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Research Article

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Blowup in $L^1(\Omega)$-norm and global existence for time-fractional diffusion equations with polynomial semilinear terms

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Abstract: This article is concerned with semilinear time-fractional diffusion equations with polynomial nonlinearity u^p in a bounded domain Ω with the homogeneous Neumann boundary condition and positive initial values. In the case of $p > 1$, we prove the blowup of solutions $u(x, t)$ in the sense that $\|u(\,\cdot\,,t)\|_{L^1(\Omega)}$ tends to ∞ as t approaches some value, by using a comparison principle for the corresponding ordinary differential equations and constructing special lower solutions. Moreover, we provide an upper bound for the blowup time. In the case of $0 < p < 1$, we establish the global existence of solutions in time based on the Schauder fixed-point theorem.

Keywords: semilinear time-fractional diffusion equation, polynomial nonlinearity, blowup, global existence

MSC 2020: 35R11, 35K58, 35B44

1 Introduction and main results

Let $d = 1, 2, 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. For $0 < \alpha < 1$, let d_t^α denote the classical Caputo derivative:

$$\mathrm{d}_t^\alpha f(t) := \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f'(s)\mathrm{d}s, \quad f \in W^{1,1}(0, T).$$

Here, $\Gamma(\,\cdot\,)$ denotes the gamma function.

For consistent discussions of semilinear time-fractional diffusion equations, we extend the classic Caputo derivative d_t^α as follows. First, for $0 < \alpha < 1$, we define the Sobolev-Slobodecki space $H^\alpha(0, T)$ with the norm $\|\cdot\|_{H^\alpha(0,T)}$ as follows:

$$\|f\|_{H^\alpha(0,T)} := \left(\|f\|_{L^2(0,T)}^2 + \int_0^T\int_0^T \frac{|f(t)-f(s)|^2}{|t-s|^{1+2\alpha}}\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{2}}$$

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(e.g., Adams [1]). Furthermore, we set $H^0(0,T) := L^2(0,T)$ and

$$H_\alpha(0,T) := \begin{cases} H^\alpha(0,T), & 0 < \alpha < \frac{1}{2}, \\ \left\{ f \in H^{\frac{1}{2}}(0,T); \int_0^T \frac{|f(t)|^2}{t} \mathrm{d}t < \infty \right\}, & \alpha = \frac{1}{2}, \\ \{ f \in H^\alpha(0,T); f(0) = 0 \}, & \frac{1}{2} < \alpha \le 1 \end{cases}$$

with the norms defined by

$$\|f\|_{H_\alpha(0,T)} := \begin{cases} \|f\|_{H^\alpha(0,T)}, & \alpha \ne \frac{1}{2}, \\ \left(\|f\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|f(t)|^2}{t} \mathrm{d}t \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

Moreover, for $\beta > 0$, we set

$$J^\beta f(t) := \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) \mathrm{d}s, \quad 0 < t < T, \quad f \in L^1(0,T).$$

Then, it was proved, e.g., in the study by Gorenflo et al. [11], that $J^\alpha : L^2(0,T) \longrightarrow H_\alpha(0,T)$ is an isomorphism for $\alpha \in (0,1)$.

Now we are ready to define the extended Caputo derivative

$$\partial_t^\alpha := (J^\alpha)^{-1}, \quad \mathcal{D}(\partial_t^\alpha) = H_\alpha(0,T).$$

Henceforth, $\mathcal{D}(\,\cdot\,)$ denotes the domain of an operator under consideration. This is the minimum closed extension of d_t^α with $\mathcal{D}(\mathrm{d}_t^\alpha) := \{v \in C^1[0,T];\, v(0) = 0\}$ and $\partial_t^\alpha v = \mathrm{d}_t^\alpha v$ for $v \in C^1[0,T]$ satisfying $v(0) = 0$. As for the details, we can refer to the studies by Gorenflo et al. [11] and Yamamoto [31].

This article is concerned with the following initial-boundary value problem for a nonlinear time-fractional diffusion equation:

$$\begin{cases} \partial_t^\alpha (u - a) = \triangle u + u^p & \text{in } \Omega \times (0,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $p > 0$ is a constant. The left-hand side of the time-fractional differential equation in equation (1.1) means that $u(x,\,\cdot\,) - a(x) \in H_\alpha(0,T)$ for almost all $x \in \Omega$. For $\frac{1}{2} < \alpha < 1$, since $v \in H_\alpha(0,T)$ implies $v(0) = 0$ by the trace theorem, we can understand that the left-hand side means that $u(x,0) = a(x)$ in the trace sense with respect to t. As a result, this corresponds to the initial condition for $\alpha > \frac{1}{2}$, whereas we do not need any initial conditions for $\alpha < \frac{1}{2}$.

There are other formulations for initial-boundary value problems for time-fractional partial differential equations (e.g., Sakamoto and Yamamoto [25] and Zacher [32]), but here we do not provide comprehensive references. In the case of $\alpha = 1$, concerning the non-existence of global solutions in time, there have been enormous works since Fujita [9], and we can refer to a comprehensive monograph by Quittner and Souplet [24]. We can refer to Fujishima and Ishige [8] and Ishige and Yagisita [13] as related results to our first main result Theorem 1 stated below. See also Chen and Tang [4], Du [6], Feng et al. [7], and Tian and Xiang [29].

For $0 < \alpha < 1$, the time-fractional diffusion equation in (1.1) is a possible model for describing anomalous diffusion in heterogeneous media, and the semilinear term u^p can describe a reaction term. There are also rapidly increasing interests for the non-existence of global solutions to semilinear time-fractional differential equations such as equation (1.1). As recent works, we refer to studies by Ahmad et al. [2], Borikhanov et al. [3], Ghergu et al. [10], Hnaien et al. [12], Kirane et al. [15], Kojima [16], Suzuki [26,27], Vergara and Zacher [30], and

Zhang and Sun [33]. In [30] and [33], the blowup is considered by $\|u(\cdot,t)\|_{L^1(\Omega)}$. Since $L^1(\Omega)$-norm is the weakest among the Lebesgue space norms, the choice $L^1(\Omega)$ as spatial norm is sharp for consideration of the blowup.

Our approach is based on the comparison of solutions to initial value problems for time-fractional ordinary differential equations, which is similar to that by Ahmad et al. [2] in the sense that the scalar product of the solution with the first eigenfunction of the Laplacian with the boundary condition is considered. Vergara and Zacher, in their study [30], discuss stability, instability, and blowup for time-fractional diffusion equations with super-linear convex semilinear terms.

To the best knowledge of the authors, there are no publications providing an upper bound of the blowup time for the time-fractional diffusion equation in $L^1(\Omega)$-norm, which is weaker than $L^q(\Omega)$-norm with $1 < q \le \infty$.

Throughout this article, we assume $\frac{3}{4} < \gamma \le 1$. First, for $p > 1$, we recall a basic result on the unique existence of local solutions in time. For $a \in H^{2\gamma}(\Omega)$ satisfying $\partial_\nu a = 0$ on $\partial\Omega$ and $a \ge 0$ in Ω, Luchko and Yamamoto [20] proved the unique existence, which is local in time t. More precisely, there exists a constant $T > 0$ depending on a such that (1.1) possesses a unique solution u such that

$$u \in C([0,T]; H^{2\gamma}(\Omega)), \quad u - a \in H_\alpha(0,T; L^2(\Omega)) \tag{1.2}$$

and $u \ge 0$ in $\Omega \times (0,T)$. The time length T of the existence of u does not depend on the choice of initial values and only depends on a bound $m_0 > 0$ such that $\|a\|_{H^{2\gamma}(\Omega)} \le m_0$, provided that $\partial_\nu a = 0$ on $\partial\Omega$.

We call $T_{\alpha,p,a} > 0$ the blowup time in $L^1(\Omega)$ of the solution to (1.1) if

$$\lim_{t \uparrow T_{\alpha,p,a}} \|u(\cdot,t)\|_{L^1(\Omega)} = \infty. \tag{1.3}$$

As the non-existence of global solutions in time, in this article, we are concerned with the blowup in $L^1(\Omega)$.

Now we are ready to state our first main results on the blowup with an upper bound of the blowup time for $p > 1$.

Theorem 1. *Let $p > 1$ and $a \in H^{2\gamma}(\Omega)$ satisfy $\partial_\nu a = 0$ on $\partial\Omega$ and $a \ge 0, \not\equiv 0$ in Ω. Then, there exists some $T = T_{\alpha,p,a} > 0$ such that the solution satisfying (1.2) exists for $0 < t < T_{\alpha,p,a}$, and (1.3) holds. Moreover, we can bound $T_{\alpha,p,a}$ from above as:*

$$T_{\alpha,p,a} \le \left(\frac{1}{(p-1)\Gamma(2-\alpha)\left(\frac{1}{|\Omega|}\int_\Omega a(x)\mathrm{d}x\right)^{p-1}} \right)^{\frac{1}{\alpha}} =: T^*(\alpha,p,a). \tag{1.4}$$

Remark 1. (1) We note that $T^*(\alpha,p,a)$ decreases as $\int_\Omega a(x)\mathrm{d}x$ increases for arbitrarily fixed p and α. Meanwhile, $T^*(\alpha,p,a)$ tends to ∞ as $p > 1$ approaches 1, which is consistent because $p = 1$ is a linear case and we have no blowup.

(2) Estimate (1.4) corresponds to the estimate in [24, Remark 17.2(i) (p.105)] for $\alpha = 1$. On the other hand, in the case of parabolic $\partial_t u = D\triangle u + u^p$ with constant $D > 0$, Ishige and Yagisita discussed the asymptotics of the blowup time $T_{p,a}(D)$ and established

$$T_{p,a}(D) = \frac{1}{(p-1)\left(\frac{1}{|\Omega|}\int_\Omega a(x)\mathrm{d}x\right)^{p-1}} + O\left(\frac{1}{D}\right) \quad \text{as } D \to \infty$$

([13, Theorem 1.1]). The principal term of the asymptotics coincides with the value obtained by substituting $\alpha = 1$ in $T^*(\alpha,p,a)$ given by (1.4). Thus, $T^*(\alpha,p,a)$ is not only one possible upper bound of equation the blowup time for $0 < \alpha < 1$ but also seems to capture some essence. Moreover, Ishige and Yagisita [13] clarified the blowup set; see also the work of Fujishima and Ishige [8]. For $0 < \alpha < 1$, there are no such detailed available results.

The second main result is the global existence of solutions to (1.1) for $0 < p < 1$.

Theorem 2. *Let* $0 < p < 1$ *and* $a \in H^{2\gamma}(\Omega)$ *satisfy* $\partial_\nu a = 0$ *on* $\partial\Omega$ *and* $a \geq 0$ *in* Ω. *For arbitrarily given* $T > 0$, *there exists a global solution* u *to* (1.1) *with* $T = \infty$ *satisfying* (1.2).

In Theorem 2, we cannot further conclude the uniqueness of the solution. This is similar to the case of $\alpha = 1$, where the uniqueness relies essentially on the Lipschitz continuity of the semilinear term u^p in $u \geq 0$. Indeed, we can easily give a counterexample by a time-fractional ordinary differential equation:

$$\partial_t^\alpha y(t) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} y(t)^{\frac{1}{2}},$$

where $y \in H_\alpha(0, T)$. Then, we can directly verify that both $y(t) = t^{2\alpha}$ and $y(t) \equiv 0$ are solutions to this initial value problem.

The key to the proof of Theorem 1 is a comparison principle [20] and a reduction to a time-fractional ordinary differential equation. Such a reduction method can be found in the studies by Kaplan [14] and Payne [21] for the case $\alpha = 1$. On the other hand, Theorem 2 is proved by the Schauder fixed-point theorem with regularity properties of solutions [31]. For a related method for Theorem 2, we refer to Díaz et al. [5].

This article is composed of five sections; in Section 2, we show lemmata that complete the proof of Theorem 1 in Section 3; we prove Theorem 2 in Section 4; finally, Section 5 is devoted to concluding remarks and discussions.

2 Preliminaries

We will prove the following two lemmata.

Lemma 1. *Let* $f \in L^2(0, T)$ *and* $c \in C[0, T]$. *Then, there exists a unique solution* $y \in H_\alpha(0, T)$ *to*

$$\partial_t^\alpha y - c(t)y = f, \quad 0 < t < T.$$

Moreover, if $f \geq 0$ *in* $(0, T)$, *then* $y \geq 0$ *in* $(0, T)$.

Proof. The unique existence of y is proved in Kubica et al. [17, Section 3.5] for example. The non-negativity $y \geq 0$ in $(0, T)$ follows from the same argument in the study by Luchko and Yamamoto [20], which is based on the extremum principle by Luchko [19]. □

Lemma 2. *Let* $c_0 > 0$, $a_0 \geq 0$, $p > 1$ *be constants and* $y - a_0, z - a_0 \in H_\alpha(0, T) \cap C[0, T]$ *satisfy*

$$\partial_t^\alpha (y - a_0) \geq c_0 y^p, \quad \partial_t^\alpha (z - a_0) \leq c_0 z^p \quad \text{in } (0, T).$$

Then, $y \geq z$ *in* $(0, T)$.

Proof. We set

$$\partial_t^\alpha (y - a_0) - c_0 y^p =: f \geq 0, \quad \partial_t^\alpha (z - a_0) - c_0 z^p =: g \leq 0.$$

Since $y - a_0, z - a_0 \in H_\alpha(0, T) \cap C[0, T]$, we see that $f, g \in L^2(0, T)$. Setting

$$\theta := y - z = (y - a_0) - (z - a_0) \in H_\alpha(0, T),$$

we have

$$\partial_t^\alpha \theta - c_0(y^p - z^p) = f - g \geq 0 \quad \text{in } (0, T).$$

We can further prove that

$$\partial_t^\alpha \theta(t) - c_0 c(t)\theta(t) \geq 0, \quad 0 < t < T, \tag{2.1}$$

where

$$c(t) := \begin{cases} \dfrac{y^p(t) - z^p(t)}{y(t) - z(t)}, & y(t) \neq z(t), \\ py^{p-1}(t), & y(t) = z(t). \end{cases} \tag{2.2}$$

Indeed, we set $\Lambda := \{t \in [0, T]; y(t) \neq z(t)\}$. For $t_0 \in \Lambda$, we immediately see that $c(t_0)\theta(t_0) = y(t_0)^p - z(t_0)^p$. For $t_0 \notin \Lambda$, i.e.,

$$\theta(t_0) = (y(t_0) - a_0) - (z(t_0) - a_0) = 0,$$

first, we assume that there does not exist any sequence $\{t_n\} \subset \Lambda$ such that $t_n \to t_0$. Then, there exists some small $\varepsilon_0 > 0$ such that $(t_0 - \varepsilon_0, t_0 + \varepsilon_0) \cap \Lambda = \emptyset$. This means $\theta(t) = 0$ for $t_0 - \varepsilon_0 < t < t_0 + \varepsilon_0$, and thus,

$$c(t)\theta(t) = p\, y^{p-1}(t)\theta(t) = 0, \quad y^p(t) - z^p(t) = 0, \quad t_0 - \varepsilon_0 < t < t_0 + \varepsilon_0.$$

Hence, we obtain $c(t_0)\theta(t_0) = y^p(t_0) - z^p(t_0)$.

Next, assume that there exists a sequence $\{t_n\} \subset \Lambda$ such that $t_n \longrightarrow t_0 \notin \Lambda$ as $n \to \infty$. By $t_n \in \Lambda$, we have $y(t_n) \neq z(t_n)$ and

$$c(t_n)\theta(t_n) = \frac{y^p(t_n) - z^p(t_n)}{y(t_n) - z(t_n)}\theta(t_n), \quad n \in \mathbb{N}.$$

Since $y, z, \theta \in C[0, T]$ and $\theta(t_0) = 0$, we employ the mean value theorem to conclude

$$\lim_{n\to\infty} \frac{y^p(t_n) - z^p(t_n)}{y(t_n) - z(t_n)}\theta(t_n) = py^{p-1}(t_0)\theta(t_0) = 0.$$

Hence, again we arrive at $c(t_0)\theta(t_0) = y^p(t_0) - z^p(t_0)$ in this case. Thus, we have verified (2.1) with (2.2). Moreover, since $y, z \in C[0, T]$, we can verify that $c \in C[0, T]$.

Therefore, a direct application of Lemma 1 to (2.1) yields $\theta \geq 0$ in $(0, T)$ or equivalently $y \geq z$ in $(0, T)$. Thus the proof of Lemma 2 is complete. □

3 Completion of proof of Theorem 1

Step 1. We set

$$\eta(t) := \int_\Omega u(x, t)\mathrm{d}x = \int_\Omega (u(x, t) - a(x))\mathrm{d}x + a_0, \quad 0 < t < T,$$

where $a_0 := \int_\Omega a(x)\mathrm{d}x$. Here, we see that $a_0 > 0$ because $a \geq 0, \not\equiv 0$ in Ω by the assumption of Theorem 1.

Remark 2. We note that $\eta(t)$ is the inner product of the solution $u(\,\cdot\,, t)$ with the first eigenfunction 1 of $-\triangle$ with the homogeneous Neumann boundary condition. As for the parabolic case, we can refer to the studies by Kaplan [14] and Payne [21].

Henceforth, we assume that the solution u to (1.1) within the class (1.2) exists for $<t < T$. By (1.2), we have $\int_\Omega (u(x, t) - a(x))\mathrm{d}x \in H_\alpha(0, T)$. Fixing $\varepsilon > 0$ arbitrarily small, we see

$$\eta(t) - a_0 = \int_\Omega (u(x, t) - a(x))\mathrm{d}x \in H_\alpha(0, T - \varepsilon),$$

and hence,

$$\partial_t^\alpha(\eta(t) - a_0) = \int_\Omega \partial_t^\alpha (u - a)(x, t)\mathrm{d}x, \quad 0 < t < T - \varepsilon.$$

Since $\partial_\nu u = 0$ on $\partial\Omega \times (0, T-\varepsilon)$, Green's formula and the governing equation $\partial_t^\alpha(u-a) = \triangle u + u^p$ yield

$$\partial_t^\alpha(\eta(t)-a_0) = \int_\Omega \triangle u(x,t)\mathrm{d}x + \int_\Omega u^p(x,t)\mathrm{d}x = \int_\Omega u^p(x,t)\mathrm{d}x, \quad 0<t<T-\varepsilon. \tag{3.1}$$

On the other hand, introducing the Hölder conjugate $q>1$ of $p>1$, i.e., $\frac{1}{q}+\frac{1}{p}=1$, it follows from $u\ge 0$ in $\Omega\times(0,T-\varepsilon)$ and the Hölder inequality that

$$\eta(t) = \int_\Omega u(x,t)\mathrm{d}x \le \left(\int_\Omega u^p(x,t)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega \mathrm{d}x\right)^{\frac{1}{q}} = |\Omega|^{\frac{1}{q}}\left(\int_\Omega u^p(x,t)\mathrm{d}x\right)^{\frac{1}{p}},$$

i.e.,

$$\int_\Omega u^p(x,t)\mathrm{d}x \ge \omega_0\,\eta^p(t), \quad \omega_0 := |\Omega|^{-\frac{p}{q}}. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$\partial_t^\alpha(\eta(t)-a_0) \ge \omega_0\eta^p(t), \quad 0<t<T-\varepsilon. \tag{3.3}$$

Step 2. This step is devoted to the construction of a lower solution $\underline{\eta}(t)$ satisfying

$$\partial_t^\alpha(\underline{\eta}(t)-a_0)(t) \le \omega_0\,\underline{\eta}^p(t), \quad 0<t<T-\varepsilon, \quad \lim_{t\uparrow T}\underline{\eta}(t)=\infty. \tag{3.4}$$

We restrict the candidates of such a lower solution to

$$\underline{\eta}(t) := a_0\left(\frac{T}{T-t}\right)^m, \quad m\in\mathbb{N}. \tag{3.5}$$

To evaluate $\partial_t^\alpha(\underline{\eta}(t)-a_0)(t) = \mathrm{d}_t^\alpha\underline{\eta}(t)$, by definition, we have to represent $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{(T-t)^m})$ in terms of the Maclaurin expansion. First, direct calculations yield

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{1}{T-t}\right) = \frac{m!}{(T-t)^{m+1}},$$

and thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{(T-t)^m}\right) = \frac{m}{(T-t)^{m+1}} = \frac{1}{(m-1)!}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{1}{T-t}\right). \tag{3.6}$$

Next, by termwise differentiation, we have

$$\frac{1}{T-t} = \sum_{k=0}^\infty \frac{t^k}{T^{k+1}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{T-t}\right) = \sum_{k=1}^\infty \frac{kt^{k-1}}{T^{k+1}}$$

for $0\le t\le T-\varepsilon$. Repeating the calculations and by induction, we reach

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{1}{T-t}\right) = \sum_{k=m}^\infty \frac{k(k-1)\cdots(k-m+1)}{T^{k+1}}t^{k-m} = \frac{1}{T^{m+1}}\sum_{k=0}^\infty\prod_{j=1}^m(k+j)\left(\frac{t}{T}\right)^k. \tag{3.7}$$

Plugging (3.7) into (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{(T-t)^m}\right) = \frac{1}{T^{m+1}(m-1)!}\sum_{k=0}^\infty\prod_{j=1}^m(k+j)\left(\frac{t}{T}\right)^k.$$

Then, by the definition of d_t^α, we calculate

$$\begin{aligned} d_t^\alpha\left(\frac{1}{(T-t)^m}\right) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{ds}\left(\frac{1}{(T-s)^m}\right)ds \\ &= \frac{1}{\Gamma(1-\alpha)T^{m+1}(m-1)!}\sum_{k=0}^\infty \frac{\prod_{j=1}^m (k+j)}{T^k}\int_0^t (t-s)^{-\alpha}s^k ds. \end{aligned}$$

Here, we employ integration by substitution $s = t\xi$ and the beta function to treat

$$\begin{aligned} \int_0^t (t-s)^{-\alpha}s^k ds &= t^{k+1-\alpha}\int_0^1 (1-\xi)^{-\alpha}\xi^k d\xi \\ &= t^{k+1-\alpha}B(1-\alpha, k+1) = \frac{\Gamma(1-\alpha)k!}{\Gamma(k+2-\alpha)}t^{k+1-\alpha}, \end{aligned}$$

which implies

$$\begin{aligned} d_t^\alpha\left(\frac{1}{(T-t)^m}\right) &= \frac{1}{T^{m+1}(m-1)!}\sum_{k=0}^\infty \frac{\prod_{j=1}^m (k+j)\; k!}{\Gamma(k+2-\alpha)}\frac{t^{k+1-\alpha}}{T^k} \\ &= \frac{t^{1-\alpha}}{T^{m+1}(m-1)!}\sum_{k=0}^\infty \frac{(k+m)!}{\Gamma(k+2-\alpha)}\left(\frac{t}{T}\right)^k. \end{aligned}$$

Since $\Gamma(s)$ is monotone increasing in $s > 2$ and $0 < \Gamma(2-\alpha) < 1$, for $k \in \mathbb{N} \cup \{0\}$, we directly estimate

$$\Gamma(k+2-\alpha) \ge \begin{Bmatrix} \Gamma(2-\alpha), & k = 0, \\ \Gamma(k+1) = k!, & k \in \mathbb{N} \end{Bmatrix} \ge \Gamma(2-\alpha)\; k!.$$

Then, we can bound $d_t^\alpha(\frac{1}{(T-t)^m})$ from above as follows:

$$\begin{aligned} d_t^\alpha\left(\frac{1}{(T-t)^m}\right) &\le \frac{T^{1-\alpha}}{T^{m+1}(m-1)!}\sum_{k=0}^\infty \frac{(k+m)!}{\Gamma(2-\alpha)\; k!}\left(\frac{t}{T}\right)^k \\ &= \frac{1}{\Gamma(2-\alpha)\; T^{m+\alpha}(m-1)!}\sum_{k=0}^\infty \prod_{j=1}^m (k+j)\left(\frac{t}{T}\right)^k. \end{aligned}$$

For the series above, we utilize (3.6) and (3.7) again to find

$$\frac{1}{(T-t)^{m+1}} = \frac{1}{m!}\frac{d^m}{dt^m}\left(\frac{1}{T-t}\right) = \frac{1}{T^{m+1}m!}\sum_{k=0}^\infty \prod_{j=1}^m (k+j)\left(\frac{t}{T}\right)^k,$$

indicating

$$d_t^\alpha\left(\frac{1}{(T-t)^m}\right) \le \frac{1}{\Gamma(2-\alpha)\; T^{m+\alpha}(m-1)!}\frac{T^{m+1}m!}{(T-t)^{m+1}} = \frac{T^{1-\alpha}m}{\Gamma(2-\alpha)}\frac{1}{(T-t)^{m+1}}.$$

Recalling the definition (3.5) of $\underline{\eta}(t)$, we eventually arrive at

$$\partial_t^\alpha(\underline{\eta}(t) - a_0) = d_t^\alpha \underline{\eta}(t) = a_0 T^m d_t^\alpha\left(\frac{1}{(T-t)^m}\right) \le \frac{a_0 T^{m+1-\alpha}m}{\Gamma(2-\alpha)}\frac{1}{(T-t)^{m+1}}. \tag{3.8}$$

Note that (3.8) holds for arbitrary $m \in \mathbb{N}$, $T > 0$, and $0 < t < T - \varepsilon$.

Finally, we claim that for any $p > 1$ and $a_0 > 0$, there exist constants $m \in \mathbb{N}$ and $T > 0$ such that

$$\frac{a_0 T^{m+1-\alpha}m}{\Gamma(2-\alpha)}\frac{1}{(T-t)^{m+1}} \le \omega_0 \underline{\eta}^p(t) = \frac{\omega_0 a_0^p T^{mp}}{(T-t)^{mp}}, \quad 0 < t < T - \varepsilon. \tag{3.9}$$

In fact, (3.9) is achieved by

$$\frac{a_0 T^{-\alpha} m}{\Gamma(2-\alpha)} \le \omega_0 a_0^p \left(\frac{T}{T-t}\right)^{mp-(m+1)} \quad \text{for } 0 < t < T,$$

which holds if

$$\frac{a_0 T^{-\alpha} m}{\Gamma(2-\alpha)} \le \omega_0 a_0^p$$

by $mp - (m+1) \ge 0$ and $\frac{T}{T-t} \ge 1$ for $0 < t < T$. Therefore, if

$$\begin{aligned} T &\ge \left(\frac{m}{\Gamma(2-\alpha)\omega_0 a_0^{p-1}}\right)^{\frac{1}{\alpha}} \ge \left(\frac{1}{(p-1)\,\Gamma(2-\alpha)\omega_0 a_0^{p-1}}\right)^{\frac{1}{\alpha}} \\ &= \left\{(p-1)\Gamma(2-\alpha)\left(\frac{1}{|\Omega|}\int_\Omega a\mathrm{d}x\right)^{p-1}\right\}^{-\frac{1}{\alpha}} =: T^*(\alpha, p, a), \end{aligned} \tag{3.10}$$

then (3.9) is satisfied.

With the above chosen m and $T^*(\alpha, p, a)$, consequently, it follows from (3.8) and (3.9) that

$$\underline{\eta}(t) = a_0\left(\frac{T^*(\alpha, p, a)}{T^*(\alpha, p, a) - t}\right)^m$$

satisfies (3.4).

Now it suffices to apply Lemma 2 to (3.4) and (3.3) on $[0,\ T^*(\alpha, p, a) - \varepsilon]$ to obtain

$$\eta(t) \ge \underline{\eta}(t), \quad 0 \le t \le T^*(p, a) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen, we obtain

$$\int_\Omega u(x,t)\mathrm{d}x = \eta(t) \ge \underline{\eta}(t) = \frac{a_0 T^*(p,a)^m}{(T^*(p,a) - t)^m}, \quad 0 < t < T^*(\alpha, p, a).$$

Since $\eta(t) = \|u(\,\cdot\,,t)\|_{L^1(\Omega)}$, this means that the solution u cannot exist for $t > T^*(\alpha, p, a)$. Hence, the blowup time $T_{p,a} \le T^*(\alpha, p, a)$. The proof of Theorem 1 is complete.

4 Proof of Theorem 2

Step 1. Henceforth, we denote the norm and the inner product of $L^2(\Omega)$ by

$$\|a\| := \|a\|_{L^2(\Omega)}, \quad (a, b) := \int_\Omega a(x)b(x)\mathrm{d}x,$$

respectively. We show the following lemma.

Lemma 3. *Let* $0 < p < 1$, $0 \le w \in L^2(\Omega)$, *and* $0 \le \eta \in L^2(0, T)$. *Then,*

$$\|w^p\| \le |\Omega|^{\frac{1-p}{2}}\|w\|^p, \quad \|\eta^p\|_{L^2(0,T)} \le T^{\frac{1-p}{2}}\|\eta\|^p_{L^2(0,T)}.$$

Proof. By $0 < p < 1$, we see that $\frac{1}{1-p} > 1$, and the Hölder inequality yields

$$\|w^p\|^2 = \int_\Omega w^{2p}\mathrm{d}x \le \left(\int_\Omega (w^{2p})^{\frac{1}{p}}\mathrm{d}x\right)^p\left(\int_\Omega 1^{\frac{1}{1-p}}\mathrm{d}x\right)^{1-p} = |\Omega|^{1-p}\|w\|^{2p},$$

which completes the proof for w. The proof for η is the same. □

Let $A = -\triangle$ with $\mathcal{D}(A) = \{w \in H^2(\Omega);\ \partial_\nu w = 0 \text{ on } \partial\Omega\}$. We number all the eigenvalues of A as

$$0 = \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_n \longrightarrow \infty \quad (n \to \infty),$$

with their multiplicities. By $\{\varphi_n\}_{n\in\mathbb{N}}$, we denote the complete orthonormal basis of $L^2(\Omega)$ formed by the eigenfunctions of A, i.e., $A\varphi_n = \lambda_n\varphi_n$ and $\|\varphi_n\| = 1$ for $n \in \mathbb{N}$. We can define the fractional power A^β for $\beta \ge 0$, and we know that $\|a\|_{H^{2\beta}(\Omega)} \le C\|A^\beta a\|$ for all $a \in \mathcal{D}(A^\beta)$, where the constant $C > 0$ depends on β, Ω (e.g., [18,22]).

We further introduce the Mittag-Leffler functions by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $0 < \alpha < 1$ and $\beta > 0$. It is known that $E_{\alpha,\beta}(z)$ is an entire function in $z \in \mathbb{C}$, and we can refer, e.g., to Podlubny [23] for further properties of $E_{\alpha,\beta}(z)$.

Henceforth, we abbreviate $u(t) = u(\,\cdot\,,t)$ and interpret $u(t)$ as a mapping from $(0, T)$ to $L^2(\Omega)$. We define

$$S(t)a \coloneqq \sum_{k=1}^{\infty} (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n,$$
$$K(t)a \coloneqq \sum_{k=1}^{\infty} (a, \varphi_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n$$

for $a \in L^2(\Omega)$ and $t > 0$. Then, as was proved in [11,31], we have the following lemma.

Lemma 4. (*i*) *Let* $0 < \gamma < 1$. *Then, there exists a constant* $C = C(\gamma) > 0$ *such that*

$$\|A^\gamma S(t)a\| \le C\ t^{-\alpha\gamma}\|a\|, \quad \|A^\gamma K(t)a\| \le C\ t^{\alpha(1-\gamma)-1}\|a\|$$

for all $a \in L^2(\Omega)$ *and all* $t > 0$.

(*ii*) *Let* $v \in L^2(0, T;\ \mathcal{D}(A))$ *satisfy* $v - a \in H_\alpha(0, T;\ L^2(\Omega))$ *and*

$$\partial_t^\alpha (v - a) = -Av + F, \quad t > 0,$$

with $a \in \mathcal{D}(A^{\frac{1}{2}})$ *and* $F(t) = F(\,\cdot\,,t) \in L^2(0, T;\ L^2(\Omega))$. *Then,* v *allows the representation*

$$v(t) = S(t)a + \int_0^t K(t-s)F(s)\mathrm{d}s, \quad t > 0.$$

(*iii*) *There holds*

$$\left\| \int_0^t K(t-s)F(s)\mathrm{d}s \right\|_{H_\alpha(0,T;\ L^2(\Omega))} \le C\|F\|_{L^2(0,T;\ L^2(\Omega))}.$$

Step 2. Let $T > 0$ be arbitrarily given. We show that there exists $u \in L^2(0, T;\ L^2(\Omega))$ such that $u \ge 0$ in $\Omega \times (0, T)$ and

$$u(t) = S(t)a + \int_0^t K(t-s)u(s)^p\mathrm{d}s, \quad 0 < t < T.$$

Henceforth, by $C > 0$, we denote generic constants depending on Ω, T, and p but independent of the choices of functions $a(x), u(x, t), v(x, t)$, etc.

Lemma 4(i) implies

$$\|S(t)a\| \le C\|a\|, \quad t \ge 0. \tag{4.1}$$

We choose a constant $M > 0$ sufficiently large such that

$$C(M + C\|a\|)^p \le M. \tag{4.2}$$

Since $0 < p < 1$, we can easily verify the existence of such $M > 0$ satisfying (4.2).

With this $M > 0$, we define a set $\mathcal{B} \subset L^2(0,T;L^2(\Omega))$ by

$$\mathcal{B} := \{v \in L^2(0,T;L^2(\Omega));\ v \ge 0 \ \text{in}\ \Omega \times (0,T), \quad \|v - S(t)a\|_{L^2(0,T;\,L^2(\Omega))} \le M\}.$$

We define a mapping L by

$$Lv(t) := S(t)a + \int_0^t K(t-s)v^p(s)\,\mathrm{d}s, \quad 0 < t < T, \quad v \in \mathcal{B}.$$

Now we will prove

$$L\mathcal{B} \subset \mathcal{B} \tag{4.3}$$

and

$$L : \mathcal{B} \longrightarrow \mathcal{B} \ \text{is a compact operator}. \tag{4.4}$$

Proof of (4.3). Let $v \in \mathcal{B}$. Then, we have

$$\|v\|_{L^2(0,T;\,L^2(\Omega))} \le \|v - S(t)a\|_{L^2(0,T;\,L^2(\Omega))} + \|S(t)a\|_{L^2(0,T;\,L^2(\Omega))} \le M + C\|a\| \tag{4.5}$$

by the definition of $\mathcal{B}$ and (4.1). On the other hand, Lemma 3 implies

$$\begin{aligned}\|v^p\|^2_{L^2(0,T;\,L^2(\Omega))} &= \int_0^T \|v^p(t)\|^2\,\mathrm{d}t \le |\Omega|^{1-p}\int_0^T \|v(t)\|^{2p}\,\mathrm{d}t\\ &\le |\Omega|^{1-p}T^{1-p}\left(\int_0^T \|v(t)\|^2\,\mathrm{d}t\right)^p = (|\Omega|\ T)^{1-p}\|v\|^{2p}_{L^2(0,T;\,L^2(\Omega))}.\end{aligned}$$

Therefore, substituting (4.5) into the above inequality yields

$$\|v^p\|_{L^2(0,T;\,L^2(\Omega))} \le (|\Omega|\ T)^{\frac{1-p}{2}}\|v\|^p_{L^2(0,T;\,L^2(\Omega))} \le C(M + C\|a\|)^p. \tag{4.6}$$

Consequently, Lemma 4(iii) and (4.2) and (4.6) imply

$$\begin{aligned}\|Lv(t) - S(t)a\|_{L^2(0,T;\,L^2(\Omega))} &= \left\| \int_0^t K(t-s)v^p(s)\,\mathrm{d}s \right\|_{L^2(0,T;\,L^2(\Omega))}\\ &\le \left\| \int_0^t K(t-s)v^p(s)\,\mathrm{d}s \right\|_{H_\alpha(0,T;\,L^2(\Omega))} \le C\|v^p\|_{L^2(0,T;\,L^2(\Omega))}\\ &\le C(M + C\|a\|)^p \le M.\end{aligned}$$

Next, by $a \ge 0$ in Ω and $v \ge 0$ in $\Omega \times (0,T)$, we can apply the comparison principle (e.g., [20]) to have

$$\int_0^t K(t-s)v^p(s)\,\mathrm{d}s \ge 0 \quad \text{in}\ \Omega \times (0,T),$$

and so, $Lv \ge 0$ in $\Omega \times (0,T)$. Hence, $Lv \in \mathcal{B}$, and thus the proof of (4.3) is complete. □

Proof of (4.4). Since $S(t)a$ is a fixed element independent of v, it suffices to verify that

$$L_0v(t) := \int_0^t K(t-s)v^p(s)\,\mathrm{d}s$$

is a compact operator from $\mathcal{B}$ to $L^2(0,T;\, L^2(\Omega))$. Let $M_0 > 0$ be an arbitrarily chosen constant and let $\|v\|_{L^2(0,T;\, L^2(\Omega))} \le M_0$, $v \ge 0$ in $\Omega \times (0,T)$. Then, Lemma 3 indicates

$$\|v^p\|_{L^2(0,T;\, L^2(\Omega))} \le C\|v\|^p_{L^2(0,T;\, L^2(\Omega))} \le CM_0^p, \tag{4.7}$$

with which we combine Lemma 4(iii) to obtain

$$\|L_0 v\|_{H_\alpha(0,T;\, L^2(\Omega))} \le CM_0^p. \tag{4.8}$$

Next, for small $\varepsilon \in (0,1)$, in view of Lemma 4(i), we estimate

$$\begin{aligned}\|A^\varepsilon L_0 v(t)\| &= \left\| \int_0^t A^\varepsilon K(t-s) v^p(s)\mathrm{d}s \right\| \le \int_0^t \|A^\varepsilon K(t-s)v^p(s)\|\mathrm{d}s \\ &\le C\int_0^t (t-s)^{(1-\varepsilon)\alpha-1}\|v^p(s)\|\mathrm{d}s.\end{aligned}$$

Hence, in terms of (4.7), Young's convolution inequality implies

$$\begin{aligned}\|A^\varepsilon L_0 v\|_{L^2(0,T;\, L^2(\Omega))} &\le C \left\| \int_0^t (t-s)^{(1-\varepsilon)\alpha-1}\|v^p(s)\|\mathrm{d}s \right\|_{L^2(0,T)} \\ &\le C\|t^{(1-\varepsilon)\alpha-1}\|_{L^1(0,T)} \left(\int_0^T \|v^p(t)\|^2 \mathrm{d}t \right)^{\frac{1}{2}} \\ &\le C\|v^p\|_{L^2(0,T;\, L^2(\Omega))} \le CM_0^p.\end{aligned}$$

Since $\mathcal{D}(A^\varepsilon) \subset H^{2\varepsilon}(\Omega)$, we have

$$\|L_0 v\|_{L^2(0,T;\, H^{2\varepsilon}(\Omega))} \le C\|A^\varepsilon L_0 v\|_{L^2(0,T;\, L^2(\Omega))} \le CM_0^p. \tag{4.9}$$

On the other hand, we know that the embedding $L^2(0,T;\, H^{2\varepsilon}(\Omega)) \cap H^\alpha(0,T;\, L^2(\Omega)) \subset L^2(0,T;\, L^2(\Omega))$ is compact (e.g., [28, Theorem 2.1, p. 271]), so that (4.8) and (4.9) imply that $L_0 : \mathcal{B} \subset L^2(0,T;\, L^2(\Omega)) \longrightarrow \mathcal{B}$ is compact. This completes the proof of (4.4). □

Since $\mathcal{B}$ is a closed and convex set in $L^2(0,T;\, L^2(\Omega))$, we can apply the Schauder fixed-point theorem to conclude that L possesses a fixed-point u satisfying

$$u(t) = S(t)a + \int_0^t K(t-s)u(s)^p \mathrm{d}s, \quad 0 < t < T, \quad u \ge 0 \quad \text{in } \Omega \times (0,T). \tag{4.10}$$

Step 3. Recalling that $\frac{3}{4} < \gamma \le 1$, we note that if $a \in H^{2\gamma}(\Omega)$ and $\partial_\nu a = 0$ on $\partial\Omega$, then $a \in \mathcal{D}(A^\gamma)$. Now it remains to prove that the fixed-point u satisfies (1.2). To this end, we separate

$$u(t) - a = (S(t)a - a) + \int_0^t K(t-s)u(s)^p \mathrm{d}s =: u_1(t) + u_2(t), \quad 0 < t < T.$$

First, we verify (1.2) for $u_1(t)$. In the same way as that for Yamamoto [31, Lemma 5(i)], we can prove that $\partial_t^\alpha u_1(t) = -AS(t)a$ in $(0,T)$ and $u_1 \in H_\alpha(0,T;\, L^2(\Omega))$ by $a \in \mathcal{D}(A^\gamma)$ with $\gamma > \frac{3}{4}$. Therefore, we obtain

$$u_1 \in H_\alpha(0,T;\, L^2(\Omega)), \quad S(t)a \in L^2(0,T;\, \mathcal{D}(A)).$$

Next, we verify (1.2) for $u_2(t)$. In terms of $u^p \in L^2(0,T;\, L^2(\Omega))$, Lemma 4(iii) implies that $u_2 \in H_\alpha(0,T;\, L^2(\Omega))$ and $\partial_t^\alpha u_2 = -Au_2 + u(t)^p$ for $0 < t < T$. Therefore, we have $-Au_2 \in L^2(0,T;\, L^2(\Omega))$ or equivalently $u_2 \in L^2(0,T;\, \mathcal{D}(A))$.

Consequently, it is verified that the fixed-point u satisfies (1.2). By (1.2) and (4.10) we see that u satisfies (1.1) in terms of [31, Lemma 5]. Thus, the proof of Theorem 2 is complete.

5 Concluding remarks and discussions

1. In this article, we consider the blowup exclusively in $L^1(\Omega)$. If we will discuss in the space $L^\infty(\Omega)$, for example, then we can more directly use a lower solution. More precisely, in (1.1) assuming that $\min_{x\in\overline{\Omega}} a(x) =: a_1 > 0$, if we can find a function $g(t)$ satisfying

$$\partial_t^\alpha (g(t) - a_1) \le g(t)^p, \quad 0 < t < T,$$

then $\underline{u}(x,t) := g(t)$ for $x \in \Omega$ and $0 < t < T$ is a lower solution to (1.1), i.e.,

$$\begin{cases} \partial_t^\alpha(\underline{u} - a_1) \le \triangle \underline{u} + \underline{u}^p & \text{in } \Omega \times (0,T), \\ \partial_\nu \underline{u} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Then, the comparison principle (e.g., [20]) yields

$$g(t) \le u(x,t), \quad x \in \Omega, \ 0 < t < T.$$

As $g(t)$, we take a similar function to (3.5):

$$g(t) := a_1\left(\frac{T}{T-t}\right)^m, \quad m \in \mathbb{N}.$$

Then, by (3.8) we have

$$\partial_t^\alpha(g(t) - a_1) \le \frac{a_1 m}{T^\alpha \Gamma(2-\alpha)}\left(\frac{T}{T-t}\right)^{m+1}, \quad 0 < t < T.$$

Therefore, for $mp - (m+1) \ge 0$, it suffices to choose $T > 0$ such that

$$\frac{a_1 m}{T^\alpha \Gamma(2-\alpha)}\left(\frac{T}{T-t}\right)^{m+1} \le a_1^p\left(\frac{T}{T-t}\right)^{mp} = g(t)^p, \quad 0 < t < T,$$

i.e.,

$$\frac{a_1^{1-p} m}{T^\alpha \Gamma(2-\alpha)} \le \xi^{mp-(m+1)} \quad \text{for all } \xi \ge 1$$

by setting $\xi := \frac{T}{T-t} \ge 1$. Hence, $g(t)$ is a lower solution if

$$\frac{a_1^{1-p} m}{T^\alpha \Gamma(2-\alpha)} \le 1, \quad \text{i.e.,} \quad T \ge \left(\frac{a_1^{1-p} m}{\Gamma(2-\alpha)}\right)^{\frac{1}{\alpha}}$$

for $mp \ge m + 1$. Choosing the minimum $m \in \mathbb{N}$ and arguing similarly to the final part of the proof of Theorem 1, we obtain an inequality for the blowup time $T_{\alpha,p,a}(\infty)$ in $L^\infty(\Omega)$:

$$T_{\alpha,p,a}(\infty) \le \left(\frac{\left[\frac{1}{p-1}\right] + 1}{\Gamma(2-\alpha) a_1^{p-1}}\right)^{\frac{1}{\alpha}} =: T_\infty^*(\alpha, p, a), \tag{5.1}$$

where $[q]$ denotes the maximum natural number not exceeding $q > 0$.

We compare $T_\infty^*(\alpha,p,a)$ with an upper bound $T^*(\alpha,p,a)$ of the blowup time in $L^1(\Omega)$. Noting that $a_1 \le \frac{1}{|\Omega|}\int_\Omega a(x)\mathrm{d}x$, we can interpret that $\frac{1}{|\Omega|}\int_\Omega a(x)\mathrm{d}x$ is comparable with a_1 and so we consider the case where $a_1 = \frac{1}{|\Omega|}\int_\Omega a(x)\mathrm{d}x$. Then, by (1.4), we have

$$T_{\alpha,p,a} \le T^*(\alpha, p, a) = \left(\frac{\frac{1}{p-1}}{\Gamma(2-\alpha)a_1^{p-1}}\right)^{\frac{1}{\alpha}}. \tag{5.2}$$

Hence, $\left[\frac{1}{p-1}\right] + 1 \ge \frac{1}{p-1}$ implies $T^*(\alpha, p, a) < T^*_\infty(\alpha, p, a)$.

To sum up, for the $L^1(\Omega)$-blowup time $T_{\alpha,p,a}$ and the $L^\infty(\Omega)$-blowup time $T_{\alpha,p,a}(\infty)$, our upper bounds $T^*(\alpha, p, a)$ and $T^*_\infty(\alpha, p, a)$ of $T_{\alpha,p,a}$ and $T_{\alpha,p,a}(\infty)$ are given by (5.2) and (5.1), respectively. Although we should expect $T^*_\infty(\alpha, p, a) \le T^*(\alpha, p, a)$ by means $T_{\alpha,p,a}(\infty) \le T_{\alpha,p,a}$, which follows from $\|u(\cdot, t)\|_{L^1(\Omega)} \le C\|u(\cdot, t)\|_{L^\infty(\Omega)}$, but our bounds do not satisfy. The upper bound depends on our choice of lower solutions, and it is a future work to discuss sharper bounds.

2. Restricting the nonlinearity to the polynomial type u^p, in this article, we investigate semilinear time-fractional diffusion with the homogeneous Neumann boundary condition. With nonnegative initial values, we obtained the blowup of solutions with $p > 1$ as well as the global-in-time existence of solutions with $0 < p < 1$. The key ingredient for the latter is the Schauder fixed-point theorem, whereas that for the former turns out to be a comparison principle for time-fractional ordinary differential (see Lemma 2) and the construction of a lower solution of the form (3.5). We can similarly discuss the blowup for certain semilinear terms like the exponential type e^u and some coupled systems. More generally, it appears plausible to consider a general convex semilinear term $f(u)$, which deserves further investigation.

Technically, by introducing

$$\eta(t) := \int_\Omega u(x, t)dx = (u(\cdot, t), 1)_{L^2(\Omega)},$$

we reduce the blowup problem to the discussion of a time-fractional ordinary differential equation. As was mentioned in Remark 2, indeed 1 is the eigenfunction for the smallest eigenvalue 0 of $-\triangle$ with $\partial_\nu u = 0$. On this direction, it is not difficult to replace $-\triangle$ with a more general elliptic operator. Actually, in place of 1, one can choose an eigenfunction φ_1 for the smallest eigenvalue λ_1 and consider $\eta(t) := (u(\cdot, t), \varphi_1)_{L^2(\Omega)}$ to follow the above arguments. In this case, it is essential that $\lambda_1 \ge 0$ and φ_1 does not change sign. We can similarly discuss the homogeneous Dirichlet boundary condition.

3. In the proof of Theorem 1, we obtained an upper bound $T^*(\alpha, p, a)$ of the blowup time T (see (1.4)), but there is no guarantee for its sharpness. Sharp estimates for the blowup time in the time-fractional case is expected to be more complicated than the parabolic case, which is postponed to a future topic.

We briefly investigate the monotonicity of

$$T^*(\alpha, p, a) = \left(\frac{1}{(p-1)\Gamma(2-\alpha)\left(\frac{1}{|\Omega|}\int_\Omega a(x)dx\right)^{p-1}}\right)^{\frac{1}{\alpha}} > 0$$

as a function of $\alpha \in (0, 1)$ with fixed p and a. Setting

$$C_{p,a} := (p-1)\left(\frac{1}{|\Omega|}\int_\Omega a(x)dx\right)^{p-1} > 0,$$

we can verify that there exist positive constants $C^* \ge 1$ and $C_* \le 1$ such that $T^*(\alpha, p, a)$ is monotone increasing in α if $C_{p,a} \ge C^*$ and monotone decreasing in α if $C_{p,a} \le C_*$.

Indeed, setting $f(\alpha) := T^*(\alpha, p, a)$ for simplicity for fixed p and a, we have

$$\log f(\alpha) = -\frac{1}{\alpha}\log(C_{p,a}\Gamma(2-\alpha)),$$

i.e.,

$$\frac{f'(\alpha)}{f(\alpha)} = -\frac{1}{\alpha}\frac{\frac{d}{d\alpha}(\Gamma(2-\alpha))}{\Gamma(2-\alpha)} + \frac{1}{\alpha^2}\log(C_{p,\alpha}\Gamma(2-\alpha)) = \frac{1}{\alpha^2}\left(\log(C_{p,\alpha}\Gamma(2-\alpha)) + \alpha\frac{\Gamma'(2-\alpha)}{\Gamma(2-\alpha)}\right)$$

for $0 < \alpha < 1$. We set $\delta_0 \coloneqq \min_{0\le\alpha\le1}\Gamma(2-\alpha) > 0$ and $M_1 \coloneqq \max_{0\le\alpha\le1}\left|\frac{\Gamma'(2-\alpha)}{\Gamma(2-\alpha)}\right|$. Then,

$$\frac{f'(\alpha)}{f(\alpha)} \ge \frac{1}{\alpha^2}(\log(C_{p,\alpha}\delta_0) - M_1) > 0$$

if $C_{p,\alpha} > 0$ is sufficiently large. On the other hand, since $\Gamma(2-\alpha) \le 1$ for $0 \le \alpha \le 1$, we see that

$$\frac{f'(\alpha)}{f(\alpha)} \le \frac{1}{\alpha^2}(\log C_{p,\alpha} + \alpha M_1) \le \frac{1}{\alpha^2}(\log C_{p,\alpha} + M_1) < 0$$

if $C_{p,\alpha} > 0$ is sufficiently small.

Since

$$\log(\Gamma(2-\alpha)^{-\frac{1}{\alpha}}) = \frac{\log\Gamma(2-\alpha) - \log\Gamma(2)}{-\alpha} \longrightarrow \frac{d}{d\beta}\Gamma(\beta)|_{\beta=2}$$

as $\alpha \to 0+$, we have $\lim_{\alpha\to0+} f(\alpha) = e^{\Gamma'(2)}$ if $C_{p,\alpha} = 1$. Therefore,

$$\lim_{\alpha\to0+} f(\alpha) = \begin{cases} +\infty, & C_{p,\alpha} < 1, \\ e^{\Gamma'(2)}, & C_{p,\alpha} = 1, \\ 0, & C_{p,\alpha} > 1. \end{cases}$$

In particular, $f(\alpha)$ cannot be monotone increasing for $C_{p,\alpha} < 1$ and cannot be monotone decreasing for $C_{p,\alpha} > 1$, which implies $C^* \ge 1$ and $C_* \le 1$.

4. Related to the blowup, we should study the following issues:

(i) Lower bounds or characterization of the blowup times.

(ii) Asymptotic behavior or lower bound of a solution near the blowup time.

(iii) Blowup set of a solution $u(x, t)$, which means the set of $x \in \Omega$, where $|u(x, t)|$ tends to ∞ as t approaches the blowup time.

For $\alpha = 1$, comprehensive and substantial works have been accomplished. We are here restricted to refer to Chapter II of Quittner and Souplet [24] and the references therein. However, for $0 < \alpha < 1$, by the memory effect of $\partial_t^\alpha u(\,\cdot\,, t)$ which involves the past value of u, several useful properties for discussing the above issues (i)–(iii) do not hold. Thus, the available results related to the blowup are still limited for $0 < \alpha < 1$, and it is up to future studies to pursue (i)–(iii).

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