AN INTRODUCTION TO THOMPSON KNOT THEORY AND TO JONES SUBGROUPS

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ABSTRACT. We review a constructions of knots from elements of the Thompson groups due to Vaughan Jones, which comes in two flavours: oriented and unoriented.

Dedicated to the memory of Vaughan F. R. Jones

CONTENTS

INTRODUCTION

On the Christmas Eve of 2014, the first article [Jon17] of Vaughan Jones on a project centred on the Thompson groups appeared on arXiv and since then it has been followed by several articles. The centre of this project was a new powerful machinery that allows one to construct actions of Thompson groups (and more generally of *groups of fractions of certain categories*) starting from suitable categories. There have been developments in several directions. A lot of efforts have been devoted to producing unitary representations of Thompson groups, mainly by means of planar algebras [Jon21] and Pythagorean C*-algebras [BJ19a], see e.g. [Jon17, Jon18, Jon21, ACJ18, ABC21, AJ21, BJ19a, BJ19b, AC19a, AC19b, BP19, AP22]. Often these representations are related to notable graph or knot invariants, such as the chromatic polynomial, the Tutte polynomial, the Jones polynomial, the Kauffman bracket, the Homflypt

polynomial, to name but a few. In another direction, groups can be produced by means of this machinery, see [Bro20a, Bro20b]. In this article, we would like to review the developments in two other directions. In particular we would like to talk about the Thompson groups as knot constructors and about Jones subgroups. Our investigations in the latter are mainly motivated by the interest in infinite index maximal subgroups of Thompson groups and in the classification of the unitary representations introduced by Jones.

In [Jon17] Jones defined a method to produce unoriented knots and links from elements of the Thompson group F, which was later extended to the Brown-Thompson group F_3. Since these links do not possess a natural orientation, Jones introduced the oriented subgroups $\vec{F} \leq F$ and $\vec{F}_3 \leq F_3$, [Jon17, Jon19]. For unoriented links Jones also proved a result analogous to Alexander theorems for braids, that is for every link L there is an element g of the Thompson group whose *closure* $\mathcal{L}(g)$ is equal to L. In the oriented case a slightly weaker result was proved, namely that the oriented link could be reproduced up to disjoint union with unknots. This result was later strengthened by the author in [Aie20], where it was shown that every link $\vec{L}$ can be exactly reproduced by choosing a suitable element of the oriented subgroup $\vec{F}$. With the Thompson group and its oriented subgroup being as good as the braid groups at producing links, it is possible to start a reboot of the theory of braids and links, but with the Thompson group replacing the braid groups. Like braids, both F and $\vec{F}$ contain interesting *positive* monoids: the monoid of positive words F_+ and the monoid of positive oriented words $\vec{F}_+$. The links produced by these monoids were studied by Sebastian Baader and the author in a couple of papers [AB21, AB22], where it was shown that the links produced by $\vec{F}_+$ are positive (in the sense that all these oriented links admit link diagram where all the crossings are positive) and those of F_+ arborescent in the sense of Conway (these knots are also known as algebraic).

Despite being introduced with knot theoretical motivations, $\vec{F}$ and $\vec{F}_3$ turned out to be interesting also from the point of view of group theory. In fact, they gave rise to new examples [GS17b, AN22] of maximal subgroups of infinite index of F and F_3, respectively.

We end this introduction with a few words on the structure of the article. Section 1 is devoted to introducing the Thompson group and the Brown-Thompson groups. Section 2 presents two equivalent methods for producing unoriented links starting from elements of F and F_3. Section 3 focuses on positive Thompson links, which are the links produced with elements of F_+. In Section 4 the binary and ternary oriented subgroups $\vec{F}$ and $\vec{F}_3$ are introduced and later in Section 5 they are used to produce oriented links. Section 6 focuses on another Jones's subgroup: the 3-colorable subgroup $\mathcal{F}$.

1. Preliminaries and notation

In this section we recall the definitions of the Thompson group F and of the Brown-Thompson groups F_k. The interested reader is referred to [CFP96] and [Bel07] for more information on F, to [Bro87] for F_k.

There are several equivalent definitions of F. One of them is the following: F is the group of all piecewise linear homeomorphisms of the unit interval $[0, 1]$ that are differentiable everywhere except at finitely many dyadic rationals numbers and such that on the intervals of differentiability the derivatives are powers of 2. We adopt the standard notation: $(f \cdot g)(t) = g(f(t))$.

The Thompson group F has the following infinite presentation

$$F = \langle x_0, x_1, x_2, \ldots \mid x_n x_k = x_k x_{n+1} \quad \forall \, k < n \rangle \, .$$

Note that x_0 and x_1 are enough to generate F. The monoid generated by $x_0, x_1, x_2, \ldots$ is denoted by F_+ and its elements are said to be positive.

Every element g of F can be written in a unique way as

$$x_0^{a_0} \cdots x_n^{a_n} x_n^{-b_n} \cdots x_0^{-b_0}$$

where $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{N}_0$, exactly one between a_n and b_n is non-zero, and if $a_i \neq 0$ and $b_i \neq 0$, then $a_{i+1} \neq 0$ or $b_{i+1} \neq 0$ for all i. This is the normal form of g.

The projection of F onto its abelianisation is denoted by $\pi : F \to F/[F, F] = \mathbb{Z} \oplus \mathbb{Z}$ and it admits a nice interpretation when F is seen as a group of homeomorphisms: $\pi(f) = (\log_2 f'(0), \log_2 f'(1))$.

A family of groups generalising the Thompson group F are the so-called Brown-Thompson groups. For any integer $k \geq 2$, the Brown-Thompson group F_k may be defined by the following presentation

$$\langle y_0, y_1, \ldots \mid y_n y_l = y_l y_{n+k-1} \quad \forall \, l < n \rangle \, .$$

The elements $y_0, y_1, \ldots, y_{k-1}$ are enough to generate F_k. Note that for $k = 2$ we have $F_2 = F$. The monoid generated by $y_0, y_1, y_2, \ldots$ is denoted by $F_{k,+}$ and it elements are said to be positive. In the present article only the monoids F_+ and $F_{3,+}$ will play a role.

Going back to F, there is still another description which is relevant to this paper: the elements of F can be seen as pairs (T_+, T_-) of planar binary rooted trees (with the same number of leaves). We draw one tree upside down on top of the other; T_+ is the top tree, while T_- is the bottom tree. Any pair of binary trees (T_+, T_-) represented in this way is called a binary tree diagram. Two pairs of binary trees are said to be equivalent if they differ by pairs of

$$x_0 = \qquad x_1 =$$
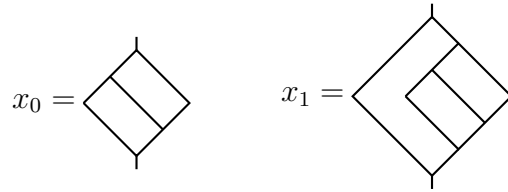
FIGURE 1. The generators of $F = F_2$.

$$y_0 = \qquad y_1 = \qquad y_2 =$$
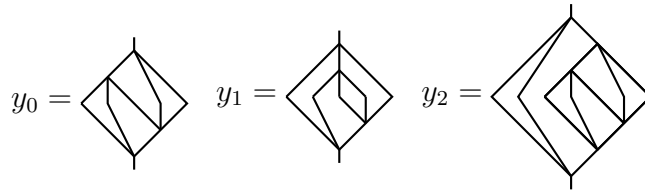
FIGURE 2. The generators of F_3.

$$y_0 = \qquad y_1 =$$

$$y_2 = \qquad y_3 =$$
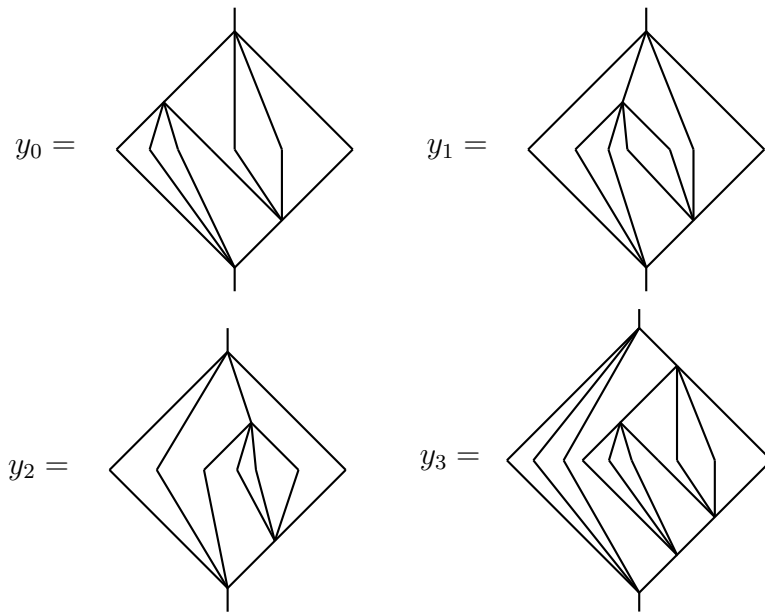
FIGURE 3. The generators of F_4.

opposing carets, namely

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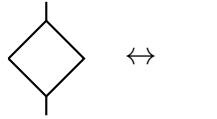
Every equivalence class of pair of binary trees (i.e. an element of F) gives rise to exactly one tree diagram which is reduced, in the sense that the number of its vertices is minimal, [Bel07]. See Figure 1 for the description of x_0 and x_1 in terms of binary trees. Thanks to this equivalence relation, the following rule defines the multiplication in F: $(T_+, T) \cdot (T, T_-) := (T_+, T_-)$. The trivial element is represented by any pair (T, T) and the inverse of (T_+, T_-) is (T_-, T_+). We illustrate how multiplication is performed with $x_0 x_1$

$$x_0 x_1 = \qquad \qquad = \qquad$$
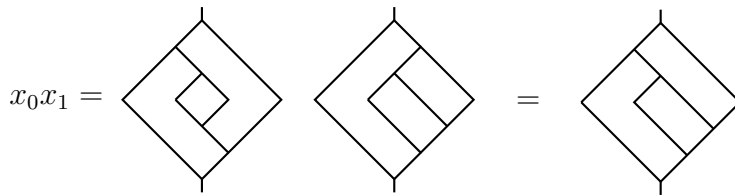
The positive elements of F may always be represented by a pair of binary trees with bottom tree having the following shape

$$\dots$$
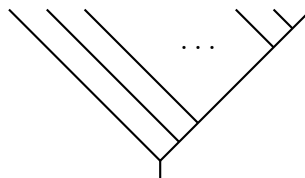
Similarly, the elements of F_k are described by pairs k-ary trees (i.e. the trees whose vertices have degree $k + 1$, except the leaves having degree 1). In this article we will be mainly interested in the Brown-Thompson F_3 and F_4. The generators of these groups are displayed in Figures 2 and 3. For F_3 and F_4 pairs of trees are equivalent if they differ by cancellations/additions of pairs of opposing carets

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The bottom ternary tree of a positive element of F_3 can be chosen with the following form
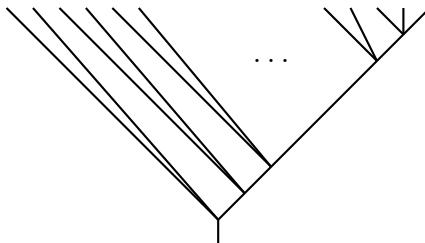

Convention 1.1. *We draw k-ary trees on the plane with the roots of our planar k-ary trees being drawn as vertices of degree $k + 1$. Each k-ary tree diagram has the uppermost and lowermost vertices of degree 1, which lie respectively on the lines $y = 1$ and $y = -1$. The leaves of the trees sit on the x-axis, precisely on the non-negative integers.*

There are an automorphism and two endomorphisms of F that will come in handy later on: the flip automorphism and the left/right shift homomorphisms $\sigma, \varphi_L, \varphi_R,: F \to F$. The flip automorphism σ is the order 2 automorphism obtained by reflecting tree diagrams about a vertical line, while the left/right shift homomorphisms φ_L, φ_R are defined graphically as

The ranges of φ_L and φ_R are those elements of F that act trivially on $[1/2, 1]$ and $[0, 1/2]$, respectively. Note that $\varphi_R(x_i) = x_{i+1}$ for every $i \in \mathbb{N}_0$. Here is $\sigma(x_1)$.
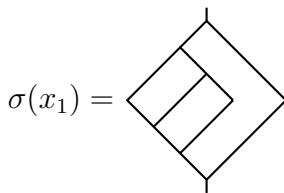
Some interesting subgroups of F are the so-called rectangular subgroups of F. They were introduced in [BW07] as

$$K_{(a,b)} := \{f \in F \mid \log_2 f'(0) \in a\mathbb{Z}, \log_2 f'(1) \in b\mathbb{Z}\} \qquad a, b \in \mathbb{N}$$

These subgroups can be characterised as the only finite index subgroups isomorphic with F [BW07, Theorem 1.1].

Denote by W_2 the set of finite binary words, i.e. finite sequences of 0 and 1. Let $\mathbb{Z}[1/2]$ be $\{a/2^k \mid ; a, k \in \mathbb{Z}\}$. There exists a map ρ between finite binary
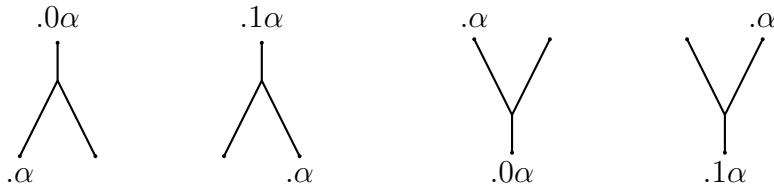
FIGURE 4. The local rules for computing the action of F on numbers expressed in binary expansion.

words and the dyadic rationals in the open unit interval $\mathcal{D} := \mathbb{Z}[1/2] \cap (0, 1)$, namely the map $\rho(a_1 \ldots a_n) := \sum_{i=1}^{n} a_i 2^{-i}$ which is bijective when restricted to finite words ending with 1 (i.e. $a_n = 1$).

The Thompson group F acts by definition on $[0, 1]$. Now we review this action on the numbers in $[0, 1]$ expressed in binary expansion. Given a number t, it enters into the top of the top tree in the binary tree diagram, follows a path towards the root of the bottom tree according to the rules portrayed in Figure 4. What emerges at the bottom is the image of t under the homeomorphism represented by the tree diagram, [BM14]. Note that there is a change of direction only when the number comes across a vertex of degree 3 (i.e., the number is unchanged when it comes across a leaf).

The action of F_3 on $[0, 1]$ can be describe in a similar way. First we express the numbers in ternary expansion (i.e. the digits are only 0, 1, 2). Then the number t enters into the top of the top tree in the ternary tree diagram, follows a path towards the root of the bottom tree according to the rules portrayed in Figure 5.

2. The construction of knots: the unoriented case

Jones introduced two equivalent methods to produce unoriented knots and links from the Thompson groups. Originally these constructions were defined for F, [Jon17], but later they were extended to F_3 in [Jon19].

We present this method by taking $x_0 x_1$ as an example. We will construct a Tait diagram $\Gamma(T_+, T_-)$ from a binary tree diagram (T_+, T_-) in F. Recall that the leaves of T_+ and T_- sit on the non-negative integers $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ on the x-axis. We place the vertices of $\Gamma(T_+, T_-)$ on the half integers, so for $x_0 x_1$ these points are $(1/2, 0)$, $(3/2, 0)$, $(5/2, 0)$, $(7/2, 0)$. We draw an edge between two of these vertices whenever there is an edge of the top tree sloping up from left to right (we call them West-North edges, or simply WN= $\diagup$) and whenever there is an edge of the bottom tree sloping down from left to right
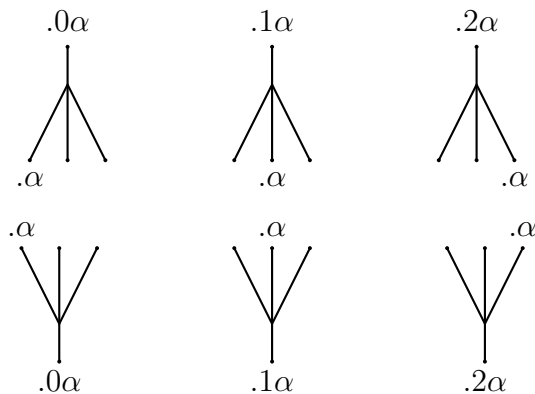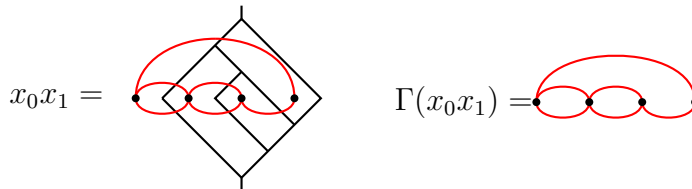
FIGURE 5. The local rules for computing the action of F_3 on numbers expressed in ternary expansion.

(we refer to them by West-South edges, or just WS=$\searrow$). This is the graph for x_0x_1

There is actually a bijection between the graphs of the form $\Gamma(T_+, T_-)$ and the pairs of trees (T_+, T_-) of F, [Jon17, Lemma 4.1.4]. We denote by $\Gamma_+(T_+)$ and $\Gamma_-(T_-)$ the subgraphs of $\Gamma(T_+, T_-)$ contained in the upper and lower-half plane, respectively. Since a Tait diagram is a signed graph, we decree that the edges of $\Gamma_+(T_+)$ and $\Gamma_-(T_-)$ are positive and negative, respectively.

Remark 2.1. *The graphs of the type $\Gamma_\pm(T_\pm)$ may always be assumed to satisfy the following properties*

   (1) the vertices are $(0,0), \ldots, (N,0)$;
   (2) each vertex other than $(0,0)$ is connected to exactly one vertex to its left;
   (3) each edge e can be parametrized by a function $(x_e(t), y_e(t))$ with $x'_e(t) > 0$, for all $t \in [0,1]$, and either $y_e(t) > 0$, for all $t \in]0,1[$ or $y_e(t) < 0$, for all $t \in]0,1[$;

FIGURE 6. A positive and a negative crossing.

see [Jon17, Proposition 4.1.3.]. *In particular, every vertex (except the leftmost) is the target of exactly two edges, one in the lower half-plane and one in the upper-half plane.*

There are two last steps to be done in order to obtain a link. First we draw the medial graph $M(\Gamma(T_+, T_-))$ of $\Gamma(T_+, T_-)$. In general, given a connected plane graph G, the vertices of its medial graph $M(G)$ sit on every edge of G and an edge of $M(G)$ connects two vertices if they are on adjacent edges of the same face. Below we will provide an example in our context. Now all the vertices of $M(\Gamma(T_+, T_-))$ have degree 4 and we may make the final step: turn the vertices into crossings and obtain a link diagram. For the vertices in the upper-half plane we use the crossing $\times$, while for those in the lower-half plane we use $\times$. We point out that in the checkerboard shading of the link diagram obtained with this procedure, the crossings corresponding to vertices on edges of $\Gamma_+(T_+)$ are positive and the crossings corresponding to vertices on edges of $\Gamma_-(T_-)$ are negative (in the sense of Figure 6). Here are $M(\Gamma(x_0 x_1))$ and $\mathcal{L}(x_0 x_1)$.
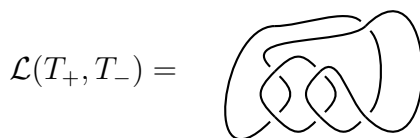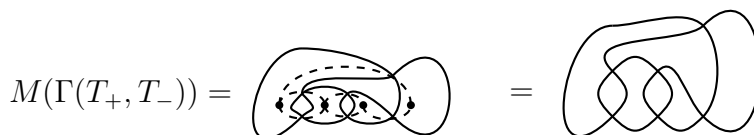
In this section we describe an equivalent procedure to obtain links from elements of F, [Jon17, Jon19]. The advantage of this description is that it can
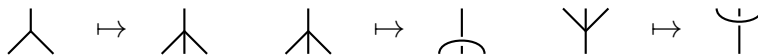
FIGURE 7. The rules needed for obtaining $\mathcal{L}(g)$.
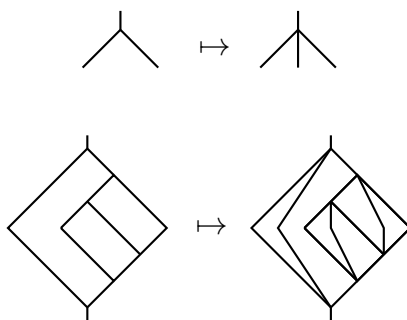
FIGURE 8. The map $\iota : F \to F_3$.

be readily extended to F_3. We start with a binary tree diagram in F. The first operation is to turn all the 3-valent vertices into 4-valent by adding additional edges below each vertex of degree 3 in the top tree and above each vertex of degree 3 of the bottom tree, which we join in the only planar possible way. The second operation is to draw an edge between the two roots of the trees. The third and last operation is to turn all the 4-valent vertices into crossings as shown in Figure 7: the vertices and the four incident edges are replaced by "forks", see leftmost illustration of Fig. 7. We exemplify this procedure with $x_0 x_1$.
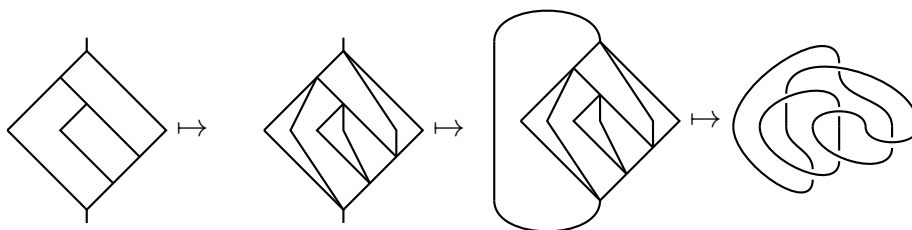

Remark 2.2. *In the first step we are actually using an injective group homomorphism $\iota : F \to F_3$, see Figure 8. This map takes a binary tree diagram, and returns 4-valent tree diagrams. This map was originally defined by Jones*

in [Jon19, Section 4]. *Therefore, the construction of knots can be extended to F_3 just by skipping the first step.*

We draw the Tait graph for the links corresponding to F_3 in the plane, the vertices sitting on the x-axis, half of the edges in the upper-half-plane, the other half in the lower-half-plane. When restricted to F, the Tait diagram of the link diagram obtained in this way is exactly the graph $\Gamma(T_+, T_-)$ described in the previous section. One of the differences between the Tait graphs of the elements F and those of F_3 is that for the elements of F_3 is that the edges in the lower-half-plane (upper-half-plane, respectively) are not necessarily negative (positive, respectively).

Example 2.3 (The 4_1 knot). *Consider the element $g = x_0 x_2^2 x_5 x_6 (x_4 x_6 x_7)^{-1} \in F$ whose image $\iota(g) = (T_+, T_-) = y_0 y_4^2 y_{10} y_{12} (y_8 y_{12} y_{14})^{-1} \in F_3$ is described by the following pair of ternary trees*
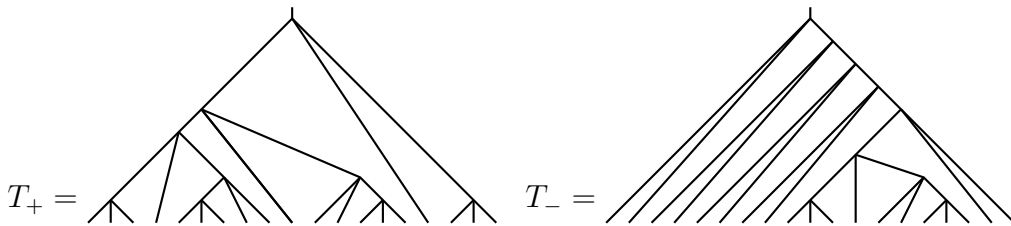
After applying some Reidemeister moves (to be precise a sequence consisting of five Reidemeister moves of type II and four of type I), one sees that $\mathcal{L}(T_+, T_-)$ is the 4_1 knot.
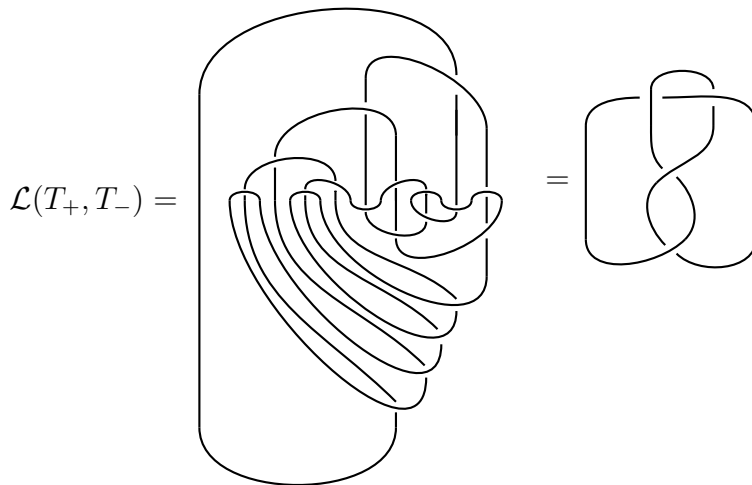
The Thompson group is just as good as the braid groups at producing links. More precisely, Jones proved a result analogous to that of the Alexander theorem.

Theorem 2.4. [Jon17] *Given an unoriented link L, there exists an element g in F such that $\mathcal{L}(g)$ is L.*

In analogy to the braid index, it is possible to define a Thompson index. The F-index of a link L is the smallest number of leaves required by each binary tree in a binary tree diagram such that L is realised as $\mathcal{L}(T_+, T_-)$. The F_3-index is defined as the smallest number of trivalent vertices plus one required by each ternary tree in a binary tree diagram such that L is realised as $\mathcal{L}(T_+, T_-)$. Note that the F_3-index is defined in terms of trivalent instead of leaves to make it compatible with the F-index. In fact, for every binary tree diagram (T_+, T_-) the number of trivalent vertices plus 1 in each ternary tree of $\iota(T_+, T_-)$ is equal to the number of leaves of T_+. The following interesting result was discovered by Golan and Sapir.

Theorem 2.5. [GS17a] *The F-index of a link containing u unlinked unknots and represented by a link diagram with n crossings does not exceed $12n + u + 3$.*

3. Positive Thompson knots

The positive Thompson knots are those produced by the elements of the monoid of positive words F_+. As explained in Section 1, each of these elements admits a representative whose bottom tree and the corresponding graph $\Gamma_-(T_-)$ have the following form

$$T_- = \qquad\qquad\qquad \Gamma_-(T_-) = \quad\cdots\quad$$

(3.1)

As the form of the bottom tree is essentially always of the same form (it depends only on the number of leaves in the upper tree), sometimes we will use the notation $\mathcal{L}(T_+)$, instead of $\mathcal{L}(T_+, T_-)$. Positive Thompson links were the object of study of [AB22], but before stating the main result of this investigation we recall some preliminary definitions.

Arborescent tangles are the minimal class of tangles closed under tangle composition, and containing all rational tangles, [Con70]. The closure of an
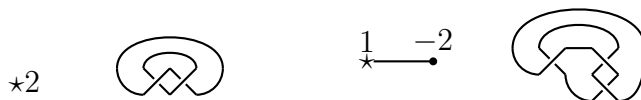
FIGURE 9. Two arborescent links: one associated with a tree consisting just of the root, another associated with a tree with a root and one leaf.

arborescent tangle is described by a finite rooted plane tree with integer vertex weights. Each weight w gives rise to a twist region with $|w|$ crossings. The orientation of these crossings, as well as the interconnections between these twist regions, are determined by the plane tree in the following way. The root vertex corresponds to a horizontal twist region, in which crossings are called positive if their strand going from the bottom left to the top right is above the other strand. If the weight is zero, then we have just two horizontal lines. The vertices adjacent to the root vertex correspond to vertical twist regions attached to this horizontal twist region. The order in which they are attached is determined by the plane cyclic arrangement of the branches around the root vertex. We keep the convention that the overcrossing strand of a positive crossing is going from the bottom left to the top right. In the end, this means that arborescent tangles whose weights carry the same sign give rise to alternating links. The vertices at distance two from the root give again rise to horizontal twist regions, and so on. Two examples are provided in Figure 9. For more details and resultst we refer to [BS, Gab86]. Finally we call a finite rooted plane tree bipartite if its vertices have weights ± 1, with the root and all the leaves carrying weight -1; all the vertices of weight 1 have degree 2; there are no edges between vertices with the same weight.

We are now in a position to state the main result of [AB22], which is kind of an Alexander theorem for F_+.

Theorem 3.1. [AB22] *The set of positive Thompson links coincides with the set of closures of bipartite arborescent tangles.*

In general links produced from F and F_3 do not posses a natural orientation, for this reason Jones introduced the so-called oriented subgroups. We will define them in the next section.

4. The oriented subgroups $\vec{F}$ and $\vec{F}_3$

In this section we introduce two new subgroups, one of F and one of F_3. They are interesting on their own, but we will need them to produce oriented links. The (binary) oriented subgroup $\vec{F} = \vec{F}_2$ is a subgroup of F, while $\vec{F}_3$ is a subgroup of F_3.

First we describe an alternative method to obtain the Tait graph associated with each ternary tree diagram. We will ignore the sign of the edges as it is not relevant for defining the oriented subgroups. Given a ternary tree diagram (T_+, T_-), we call this graph $\Gamma(T_+, T_-)$ the planar graph of (T_+, T_-). We imagine (T_+, T_-) sitting in the strip bounded by the lines $y = 1$ and $y = -1$. This strip is 2-colourable. We use two colors: black and white, the left-most region is black. The vertices of $\Gamma(T_+, T_-)$ sit on the x-axis, precisely on $-1/2 + 2\mathbb{N}_0 := \{-1/2, 1 + 1/2, 3 + 1/2, \ldots\}$ and there is precisely one vertex for every black region. We draw an edge between two black regions whenever they meet at a 4-valent vertex.


$$y_1 = \qquad\qquad \Gamma(y_1) =$$
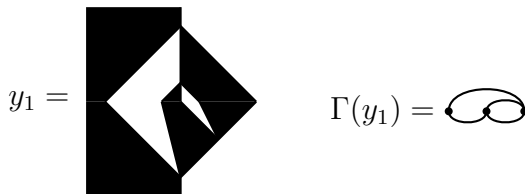
The binary and the ternary oriented subgroup $\vec{F}_3$ can be defined as

$$(4.1) \qquad \vec{F} = \vec{F}_2 := \{(T_+, T_-) \in F \mid \Gamma(\iota(T_+, T_-)) \text{ is 2-colorable}\}$$

$$(4.2) \qquad \vec{F}_3 := \{(T_+, T_-) \in F_3 \mid \Gamma(T_+, T_-) \text{ is 2-colorable}\}$$

Convention 4.1. *We denote the colors used for the vertices of $\Gamma(T_+, T_-)$ by $+$ and $-$. The graph $\Gamma(T_+, T_-)$ is always connected and, therefore, if it is 2-colorable there are only 2 possible colorings: one where the leftmost vertex has color $+$, one with color $-$. By convention we choose always choose the first of these colorings.*

These groups were introduced by Jones in 2014 [Jon17] and 2018 [Jon19], respectively. The (binary) oriented subgroup was first studied by Golan and Sapir in [GS17a, GS17b], who determined its generators x_0x_1, x_1x_2, x_2x_3.
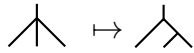
FIGURE 10. Ren's map for $\vec{F}$.
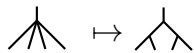
FIGURE 11. The map for $\mathcal{F}$.

Moreover, they also discovered that the map induced by

$$\alpha : \vec{F} \to F_3$$
$$x_0 x_1 \mapsto y_0$$
$$x_1 x_2 \mapsto y_1$$
$$x_2 x_3 \mapsto y_2$$

is an isomorphism. A pictorial interpretation of this isomorphism was later found by Ren in [Ren18], where he realised that this map can be obtained by taking a ternary tree diagram and replacing each trivalent vertex with a suitable tree with 3 leaves, see Figure 10.

One may define a *weight* ω set of finite binary words W_2 with values in $\mathbb{Z}_2$ by the formula $\omega(a_1 \ldots a_n) := \sum_{i=0}^{n} a_i$ and a subset of dyadic rationals

$$S := \{ t \in W_2 \mid \omega(t) = 0 \}.$$

Theorem 4.1. [GS17a] *The oriented subgroup $\vec{F}$ is the stabiliser subgroup* Stab(S).

Thanks to this characterisation Golan and Sapir were able to prove that $\vec{F}$ coincided with its commensurator and thus the corresponding quasi-regular representation is irreducible.

The oriented subgroup F is also interesting because it gave rise to a novel example of maximal subgroup of infinite index in F. Before this group was defined, the only known subgroups of this type were the so-called parabolic subgroups, that is the stabilisers of points Stab(t), $t \in (0,1)$, under the natural action of F on $(0,1)$ that were studied in [Sav15, Sav10]. Golan and Sapir in [GS17b] proved that $\vec{F}$ sits inside the rectangular subgroup $K_{(1,2)}$ and it is maximal and of infinite index in it. By exhibiting the explicit isomorphism $\beta : K_{(1,2)} \to F$ induced by $\beta(x_0 x_2) = x_0$, $\beta(x_1 x_2) = x_1$, they were able to show

that $\beta(\vec{F})$ was a maximal subgroup of infinite index in F distinct from the parabolic subgroups.

Once the ternary oriented subgroup was introduced, we extended some of these results to it. More precisely, we found a set of generators for $\vec{F}_3$.

Theorem 4.2. [AN22] *The ternary oriented Thompson group $\vec{F}_3$ is generated by the following elements*

$$y_{2i+1}^2, y_{2i}y_{2i+2}, y_{2i}y_{2i+3} \qquad i = 0, 1, 2.$$

Question 1. *Is $\vec{F}_3$ isomorphic to a Brown-Thompson group or other known groups?*

One might also ask the following question (which was originally asked in [AN22, Problem 1]).

Question 2. *Is $\vec{F}_3$ finitely presented?*

Then we also realised that $\vec{F}_3$ is the stabiliser of a suitable subset. For this we needed a new weight on the set of ternary words.

Consider a tree in the upper half-plane and its leaves on the x-axis as usual. To each vertex v of a tree we associate a natural number $c(v)$ which we call its **weight**, as follows. Given a vertex, there exists a unique minimal path from the root of the tree to the vertex. This path is made by a collection of left, middle, right edges, and may be represented by a word $w_1 1 w_2 1 \cdots 1 w_n$ in the letters $\{0, 1, 2\}$ (0 stands for a left edge, 1 for a middle edge, 2 for a right edge), where $w_1, \ldots, w_{n-1}$ are words that do not contain the letter 1, w_n can have 1 only as its last letter. We call $\{w_{2k+1}\}_{k\geq 0}$ the odd words and $\{w_{2k}\}_{k\geq 0}$ the even words. The weight of v is the sum of the number of digits equal to 1, plus the number of digits equal to 2 in the odd words, plus the number of digits equal to 0 in the even words. When we compute the weight of a leaf in a tree diagram, sometimes we use the symbol c_+ or c_- to distinguish which tree we are considering (c_+ for the top tree, c_- for the reflected bottom tree).

Theorem 4.3. [AN22] *The ternary oriented Thompson group $\vec{F}_3$ is the stabilizer of the following subset of the triadic fractions*

$$Z := \{.a_1 a_2 \cdots a_n \mid \# \ of \ 1\text{'s is even}, c(.a_1 a_2 \cdots a_n) \ is \ even\} \, .$$
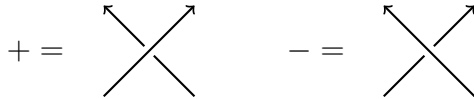
$$+ = \qquad - =$$

FIGURE 12. A positive and a negative crossing in an oriented link.

5. THE CONSTRUCTION OF KNOTS: THE ORIENTED CASE

Links produced from the oriented subgroups admit a natural orientation. Recall that the Tait diagrams $\Gamma(T_+, T_-)$ of elements in $\vec{F}$ are 2-colorable (the colors being $\{+, -\}$ and the left-most vertex having colour $+$).

Given (T_+, T_-), if we shade the link diagram $\mathcal{L}(T_+, T_-)$ in black and white (we adopt the convention that the colour of the unbounded region is white), by construction the vertices of the graph $\Gamma(T_+, T_-)$ sit in the black regions and each one has been assigned with a colour $+$ or $-$. These colours determine an orientation of the surface and of the boundary ($+$ means that the region is positively oriented). It can be easily seen that the graph $\Gamma(\iota(x_0 x_1))$ is 2-colorable and thus $x_0 x_1$ is in $\vec{F}$. Here is the oriented link associated with $x_0 x_1$.

$$\vec{\mathcal{L}}(T_+, T_-) = $$
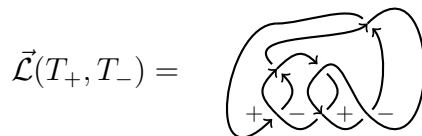
Also in the setting of the oriented subgroups, Jones proved a result analogous to Alexander's theorem.

Theorem 5.1. [Jon17] *Given an oriented link $\vec{L}$, there exists an element g in $\vec{F}$ such that $\vec{\mathcal{L}}(g)$ is $\vec{L}$ up to disjoint union with unknots.*

In [Jon19] Jones asked whether the previous theorem could be improved and each oriented link could be exactly reproduced. An answer was provided in [Aie20].

Theorem 5.2. [Aie20] *Given an oriented link $\vec{L}$, there exists an element g in $\vec{F}$ such that $\vec{\mathcal{L}}(g)$ is $\vec{L}$.*

There is an interesting monoid in $\vec{F}$, namely the monoid $\vec{F}_+ := \vec{F} \cap F_+$. It is well known that every element of the braid group may be expressed as

the product of a positive braid and the inverse of a positive braid. Similarly, for the oriented Thompson group we have the following result which is due to Ren, [Ren18].

Proposition 5.3. [Ren18] *For every $g \in \vec{F}$, there exist g_+, $g_- \in \vec{F}_+$ such that $g = g_+(g_-)^{-1}$.*

We call the links produced by the monoid $\vec{F}_+$ the positive oriented Thompson links. Recall that an oriented link is called positive if it admits a link diagram where all it crossings are positive in the sense of Figure 12.

Theorem 5.4. [AB21] *The positive oriented Thompson links are positive.*

As in for F and F_3, it is possible to define the $\vec{F}$ and $\vec{F}_3$ indices.

Question 3. *Does Theorem 2.5 extend to the $\vec{F}$ and $\vec{F}_3$ indices?*

6. The 3-colorable subgroup $\mathcal{F}$

Another subgroup introduced by Jones is the so-called 3-colorable subgroup $\mathcal{F}$. As before, any binary tree diagram partitions the strip bounded by the lines $y = 1$ and $y = -1$ in regions. This strip may or may not be 3-colorable, i.e., it may or may not be possible to assign the colors $\mathbb{Z}_3 = \{0, 1, 2\}$ to the regions of the strip in such a way that if two regions share an edge, they have different colors.

Convention 6.1. *If the strip is 3-colorable, we adopt this convention: we assign the following colors to the regions near the roots*

(6.1)

$$\begin{array}{cc} 0 \diagdown\!\!\!\diagup 1 \\ 2 \end{array} \qquad \begin{array}{cc} 2 \\ 0 \diagup\!\!\!\diagdown 1 \end{array}$$

Once we make this convention, if the strip is 3-colourable, there exists a unique colouring.

The **3-colorable subgroup** $\mathcal{F}$ consists of the elements of F whose corresponding strip is 3-colorable. For example, this is the strip corresponding to
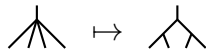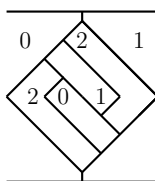
FIGURE 13. Ren's map Φ from the set of 4-ary trees to binary trees.

$w_0 := x_0^2 x_1 x_2^{-1}$ (which is 3-colorable)

The 3-colorable subgroup actually provides a copy of a Borwn-Thompson group inside F.

Theorem 6.1. [Ren18] *The subgroup $\mathcal{F}$ is isomorphic to the Brown-Thompson group F_4 by means of the isomorphism (with domain F_4 and range $\mathcal{F}$) obtained by replacing every 5-valent vertex of 4-ary trees by the complete binary tree with 4 leaves (see Figure 13). The images of y_0, y_1, y_2, y_3 yield the following elements $w_0 := x_0^2 x_1 x_2^{-1}$, $w_1 := x_0 x_1^2 x_0^{-1}$, $w_2 := x_1^2 x_3 x_2^{-1}$, $w_3 := x_2^2 x_3 x_4^{-1}$ (see Figure 14).*

As the oriented subgroup, $\mathcal{F}$ is also the stabiliser of subset of dyadic rationals and the weight function is particularly simple.

Theorem 6.2. [AN21] *For a binary word $a_1 a_2 \ldots a_n$ set*
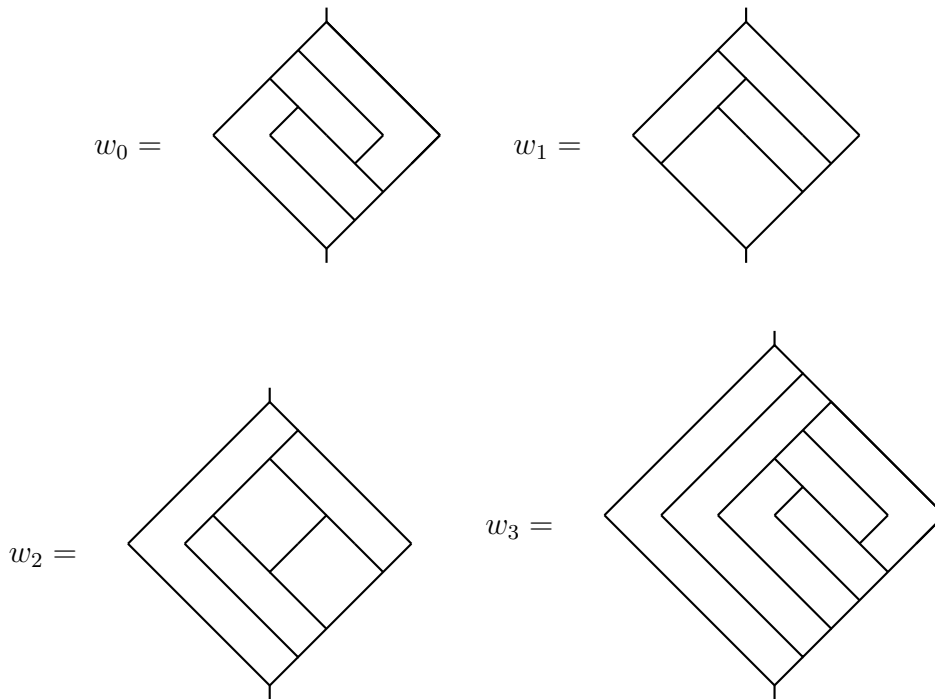
$$\omega(a_1 a_2 \ldots a_n) := \sum_{i=1}^n (-1)^i a_i \in \mathbb{Z}_3 \,.$$

$$S_i := \{t \in (0,1) \cap \mathbb{Z}[1/2] \mid \omega(t) = i\} \qquad i \in \mathbb{Z}_3$$

where $\equiv_3$ is the equivalence modulo 3. Then it holds

$$\mathcal{F} = \mathrm{Stab}(S_i) \qquad i \in \mathbb{Z}_3 \,.$$

Simple computations show that the 3-colorable subgroup $\mathcal{F}$ is contained in the rectangular subgroup $K_{(2,2)} \cong F$, but unlike $\vec{F}$ it is not maximal in it. However, there are still three maximal subgroups of $K_{(2,2)}$ of infinite index, namely $M_0 := \langle x_0^2, \mathcal{F} \rangle$, $M_1 := \langle x_1^2, \mathcal{F} \rangle$, and $M_2 := \langle \sigma(x_1)^2, \mathcal{F} \rangle$, [AN21].

FIGURE 14. The generators of $\mathcal{F}$.

Connections between $\mathcal{F}$ and Jones's construction of knots have not been explored yet, nevertheless it is natural to ask the following question.

Question 4. *Do the elements of $\mathcal{F}$ produce all unoriented knots and links?*

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