|\nabla w|^p\varphi = \int_\Omega h_n(|v|) f_n\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{3.11}$$

We claim that w is nonnegative; indeed, one can fix in (3.11) $\varphi = -w^- e^{-tw}$, for some $t > 0$ to be chosen later, and, here, $w^- \ge 0$ denotes the negative part of w. This yields to

$$-\int_\Omega a(x,w,\nabla w)\cdot\nabla w^- e^{-tw} + \int_\Omega t a(x,w,\nabla w)\cdot\nabla w w^- e^{-tw} - \int_\Omega g_n(w)|\nabla w|^p w^- e^{-tw}$$
$$= -\int_\Omega h_n(|v|) f_n w^- e^{-tw} \le 0,$$

which, recalling (3.1), implies

$$\alpha \int_\Omega |\nabla w^-|^p e^{-tw} + \int_\Omega |\nabla w|^p w^- e^{-tw}(\alpha t - g_n(w)) \le 0.$$

Hence it is sufficient requiring $t > \frac{n}{\alpha}$ in order to deduce that $w \ge 0$ almost everywhere in Ω.

Now we show that the map S has an invariant ball, it is continuous and relatively compact in $L^p(\Omega)$, so that the Schauder fixed point Theorem can be applied.

Let us fix $\varphi = w$ in (3.11) yielding to

$$\begin{aligned} \alpha \int_\Omega |\nabla w|^p &\overset{(3.1)}{\le} \int_\Omega a(x,w,\nabla w)\cdot\nabla w + \int_\Omega g_n(w)|\nabla w|^p w \\ &= \int_\Omega h_n(|v|) f_n w \le n^2 |\Omega|^{\frac{1}{p'}} \|w\|_{L^p(\Omega)}, \end{aligned} \tag{3.12}$$

after an application of the Hölder inequality on the right-hand. Using the Poincaré inequality on the left-hand, one deduces

$$\|w\|_{L^p(\Omega)} \le \left(\frac{c^p(p,\Omega)n^2}{\alpha}\right)^{\frac{1}{p-1}} |\Omega|^{\frac{1}{p}},$$

where $c(p,\Omega)$ is the Poincaré constant; observe that these estimates are independent of v. Thus, we can affirm that the ball of radius $\left(\frac{c^p(p,\Omega)n^2}{\alpha}\right)^{\frac{1}{p-1}} |\Omega|^{\frac{1}{p}}$ is invariant for S.

Moreover from (3.12) one deduces that

$$\|w\|_{W_0^{1,p}(\Omega)} \le C, \tag{3.13}$$

where C is independent of v. This is sufficient to deduce that $S(L^p(\Omega))$ is relatively compact in $L^p(\Omega)$ by Rellich-Kondrachov Theorem.

It is left to show that S is continuous in $L^p(\Omega)$. Let consider $v_k \in L^p(\Omega)$ converging to $v \in L^p(\Omega)$ as $k \to \infty$.

If we denote by $w_k = S(v_k)$ then w_k is bounded in $W_0^{1,p}(\Omega)$ with respect to k thanks to (3.13). Moreover, exists $w \in W_0^{1,p}(\Omega)$ to which w_k, up to subsequences, converges weakly in $W_0^{1,p}(\Omega)$. Now, as $h_n(|v_k|)f_n \le n^2$ by the classical Stampacchia's argument (see for instance [9, Lemma 2]), one has that $w_k \le C$ almost everywhere in Ω where C is independent of k, i.e. $w \in L^\infty(\Omega)$.

Now we need to show that $w = S(v)$; i.e. we need to pass to the limit with respect to k in the following formulation

$$\int_\Omega a(x,w_k,\nabla w_k)\cdot\nabla\phi + \int_\Omega g_n(w_k)|\nabla w_k|^p\phi = \int_\Omega h_n(|v_k|)f_n\phi, \ \ \forall \phi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{3.14}$$

For the right-hand of (3.14) one can apply the Lebesgue Theorem since $h_n(|v_k|)f_n\varphi \le n^2\varphi \in L^1(\Omega)$. Now observe that, once one proves that w_k converges strongly to w as $k \to \infty$ in

$W_0^{1,p}(\Omega)$, one can safely pass to the limit on the left-hand of (3.14). Indeed it will be sufficient recall that g_n is bounded and that a satisfies (3.2).

The proof of the strong convergence in $W_0^{1,p}(\Omega)$ of w_k is also quite classical under the above assumptions. Anyway, for the sake of completeness, we will sketch it.

Let consider $\varphi_\rho(s) := se^{\rho s^2}$ $(\rho > 0)$ which satisfies

$$\eta\varphi_\rho'(s) - \mu|\varphi_\rho(s)| \geq \frac{\eta}{2}, \quad \forall s \in \mathbb{R}, \quad \forall \eta, \mu > 0, \quad \forall \rho \geq \frac{\mu^2}{4\eta^2}. \tag{3.15}$$

We fix $\phi = \varphi_\rho(u_k)$ in (3.14) where $u_k := w_k - w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then we have

$$\int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla u_k \varphi_\rho'(u_k) + \int_\Omega g_n(w_k)|\nabla w_k|^p \varphi_\rho(u_k) = \int_\Omega h_n(|v_k|) f_n \varphi_\rho(u_k). \tag{3.16}$$

By (3.1) we find

$$-\int_\Omega g_n(w_k)|\nabla w_k|^p \varphi_\rho(u_k) \leq n \int_\Omega |\nabla w_k|^p |\varphi_\rho(u_k)| \leq \frac{n}{\alpha} \int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla w_k |\varphi_\rho(u_k)|,$$

which gives

$$\begin{aligned} -\int_\Omega g_n(w_k)|\nabla w_k|^p \varphi_\rho(u_k) \leq \frac{n}{\alpha} &\int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla u_k |\varphi_\rho(u_k)| \\ + \frac{n}{\alpha} &\int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla w |\varphi_\rho(u_k)|. \end{aligned} \tag{3.17}$$

Observe now that, since $a(x, w_k, \nabla w_k)$ is bounded in $L^{p'}(\Omega)^N$ and since $\nabla w|\varphi_\rho(u_k)|$ strongly converges to zero in $L^p(\Omega)^N$ as $k \to \infty$, one has

$$\lim_{k\to\infty} \int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla w |\varphi_\rho(u_k)| = 0. \tag{3.18}$$

Moreover, an application of the Lebesgue Theorem gives that

$$\lim_{k\to\infty} \int_\Omega h_n(|v_k|) f_n \varphi_\rho(u_k) = 0. \tag{3.19}$$

Gathering together (3.16), (3.17), (3.18), and (3.19) we have

$$\int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla u_k \varphi_\rho'(u_k) \leq \frac{n}{\alpha} \int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla u_k |\varphi_\rho(u_k)| + \omega_k,$$

where ω_k denotes an infinitesimal quantity as $k \to \infty$.

Thus, using (3.15) fixing $\rho = \frac{n^2}{4\alpha^2}$, one has

$$\int_\Omega a(x, w_k, \nabla w_k) \cdot \nabla u_k \le \omega_k,$$

which implies

$$\int_\Omega (a(x, w_k, \nabla w_k) - a(x, w_k, \nabla w)) \cdot \nabla(w_k - w) \le - \int_\Omega a(x, w_k, \nabla w) \cdot \nabla u_k + \omega_k. \quad (3.20)$$

It follows from (3.2) and from the fact that w_k converges to w in $L^p(\Omega)$ as $k \to \infty$ that $a(x, w_k, \nabla w)$ strongly converges in $L^{p'}(\Omega)^N$ to $a(x, w, \nabla w)$ as $k \to \infty$. Moreover, as u_k weakly converges to 0 in $W_0^{1,p}(\Omega)$ as $k \to \infty$, from (3.20) and (3.3) one has that

$$\lim_{k\to\infty} \int_\Omega (a(x, w_k, \nabla w_k) - a(x, w_k, \nabla w)) \cdot \nabla(w_k - w) = 0,$$

which allows to apply Lemma 5 of [9] in order to deduce that

$$w_k \to w \quad \text{strongly in } W_0^{1,p}(\Omega).$$

This is sufficient to conclude that $w = S(v)$, i.e. S is continuous.

We finally apply the Schauder fixed point theorem to conclude that S has a nonnegative fixed point $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ that is a solution to (3.10). □

3.2. A priori estimates

In this section we collect all the estimates from which one could derive the existence of a limit function for u of u_n.

Lemma 3.5. *Let a satisfy (3.1)-(3.3) with $1 < p < N$, let u_n be a nonnegative solution to problem (3.9). Then u_n is bounded in $W_0^{1,p}(\Omega)$ with respect to n. Moreover $g_n(u_n)|\nabla u_n|^p$ is bounded in $L^1(\Omega)$ and $h_n(u_n) f_n$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$ with respect to n.*

Proof. We start proving that u_n is bounded in $W_0^{1,p}(\Omega)$. Let us first observe that, from (3.5), there exists $\overline{k} > 0$ such that for all $s \in [\overline{k}, \infty)$ one has $g(s) \ge \eta > 0$ for some $\eta > 0$. Then we choose $T_{\overline{k}}(u_n)$ as test function in the weak formulation of (3.9) in order to deduce that

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla T_{\overline{k}}(u_n) + \int_\Omega g_n(u_n)|\nabla u_n|^p T_{\overline{k}}(u_n) = \int_\Omega h_n(u_n) f_n T_{\overline{k}}(u_n),$$

which implies that

$$\int_{\{u_n\le\overline{k}\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n + \int_{\{u_n\le\overline{k}\}} g_n(u_n)|\nabla u_n|^p u_n + \overline{k}\int_{\{u_n>\overline{k}\}} g_n(u_n)|\nabla u_n|^p$$
$$\le \int_{\{u_n\le s_1\}} h_n(u_n) f_n u_n + \overline{k}\int_{\{u_n>s_1\}} h_n(u_n) f_n.$$

Without loosing generality let assume $n\ge\eta$ and, using that $g_n(u_n)>\eta$ on $\{u_n>\overline{k}\}$, (3.1) and (3.6), one yields to

$$\alpha\int_{\{u_n\le\overline{k}\}}|\nabla u_n|^p+\overline{k}\eta\int_{\{u_n>\overline{k}\}}|\nabla u_n|^p\le c_1\int_{\{u_n\le s_1\}}u_n^{1-\gamma}f_n+\overline{k}\sup_{s\in[s_1,\infty)}h(s)\int_{\{u_n>s_1\}}f_n.$$

From the previous it follows

$$\min\{\alpha,\overline{k}\eta\}\int_\Omega|\nabla u_n|^p\le\left(c_1 s_1^{1-\gamma}+\overline{k}\sup_{s\in[s_1,\infty)}h(s)\right)\|f\|_{L^1(\Omega)},$$

namely u_n is bounded in $W_0^{1,p}(\Omega)$ with respect to n since $\gamma\le 1$ and thanks to (3.7).

Now we focus on proving the L^1-estimate on $g_n(u_n)|\nabla u_n|^p$ in n. Let us take $T_1(u_n)$ as test function in the weak formulation of (3.9), obtaining

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_1(u_n)+\int_\Omega g_n(u_n)|\nabla u_n|^p T_1(u_n)=\int_\Omega h_n(u_n)f_nT_1(u_n),$$

which, recalling (3.1) and (3.6), gives

$$\begin{aligned}\int_{\{u_n\ge 1\}} g_n(u_n)|\nabla u_n|^p &\le c_1\int_{\{u_n\le s_1\}}u_n^{1-\gamma}f_n+\sup_{s\in[s_1,\infty)}h(s)\int_{\{u_n>s_1\}}f_n\\ &\le\left(c_1 s_1^{1-\gamma}+\sup_{s\in[s_1,\infty)}h(s)\right)\|f\|_{L^1(\Omega)}.\end{aligned}\tag{3.21}$$

Moreover one can observe that

$$\int_{\{u_n<1\}} g_n(u_n)|\nabla u_n|^p\le\max_{s\in[0,1]}g(s)\int_\Omega|\nabla u_n|^p\le C,\tag{3.22}$$

where C does not depend on n since u_n is bounded in $W_0^{1,p}(\Omega)$ with respect to n. Then it follows from (3.21) and (3.22) that $g_n(u_n)|\nabla u_n|^p$ is bounded in $L^1(\Omega)$ with respect to n.

We finally show that $h_n(u_n)f_n$ is bounded in $L^1_{\text{loc}}(\Omega)$ with respect to n.

Let consider $0 \le \varphi \in C_c^1(\Omega)$ as a test function in the weak formulation of (3.9), obtaining

$$\int_\Omega h_n(u_n) f_n \varphi = \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \varphi + \int_\Omega g_n(u_n)|\nabla u_n|^p \varphi \le C,$$

where C does not depend on n. Indeed we have already proven that $g_n(u_n)|\nabla u_n|^p$ is bounded in $L^1(\Omega)$. Moreover, as u_n is bounded in $W_0^{1,p}(\Omega)$ and thanks to (3.2), it is easy to check that $a(x, u_n, \nabla u_n)$ is bounded in $L^{p'}(\Omega)^N$ with respect to n. $\square$

In the next lemma we show the existence of a limit function for u_n with respect to n and we show that any truncation of u_n strongly converges in $W_0^{1,p}(\Omega)$ as $n \to \infty$.

Lemma 3.6. *Let a satisfy (3.1)-(3.3) with $1 < p < N$, let u_n be a nonnegative solution to problem (3.9). Then there exists $u \in W_0^{1,p}(\Omega)$ to which u_n, up to subsequences, converges as $n \to \infty$ almost everywhere in Ω. Moreover $T_k(u_n)$ converges, up to subsequences, strongly in $W_0^{1,p}(\Omega)$ to $T_k(u)$ as $n \to \infty$ for any $k > 0$. Finally $g(u)|\nabla u|^p \in L^1(\Omega)$ and $h(u)f \in L^1_{\mathrm{loc}}(\Omega)$.*

Proof. The existence of a limit function u follows from a standard compactness argument once that Lemma 3.5 is in force. From the same lemma one also has that $h_n(u_n)f_n$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$ with respect to n; then an application of the Fatou Lemma as $n \to \infty$ gives that $h(u)f \in L^1_{\mathrm{loc}}(\Omega)$.

To show the strong convergence of $T_k(u_n)$ in n, we re-adapt a classical idea of [8].

We recall that the function $\varphi_\rho(s) := se^{\rho s^2}$ ($\rho > 0$) satisfies (3.15) and we define for any $k > 0$

$$w_{n,k} := T_k(u_n) - T_k(u)\,.$$

We take $\varphi_\rho(w_{n,k})$ as a test function in the weak formulation of (3.9); one has

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla w_{n,k} \varphi_\rho'(w_{n,k}) + \int_\Omega g_n(u_n)|\nabla u_n|^p \varphi_\rho(w_{n,k}) = \int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}). \tag{3.23}$$

We can write the first term on the left-hand of the previous as

$$\begin{aligned}
&\int_{\{u_n \le k\}} a(x, u_n, \nabla u_n) \cdot \nabla w_{n,k} \varphi_\rho'(w_{n,k}) - \int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \cdot \nabla T_k(u) \varphi_\rho'(w_{n,k}) \\
&\ge \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla w_{n,k} \varphi_\rho'(w_{n,k}) \\
&- \beta \int_{\{u_n > k\}} \left(b(x) + |u_n|^{p-1} + |\nabla u_n|^{p-1}\right) |\nabla T_k(u)||\varphi_\rho'(w_{n,k})|\,,
\end{aligned}$$

where in the last step we used (3.2). Gathering the previous inequality into (3.23) one has

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u))\varphi_\rho'(w_{n,k}) + \int_\Omega g_n(u_n)|\nabla u_n|^p \varphi_\rho(w_{n,k})$$
$$\leq \int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}) + \beta \int_{\{u_n>k\}} \left(b(x) + |u_n|^{p-1} + |\nabla u_n|^{p-1}\right) |\nabla T_k(u)||\varphi_\rho'(w_{n,k})|.$$

Since $|\nabla T_k(u)|\chi_{\{u_n>k\}} \to 0$ strongly in $L^p(\Omega)$ as $n \to \infty$ while $\beta \left(b(x) + |u_n|^{p-1} + |\nabla u_n|^{p-1}\right)$ $|\varphi_\rho'(w_{n,k})|$ is bounded in $L^{p'}(\Omega)$ since u_n is bounded in $W_0^{1,p}(\Omega)$ with respect to n, then the last term in previous inequality tends to zero as n tends to infinity. Hence,

$$\begin{aligned}&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u))\varphi_\rho'(w_{n,k}) \\ &\leq -\int_\Omega g_n(u_n)|\nabla u_n|^p \varphi_\rho(w_{n,k}) + \int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}) + \omega_n,\end{aligned} \tag{3.24}$$

where, again, ω_n is a quantity that tends to 0 as $n \to \infty$.

Now observe that for the first term on the right-hand of (3.24), one has

$$-\int_\Omega g_n(u_n)|\nabla u_n|^p \varphi_\rho(w_{n,k}) \leq \int_{\{u_n\leq k\}} g_n(u_n)|\nabla u_n|^p |\varphi_\rho(w_{n,k})|,$$

since $\varphi_\rho(w_{n,k}) \geq 0$ on $\{u_n > k\}$. Furthermore, using (3.1), one deduces

$$-\int_\Omega g_n(u_n)|\nabla u_n|^p \varphi_\rho(w_{n,k}) \leq \frac{\max_{s\in[0,k]} g(s)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)|\varphi_\rho(w_{n,k})|.$$

Since $a(x, T_k(u_n), \nabla T_k(u_n))$ is bounded in $L^{p'}(\Omega)^N$ while $\nabla T_k(u)|\varphi_\rho(w_{n,k})|$ strongly converges to zero in $L^p(\Omega)^N$ as $n \to \infty$, one yields to

$$\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u)|\varphi_\rho(w_{n,k})| = 0,$$

which implies

$$\begin{aligned}&-\int_\Omega g_n(u_n)|\nabla u_n|^p \varphi_\rho(w_{n,k}) \\ &\leq \frac{\max_{s\in[0,k]} g(s)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u))|\varphi_\rho(w_{n,k})| + \omega_n.\end{aligned}$$

Then gathering the previous into (3.24) one has

$$
\begin{aligned}
&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u))\varphi'_\rho(w_{n,k})\\
&\le \frac{\max_{s\in[0,k]} g(s)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u))|\varphi_\rho(w_{n,k})|\\
&+ \int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}) + \omega_n,
\end{aligned}
$$

which, using (3.15) with $\rho = \left(\frac{\max_{s\in[0,k]} g(s)}{2\alpha}\right)^2$, implies that

$$
\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u)) \le 2 \int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}) + \omega_n. \tag{3.25}
$$

Now observe that it follows from (3.2) and from having u_n weakly converging in $W_0^{1,p}(\Omega)$ that

$$
\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla (T_k(u_n) - T_k(u)) = 0.
$$

Then by adding and subtracting this quantity into (3.25), one has

$$
\begin{aligned}
&\int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot \nabla (T_k(u_n) - T_k(u))\\
&\le 2 \int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}) + \omega_n.
\end{aligned}
$$

We claim that the right-hand of the previous inequality converges to 0 as $n \to \infty$, for every fixed $k > 0$. If $h(0) < \infty$ then this passage to the limit follows from the convergence in $L^1(\Omega)$ of f_n coupled with the *-weak convergence of $\varphi_\rho(w_{n,k})$ to zero in $L^\infty(\Omega)$.

Hence, we assume $h(0) = \infty$. We fix $0 < \delta < s_1$. Then, using (3.6) one yields to

$$
\int_\Omega h_n(u_n) f_n \varphi_\rho(w_{n,k}) \le c_1 \int_{\{u_n\le\delta\}} \frac{f_n}{u_n^\gamma} \varphi_\rho(w_{n,k}) + \sup_{s\in[\delta,\infty)} h(s) \int_{\{u_n>\delta\}} f_n \varphi_\rho(w_{n,k}). \tag{3.26}
$$

The second term on the right-hand of (3.26) converges to 0 as $n \to \infty$ since f_n converges in $L^1(\Omega)$ while $\varphi_\rho(w_{n,k})$ converges *-weakly in $L^\infty(\Omega)$ to 0 as $n \to \infty$. For the first term of (3.26) one reason as

$$
c_1 \int_{\{u_n\le\delta\}} \frac{f_n}{u_n^\gamma} \varphi_\rho(w_{n,k}) \le 2c_1 \int_{\{u_n\le\delta\}} \delta^{1-\gamma} f \, e^{\rho w_{n,k}^2}.
$$

Applying the Lebesgue Theorem we can say that (here we may assume $\delta \le 1$)

$$\lim_{n\to\infty} 2c_1 \int_{\{u_n\le\delta\}} \delta^{1-\gamma} f\, e^{\rho w_{n,k}^2} = 2c_1 \int_{\{u\le\delta\}} \delta^{1-\gamma} f \le 2c_1 \int_{\{u\le\delta\}} f,$$

since $\gamma \le 1$. We have already shown that $h(u)f \in L^1_{\mathrm{loc}}(\Omega)$. This implies that $\{u=0\} \subset \{f=0\}$ up to a set of zero Lebesgue measure, which gives

$$\lim_{\delta\to 0} \int_{\{u\le\delta\}} f = \int_{\{u=0\}} f = 0.$$

This allows to deduce that for every $k>0$

$$\lim_{n\to\infty} \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) = 0,$$

which is sufficient to apply [9, Lemma 5], deducing

$$T_k(u_n) \to T_k(u) \text{ strongly in } W^{1,p}_0(\Omega),$$

for every $k>0$. This also implies that ∇u_n converges almost everywhere in Ω to ∇u as $n\to\infty$. Moreover, as Lemma 3.6 guarantees that $g_n(u_n)|\nabla u_n|^p$ is bounded in $L^1(\Omega)$ with respect to n, an application of the Fatou Lemma in n gives that $g(u)|\nabla u|^p \in L^1(\Omega)$. This concludes the proof. □

Remark 3.7. In Lemma 3.6 we have shown $T_k(u_n)$ converges, up to subsequences, strongly in $W^{1,p}_0(\Omega)$ to $T_k(u)$ as $n\to\infty$ for any $k>0$. From this fact we have that ∇u_n converges almost everywhere in Ω to ∇u as $n\to\infty$. This can be used to deduce below that $a(x,u_n,\nabla u_n)$ strongly converges to $a(x,u,\nabla u)$ in $L^{p'}(\Omega)^N$ as $n\to\infty$ thanks to (3.2).

3.3. *Proof of the existence result*

We are now ready to prove the main existence result of this section, i.e. Theorem 3.3.

Proof of Theorem 3.3. Let u_n be a solution to (3.9) whose existence is guaranteed by Lemma 3.4. Moreover it follows from Lemma 3.6 that there exists $u \in W^{1,p}_0(\Omega)$ which is, up to subsequences, the almost everywhere limit in Ω of u_n as $n\to\infty$. Moreover the same lemma gives that $g(u)|\nabla u|^p \in L^1(\Omega)$ and that $h(u)f \in L^1_{\mathrm{loc}}(\Omega)$.

To conclude the proof, it remains to show that u satisfies (3.8), namely that one can pass to the limit with respect to n:

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi + \int_\Omega g_n(u_n)|\nabla u_n|^p\varphi = \int_\Omega h_n(u_n) f_n\varphi, \tag{3.27}$$

where $\varphi \in C^1_c(\Omega)$.

It is a consequence of both (3.2) and Lemma 3.5 that $a(x,u_n,\nabla u_n)$ strongly converges to $a(x,u,\nabla u)$ in $L^{p'}(\Omega)^N$ as $n\to\infty$. This is sufficient to take $n\to\infty$ in the first term of (3.27) (see also Remark 3.7).

For the second term we show the equi-integrability of the sequence $g_n(u_n)|\nabla u_n|^p$.

To this aim we introduce the following function:

$$S_{\eta,k}(s) := \begin{cases} 0 & s\le k, \\ \frac{s-k}{\eta} & k<s<k+\eta, \\ 1 & s\ge k+\eta, \end{cases}$$

where $k>0$ is a fixed parameter. Let us take $S_{\eta,k}(u_n)$ with $k>0$ as a test function in the weak formulation of (3.9), yielding to

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n S'_{\eta,k}(u_n) + \int_\Omega g_n(u_n)|\nabla u_n|^p S_{\eta,k}(u_n) = \int_\Omega h_n(u_n) f_n S_{\eta,k}(u_n),$$

which, thanks to (3.1), implies

$$\int_\Omega g_n(u_n)|\nabla u_n|^p S_{\eta,k}(u_n) \le \sup_{s\in[k,\infty)} h(s) \int_{\{u_n>k\}} f.$$

We take $\eta\to 0$ applying the Fatou Lemma, obtaining

$$\int_{\{u_n>k\}} g_n(u_n)|\nabla u_n|^p \le \sup_{s\in[k,\infty)} h(s) \int_{\{u_n>k\}} f.$$

Recalling that $T_k(u_n)$ strongly converges in $W_0^{1,p}(\Omega)$ with respect to n and that g is continuous, the former inequality gives the equi-integrability of the sequence $g_n(u_n)|\nabla u_n|^p$ with respect to n. This fact, jointly with

$$g_n(u_n)|\nabla u_n|^p \to g(u)|\nabla u|^p \text{ a.e. in } \Omega \text{ as } n\to\infty,$$

and recalling that $g(u)|\nabla u|^p\in L^1(\Omega)$, allows to apply the Vitali Theorem to deduce that

$$g_n(u_n)|\nabla u_n|^p \to g(u)|\nabla u|^p \text{ strongly converges in } L^1(\Omega) \text{ as } n\to\infty.$$

The proof is then concluded once we pass to the limit in the right-hand side of (3.27) for any $0\le\varphi\in C_c^1(\Omega)$, as the case of φ with general sign will easily follow. If $h(0)<\infty$ the passage to the limit is trivial. Hence, without loss of generality, we assume that $h(0)=\infty$.

We choose $\delta>0$ such that $\delta\notin\{\eta : |\{u=\eta\}|>0\}$ and we split (3.27) as follows

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi + \int_\Omega g_n(u_n)|\nabla u_n|^p\varphi = \int_{\{u_n\le\delta\}} h_n(u_n)f_n\varphi + \int_{\{u_n>\delta\}} h_n(u_n)f_n\varphi. \quad (3.28)$$

We first show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\int\limits_{\{u_n\le\delta\}} h_n(u_n)f_n\varphi = 0. \tag{3.29}$$

Let fix $V_\delta(u_n)\varphi$ (V_δ is defined in (2.2)) with $0\le\varphi\in C^1_c(\Omega)$ as test function in the weak formulation of (3.9), and we deduce

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\left(V_\delta(u_n)\varphi\right) + \int\limits_\Omega g_n(u_n)|\nabla u_n|^p V_\delta(u_n)\varphi = \int\limits_\Omega h_n(u_n)f_n V_\delta(u_n)\varphi.$$

Then one has

$$\begin{aligned}\int\limits_{\{u_n\le\delta\}} h_n(u_n)f_n\varphi &\le \int\limits_\Omega h_n(u_n)f_n V_\delta(u_n)\varphi \\ &\le \int\limits_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi V_\delta(u_n) + \int\limits_\Omega g_n(u_n)|\nabla u_n|^p V_\delta(u_n)\varphi.\end{aligned}$$

Now we can take the limsup as $n\to\infty$ in the previous inequality; indeed for the first term on the right-hand one recalls Remark 3.7 and the fact that $V_\delta\le 1$. For the second term on the right-hand we have already proven that $g_n(u_n)|\nabla u_n|^p$ strongly converges in $L^1(\Omega)$ to $g(u)|\nabla u|^p$ as $n\to\infty$. Then one has

$$\limsup_{n\to\infty}\int\limits_{\{u_n\le\delta\}} h_n(u_n)f_n\varphi \le \int\limits_\Omega a(x,u,\nabla u)\cdot\nabla\varphi V_\delta(u) + \int\limits_\Omega g(u)|\nabla u|^p V_\delta(u)\varphi. \tag{3.30}$$

Now let $\delta\to 0$ in (3.30) applying the Lebesgue Theorem, deducing that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\int\limits_{\{u_n\le\delta\}} h_n(u_n)f_n\varphi \le \int\limits_{\{u=0\}} a(x,u,\nabla u)\cdot\nabla\varphi + \int\limits_{\{u=0\}} g(u)|\nabla u|^p\varphi = 0,$$

since $u\in W^{1,p}_0(\Omega)$ and $a(x,0,0)=0$ for all $x\in\Omega$ (recall $\nabla u=0$ a.e. on $\{u=0\}$).

Now we focus on the second term on the right-hand of (3.28). Observing that

$$h_n(u_n)f_n\varphi\chi_{\{u_n>\delta\}} \le \sup_{s\in[\delta,\infty)} h(s)f\varphi \in L^1(\Omega)$$

then one can take $n\to\infty$, yielding to

$$\lim_{n\to\infty}\int\limits_{\{u_n>\delta\}} h_n(u_n)f_n\varphi = \int\limits_{\{u>\delta\}} h(u)f\varphi,$$

since $|\{u=\delta\}|=0$.

Moreover, as $h(u)f \in L^1_{\text{loc}}(\Omega)$, one can take $\delta \to 0$

$$\lim_{\delta\to 0}\lim_{n\to\infty} \int_{\{u_n>\delta\}} h_n(u_n) f_n \varphi = \int_{\{u>0\}} h(u) f \varphi = \int_\Omega h(u) f \varphi, \tag{3.31}$$

where the last equality follows from the fact $\{u = 0\} \subset \{f = 0\}$ up to a set of zero Lebesgue measure since $h(u)f$ is locally integrable.

Finally observe that (3.29) and (3.31) allow us to pass to the limit as $n \to \infty$, for fixed $\delta > 0$, and then as $\delta \to 0$ in (3.28) and the proof is concluded. □

Remark 3.8. In Theorem 3.3 we found a distributional solution $u \in W^{1,p}_0(\Omega)$ satisfying $g(u)|\nabla u|^p \in L^1(\Omega)$. Then it is worth mentioning that in this case we can extend the class of test function given in (3.8) to $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

Indeed, for any $0 \le v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, there exists $\varphi_n \in C^1_c(\Omega)$ such that $\varphi_n \to v$ strongly in $W^{1,p}_0(\Omega)$ as $n \to \infty$. Moreover let ρ_η be a standard mollifier. We note that $\psi_{n,\eta} = \rho_\eta * \min\{v, \varphi_n\} \in C^1_c(\Omega)$ for $\eta > 0$ small enough, hence it is an admissible test function for the problem (3.4), so we can write

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla \psi_{n,\eta} + \int_\Omega g(u)|\nabla u|^p \psi_{n,\eta} = \int_\Omega h(u) f \psi_{n,\eta}. \tag{3.32}$$

We recall that $\psi_{n,\eta} \to \psi_n = \min\{\varphi_n, v\}$ as $\eta \to 0$ strongly in $W^{1,p}(\Omega)$ and $*$-weak in $L^\infty(\Omega)$. Then, as $a(x, u, \nabla u) \in L^{p'}(\Omega)^N$, $g(u)|\nabla u|^p \in L^1(\Omega)$ and $h(u)f \in L^1_{\text{loc}}(\Omega)$, one can take $\eta \to 0$ in (3.32) obtaining

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla \psi_n + \int_\Omega g(u)|\nabla u|^p \psi_n = \int_\Omega h(u) f \psi_n. \tag{3.33}$$

Now note that $\psi_n \to v$ strongly in $W^{1,p}(\Omega)$ and $*$-weak in $L^\infty(\Omega)$ as $n \to \infty$, so we can take $n \to \infty$ in the first two terms of (3.33).

For the term on the right-hand of (3.33) one can reason as follows. Firstly observe that an application of the Fatou Lemma with respect to n gives that

$$\int_\Omega h(u) f v \le \liminf_{n\to\infty} \int_\Omega h(u) f \psi_n = \int_\Omega a(x, u, \nabla u) \cdot \nabla v + \int_\Omega g(u)|\nabla u|^p v,$$

whose right-hand is finite. Then, as $h(u)fv \in L^1(\Omega)$, one can apply the Lebesgue Theorem to obtain

$$\lim_{n\to\infty} \int_\Omega h(u) f \psi_n = \int_\Omega h(u) f v.$$

Therefore, as the case of a function v with generic sign easily follows, we have proven that the solution u to (3.4) found in Theorem 3.3 satisfies

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v+\int_\Omega g(u)|\nabla u|^p v=\int_\Omega h(u)fv,$$

for all $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

4. Main assumptions and existence result for $p=1$

In this section we address the limit case $p=1$. In particular we are interested in proving existence of nonnegative solutions to the following Dirichlet boundary value problem

$$\begin{cases}-\Delta_1 u+g(u)|Du|=h(u)f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{4.1}$$

where f is a positive function in $L^1(\Omega)$, $g:[0,\infty)\to[0,\infty)$ is a positive, bounded and continuous function such that (3.5) is in force. The function $h:[0,\infty)\to[0,\infty]$ is continuous and possibly singular with $h(0)\neq 0$, it is finite outside the origin and such that (3.6) and (3.7) hold.

Here is how the notion of solution for problem (4.1) is intended.

Definition 4.1. Let $0<f\in L^1(\Omega)$. A nonnegative $u\in BV(\Omega)$ is a solution of problem (4.1) if $D^j u=0$, $g(u)\in L^1_{\rm loc}(\Omega,|Du|)$ and $h(u)f\in L^1_{\rm loc}(\Omega)$ and if there exists a vector field $z\in\mathcal{DM}^\infty_{\rm loc}(\Omega)$, with $\|z\|_{L^\infty(\Omega)^N}\le 1$ satisfying

$$-\operatorname{div} z+g(u)|Du|=h(u)f \quad \text{in } \mathcal{D}'(\Omega),\tag{4.2}$$

$$(z,DT_k(u))=|DT_k(u)| \quad \text{as measures in } \Omega \text{ for any } k>0,\tag{4.3}$$

and

$$u(x)=0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x\in\partial\Omega.\tag{4.4}$$

Remark 4.2. Let us spend a few words on Definition 4.1. Formula (4.3) is the weak way in which the vector field z plays the role of the singular quotient $Du|Du|^{-1}$. Hence the (4.2) and (4.3) represent the weak way the first term in (4.1) is intended.

Furthermore, the boundary datum is given by (4.4) which is something strongly related to the presence of g. Indeed, it is classical nowadays that solutions to 1-Laplace Dirichlet problems do not attain, in general, the boundary datum pointwise when $g\equiv 0$, and in this case, a weaker condition involving $[z,\nu]$ is usually required (see for instance [3,4,16]).

Let us finally explicitly observe that if $h(0)=\infty$, as $h(u)f\in L^1_{\rm loc}(\Omega)$, then, again, $\{u=0\}\subset\{f=0\}$. So that in this case, as $f>0$, then $u>0$.

We define the following function which will be widely used in the sequel

$$\Gamma_p(s):=\int_0^s g^{\frac{1}{p}}(\sigma)\,d\sigma.$$

Moreover, we denote by $\Gamma(s) := \Gamma_1(s)$. Let explicitly observe that, as g is bounded, $\Gamma(s)$ is a Lipschitz function satisfying assumptions of Lemma 2.2. In Section 5.2 we briefly discuss the case of a g possibly unbounded at infinity.

A very similar reasoning to the one of [26, Remark 3.4] gives the following result.

Proposition 4.3. *Let u be a solution of the problem (4.1) in the sense of Definition 4.1, then*

$$-\operatorname{div}(ze^{-\Gamma(u)}) = h(u) f e^{-\Gamma(u)} \quad \text{in } \mathcal{D}'(\Omega). \tag{4.5}$$

Proof. By (4.3) and [23, Proposition 3.3] we have that

$$\theta(z, D(1-e^{-\Gamma(u)}), x) = 1 \quad |D(1-e^{-\Gamma(u)})|\text{–a.e. in } \Omega,$$

where $\theta(z, Dv, x)$ is the Radon–Nikodým derivative of (z, Dv) with respect to $|Dv|$ provided $v \in BV(\Omega)$. Consequently, for all Borel sets $B \subset \Omega$,

$$\int_B (z, D(1-e^{-\Gamma(u)})) = \int_B \theta(z, D(1-e^{-\Gamma(u)}), x)\, |D(1-e^{-\Gamma(u)})| = \int_B |D(1-e^{-\Gamma(u)})|.$$

Therefore

$$(z, D(1-e^{-\Gamma(u)})) = |D(1-e^{-\Gamma(u)})| \quad \text{as Radon measures in } \Omega. \tag{4.6}$$

On the other hand, by (2.5), (4.6), (4.2) and Lemma 2.2, we have

$$\begin{aligned}
-\operatorname{div}(e^{-\Gamma(u)} z) &= \operatorname{div}((1-e^{-\Gamma(u)})z) - \operatorname{div} z = (z, D(1-e^{-\Gamma(u)})) + (1-e^{-\Gamma(u)}) \operatorname{div} z - \operatorname{div} z \\
&= |D(1-e^{-\Gamma(u)})| - (e^{-\Gamma(u)}) \operatorname{div} z = (e^{-\Gamma(u)}) g(u)|Du| - (e^{-\Gamma(u)})(-h(u)f + g(u)|Du|) \\
&= e^{-\Gamma(u)} h(u) f,
\end{aligned}$$

i.e. we obtain (4.5). □

Let us then state the main result of this section.

Theorem 4.4. *Let g be positive, bounded and satisfying (3.5) and let h satisfy (3.6) and (3.7). Finally let $0 < f \in L^1(\Omega)$. Then there exists a solution to (4.1) in the sense of Definition 4.1.*

4.1. Approximation scheme and existence of a limit function

The proof of Theorem 4.4 will be presented as an application of a series of lemmas. We introduce the following approximation scheme:

$$\begin{cases} -\Delta_p u_p + g(u_p)|\nabla u_p|^p = h(u_p) f & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

whose existence of $u_p \in W_0^{1,p}(\Omega)$ in the sense of Definition 3.1 has been proven in Theorem 3.3. Let us explicitly observe that, as long as we deal with the solution found in the mentioned theorem, Remark 3.8 is in force; this means that the set of test functions is enlarged to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We firstly look for some uniform estimates on u_p for $p > 1$ small enough. Without loss of generality and for the sake of exposition, by *uniformly bounded with respect to* p we mean the existence of $p_0 > 1$ with some estimate holding for any $1 < p \le p_0$.

Lemma 4.5. *Let g be positive, bounded and satisfying* (3.5)*, let h satisfy* (3.6) *and* (3.7) *and let $0 < f \in L^1(\Omega)$. Let u_p be the solution to* (4.7) *obtained in Theorem 3.3. Then u_p and $\Gamma_p(u_p)$ are uniformly bounded with respect to p in $BV(\Omega)$. Moreover there exists $u \in BV(\Omega)$ such that u_p converges to u (up to a subsequence) in $L^q(\Omega)$ for every $q < \frac{N}{N-1}$ and ∇u_p converges to Du *-weakly as measures. Finally, $\Gamma(u) \in BV(\Omega)$ and $h(u)f \in L^1_{\text{loc}}(\Omega)$.*

Proof. We observe again that, as (3.5) is in force, there exists $\overline{k} > 0$ such that for all $s \in [\overline{k}, \infty)$ one has $g(s) \ge \eta > 0$ for some $\eta > 0$.

We choose $T_{\overline{k}}(u_p) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in the weak formulation of (4.7) (recall Remark 3.8), so that

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla T_{\overline{k}}(u_p) + \int_\Omega g(u_p)|\nabla u_p|^p T_{\overline{k}}(u_p) = \int_\Omega h(u_p) f T_{\overline{k}}(u_p),$$

from which

$$\min\{\alpha, \overline{k}\eta\} \int_\Omega |\nabla u_p|^p \le \left(c_1 s_1^{1-\gamma} + \sup_{s\in[s_1,\infty)} h(s)\overline{k} \right) \|f\|_{L^1(\Omega)}. \tag{4.8}$$

It follows from (4.8) and from an application of the Young inequality that

$$\int_\Omega |\nabla u_p| \le \frac{1}{p}\int_\Omega |\nabla u_p|^p + \frac{1}{p'}|\Omega| \le \frac{1}{p\min\{\alpha, \overline{k}\eta\}} \left(c_1 s_1^{1-\gamma} + \sup_{s\in[s_1,\infty[} h(s)\overline{k} \right) \|f\|_{L^1(\Omega)} + \frac{1}{p'}|\Omega|,$$

which shows that u_p is bounded in $BV(\Omega)$ with respect to p since the right-hand of the previous is bounded with respect to p and u_p has zero trace on $\partial\Omega$.

Then standard compactness result for BV functions assures that there exists $u \in BV(\Omega)$ such that, up to a subsequence, u_p converges in $L^q(\Omega)$ for any $q < \frac{N}{N-1}$, almost everywhere in Ω and ∇u_p converges to Du *-weakly as measures as $p \to 1^+$.

To show that $\Gamma(u_p)$ is bounded in $BV(\Omega)$ it is sufficient to reason as for (3.21) and (3.22) in order to deduce

$$\int_\Omega g(u_p)|\nabla u_p|^p \le C, \tag{4.9}$$

where C is a constant independent of p, and to use the Young inequality to get

$$\int_\Omega |\nabla\Gamma_p(u_p)| = \int_\Omega g(u_p)^{\frac{1}{p}}|\nabla u_p| \le \int_\Omega g(u_p)|\nabla u_p|^p + |\Omega| \le C + |\Omega|; \tag{4.10}$$

observe that, in particular, by weak lower semicontinuity one has that $\Gamma(u) \in BV(\Omega)$.

Now we focus on showing that $h(u)f$ is locally integrable. We choose $0 \le \varphi \in C_c^1(\Omega)$ as a test function in the weak formulation of (4.7); this yields to

$$\begin{aligned} \int_\Omega h(u_p)f\varphi &= \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi + \int_\Omega g(u_p)|\nabla u_p|^p\varphi \\ &\le \frac{1}{p'}\int_\Omega |\nabla u_p|^p + \frac{1}{p}\int_\Omega |\nabla\varphi|^p + \int_\Omega g(u_p)|\nabla u_p|^p\varphi. \end{aligned} \tag{4.11}$$

As the right-hand of (4.11) is bounded with respect to p thanks to (4.8) and (4.9), one has that $h(u_p)f$ is locally bounded in $L^1(\Omega)$ with respect to p.

Finally an application of the Fatou Lemma as $p \to 1^+$ gives that $h(u)f \in L^1_{\mathrm{loc}}(\Omega)$. The proof is concluded. □

In the next Lemma we find a vector field z which is the weak limit of $|\nabla u_p|^{p-2}\nabla u_p$ as $p \to 1^+$.

Lemma 4.6. *Let g be positive, bounded and satisfying (3.5), let h satisfy (3.6) and (3.7) and let $0 < f \in L^1(\Omega)$. Moreover, let u be the function found in Lemma 4.5. Then there exists a vector field $z \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ with $\|z\|_{L^\infty(\Omega)^N} \le 1$, such that*

$$ze^{-\Gamma(u)} \in \mathcal{DM}^\infty(\Omega), \tag{4.12}$$

$$-\operatorname{div}(ze^{-\Gamma(u)}) = h(u)fe^{-\Gamma(u)} \quad \text{in } \mathcal{D}'(\Omega), \tag{4.13}$$

and

$$-\operatorname{div} z + |D\Gamma(u)| \le h(u)f \quad \text{in } \mathcal{D}'(\Omega). \tag{4.14}$$

Proof. Let u_p be the solution to (4.7) obtained in Theorem 3.3. Then it follows from (4.8) and from an application of the Hölder inequality that, for $1 \le q < p'$, it holds

$$\begin{aligned} \int_\Omega \left||\nabla u_p|^{p-2}\nabla u_p\right|^q &= \int_\Omega |\nabla u_p|^{q(p-1)} \le \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{q(p-1)}{p}} |\Omega|^{1-\frac{q(p-1)}{p}} \\ &\le C^{\frac{q(p-1)}{p}}|\Omega|^{1-\frac{q(p-1)}{p}}. \end{aligned} \tag{4.15}$$

Hence $|\nabla u_p|^{p-2}\nabla u_p$ is bounded in $L^q(\Omega)^N$ with respect to p and there exists $z_q \in L^q(\Omega)^N$ such that $|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z_q$ weakly in $L^q(\Omega)^N$, for all $q < \infty$. Moreover it follows from the lower semicontinuity in (4.15) with respect to p that

$$\|z\|_{L^q(\Omega)^N} \le |\Omega|^{\frac{1}{q}}, \quad \forall q < \infty,$$

and thus letting $q \to \infty$ then $z \in L^\infty(\Omega)^N$ with $\|z\|_{L^\infty(\Omega)^N} \le 1$.

Let us now show (4.13); let us take $e^{-\Gamma(u_p)}\varphi$ as a test function in the weak formulation of (4.7) where $0 \le \varphi \in C_c^1(\Omega)$, yielding to

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi e^{-\Gamma(u_p)} = \int_\Omega h(u_p) f e^{-\Gamma(u_p)}\varphi.$$

We can pass to the limit in the left-hand of the previous since $e^{-\Gamma(u_p)}|\nabla u_p|^{p-2}\nabla u_p$ converges to $e^{-\Gamma(u)}z$ in $L^q(\Omega)^N$ for any $q < \infty$ as $p \to 1^+$. This shows that

$$\lim_{p\to 1^+} \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi e^{-\Gamma(u_p)} = \int_\Omega z \cdot \nabla\varphi\, e^{-\Gamma(u)}.$$

For the right-hand we distinguish two cases: if h is finite at the origin then the passage to the limit is trivial. Hence, without losing generality we assume that $h(0) = \infty$. We first split the integral as

$$\int_\Omega h(u_p) f e^{-\Gamma(u_p)}\varphi = \int_{\{u_p \le \delta\}} h(u_p) f e^{-\Gamma(u_p)}\varphi + \int_{\{u_p > \delta\}} h(u_p) f e^{-\Gamma(u_p)}\varphi, \tag{4.16}$$

where $\delta \notin \{\eta : |\{u = \eta\}| > 0\}$ which is at most a countable set.

Observe that it follows from Lemma 4.5 that $h(u)f$ is locally integrable. Since $h(0) = \infty$ and $f > 0$, then $u > 0$ almost everywhere in Ω. Moreover, since

$$\chi_{\{u_p > \delta\}} h(u_p) f e^{-\Gamma(u_p)}\varphi \le \sup_{s\in[\delta,\infty)} h(s) f\varphi \in L^1(\Omega),$$

and

$$\chi_{\{u > \delta\}} h(u) f e^{-\Gamma(u)}\varphi \le h(u) f\varphi \in L^1(\Omega),$$

one can apply twice the Lebesgue Theorem, deducing that

$$\lim_{\delta\to 0}\lim_{p\to 1^+} \int_{\{u_p > \delta\}} h(u_p) f e^{-\Gamma(u_p)}\varphi = \lim_{\delta\to 0} \int_{\{u > \delta\}} h(u) f e^{-\Gamma(u)}\varphi \overset{u>0}{=} \int_\Omega h(u) f e^{-\Gamma(u)}\varphi. \tag{4.17}$$

Now we analyze the first term on the right-hand of (4.16), we fix $V_\delta(u_p)e^{-\Gamma(u_p)}\varphi$ (V_δ is defined in (2.2)) with $0 \le \varphi \in C_c^1(\Omega)$ as test function in the weak formulation of (4.7), obtaining

$$\int_\Omega |\nabla u_p|^p V_\delta'(u_p) e^{-\Gamma(u_p)}\varphi - \int_\Omega g(u_p)|\nabla u_p|^p V_\delta(u_p) e^{-\Gamma(u_p)}$$

$$+\int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi V_\delta(u_p)e^{-\Gamma(u_p)} + \int_\Omega g(u_p)|\nabla u_p|^p V_\delta(u_p)e^{-\Gamma(u_p)}\varphi$$
$$=\int_\Omega h(u_p)fV_\delta(u_p)e^{-\Gamma(u_p)}\varphi,$$

which implies

$$\int_{\{u_p\le\delta\}} h(u_p)fe^{-\Gamma(u_p)}\varphi \le \int_\Omega h(u_p)fV_\delta(u_p)e^{-\Gamma(u_p)}\varphi \le \int_\Omega |\nabla u_p|^{p-2}\nabla u_p\cdot\nabla\varphi V_\delta(u_p)e^{-\Gamma(u_p)}.$$

Through the Lebesgue Theorem we deduce

$$\lim_{\delta\to 0}\limsup_{p\to 1^+}\int_{\{u_p\le\delta\}} h(u_p)fe^{-\Gamma(u_p)}\varphi \le \int_{\{u=0\}} z\cdot\nabla\varphi e^{-\Gamma(u)} \overset{u\ge 0}{=} 0. \tag{4.18}$$

From (4.17) and (4.18), one gets

$$\lim_{p\to 1^+}\int_\Omega h(u_p)fe^{-\Gamma(u_p)}\varphi = \int_\Omega h(u)fe^{-\Gamma(u)}\varphi. \tag{4.19}$$

Hence we have shown (4.13).

Observe that as $h(u)fe^{-\Gamma(u)}\ge 0$ then, by [18, Lemma 2.3], $ze^{-\Gamma(u)}\in\mathcal{DM}^\infty(\Omega)$, namely (4.12).

Finally we show (4.14). Recalling (4.10) one has that, up to subsequences, $\Gamma_p(u_p)\to\Gamma(u)$ a.e. as $p\to 1^+$, and

$$\int_\Omega |D\Gamma(u)|\varphi \le \liminf_{p\to 1^+}\int_\Omega |\nabla\Gamma_p(u_p)|^p\varphi \le C,$$

by weak lower semicontinuity (again we also used Young's inequality as for (4.10)). Observe that C does not depend on p thanks to (4.8).

Hence this allows to take $p\to 1^+$ in (3.4) obtaining

$$\int_\Omega z\cdot\nabla\varphi + \int_\Omega |D\Gamma(u)|\varphi \le \int_\Omega h(u)f\varphi, \tag{4.20}$$

where for the right-hand we have reasoned analogously as to proven (4.19). Indeed, one has that ($\delta\le 1$)

$$\lim_{\delta\to 0}\limsup_{p\to 1^+}\int_{\{u_p\le\delta\}} h(u_p)f\varphi \le \lim_{\delta\to 0}\limsup_{p\to 1^+}\frac{1}{e^{-\Gamma(1)}}\int_{\{u_p\le\delta\}} h(u_p)fe^{-\Gamma(u_p)}\varphi = 0.$$

Let also note that (4.20) gives $z\in\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$. This concludes the proof. □

4.2. Identification of the vector field z and boundary datum

For the next two lemmas we need to define the following function

$$\tilde{\Gamma}_p(s) := \int_0^s (T_k(\sigma)g(\sigma))^{\frac{1}{p}}\, d\sigma, \tag{4.21}$$

where $\tilde{\Gamma}(s) := \tilde{\Gamma}_1(s)$.

The next result clarifies the role of z.

Lemma 4.7. *Let g be positive, bounded and satisfying (3.5), let h satisfy (3.6) and (3.7) and let $0 < f \in L^1(\Omega)$. Moreover, let u be the function found in Lemma 4.5 and let z be the vector field found in Lemma 4.6. It holds both*

$$-\operatorname{div} z + g(u)|Du| = h(u)f \quad \text{in } \mathcal{D}'(\Omega), \tag{4.22}$$

and

$$-T_k(u)\operatorname{div} z + T_k(u)|D\Gamma(u)| = h(u)fT_k(u) \ \text{ in } \mathcal{D}'(\Omega) \text{ for any } k > 0. \tag{4.23}$$

Finally it also holds

$$D^j u = 0, \tag{4.24}$$

and

$$(z, DT_k(u)) = |DT_k(u)| \ \text{ as measures in } \Omega \text{ for any } k > 0. \tag{4.25}$$

Proof. We highlight that here we need to make use of the new pairing introduced in (2.9) applied to $\beta(s) = -e^{-s}$ and $v = \Gamma(u)$.

One has

$$\begin{aligned}(e^{-\Gamma(u)})^{\#}|D\Gamma(u)| &\overset{(4.14)}{\le} (e^{-\Gamma(u)})h(u)f + (e^{-\Gamma(u)})^{\#}\operatorname{div} z \overset{(4.13)}{=} -\operatorname{div}(e^{-\Gamma(u)}z) + (e^{-\Gamma(u)})^{\#}\operatorname{div} z \\ &\overset{(2.9)}{=} \left(z, D[(-e^{-\Gamma(u)})^{\#}]\right) \overset{(2.10)}{\le} |D(-e^{-\Gamma(u)})| \overset{(2.11)}{=} (e^{-\Gamma(u)})^{\#}|D\Gamma(u)|\,,\end{aligned} \tag{4.26}$$

which implies that all the inequalities in (4.26) are actually equalities.

In particular,

$$\left(z, D[(-e^{-\Gamma(u)})^{\#}]\right) = |D(-e^{-\Gamma(u)})|\,.$$

Recalling (4.14), we observe that one can apply Lemma 2.3 with $\alpha(s) = s$, $\beta(s) = -e^{-s}$, and $w = \Gamma(u)$ in order to deduce that $|D^j\Gamma(u)| = 0$ and, as Γ is increasing, that (4.24) holds.

Now we want to show the reverse inequality of (4.14) in order to get (4.22). For a fixed $\varepsilon > 0$ and for any $k > 0$ one has

$$
\begin{aligned}
e^{-\Gamma(T_k(u))}|D\Gamma(u)| &\overset{(4.14)}{\leq} e^{-\Gamma(T_k(u))}(h(u)f + \operatorname{div} z) \\
&\overset{(4.13)}{=} \frac{e^{-\Gamma(T_k(u))}}{e^{-\Gamma(u)}+\varepsilon}\Big(-\operatorname{div}(e^{-\Gamma(u)}z) + (e^{-\Gamma(u)})\operatorname{div} z\Big) \\
&+ \varepsilon e^{-\Gamma(T_k(u))}(h(u)f + \operatorname{div} z) \overset{(2.3)}{=} \frac{e^{-\Gamma(T_k(u))}}{e^{-\Gamma(u)}+\varepsilon}\Big(z, D[(-e^{-\Gamma(u)})]\Big) + \eta_\varepsilon \\
&\overset{(2.4)}{\leq} \frac{e^{-\Gamma(T_k(u))}}{e^{-\Gamma(u)}+\varepsilon}|D(-e^{-\Gamma(u)})| + \eta_\varepsilon \overset{(2.11)}{\leq} e^{-\Gamma(T_k(u))}|D\Gamma(u)| + \eta_\varepsilon\,,
\end{aligned}
\tag{4.27}
$$

where $\eta_\varepsilon = \varepsilon e^{-\Gamma(T_k(u))}(h(u)f + \operatorname{div} z)$ is a sequence of measures that vanish as $\varepsilon \to 0$. So that, letting ε go to zero in (4.27) one gets a chain of equalities between measures that in particular implies that for any $k > 0$

$$
e^{-\Gamma(T_k(u))}(-\operatorname{div} z + |D\Gamma(u)|) = e^{-\Gamma(T_k(u))}h(u)f \ \text{ in } \mathcal{D}'(\Omega).
$$

As $e^{-\Gamma(T_k(u))} > 0$ one in particular gets the reverse inequality of (4.14), so that

$$
-\operatorname{div} z + |D\Gamma(u)| = h(u)f \quad \text{in } \mathcal{D}'(\Omega)\,,
\tag{4.28}
$$

which, in turn, by applying Lemma 2.2 gives (4.22).

To prove (4.23) we test (4.28) with $(\rho_\epsilon * T_k(u))\varphi$ where $k > 0$, $0 \leq \varphi \in C^1_c(\Omega)$ and ρ_ϵ is a sequence of smooth mollifiers. For ε small enough, this takes to

$$
-\int_\Omega (\rho_\epsilon * T_k(u))\varphi \operatorname{div} z + \int_\Omega (\rho_\epsilon * T_k(u))\varphi|D\Gamma(u)| = \int_\Omega h(u)f(\rho_\epsilon * T_k(u))\varphi.
$$

Now observe that $(\rho_\epsilon * T_k(u))$ converges $\mathcal{H}^{N-1}$ a.e. to $T_k(u)^*$ as $\epsilon \to 0$ and $T_k(u)^* \leq k$. Then, as it follows from Lemma 4.5 that $h(u)f \in L^1_{\rm loc}(\Omega)$ and $\Gamma(u) \in BV(\Omega)$, one can take $\epsilon \to 0$ applying the Lebesgue Theorem. This implies that (4.23) holds.

It is left to show (4.25); we take $T_k(u_p)\varphi$ with $0 \leq \varphi \in C^1_c(\Omega)$ as a test function in the weak formulation of (4.7); this takes to

$$
\int_\Omega |\nabla T_k(u_p)|^p\varphi + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi T_k(u_p) + \int_\Omega |\nabla\tilde{\Gamma}_p(u_p)|^p\varphi = \int_\Omega h(u_p)fT_k(u_p)\varphi,
\tag{4.29}
$$

where $\tilde{\Gamma}_p$ is defined in (4.21). Now observe that an application of the Young inequality gives

$$
\begin{aligned}
\int_\Omega |\nabla T_k(u_p)|\varphi + \int_\Omega |\nabla\tilde{\Gamma}_p(u_p)|\varphi \leq \frac{1}{p}\int_\Omega |\nabla T_k(u_p)|^p\varphi \\
+ \frac{1}{p}\int_\Omega |\nabla\tilde{\Gamma}(u_p)|^p\varphi + \frac{2(p-1)}{p}\int_\Omega \varphi.
\end{aligned}
\tag{4.30}
$$

Hence gathering (4.30) into (4.29) yields to

$$\begin{aligned}&\int_\Omega |\nabla T_k(u_p)|\varphi + \int_\Omega |\nabla \tilde{\Gamma}_p(u_p)|\varphi \\ &\le \int_\Omega h(u_p) f T_k(u_p)\varphi - \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi T_k(u_p) + \frac{2(p-1)}{p}\int_\Omega \varphi.\end{aligned} \tag{4.31}$$

Hence we can take the liminf as $p \to 1^+$ in (4.31) using weak lower semicontinuity for the left-hand. For the first term on the right-hand one can use the Lebesgue Theorem since ($\delta < s_1 < k$)

$$h(u_p) f T_k(u_p) \le c_1 \delta^{1-\gamma} f \chi_{\{u_p \le \delta\}} + k \sup_{s\in[\delta,\infty)} h(s) f \chi_{\{u_p > \delta\}},$$

which is strongly compact in $L^1(\Omega)$ with respect to p. The second term on the right-hand passes to the limit while the third term degenerates as $p \to 1^+$. Hence one has

$$\begin{aligned}\int_\Omega |DT_k(u)|\varphi + \int_\Omega |D\tilde{\Gamma}(u)|\varphi &\le \int_\Omega h(u) f T_k(u)\varphi - \int_\Omega z\cdot\nabla\varphi T_k(u) \\ &\overset{(4.23)}{=} -\int_\Omega T_k(u)\,\mathrm{div}\, z\varphi + \int_\Omega |D\tilde{\Gamma}(u)|\varphi - \int_\Omega z\cdot\nabla\varphi T_k(u) \\ &= \int_\Omega (z, DT_k(u))\varphi + \int_\Omega |D\tilde{\Gamma}(u)|\varphi,\end{aligned}$$

where we also got advantage of $D^j u = 0$, by writing $T_k(u)|D\Gamma(u)| = |D\tilde{\Gamma}(u)|$ In particular this means

$$\int_\Omega |DT_k(u)|\varphi \le \int_\Omega (z, DT_k(u))\varphi,$$

and, being the reverse inequality trivial, this shows (4.25).

The proof is concluded. □

Remark 4.8. We explicitly remark that the request of positivity on g is needed to deduce that Γ is an increasing function. It is worth mentioning that Theorem 4.4 continues to hold in case g is only nonnegative but Γ is still a well defined increasing function.

Finally we deal with the boundary datum.

Lemma 4.9. *Let g be positive, bounded and satisfying (3.5), let h satisfy (3.6) and (3.7) and let $0 < f \in L^1(\Omega)$. Moreover, let u be the function found in Lemma 4.5 and let z be the vector field found in Lemma 4.6. Then $u(x) = 0$ $\mathcal{H}^{N-1}$ almost everywhere in Ω.*

Proof. Let u_p be the solution to (4.7) obtained in Theorem 3.3. Then let us take $T_k(u_p)$ as test function in (4.7), obtaining

$$\int_\Omega |\nabla T_k(u_p)|^p + \int_\Omega T_k(u_p)g(u_p)|\nabla u_p|^p = \int_\Omega h(u_p)fT_k(u_p).$$

For the right-hand once again one observes that ($k > \delta$)

$$h(u_p)fT_k(u_p) \le \delta^{1-\gamma} f\chi_{\{u_p\le\delta\}} + k \sup_{s\in[\delta,\infty)} h(s)f\chi_{\{u_p>\delta\}},$$

which is strongly compact in $L^1(\Omega)$ with respect to p. This allows to apply the generalized Lebesgue Theorem for the right-hand. Hence one can take the liminf as $p \to 1^+$ in the previous; indeed one can use weak lower semicontinuity on the left-hand after an application of the Young inequality, recalling also that u_p is zero on the boundary of Ω.

This proves that

$$\int_\Omega |DT_k(u)| + \int_{\partial\Omega} T_k(u)\, d\mathcal{H}^{N-1} + \int_\Omega |D\tilde\Gamma(u)| + \int_{\partial\Omega} \tilde\Gamma(u)\, d\mathcal{H}^{N-1} \le \int_\Omega h(u)fT_k(u), \tag{4.32}$$

where $\tilde\Gamma$ is defined in (4.21).

Since $h(u)fT_k(u) \in L^1(\Omega)$ and $\Gamma(u) \in BV(\Omega)$, one has

$$\begin{aligned}\int_\Omega h(u)fT_k(u) &\overset{(4.23)}{=} -\int_\Omega T_k(u)\operatorname{div} z + \int_\Omega |D\tilde\Gamma(u)| \\ &\overset{(2.7)}{=} \int_\Omega (z, DT_k(u)) - \int_{\partial\Omega} [T_k(u)z,\nu]\, d\mathcal{H}^{N-1} + \int_\Omega |D\tilde\Gamma(u)| \\ &= \int_\Omega |DT_k(u)| - \int_{\partial\Omega} [T_k(u)z,\nu]\, d\mathcal{H}^{N-1} + \int_\Omega |D\tilde\Gamma(u)|.\end{aligned} \tag{4.33}$$

Then gathering (4.33) into (4.32), one yields to

$$\int_{\partial\Omega} ([T_k(u)z,\nu] + T_k(u))\, d\mathcal{H}^{N-1} + \int_{\partial\Omega} \tilde\Gamma(u)\, d\mathcal{H}^{N-1} = 0,$$

which, since $|[T_k(u)z,\nu]| \le T_k(u)$ on $\partial\Omega$ (recall (2.8)), it gives that $\tilde\Gamma(u)$ (and so u) is identically null on $\partial\Omega$. This concludes the proof. □

As consequence of the previous results we can now prove Theorem 4.4.

Proof of Theorem 4.4. Let u_p be the solution to (4.7) obtained in Theorem 3.3. Then the proof follows from Lemmas 4.5, 4.6, 4.7 and 4.9. □

5. Some extensions and remarks

5.1. The case of a nonnegative datum f

Up to now we have required the positivity of the datum f. Now we want to consider the case of a datum f which is only nonnegative; i.e. we consider

$$\begin{cases} -\Delta_1 u + g(u)|Du| = h(u)f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with $f \in L^1(\Omega)$ being a nonnegative function, and h, g as before.

Let explicitly underline that the extension of Theorem 4.4 is straightforward in the case $h(0) < \infty$ and f nonnegative the proof being de facto the one already presented. Hence, without loosing generality, here we assume $h(0) = \infty$.

As we will see the existence of a solution can be obtained with some technical modifications both in the definition of solution of (5.1), which needs to be properly intended, and in the proof that is a suitable adaptation of the one of Theorem 4.4.

Here is how the notion of solution to (5.1) has to be intended.

Definition 5.1. A nonnegative $u \in BV(\Omega)$ is a solution to (5.1) if $\chi_{\{u>0\}} \in BV_{\rm loc}(\Omega)$, $D^j u = 0$, $g(u) \in L^1_{\rm loc}(\Omega, |Du|)$, $h(u)f \in L^1_{\rm loc}(\Omega)$, and if there exists $z \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ with $||z||_{L^\infty(\Omega)^N} \le 1$ such that

$$-\chi^*_{\{u>0\}} \operatorname{div} z + g(u)|Du| = h(u)f \quad \text{in } \mathcal{D}'(\Omega), \tag{5.2}$$

$$(z, DT_k(u)) = |DT_k(u)| \quad \text{as measures in } \Omega \text{ for any } k > 0, \tag{5.3}$$

$$u(x) = 0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega. \tag{5.4}$$

Remark 5.2. As one can see the main difference with respect to Definition 4.1 consists in the presence of the characteristic function $\chi_{\{u>0\}}$ in (5.2) which is a natural request as, in this case, we cannot infer $u > 0$ from $h(u)f \in L^1_{\rm loc}(\Omega)$ as in Remark 4.2

Let us state the existence result for this section.

Theorem 5.3. *Let g be positive, bounded and satisfying* (3.5) *and let h satisfy* (3.6) *and* (3.7) *with $h(0) = \infty$. Finally let $0 \le f \in L^1(\Omega)$. Then there exists a solution to* (5.1) *in the sense of Definition 5.1.*

Sketch of the proof. Here we only highlight the authentic differences with the proof of Theorem 4.4. We consider $u_p \in W^{1,p}_0(\Omega)$, solution to the approximating problems in (4.7) and whose existence is proven in Theorem 3.3.

First observe that Lemma 4.5 continues to hold in this case. Therefore, there exists a nonnegative limit function $u \in BV(\Omega)$ for u_p, as $p \to 1^+$, such that $g(u) \in L^1(\Omega, |Du|)$ and $h(u)f \in L^1_{\rm loc}(\Omega)$. Moreover, reasoning as in the first part of the proof of Lemma 4.6, one gains the existence of a bounded vector field z such that $|z| \le 1$ in Ω with $|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z$ weakly in

$L^q(\Omega)^N$ for all $q<\infty$. Let also underline that, as $-\Delta_p u_p = h(u_p)f - g(u_p)|\nabla u_p|^p$ is bounded with respect to p as measures, so that one deduces that $z \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$.

Now we focus on showing both $D^j u = 0$ and (5.2).

Let $0 \le \varphi \in C^1_c(\Omega)$ and let us take $(1 - V_\delta(u_p))\varphi$ (V_δ is defined in (2.2)) as a test function in the weak formulation of (4.7). Then, taking the liminf as $p \to 1^+$, one gains $\chi_{\{u>0\}} \in BV_{\rm loc}(\Omega)$, yielding to

$$-\operatorname{div}(z\chi_{\{u>0\}}) + |D\chi_{\{u>0\}}| + \chi^*_{\{u>0\}}|D\Gamma(u)| \le h(u)f\chi_{\{u>0\}}, \tag{5.5}$$

in $\mathcal{D}'(\Omega)$. Moreover, as $|z| \le 1$ in Ω, one has

$$-\operatorname{div}(z\chi_{\{u>0\}}) + |D\chi_{\{u>0\}}| \ge -\operatorname{div}(z\chi_{\{u>0\}}) + (z, D\chi_{\{u>0\}}) \stackrel{(2.5)}{=} -\chi^*_{\{u>0\}}\operatorname{div} z,$$

which, gathered into (5.5), gives

$$-\chi^*_{\{u>0\}}\operatorname{div} z + \chi^*_{\{u>0\}}|D\Gamma(u)| \le h(u)f\chi_{\{u>0\}}, \tag{5.6}$$

in $\mathcal{D}'(\Omega)$. Let us stress that (5.5) gives that $z\chi_{\{u>0\}} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ (recall (2.5)).

Now let $0 \le \varphi \in C^1_c(\Omega)$ and let us take $e^{-\Gamma(u_p)}\varphi$ as a test function in the weak formulation of (4.7). Then, taking the liminf as $p \to 1^+$ applying the Fatou Lemma, it allows to deduce

$$-\operatorname{div}(ze^{-\Gamma(u)}) \ge h(u)fe^{-\Gamma(u)}. \tag{5.7}$$

One has

$$\begin{aligned}
(e^{-\Gamma(u)})^\# \chi^*_{\{u>0\}}|D\Gamma(u)| &\stackrel{(5.6)}{\le} e^{-\Gamma(u)}h(u)f\chi_{\{u>0\}} + (e^{-\Gamma(u)})^\#\chi^*_{\{u>0\}}\operatorname{div} z \\
&\stackrel{(5.7)}{\le} -\operatorname{div}(ze^{-\Gamma(u)})\chi^*_{\{u>0\}} + (e^{-\Gamma(u)})^\#\chi^*_{\{u>0\}}\operatorname{div} z \\
&\stackrel{(2.9)}{=} \chi^*_{\{u>0\}}\left(z, D(-e^{-\Gamma(u)})^\#\right) \\
&\stackrel{(2.10)}{\le} \chi^*_{\{u>0\}}|De^{-\Gamma(u)}| = (e^{-\Gamma(u)})^\#\chi^*_{\{u>0\}}|D\Gamma(u)|,
\end{aligned}$$

where in the last equality we used Lemma 2.2. This proves that

$$\chi^*_{\{u>0\}}\left(z, D(-e^{-\Gamma(u)})^\#\right) = \chi^*_{\{u>0\}}|De^{-\Gamma(u)}|. \tag{5.8}$$

Now, as in the proof of Lemma 4.7, we want to apply an easy variation of Lemma 2.3. In fact, as $\chi^*_{\{u>0\}} > 0$ $\mathcal{H}^{N-1}$-a.e. on $J_{\Gamma(u)}$, then the very same proof is still valid, and then, using both (5.6) and (5.8) one deduces that $D^j u = 0$, and that (5.6) is equivalent to

$$-\chi^*_{\{u>0\}}\operatorname{div} z + |D\Gamma(u)| \le h(u)f\chi_{\{u>0\}}. \tag{5.9}$$

From now on the proof follows step-by-step the proof of Lemma 4.7; in particular, a suited version of (4.27) involving $\chi^*_{\{u>0\}}$ holds allowing us to prove the validity of (5.2). Also as for (4.23) one readily gets

$$-T_k(u)\chi^*_{\{u>0\}}\operatorname{div} z + T_k(u)|D\Gamma(u)| = h(u)fT_k(u) \ \text{ in } \mathcal{D}'(\Omega) \text{ for any } k>0.$$

In particular, this means that $T_k(u) \in L^1(\Omega, \operatorname{div} z)$.

The proof of (5.3) is not affected by the sign of the datum and follows as for (4.25).

Finally one has that Lemma 4.9 applies without any modification in the proof. Indeed, as $T_k(u) \in L^1(\Omega, \operatorname{div} z)$, one uses that $T_k(u)z \in \mathcal{DM}^\infty(\Omega)$ from Lemma 2.1. This shows (5.4). The proof is concluded. □

5.2. The case of a nonnegative g possibly blowing-up at infinity

In Section 4 we only dealt with positive bounded functions g; this has been necessary to apply Lemma 2.2, i.e. to deduce

$$|D\Gamma(u)| = g(u)|Du|.$$

By the way one can suitably modify Definition 4.1 in order to gain the existence of a solution in a weaker sense. Let explicitly fix the notion of solution in which, due to what we just said, do not need to ask for $D^j u = 0$.

Definition 5.4. Let $0 < f \in L^1(\Omega)$. A nonnegative $u \in BV(\Omega)$ is a solution to (5.1) if $\Gamma(u) \in BV_{\text{loc}}(\Omega)$, $h(u)f \in L^1_{\text{loc}}(\Omega)$, and if there exists a vector field $z \in \mathcal{DM}^\infty_{\text{loc}}(\Omega)$, with $\|z\|_{L^\infty(\Omega)^N} \leq 1$ satisfying

$$-\operatorname{div} z + |D\Gamma(u)| = h(u)f \quad \text{in } \mathcal{D}'(\Omega), \tag{5.10}$$

$$(z, DT_k(u)) = |DT_k(u)| \quad \text{as measures in } \Omega \text{ for any } k>0, \tag{5.11}$$

and

$$u(x) = 0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega. \tag{5.12}$$

In case of a nonnegative g possibly blowing-up at infinity we then have the following result.

Theorem 5.5. *Let g satisfy (3.5) and let h satisfy (3.6)-(3.7). Finally let $0 < f \in L^1(\Omega)$. Then there exists a solution $u \in BV(\Omega)$ to problem (5.1) in the sense of Definition 5.4. Moreover, if g is positive, then $D^j u = 0$.*

Proof. The proof is identical to the one of Theorem 4.4 apart from the application of Lemma 2.2 which is not needed in order to get (5.10). □

Data availability

No data was used for the research described in the article.

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