# The Geometry of the Bismut Connection 

Scuola di Dottorato in Matematica
Dottorato di Ricerca in Matematica (XXXV cycle)

## Giuseppe Barbaro

ID number 1893953

Advisor
Prof. Daniele Angella

Thesis defended on 12 May 2023
in front of a Board of Examiners composed by:
Prof. Simone Diverio (chairman)
Prof. Stefano Trapani
Prof. Luis Ugarte

The manuscript has been reviewed by:
Prof. Mario Garcia-Fernandez
Prof. Raquel Villacampa

The Geometry of the Bismut Connection
Ph.D. thesis. Sapienza University of Rome
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This thesis has been typeset by $\mathrm{AT}_{\mathrm{E}} \mathrm{X}$ and the Sapthesis class.
Author's email: giuseppe95barbaro@gmail.com


#### Abstract

This thesis concerns the study of special metrics in Hermitian and almost-Hermitian geometry, characterized by classical constraints on the curvature of their Bismut, Chern, or Gauduchon connection. More precisely, we intend to study the analogs in Hermitian and almost-Hermitian geometry of constant scalar curvature metrics, Einstein metrics, and metrics whose curvature tensor satisfies some positivity notion.

We study the existence of metrics with constant scalar curvature with respect to the Gauduchon connection, which can be interpreted as a Yamabe-type problem. We then analyze the geometry of 4-dimensional compact almost-complex manifolds that carry a second-ChernEinstein metric and we produce new examples of such spaces. With the aim of investigating the geometry of the Bismut connection, we describe the Calabi-Yau with torsion metrics of submersion type on toric bundles over Hermitian manifolds. Moreover, we analyze the cohomological properties of compact complex manifolds equipped with a Bismut flat metric. This leads to a better understanding of the evolution of the pluriclosed flow on Bismut flat manifolds. Finally, we consider a new notion of positivity for Hermitian manifolds which involves the Bismut curvature tensor, and we investigate its behavior under the action of the Hermitian curvature flows.


## Contents

Introduction ..... vii
1 Preliminaries on Hermitian and almost-Hermitian geometry ..... 1
1.1 Almost-Hermitian structures and non-Kähler metrics ..... 1
1.1.1 Almost-complex structures ..... 2
1.1.2 Almost-Hermitian structures ..... 3
1.1.3 Special almost-Hermitian metrics ..... 4
1.1.4 Hodge-de Rham Laplacian and Chern Laplacian ..... 5
1.2 Hermitian and conformal connections ..... 6
1.2.1 Chern connection ..... 7
1.2.2 Bismut connection ..... 8
1.2.3 Gauduchon connections ..... 9
1.2.4 Canonical Weyl connection ..... 10
1.2.5 Curvature tensors ..... 10
1.2.6 Further relations between Gauduchon curvature tensors ..... 14
1.3 Double complex and cohomologies of complex manifolds ..... 15
1.3.1 Dolbeault, Bott-Chern, and Aeppli cohomologies ..... 16
1.3.2 Structure of the double complex of forms ..... 17
2 Symmetries of the curvature of Hermitian connections and examples ..... 21
2.1 Kähler-like condition ..... 21
2.2 Hopf manifolds ..... 24
2.2.1 Diagonal Hopf manifolds ..... 24
2.2.2 Hopf surfaces ..... 26
2.3 Calabi-Eckmann manifolds ..... 27
2.3.1 Calabi-Eckmann threefold ..... 30
3 The scalar curvature of Gauduchon connections ..... 31
3.1 Gauduchon degree and conformal changes ..... 31
3.2 Yamabe problem for Gauduchon connections ..... 33
3.3 Existence of constant scalar curvature metrics ..... 35
3.3.1 Linear case ..... 35
3.3.2 Non-linear case ..... 35
4 The Ricci curvature of the Chern connection on almost-Hermitian 4-manifolds ..... 41
4.1 Second-Chern-Einstein and Weyl-Einstein metrics ..... 41
4.2 Lee form of a second-Chern-Einstein metric ..... 44
4.3 Second-Chern-Einstein metrics with constant scalar curvature ..... 46
4.4 Geometry of almost-Hermitian second-Chern-Einstein manifolds ..... 47
4.4.1 Second-Chern-Einstein locally conformally almost-Kähler metrics, ..... 49
4.4.2 Second-Chern-Einstein metrics with vanishing $N_{\theta^{\sharp}}$ ..... 51
4.5 Examples of compact almost-Hermitian second-Chern-Einstein 4-manifolds ..... 53
4.6 Second-Chern-Einstein metrics on almost-Abelian Lie algebras ..... 56
4.6.1 Second Chern-Ricci and Bismut-Ricci tensors ..... 56
4.6.2 Almost-Abelian Lie groups ..... 57
4.6.3 Classification of second-Chern-Einstein almost-Abelian Lie algebras ..... 58
5 The Ricci curvature of the Bismut connection ..... 61
5.1 Calabi-Yau with torsion manifolds ..... 61
5.2 Calabi-Yau with torsion metrics on toric bundles over Hermitian manifolds ..... 63
5.3 Calabi-Yau with torsion metrics on class $\mathcal{C}$ manifolds ..... 66
6 The curvature of the Bismut connection ..... 69
6.1 Flat connections with torsion ..... 69
6.2 Flat Bismut connection ..... 70
6.3 Complex structures on Bismut flat manifolds ..... 71
6.3.1 Cartan decomposition ..... 72
6.3.2 Samelson construction ..... 72
6.3.3 Isotropic complex structures on Bismut flat manifolds of rank two ..... 73
6.4 Bott-Chern cohomology of Bismut flat manifolds of rank two ..... 76
6.4.1 Structure of the double complexes ..... 77
6.4.2 Harmonic Bott-Chern representatives ..... 80
7 The Pluriclosed flow and the stability of the Bismut flat metrics ..... 87
7.1 Hermitian curvature flows ..... 87
7.2 Pluriclosed flow ..... 89
7.3 Stability of the Bismut flat metrics for the pluriclosed flow ..... 92
7.4 Further analysis on higher rank cases ..... 94
8 Bismut positivity along Hermitian Curvature Flows ..... 97
8.1 Positivity notions for Hermitian manifolds ..... 97
8.1.1 Griffiths positivity ..... 97
8.1.2 Bismut positivity ..... 98
8.1.3 Bismut positivity on 6 -dimensional Calabi-Yau solvmanifolds ..... 99
8.1.4 Bismut positivity on Hopf manifolds ..... 102
8.2 Evolution of the Hermitian curvature flows on Hopf manifolds ..... 103
8.3 Hermitian curvature flows preserving Bismut positivity ..... 105
8.4 Computations on 6-dimensional Calabi-Yau solvmanifolds ..... 106
8.4.1 Nilmanifolds ..... 106
8.4.2 Solvmanifolds ..... 109
Bibliography ..... 115

## Introduction

Kähler geometry lies at the intersection of complex, Riemannian and symplectic geometry. Indeed, a Kähler manifold ( $M, J, g$ ) is endowed with a complex structure $J$ and a compatible metric $g$ which yields a symplectic structure, i. e. $g$ is $J$-invariant and $\omega:=g(J \cdot, \cdot)$ is a closed non-degenerate 2 -form. The interaction between these structures allows us to derive the very special properties of Kähler manifolds. In order to better understand these properties, it is hence natural to study the contribution of any of the three structures separately. A classical approach to do so is by weakening the symplectic assumption. For example, various notions generalizing the Kähler condition have been introduced and studied: balanced in the sense of Michelsohn [229], pluriclosed (49], Gauduchon [136, locally conformally Kähler and locally conformally symplectic 312, 314, and so on. On the other hand, one can eventually relax also the complex assumption, which leads to study almost-Hermitian manifolds.

Several natural connections have been introduced to deal with the non-Kähler case. As a matter of fact, on an almost-Hermitian manifold $(M, J, g)$ the Levi-Civita connection $\nabla^{L C}$ preserves the complex structure, i.e. $\nabla^{L C} J=0$, if and only if $J$ is integrable and the metric is Kähler. Henceforth, one considers other connections, possibly with torsion, that preserve the almost-Hermitian structure. These connections $\nabla$ are called Hermitian, and satisfy $\nabla J=\nabla g=0$. The most important class of Hermitian connections is given by the canonical connections. They were firstly considered by Libermann in [214], and include the first and second canonical connections of Lichnerowicz [215]. In [141] Gauduchon gave a unified presentation of them as a 1-parameter family of Hermitian connections $\left\{\nabla^{t}\right\}_{t \in \mathbb{R}}$ with a specific ansatz on the torsion. In particular, their torsion vanishes if and only if the Hermitian structure is Kähler.

Given an almost-Hermitian manifold ( $M, J, g$ ), we denote the Nijenhuis tensor characterizing the almost-complex structures $J$ with the symbol $N_{J}$. Then the canonical Gauduchon connections are described with respect to the Levi-Civita connection as

$$
\begin{equation*}
g\left(\nabla_{x}^{t} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right)+\frac{1-t}{4} d \omega(J x, J y, J z)+\frac{1+t}{4} d \omega(J x, y, z)+\frac{1}{2} g\left(N_{J}(x, y), z\right), \tag{0.1}
\end{equation*}
$$

for $x, y, z$ vector fields on $M$. The importance of this family lies in the fact that it contains both the Chern connection $\nabla^{C h}$, corresponding to the Gauduchon parameter $t=1$, and the Bismut connection $\nabla^{B}$ (also known as Strominger or Strominger-Bismut connection), corresponding to $t=-1$. We recall that for a Hermitian manifold, $\nabla^{C h}$ is the Chern connection of the holomorphic tangent bundle, namely the unique Hermitian connection whose $(0,1)$-component equals the Cauchy-Riemann operator of the holomorphic tangent bundle; it is also characterized as the unique Hermitian connection whose torsion has vanishing (1,1)-component (see Section 1.2.1). On the other hand, $\nabla^{B}$ is the unique Hermitian connection with skew-symmetric torsion (see Section 1.2.2. Since all these connections are equal in the Kähler case, it is hence natural to try to understand the contribution of any of them to the properties of Kähler geometry. In particular, in this thesis, we study some results on the geometry of the Bismut connection, while, for some problems, we keep our results general including also the other canonical Gauduchon connections.

The importance of the Bismut connection comes from its wide applications in both Mathematics and Physics. For the former, the Bismut connection appears in non-Kähler index theory
[49, 51, 52 and in the context of geometrization of complex surfaces through the pluriclosed flow 284, 285, 287]; for the latter, the Bismut connection is also a very powerful tool in describing models of string theory $\backslash 72,134,175,288$. The crucial peculiarity of the Bismut connection that makes it useful in these areas is that it is a connection with skew-symmetric torsion chosen to adapt to the Hermitian context (see Section 1.2 .2 for more details). Indeed, on a Hermitian manifold $(M, J, g)$, the torsion of the Bismut connection is given by (see (0.1) with parameter $t=-1$ )

$$
\begin{equation*}
g\left(T^{B}(\cdot, \cdot), \cdot\right)=d \omega(J \cdot, J \cdot, J \cdot)=-J d \omega \tag{0.2}
\end{equation*}
$$

We shall remark that connections with prescribed skew-symmetric torsion have appeared implicitly in the last century's mathematical literature before the introduction of the Bismut connection. For instance, in 330 Yano considered the connections $\nabla^{ \pm}$in its study of almost-product-structures. Here, $\nabla^{ \pm}$are the unique metric connections with skew-symmetric torsion given by a fixed 3 -form $\pm H$, namely

$$
\begin{equation*}
g\left(\nabla_{x}^{ \pm} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right) \pm \frac{1}{2} H(x, y, z) \tag{0.3}
\end{equation*}
$$

Remarkably, Cartan and Schouten $[82,83$ also encountered these connections in their study of Riemannian manifolds admitting an "absolute parallelism", i.e. a flat metric connection. In particular, they showed that, beyond classical flat Riemannian metrics, the only examples are semisimple Lie groups with bi-invariant metric and torsion determined canonically by the Lie algebra structure, as well as an exceptional example on the seven-sphere $\mathbb{S}^{7}$. It turns out that, in the Hermitian case, these are precisely the Bismut flat manifolds, namely those Hermitian manifolds whose Bismut curvature tensor vanishes identically, i. e. $R^{B} \equiv 0$. This fact is intimately related to the very basic idea behind the notion of torsion. Indeed, the notion of torsion of a connection was invented by Elie Cartan (it appeared for the first time in a short note at the Académie des Sciences de Paris in 1922 [78]), and Cartan's fundamental idea was that the connection should be adapted to the geometry of the manifold. As an example of this insight, the characteristic connection 122 of a non-symmetric homogeneous space does not coincide with the Levi-Civita connection [302], but it is a connection with skew-symmetric torsion given by the Lie brackets (see also Section 6.1 and Example 6.1.1. In particular, in the Hermitian case, it is the Bismut connection.

Let us now explain in more detail the importance of the Bismut connection and in general of the connections with skew-symmetric torsion in the mathematical and physical developments of the last century. The torsion of a connection made its first appearance in Physics in the 1920s through the work of Cartan [79], who proposed a modified version of Einstein's General Relativity. This led to the Einstein-Cartan theory formulated by Kibbe and Sciama [84, 188] in the late fifties where the torsion is an additional data for describing an intrinsic angular momentum of space. More recently, the interest in geometries with torsion was revived again in a physical context through developments of superstring theory. Indeed, if one deforms the vacuum equations of the Strominger model and looks for solutions with non-vanishing B-field, then these can be constructed from geometries with torsion. In this theory, the traditional Yang-Mills approach needs to be modified, and that is where skew-symmetric torsion enters the picture. As a matter of fact, in string theory one substitutes the point with a string. Consequently, the strength of the B-field, which classically is a 2 -form (hence described through the curvature), becomes a 3 -form. Therefore, it is now described as the torsion tensor of a connection with skew-symmetric torsion. We refer to $[100, \sqrt[114]{ }, \sqrt{146}, \sqrt{148}, \sqrt{172}, \sqrt{173}, 225,241,248,250]$ and the references therein for further discussions on the use of connections with skew-symmetric torsion, and in particular the Bismut connection, to describe models in string theory. After the pioneering works of Candelas, Gates, Horowitz, Hull, Roček, Strominger, and Witten 72, 134, 175, 288 the Bismut connection is now deeply used in superstring compactifications. In particular, the explicit expression of the Bismut connection appeared in Strominger's paper [288] in 1986, where he described the basic model in heterotic superstring theory; there, it was called the $H$-connection.

In dimension four, the Strominger model leads to hyper-Kähler with torsion (HKT) structures, i. e. a hyper-Hermitian structure that is parallel with respect to the Bismut connection. HKT manifolds also appear in the description of other physical structures, see for example 148,158 , [159, 248. From the mathematical point of view, hyper-Kähler with torsion manifolds were introduced by Howe and Papadopoulos [173]. Since then they have been deeply studied by several authors, see for example [30, 107, 115, 119, 178, 179, 265, 266, 316].

Independently from the works of Strominger, in the late 1980s, Bismut came across the Bismut connection in the context of index theory problems in complex non-Kähler geometry [49. In particular, he characterized the Dirac operators associated with metric connections that differ from the Levi-Civita connection in terms of their torsion and obtained a local index theorem for them. More precisely, given a metric connection $\nabla \neq \nabla^{L C}$, if the three-form given by the antisymmetrization of the torsion of $\nabla$ is closed, a local index theorem still holds 49, Theorem 1.11]. However, the corresponding Atiyah-Singer polynomial is now calculated with respect to the Bismut connection. It turns out that this result specializes to the complex Dirac operator obtained by taking the sum of the Dolbeault operator and its adjoint giving a local index theorem for it. Indeed, it is known that when the metric on the manifold is Kähler, $\bar{\partial}+\bar{\partial}^{*}$ is the Dirac operator associated to the Levi-Civita connection, see for example the proof of Theorem 2.1 in [164] or [202, Theorem 5.12]. However, this is not the case for non-Kähler metrics 141, Proposition 7], but, for general Hermitian manifolds, $\bar{\partial}+\bar{\partial}^{*}$ is the Dirac operator associated to the Bismut connection [49, (2.33)]. This means that if the torsion of the Bismut connection is closed, namely, thanks to (0.2), if the metric is $d J d$-closed (or equivalently $\partial \bar{\partial}$-closed), there is a local index theorem for $\bar{\partial}+\bar{\partial}^{*}$. In [49] Bismut first defined a Hermitian manifold pluriclosed when the canonical 2-form associated to the Hermitian metric satisfies $d J d \omega=0$. To summarize, the Bismut connection allows us to incorporate the Dolbeault operator in this framework even if the metric is not Kähler. Consequently, if the manifold cannot be equipped with a Kähler metric, but with a pluriclosed one, it is still possible to deduce a local Riemann-Roch-Hirzebruch theorem [49, Theorem 2.11], as like as, in the Kähler case, the local index theorem gives the formula of Riemann-Roch-Hirzebruch for the Euler characteristic of a holomorphic vector bundle. However, the local limit is no longer locally given by a Riemann-Roch-Hirzebruch polynomial since the limit index polynomial is now evaluated by means of the curvature of the Bismut connection. See also $[51,52]$ and the references therein for recent developments on this topic.

In 49 Bismut also proved the existence and uniqueness of the Bismut connection. Moreover, in that article, the Bismut connection is related to the Chern connection of an "exotic" holomorphic structure on $T_{\mathbb{C}} M$. Let us briefly motivate and explain this crucial construction. First of all, we recall that the essential property of the connections $\nabla^{ \pm}$(defined in (0.3p) is that if their torsion is closed then their curvatures are equal up to switching the endomorphism with the form components. In detail [49, Theorem 1.6], if $d H=0$ then for all vector fields $x, y, z, w$,

$$
g\left(R_{x, y}^{+} z, w\right)=g\left(R_{z, w}^{-} x, y\right) .
$$

Then we denote by $\nabla^{-B}$ the metric connection with skew-symmetric torsion opposite to the Bismut connection, that is

$$
g\left(T^{-B}(\cdot, \cdot), \cdot\right)=-d \omega(J \cdot, J \cdot, J \cdot)=J d \omega .
$$

We notice that for the connections $\nabla^{ \pm B}$ the differential of the torsion is $\mp d J d \omega$. Therefore, the above symmetry is realized by the curvature tensors of $\nabla^{ \pm B}$ when the Hermitian metric is pluriclosed. Since $\nabla^{B}$ preserves the complex structure of $M$, the curvature $R^{B}$ is a two-form taking values in complex automorphisms of $T_{\mathbb{C}} M$. Then, when the metric is $\partial \bar{\partial}$-closed, $R^{-B}$ becomes a (1,1)-form with values in End $\left(T_{\mathbb{C}} M\right)$. Hence, thanks to [24, Theorem 5.1], there must be a holomorphic structure on $T_{\mathbb{C}} M$ such that $\nabla^{-B}$ is the associated Chern connection. In [49] it is shown that given a Hermitian manifold ( $M, J, g$ ), this new holomorphic structure on $T_{\mathbb{C}} M$ is
given by the canonical holomorphic structure on it coming from $J$ twisted by an element which depends explicitly on the Kähler form $\omega$. In more detail, the connection $\nabla^{-B}$ on $M$ associated to $(J, g)$ equals the Chern connection of the vector bundle $T_{\mathbb{C}} M$ whose holomorphic structure is given by

$$
\bar{\partial}^{J}+\beta
$$

where $\bar{\partial}^{J}$ is the Dolbeault operator induced by $J$, and $\beta$ is the one form with values in the endomorphisms of $T^{1,0} M \oplus\left(T^{1,0} M\right)^{*}$ given by

$$
\beta(x)\binom{y}{\eta}=\binom{0}{\mathrm{i} \partial \omega(x, y, \cdot)}
$$

after the identification $T_{\mathbb{C}} M \cong T^{1,0} M \oplus\left(T^{1,0} M\right)^{*}$ given by the metric $g$. This construction can be nicely expressed in the language of Generalized Complex Geometry 155,165 where one replaces the tangent bundle $T M$ with the tangent plus cotangent bundle $T M \oplus T^{*} M$. Conversely, the connections on an exact Courant algebroid which preserve a generalized metric may be expressed as connections with prescribed skew-symmetric torsion $\nabla^{ \pm}$. An explicit formulation of this relation is in 132 . In that work, for a given pluriclosed Hermitian manifold $(M, J, g)$, Garcia-Fernandez, Jordan, and Streets constructed a holomorphic Courant algebroid $\mathcal{Q}_{\partial \omega}$ on $M$ depending on the torsion class $[\partial \omega] \in H_{\bar{\partial}}^{2,1}(M, J)$, and a generalized Hermitian metric $\mathcal{G}$ on it which depends on the Hermitian structure $(J, g)$. They are such that $\nabla^{-B}$ of $g$ is equivalent in a canonical way to the Chern connection of $\mathcal{G}$. Therefore, there is a striking relation between the curvature of the Chern connection associated to the generalized Hermitian metric $\mathcal{G}$ and the curvature of the Bismut connection associated to the underlying pluriclosed structure. In particular, the Ricci form $R i c^{B}$ obtained by tracing $R^{B}$ in the endomorphism components is related to the second Ricci curvature of the Chern connection of $\mathcal{G}$ obtained by tracing the Chern curvature tensor in the form components.

Finally, we want to shortly highlight how the Ricci curvature tensor of the Bismut connection is used to produce a geometric flow which is expected to have crucial implications in the classification of compact complex surfaces. Let us start by noticing that in recent years geometric flows proved to be powerful tools in geometry. As a matter of fact, after Perelman's landmark resolution of Thurston's Geometrization Conjecture for 3-manifolds by using Ricci flow 74 , $75,161,190,257,259$, their use has been extended to many others contexts. For example, the Kähler-Ricci flow, evolving a Kähler metric by its Ricci tensor Ric ${ }^{L C}$ with respect to the Levi-Civita connection is now one of the most powerful tools in Kähler geometry, see [63] and the references therein. In general, starting at a Hermitian non-Kähler metric the Ricci flow does not preserve the Hermitian condition since $\operatorname{Ric}^{L C}$ is not $J$-invariant. Henceforth, other flows have been introduced to generalize the Ricci flow to the Hermitian context. One of the first attempts in this direction was made by Gill $\sqrt[149]{ }$, who first investigated the Chern-Ricci flow in the context of non-Kähler Calabi-Yau manifolds, and then also by Tosatti and Weinkove (see 299 and the reference therein), who studied it in general on Hermitian non-Kähler manifolds. Later, the Hermitian curvature flows (HCF in short) were introduced by Streets and Tian in [286]. They evolve a starting Hermitian metric $g_{0}$ as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t))+Q(g(t))  \tag{0.4}\\
g(0)=g_{0}
\end{array}\right.
$$

where $R i c^{C h, 2}$ is the second Chern-Ricci tensor given by the trace of the Chern curvature tensor in the form components (see Section 1.2 .5 , and $Q$ is a real symmetric ( 1,1 )-tensor obtained as a linear combination of quadratic terms in the torsion of the Chern connection $Q=a Q^{1}+b Q^{2}+c Q^{3}+d Q^{4}$. See Section 7.1 for a more detailed overview of the Hermitian curvature flows. Different choices of $Q$ can be performed in order to suit the flow to specific problems. Indeed, one looks for flows that are adapted to particular geometric issues, and this is
usually made by choosing flows that preserve some natural geometric condition (as the Ricci flow preserves the Kähler condition, or the HCF in [310] preserves the Griffiths positivity). In this spirit, in [285] a particular choice of quadratic term $\mathcal{Q}$ was detected, which leads to an evolution equation for Hermitian metrics that preserves the pluriclosed condition. It is hence called pluriclosed flow, and it evolves a starting pluriclosed metric $\omega_{0}$ as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} \omega(t)=-\left(\operatorname{Ric}^{B}(\omega(t))\right)^{1,1}, \\
\omega(0)=\omega_{0},
\end{array}\right.
$$

where $(\cdot)^{1,1}$ denotes the ( 1,1 )-component of the form. The pluriclosed flow is particularly wellsuited for compact complex surfaces since there is a pluriclosed representative in any conformal class of Hermitian metrics [136, Théorème 1]. As a matter of fact, it has been introduced as an analytic tool to understand the topology and geometry of compact complex surfaces. In particular, it is conjectured [287, Section 5] that on a minimal class VII surface with $b_{2}>0$ it should detect a rational curve, leading to a parabolic proof of the Global Spherical Shell conjecture [235] for $b_{2}=1$. We present a more general overview of the pluriclosed flow in Section [7.2, see also [35, Section 7.3] and the references therein for a survey on its possible application to the classification of complex surfaces.

One of the main topics in Kähler geometry is the study of special metrics, namely, metrics with specific conditions on the curvature tensors. Among these, we recall the constant scalar curvature Kähler metrics, the Kähler-Einstein metrics, and the metrics with positive holomorphic bisectional curvature (which appear for instance in the context of Frankel conjecture). Any of these geometries satisfies strong properties which have been deeply studied in the past decades and still are important topics of research. In this thesis, we are concerned with studying some analogs of these classical problems in Hermitian and almost-Hermitian geometry. Specifically, we study metrics with constant Gauduchon scalar curvature, two different Einstein-type equations for the Bismut and Chern connection respectively, Bismut flat manifolds and the evolution of the pluriclosed flow on them, and finally a notion of positivity for Hermitian manifolds which involves the curvature of the Bismut connection.
In the following, we summarize the main results of this thesis, which are collected in $31,32,34$, 36.

The existence of metrics characterized by special curvature properties becomes a useful property in non-Kähler geometry to contrast the abundance of Hermitian metrics. Furthermore, one intends to tie properties of almost complex structures with the existence of canonical metrics, in order to understand the former in terms of the latter. For these reasons, metrics with constant scalar curvature are a very natural object to look for. In the Riemannian setting, they have been studied by many authors since in [327] Yamabe stated the problem of finding constant scalar curvature metrics in the conformal class of a given Riemannian metric. Eventually, Yamabe, Trudinger, Aubin, and Schoen proved that on compact manifolds, a metric with constant scalar curvature exists in any conformal class [26, 203, 272, 303, 327]. Moreover, in the Riemannian setting, the problem of prescribed scalar curvature was also entirely solved, in the general case by Kazdan and Warner [185, 186, while for conformal variation of the metric see the references in [27, Chapter 6].

The first problem to face when extending the constant scalar curvature condition to the Hermitian or almost-Hermitian non-Kähler setting is to understand which connection is a good analog to be considered. In this thesis, following what we did in [34], we choose to study Hermitian metrics with constant Bismut scalar curvature. Our motivation for considering exactly such scalar curvature comes from the importance of the Bismut-Ricci curvature in non-Kähler Calabi-Yau problems, precisely in Calabi-Yau geometries with torsion (we will come back on this argument later, see also Chapter 5). We actually study a more general problem which
includes the analysis of constant Bismut scalar curvature metrics as a special case. Namely, we study an analog of the Yamabe problem for all the canonical Gauduchon connections. Precisely, given a conformal class of almost-Hermitian metrics, we look for a representative with constant Gauduchon scalar curvature $s^{t}$, where the scalar curvature is obtained by tracing the curvature tensor of $\nabla^{t}$ both in the form and endomorphism components. Note that this goes in a different direction with respect to both the classical Yamabe problem stated above and the Yamabe problem considered by del Rio and Simanca. Indeed, the former refers to the scalar curvature of the Levi-Civita connection, while the latter is an analog of the Yamabe problem for almostHermitian manifolds [268, consisting in looking for metrics with constant $\star$-scalar curvature (obtained by tracing the Riemannian curvature tensor after twisting it with the almost-complex structure), and these tensors are in general different from $s^{t}$ for non-Kähler manifolds.

The results about the Gauduchon-Yamabe problem collected in this thesis are a generalization to the almost-Hermitian case of the ones we obtained in [34. First of all, Angella, Calamai, and Spotti introduced and studied the so-called Chern-Yamabe problem in 14 , which is an analog of the Yamabe problem on Hermitian manifolds which involves the curvature of the Chern connection. They proved that, on a compact Hermitian manifold, a metric with constant Chern scalar curvature exists unique in a given conformal class if its expected value is non-positive. Afterward, in 207 Lejmi and Upmeier extended their results to the non-integrable case. Here, we prove that, on a compact almost-complex manifold, in the conformal class of a given almostHermitian metric, there exists a unique (up to a multiplicative constant) metric such that it has constant scalar curvature with respect to the Gauduchon connection $\nabla^{t}$ whenever the expected constant scalar curvature is non-positive or non-negative depending on $t>\frac{1}{1-n}$ or $t<\frac{1}{1-n}$ respectively, obtaining the results in $[14,207$ as a special case. In detail, we prove the following theorem, which extends the results we obtained in [34, Theorems 3.1 and 3.2].

Theorem (Theorems 3.3.1 and 3.3.2). Let $M$ be a $2 n$-dimensional compact manifold with almost-Hermitian structure $(J, \omega)$. Fix a Gauduchon representative $\eta$ in the conformal class of $\omega$. If the Gauduchon parameter $t$ is such that $1+n t-t \neq 0$ and

$$
(1+n t-t) \int_{M} s^{t}(\eta) \operatorname{Vol}_{\eta} \leq 0
$$

then there exists a unique metric $\widetilde{\omega}$, up to multiplicative constant, conformal to $\omega$ and such that it has constant scalar curvature with respect to the Gauduchon connection $\nabla^{t}$.

Notice that if $\Delta_{\omega}^{C h}$ represents the Chern Laplacian associated to $g$ (definition in Section 1.1.4), then the Gauduchon-Yamabe problem reduces to solve the semi-linear elliptic equation of the second order in $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$ given by

$$
C \Delta_{\omega}^{C h} f+s^{t}(\omega)=\lambda e^{f}
$$

Here, $C$ is a constant that depends on the dimension $2 n$ and the Gauduchon parameter $t$, while $\lambda$ is the expected value of the constant scalar curvature and it is a conformal invariant of the metric (see Proposition 3.1.1). It turns out that whenever $\lambda=0$ or $C \lambda<0$, standard analytic techniques apply to give the above result. On the contrary, if $C \lambda>0$ many of the good analytical properties of the above elliptic equation disappear, mostly the ones related to the maximum principle.

The case of $C \lambda>0$ remains one of the main open problems of this topic. For the Chern connection, it corresponds to positive Gauduchon degree (definition in Section 3.1), and, of course, the solution to the Chern-Yamabe problem with positive Gauduchon degree would probably lead to a better understanding of the critical case for the other Gauduchon connections. Vice versa, the solution to the Yamabe problem for the whole line of canonical Gauduchon connections out of the Chern connection might help in completing the analysis of the Chern-Yamabe problem. We observe that, up to now, some partial results have been obtained, see for example [14, Section 5]. Moreover, a flow approach through the so-called Chern-Yamabe flows has been proposed to
tackle the problem $14,70,206$, and also these parabolic techniques may be extended to Bismut, and in general Gauduchon, cases.

Our analysis of metrics with constant Gauduchon scalar curvature in a given conformal class on compact manifolds has been recently enlarged in 213 . In that work, the authors study metrics with prescribed Gauduchon scalar curvature in a given conformal class on compact complex manifolds, while it was done in 129 for the Chern connection, and in 44 for compact Kähler manifolds. We should also mention that the general problem of prescribed scalar curvature was faced in the case of Chern scalar curvature by analyzing its linearized operator 20. We believe that also these techniques may be adapted to any canonical Gauduchon connection. Moreover, the knowledge of the linearized operators associated to the Gauduchon scalar curvatures might lead to a generalization of the following proposition to all the Gauduchon parameters $t \neq \pm 1$ by performing a continuity method.
Proposition (Proposition 2.1.1). Let $(M, J, g)$ be a Hermitian manifold. If $s^{t}=s_{2}^{t}$ for $t \geq$ $-3+2 \sqrt{3}$ or $t \leq-3-2 \sqrt{3}$ and $t \neq 1$, then $g$ is Kähler.

Here $s_{2}^{t}$ is the second-scalar curvature which is obtained by tracing the curvature tensor of $\nabla^{t}$ in the mixed form-endomorphism terms (see Section 1.2.5).

Another very natural constraint to impose on the curvature of a metric is that its Ricci tensor is "constant". This condition has its origin in physics, precisely, it arises from the vacuum equations of Einstein's General Relativity. Given a Riemannian manifold $(M, g)$, the metric $g$ is said Einstein if the Ricci tensor obtained by tracing the Levi-Civita curvature tensor satisfies

$$
\operatorname{Ric}^{L C}(g)=\lambda g, \quad \text { for some } \lambda \in \mathbb{R}
$$

Then, a Kähler-Einstein metric $g$ on a complex manifold $(M, J)$ is a Kähler metric which satisfies the Einstein condition. Kähler-Einstein manifolds have been deeply studied in the past decades, see for example $[46,156,289]$ and the references therein. In particular, results on existence and uniqueness have been achieved when the first Chern class is non-positive $25,331,332$, while for Fano manifolds the Kähler-Einstein equation has been related to the notion of $K$-stability [87, 90, 106, 294, 295, see also [45, 92, 101, 290.

In (almost-)Hermitian non-Kähler geometry there are several analogous problems that can be defined by relaxing the Kähler-Einstein equation. For example, one can replace the Levi-Civita connection with any Gauduchon connection. In this case, there are many ways of tracing the Gauduchon curvature tensor to get Ricci tensors. In general, these lead to different Einstein-type equations since the Gauduchon curvature tensors do not satisfy the same symmetries as in the Kähler case. Indeed, due to the lack of the first Bianchi symmetry, there actually are three different ways of contracting the curvature tensors. These are the trace in the endomorphism components, the trace in the form components, and the trace in the mixed entries (the two possible traces in the mixed components give complex conjugates results). In this thesis, we consider two Einstein-type equations that involve respectively the Bismut curvature tensor and the Chern curvature tensor. Let us now motivate these problems and present our results.

Given a complex manifold $(M, J)$, a Hermitian metric $g$ is said to be Calabi-Yau with torsion (CYT in short) if the associated Bismut-Ricci tensor vanishes, i.e. $\operatorname{Ric}^{B}(g)=0$. CYT geometry plays a role in Physics after the works of Strominger [288] and Hull 174 . Recently CYT structures on non-Kähler manifolds attracted attention as models for string compactifications, see e.g. $39,40,145,150,220,221]$. Moreover, there is a general interest in finding explicit nontrivial examples of pluriclosed Hermitian structures whose ( 1,1 )-component of the Bismut-Ricci form satisfies

$$
\left(\operatorname{Ric}^{B}(\omega)\right)^{1,1}=\lambda \omega, \quad \text { for some } \lambda \in \mathbb{R}
$$

since these are static points of the pluriclosed flow (see Section 7.2 for more details). In particular this motivates the search for pluriclosed CYT metrics, which are known in the literature 133 Definition 8.11] as Bismut Hermitian-Einstein metrics.

Since the Bismut-Ricci form is a representative of the first Chern class in de Rham cohomology (see for example (1.8)), the existence of Calabi-Yau with torsion structures is obstructed by having vanishing first Chern class. Furthermore, there is a canonical way to obtain a manifold with vanishing first Chern class out of a given Hermitian manifold. This is made by constructing a toric fibration on it, see Section 5.1 for details. Therefore, toric bundles over complex manifolds represent a natural environment where examples of Calabi-Yau with torsion metrics can be constructed. Moreover, the easiest non-Kähler examples of Calabi-Yau with torsion manifolds are given by the Hopf surface $\mathbb{S}^{1} \times \mathrm{SU}(2)$ and the Calabi-Eckmann threefold $\mathrm{SU}(2) \times \mathrm{SU}(2)$, which are both $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over the product of complex projective spaces. Therefore, also inspired by the results of $[126,127,150,151,255,320$, we investigate the Calabi-Yau with torsion condition on the total spaces of 2-dimensional toric bundles over Hermitian manifolds. Indeed, in 320 and in 255, Wang, Ziller, Pedersen, and Swann proved the existence of families of Einstein and respectively Weyl-Einstein structures on certain principal toric bundles over products of Kähler-Einstein manifolds. Moreover, toric fibrations over Calabi-Yau surfaces were used by Fu and Yau 126,127 to show explicit solutions to the Hull-Strominger system, and by Goldstein and Prokushkin 150 to construct non-Kähler $\mathrm{SU}(3)$-structures satisfying a supersymmetry condition. Then in 151 D. Grantcharov, G. Grantcharov, and Poon studied the CYT condition on the total spaces of toric bundles over Kähler manifolds, producing a family of non-homogeneous examples.

In this thesis, we describe the Calabi-Yau with torsion condition for metrics of submersion type on the total spaces of $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over Hermitian manifolds. Specifically, these are metrics for which the projection over the Hermitian base-space is a Riemannian submersion. In particular, we characterize the submersion metrics which give a CYT structure in terms of the metric on the base-space and the characteristic class of the principal bundle. We also remark that since the work of Bérard-Bergery [42] looking at submersion metrics is the first natural attempt if one seeks to construct Einstein manifolds, see 46, Chapter 9.H] for a general account on the subject.

Proposition (Proposition 5.2.1, or Proposition 4.2 in [34]). Let $M$ be the total space of a principal toric bundle $\mathbb{S}^{1} \times \mathbb{S}^{1} \hookrightarrow M \xrightarrow{\pi} X$ over a Hermitian manifold $\left(X, \omega_{X}\right)$ with connection one forms $\left(\theta_{1}, \theta_{2}\right)$, and curvature forms $\pi^{*} \omega_{i}=d \theta_{i}$. Consider a positive function $f \in \mathcal{C}^{\infty}\left(X ; \mathbb{R}_{+}\right)$. Then the metric $\omega=\pi^{*}\left(\omega_{X}\right)+\pi^{*}(f) \theta_{1} \wedge \theta_{2}$ defines a Calabi-Yau with torsion structure if there exist constants $c_{1}, c_{2}$ such that

$$
\left\{\begin{array}{l}
\operatorname{Ric}^{B}\left(\omega_{X}\right)=d J d f+\left(c_{1} \omega_{1}+c_{2} \omega_{2}\right) \\
f \operatorname{tr}_{\omega_{X}} \omega_{i}=c_{i}
\end{array}\right.
$$

In Section 5.3, we specialize the results of Proposition 5.2.1 to class $\mathcal{C}$ manifolds in the sense of [263]. These are homogeneous manifolds which lie between the total spaces of 2-dimensional toric bundles over compact Hermitian manifolds and Calabi-Eckmann manifolds. Indeed, they are the total spaces of homogeneous $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over the product of two compact irreducible Hermitian symmetric spaces. We thus explicitly construct Calabi-Yau with torsion metrics on class $\mathcal{C}$ manifolds, and we prove that, except for some particular cases, they are the unique homogeneous CYT metrics. The interest in this kind of result comes from the fact that a compact simply connected homogeneous manifold $G / H$ with an invariant complex structure $J$ (called C-space in $[319]$ ) is Kählerian if and only if it is a generalized flag manifold, namely when $G$ is a semisimple Lie group and $H$ is the centralizer of a torus in $G$, as stated in [58]. In such case, in 226 it is proved that they can be endowed with a (unique) invariant Kähler-Einstein metric, while there is a general interest in finding special invariant metrics on the non-Kähler C-spaces $G / H$.

Theorem (Theorems 5.3.1 and 5.3.2, or Theorems 4.6 and 4.7 in 34 ). Let $M$ be a class $\mathcal{C}$ manifold as in [263], that is a product $M=M_{1} \times M_{2}$ of manifolds which fibers through the Tits fibrations $\phi_{i}$ over two generalized flag manifolds $X_{1}=G_{1} / H_{1}$ and $X_{2}=G_{2} / H_{2}$ with $\mathbb{S}^{1}$-fibers,
and equip it with a standard complex structure (as in Section5.3). Set $\omega_{i}$ the unique invariant Kähler-Einstein metrics on $X_{i}$ with Einstein constants $n_{i}=\operatorname{dim}\left(X_{i}\right)$. Then the metric on $M$ given by

$$
\omega=\phi_{1}^{*}\left(\omega_{1}\right)+\phi_{2}^{*}\left(\omega_{2}\right)+\theta_{1} \wedge \theta_{2}
$$

where $\theta_{1}$ and $\theta_{2}$ are the connection one-forms on the fiber bundles such that $d \theta_{i}=\phi_{i}^{*} \omega_{i}$, defines a CYT structure on $M$. Furthermore, if none of the $X_{i}$ 's is $S O(k+2) / S O(2) \times S O(k)$ for $k \geq 3$, then the metric defined above is the unique (up to homothety) homogeneous CYT metric on $M$.

Notice that Theorem 3 in 152 ensures the existence of a Calabi-Yau with torsion metric on compact simply connected homogeneous manifolds $G / H$ with a G-invariant complex structure of vanishing first Chern class, after an appropriate deformation of the complex structure. However, the following result implies that, in general, it is not unique, while the standard Hermitian structures on class $\mathcal{C}$ manifolds are the unique invariant CYT structure on them (with the exception of some specific cases).

Corollary (Corollary 5.2.1, or Corollary 4.5 in 34 ). Let $(X, J)$ be a complex manifold of complex dimension $n$ endowed with a CYT metric $g_{X}$. Consider the product space $M:=\mathbb{S}^{1} \times \mathbb{S}^{1} \times X$ equipped with the product complex structure. Then for any positive function $f \in \mathcal{C}^{\infty}\left(X ; \mathbb{R}_{+}\right)$, the submersion metric

$$
\omega=\pi^{*}\left(e^{f} \omega_{X}\right)+(n-2) \pi^{*}(f) \theta_{1} \wedge \theta_{2}
$$

is a CYT metric on $M$.
As a particular case, our results of existence and uniqueness apply to the Calabi-Eckmann manifolds, showing that their standard homogeneous metrics are the unique (among the invariant ones) CYT metrics. However, we found a cohomological obstruction to the existence of pluriclosed non-Kähler metrics.

Proposition (Corollary 2.3.1). Let $(M, J)$ be a complex manifold. If the Dolbeault cohomology ring satisfies

$$
H_{\bar{\partial}}^{2,1}(M, J)=H_{\bar{\partial}}^{0,2}(M, J)=H_{\bar{\partial}}^{3,0}(M, J)=0
$$

then any pluriclosed metric is either Kähler or $(\partial+\bar{\partial})$-exact.
Since the torus fibers in the Calabi-Eckmann manifolds give compact complex submanifolds, by integrating the pluriclosed metrics on them one sees that they can not be $(\partial+\bar{\partial})$-exact. This argument (see also 85 , Theorem 5.16 and Example 5.17]) provides a classification of the Calabi-Eckmann manifolds which can be equipped with a pluriclosed non-Kähler metric. Namely, the only pluriclosed non-Kähler metrics on the Calabi-Eckmann manifolds are actually the ones on the Hopf surface and the Calabi-Eckmann threefold. This answers negatively a question raised by Garcia-Fernandez and Streets in 133, Question 8.36] asking whether higher dimensional Calabi-Eckmann manifolds admit Bismut Hermitian-Einstein metrics. However, since it is relatively easy to construct explicit CYT structures on toric bundles over Hermitian manifolds we wonder whether they can actually be equipped with pluriclosed non-Kähler Calabi-Yau with torsion structures, and if directly checking this cohomological obstruction may help us find them. We shall finally remark that there are also other conditions that a Hermitian manifold should satisfy in order to admit a Bismut Hermitian-Einstein metric. Specifically, CYT manifolds not only have vanishing first Chern class, but the holonomy of their Bismut connection is in $\mathrm{SU}(n)$ (see Section 5.1 for details). Furthermore, in 132 some other obstructions are deduced by exploiting the identification between $\nabla^{-B}$ and the Chern connection of the exotic complex structure on $T_{\mathbb{C}} M$. Indeed, as we already outlined, in 132 an explicit construction in the language of generalized geometry is performed, which allows us to relate the Bismut-Ricci form of a pluriclosed metric with the second Chern-Ricci curvature of a suitable Hermitian metric on $\mathcal{Q}_{\partial \omega}$. Consequently, Bismut Hermitian-Einstein metrics are Hermitian-Einstein metrics on the relevant holomorphic Courant algebroid. We recall that given a holomorphic bundle $E$ over a

Hermitian manifold $(M, J, g)$, a Hermitian metric $h$ on $E$ is said to be Hermitian-Einstein if the trace of its curvature tensor with respect to $g$ is proportional to the identity in $\operatorname{End}(E)$. These metrics were introduced in 192 as a differential-geometry counterpart of an algebraic stability condition for vector bundles over complex surfaces [66, 105, 193, 209, 222, 308]. Eventually, the relation between slope stability and the Hermitian-Einstein equation on general Hermitian manifolds is provided in 223]. Therefore, obstructions to the existence of Bismut Hermitian-Einstein metrics in terms of an algebro-geometric condition on $\mathcal{Q}_{\partial \omega}$ follow [132, Corollary 4.5].

In the complex case, the second-Chern-Einstein metrics are precisely those Hermitian metrics on the holomorphic tangent bundle that are Hermitian-Einstein with respect to themselves. More precisely, given an almost-complex manifold $(M, J)$, an almost-Hermitian metric $g$ is said to be second-Chern-Einstein if it satisfies

$$
R i c^{C h, 2}(g)=\frac{s^{C h}(g)}{n} g
$$

where $s^{C h}(g)$ is the Chern scalar curvature. In the Hermitian case, second-Chern-Einstein metrics were studied, for example, in $\sqrt[{15, ~ 19, ~ 137, ~ 138, ~ 143, ~ 216, ~ 263] . ~ I n ~ p a r t i c u l a r, ~ v a r i o u s} ~]{\text {, }}$ examples of second-Chern-Einstein metrics have been constructed; see [15, Section 3.3] for an account of the homogeneous examples, and 19 for cohomogeneity-one examples.

The second-Chern-Einstein problem turns out to be conformally invariant. Indeed, if $\widetilde{\omega}=e^{2 f} \omega$ for $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$ is a metric conformal to $\omega$, then the equations for the conformal variations of the second Chern-Ricci form is

$$
\operatorname{Ric}^{C h, 2}(\widetilde{\omega})=\operatorname{Ric}^{C h, 2}(\omega)+\left(\Delta_{\omega}^{C h} f\right) \omega
$$

and consequently,

$$
s^{C h}(\widetilde{\omega})=e^{-2 f}\left(s^{C h}(\omega)+n \Delta_{\omega}^{C h} f\right)
$$

This leads us to explore the relation between the second-Chern-Einstein problem and an Einsteintype condition for the Weyl connection $\nabla^{W}$ (see Section 4.1 for details), which is also invariant under conformal change of the metric (see definition in Section 1.2.4. It turns out that the second Chern-Ricci form and the $J$-twisted Weyl-Ricci tensor Ric ${ }^{W, J}$ (see the precise definition in Section 1.2.5 on an almost-Hermitian 4-manifold $(M, J, g)$ are related by (Theorem 4.1.1)

$$
\operatorname{Ric}^{C h, 2}(\omega)=\left(\operatorname{Ric}^{W, J}(\omega)\right)^{1,1}+\Xi_{1}(J, \omega) \omega
$$

where $\Xi_{1}$ is a function depending on the Nijenhuis tensor $N_{J}$ and the metric $\omega$ itself. It follows that, for complex surfaces, the second-Chern-Einstein condition and the Weyl-Einstein condition are equivalent $[143$, Theorem 1]. This yields [143, Theorem 2] a classification of second-ChernEinstein complex surfaces: they are either Kähler-Einstein surfaces or the Hopf surface with its standard Hermitian structure. On the other hand, exploiting again the relation between the second Chern-Ricci form and Ric ${ }^{W, J}$ in the almost-Hermitian setting, we obtain, in a joint work with Lejmi [36], the following result. Before stating it, we recall that locally conformally almost-Kähler manifold, are almost-Hermitian manifold such that a conformal change can be locally performed in order to obtain a closed canonical 2-form.

Theorem (Theorem 4.4.1, or Theorem 15 in [36]). Let $(M, J, \tilde{g})$ be a 4-dimensional compact locally conformally almost-Kähler manifold. Suppose that $\tilde{g}$ is a second-Chern-Einstein metric, and that the Gauduchon metric $g$ in the conformal class of $\tilde{g}$ satisfies $\left(\nabla^{L C} \theta\right)^{\text {sym,J,- }}=0$, where $\theta$ is the Lee form of $(g, J)$, and $(\cdot)^{\text {sym, } J,-}$ denotes the $g$-symmetric $J$-anti-invariant part. Then, either

1. $(M, g, J)$ is a second-Chern-Einstein almost-Kähler manifold, meaning that the the canonical 2-form $\omega$ is closed, or
2. $\theta$ is $\nabla^{L C}$-parallel and the conformal scalar curvature $s^{W}$ obtained by tracing the curvature tensor of $\nabla^{W}$ is non-positive. Moreover, $s^{W}$ is identically zero if and only if $J$ is integrable and so $(M, J)$ is a Hopf surface. Furthermore, if $s^{W}$ is nowhere zero then $\chi=\sigma=0$, where $\chi$ and $\sigma$ are the Euler class and signature of $M$ respectively.

Notice that the locally conformally almost-Kähler assumption is a natural replacement of the integrability condition, compare (4.15). Furthermore, if one seeks to extend these arguments to higher dimensions, the locally conformally almost-Kähler condition is needed. Indeed, as explained in Section 4.1, even in the integrable case, the second-Chern-Einstein problem is well understood through its relation with the Weyl-Einstein problem under the additional assumption of locally conformally Kähler. Moreover, since the second-Chern-Einstein condition is preserved by conformal changes and Théorème 1 in 136 proves that there is a Gauduchon representative in any conformal class, it is very natural to pair this Einstein condition with the Gauduchon one in order to obtain geometric characterizations. Finally, the examples collected in Section 4.5 show that the condition $\left(\nabla^{L C} \theta\right)^{s y m, J,-}=0$ is necessary for the $g$-Riemannian dual of the Lee form $\theta^{\sharp}$ to be a Killing vector field. As a matter of fact, similarly to the integrable case, this classification result relies on the fact that $\theta^{\sharp}$ is a Killing vector field (compare with 143 and [252, Proposition 3.2]).

In Section 4.6.1 we observe that, on complex surfaces, the second-Chern-Einstein condition is equivalent, up to conformal change, to Bismut Hermitian-Einstein condition since

$$
\operatorname{Ric}^{C h, 2}(\omega)=\left(\operatorname{Ric}^{B}(\omega)\right)^{1,1}+\Xi_{2}(\omega) \omega
$$

where $\Xi_{2}$ is a term depending on the metric which vanishes if and only if it is Kähler. This leads to another proof [283, Theorem 1.1] of the classification of the Bismut Hermitian-Einstein surfaces as either Kähler-Einstein surfaces or the Hopf surface. This correspondence is no more true in the non-integrable case because a term depending on $N_{J}$ pops up in the above formula (see Proposition 4.6.1. However, using the results in 317 and its relation with $\left(\operatorname{Ric}^{B}\right)^{1,1}$, it is possible to deduce explicit formulas for the second Chern-Ricci form on Lie algebras. In particular, we are able to classify 4-dimensional, unimodular, almost-abelian Lie groups that admit a second-Chern-Einstein metric with $\nabla^{L C}$-parallel, non-zero Lee form. Recall that the almost-abelian Lie algebras are those with a codimension-one abelian ideal, and the unimodular condition is necessary to get a compact quotient. Thus, using the notation of Lie algebras as 251 we obtain the following result.

Theorem (Theorem 4.6.1, or Theorem 32 in 36 ). Let $\mathfrak{g}$ be a 4-dimensional unimodular almostAbelian Lie algebra equipped with a left-invariant almost-Hermitian non-Hermitian structure $(J, g)$ such that the Lee form $\theta$ is $\nabla^{L C}$-parallel and non-zero. Suppose that $(J, g)$ is a solution to the second-Chern-Einstein problem. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras

1. $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}:\left[e_{1}, e_{3}\right]=-e_{2}, \quad\left[e_{2}, e_{3}\right]=e_{1} ;$
2. $\mathcal{A}_{3,4} \oplus \mathcal{A}_{1}:\left[e_{1}, e_{3}\right]=e_{1}, \quad\left[e_{2}, e_{3}\right]=-e_{2}$.

Both Lie algebras admit compact quotients.
We finally discuss the parabolic version of the second-Chern-Einstein equation. This is given by the Hermitian curvature flow (with vanishing quadratic term $Q \equiv 0$ ) which evolves a Hermitian metric $g_{0}$ as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t)) \\
\mathfrak{g}(0)=g_{0}
\end{array}\right.
$$

From the explicit computations performed in Section 8.2 it follows that the standard metric on any Hopf manifold is the unique (up to homotheties) second-Chern-Einstein metric among
the homogeneous ones. We shall remark here that the existence and uniqueness of a second-Chern-Einstein metric on the Hopf manifolds follows as a particular case of the analogous result on the class $\mathcal{C}$ manifolds [263, Theorem 3.4]. Notice also that second-Chern-Einstein metrics are static points of the above Hermitian curvature flow. Moreover, Theorem 8.2.1 completely describes the evolution of this flow on the Hopf manifolds, showing that it exists at all times and always converges to these standard metrics. It would be interesting to understand the behavior of this flow in general, also on almost-Hermitian manifolds. For example, on 4-dimensional almost-abelian Lie groups (where a classification of second-Chern-Einstein metrics has been obtained) it could be studied by describing it as a bracket flow on the algebras. This technique has already proved to be a powerful tool in studying, for example, the pluriclosed flow on almost-abelian Lie algebras 23 .

As we outlined above, there are very few examples of Bismut Hermitian-Einstein manifolds. Indeed, the only known non-Kähler metrics which are Bismut Hermitian-Einstein are actually Bismut flat. Notice that the Bismut flat metrics obviously have vanishing Ricci tensor, and they are pluriclosed thanks to Theorem 1 in [335]. Moreover, the existence of other Bismut Hermitian-Einstein metrics on Bismut flat manifolds has recently been studied in [116]. Since the Bismut connection represents a particular case of connection with skew-symmetric torsion, the Riemannian structure of Bismut flat manifolds is prescribed by the results of Cartan and Schouten in 82 (see Theorem 6.1.1]; namely, up to take the universal cover, they are semisimple Lie groups with a bi-invariant metric. Then, their complex structures are given by the Samelson construction in 271, which we present in Section 6.3.2. Finally, thanks to the works of Alexandrov, Ivanov, Wang, Yang, and Zheng [6, 321], compact Bismut flat manifolds are characterized as local Samelson spaces, which are compact quotients of connected and simply-connected, even-dimensional Lie groups with a left-invariant Hermitian structure $(J, g)$ whose metric $g$ is also right-invariant. See Sections 6.1, 6.2, and 6.3 for the precise definitions and an overview of the geometry of Samelson spaces.

Thanks to their characterization, the Bismut flat manifolds can be distinguished by means of their rank, that is the dimension of the maximal torus in their universal cover. Thus, other than tori, which are Kähler flat, the easiest examples occur in rank two and are the Hopf surfaces, and the Calabi-Eckmann threefold. Excluding the trivial case of tori, the Dolbeault cohomologies of these manifolds were computed in [163, Appendix II, Theorem 9.5] exploiting their structure of toric fibration over complex projective spaces. Then, studying their left-invariant forms, the harmonic representatives of the Bott-Chern cohomologies were given in [21, Section 3.3]. In this thesis, we complete the picture by computing the Bott-Chern cohomologies of the remaining Bismut flat manifolds of rank two. These are given by the simple Lie groups $\mathrm{SU}(3)$, $\operatorname{Spin}(5)$, and $\mathrm{G}_{2}$ equipped with Samelson complex structures compatible with the bi-invariant metrics given by the Killing forms. We use the fact that the de Rham cohomologies of compact simple Lie groups have been computed in [64, 81, 94, 96, 99, 170, 264, 267, 270, 334]. Moreover, a model for the Dolbeault cohomology ring of compact simply-connected simple Lie groups of rank two equipped with a left-invariant complex structure compatible with the Killing metric was given in 261, Proposition 4.5]. It is

$$
H_{\bar{\partial}}^{\bullet \bullet}(G) \cong \mathbb{C}\left[y_{1,1}\right] /\left(\left(y_{1,1}\right)^{n-1}\right) \otimes \wedge^{\bullet \bullet \bullet}\left(\mathbb{C}\left\langle\left[u_{2,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle\right)
$$

where the subscripts denote the bi-degree of the generators $x, y, u$. Thanks to this information, we are able to recover the structure of their double complexes as in Figure 0.1.

From this, we obtain the dimension of any cohomology group. On the other hand, we also combine the results of Pittie [261] and Stelzig [281] to prove that the left-invariant forms compute all the cohomologies. Notice that this is not always the case for homogeneous spaces as outlined in 97 (see also $[4,183,208,262$ for an analysis of the behavior of the Dolbeult cohomology of homogeneous manifolds under group actions). Thus, we can perform explicit computations to obtain the harmonic representatives of the Dolbeault and Bott-Chern cohomologies.


Figure 0.1. Sketch of the double complex of a compact simply-connected simple Lie group of rank two and complex dimension $n$. See Section 1.3 .2 for the description of the double complex diagrams.

The understanding of the cohomology of these manifolds leads to a stability property of the Bismut flat metrics for the pluriclosed flow. Let us explain this more precisely. In [132], Garcia-Fernandez, Jordan, and Streets exploit the relationship between the connection $\nabla^{-B}$ and the Chern connection to study the long-time behavior of the pluriclosed flow. Indeed, thanks to the equivalence of the second Chern-Ricci curvature of $\left(\mathcal{Q}_{\partial \omega}, \mathcal{G}\right)$ and the Bismut-Ricci tensor of the underlying pluriclosed Hermitian structure, they can apply the Schwarz Lemma to compare the Bismut connection of metrics with the same torsion class. Hence they derive a result about the long-time existence and convergence of the pluriclosed flow that only depends on $[\partial \omega] \in H_{\bar{\partial}}^{2,1}(M, J)$ for $\omega$ the starting point of the flow. In details, 132 . Theorem 1.2] proves that given a compact Bismut flat manifold ( $M, J, \omega_{B F}$ ), and a pluriclosed metric $\omega_{0}$ such that $\left[\partial \omega_{0}\right]=\left[\partial \omega_{B F}\right] \in H_{\bar{\partial}}^{2,1}(M, J)$, the solution of the pluriclosed flow with initial data $\omega_{0}$ exists on $[0, \infty)$ and converges to a Bismut flat metric $\omega_{\infty}$. Therefore, even if there is no precise control on the limit metric $\omega_{\infty}$, the Bismut flat metrics are attractive for the pluriclosed flow in their torsion class. Consequently, the global stability of this family of special metrics follows by checking a cohomological condition. Namely, if there exists a representative coming from a Bismut flat metric in any torsion class, then the pluriclosed flow starting from any pluriclosed metric will exist for all times and converge to a Bismut flat metric. This global stability property was proved to hold true for the tori [282, Theorem 1.1], the Hopf surface [132, Example 2.7], and the Calabi-Eckmann threefold [132, Example 2.8], see also our Examples 7.3.1 and 7.3.2 In this thesis, we show that the Bismut flat metrics are globally stable for the pluriclosed flow on all the Bismut flat manifolds of rank two. We do it by combining the knowledge of their Bott-Chern cohomology and the fact that they have a one-dimensional complex submanifold given by the maximal torus. In detail, we prove the following result.

Theorem (Theorem 7.3.2, or Theorems 4.1, 4.2, and 4.3 in 32 ). Given a compact simplyconnected simple Lie group $G$ of rank 2, consider a Bismut flat Hermitian structure $\left(J, \omega_{B F}\right)$ coming from the Killing form (as in Section 6.3). Then for any pluriclosed metric $\omega_{0}$ on ( $G, J$ ) there exists a positive $\lambda$ such that the solution to the pluriclosed flow with initial data $\omega_{0}$ exists on $[0, \infty)$ and converges to $\lambda \omega_{B F}$ up to diffeomorphism.

Notice that, in this class of manifolds we also have good control of the limit metric $\omega_{\infty}$. On the other hand, for higher ranks, the situation is rather unclear. In fact, our argument for the theorem above starts from the computation of the cohomology groups of these manifolds, and this is far from being done in general. Moreover, for semisimple Lie groups, the dimension of the cohomology groups increases following a Künneth-type formula; henceforth, the torsion classes
to be controlled became accordingly more. A sample of this behavior is given in Example 7.4.1.

Finally, we introduce and investigate a new positivity notion for complex Hermitian nonKähler manifolds which we call Bismut positivity. The solution of the Frankel conjecture 231, 232, 275], as well as its extension to the Hermitian setting by Ustinovskiy [310], are evidence of the general principle which states that positive curvature conditions impose important geometric and topological constraints on the manifold. In particular, the Frankel conjecture proved by Mori [232, Siu-Yau [275], and then extended by Mok [231], states that a compact Kähler manifold with Griffiths positive curvature must be biholomorphic to the complex projective space. Moreover, Ustinovskiy proved that a compact Hermitian manifold of complex dimension $n$ such that its curvature is Griffiths-non-negative everywhere and strictly positive somewhere must be biholomorphic to the projective space $\mathbb{C P}^{n}$ 310, Proposition 0.3]. See Section 8.1.1 for a more detailed exposition of the subject. Since, in recent years, the study of non-Kähler Hermitian geometry has received a lot of attention, partly due to its connection with theoretical physics as previously outlined, it is natural to look for curvature positivity conditions also in non-Kähler geometry. In this spirit, we introduce and study the Bismut positivity notion for complex Hermitian non-Kähler manifolds. It involves the curvature of the Bismut connection, naturally emulating the Griffiths positivity of the holomorphic tangent bundle, which is the holomorphic bisectional curvature associated to the Chern connection. In detail, given a Hermitian manifold ( $M, J, g$ ), it is said to have Bismut-positive (resp. Bismut-non-negative) curvature if its Bismut curvature tensor $R^{B}$ satisfies for any non-zero $x, y \in T^{1,0} M$,

$$
R_{x, y}^{B} \equiv 0, \quad \text { and } \quad g\left(R_{x, \bar{x}}^{B} y, \bar{y}\right)>0 \quad(\text { resp. } \geq 0)
$$

Notice that the first condition, known in the literature as complex condition (Cplx), is just a technical one ensuring that the Bismut curvature has no $(2,0)$ and $(0,2)$ components. The motivation for this is that the second condition only controls the $(1,1)$ component, hence we want it to represent the whole tensor.

Bando 29, and Mok 231 proved that the positivity of the holomorphic bisectional curvature on Kähler manifolds is preserved under the evolution of the Kähler-Ricci flow. Then, Chen, Sun, and Tian 91 used the Kähler-Ricci flow to obtain an alternative proof of the Frankel conjecture avoiding the Siu-Yau and Mori result. Furthermore, the result of Ustinovskiy [310] relies on the fact that he detected a quadratic term $Q$ in the Chern torsion such that the associated Hermitian curvature flow preserves and regularizes the Griffiths positivity (non-negativity). We thus study the behavior of the Bismut positivity under the evolution of all the Hermitian curvature flows seeking to understand if there is a particular choice of $Q$ which behaves well with it. We perform our analysis on two concrete classes of examples, namely, Hopf manifolds and 6-dimensional Calabi-Yau solvmanifolds with holomorphically-trivial canonical bundle. The latter are compact quotients of solvable Lie groups endowed with an invariant complex structure and a holomorphically-trivial canonical bundle. We classify the manifolds in this class which admit invariant metrics satisfying (Cplx) (see Theorem 8.1.1). In particular, we find examples of manifolds that do not satisfy (Cplx); among those whose Bismut curvature is $J$-invariant, we find Bismut-non-negative manifolds (see Theorem 8.1.2). We thus prove that the symmetry (Cplx) as well as our positivity notion are preserved by the Hermitian curvature flows on these manifolds.

Theorem (Theorem 8.1.3, or Theorem 3 in [31]). Let $M$ be a 6-dimensional Calabi-Yau solvmanifold endowed with an invariant Hermitian structure such that the canonical bundle is holomorphically-trivial. Then the symmetries of (Cplx are preserved by any Hermitian curvature flow. Moreover, the Hermitian curvature flows preserve Bismut non-negativity on these manifolds.

The Hopf surface with its standard Hermitian structure has flat Bismut curvature, therefore, it is trivially Bismut-non-negative. Among the Hopf manifolds of higher dimensions, we find
other (non-flat) examples of metrics with Bismut-non-negative curvature. In particular, our results come from the analysis of the homogeneous metrics on these manifolds. We describe them as a family of metrics $g(\alpha, \beta)$ depending on two real parameters $\alpha$ and $\beta$. Thanks to Proposition 2.2.1 they are all the $\mathrm{U}(1) \times \mathrm{SU}(n)$-invariant metrics. Hence, in particular, in complex dimensions greater than three they are all the homogeneous metrics on $M$. Consequently, this class of metrics is naturally closed by the action of the Hermitian curvature flows and contains the standard metrics on the Hopf manifolds. Moreover, these metrics always satisfy the (Cplx) condition (computations in Section 2.2). Then, in Proposition 2.2.1 we characterize those which have Bismut-non-negative curvature. Furthermore, the HCF equation for homogeneous metrics reduces to an ODE, see Proposition 8.5. Hence, we can solve it and prove that the static homogeneous metrics on the Hopf manifolds are globally stable for the Hermitian curvature flows. In detail, we prove the following theorem.

Theorem (Theorem 8.2.1. or Theorem 4 in [31]). Consider the Hermitian curvature flow with parameters $a, b, c, d \in \mathbb{R}$ evolving the homogeneous metric $g_{0}=g\left(\alpha_{0}, \beta_{0}\right)$ on an $n$-dimensional diagonal Hopf manifold. Suppose that the coefficients ( $a, b, c, d$ ) are such that

$$
(n-2) a-2 b+(n-1)^{2} c-(n-1) d<n
$$

Then there exist static metrics for the flow and the metric $g_{0}$ evolves along the flow so that it converges to one of them.

This stability result gives a complete understanding of the evolution of the Hermitian curvature flows along homogeneous metrics on the Hopf manifolds. Thus, together with the characterization of the Bismut-non-negative metrics, it enables us to detect a set of Hermitian curvature flows which preserve Bismut non-negativity on homogeneous metrics on the Hopf manifolds.

Theorem (Theorem 8.3.1, or Theorem 5 in [31]). Let $(M, J)$ be a Hopf manifold of complex dimension $n$. Suppose that $a, b, c, d \in \mathbb{R}$ are such that

$$
\begin{array}{ll}
2 b-c+d \geq 0 & \text { if } n=2 ; \\
(n-2) a-2 b+(n-1)^{2} c-(n-1) d \leq-n & \text { if } n>2 .
\end{array}
$$

Then if the metric $g_{0}=g\left(\alpha_{0}, \beta_{0}\right)$ is Bismut-non-negative, the Hermitian curvature flow (0.4) preserves the Bismut non-negativity.

The characterization we obtain is the best possible for this class of examples since all the other Hermitian curvature flows do not preserve Bismut non-negativity in general.

Proposition (Proposition 8.3.1, or Proposition 5 in [31]). Let $(M, J)$ be a Hopf manifold of complex dimension $n$. If the coefficients $(a, b, c, d)$ do not satisfy the inequality (0.5) in the above theorem, then there exists a Bismut-non-negative metric $g_{0}$ on $M$ such that the Hermitian curvature flow (0.4) evolves it into a metric that is no more Bismut-non-negative.

We wonder whether among the Hermitian curvature flows detected by the above results, there is one which preserves the Bismut non-negativity (positivity) on general manifolds, as like as Ustinovskiy's flow preserves the Griffiths non-negativity (positivity). We remark that the pluriclosed flow behaves well with our positivity notion only in complex dimension 2 , namely on the Hopf surfaces. On the one hand, this is interesting because those are the only Hopf manifolds that can be equipped with pluriclosed metrics. On the other hand, for pluriclosed metrics the Bismut positivity is stronger than the Griffiths positivity thanks to the relations in 8.1) and (8.2); henceforth, it would imply that the manifold is a complex projective space.

The thesis is organized as follows.

In Chapter 1. we collect the basic notions concerning almost-Hermitian and Hermitian structures. Precisely, in Section 1.1.3, we give a brief overview on some classes of special metrics generalizing the Kähler condition. In Section 1.2, we introduce the main connections that naturally arise for non-Kähler manifolds, and their associated curvature tensors, showing the relations among them. Finally, in Section 1.3, we give an overview of the Dolbeault, Bott-Chern, and Aeppli cohomologies, also clarifying their relation with the structure of the underlying double complex.

In Chapter 2, we introduce the Hopf manifolds and the Calabi-Eckmann manifolds. Sections 2.2 and 2.3 are dedicated to highlighting their properties, with a focus on the cases of Hopf surfaces in Sections 2.2 .2 and Calabi-Eckman threefold in 2.3.1. In Section 2.3, we deduce a cohomological obstruction to the existence of non-Kähler pluriclosed metrics (Lemma 2.3.1), and thanks to it we characterize the Calabi-Eckmann manifolds admitting a pluriclosed metric in Theorem 2.3.1. In Section 2.1, we study the Hermitian structures whose canonical Gauduchon connections satisfy the symmetries known as Kähler-like condition. Particular interest is on the Bismut-Kähler-like metrics.

Chapter 3 is dedicated to the study of the Gauduchon-Yamabe problem. Precisely, in Section [3.1, we describe the variation of the curvature tensors associated to the canonical Gauduchon connections under conformal change of the metric, and we define a conformal invariant generalizing the Gauduchon degree. Then, in section 3.2 we can express this Yamabe problem for the Gauduchon connections as a semi-linear elliptic equation of the second order. We solve it in Section 3.3 depending on the sign of the parameter as explained above.

In Chapter 4 , we study 4 -dimensional second-Chern-Einstein almost-Hermitian manifolds. In particular, in Section 4.1, we highlight the relation between the second-Chern-Einstein equation and an Einstein-type equation for the Weyl connection. Thanks to it, in Section 4.2, we prove that in the compact case, under a natural hypothesis, the Riemannian dual of the Lee form is a Killing vector field. Thus, we use this observation in Section 4.4 to describe 4 -dimensional compact second-Chern-Einstein locally conformally almost-Kähler manifolds. We give some examples of such manifolds in Section 4.5. Finally, in Section 4.6, we study the second-Chern-Einstein problem on unimodular almost-abelian Lie algebras, classifying those that admit a left-invariant second-Chern-Einstein metric with a parallel non-zero Lee form.

In Chapter 5 we study Calabi-Yau with torsion metrics on principal bundles over Hermitian manifolds with complex tori as fibers. We start with an overview of the geometry of CYT manifolds in Section 5.1. There, we observe that toric bundles over Hermitian manifolds represent a perfect environment where to construct examples of Calabi-Yau with torsion metrics. Thus, in Section 5.2, we describe the CYT condition for the submersion metrics on the total spaces of $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over Hermitian manifolds. Thanks to this analysis, in Section 5.3, we construct explicit examples of Calabi-Yau with torsion Hermitian structures on class $\mathcal{C}$ manifolds and prove the uniqueness result we stated above.

In Chapters 6 and 7 we study respectively the geometry of Bismut flat manifolds and the evolution of the pluriclosed flow on them. Precisely, in Sections 6.1 and 6.2, we give an account of the Riemannian and complex structures of these manifolds. For the special case of rank- 2 compact Bismut flat manifolds, we give an explicit description of their geometry in Section 6.3.3 Moreover, for this class, we compute the Bott-Chern numbers, in Section 6.4.1, and present explicit harmonic representatives for the Dolbeault and Bott-Chern cohomologies, in Section 6.4.2 In particular, we verify that the $(1,1)$-Aeppli cohomology groups are of dimension one and generated by the classes of the Bismut flat metrics coming from the Killing forms. This yields, in Section 7.3 , the result on the stability of the pluriclosed flow on compact Bismut flat manifolds of rank 2 that we stated above. On the other hand, further considerations on the behavior of the pluriclosed flow on Bismut flat manifolds of higher rank are collected in Section 7.4 In Chapter 7. precisely in Section 7.1, we also give a general introduction to the Hermitian curvature flows.

In the last chapter of the thesis, Chapter 8, we introduce and study the notion of Bismut positivity for Hermitian non-Kähler manifolds. In particular, we study Hopf manifolds and 6-dimensional Calabi-Yau solvmanifolds with holomorphically-trivial canonical bundle, respec-
tively in Sections 8.1.4 and 8.1.3. In Section 8.2, the evolution of the Hermitian curvature flows on the Hopf manifolds is completely described through a global stability result (Theorem 8.2.11. Then, in Section 8.3 we use this result to characterize those HCFs which preserve or do not preserve Bismut non-negativity when evolving homogeneous metrics on the Hopf manifolds.

The original results we present in Chapter 8 have been published in [31], while those in Chapters 6 and 7 have been published in 32 . The original results of Chapters 3 and 5 have been accepted for publication in [34]. Furthermore, the original results in Chapter 4 have been obtained in collaboration with Mehdi Lejmi and have been collected in the preprint [36]. Finally, in the thesis, there are also some other original results and observations that will be submitted elsewhere.

## Chapter 1

## Preliminaries on Hermitian and almost-Hermitian geometry

In this preliminary chapter, we recall the notions on Hermitian and almost-Hermitian manifolds that we will need in the thesis. Particular interest will be given to the conditions which extend the Kähler one and to the natural families of connections and cohomologies which arise in the non-Kähler context.

In Section 1.1, we recall some basic notions in almost-complex and almost-Hermitian geometry. These geometries represent perfect environments to get an insight into the origin of properties of Kähler manifolds. Indeed, in the decades, various weaker conditions generalizing the Kähler one, by relaxing the symplectic or complex assumptions, have been introduced and studied, such as Gauduchon, balanced, pluriclosed, locally conformally Kähler, symplectic, or almost-Kähler and so on. We present them in Section 1.1.3. At the same time, several natural connections and cohomological invariants have also been introduced and studied to deal with the Hermitian non-Kähler case. Thus in Section 1.2, we recall the definitions of the Hermitian and conformal connections together with various curvature tensors associated to them and some of the known relations between them. Then, in Section 1.3 , we summarize the notions of Dolbeault, Bott-Chern, and Aeppli cohomologies and their dependence on the structure of the double complex.

The contents of this chapter follow $46,154,162,176,194,195,336]$. Several further references are cited during the exposition.

In this chapter, we also fix the notation which will be used in the rest of the thesis. Firstly, by "manifold" we mean "connected differentiable manifold without border". Then, given a manifold $M$, we consider $k$-tensors of covariant type, namely sections in $\mathcal{C}^{\infty}\left(M ; \otimes^{k} T^{*} M\right)$, and differential $k$-forms, whose space is $\mathcal{A}^{k}(M)$. Moreover, given a Riemannian manifold $(M, g)$, its associated Levi-Civita connection is indicated with the symbol $\nabla^{L C}$, while the isomorphisms induced by $g$ between the tangent and co-tangent bundle are

$$
(\cdot)^{b}: \mathcal{C}^{\infty}(M ; T M) \rightarrow \mathcal{A}^{1}(M) \quad \text { and } \quad(\cdot)^{\sharp}: \mathcal{A}^{1}(M) \rightarrow \mathcal{C}^{\infty}(M ; T M)
$$

### 1.1 Almost-Hermitian structures and non-Kähler metrics

In studying almost-Hermitian geometry, the existence of almost-complex structures on a compact manifold is a very interesting subject. In real dimension greater than 4 , other than the obvious conditions of being even-dimension and orientable, it is not easy to see that a given manifold does or does not carry almost-complex structures, and when it does, whether it also admits integrable structures. For example, in [171 Hopf exhibited infinitely many orientable even-dimensional manifolds that do not admit a complex structure. Furthermore, it has been a long time since we have known that of all Euclidean spheres, only $\mathbb{S}^{2}$ and $\mathbb{S}^{6}$ carry almost-complex structures 61,

189, and we do not yet know if the latter carries an integrable complex structure 111, 125 (see also [124, 2. Kapitel]).

### 1.1.1 Almost-complex structures

Given a manifold $M$ of even dimension, an almost-complex structure $J$ on it is a smooth endomorphism of the tangent bundle, $J \in \operatorname{End}(T M)$, squaring to minus the identity, $J^{2}=-\mathrm{id}$. The Nijenhuis tensor associated to the almost-complex structure is defined by: for any $x, y \in$ $\mathcal{C}^{\infty}(M ; T M)$

$$
4 N_{J}(x, y)=[J x, J y]-[x, y]-J[J x, y]-J[x, J y] .
$$

A complex manifold is a manifold that is equipped with a holomorphic atlas, meaning that the charts are in $\mathbb{C}^{n}$, i.e. the local coordinates are $\left\{z_{i}=x_{i}+\mathrm{i} y_{i}: M \supset U_{i} \rightarrow \mathbb{C}\right\}_{i=1, \ldots, n}$, and the transition functions are holomorphic. A natural almost-complex structure can be associated to it as $J\left(\frac{\partial}{\partial x_{i}}\right) \stackrel{\text { loc }}{=} \frac{\partial}{\partial y_{i}}$ for $i=1, \ldots, n$. Then an almost-complex structure $J$ on $M$ is said to be integrable if there exists a holomorphic atlas of $M$ such that $J$ is the natural almost-complex structure associated to it. Thanks to the celebrated Newlander-Nirenberg theorem, this is equivalent to the vanishing of the Nijenhuis tensor associated to $J$.

Theorem (Theorem 1.1 in 237 ). Let $M$ be a manifold. An almost-complex structure $J$ on $M$ is integrable if and only if $N_{J}=0$.

The $\mathbb{C}$-linear extension of an almost-complex structure $J: T M \rightarrow T M$ to the complexified tangent bundle $T_{\mathbb{C}} M:=T M \otimes_{\mathbb{R}} \mathbb{C}$ of a manifold $M$ yields a bundle decomposition into $\pm \mathrm{i}$ eigenspaces $T_{\mathbb{C}}^{1,0} M$ and $T_{\mathbb{C}}^{0,1} M$. In detail, for every $p \in M$,

$$
\left(T_{\mathbb{C}}^{1,0} M\right)_{p}=\left\{v-\mathrm{i} J_{p}(v) \mid v \in T_{p} M\right\}, \quad\left(T_{\mathbb{C}}^{0,1} M\right)_{p}=\left\{v+\mathrm{i} J_{p}(v) \mid v \in T_{p} M\right\} .
$$

Their dual $\left(T_{\mathbb{C}}^{1,0} M\right)^{*}$ and $\left(T_{\mathbb{C}}^{0,1} M\right)^{*}$ can be also characterized as the $\pm \mathrm{i}$-eigenspaces of the $\mathbb{C}$-linearization of the dual of $J$ acting on $T^{*} M \otimes \mathbb{C}$. Indeed, the action of $J$ on $T^{*} M$ is defined in such a way that the isomorphisms $(\cdot)^{\sharp}$ and $(\cdot)^{b}$ are $J$-linear: for any 1 -form $\alpha$

$$
J(\alpha)(\cdot)=-\alpha(J \cdot) .
$$

This decomposition induces a bi-grading on the complex de Rham algebra of differential forms

$$
\mathcal{A}_{\mathbb{C}}^{k}(M):=\mathcal{A}^{k}(M) \otimes_{\mathbb{R}} \mathbb{C}=\bigoplus_{p+q=k} \mathcal{A}_{j}^{p, q}(M),
$$

with

$$
\mathcal{A}_{J}^{p, q}(M):=\mathcal{C}^{\infty}\left(M ; \wedge^{p, q}(M, J)\right),
$$

and

$$
\wedge^{p, q}(M, J):=\underbrace{\left(T_{\mathbb{C}}^{1,0} M\right)^{*} \wedge \cdots \wedge\left(T_{\mathbb{C}}^{1,0} M\right)^{*}}_{p \text { times }} \wedge \underbrace{\left(T_{\mathbb{C}}^{0,1} M\right)^{*} \wedge \cdots \wedge\left(T_{\mathbb{C}}^{0,1} M\right)^{*}}_{q \text { times }}
$$

The $(p, q)$-component of a $(p+q)$-form $\eta$ will be henceforth indicated with $\eta^{(p, q)}$, and the notation $\eta^{(p, q)+\left(p^{\prime}, q^{\prime}\right)}=\eta^{(p, q)}+\eta^{\left(p^{\prime}, q^{\prime}\right)}$ will also be used. As consequence of this decomposition and the properties of $d$, the $\mathbb{C}$-linear extension of the exterior derivative $d: \mathcal{A}_{\mathbb{C}}^{k}(M) \rightarrow \mathcal{A}_{\mathbb{C}}^{k+1}(M)$ decomposes as

$$
d=\mu+\partial+\bar{\partial}+\bar{\mu},
$$

where $\bar{\partial}$ has bidegree $(0,1)$ and $\partial$ is its complex conjugate, while $\bar{\mu}$ has bidegree $(-1,2)$ and $\mu$ is its complex conjugate. These differential operators are obtained by composing the exterior
derivative $d$ with the natural projections.


In terms of these components, the condition $d^{2}=0$ becomes

$$
\left\{\begin{array}{l}
\mu^{2}=0 \\
\mu \partial+\partial \mu=0 \\
\mu \bar{\partial}+\partial^{2}+\bar{\partial} \mu=0 \\
\mu \bar{\mu}+\bar{\mu} \mu+\partial \bar{\partial}+\bar{\partial} \partial=0
\end{array}\right.
$$

while the integrability condition is equivalent to $\mu=0$, see e.g. 233. Proposition 2.2].
Finally, the exterior differential $d$ can be twisted with the almost-complex structure $J$ defining, for every $k$-form $\alpha$, the operator $d^{c} \alpha=J d J^{-1} \alpha$, where $J^{-1}=(-1)^{k} J$ is the inverse of $J$ acting on $k$-forms, and $J$ acts on $k$-forms as

$$
J \alpha=\alpha\left(J^{-1} \cdot, \ldots, J^{-1} \cdot\right)=(-1)^{k} \alpha(J \cdot, \cdots, J \cdot)
$$

Then the almost-complex structure $J$ is integrable if and only if $d$ and $d^{c}$ anticommute 142 , Proposition 1.11.2], i.e. if and only if

$$
d d^{c}+d^{c} d=0 .
$$

In particular, if $J$ is integrable it holds:

$$
d^{c}=\mathrm{i}(\bar{\partial}-\partial), \quad \text { and } \quad d d^{c}=2 \mathrm{i} \partial \bar{\partial} .
$$

### 1.1.2 Almost-Hermitian structures

An almost-Hermitian manifold $(M, J, g)$ is an almost-complex manifold $(M, J)$ equipped with a Riemannian metric $g$ such that the almost-complex structure $J$ is $g$-orthogonal, that is

$$
g(J \cdot, J \cdot)=g(\cdot, \cdot)
$$

It is called Hermitian when the almost-complex structure is integrable. The linear extension of $g$ to $T_{\mathbb{C}} M$ is given in local holomorphic coordinates $\left\{z_{i}\right\}_{i}$ by

$$
g=g_{i \bar{j}} d z^{i} \otimes d \bar{z}^{j}
$$

with coefficients $g_{i \bar{j}}=g\left(\frac{\partial}{\partial z_{i}}, \frac{\partial}{\partial \bar{z}_{j}}\right)$. Then the matrix $\left(g_{i \bar{j}}\right)_{i, j}$ is Hermitian and its inverse will henceforth be denoted by $\left(g^{i \bar{j}}\right)_{i, j}$. The almost-Hermitian structure $(g, J)$ induces the fundamental 2 -form $\omega(\cdot, \cdot)=g(J \cdot, \cdot)$. In particular, in local holomorphic coordinates $\left\{z_{i}\right\}_{i}$,

$$
\omega=\sqrt{-1} g_{i \bar{j}} d z^{i} \wedge d \bar{z}^{j}
$$

The fundamental form $\omega$ is a real non-degenerate positive ( 1,1 )-form, meaning that for any holomorphic tangent vector $\xi \in T^{1,0} M$,

$$
-\sqrt{-1} \omega(\xi, \bar{\xi})>0
$$

Conversely, on an almost-complex manifold ( $M, J$ ), given a real non-degenerate positive $(1,1)$ form, one can construct an almost-Hermitian metric as $g(\cdot, \cdot)=\omega(\cdot, J \cdot)$. Thus in this thesis, $g$ and $\omega$ will be both addressed as Hermitian metrics. Let $(M, J, g)$ be a compact almostHermitian manifold, then $g$ induces a Hermitian product $\langle\cdot, \cdot\rangle_{g}$ on the spaces $\wedge^{p, q}(M, J)$. In local holomorphic coordinates $\left\{z_{i}\right\}_{i}$ around $z \in M$, given $\alpha, \beta \in \wedge_{z}^{p, q}(M, J)$,

$$
\langle\alpha, \beta\rangle_{g}=\alpha_{i_{1}, \ldots, i_{p}, \bar{j}_{1}, \ldots, \bar{j}_{q}} \overline{\beta_{k_{1}, \ldots, k_{p}, \bar{l}_{1}, \ldots, \bar{l}_{q}}} g^{i_{1}, \bar{k}_{1}} \cdots g^{i_{p}, \bar{k}_{p}} g^{l_{1}, \bar{j}_{1}} \cdots g^{l_{q}, \bar{j}_{q}}
$$

Then a Hermitian product on $\mathcal{A}_{J}^{p, q}(M)$ can be defined as, for any $\alpha, \beta \in \mathcal{A}_{J}^{p, q}(M)$,

$$
(\alpha, \beta)_{g}=\int_{M}\langle\alpha, \beta\rangle_{g} \operatorname{Vol}_{\omega},
$$

where $\operatorname{Vol}_{\omega}$ is the volume form defined by $\operatorname{Vol}_{\omega}=\frac{1}{n} \omega^{n}$. We indicate with $\|\cdot\|_{g}$ and $|\cdot|_{g}$ the norms of $\langle\cdot, \cdot\rangle_{g}$ and $(\cdot, \cdot)_{g}$ respectively. The Hodge-t-operator associated to $g$ is the $\mathbb{C}$-linear map

$$
\star_{g}: \mathcal{A}_{J}^{p, q}(M) \rightarrow \mathcal{A}_{J}^{n-q, n-p}(M)
$$

defined by the following property: for every $\alpha, \beta \in \mathcal{A}_{j}^{p, q}(M)$

$$
\alpha \wedge \star \beta=\langle\alpha, \beta\rangle_{g} \operatorname{Vol}_{\omega} .
$$

Define $\partial_{g}^{*}: \mathcal{A}^{\bullet \bullet} \rightarrow \mathcal{A}^{\bullet-1, \bullet}$ the adjoint operator of $\partial$ with respect to $(\cdot, \cdot)_{g}$. It holds

$$
\partial_{g}^{*}=-\star_{g} \partial \star_{g} .
$$

Similarly, for the adjoint operators of $\bar{\partial}$ and $d$

$$
\bar{\partial}_{g}^{*}=-\star_{g} \bar{\partial} \star_{g} \quad \text { and } \quad d_{g}^{*}=-\star_{g} d \star_{g} .
$$

For the sake of simplicity, the subscript referring to the metric is omitted when clear. The Lee (or torsion) form $\theta$ associated to the almost-Hermitian structure $(g, J)$ is defined as

$$
d \omega^{n-1}=\theta \wedge \omega^{n-1}
$$

The above definition is well-posed since wedging with $\omega^{n-1}$ gives an isomorphism between the spaces of 1 -forms and ( $2 n-1$ )-forms. The Lee form can equivalently be expressed as $\theta=J d^{*} \omega$, see e.g. [136].

### 1.1.3 Special almost-Hermitian metrics

An almost-Hermitian manifold ( $M, J, g$ ) is called almost-Kähler if the associated fundamental form $\omega$ is symplectic, i.e. $d \omega=0$. It is called Kähler if $J$ is also integrable. The almostKähler condition can be relaxed in many different ways. For example, an almost-Hermitian manifold is said balanced in the sense of Michelsohn [229] (or semi-Kähler) if $\theta=0$, that is $d \omega^{n-1}=0$ by definition. In 49 Bismut firstly defined a Hermitian manifold pluriclosed when $d J d \omega=d d^{c} \omega=0$. The pluriclosed condition is also known as strong Kähler with torsion (SKT); this name is inherited from physics nomenclature and relates with the torsion of the Bismut connection, see Section 1.2.2. The balanced and pluriclosed conditions are transverse in the sense that if a metric is both balanced and pluriclosed then it must be Kähler, see 216, Proposition 3.8] or [6, Remark 1]. Even in the non-integrable case, a balanced metric that satisfies $d d^{c} \omega=0$ is almost-Kähler. Moreover, it is conjectured that the existence of a balanced metric together
with a (possibly different) pluriclosed metric on the same complex manifold implies that the manifold admits a Kähler metric, see 121 .

Recently, there has been great interest into locally conformally symplectic (LCS) structures and locally conformally Kähler (LCK) metrics 312,314 (for a general introduction to the subject see $38,108,245)$. A locally conformal symplectic manifold is a pair $(M, \omega)$ where $M$ is a manifold, and $\omega$ a non-degenerate 2 -form on $M$ such that locally it is possible to perform a conformal change that makes it symplectic. Namely, given a LCS form $\omega$ on $M$, if $U \subset M$ is a sufficiently small open set, there exists a function $f_{U}$ such that $e^{f_{U}} \omega$ is a closed form. Thus, $d \omega=-d f_{U} \wedge \omega$ on $U$, and on $U \cap V$ we have $\left(d f_{U}-d f_{V}\right) \wedge \omega=0$. Since $\omega$ is non-degenerate, this implies that the 1-form $\mu:=\left\{d f_{U}\right\}_{U}$ is well-defined globally and closed, and that $d \omega=\mu \wedge \omega$. Given an almost-Hermitian manifold $(M, J, g)$, the metric $g$ is locally conformally almost-Kähler (LCaK) if its fundamental 2-form is LCS, and in this case $\mu=\frac{1}{n-1} \theta$ where $2 n$ is the real dimension of $M$, and $\theta$ is the Lee form. Therefore, an almost-Hermitian manifold $(M, J, g)$ is locally conformally almost-Kähler if $d \theta=0$ and

$$
\begin{equation*}
d \omega=\frac{1}{n-1} \theta \wedge \omega \tag{1.1}
\end{equation*}
$$

In particular, a 4-dimensional almost-Hermitian manifold $(M, J, g)$ is LCaK if $d \theta=0$ since for $n=2$, condition (1.1) is satisfied by definition. On the other hand, in higher dimensions, the condition in 1.1 implies that $d \theta=0$. As a matter of fact, if $M$ has real dimension $2 n>4$, multiplication by $\omega$ is an injective map from $\mathcal{A}^{2}(M)$ into $\mathcal{A}^{4}(M)$ because multiplication by $\omega^{n-2}$ gives an isomorphism between $\mathcal{A}^{2}(M)$ and $\mathcal{A}^{2 n-2}(M)$. This implies that if $d \omega=\frac{1}{n-1} \theta \wedge \omega$ then

$$
0=d^{2} \omega=\frac{1}{n-1} d(\theta \wedge \omega)=\frac{1}{n-1} d \theta \wedge \omega
$$

hence $d \theta=0$. In the integrable case, locally conformally almost-Kähler is locally conformally Kähler.

An almost-Hermitian structure $(J, g)$ is called Gauduchon (or standard, in the notation of 136) if $d^{*} \theta=0$. This condition is equivalent to $d d^{c} \omega^{n-1}=0$ since

$$
d^{*} \theta=d^{*} J d^{*} \omega=\star d \star J \star d \omega^{n-1}
$$

and the Hodge- $\star$-operator commutes with $J$ (see, e.g., 142 , Lemma 1.10.1]). In particular, any balanced metric is a Gauduchon metric, and in complex dimension 2 the Gauduchon condition agrees with the pluriclosed condition. The following fundamental result by Gauduchon proves the existence of a Gauduchon metric in any conformal class

$$
\{\omega\}:=\left\{\exp (f) \omega \mid f \in \mathcal{C}^{\infty}(M ; \mathbb{R})\right\}
$$

of a compact almost-Hermitian manifold; while there are many manifolds which can not support balanced metrics (consider for example the non-Kähler surfaces, such as the Hopf surface).

Theorem 1.1.1 ([136], Théorème 1). Let $M$ be a compact almost-complex manifold of real dimension $\operatorname{dim}_{\mathbb{R}} M \geq 4$, and fix a conformal structure $\{\omega\}$ on it. Then there exists a unique Gauduchon metric $\eta$ in $\{\omega\}$ such that $\int_{M} \operatorname{Vol}_{\eta}=1$.

### 1.1.4 Hodge-de Rham Laplacian and Chern Laplacian

The Hodge-de Rham Laplacian associated to $g$ will be denoted by $\Delta_{g}$. On smooth functions $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$, it is well-known that

$$
\Delta_{g} f=d^{*} d f
$$

With this convention, $\Delta_{g} f_{\left.\right|_{p}} \geq 0$ whenever $f$ is a smooth real function on $M$ which attains a local maximum at $p \in M$. Moreover, integration by part is written as

$$
\int_{M} u \Delta_{g} v \operatorname{Vol}_{\omega}=\int_{M}\langle d u, d v\rangle_{g} \operatorname{Vol}_{\omega}
$$

for $u, v$ smooth functions on a compact manifold $M$.
The Chern Laplacian $\Delta_{\omega}^{C h}$ associated to the Hermitian metric $\omega$ on a smooth function $f$ is defined as

$$
\Delta_{\omega}^{C h} f=2 \sqrt{-1} \operatorname{tr}_{\omega} \bar{\partial} \partial f,
$$

where the trace with respect to $\omega$ of a 2 -form $\alpha$ is

$$
\operatorname{tr}_{\omega} \alpha:=\frac{\alpha \wedge \omega^{n-1}}{\omega^{n}}
$$

The Chern Laplacian is equivalent, in local holomorphic coordinates $\left\{z_{i}\right\}_{i}$, to

$$
\Delta_{\omega}^{C h} \stackrel{\text { loc }}{=}-2 g^{i \bar{j}} \partial_{i} \partial_{\bar{j}} .
$$

In [139], Gauduchon made explicit the relation between the Hodge-de Rham Laplacian $\Delta_{d, \omega}$ and the Chern Laplacian $\Delta_{\omega}^{C h}$ on smooth functions through the Lee form $\theta$.

Lemma 1.1.1 (Page 502 in [139]). Let $(M, J, g)$ be a compact almost-Hermitian manifold. The Chern Laplacian $\Delta_{\omega}^{C h}$ on smooth functions $f$ and its formal adjoint $\left(\Delta_{\omega}^{C h}\right)^{*}$ have the form

$$
\Delta_{\omega}^{C h} f=\Delta_{g} f+\langle d f, \theta\rangle_{g} \quad \text { and } \quad\left(\Delta_{\omega}^{C h}\right)^{*} f=\Delta_{g} f-\langle d f, \theta\rangle_{g}+d^{*} \theta f .
$$

In particular, the Chern Laplacian is a differential elliptic operator of 2 nd order without terms of order 0 and its index equals the index of the Hodge-de Rham Laplacian, as it is observed in [136]. Moreover, the Chern Laplacian and the Hodge-de Rham Laplacian on smooth functions coincide when $\omega$ is balanced, and $\Delta_{\omega}^{C h} f_{\left.\right|_{p}} \geq 0$ whenever $f$ is a smooth real function on $M$ which attains a local maximum at $p \in M$.

### 1.2 Hermitian and conformal connections

Throughout this thesis, all the connections on an almost-complex manifold ( $M, J$ ) will be automatically extended (with the same symbols) to $\mathbb{C}$-linear connections on $T_{\mathbb{C}} M$ and to connections on all the associated vector bundles, e.g., $T^{*} M, \mathcal{A}^{k}(M), \operatorname{End}(T M)$, etc. in a natural way.

In the non-Kähler setting, $\nabla^{L C} J \neq 0$. As a matter of fact, since the Levi-Civita connection is torsion-free, if an almost-complex structure is parallel with respect to $\nabla^{L C}$ then it is integrable. Moreover, on a Hermitian manifold $(M, J, g)$, for every $x, y, z \in \mathcal{C}^{\infty}(M ; T M)$ (see for example 28, Proposition 4.16])

$$
d \omega(x, y, z)=g\left(\left(\nabla_{x}^{L C} J\right) y, z\right)+g\left(\left(\nabla_{y}^{L C} J\right) z, x\right)+g\left(\left(\nabla_{z}^{L C} J\right) x, y\right) .
$$

It follows that $\nabla^{L C} J=0$ implies that the metric is Kähler. On the other hand, for every $x, y, z \in \mathcal{C}^{\infty}(M ; T M)$ (see again [28, Proposition 4.16])

$$
2 g\left(\left(\nabla_{x}^{L C} J\right) y, z\right)=d \omega(x, y, z)-d \omega(x, J y, J z)-g\left(N_{J}(y, J z), x\right) .
$$

Hence also the reverse holds, that is $\nabla^{L C} J=0$ if and only if $g$ is Kähler 28, Theorem 4.17]. For this reason, in Hermitian and almost-Hermitian geometry, one also works with different connections than the Levi-Civita one. In particular, here the interest is on Hermitian connections, which, for a given almost-Hermitian manifold ( $M, J, g$ ), are linear connections on $M$ that preserve the almost-Hermitian structure $(J, g)$ (the term Hermitian for the connection does not suppose $J$ being integrable). In details,

Definition 1.2.1. Let $(M, J, g)$ be an almost-Hermitian manifold. A linear connection $\nabla$ on $M$ is Hermitian if it preserves both the metric $g$ and the almost-complex structure $J$, i.e. $\nabla g=\nabla J=0$.

Given a connection $\nabla$ which satisfies $\nabla J=0$, then $\nabla$ induces a connection on the tangent bundle $T^{1,0}$ (which we indicate with the same symbol) $\nabla: \mathcal{C}^{\infty}\left(M ; T^{1,0} M\right) \rightarrow \mathcal{A}^{1}\left(M ; T^{1,0} M\right)$. Indeed, for any $x \in \mathcal{C}^{\infty}(M ; T M)$ and $y \in \mathcal{C}^{\infty}\left(M ; T^{1,0} M\right)$ it holds

$$
J\left(\nabla_{x} y\right)=\nabla_{x}(J y)=\mathrm{i} \nabla_{x} y .
$$

Finally, on an almost-Hermitian manifold ( $M, J, g$ ), the metric $g$ induces a Hermitian metric (still indicated with $g$ ) on the tangent bundle $T^{1,0} M \rightarrow M$.

### 1.2.1 Chern connection

The Chern connection on an almost-Hermitian manifold ( $M, J, g$ ) can be defined through its relation with the Levi-Civita connection as follows.

Definition 1.2.2. Let $(M, J, g)$ be an almost-Hermitian manifold. Then its associated Chern connection $\nabla^{C h}$ is defined as

$$
g\left(\nabla_{x}^{C h} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right)+\frac{1}{2} d \omega(J x, y, z)-g\left(x, N_{J}(y, z)\right),
$$

for $x, y, z \in \mathcal{C}^{\infty}(M ; T M)$.
It can be proved to be a Hermitian connection, in particular, $\nabla^{C h}: \mathcal{C}^{\infty}\left(M ; T^{1,0} M\right) \rightarrow$ $\mathcal{A}^{1}\left(M ; T^{1,0} M\right)$. Moreover, in the Hermitian case, it coincides with the Chern connection of the holomorphic tangent bundle $T^{1,0} M$ equipped with the extension of $g$, as explained in the following. Given a holomorphic vector bundle $E \rightarrow M$ over a complex manifold ( $M, J$ ), together with a smooth Hermitian fiber metric $(\cdot, \cdot)$ on it, a connection $\nabla$ on $E$ is Hermitian if

$$
x(\eta, \nu)=\left(\nabla_{x} \eta, \nu\right)+\left(\eta, \nabla_{x} \nu\right),
$$

for any $x \in \mathcal{C}^{\infty}(M ; T M)$ and $\eta, \nu \in \mathcal{C}^{\infty}(M ; E)$. Note, in particular, that the Hermitian connections on $(M, J, g)$ are Hermitian connections on $T^{1,0} M$ with respect to the Hermitian metric induced by $g$ on $T^{1,0} M$. A connection on $E$

$$
\nabla: \mathcal{C}^{\infty}(M ; E) \rightarrow \mathcal{A}^{1}(M ; E):=\mathcal{C}^{\infty}\left(M ; T^{*} M \otimes E\right)
$$

can be $\mathbb{C}$-linearly extended, and then decomposed as

$$
\nabla \sigma=\nabla^{\prime} \sigma+\nabla^{\prime \prime} \sigma
$$

for any smooth section $\sigma$ of $E$ with $\nabla^{\prime} \sigma$ in $\mathcal{A}^{1,0}(M ; E):=\mathcal{C}^{\infty}\left(M ;\left(T^{1,0} M\right)^{*} \otimes E\right)$ and $\nabla^{\prime \prime} \sigma$ in $\mathcal{A}^{0,1}(M ; E):=\mathcal{C}^{\infty}\left(M ;\left(T^{0,1} M\right)^{*} \otimes E\right)$. Given a local frame of sections $\left\{e_{i}\right\}_{i}$ for $E \rightarrow M$, the Cauchy-Riemann operator

$$
\bar{\partial}: \mathcal{C}^{\infty}(M ; E) \rightarrow \mathcal{A}^{0,1}(M ; E)
$$

is defined locally as

$$
\bar{\partial} \sigma \stackrel{\text { loc }}{=}\left(\bar{\partial} \sigma^{j}\right) e_{j}
$$

for a section $\sigma \stackrel{\text { loc }}{=} \sigma^{j} e_{j}$ of $E$. Indeed, the transition functions are holomorphic, hence the definition is well-posed.

Proposition 1.2.1. Let $E \rightarrow M$ be a holomorphic vector bundle over the complex manifold $(M, J)$, and consider a smooth Hermitian metric $h=(\cdot, \cdot)$ on it. Then there exists a unique Hermitian connection $\nabla$ such that $\nabla^{\prime \prime}=\bar{\partial}$. This connection is called Chern connection with respect to $(E, h)$.

Then one can verify the following property which characterizes the Chern connection.

Proposition 1.2.2. Let $(M, J, g)$ be a Hermitian manifold. The Chern connection $\nabla^{C h}$ on $(M, J, g)$ is the only Hermitian connection whose extension to the holomorphic tangent bundle coincides with the Chern connection on $\left(T^{1,0} M, g\right)$.

Alternatively one can also characterize the Chern connection on a complex manifold as the unique Hermitian connection on $(M, J, g)$ such that the torsion tensor $T^{C h}(x, y):=\nabla_{x}^{C h} y-$ $\nabla_{y}^{C h} x-[x, y]$ satisfies $T^{C h}(J x, y)=T^{C h}(x, J y)$ for any $x, y \in \mathcal{C}^{\infty}(M ; T M)$. This means that $T^{C h}(x, \bar{y})=0$ for the natural extension of the torsion tensor to $T_{\mathbb{C}} M$ and $x, y \in T^{1,0} M$; or in other words, the complex $(1,1)$-part of $T^{C h}$ vanishes. This property is usually used to define the Chern connection in the almost-Hermitian case.

Proposition 1.2.3. Let $(M, J, g)$ be an almost-Hermitian manifold. Then, there exists a unique Hermitian connection such that its torsion tensor is J-skew-invariant. This connection is the Chern connection $\nabla^{C h}$ on $(M, J, g)$.

We remark that $\nabla^{C h}$ was first defined using this characterizing property by Ehresmann and Libermann in 111 . Since in the complex case it coincides with the connection used by Chern in [93], it is called the Chern connection.

### 1.2.2 Bismut connection

Let $(M, g)$ be a Riemannian manifold. Consider a connection $\nabla$ on $M$, and its torsion tensor $T(x, y):=\nabla_{x} y-\nabla_{y} x-[x, y]$. It is possible to associate to $T$ a 3 -tensor by using $g$ as $g(T(\cdot, \cdot), \cdot)$. This tensor will also be called torsion tensor.

Definition 1.2.3. Let $(M, g)$ be a Riemannian manifold and $H \in \mathcal{A}^{3}(M)$. The $\pm$-connections associated to the pair $(g, H)$ are defined via

$$
g\left(\nabla_{x}^{ \pm} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right) \pm \frac{1}{2} H(x, y, z)
$$

Since the Levi-Civita connection is torsion-free, these connections have torsion equal to $\pm H$. Moreover, this property characterizes them in the following sense.

Proposition 1.2.4. Given a Riemannian manifold $(M, g)$ and a 3-form $H$, the $\nabla^{+}$connection is the unique metric connection with torsion $H$.

Proof. Any metric connection $\nabla$ on $(M, g)$ can be expressed in terms of the Levi-Civita connection as

$$
\nabla_{x} y=\nabla_{x}^{L C} y+A(x, y)
$$

where $A \in \mathcal{A}^{1}(M ; \operatorname{End}(M))$ satisfies

$$
g(A(x, y), z)+g(A(x, z), y)=0
$$

Hence a metric connection is uniquely determined by its torsion since the following relation between $A$ and $T$ holds:

$$
\left\{\begin{array}{l}
2 A(x, y, z)=T(x, y, z)-T(y, z, x)+T(z, x, y)  \tag{1.2}\\
T(x, y, z)=A(x, y, z)-A(y, x, z)
\end{array}\right.
$$

(see, for example, $141,(2.1 .3)$ and (2.1.4)]). Then the statement follows.

In the Hermitian context, there is a preferred choice of 3 -form $H$ which involves the complex structure, namely $H=-J d \omega=-d^{c} \omega$.

Definition 1.2.4. Let $(M, J, g)$ be an Hermitian manifold. The Bismut connection $\nabla^{B}$ is the $\nabla^{+}$connection associated to $(g,-J d \omega)$. In particular, it holds

$$
g\left(\nabla_{x}^{B} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right)-\frac{1}{2} J d \omega(x, y, z), \quad \text { and } \quad T^{B}=-J d \omega=-d^{c} \omega
$$

for $T^{B}$ the torsion tensor of $\nabla^{B}$.
The explicit expression of the above Bismut connection appeared in Strominger's paper [288] in 1986, where he called it the H-connection. Independently, Bismut came across this connection in the context of index theory problems in complex non-Kähler geometry [49] there he also established its existence and uniqueness. For the sake of simplicity, in this thesis, we choose to call it the Bismut connection, while in some literature it is also called Strominger or StromingerBismut connection. A nomenclature inherited from physics would be KT connection (Kähler with torsion). The name SKT for pluriclosed metrics has its origin in the Bismut connection. Indeed, for a Hermitian metric, the torsion of the KT connection is $d$-closed if and only if the metric is pluriclosed.

In extending the definition to an almost-Hermitian manifold one has to take into account the Nijenhuis tensor. Thus finally we define the Bismut connection associated to an almost-Hermitian manifold $(M, J, g)$ as follows.

Definition 1.2.5. Let $(M, J, g)$ be an almost-Hermitian manifold. Then its associated Bismut connection $\nabla^{B}$ is defined as

$$
g\left(\nabla_{x}^{B} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right)-\frac{1}{2} J d \omega(x, y, z)+\frac{1}{2} g\left(N_{J}(x, y), z\right)
$$

for $x, y, z \in \mathcal{C}^{\infty}(M ; T M)$.
Notice that, in general, on an almost-Hermitian manifold, $\nabla^{B}$ has not skew-symmetric torsion [123. Theorem 10.1]. When it happens, the connection is called the characteristic connection of the almost Hermitian structure (see [141] for a survey).

Proposition 1.2.5 (Theorem 10.1 in 123 ). Let $(M, J, g)$ be an almost-Hermitian manifold. Then there exists a linear connection with totally skew-symmetric torsion preserving the Hermitian structure $(J, g)$ if and only if the Nijenhuis tensor $g\left(N_{J}(\cdot, \cdot), \cdot\right)$ is a 3-form. In this case, the connection is unique and is determined by

$$
T(x, y, z)=d \omega(J x, J y, J z)+g\left(N_{J}(x, y), z\right)
$$

### 1.2.3 Gauduchon connections

In 141] Gauduchon introduced a family of canonical Hermitian connections with prescribed torsion depending on a real parameter $t \in \mathbb{R}$. Afterward this family was further extended by the authors of 247 to a 2-parameters family in order to include also the Levi-Civita connection. The canonical Hermitian connection corresponding to the value $t$ of the Gauduchon parameter is denoted by $\nabla^{t}$. Given an almost-Hermitian manifold $(M, J, g)$, these are described with respect to the Levi-Civita connection $\nabla^{L C}$ as

$$
\begin{equation*}
g\left(\nabla_{x}^{t} y, z\right)=g\left(\nabla_{x}^{L C} y, z\right)+\frac{t-1}{4} J d \omega(x, y, z)+\frac{t+1}{4} d \omega(J x, y, z)+\frac{1}{2} g\left(N_{J}(x, y), z\right) \tag{1.3}
\end{equation*}
$$

Hence, by 1.2 , the above formula prescribes the torsion $T^{t}$ of the Gauduchon connections as

$$
T^{t}(x, y, z)=\frac{t-1}{2} J d \omega(x, y, z)+\frac{t+1}{4}(d \omega(J x, y, z)+d \omega(x, J y, z))+g\left(N_{J}(x, y), z\right)
$$

For any almost-Hermitian structure, the canonical Hermitian connection $\nabla^{t}$ satisfies 141 , Proposition 4]

$$
\nabla^{t}=\frac{1}{2}\left(\nabla^{1}+\nabla^{-1}\right)+\frac{t}{4}\left(\left(d^{c} \omega\right)^{(2,1)+(1,2)}+\left(d^{c} \omega\right)^{(2,1)+(1,2)}(\cdot, J \cdot, J \cdot)\right),
$$

In particular, if $(J, g)$ is $(2,1)$-symplectic, meaning that $\left(d^{c} \omega\right)^{(2,1)+(1,2)}=0$, the canonical set $\left\{\nabla^{t}\right\}_{t \in \mathbb{R}}$ reduces to just one element. On the other hand, if $(J, g)$ is not $(2,1)$-symplectic, elements of $\left\{\nabla^{t}\right\}_{t \in \mathbb{R}}$ corresponding to different Gauduchon parameters are distinct and the set of canonical connections forms an affine line in the space of Hermitian connections. Here, $\frac{1}{2}\left(\nabla^{1}+\nabla^{-1}\right)=\nabla^{0}$ represents the orthogonal projection of the Levi-Civita connection into the affine space of Hermitian connections. Hence $\nabla^{0}$ coincides with the first canonical connection of 215]. Moreover, the connection $\nabla^{1}$ is the Chern connection $\nabla^{C h}$ of $(J, g)$ as defined in Definition 1.2.2, also known as the second canonical connection of 215; while $\nabla^{-1}$ is the Bismut connection $\nabla^{B}$ as in Definition 1.2.5

Given a Hermitian manifold ( $M, J, g$ ), the Christoffel symbols $\Gamma^{t}$ of the canonical Gauduchon connection $\nabla^{t}$ associated to the Hermitian structure $(J, g)$ can be easily computed from (1.3), and in local holomorphic coordinates $\left\{z_{1}, \ldots, z_{n}\right\}$ are:

$$
\begin{align*}
& \left(\Gamma^{t}\right)_{i j}^{k}=g^{k \bar{s}}\left(\frac{1+t}{2} \frac{\partial g_{j \bar{s}}}{\partial z_{i}}+\frac{1-t}{2} \frac{\partial g_{i \bar{s}}}{\partial z_{j}}\right) ; \\
& \left(\Gamma^{t}\right)_{\bar{i} j}^{k}=\frac{1-t}{2} g^{k \bar{s}}\left(\frac{\partial g_{j \bar{s}}}{\partial \bar{z}_{i}}-\frac{\partial g_{j \bar{i}}}{\partial \bar{z}_{s}}\right) ;  \tag{1.4}\\
& \left(\Gamma^{t}\right)_{i \bar{j}}^{k}=\left(\Gamma^{t}\right)_{\overline{i j}}^{k}=0 .
\end{align*}
$$

### 1.2.4 Canonical Weyl connection

Let $(M, J, g)$ be an almost-Hermitian manifold. The canonical Weyl connection $\nabla^{W}$ associated to ( $M, J, g$ ) is the only torsion-free connection which satisfies

$$
\nabla^{W} g=\theta \otimes g
$$

where $\theta$ is the Lee form. It follows from a direct computation that it is related to the Levi-Civita connection $\nabla^{L C}$ associated to $g$ by

$$
\nabla_{x}^{W} y=\nabla_{X}^{L C} Y-\frac{1}{2} \theta(x) y-\frac{1}{2} \theta(y) x+\frac{1}{2} g(x, y) \theta^{\sharp} .
$$

The condition $\nabla^{W} J=0$ is intimately related to the integrability of $J$ and the locally conformally Kähler condition. This is clarified by a result of Vaismann [312, Theorem 2.2]. We refer to Section 4.1 for more details.

### 1.2.5 Curvature tensors

In the following, given an almost-Hermitian manifold ( $M, J, g$ ) and a 2 -tensor $\psi$, we denote by $\psi^{J,+}$ its $J$-invariant part, and by $\psi^{J,-}$ its $J$-skew-invariant part; $\psi^{\text {sym }}$ denotes its symmetric part, while $\psi^{\text {skew-sym }}$ its skew-symmetric part. Moreover, in real dimension 4, a 2 -form $\phi$ can be decomposed into a $g$-orthogonal sum $\phi=\phi^{+}+\phi^{-}$under the action of the Riemannian Hodge- $\star$-operator, where $\phi^{+}$is self dual, i.e. $\star \phi^{+}=\phi^{+}$, and $\phi^{-}$is anti-self dual, i.e. $\star \phi^{-}=-\phi^{-}$.

It is known [286, Lemma 2.9], that on a Hermitian manifold $(M, J, g)$, at a fixed point $p \in M$ there exist special holomorphic coordinates $\left\{z_{i}\right\}$, such that

$$
g_{i \bar{j}}(p)=\delta_{i j}
$$

(where $\delta$ is the Kronecker delta), and the complexified Christoffel symbols of the Levi-Civita connection vanish at $p$, i.e.

$$
\left(\Gamma^{L C}\right)_{i j}^{k}(p)=\frac{1}{2} g^{k \bar{l}}\left(\partial_{i} g_{j \bar{l}}+\partial_{j} g_{i \bar{l}}\right)=0
$$

Notice also that the existence of holomorphic coordinates which osculate the metric at the first order (i.e. $g_{i \bar{j}}(p)=\delta_{i j}$ and $\partial_{i} g_{j \bar{l}}=0$ ) is equivalent to the metric being Kähler.

Fix an almost-Hermitian manifold ( $M, J, g$ ). In this thesis, the curvature tensor

$$
R_{x, y}^{\nabla}:=\left[\nabla_{x}, \nabla_{y}\right]-\nabla_{[x, y]}
$$

of a given connection $\nabla$ on $M$ will be considered as linearly extended to $T_{\mathbb{C}} M$, hence

$$
R^{\nabla} \in \mathcal{C}^{\infty}\left(M ; T_{\mathbb{C}}^{*} M \wedge T_{\mathbb{C}}^{*} M \otimes \operatorname{End}\left(T_{\mathbb{C}} M\right)\right)
$$

If the connection is Hermitian, then the conditions $\nabla g=\nabla J=0$ imply

$$
g\left(R_{\cdot, \cdot}^{\nabla}, \cdot\right) \in \mathcal{A}_{\mathbb{C}}^{2}(M) \otimes \mathcal{A}_{j}^{1,1}(M)
$$

The 4-tensor $g\left(R_{,, \cdot}^{\nabla}, \cdot\right)=: R^{\nabla}(\cdot, \cdot, \cdot, \cdot)$ will be indicated with the symbol $R_{\text {}}^{\nabla}$.. In the Hermitian case, the curvature tensor $R^{t}$ of the canonical Gauduchon connection $\nabla^{t}$ in special local holomorphic coordinates has coefficients (directly computed from (1.4)):

$$
\begin{align*}
R_{i \bar{j} k \bar{l}}^{t}(g)=\frac{1-t}{2}\left(\frac{\partial^{2} g_{k \bar{l}}}{\partial z_{i} \partial \bar{z}_{j}}-\frac{\partial^{2} g_{k \bar{j}}}{\partial z_{i} \partial \bar{z}_{l}}-\frac{\partial^{2} g_{i \bar{l}}}{\partial z_{k} \partial \bar{z}_{j}}\right) & -\frac{1+t}{2} \frac{\partial^{2} g_{k \bar{l}}}{\partial z_{i} \partial \bar{z}_{j}} \\
& +\sum_{q}\left((1-t)^{2} \frac{\partial g_{q \bar{l}}}{\partial z_{i}} \frac{\partial g_{k \bar{j}}}{\partial \bar{z}_{q}}-t^{2} \frac{\partial g_{i \bar{q}}}{\partial z_{k}} \frac{\partial g_{q \bar{l}}}{\partial \bar{z}_{j}}\right), \tag{1.5}
\end{align*}
$$

and

$$
\begin{equation*}
R_{i j k \bar{l}}^{t}(g)=\frac{1-t}{2}\left(\frac{\partial^{2} g_{j \bar{l}}}{\partial z_{i} \partial z_{k}}-\frac{\partial^{2} g_{i \bar{l}}}{\partial z_{j} \partial z_{k}}\right)+t(1-t) \sum_{q}\left(\frac{\partial g_{q \bar{l}}}{\partial z_{j}} \frac{\partial g_{k \bar{q}}}{\partial z_{i}}-\frac{\partial g_{q \bar{l}}}{\partial z_{i}} \frac{\partial g_{k \bar{q}}}{\partial z_{j}}\right) \tag{1.6}
\end{equation*}
$$

In particular, the only non-vanishing coefficients of the Chern curvature tensor $R^{C h}$ are

$$
R_{i \bar{k} \bar{l}}^{C h}(g)=-\frac{\partial^{2} g_{k \bar{l}}}{\partial z_{i} \partial \bar{z}_{j}}-\frac{\partial g_{i \bar{q}}}{\partial z_{k}} \frac{\partial g_{q \bar{l}}}{\partial \bar{z}_{j}}
$$

while for the Bismut connection

$$
\begin{aligned}
& R_{i \overline{j k} \bar{l}}^{B}(g)=\left(\frac{\partial^{2} g_{k \bar{l}}}{\partial z_{i} \partial \bar{z}_{j}}-\frac{\partial^{2} g_{k \bar{j}}}{\partial z_{i} \partial \bar{z}_{l}}-\frac{\partial^{2} g_{i \bar{l}}}{\partial z_{k} \partial \bar{z}_{j}}\right)+4 \sum_{q}\left(\frac{\partial g_{q \bar{l}}}{\partial z_{i}} \frac{\partial g_{k \bar{j}}}{\partial \bar{z}_{q}}-\frac{\partial g_{i \bar{q}}}{\partial z_{k}} \frac{\partial g_{q \bar{l}}}{\partial \bar{z}_{j}}\right) ; \\
& R_{i j k \bar{l}}^{B}(g)=\left(\frac{\partial^{2} g_{j \bar{l}}}{\partial z_{i} \partial z_{k}}-\frac{\partial^{2} g_{i \bar{l}}}{\partial z_{j} \partial z_{k}}\right)-2 \sum_{q}\left(\frac{\partial g_{q \bar{l}}}{\partial z_{j}} \frac{\partial g_{k \bar{q}}}{\partial z_{i}}-\frac{\partial g_{q \bar{l}}}{\partial z_{i}} \frac{\partial g_{k \bar{q}}}{\partial z_{j}}\right) .
\end{aligned}
$$

## Ricci curvatures

Fix an almost-Hermitian manifold ( $M, J, g$ ), and consider a $J$-adapted $g$-orthonormal frame of the tangent bundle $\left\{e_{1}, e_{2}=J e_{1}, e_{3}, e_{4}=J e_{3}, \ldots, e_{2 n-1}, e_{2 n}=J e_{2 n-1}\right\}$. The first Chern-Ricci form $R i c^{C h, 1}$ (also called the Hermitian Ricci form) of the Chern connection $\nabla^{C h}$ is defined by

$$
R i c^{C h, 1}(x, y)=-\frac{1}{2} \sum_{i=1}^{2 n} g\left(R_{x, y}^{C h} e_{i}, J e_{i}\right) .
$$

It is a $J$-invariant 2 -form and the symmetric 2 -tensor obtained by twisting with $J, R i c^{C h, 1}(\cdot, J \cdot)$, will be indicated with the same symbol. Similarly, the Bismut-Ricci 2 -form Ric ${ }^{B}$ is defined by

$$
\operatorname{Ric}^{B}(x, y)=-\frac{1}{2} \sum_{i=1}^{2 n} g\left(R_{x, y}^{B} e_{i}, J e_{i}\right)
$$

but it is usually not $J$-invariant: there are also $(2,0)$ and $(0,2)$ components. Finally, the $\nabla^{t}$-Ricci 2 -form Ric $^{t}$ of the canonical Gauduchon connection $\nabla^{t}$ is defined by

$$
\operatorname{Ric}^{t}(x, y)=-\frac{1}{2} \sum_{i=1}^{2 n} g\left(R_{x, y}^{t} e_{i}, J e_{i}\right)
$$

Again, these are not $J$-invariant in general. As a matter of fact, it is $J$-invariant if and only if $t=1$ or the metric is balanced (see 1.7) and Proposition 1.2.6). The 2 -forms Ric ${ }^{t}$ are closed and they are representatives of the first Chern class $2 \pi c_{1}(T M, J)$ in de Rham cohomology. Indeed, they differ from the first Chern-Ricci form by an exact factor multiple of $d J \theta$, i.e.

$$
\begin{equation*}
R i c^{t}=R i c^{C h, 1}-\frac{t-1}{2} d J \theta . \tag{1.7}
\end{equation*}
$$

This relation can be directly verified in the Hermitian case by using 1.5 and 1.6, and computing $d J \theta=-d d^{*} \omega$ as in [286, Lemma 2.6]; while for the almost-Hermitian case see [141, Equation (2.7.6)]. In particular,

$$
\begin{equation*}
R i c^{B}=R i c^{C h, 1}+d J \theta \tag{1.8}
\end{equation*}
$$

The second Chern-Ricci form $R i c^{C h, 2}$ of $\nabla^{C h}$ is defined by

$$
\operatorname{Ric}^{C h, 2}(x, y)=-\frac{1}{2} \sum_{i=1}^{2 n} g\left(R_{e_{i}, J e_{i}}^{C h} x, y\right) .
$$

It is a $J$-invariant 2-form but not closed in general, and the symmetric $J$-invariant 2-tensor $\operatorname{RiC}^{C h, 2}(\cdot, J \cdot)$, obtained by twisting with $J$, will be indicated with the same symbol. In general, the first and the second Chern-Ricci forms are not equal. It happens, for example, if the metric is Kähler, or more generally if the Chern curvature tensor satisfies the torsionless first Bianchi identity (1-Bnc), see the next section. Similarly, $R i c^{W, J}$ is given by the formula

$$
\operatorname{Ric}^{W, J}(x, y)=-\frac{1}{2} \sum_{i=1}^{2 n} g\left(R_{e_{i}, J e_{i}}^{W} x, y\right)
$$

where $R^{W}$ is the curvature tensor of the Weyl connection $\nabla^{W}$. The 2 -tensor $R i c^{W, J}$ is a 2 -form and it is not $J$-invariant in general. However, in real dimension $\operatorname{dim}_{\mathbb{R}} M=4$, and when $J$ is integrable, $\nabla^{W}$ preserves $J$ and so $R i c^{W, J}=\left(R i c^{W, J}\right)^{J,+}$ is $J$-invariant. Moreover,

$$
\begin{equation*}
R i c^{W, J}=\operatorname{Ric}^{B}+(d \theta)^{(2,0)+(0,2)}(J \cdot, \cdot)=R i c^{B}-(d J \theta)^{(2,0)+(0,2)}=\left(R i c^{B}\right)^{J,+} \tag{1.9}
\end{equation*}
$$

see Lemma 4.4.2 and compare with (1.8). In higher dimensions, if the canonical Weyl connection preserves the complex structure, then $R i c^{W, J}$ is a $(1,1)$-form. Furthermore, if $(M, J, g)$ is a locally conformally Kähler manifold of complex dimension $n$, then (see Remark 4.6.1)

$$
R i c^{W, J}=R i c^{\frac{1}{1-n}}
$$

The Weyl-Ricci tensor $\mathrm{Ric}^{W}$ is defined in 143 as

$$
\operatorname{Ric}^{W}(x, y)=-\sum_{i=1}^{2 n} g\left(R_{e_{i}, x}^{W} e_{i}, y\right)
$$

In real dimension $\operatorname{dim}_{\mathbb{R}} M=4$, the tensor $R i c^{W}$ is symmetric (this is only true in dimension 4). On the other hand, the tensor $\widetilde{\text { Ric }}^{W}$, defined as (see for example 255)

$$
\widetilde{\operatorname{Ric}}^{W}(x, y)=-\sum_{i=1}^{2 n} g\left(R_{x, e_{i}}^{W} y, e_{i}\right),
$$

is not symmetric in general. Its skew-symmetric part is $d \theta$ while its symmetric part is $R i c^{W}$, that is ( $\operatorname{see}[7])$

$$
\widetilde{\operatorname{Ric}}^{W}=R i c^{W}+d \theta
$$

The Riemannian Ricci tensor $R i c^{L C}$ is

$$
\operatorname{Ric}^{L C}(x, y)=-\sum_{i=1}^{2 n} g\left(R_{e_{i}, x}^{L C} e_{i}, y\right)
$$

and the $\star$-Ricci tensor $\rho^{\star}$ is defined as

$$
\rho^{\star}(x, y)=-\frac{1}{2} \sum_{i=1}^{2 n} g\left(R_{e_{i}, J e_{i}}^{L C} x, J y\right) .
$$

From the definition, it follows that $\rho^{\star}(x, y)=\rho^{\star}(J y, J x)$ so $\rho^{\star}$ is symmetric if and only it is $J$-invariant.

## Scalar curvatures

With the same notation as before, the Chern scalar curvature $s^{C h}$ (also called the Hermitian scalar curvature) of $\nabla^{C h}$ is defined by

$$
s^{C h}=\sum_{i=1}^{2 n} R i c^{C h, 1}\left(e_{i}, J e_{i}\right)=\sum_{i=1}^{2 n} R i c^{C h, 2}\left(e_{i}, J e_{i}\right) .
$$

In general, the $\nabla^{t}$-scalar curvature $s^{t}$ of the canonical Gauduchon connection $\nabla^{t}$ is defined as

$$
s^{t}=\sum_{i=1}^{2 n} R i c^{t}\left(e_{i}, J e_{i}\right)
$$

Hence, taking the trace in (1.7), the following relation holds:

$$
\begin{equation*}
s^{t}=s^{C h}+\frac{t-1}{2} \operatorname{tr}_{\omega} d d^{*} \omega=s^{C h}+\frac{t-1}{2} \operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega+\overline{\partial \partial}^{*} \omega\right) . \tag{1.10}
\end{equation*}
$$

The second- $\nabla^{t}$-scalar curvature $s_{2}^{t}$ of the canonical Gauduchon connection $\nabla^{t}$ is defined tracing in the other way around, namely

$$
s_{2}^{t}=\sum_{i=1}^{2 n} g\left(R_{e_{i}, e_{j}}^{t} J e_{j}, J e_{i}\right)
$$

The Riemannian scalar curvature $s^{L C}$ is defined as

$$
s^{L C}=\sum_{i=1}^{2 n} \operatorname{Ric}^{L C}\left(e_{i}, e_{i}\right)
$$

the conformal scalar curvature $s^{W}$ as

$$
s^{W}=\sum_{i=1}^{2 n} \operatorname{Ric}^{W}\left(e_{i}, e_{i}\right)
$$

and the $\star$-scalar curvature $s^{\star}$ as

$$
s^{\star}=\sum_{i=1}^{2 n} \rho^{\star}\left(e_{i}, e_{i}\right)
$$

The conformal scalar curvature $s^{W}$ is then related to the Riemannian scalar curvature $s^{L C}$ by (see for example [7, Equation (2.4)])

$$
\begin{equation*}
s^{W}=s^{L C}-(2 n-1) d^{*} \theta-\frac{(2 n-1)(n-1)}{2}\|\theta\|_{g}^{2} \tag{1.11}
\end{equation*}
$$

Furthermore, for any almost-Hermitian manifold of dimension $\operatorname{dim}_{\mathbb{R}} M=4$, by 301 it holds

$$
\begin{equation*}
\left(\rho^{\star}\right)^{s y m}-\left(R i c^{L C}\right)^{J,+}=\frac{s^{\star}-s^{L C}}{4} g \tag{1.12}
\end{equation*}
$$

On the other hand, $s^{\star}$ is related to the Riemannian scalar curvature $s^{L C}$ by (see 204, 274, and 313 in the integrable case)

$$
\begin{equation*}
s^{\star}-s^{L C}=-2 d^{*} \theta-\|\theta\|_{g}^{2}+2\left\|N_{J}\right\|_{g}^{2} \tag{1.13}
\end{equation*}
$$

Several more relations (mostly in real dimension 4) involving $\rho^{\star}$ and $s^{\star}$ will be obtained in Chapter 4.

### 1.2.6 Further relations between Gauduchon curvature tensors

Given a Hermitian (non-Kähler) manifold, the relations among curvature tensors have been studied, for example, in 216] and 217] where the authors focused on Levi-Civita, Chern, and Bismut connections. Very recently, the relations between the curvature tensors of the Gauduchon connections have been studied in [65]. In presence of Kähler symmetries (such as the torsionless first Bianchi identity (1-Bnc), see Section 2.1) there turns out to exist stronger relations among the curvature tensors (for example, the first and second Ricci tensors are equal). The condition $R^{C h}=R^{L C}$ forces the metric to be Kähler, as proven in $[329$, Theorem 1.1]. In the same spirit, in [217, Corollary 4.5], the authors prove that if $s^{C h}=s^{L C}$, then the metric is balanced. In the following, we prove some other equivalences between relations among curvature tensors and properties of the Hermitian structures. For example, if the first scalar curvatures with respect to different Gauduchon parameters are equal, that is $s^{t_{1}}=s^{t_{2}}$ for $t_{1} \neq t_{2}$, then the Hermitian structure must be balanced. This is also equivalent to the $\nabla^{t}$-Ricci curvature forms being equal. In details,

Proposition 1.2.6 (Proposition 2.2 of $[34])$. Let $(M, J, g)$ be a compact Hermitian manifold and take two Gauduchon parameters $t_{1} \neq t_{2}$. Then the following conditions are equivalent:
i. Ric $^{t_{1}}=$ Ric $^{t_{2}}$;
ii. $s^{t_{1}}=s^{t_{2}}$;
iii. $g$ is balanced.

Proof. Obviously, $(i) \Rightarrow(i i)$; while $(i i i) \Rightarrow(i)$ is a simple consequence of (1.7). As for $(i i) \Rightarrow(i i i)$, taking the trace in 1.7) we have that $s^{t_{1}}=s^{t_{2}}$ if and only if

$$
\operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega+{\left.\overline{\partial \partial_{g}} \omega\right)=0 . . . ~}_{*}\right.
$$

However, by integrating over $M$ we get that

$$
\int_{M} \operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega\right) \operatorname{Vol}_{g}=\left(\partial \partial^{*} \omega, \omega\right)_{g}=\left(\partial^{*} \omega, \partial^{*} \omega\right)_{g}=\left|\partial^{*} \omega\right|_{g}^{2}
$$

and similarly for $\operatorname{tr}_{\omega}\left(\overline{\partial \partial}_{g}^{*} \omega\right)$. Thus both $\partial^{*} \omega$ and $\bar{\partial}_{g}^{*} \omega$ vanish, which means that $\theta=J d^{*} \omega=0$ and $g$ is balanced.

A stronger restriction is obtained by asking both the first and second scalar curvatures to be equal. This turns out to be equivalent to the $\nabla^{t}$-curvature tensors to be equal. Precisely,

Proposition 1.2.7. Let $(M, J, g)$ be a Hermitian manifold. Then for $t_{1} \neq t_{2}$ and $t_{1}+t_{2} \neq 2$ the following conditions are equivalent:
i. $R^{t_{1}}=R^{t_{2}}$;
ii. $s^{t_{1}}=s^{t_{2}}$ and $s_{2}^{t_{1}}=s_{2}^{t_{2}}$;
iii. $g$ is Kähler.

Proof. There is only one non-trivial implication to be proved, namely (ii) $\Rightarrow$ (iii). Hence, suppose that $s^{t_{1}}=s^{t_{2}}$ and $s_{2}^{t_{1}}=s_{2}^{t_{2}}$, then, thanks to Proposition 1.2.6, $g$ must be balanced, which implies that

$$
0=\theta_{i}=\sum_{k}\left(T^{C h}\right)_{i k}^{k}
$$

This identity simplifies the expression of $s_{2}^{t}$, which can be computed in special holomorphic coordinates as in Proposition 2.1 .1 (where $T=T^{C h}$ is the torsion of the Chern connection $\nabla^{C h}$ ):

$$
\begin{aligned}
s_{2}^{t} & =\frac{1-t}{2} \operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega+{\left.\overline{\partial \partial_{g}^{*}} \omega\right)-\frac{1}{4}(t-1)^{2}\left\|T_{s}^{s} \cdot\right\|_{g}^{2}-\frac{1}{4}\left(t^{2}-2 t+2\right)\|T\|_{g}^{2}-\sum_{i j} \partial_{i} \bar{\partial}_{j} g_{j \bar{i}}}=-\frac{1}{4}\left(t^{2}-2 t+2\right)\|T\|_{g}^{2}-\sum_{i j} \partial_{i} \bar{\partial}_{j} g_{j \bar{i}}\right.
\end{aligned}
$$

Finally, $s_{2}^{t_{1}}=s_{2}^{t_{2}}$ becomes

$$
-\frac{1}{4}\left(t_{1}^{2}-2 t_{1}+2\right)\|T\|_{g}^{2}=-\frac{1}{4}\left(t_{2}^{2}-2 t_{2}+2\right)\|T\|_{g}^{2}
$$

that is,

$$
0=\frac{\left(t_{1}-1\right)^{2}-\left(t_{2}-1\right)^{2}}{4}\|T\|_{g}^{2}
$$

Since by hypothesis $\left(1-t_{1}\right)^{2} \neq\left(1-t_{2}\right)^{2}$, the Chern torsion $T$ vanishes, and hence the manifold is Kähler.

### 1.3 Double complex and cohomologies of complex manifolds

The Dolbeault cohomology is a natural cohomological invariant in complex geometry. It is the cohomology of the holomorphic tangent bundle, while the Bott-Chern cohomology and the Aeppli cohomology provide a bridge between the de Rham cohomology and the Dolbeault cohomology of a complex manifold. They represent important invariants for the study of the geometry of compact complex manifolds, especially non-Kähler. In fact, in Kähler geometry they both equal the Dolbeault cohomology, while they capture different information in the non-Kähler context. The Bott-Chern cohomology and the Aeppli cohomology groups were introduced respectively by Bott and Chern in [60], and by Aeppli in [2], and hence studied by many authors [1, 5, 12, 47, $48,50,51,62,95,103,199,224,239,240,273,304,315$ showing their geometric meaning. In this thesis, we are interested in the Aeppli, Dolbeault, and Bott-Chern cohomologies because they represent the natural environments to study torsion classes. As a matter of fact, as we will see in Chapter 7 (see also [32, 132]) the stability of a particular flow of metrics of our interest can be reduced to "qualitative" properties of the Aeppli cohomology in bi-degree $(1,1)$.

In this section, we recall the definitions of the Dolbeault, Bott-Chern, and Aeppli cohomologies. Then we recall some basic results on Hodge theory, referring to $\sqrt[273]{ }$, and on the $\partial \bar{\partial}$-lemma, referring to 102 . Moreover, we briefly describe the structure of the double complex of forms associated to a complex manifold following [281], and we use this description to clarify the relations among the above cohomologies.

### 1.3.1 Dolbeault, Bott-Chern, and Aeppli cohomologies

Fix a complex manifold $(M, J)$. The differential $d$ decomposes as

$$
d=\partial+\bar{\partial},
$$

and the equations $\partial^{2}=\bar{\partial}^{2}=0$ and $\partial \bar{\partial}+\bar{\partial} \partial=0$ are satisfied. Hence one defines the Dolbeault cohomology and the conjugate-Dolbeault cohomology of $(M, J)$ respectively as the cohomologies with respect to the operators $\bar{\partial}$ and $\partial$ :

$$
H_{\bar{\partial}}^{p, q}(M, J)=\frac{\operatorname{ker}\left\{\bar{\partial}: \mathcal{A}_{J}^{p, q}(M) \rightarrow \mathcal{A}_{J}^{p, q+1}(M)\right\}}{\operatorname{Im}\left\{\bar{\partial}: \mathcal{A}_{J}^{p, q-1}(M) \rightarrow \mathcal{A}_{J}^{p, q}(M)\right\}},
$$

and

$$
H_{\partial}^{p, q}(M, J)=\frac{\operatorname{ker}\left\{\partial: \mathcal{A}_{J}^{p, q}(M) \rightarrow \mathcal{A}_{J}^{p+1, q}(M)\right\}}{\operatorname{Im}\left\{\partial: \mathcal{A}_{J}^{p-1, q}(M) \rightarrow \mathcal{A}_{J}^{p, q}(M)\right\}}
$$

The Bott-Chern cohomology groups of $(M, J)$ are defined as (see 60])

$$
H_{B C}^{p, q}(M, J)=\frac{\operatorname{ker}\left\{\partial: \mathcal{A}_{J}^{p, q}(M) \rightarrow \mathcal{A}_{J}^{p+1, q}(M)\right\} \cap \operatorname{ker}\left\{\bar{\partial}: \mathcal{A}_{J}^{p, q}(M) \rightarrow \mathcal{A}_{J}^{p, q+1}(M)\right\}}{\operatorname{Im}\left\{\partial \circ \bar{\partial}: \mathcal{A}_{J}^{p-1, q-1}(M) \rightarrow \mathcal{A}_{J}^{p, q}(M)\right\}} .
$$

On the other hand, the Aeppli cohomology groups of $(M, J)$ are defined as (see [2])

$$
H_{A}^{p, q}(M, J)=\frac{\operatorname{ker}\left\{\partial \circ \bar{\partial}: \mathcal{A}_{J}^{p, q}(M) \rightarrow \mathcal{A}_{J}^{p+1, q+1}(M)\right\}}{\operatorname{Im}\left\{\partial: \mathcal{A}_{J}^{p-1, q}(M) \rightarrow \mathcal{A}_{J}^{p, q}(M)\right\}+\operatorname{Im}\left\{\bar{\partial}: \mathcal{A}_{J}^{p, q-1}(M) \rightarrow \mathcal{A}_{J}^{p, q}(M)\right\}} .
$$

Notice that these cohomologies are well-defined on a complex manifold because it holds $\partial \bar{\partial}+\bar{\partial} \partial=$ 0 . The wedge product induces a structure of $\mathbb{Z}^{2}$-graded algebra on $\bigoplus_{p, q} H_{B C}^{p, q}(M, J)$ and a structure of $\mathbb{Z}^{2}$-graded $\left(\oplus_{p, q} H_{B C}^{p, q}(M, J)\right)$-module on $\bigoplus_{p, q} H_{A}^{p, q}(M, J)$.

The Bott-Chern and Aeppli cohomologies provide, in a sense, a bridge between the holomorphic contents of the Dolbeault cohomology, and the topological contents of the de Rham cohomology. In fact, the identity induces natural maps of either $\mathbb{Z}^{2}$-graded or $\mathbb{Z}$-graded vector spaces.


A complex manifold $(M, J)$ satisfies the $\partial \bar{\partial}$-lemma 102 , (5.11)] if given a $d$-closed $(p, q)$-form $\alpha \in \mathcal{A}_{J}^{p, q}(M)$ the following implication holds:

$$
\alpha \text { is }(\partial+\bar{\partial}) \text {-exact } \Rightarrow \alpha \text { is } \partial \bar{\partial} \text {-exact. }
$$

In this case, for all $(p, q) \in \mathbb{Z}^{2}$, the diagonal maps in the above diagram are isomorphisms, and the vertical ones are injective, respectively surjective, see e.g. [102, (5.16)]. The $\partial \bar{\partial}$-lemma holds, for example, for compact Kähler manifolds [102, Main Theorem]. Thus, Bott-Chern and Aeppli cohomologies are expected to provide more information on the complex structure when $(M, J)$ does not admit a Kähler metric, see for example [273]. On the other hand, the Bott-Chern and Aeppli cohomologies are dual. For example, on a Hermitian manifold, the Hodge- $\star$-operator
provides a natural isomorphism between the Bott-Chern and the Aeppli cohomologies, as explained in the following. Given a Hermitian metric $g$ on a compact complex manifold $(M, J)$, define the Laplacians $\Delta_{B C}^{g}$ and $\Delta_{A}^{g}$ as the 4-th order elliptic self-adjoint differential operators given respectively by the formulas (see [273])

$$
\Delta_{B C}^{g}:=(\partial \bar{\partial})(\partial \bar{\partial})^{*}+(\partial \bar{\partial})^{*}(\partial \bar{\partial})+\left(\bar{\partial}^{*} \partial\right)\left(\bar{\partial}^{*} \partial\right)^{*}+\left(\bar{\partial}^{*} \partial\right)^{*}\left(\bar{\partial}^{*} \partial\right)+\bar{\partial}^{*} \bar{\partial}+\partial^{*} \partial
$$

and

$$
\Delta_{A}^{g}:=\partial \partial^{*}+\overline{\partial \bar{\partial}}^{*}+(\partial \bar{\partial})^{*}(\partial \bar{\partial})+(\partial \bar{\partial})(\partial \bar{\partial})^{*}+\left(\bar{\partial} \partial^{*}\right)^{*}\left(\bar{\partial} \partial^{*}\right)+\left(\bar{\partial} \partial^{*}\right)\left(\bar{\partial} \partial^{*}\right)^{*}
$$

It turns out $\left[273\right.$, Théorème 2.2] that $H_{B C}^{\bullet, \bullet}(M, J) \simeq \operatorname{ker} \Delta_{B C}^{g}$ and $H_{A}^{\bullet \bullet}(M, J) \simeq \operatorname{ker} \Delta_{A}^{g}$, (the representatives in the kernel of the Laplacians are called harmonic). As a consequence 273 , Corollaire 2.3], for every $(p, q) \in \mathbb{Z}^{2}$ the cohomology spaces $H_{B C}^{p, q}(M, J)$ and $H_{A}^{p, q}(M, J)$ are finite-dimensional. Furthermore, taking a $(p, q)$-form $\alpha \in \mathcal{A}_{J}^{p, q}(M)$, the following chain of equivalences holds:

$$
\alpha \in \operatorname{ker} \Delta_{B C}^{g} \Longleftrightarrow\left\{\begin{array}{l}
\partial \alpha=0, \\
\bar{\partial} \alpha=0, \\
\partial \bar{\partial} \star_{g} \alpha=0,
\end{array} \Longleftrightarrow \star_{g} \alpha \in \operatorname{ker} \Delta_{A}^{g}\right.
$$

Therefore, the Hodge- $\star$-operator induces an isomorphism between $H_{B C}^{p, q}(M, J)$ and $H_{A}^{n-q, n-p}(M, J)$.
By "quantitative" properties of Dolbeault, Bott-Chern and Aeppli cohomologies we mean the dimensions of the spaces $h_{\bar{\partial}}^{p, q}=\operatorname{dim} H_{\bar{\partial}}^{p, q}(M, J), h_{B C}^{p, q}=\operatorname{dim} H_{B C}^{p, q}(M, J)$ and $h_{A}^{p, q}=$ $\operatorname{dim} H_{A}^{p, q}(M, J)$ respectively. Given a complex manifold of complex dimension $n$, these are collected in Hodge diamonds in the following fashion:


### 1.3.2 Structure of the double complex of forms

A double complex $\left(\mathcal{X}, \partial_{1}, \partial_{2}\right)$ over a field $K$ is a bigraded $K$-vector space $\mathcal{X}=\bigoplus_{(p, q) \in \mathbb{Z}^{2}} \mathcal{X}^{p, q}$ with two endomorphisms $\partial_{1}$ and $\partial_{2}$ of bidegree $(0,1)$ and $(1,0)$ that satisfy the boundary condition $\partial_{1} \circ \partial_{1}=0=\partial_{2} \circ \partial_{2}$ and anticommute, i. e., $\partial_{1} \circ \partial_{2}+\partial_{2} \circ \partial_{1}=0$. Given a double complex $\left(\mathcal{X}, \partial_{1}, \partial_{2}\right)$ the Dolbeault (and conjugate-Dolbeault) cohomologies are computed as $H_{\partial_{i}}^{\bullet \bullet \bullet}=\frac{\operatorname{ker} \partial_{i}}{\operatorname{Im} \partial_{i}}$ respectively for $i=1,2$; the Bott-Chern cohomology is computed as $H_{B C}^{\bullet \bullet \bullet}=\frac{\operatorname{ker} \partial_{1} \cap \operatorname{ker} \partial_{2}}{\operatorname{Im} \partial_{1} \partial_{2}}$; and finally, the Aeppli cohomology is computed as $H_{A}^{\bullet \bullet \bullet}=\frac{\operatorname{ker} \partial_{1} \partial_{2}}{\operatorname{Im} \partial_{1}+\operatorname{Im} \partial_{2}}$. A map of double complexes $\left(\mathcal{X}, \partial_{1}, \partial_{2}\right) \xrightarrow{\varphi}\left(\mathcal{Y}, d_{1}, d_{2}\right)$ is a collection $\left\{\mathcal{X}^{p, q} \xrightarrow{\varphi^{p, q}} \mathcal{Y}^{p, q}\right\}$ of $K$-linear maps which commute with the differentials, that is $d_{i} \circ \varphi=\varphi \circ \partial_{i}$, for $i=1,2$. It is called a quasi-isomorphism with respect to a chosen cohomology if it induces an isomorphism in that cohomology. A double complex $\left(\mathcal{X}, \partial_{1}, \partial_{2}\right)$ is bounded if $\mathcal{X}^{p, q}=0$ except that for finite $(p, q) \in \mathbb{Z}^{2}$. This is the case for the double complex of differential forms associated to an almost-complex manifold $(M, J)$ which we indicate with $(\mathcal{A}, \bar{\partial}, \partial)$. Thanks to Theorem A in 281] (see also 187) any bounded double complex can be decomposed into irreducible pieces, which are

- squares of isomorphisms:

- zig-zags of length $l$, where $l \in \mathbb{N}$ counts the number of nodes:

- dots, i. e., 1-dimensional complexes, being concentrated in a single bi-degree, with all maps equal to zero (these are zig-zags of length 1 ).

Since they consist of just isomorphisms, the squares do not contribute to any cohomology. On the other hand, dots always contribute to cohomologies. With regard to the contribution of the zig-zags of length $l>1$, it depends on the chosen cohomology.

- The Dolbeault cohomology counts the endpoints of the zig-zags whenever they are "horizontal":


In the diagram, the filled dots are generators of the Dolbeault cohomology. Indeed, they are $\partial_{1}$-closed since the vertical outgoing arrows (which are not pictured) are zero. Evidently, they are not $\partial_{1}$-exact, even if they might be $\partial_{2}$-exact.

- The Bott-Chern cohomology counts the corners not having out-going arrows:


These are both $\partial_{1}$-closed and $\partial_{2}$-closed since the outgoing arrows are zero. Even if they might be $\partial_{1}$-exact and $\partial_{2}$-exact they are not $\partial_{1} \partial_{2}$-exact, that is, they are not the top-right corner of a square.

- Dually, the Aeppli cohomology counts the corners not having in-going arrows:


These are not $\partial_{1}$-closed and $\partial_{2}$-closed at the same time, but they are $\partial_{1} \partial_{2}$-closed since they are not the bottom-left corner of a square. Moreover, they have no non-zero incoming arrows, thus they are not $\left(\partial_{1}+\partial_{2}\right)$-exact.

If $M$ is compact, by Hodge theory and elliptic PDE theory, the cohomologies have finite dimensions. Then, the number of zig-zags (hence dots) in the decomposition of $(\mathcal{A}, \partial, \bar{\partial})$ is finite, while the number of squares is infinite.

Any compact complex manifold admits two natural structures, namely, a real structure, and a non-degenerate pairing structure given by a fixed Hermitian metric. These lead to symmetries of the double complex $(\mathcal{A}, \partial, \bar{\partial})$. In particular, conjugation yields symmetry around the bottom-left/top-right diagonal (red one in the picture), while duality yields symmetry around the bottom-right/top-left diagonal (blue one).


These symmetries reflect into symmetries of the Hodge diamonds. In particular, conjugation implies that $H_{B C}^{p, q}(M) \cong H_{B C}^{q, p}(M)$, as well as $H_{A}^{p, q}(M) \cong H_{A}^{q, p}(M)$.

The Dolbeault, Bott-Chern, and Aeppli cohomology's quantitative properties reflect the double complex's structure. Consequently, by looking at it, some relation among these cohomologies can be outlined, and in some particular cases, the information of one cohomology can be derived from the others (see for example Section 6.4.1 and [281, pg 29-30]). Moreover, given a map of double complexes $\left(\mathcal{X}, \partial_{1}, \partial_{2}\right) \rightarrow\left(\mathcal{Y}, d_{1}, d_{2}\right)$, if it is a quasi-isomorphism with respect to Dolbeault and conjugate-Dolbeault cohomologies, then it is an isomorphism on zig-zags, and hence an isomorphism in all cohomologies. More precisely, the following result holds.

Theorem 1.3.1 (Proposition E of 281]). Let $(M, J)$ be a complex manifold and let $(\mathcal{X}, \partial, \bar{\partial}) \hookrightarrow$ $(\mathcal{A}, \partial, \bar{\partial})$ be a sub-double complex of the double complex of differential forms of $(M, J)$. Suppose that the inclusion is a quasi-isomorphism with respect to both Dolbeault cohomology and conjugateDolbeault cohomology, then it also induces an isomorphism in Bott-Chern and Aeppli cohomology.

Furthermore, it is clear that being an isomorphism in Dolbeault cohomology or conjugateDolbeault cohomology is equivalent once the sub-double complex is compatible with the real structure.

## Chapter 2

## Symmetries of the curvature of Hermitian connections and examples

In this chapter, we introduce the notion of Kähler-like connections 153, 329, which are connections whose curvature tensors satisfy the same symmetries for curvature as the Levi-Civita and Chern connections. We focus on the canonical Gauduchon connections satisfying these symmetries referring to 18,200 . In particular, we are interested in the Kähler-like condition for the Bismut connection. Notice that symmetries are trivially satisfied when the curvature tensors are identically zero. Thus, in particular, by studying Bismut Kähler-like metrics one gets information also on the geometry of Bismut flat manifolds.

Some original results and examples are presented in this chapter. Namely, in section 2.1. we prove that, for specific values of the Gauduchon parameter $t$, if the first and second scalar curvatures of $\nabla^{t}$ are equal, the metric must be Kähler. Then, in Sections 2.2 and 2.3, we describe the constructions and the principal properties of the Hopf manifolds [171] and the Calabi-Eckmann manifolds 69, as we will use them to produce basics examples in the current and later chapters. Specifically, the 2-dimensional Hopf manifolds and the 3-dimensional CalabiEckmann manifolds are the simplest examples of non-Kähler Bismut flat manifolds [6, 180]. The Hopf surface is also the only non-Kähler second-Chern-Einstein Hermitian surface, see Chapter 4 Moreover, we construct non-Kähler examples of Bismut Kähler-like, non-Bismut flat metrics on the Hopf surfaces. Finally, the Hopf and Calabi-Eckmann manifolds will be used as toy spaces to study respectively a positivity condition for the Bismut curvature tensor in Chapter 8, and the Calabi-Yau with torsion condition in Chapter 5. Finally, we study cohomological obstructions to the existence of pluriclosed non-Kähler metrics. This allows us to classify the pluriclosed (hence also the Bismut Kähler-like and Bismut flat) metrics on the Calabi-Eckmann manifolds.

### 2.1 Kähler-like condition

The study of Hermitian manifolds whose curvature tensor, with respect to Levi-Civita or Chern connections, satisfy further symmetries, was initiated by Gray [153], and then it was recently studied also by Yang and Zheng [329. The analysis was extended by Angella, Otal, Ugarte, and Villacampa 18 to the canonical Gauduchon connections which satisfy symmetries that belong to the Kähler context, meaning that they are satisfied by the curvature tensor of Kähler metrics.

Fix a Hermitian manifold $(M, J, g)$. Given a Hermitian connection $\nabla$ on it, its curvature tensor $R^{\nabla}$ satisfies

$$
R^{\nabla} \in \mathcal{A}_{\mathbb{C}}^{2}(M) \otimes \mathcal{A}_{J}^{1,1}(M)
$$

The curvature tensor of the Chern connection $R^{C h}$ satisfies one more symmetry:

$$
R^{C h} \in \mathcal{A}^{1,1}\left(M ; \operatorname{End}\left(T^{1,0} M\right)\right) .
$$

This condition is known in the literature as complex condition. Namely, a connection $\nabla$ on $(M, J, g)$ is said to satisfy the complex condition if it is $J$-invariant both in the first two and last two entries. In formula, for $x, y, z, w \in \mathcal{C}^{\infty}(M ; T M)$,

$$
\begin{equation*}
R^{\nabla}(x, y, z, w)=R^{\nabla}(x, y, J z, J w)=R^{\nabla}(J x, J y, z, w) \tag{Cplx}
\end{equation*}
$$

For a Hermitian connection $\nabla$, (Cplx) reduces to the $J$-invariance

$$
R^{\nabla}(x, y, z, w)=R^{\nabla}(J x, J y, z, w)
$$

that is satisfied by the Chern connection since $\nabla^{C h}$ is real and $\left(\nabla^{C h}\right)^{0,1}$ equals the CauchyRiemann operator $\bar{\partial}$ which then squares to zero.

In general, a metric connection $\nabla$ with possibly non-zero torsion $T$ satisfies the first Bianchi identity with torsion (see e.g. [142, § 1.18] or [194, Ch III, Theorem 5.3]):

$$
\sum_{\sigma \in \mathfrak{G}} R^{\nabla}(\sigma x, \sigma y) \sigma z=\nabla T(x, y, z),
$$

$\mathfrak{G}$ being the group of permutations.
On a Kähler manifold, all the canonical Gauduchon connections equal the Levi-Civita connection. Thus, in particular, all of them satisfy (Cplx) and the torsionless Bianchi identity:

$$
\begin{equation*}
\sum_{\sigma \in \mathfrak{G}} R^{\nabla}(\sigma x, \sigma y) \sigma z=0 . \tag{1-Bnc}
\end{equation*}
$$

Definition 2.1.1 (Definition 4 of $[18]$ ). Let $(M, J, g)$ be a Hermitian manifold. Let $\nabla$ be a Hermitian connection on it. $\nabla$ is called Kähler-like if it satisfies both (1-Bnc) and (Cplx).

It is natural to ask when, in the non-Kähler setting, the canonical Gauduchon connections satisfy these symmetries. For example, it is known that, if a Hermitian metric on a compact complex manifold has Chern connection being Kähler-like, then it is balanced ( $\sqrt{329}$, Theorem 1.3], see also Corollary 2.1.2). Moreover, if a Hermitian metric on a compact complex manifold has Bismut connection being Kähler-like, then it is pluriclosed [335, Theorem 1]. More precisely, the following holds.

Theorem 2.1.1 (Theorem 1 of [335]). Consider a compact complex manifold ( $M, J$ ) endowed with a Hermitian metric $g$. The Bismut connection $\nabla^{B}$ associated to $(J, g)$ is Kähler-like if and only if it has parallel torsion and $g$ is pluriclosed, namely $\nabla^{B} T^{B}=0=d T^{B}$.

Notice that when a connection $\nabla$ is flat, meaning that $R^{\nabla} \equiv 0$, then trivially it is also Kähler-like. As a consequence,

Corollary 2.1.1. Consider a compact complex manifold endowed with a Hermitian metric. If the Bismut connection is flat, then the metric is pluriclosed.

We construct non-Kähler examples of Bismut Kähler-like non-Bismut flat manifolds in Section 2.2. Specifically, the Bismut curvature tensor of any homogeneous metric $g(\alpha, \beta)$ on the Hopf surface is Kähler-like but only one of these metrics is Bismut flat, see Remark 2.2.2.

In general, for Gauduchon connections it is conjectured that, as for the Chern and the Bismut connections, the symmetries of the Gauduchon curvature tensors should be related to the existence of special metrics. In particular, in [18] the authors conjectured that Hermitian structure with Kähler-like Gauduchon connection with parameter $t \neq \pm 1$ should be Kähler, as they would be both balanced and pluriclosed. As a matter of fact, it was expected that the Gauduchon connections, different from Chern and Bismut, should behave like both the Chern and the Bismut connections. Very recently, a solution to this conjecture was proposed in 200.

Theorem 2.1.2 (Theorem 3.1 of [200]). Consider a compact complex manifold endowed with a Hermitian metric, and a canonical connection in the Gauduchon family $\nabla^{t}$, different from the Bismut and the Chern connection. If it is Kähler-like, then the metric is Kähler.

For a particular set of Gauduchon parameters, namely $t \in(-\infty,-3-2 \sqrt{3}) \cup(-3+2 \sqrt{3},+\infty)$ but $t \neq 1$, the statement of Theorem 2.1 .2 can be proved even relaxing the Kähler-like hypothesis. Indeed, it is enough to ask for the two scalar curvatures $s^{t}$ and $s_{2}^{t}$ to be equal. We prove it here (for the Gauduchon flat case see [328, Theorem 1.6] while for the almost-Hermitian case see [128, Theorem 5.5]).

Proposition 2.1.1. Let $(M, J, g)$ be a Hermitian manifold. If $s^{t}=s_{2}^{t}$ for $t \geq-3+2 \sqrt{3}$ or $t \leq-3-2 \sqrt{3}$ and $t \neq 1$, then $g$ is Kähler.

Proof. We fix a point $p \in M$ and compute $s^{t}$ and $s_{2}^{t}$ in special holomorphic coordinates around $p$ tracing (1.5). Thus, first of all, we rewrite (1.5) as

$$
\begin{aligned}
R_{i \bar{j} k \bar{l}}^{t}=\frac{1-t}{2}\left(\partial_{i} \bar{\partial}_{j} g_{k \bar{l}}-\partial_{i} \bar{\partial}_{l} g_{k \bar{j}}-\partial_{k} \bar{\partial}_{j} g_{i \bar{l}}\right)-\frac{1+t}{2} \partial_{i} \bar{\partial}_{j} g_{k \bar{l}} & \\
& +\frac{(1-t)^{2}}{4} \sum_{q} T_{i q}^{l} \overline{T_{q j}^{k}}+\frac{t^{2}}{4} \sum_{q} T_{i k}^{q} \overline{T_{j l}^{q}}
\end{aligned}
$$

Here $T=T^{C h}$ represents the torsion of the Chern connection $\nabla^{C h}$, which is of type (2,0), and in local coordinates it reads as

$$
T_{i j}^{k}=g^{k \bar{s}}\left(\partial_{i} g_{j \bar{s}}-\partial_{j} g_{i \bar{s}}\right) .
$$

Then, we take the traces $s^{t}=g^{k \bar{l}} g^{i \bar{j}} R_{i \bar{j} k \bar{l}}^{t}$ and $s_{2}^{t}=g^{i \bar{l}} g^{k \bar{j}} R_{i \bar{k} k \bar{l}}^{t}$ obtaining respectively

$$
\begin{aligned}
& s^{t}=\frac{t-1}{2} \operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega+\overline{\partial \partial}_{g}^{*} \omega\right)+\operatorname{tr}_{\omega} \partial \partial^{*} \omega-\frac{1}{4}\|T\|_{g}^{2}-\sum_{i j} \partial_{i} \bar{\partial}_{j} g_{j \bar{i}}, \\
& s_{2}^{t}=\frac{1-t}{2} \operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega+\bar{\partial}_{g}^{*} \omega\right)-\frac{1}{4}(t-1)^{2}\left\|T_{s .}^{s}\right\|_{g}^{2}-\frac{1}{4}\left(t^{2}-2 t+2\right)\|T\|_{g}^{2}-\sum_{i j} \partial_{i} \bar{\partial}_{j} g_{j \bar{i}},
\end{aligned}
$$

where

$$
\|T\|_{g}^{2}=\sum_{i j k} T_{i j}^{k} \overline{T_{i j}^{k}} \quad \text { and } \quad\left\|T_{s .}^{s}\right\|_{g}^{2}=\sum_{q}\left(\sum_{s} T_{s q}^{s}\right)\left(\sum_{s} \bar{T}_{s q}^{s}\right) .
$$

Taking the difference we get

$$
(t-1) \operatorname{tr}_{\omega}\left(\partial \partial^{*} \omega+{\left.\overline{\partial \partial_{g}}{ }_{g}^{*} \omega\right)+\operatorname{tr}_{\omega} \partial \partial^{*} \omega=-\frac{1}{4}(t-1)^{2}\left(\|T\|_{g}^{2}+\left\|T_{s \cdot}^{s}\right\|_{g}^{2}\right), \quad, \quad \text {, }}\right.
$$

which we can integrate on $M$ since it is independent by the choice of coordinates, obtaining

$$
\begin{equation*}
(2 t-1)\left|\partial^{*} \omega\right|_{g}^{2}=-\frac{1}{4}(t-1)^{2} \int_{M}\|T\|_{g}^{2}+\left\|T_{s \cdot}^{s}\right\|_{g}^{2} d \mu_{g} . \tag{2.1}
\end{equation*}
$$

Notice that $\int_{M}\left\|T_{s}^{s} .\right\|_{g}^{2} d \mu_{g}=\left|\partial^{*} \omega\right|_{g}^{2}$, hence we get

$$
\begin{equation*}
\left(t^{2}+6 t-3\right)\left|\partial^{*} \omega\right|_{g}^{2}=-(t-1)^{2} \int_{M}\|T\|_{g}^{2} d \mu_{g} \leq 0 . \tag{2.2}
\end{equation*}
$$

If $t^{2}+6 t-3 \geq 0$ then $\|T\|_{g}^{2}$ must vanish, proving the theorem.
Notice that, as far as we know, there is no evidence in general that the Kähler-like condition is strictly stronger than having equal first and second scalar curvature. For example, for the Bismut connection of homogeneous metrics on the Hopf manifolds they are equivalent, see Remark 2.2.1. Furthermore, for the Gauduchon connections with parameter $t \neq \pm 1$, we expect that having equal scalar curvatures, $s^{t}=s_{2}^{t}$, is as strong as the Gauduchon Kähler-like condition. Indeed,
we believe that the statement of Proposition 2.1.1 should hold for any Gauduchon parameter $t \neq \pm 1$.

Specializing equation (2.1) (or equivalently (2.2) in the above proof to $t=1$ we obtain the following result about the Chern curvatures which generalizes the statement of Proposition 1.9 in 328.

Corollary 2.1.2. Let $(M, J, g)$ be a Hermitian manifold. If $s^{C h}=s_{2}^{C h}$, then $g$ is balanced. Furthermore, a balanced Hermitian manifold $(M, J, g)$, with $s^{t}=s_{2}^{t}$ for $t \neq 0$ is Kähler.

### 2.2 Hopf manifolds

A Hopf manifold 171 is a compact complex manifold obtained as quotient of $\mathbb{C}^{n} \backslash\{0\}$ by a free action of the cyclic group $\mathbb{Z}\langle\gamma\rangle$ generated by a holomorphic contraction. A holomorphic contraction $\gamma$ is a holomorphic endomorphism of $\mathbb{C}^{n} \backslash\{0\}$ such that a sufficiently big iteration $\gamma^{N}$ maps any given compact subset of $\mathbb{C}^{n} \backslash\{0\}$ into an arbitrarily small neighborhood of 0 . Consequently, Hopf manifolds are all diffeomorphic to $\mathbb{S}^{2 n-1} \times \mathbb{S}^{1}$. It follows that $b_{1}=1$ and $b_{2}=0$ if $n \geq 2$, and hence they are non-Kähler manifolds 167,205 unless $n=1$, in which case one gets the complex tori. However, they always carry LCK metrics $41,144,182,243,244,312$.

Historically, the Hopf manifolds were the first known varieties that are not embedded submanifolds of complex projective spaces. As a matter of fact, they represent the first examples of non-Kähler manifolds known in complex geometry. Hopf manifolds and their generalizations have been studied by Calabi, Eckmann, López de Medrano, Meersseman, Nicolau, Verjovsky, Bosio, and many other authors; see for example [59, 69, 160, 218, 219, 227, 228] and the references therein. Recently, they have been used in [11] as a class of examples where to analyze the holonomy of the Bismut connection.

### 2.2.1 Diagonal Hopf manifolds

A linear Hopf manifold is a Hopf manifold whose contraction $\gamma$ is a linear operator $\gamma \in G L(n ; \mathbb{C})$. It is called diagonal Hopf manifold if $\gamma \in G L(n ; \mathbb{C})$ is diagonal. As a consequence, all the eigenvalues satisfy $\left|a_{i}\right|<1$.

Fix a diagonal Hopf manifold $M=\left(\mathbb{C}^{n} \backslash\{0\}\right) / \mathbb{Z}$. It has the structure of homogeneous space given by

$$
M \cong \frac{\mathrm{U}(1) \times \mathrm{SU}(n)}{\mathrm{SU}(n-1)}
$$

where $\mathrm{SU}(n-1)$ is acting trivially on the first component. Then the bi-invariant metric given by the Killing form on $\mathrm{U}(1) \times \mathrm{SU}(n)$ restricts to a metric $g_{H}$ on $M$. It can be proved that $g_{H}$ is a Hermitian metric with respect to the natural complex structure on $M$ inherited by $\mathbb{C}^{n}$, and that it is defined in local coordinates $\left\{z_{i}\right\}_{i=1, \ldots, n}$ as

$$
g_{H}:=\frac{\delta_{i j}}{|z|^{2}} d z^{i} \otimes d \bar{z}^{j}
$$

The metric $g_{H}$ belongs to a class of Hermitian metrics on $M$, given in coordinates by

$$
\begin{equation*}
g(\alpha, \beta)_{i \bar{j}}:=\alpha \frac{\delta_{i j}}{|z|^{2}}+\beta \frac{\bar{z}_{i} z_{j}}{|z|^{4}} \tag{2.3}
\end{equation*}
$$

where $\alpha, \beta$ are real parameters such that $\alpha>0$ and $\beta>-\alpha$. By a straightforward computation one sees that the metric $g_{H}$ on the Hopf surfaces is the only pluriclosed metric among them. We shall also remark that this family of metrics naturally arises in studying the evolution of flows of metrics on diagonal Hopf manifolds, for example, for the evolution of $g_{H}$ by the Chern-Ricci flow see $[298]$. Moreover, the $g(\alpha, \beta)$ metrics appeared in [217] where the authors used them to produce examples of Levi-Civita Ricci flat Hermitian metrics on Hopf manifolds.

Proposition 2.2.1. Given an $n$-dimensional diagonal Hopf manifold $M$, the $g(\alpha, \beta)$ metrics are all the $\mathrm{U}(1) \times \mathrm{SU}(n)$-invariant metrics on $M$; hence, in particular, they are homogeneous. Moreover, if $n \geq 3$ they are all the homogeneous metrics on $M$.

Proof. On a generic Hermitian metric on $M$,

$$
g=g_{i \bar{j}}(z) d z^{i} \otimes d \bar{z}^{j}
$$

the $\mathrm{SU}(n)$-invariant condition is

$$
\bar{U}\left(g_{i \bar{j}}(z)\right) U^{t}=\left(g_{i \bar{j}}(U z)\right),
$$

for any $U \in \operatorname{SU}(n)$ and $z \in M$. Notice that it is satisfied by the metrics $g(\alpha, \beta)$. Moreover, these metrics are also $\mathrm{U}(1)$-invariant, hence homogeneous on $M$.

Now suppose that $n>2$ and $g$ is a Hermitian $\mathrm{U}(1) \times \mathrm{SU}(n)$-invariant metric on $M$. Fixed a point $z \in M$ its isotropy group is $(\mathrm{U}(1) \times \mathrm{SU}(n))_{z} \cong \mathrm{SU}(n-1)$, which is not trivial since $n \geq 3$. The $\mathrm{U}(1) \times \mathrm{SU}(n)$-invariance of $g$ translates in

$$
\bar{U}\left(g_{i \bar{j}}(z)\right) U^{t}=\left(g_{i \bar{j}}(z)\right),
$$

for any $U \in(\mathrm{U}(1) \times \mathrm{SU}(n))_{z}$. Moreover, we can take $e_{1}$ as point $z$; hence we get that the matrix

$$
\left(\begin{array}{ll}
1 & \\
& U
\end{array}\right)\left(g_{i \bar{j}}\left(e_{1}\right)\right)\left(\begin{array}{ll}
1 & \\
& \bar{U}^{t}
\end{array}\right)
$$

must be independent on $U \in \operatorname{SU}(n-1)$. Consequently, $g_{i \bar{j}}\left(e_{1}\right)$ is forced to be of the form

$$
g_{i \bar{j}}\left(e_{1}\right)=\left(\begin{array}{ll}
a & \\
& \lambda I d
\end{array}\right),
$$

where $a$ and $\lambda$ are positive real numbers. This means that $g$ agrees with $g(\lambda, a-\lambda)$ in $e_{1}$. Finally, the $\mathrm{U}(1) \times \mathrm{SU}(n)$-invariance ensures that they agree all over $M$.

The inverse of $g(\alpha, \beta)$ is

$$
g(\alpha, \beta)^{i \bar{j}}=\frac{|z|^{2}}{\alpha}\left(\delta^{i j}-\frac{\beta}{\alpha+\beta} \frac{\bar{z}^{j} z^{i}}{|z|^{2}}\right) .
$$

Then the Christoffel symbols for the Bismut connection are

$$
\begin{aligned}
& \left(\Gamma^{-1}\right)_{i j}^{k}=g^{k \bar{s}} \partial_{j} g_{i \bar{s}}=\frac{1}{|z|^{2}}\left(\frac{\beta}{\alpha} \delta_{j}^{k} \bar{z}_{i}-\delta_{i}^{k} \bar{z}_{j}\right)-\frac{\beta}{\alpha} \bar{z}_{i} \bar{z}_{j} z^{k} \\
& |z|^{4} \\
& \left(\Gamma^{-1}\right)_{\bar{i} j}^{k}=g^{k \bar{s}}\left(\bar{\partial}_{i} g_{j \bar{s}}-\bar{\partial}_{s} g_{j \bar{i}}\right)=\frac{1}{|z|^{2}}\left(\delta_{i j} z^{k}-\frac{\alpha+\beta}{\alpha} \delta_{j}^{k} z_{i}\right)+\frac{\beta}{\alpha} \frac{\bar{z}_{z} \bar{z}_{j} z^{k}}{|z|^{4}} .
\end{aligned}
$$

A direct computation shows that the Bismut curvature tensor associated to a metric $g(\alpha, \beta)$ satisfies various symmetries, including (Cplx). Indeed, for $z \in M$, its non-vanishing coefficients are

$$
\begin{align*}
R_{i \bar{j} k l}^{B}(g(\alpha, \beta))_{\mid z}= & \underbrace{U_{i \bar{j} k \bar{l}}^{\alpha}(z)}_{U^{\left[\frac{\delta_{i l} \delta_{j k}-\delta_{i j} \delta_{k l}}{|z|^{4}}+\frac{\delta_{i j} \bar{z}_{k} z_{l}+\delta_{k l} \bar{z}_{i} z_{j}-\delta_{i l} z_{j} \bar{z}_{k}-\delta_{j k} \bar{z}_{i} z_{l}}{|z|^{6}}\right]}}  \tag{2.4}\\
& +2 \beta \underbrace{\left[\frac{-\delta_{i j} \delta_{k l}}{|z|^{4}}+\frac{\delta_{i j} \bar{z}_{k} z_{l}+\delta_{k l} \bar{z}_{i} z_{j}}{|z|^{6}}+\frac{\left.-\bar{z}_{i} z_{j} \bar{z}_{k} z_{l}\right]}{|z|^{8}}\right]}_{U_{i \bar{j} k l}^{\beta}(z)} .
\end{align*}
$$

In particular, the term $U^{\alpha}$ corresponds to the Bismut curvature tensor of $g_{H}$ which is (see also [216):

$$
\begin{equation*}
R_{i \bar{i} k \bar{l}}^{B}\left(g_{H}\right)_{\left.\right|_{z}}=\frac{\delta_{i l} \delta_{j k}-\delta_{i j} \delta_{k l}}{|z|^{4}}+\frac{\delta_{i j} \bar{z}_{k} z_{l}+\delta_{k l} \bar{z}_{i} z_{j}-\delta_{i l} z_{j} \bar{z}_{k}-\delta_{j k} \bar{z}_{i} z_{l}}{|z|^{6}} . \tag{2.5}
\end{equation*}
$$

Proposition 2.2.2. Let $M$ be an n-dimensional diagonal Hopf manifold, and consider the metric $g(\alpha, \beta)$. Then the Bismut curvature tensor satisfies, for any $\xi, \eta, \mu, \nu \in T_{z} M, z \in M$

$$
R_{z}^{B}(\xi, \eta, \mu, \nu)=R_{z}^{B}(\mu, \nu, \xi, \eta)
$$

However, it is Kähler-like if and only if $n=2$.
Proof. The first statement follows directly by computing the coefficients of $R^{B}$ in local coordinates (2.4). Then, since the Bismut curvature tensor of $g(\alpha, \beta)$ satisfies (Cplx), to verify the Kähler-like condition it remains to check 1 -Bnc). This is equivalent to

$$
R_{z}^{B}(\xi, \bar{\eta}, \nu, \bar{\mu})-R_{z}^{B}(\nu, \bar{\eta}, \xi, \bar{\mu})=0
$$

for any $\xi, \eta, \mu, \nu \in T_{z}^{1,0} M, z \in M$. Thus we compute it in coordinates. Using (2.4,

$$
\begin{equation*}
R_{i, \bar{j}, k, \bar{l}}^{B}-R_{k, \bar{j}, i, \bar{l}}^{B}=2(\alpha+\beta) U_{i \bar{j} k \bar{l}}^{\alpha} \tag{2.6}
\end{equation*}
$$

This vanishes if and only if $U^{\alpha}=0$ since $\alpha+\beta$ is always positive. Finally, $U^{\alpha}$ vanishes identically if and only if $n=2$.

Remark 2.2.1. Taking the traces with $g(\alpha, \beta)^{i \bar{j}}$ and $g(\alpha, \beta)^{k \bar{l}}$ in (2.6), we obtain

$$
s_{1}^{B}(g(\alpha, \beta))-s_{2}^{B}(g(\alpha, \beta))=2 \frac{\alpha+\beta}{\alpha^{2}}(n-1)(2-n)
$$

Therefore, the two scalar curvatures of a homogeneous metric on an n-dimensional Hopf manifold are equal if and only if $n=2$. However, this is precisely the case when the Bismut connection of the homogeneous metrics is Kähler-like.

Tracing out $(2.4)$ with $g(\alpha, \beta)^{k \bar{l}}$ we compute the Bismut-Ricci curvature of the homogeneous metrics $g(\alpha, \beta)$ as

$$
\begin{equation*}
\operatorname{Ric}^{B}(g(\alpha, \beta))_{i \bar{j}}=\left(2-n+2 \frac{\beta}{\alpha}(1-n)\right)\left(\frac{\delta_{i j}}{|z|^{2}}-\frac{\bar{z}_{i} z_{j}}{|z|^{4}}\right) \tag{2.7}
\end{equation*}
$$

We observe that the only way $\operatorname{Ric}^{B}(g(\alpha, \beta))$ can be a multiple of $g(\alpha, \beta)$ is when it vanishes. In particular, this happens precisely when the ratio $\frac{\beta}{\alpha}=-\frac{1}{2} \frac{n-2}{n-1}$, which is an admissible value.

### 2.2.2 Hopf surfaces

The Hopf manifolds of complex dimension 2 are called Hopf surfaces. In these cases, there is a normal form for the contraction $\gamma$ [197, Theorem 1], which in appropriate coordinates can be written as

$$
\gamma(x, y)=\left(a x+\lambda y^{n}, b y\right)
$$

where $a, b \in \mathbb{C}$ are such that $0<|a|<1,0<|b|<1$ and either $\lambda=0$ or $a=b^{n}$. The image of the $x$-axis gives an elliptic curve on these surfaces, and there is the following characterization due to Kodaira.

Theorem (Theorem 34 in [198]). Any complex surface with $b_{1}=1, b_{2}=0$, and no non-constant meromorphic functions is a Hopf surface if it contains a curve.

Moreover, when $\lambda$ vanishes they become diagonal Hopf surfaces. In this case, the image of the $y$-axis gives a second elliptic curve on them, and they were characterized by Kato as follows, see e.g. [236, (5.2)].

Theorem. Any minimal non-Kähler surface $S$ with Kodaira dimension $-\infty$ and no non-constant meromorphic functions is a Hopf surface if it has exactly two elliptic curves. If moreover $H^{1}(S, \mathbb{Z}) \cong \mathbb{Z}$, then $S$ is a diagonal Hopf surface.

The diagonal Hopf surfaces are the first and easiest examples of Bismut flat manifolds. Indeed, for $z \in M$ one can identify $T_{z}^{1,0} M \cong \mathbb{C}^{n}$; then taking $\xi, \eta, \nu, \mu \in T_{z}^{1,0} M$, 2.5 reads as

$$
\begin{aligned}
& R_{z}^{B}(\xi, \bar{\eta}, \nu, \bar{\mu})=\frac{1}{|z|^{6}}\left\{(\xi \cdot \mu)(\nu \cdot \eta)|z|^{2}-(\xi \cdot \eta)(\nu \cdot \mu)|z|^{2}+(\xi \cdot \eta)(\nu \cdot z)(z \cdot \mu)\right. \\
&+(\nu \cdot \mu)(\xi \cdot z)(z \cdot \eta)-(\xi \cdot \mu)(z \cdot \eta)(\nu \cdot z)-(\nu \cdot \eta)(\xi \cdot z)(z \cdot \mu)\}
\end{aligned}
$$

where $(\cdot)$ is the Hermitian product in $\mathbb{C}^{n}$. It can be checked that this identically vanishes when $\xi, \eta, \nu, \mu \in \mathbb{C}^{2}$, while it does not for $n \geq 3$. For the latter, consider three orthogonal vectors $\xi, \nu, z \in \mathbb{C}^{3}$; then $R_{z}^{B}(\xi, \bar{\xi}, \nu, \bar{\nu})=|\xi|^{2}|\nu|^{2}|z|^{2}$. Thus the standard metric $g_{H}$ on the Hopf surface is Bismut flat (see also [6, 180]). Moreover, up to scaling it is the only one among the $g(\alpha, \beta)$ 's with this property. Indeed, evaluating $U^{\beta}$ in $z \in M$ on vectors $\xi, \nu \in T^{1,0} M \cong \mathbb{C}^{n}$, we get

$$
\begin{equation*}
U^{\beta}(\xi, \bar{\xi}, \nu, \bar{\nu})=\frac{1}{|z|^{8}}\left(|\xi|^{2}|z|^{2}-|\xi \cdot z|^{2}\right)\left(|\nu \cdot z|^{2}-|\nu|^{2}|z|^{2}\right) \leq 0 \tag{2.8}
\end{equation*}
$$

and the equality holds if and only if $\xi=\lambda z$ or $\nu=\lambda z$ for $\lambda \in \mathbb{C}$.
Remark 2.2.2. The Hermitian metrics $g(\alpha, \beta)$ on a Hopf surface are all Bismut Kähler-like thanks to Proposition 2.2.2. However, $g_{H}$ is the only one which is also Bismut flat.

We finally recall that the Dolbeault cohomology of these manifolds is known since 163 , Appendix II, Theorem 9.5], while the Bott-Chern cohomologies were computed in [21, Theorem 3.3], where harmonic representatives were also given. In particular, the Dolbeault and Bott-Chern diamonds respectively are


### 2.3 Calabi-Eckmann manifolds

A Calabi-Eckmann manifold [69] is a compact complex manifold constructed as quotient of $\mathbb{C}^{n+1} \backslash\{0\} \times \mathbb{C}^{m+1} \backslash\{0\}$ with $n, m \geq 0$, by the free, proper and holomorphic $\mathbb{C}$-action

$$
\xi \in \mathbb{C}, \quad \mathbb{C}^{n+1} \backslash\{0\} \times \mathbb{C}^{m+1} \backslash\{0\} \ni(x, y) \mapsto\left(e^{\xi} x, e^{\lambda \xi} y\right)
$$

for some fixed $\lambda \in \mathbb{C} \backslash \mathbb{R}$. We indicate the quotient as

$$
M_{n, m}:=\frac{\mathbb{C}^{n+1} \backslash\{0\} \times \mathbb{C}^{m+1} \backslash\{0\}}{\sim_{\mathbb{C}}}
$$

This construction generalizes that of Hopf manifolds, obtained for $n m=0$. It is easy to check that $M_{n, m}$ is diffeomorphic to the product of two odd-dimensional spheres $\mathbb{S}^{2 n+1} \times \mathbb{S}^{2 m+1}$. Therefore, for the Calabi-Eckmann manifolds $b_{2}=0$ if $n m>0$, and hence they are not Kähler unless $n=m=0$, in which case one gets complex tori.

The Calabi-Eckmann manifold $M_{n, m}$ has the structure of a principal toric bundle 69 , Theorem II] given by taking the product of two Hopf fibrations. Namely,

$$
\mathbb{S}^{1} \times \mathbb{S}^{1} \longrightarrow \mathbb{S}^{2 n+1} \times \mathbb{S}^{2 m+1} \xrightarrow{\pi_{1} \times \pi_{2}} \mathbb{C P}^{n} \times \mathbb{C} \mathbb{P}^{m}
$$

Then the standard complex structure on $M_{n, m}$ inherited by that of $\mathbb{C}^{n+1} \backslash\{0\} \times \mathbb{C}^{m+1} \backslash\{0\}$ is the same as the one coming from the fibration structure. In detail, consider $Z_{1}, Z_{2}$ the canonical vector fields associated to the two factors of the fibration, with $\mu_{i}$ the associated canonical
connections satisfying $d \mu_{i}=\pi_{i}^{*} \omega_{F S}, \omega_{F S}$ being the Fubini-Study metrics on the projective spaces. Then $J$ on

$$
T M_{n, m}=H_{\mathbb{C P}^{n}} \oplus\left\langle Z_{1}, Z_{2}\right\rangle \oplus H_{\mathbb{C P}^{m}}
$$

is $J=J_{\mathbb{C P}^{n}} \oplus I \oplus J_{\mathbb{C P}^{m}}$, where $J_{\mathbb{C P}^{n}}$ and $J_{\mathbb{C P}^{m}}$ are the complex structures of $\mathbb{C P}^{n}$ and $\mathbb{C P}^{m}$ respectively pulled-back on the horizontal spaces $H_{\mathbb{C P}}$ and $H_{\mathbb{C P}}$, and $I\left(Z_{1}\right)=Z_{2}$. Moreover, the Calabi-Eckmann complex structure on $S^{2 n+1} \times \mathbb{S}^{1}$ corresponds to the Hopf complex structure.

Exploiting the structure of toric fibration, the Dolbeault cohomology of the Calabi-Eckmann manifolds was computed in [163, Appendix II, Theorem 9.5]. In particular, a model for the Dolbeault cohomology was given. It is, for $n \leq m$,

$$
\begin{equation*}
H_{\bar{\partial}}^{\bullet \bullet}\left(M_{n, m}, J\right) \cong \mathbb{C}\left[y_{1,1}\right] /\left(\left(y_{1,1}\right)^{n+1}\right) \otimes \wedge^{\bullet \bullet}\left(\mathbb{C}\left\langle\left[u_{m+1, m}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle\right) \tag{2.9}
\end{equation*}
$$

where subscripts denote the bi-degree of the generators $x, y, u$. Using this model we can prove that the only Calabi-Eckmann manifolds that can be equipped with a pluriclosed metric occur in low dimensions. First of all, we need the following lemma.

Lemma 2.3.1. Let $(M, J)$ be a complex manifold. If

$$
H_{\bar{\partial}}^{2,1}(M, J)=H_{\bar{\partial}}^{0,2}(M, J)=H_{\bar{\partial}}^{3,0}(M, J)=0
$$

then any $\partial \bar{\partial}$-closed not d-closed $(1,1)$-form is $(\partial+\bar{\partial})$-exact.
Proof. Consider a $(1,1)$-form $\omega$ that is $\partial \bar{\partial}$-closed but not $d$-closed. Then at least one of the following holds: $\partial \omega \neq 0$ or $\bar{\partial} \omega \neq 0$. Let us suppose that $\partial \omega \neq 0$ since the other case is symmetric. Hence we have a piece of zig-zag in the double complex of $(M, J)$ as in Figure (a), with possibly $\omega=\bar{\omega}$.

(a)

(b)

Since $H_{\bar{\partial}}^{2,1}(M, J)=0$ and $\partial \bar{\partial} \omega=0$, there must be primitives $\eta_{2,0}, \overline{\eta_{2,0}}$ in bi-degrees $(2,0)$ and $(0,2)$ (by conjugation) as in Figure (b).

Moreover, $H_{\bar{\partial}}^{0,2}(M, J)=0$ implies that either there exists a $\eta_{0,3} \neq 0$ such that $\bar{\partial} \overline{\eta_{2,0}}=\eta_{0,3}$ or there exists $\eta_{0,1}$ such that $\bar{\partial} \eta_{0,1}=\overline{\eta_{2,0}}$. Suppose the former holds, we are in the situation of Figure (c). But then since $H_{\bar{\partial}}^{0,3}(M, J)=0$ there must be $\bar{\partial} \overline{\eta_{0,3}} \neq 0$, which is impossible because it would imply $\partial \partial \omega \neq 0$. Thus there exists $\eta_{0,1}$ such that $\bar{\partial} \eta_{0,1}=\overline{\eta_{2,0}}$. Hence we have two conjugate squares as in Figure (d).

This leads to a cohomological obstruction to the existence of non-Kähler pluriclosed metrics; while in [95] cohomological obstructions to the existence of astheno-Kähler metrics have also been shown.


Corollary 2.3.1. Let $(M, J)$ be a complex manifold. If

$$
H_{\bar{\partial}}^{2,1}(M, J)=H_{\bar{\partial}}^{0,2}(M, J)=H_{\bar{\partial}}^{3,0}(M, J)=0
$$

then any pluriclosed metric is either Kähler or $(\partial+\bar{\partial})$-exact.
We then obtain the following result. It was obtained with a similar argument involving symplectic structures in 85.

Theorem 2.3.1 (Theorem 5.16 and Example 5.17 in [85]). The only Calabi-Eckmann manifolds that can be equipped with a pluriclosed metric are $M_{0,0}, M_{0,1}$ and $M_{1,1}$.

Proof. We provide a different proof than [85]. First of all, the manifold $M_{0,0}$ is isomorphic to a torus so it has a Kähler (hence pluriclosed) metric. Now consider a Calabi-Eckmann manifold $M_{n, m}$ with $n \leq m$. From the Dolbeault cohomology model (2.9) we can deduce that

- $H_{\bar{\partial}}^{0,2}\left(M_{n, m}, J\right)=H_{\bar{\partial}}^{3,0}\left(M_{n, m}, J\right)=0$ for any $n, m \in \mathbb{N}$, while
- $H_{\bar{\partial}}^{2,1}\left(M_{n, m}, J\right) \neq 0$ if and only if $m=1$, which correspond to two possible cases, namely $M_{0,1}$ and $M_{1,1}$.

The manifold $M_{0,1}$ is the Hopf surface whose standard metric $g_{H}$ is Bismut flat (see Section 2.2.2 and hence pluriclosed by Corollary 2.1.1. Moreover, also the standard metric on $M_{1,1}$ is Bismut flat (see the next section).

Now suppose to have a pluriclosed metric $\omega$ on a Calabi-Eckmann manifold $M_{n, m}$ with $m \geq 2$. Since $M_{n, m}$ is not Kähler, thanks to Corollary 2.3.1, $\omega$ must be $(\partial+\bar{\partial})$-exact. However, the fibers $\mathbb{S}^{1} \times \mathbb{S}^{1}$ are compact complex submanifolds in $M_{n, m}$. Moreover, since $\omega$ is a metric, integrating it over the fibers can not be zero. Thus $[\omega] \neq 0$ in $H_{A}^{1,1}(M, J)$, a contradiction.

This result answers negatively to a question raised in [133, Question 8.36] about the existence of pluriclosed metrics (with $R i c^{B} \equiv 0$ ) on the Calabi-Eckmann manifolds which are not $M_{0,0}, M_{0,1}$ or $M_{1,1}$. For more details see the discussion on Bismut Hermitian-Einstein metrics in Section 7.2.

### 2.3.1 Calabi-Eckmann threefold

The Calabi-Eckmann manifold $M_{1,1}$ is called Calabi-Eckmann threefold. All the Calabi-Eckmann manifolds trivially have homogeneous structures being the product of spheres. However, $M_{1,1}$ (together with the torus and the Hopf surface) are the only ones in the family that have a Lie group structure. This is not a coincidence but relies on the existence of Bismut flat metrics (refer to Section 6). Indeed, thanks to Corollary 2.1.1 Bismut flat metrics are pluriclosed, and hence their existence is obstructed by Theorem 2.3.1. The Lie group structure on the Calabi-Eckmann threefold is given by

$$
M_{1,1} \cong \mathrm{SU}(2) \times \mathrm{SU}(2),
$$

and the bi-invariant metric coming from the Killing form is Hermitian with respect to the standard complex structure and Bismut flat.

The Bott-Chern cohomologies of the Calabi-Eckmann manifolds $M_{n, m}$ were described (when $m \neq n$ ) in [281, Theorem H], while the Bott-Chern cohomology of the Calabi-Eckmann threefold $M_{1,1}$ was studied in [21, Section 3.3] through left-invariant forms. In particular, the Dolbeault and Bott-Chern diamonds of $M_{1,1}$ respectively are


## Chapter 3

## The scalar curvature of Gauduchon connections

In this chapter, we analyze metrics of constant Gauduchon scalar curvature. In particular, we introduce and study an analog of the Yamabe problem for Gauduchon connections.

We firstly define the conformal invariant $\Gamma_{M}^{t}(\{\omega\})$ associated to a conformal class $\{\omega\}$ on an almost-complex manifold ( $M, J$ ), in Section 3.1. This coincides with the Gauduchon degree when $t=1$, and, in general, it is related with the expected constant scalar curvature. Then, in Section 3.2, we introduce and motivate the Gauduchon-Yamabe problem, and, in Section 3.3, we provide a positive answer depending on whether $\Gamma_{M}^{t}(\{\omega\})$ is non-positive or not. Namely, we prove that once fixed an almost-Hermitian conformal structure $\{\omega\}$ on a compact almost-complex manifold $(M, J)$, there exists a metric with constant $\nabla^{t}$-scalar curvature in that class when $\Gamma_{M}^{t}(\{\omega\}) \leq 0$ or $\Gamma_{M}^{t}(\{\omega\}) \geq 0$ depending on $t>\frac{1}{1-n}$ or $t<\frac{1}{1-n}$, for $2 n=\operatorname{dim}_{\mathbb{R}} M$.

The original results of this chapter have been obtained in [34 in the Hermitian case. We provide here generalizations to the almost-Hermitian setting.

### 3.1 Gauduchon degree and conformal changes

We introduce here the notion of Gauduchon degree and its generalizations which are going to be crucial for the solution of the Gauduchon Yamabe problem. With the same intent, we also summarize the formulas of conformal change for the scalar curvature functions.

## Conformal change of scalar curvatures

Let $(M, J, g)$ be an almost-Hermitian manifold. Under a conformal change $\tilde{g}=e^{2 f} g$ for $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$, the conformal variation of the second Chern-Ricci form of $(g, J)$ is given by (207, Corollary 4.4], in the Hermitian case see [139])

$$
\begin{equation*}
\operatorname{Ric}^{C h, 2}(\widetilde{\omega})=\operatorname{Ric}^{C h, 2}(\omega)+\left(\Delta_{g} f+g(\theta, d f)\right) \omega=\operatorname{Ric}^{C h, 2}(\omega)+\left(\Delta_{\omega}^{C h} f\right) \omega \tag{3.1}
\end{equation*}
$$

where the second equality comes from Lemma 1.1.1. As a consequence, the conformal variation of the Chern scalar curvature is

$$
s^{C h}(\widetilde{\omega})=e^{-2 f}\left(s^{C h}(\omega)+n \Delta_{\omega}^{C h} f\right) .
$$

Moreover, using 1.7) and 207, Corollary 4.4], the conformal variation of the $\nabla^{t}$-Ricci curvature is

$$
\begin{equation*}
\operatorname{Ric}^{t}(\widetilde{\omega})=\operatorname{Ric}^{t}(\omega)-(1+n t-t) d d^{c} f, \tag{3.2}
\end{equation*}
$$

and consequently, the conformal variation of the $\nabla^{t}$-scalar curvature is given by

$$
\begin{equation*}
s^{t}(\widetilde{\omega})=e^{-2 f}\left(s^{t}(\omega)+(1+n t-t) \Delta_{\omega}^{C h} f\right) . \tag{3.3}
\end{equation*}
$$

## Gauduchon degree

Given a Hermitian manifold $(M, J, g)$, the Gauduchon degree 14 (also known as fundamental constant), $\Gamma_{M}(\{\omega\}) \in \mathbb{R}$, is an invariant of the conformal class $\{\omega\}$. We denote by $K_{M}:=\wedge^{n} T^{*} M$ the canonical line bundle, and by $K_{M}^{-1}$ the anti-canonical line bundle. Then the Gauduchon degree is defined as

$$
\Gamma_{M}(\{\omega\}):=\frac{1}{(n-1)!} \int_{M} c_{1}^{B C}\left(K_{M}^{-1}\right) \wedge \eta^{n-1}=\int_{M} s^{C h}(\eta) \operatorname{Vol}_{\eta}
$$

where $c_{1}^{B C}\left(K_{M}^{-1}\right)$ is the Chern class of the anti-canonical line bundle in the Bott-Chern cohomology, and $\eta \in\{\omega\}$ is the unique Gauduchon representative of volume one. Hence, by definition, $\Gamma_{M}(\{\omega\})$ only depends on the conformal class of $\omega$ and the complex structure of $M$. On an almost-Hermitian manifold $(M, J, g)$, for any Gauduchon parameter $t \in \mathbb{R}$, we define (see 34 , 207|)

$$
\Gamma_{M}^{t}(\{\omega\}):=\int_{M} s^{t}(\eta) \operatorname{Vol}_{\eta}
$$

Using Theorem 1.1.1, we also define the following normalized conformal class, which will turn out to be an appropriate tool to face the Gauduchon Yamabe problem (see Proposition 3.1.1 and the next sections).

$$
\{\omega\}_{1}:=\left\{e^{f} \eta \in\{\omega\} \mid \int_{M} e^{f} \operatorname{Vol}_{\eta}=1\right\} \subset\{\omega\}
$$

where $\eta \in\{\omega\}$ denotes the unique Gauduchon representative of volume 1 . Now we can prove the following result, which explains the relation between the expected constant $\nabla^{t}$-scalar curvature and $\Gamma_{M}^{t}(\{\omega\})$.

Proposition 3.1.1 (Proposition 2.6 of 34$)$. Let $(M, J, g)$ be a compact almost-Hermitian manifold. Assume that $\widetilde{\omega} \in\{\omega\}$ has constant $\nabla^{t}$-scalar curvature equal to $\lambda \in \mathbb{R}$. Then $\widetilde{\omega} \in\{\omega\}_{1}$ if and only if

$$
\Gamma_{M}^{t}(\{\omega\})=\lambda
$$

In particular, the $\nabla^{t}$-scalar curvature of a possible constant $\nabla^{t}$-scalar curvature metric in $\{\omega\}_{1}$ must equal $\Gamma_{M}^{t}(\{\omega\})$; while the sign of the $\nabla^{t}$-scalar curvature of a possible constant $\nabla^{t}$-scalar curvature metric in $\{\omega\}$ equals the sign of $\Gamma_{M}^{t}(\{\omega\})$.

Proof. As a representative in $\{\omega\}$, fix the unique Gauduchon metric $\eta \in\{\omega\}$ of volume 1 and denote by $\theta$ its Lee form. Suppose that, for some $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$, the conformal metric $\widetilde{\omega}=e^{f} \eta \in\{\omega\}_{1}$ has constant $\nabla^{t}$-scalar curvature equal to $\lambda$. Equation (3.3) yields

$$
\begin{aligned}
\lambda \int_{M} e^{f} \operatorname{Vol}_{\eta}=\int_{M} e^{f} s^{t}(\widetilde{\omega}) \operatorname{Vol}_{\eta} & =\int_{M}\left(s^{t}(\eta)+\frac{1}{2}(1+n t-t) \Delta_{\eta}^{C h} f\right) \operatorname{Vol}_{\eta} \\
& =\frac{1}{2}(1+n t-t) \int_{M} \Delta_{\eta}^{C h} f \operatorname{Vol}_{\eta}+\int_{M} s^{t}(\eta) \operatorname{Vol}_{\eta}
\end{aligned}
$$

Moreover, the first term on the right-hand side vanishes. Indeed,

$$
\int_{M} \Delta_{\eta}^{C h} f \operatorname{Vol}_{\eta}=\int_{M} \Delta_{d} f \operatorname{Vol}_{\eta}+\int_{M}(d f, \theta)_{\eta} \operatorname{Vol}_{\eta}=\int_{M} \Delta_{d} f \operatorname{Vol}_{\eta}+\int_{M}\left(f, d^{*} \theta\right)_{\eta} \operatorname{Vol}_{\eta}=0
$$

since $\eta$ is Gauduchon. Therefore,

$$
\Gamma_{M}^{t}(\{\omega\})=\int_{M} s^{t}(\eta) \operatorname{Vol}_{\eta}=\lambda \int_{M} e^{f} \operatorname{Vol}_{\eta}=\lambda
$$

yielding the first implication.

On the other hand, if $e^{f} \omega \in\{\omega\}$ is a metric with constant $\nabla^{t}$-scalar curvature equal to $\Gamma_{M}^{t}(\{\omega\})$, it can be scaled by a constant $e^{c}$ so that $e^{f+c} \omega$ stays in the normalized conformal class $\{\omega\}_{1}$ and its Gauduchon scalar curvature becomes $e^{-c} \Gamma_{M}^{t}(\{\omega\})$. Here $c$ is such that

$$
e^{-c}=\int_{M} e^{f} \mathrm{Vol}_{\eta}
$$

Since $e^{f+c} \omega$ is a constant $\nabla^{t}$-scalar curvature metric in $\{\omega\}_{1}$, it has scalar curvature equal to $\Gamma_{M}^{t}(\{\omega\})$. Thus, finally, $c=0$.

Proposition 3.1.2. Let $(M, J, g)$ be a compact Hermitian manifold, and fix $t \in \mathbb{R}$. The value $\Gamma_{M}^{t}(\{\omega\})$ as defined above is a non-decreasing function in $t$.

Proof. Thanks to (1.10), it is sufficient to show that $\int_{M} \operatorname{tr}_{\omega} d d^{*} \omega \operatorname{Vol}_{\omega}$ is non-negative. We have that

$$
\begin{aligned}
\int_{M} \operatorname{tr}_{\omega} d d^{*} \omega \operatorname{Vol}_{\omega} & =\frac{1}{n} \int_{M} d d^{*} \omega \wedge(\star \omega) \\
& =\frac{1}{n} \int_{M}\left\langle d d^{*} \omega, \omega\right\rangle \operatorname{Vol}_{\omega} \\
& =\frac{1}{n}\left(d d^{*} \omega, \omega\right)_{g}=\frac{1}{n}\left(d^{*} \omega, d^{*} \omega\right)_{g}=\left|d^{*} \omega\right|_{g}^{2} \geq 0
\end{aligned}
$$

hence the thesis follows.

### 3.2 Yamabe problem for Gauduchon connections

With the aim of solving the Poincaré conjecture, Yamabe thought to exhibit a metric with constant scalar curvature as a preliminary step. In [327] he proposed a proof of the following problem (which now takes his name): finding a constant scalar curvature metric in the conformal class of a given Riemannian metric. The Yamabe problem was born because there was a gap in Yamabe's proof. Nowadays, it is well understood in the Riemannian setting, since the works of Yamabe, Trudinger, Aubin, and Schoen [26, 203, 272, 303, 327] give a solution on compact manifolds (see [27, Chapter 5] for an overview).

## The Chern-Yamabe problem

An analog of the Yamabe problem for almost-Hermitian manifolds was studied by del Rio and Simanca [268, which were looking for metrics with constant $\star$-scalar curvature $s^{\star}$ (they call it $s^{J}$ in 268]; see their footnote on page 187). A different direction in the Hermitian non-Kähler context was taken by Angella, Calamai, and Spotti who introduced and studied the Chern-Yamabe problem in [14. They stated the following conjecture and proved it in the case of non-positive Gauduchon degree.

Conjecture (Conjecture 2.1 in [14]). Let $(M, J, \omega)$ be a compact Hermitian manifold. Then there exists a metric with constant Chern scalar curvature in the conformal class $\{\omega\}$.
Theorem (Theorem 3.1 and Theorem 4.1 in (14). Let $(M, J, \omega)$ be a compact Hermitian manifold. If $\Gamma_{M}(\{\omega\}) \leq 0$, then there exists a unique metric $\widetilde{\omega} \in\{\omega\}_{1}$ with constant Chern scalar curvature. Moreover, $s^{C h}(\widetilde{\omega})=\Gamma_{M}(\{\omega\})$.

In [207] the authors extended the above results to the non-integrable case. On the other hand, in $[34$ we generalized the arguments of [14] to analyze all the Gauduchon scalar curvatures. We present here those results extended to the non-integrable case. We finally remark that in [129] the author studied the problem of prescribed Chern scalar curvature, while the prescribed Gauduchon scalar curvature problem has recently been studied in the almost-Hermitian setting in 213.

## The Gauduchon-Yamabe problem

In this chapter, we study the Yamabe problem for the Gauduchon connections in the almostHermitian context. Precisely, given an almost-Hermitian manifold ( $M, J, g$ ) , and a fixed parameter $t \in \mathbb{R}$, we look for constant $\nabla^{t}$-scalar curvature metrics $\widetilde{\omega}$ in the conformal class $\{\omega\}$. Thanks to the equation (3.3) on the conformal change of the scalar curvature of $\nabla^{t}$, this problem reduces to solve a semi-linear elliptic equation of 2 nd order. In detail, given an almost-complex manifold $(M, J)$ of real dimension $2 n$ and an almost-Hermitian conformal class $\{\omega\}$ on it, fix $\lambda \in \mathbb{R}$ and define the constant $C_{t}:=1+n t-t$. Then, given $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$ such that $\widetilde{\omega}=e^{2 f} \omega \in\{\omega\}_{1}, \widetilde{\omega}$ has constant scalar curvature equal to $\lambda$ if and only if $f$ solves

$$
\begin{equation*}
C_{t} \Delta_{\omega}^{C h} f+s^{t}(\omega)=\lambda e^{2 f} . \tag{3.4}
\end{equation*}
$$

A posteriori, if such a solution exists, $\lambda$ must equal $\Gamma_{M}^{t}(\{\omega\})$ by Proposition 3.1.1.
For $t=1$, (3.4) corresponds to the elliptic equation associated to the Chern-Yamabe problem as in [14, 207]. We thus obtain the results of [14, 207] on the Chern connection as a particular choice of Gauduchon connection in our Theorems 3.3.1 and 3.3.2. In particular, depending on the signs of $C_{t}$ and $\Gamma_{M}^{t}(\{\omega\})$, standard analytic techniques apply to give the following result.

Theorem 3.2.1 (Theorems 3.3.1 and 3.3.2). Let M be a compact manifold with almost-Hermitian structure $(J, \omega)$. If the Gauduchon parameter $t$ is such that $C_{t} \neq 0$ and $C_{t} \Gamma_{M}^{t}(\{\omega\}) \leq 0$, then there exists a unique metric $\widetilde{\omega} \in\{\omega\}_{1}$ such that it has constant scalar curvature with respect to the Gauduchon connection $\nabla^{t}$. Moreover, $s^{t}(\widetilde{\omega})=\Gamma_{M}^{t}(\{\omega\})$.

We remark that the "critical" Gauduchon connection for which the constant $C_{t}$ vanishes are left out by these theorems. In fact, in these cases, (3.3) reduces to

$$
s^{t}(\widetilde{\omega})=e^{-2 f} s^{t}(\omega) .
$$

Consequently, in a given almost-Hermitian conformal class $\{\omega\}$ there exist metrics with constant $\nabla^{t}$-scalar curvature if and only if $s^{t}(\omega)$ is never zero or it vanishes identically. In complex dimension 2, the "critical" Gauduchon parameter $t=\frac{1}{1-n}$ is $t=-1$, which corresponds to the Bismut connection. As a matter of fact, on a Hermitian manifold of complex dimension 2, the $J$-invariant part of the Bismut-Ricci form $\left(R i c^{B}\right)^{J,+}$ equals the Ricci tensor of the Weyl connection Ric ${ }^{W, J}$ (see $(\overline{1.9}$ ), which is invariant under conformal transformations. Moreover, for a Hermitian manifold $R i c^{W, J}(x, y)=\operatorname{Ric}^{W}(x, y)$ since $R_{J x, y}^{W}$ commutes with $J$. Hence, the Bismut scalar curvature equals the conformal scalar curvature

$$
s^{B}=s^{W} .
$$

Consequently, thanks to (1.11) $s^{B}$ can be related to the Riemannian scalar curvature $s^{L C}$ as (see also [6. Equation (2.12)])

$$
s^{B}=s^{L C}-3 d^{*} \theta-\frac{3}{2}\|\theta\|^{2}
$$

Therefore, if $s^{B}(\omega) \geq 0$ (respectively $>0$ ), then taking a Gauduchon representative $\eta \in\{\omega\}$ it holds $s^{L C}(\eta) \geq 0$ (respectively $>0$ ). Furthermore, Proposition 3.1 in 6 gives cohomological obstructions to this condition. In detail, the following result holds.

Proposition (Proposition 3.1 in [6]). Let $(M, J, g)$ be a compact Hermitian surface. If $s^{B}(g) \geq 0$, then for all $m>0$ the $m$-th plurigenus $p_{m}:=\operatorname{dim} H^{0}\left(M ;\left(K_{M}\right)^{m}\right)$ satisfies $p_{m} \leq 1$. Furthermore, if the inequality is strict at some point or the Gauduchon metric of the Hermitian structure is not Kähler, then $p_{m}=0$ for all $m>0$.

Consequently, if $s^{B}(\omega)>0$ then $M$ has Kodaira dimension $-\infty$, and in particular $H_{\bar{\partial}}^{2,0}(M, J)=$ 0 . The same consequences hold also if $s^{B}(\omega)$ vanishes identically and given $\eta \in\{\omega\}$ a Gauduchon (hence pluriclosed) representative, it is not Kähler.

### 3.3 Existence of constant scalar curvature metrics

Let $(M, J)$ be an almost-complex manifold of real dimension $2 n$, and consider an almostHermitian conformal structure $\{\omega\}$ on it. Fix a Gauduchon parameter $t \in \mathbb{R}$. The non-linearity of the elliptic equation (3.4) depends on the parameter $\lambda$, which in turns depends on $\Gamma_{M}^{t}(\{\omega\})$ by Proposition 3.1.1. Hence, we distinguish two cases, that are the linear case and the non-linear case, which respectively correspond to $\Gamma_{M}^{t}(\{\omega\})=0$ and $\Gamma_{M}^{t}(\{\omega\}) \neq 0$.

### 3.3.1 Linear case

In case of $\Gamma_{M}^{t}(\{\omega\})=0$, the semi-linear elliptic differential equation (3.4) becomes just linear, and so we get a solution for the corresponding Gauduchon-Yamabe problem whenever $C_{t} \neq 0$.

Theorem 3.3.1. Let $M$ be a compact manifold with almost-Hermitian structure $(J, \omega)$. If the Gauduchon parameter $t$ is such that $C_{t} \neq 0$ and $\Gamma_{M}^{t}(\{\omega\})=0$, then there exists a unique metric $\widetilde{\omega} \in\{\omega\}_{1}$ such that it has constant scalar curvature with respect to the Gauduchon connection $\nabla^{t}$. Moreover, $s^{t}(\widetilde{\omega})=\Gamma_{M}^{t}(\{\omega\})=0$.

Proof. Fix $\eta \in\{\omega\}$ the unique Gauduchon representative in $\{\omega\}$ with volume 1. Equation (3.4) with $\lambda=0$ becomes

$$
\begin{equation*}
C_{t} \Delta_{\eta}^{C h} f=-s^{t}(\eta) . \tag{3.5}
\end{equation*}
$$

Hence, we should prove the existence of $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$ solving (3.5), and its uniqueness up to additive constants.

We start with uniqueness. Using the relation in Lemma 1.1.1, it can be shown that the kernel of the Chern Laplacian consists of just the constant functions. Indeed, we recall from Lemma 1.1.1 that

$$
\Delta_{\eta}^{C h} f=\Delta_{\eta} f+(d f, \theta)_{\eta} .
$$

Thus taking a function $u$ in $\operatorname{ker} \Delta_{\eta}^{C h}$ one obtains

$$
0=\int_{M} u \Delta_{\eta}^{C h} u \operatorname{Vol}_{\eta}=\int_{M}\left(|\nabla u|_{\eta}^{2}+\frac{1}{2}\left(d u^{2}, \theta\right)_{\eta}\right) \operatorname{Vol}_{\eta}=\int_{M}|\nabla u|_{\eta}^{2} \operatorname{Vol}_{\eta},
$$

since $d_{\eta}^{*} \theta=0$ because $\eta$ is Gauduchon. It follows that two conformal metrics with zero Gauduchon scalar curvature differ by a multiplicative constant. Notice also that, this constant must be 1 if they are both in $\{\omega\}_{1}$.

Now the existence follows again by using the relation in Lemma 1.1.1. As a matter of fact, the adjoint of $\Delta_{\eta}^{C h}$ on smooth functions $u$ is

$$
\left(\Delta_{\eta}^{C h}\right)^{*} u=\Delta_{\eta} u-(d u, \theta)_{\eta}+d_{\eta}^{*} \theta u=\Delta_{\eta} u-(d u, \theta)_{\eta},
$$

$d_{\eta}^{*} \theta$ being zero since $\eta$ is Gauduchon. Thus the same computation as before applies, and hence also the kernel of the adjoint of the Chern Laplacian of a Gauduchon metric consists of just the constants. Since the integral of $-s^{t}(\eta)$ is zero by hypothesis,

$$
-C_{t}^{-1} s^{t}(\eta) \in\left(\operatorname{ker}\left(\Delta_{\eta}^{C h}\right)^{*}\right)^{\perp}=\operatorname{Im} \Delta_{\eta}^{C h}
$$

We thus achieve the existence of a metric of zero Gauduchon scalar curvature.

### 3.3.2 Non-linear case

Here we provide a positive answer for the Gauduchon-Yamabe problem when $C_{t} \Gamma_{M}^{t}(\{\omega\})<0$. As a particular case, we will obtain the solution to the Bismut-Yamabe problem when the Gauduchon degree $\Gamma_{M}^{-1}(\{\omega\})$ is strictly positive and the dimension of the manifold is greater than 6.

Theorem 3.3.2. Let $M$ be a compact manifold with almost-Hermitian structure $(J, \omega)$. Fix a Gauduchon parameter $t$ and suppose $C_{t} \Gamma_{M}^{t}(\{\omega\})<0$. Then there exists a unique $\widetilde{\omega} \in\{\omega\}_{1}$ with constant $\nabla^{t}$-scalar curvature. Moreover, its Gauduchon scalar curvature satisfies $s^{t}(\widetilde{\omega})=$ $\Gamma_{M}^{t}(\{\omega\})$.

Proof. Fix $\eta \in\{\omega\}$ the unique Gauduchon representative in $\{\omega\}$ with volume 1. By hypothesis, it holds

$$
C_{t} \Gamma_{M}^{t}(\{\omega\})=C_{t} \int_{M} s^{t}(\eta) \operatorname{Vol}_{\eta}<0
$$

The proof of the existence of a constant $\nabla^{t}$-scalar curvature metric consists of two steps. We apply a continuity method to prove the existence of a constant $\nabla^{t}$-scalar curvature metric in $\{\omega\}$ of class $\mathcal{C}^{2, \alpha}$; then we exploit the structure of the elliptic equation by a standard bootstrap argument to prove that it is actually smooth.

Before starting with the continuity method, we need a preliminary step. Namely, we prove that in the normalized conformal class $\{\omega\}_{1}$, there is a metric that has $\nabla^{t}$-scalar curvature of constant sign equal to $-\operatorname{sgn}\left(C_{t}\right)$. By this, we can assume that $C_{t} S^{t}(\omega)<0$ at every point. Consider the equation

$$
\Delta_{\eta}^{C h} f=-s^{t}(\eta)+\int_{M} s^{t}(\eta) \operatorname{Vol}_{\eta} .
$$

Since $\eta$ is Gauduchon, arguing as in the proof of Theorem 3.3.1, the above equation has a solution $f \in \mathcal{C}^{\infty}(M ; \mathbb{R})$, which is unique once we require $\int_{M} \exp \left(2 f / C_{t}\right) \operatorname{Vol}_{\eta}=1$. Then $\exp \left(2 f / C_{t}\right) \eta \in\{\omega\}_{1}$ satisfies

$$
\begin{aligned}
C_{t} s^{t}\left(e^{2 f / C_{t}} \eta\right) & =C_{t} e^{-2 f / C_{t}}\left(s^{t}(\eta)+\Delta_{\eta}^{C h} f\right) \\
& =C_{t} e^{-2 f / C_{t}} \int_{M} s^{t}(\eta) \mathrm{Vol}_{\eta} \\
& =C_{t} e^{-2 f / C_{t}} \Gamma_{M}^{t}(\{\omega\})<0 .
\end{aligned}
$$

Now we can set up the following continuity path using as a reference metric in the conformal class of $\eta$ the above metric $\omega$ with $C_{t} s^{t}(\omega)<0$. For $\alpha \in(0,1)$, consider the map

$$
\text { GaYa: }[0,1] \times \mathcal{C}^{2, \alpha}(M ; \mathbb{R}) \rightarrow \mathcal{C}^{0, \alpha}(M ; \mathbb{R}),
$$

such that

$$
\operatorname{GaYa}(\xi, f):=\Delta_{\omega}^{C h} f+\xi s^{t}(\omega)-\lambda e^{2 f / C_{t}}+\lambda(1-\xi) .
$$

Let us define the set

$$
\Xi:=\left\{\xi \in[0,1] \mid \exists f_{\xi} \in \mathcal{C}^{2, \alpha}(M ; \mathbb{R}) \text { such that } \operatorname{GaYa}\left(\xi, f_{\xi}\right)=0\right\},
$$

which trivially is non-empty since $\operatorname{GaYa}(0,0)=0$. Thus, we should prove that it is also open and closed since the solution to the Gauduchon-Yamabe problem is achieved when $\xi=1$.

We start showing that $\Xi$ is open. The implicit function theorem for Hilbert spaces guarantees that $\Xi$ is open as long as the linearization of GaYa with respect to the second variable is bijective. Hence, we prove that, for a fixed solution $\mathrm{GaYa}\left(\xi_{0}, f_{\xi_{0}}\right)=0$, the linearized operator of GaYa ,

$$
D: \mathcal{C}^{2, \alpha}(M ; \mathbb{R}) \rightarrow \mathcal{C}^{0, \alpha}(M ; \mathbb{R})
$$

defined by

$$
v \mapsto D v:=\Delta_{\omega}^{C h} v-\lambda \exp \left(2 f_{\xi_{0}} / C_{t}\right) \cdot 2 v / C_{t}
$$

is bijective. Let us remark that $D$ differs from the Chern Laplacian by a compact operator, thus they have the same index, equal to zero. This means that injectivity directly implies surjectivity, hence we are reduced to proving the former.

If $v$ belongs to ker $D$, then at a maximum point $p \in M$ for $v$ there holds

$$
-\lambda \exp \left(2 f_{\xi_{0}}(p) / C_{t}\right) \cdot 2 v(p) / C_{t} \leq 0,
$$

and hence $v(p) \leq 0$, since $-\lambda / C_{t}>0$. Similarly, at a minimum point $q$ for $v$, there holds $v(q) \geq 0$. Thus, $\operatorname{ker} D=\{0\}$, and $\Xi$ is therefore open.
To show that $\Xi$ is also closed we argue as follows. Take $\left\{\xi_{n}\right\}_{n} \subset \Xi$ a sequence converging to $\xi_{\infty}$ and $f_{\xi_{n}} \in \mathcal{C}^{2, \alpha}(M ; \mathbb{R})$ such that $\operatorname{GaYa}\left(\xi_{n}, f_{\xi_{n}}\right)=0$ for any $n$; we will use the Ascoli-Arzelà theorem to prove that the sequence $\left\{f_{\xi_{n}}\right\}_{n}$ converges in $\mathcal{C}^{2, \alpha}(M ; \mathbb{R})$ to a function $f_{\infty}$ such that $\mathrm{GaYa}\left(\xi_{\infty}, f_{\infty}\right)=0$. To use that theorem, we first need uniform $L^{\infty}$ estimates of the solutions $f_{\xi_{n}}$.

Lemma 3.3.1. There exists a positive constant $K$, depending only on $M, \omega, \lambda$, and $t$ such that, for any $n$, it holds

$$
\left\|f_{\xi_{n}}\right\|_{L^{\infty}} \leq K
$$

Proof. By hypothesis the functions $f_{\xi_{n}}$ satisfy $\operatorname{GaYa}\left(\xi_{n}, f_{\xi_{n}}\right)=0$, which means that the following equality holds:

$$
\begin{equation*}
\Delta_{\omega}^{C h} f_{\xi_{n}}+\xi_{n} s^{t}(\omega)-\lambda \exp \left(2 f_{\xi_{n}} / C_{t}\right)+\lambda\left(1-\xi_{n}\right)=0 . \tag{3.6}
\end{equation*}
$$

We distinguish the two cases $C_{t}>0$ and $C_{t}<0$ which correspond to $\lambda<0$ or $\lambda>0$ respectively. We also recall that, by the preliminary step in the proof, $s^{t}(\omega)$ can be supposed to be a negative function when $C_{t}>0$ and positive when $C_{t}<0$.

Suppose $C_{t}>0$ and take a maximum point $p \in M$ for $f_{\xi_{n}}$. Then, at $p$, there holds

$$
-\lambda \exp \left(2 f_{\xi_{n}}(p) / C_{t}\right) \leq-\xi_{n} s^{t}(\omega)(p)-\lambda\left(1-\xi_{n}\right) \leq-\left(\min _{M} s^{t}(\omega)\right)-\lambda
$$

On the other hand, at a minimum point for $f_{\xi_{n}}$, say $q \in M$, there holds

$$
\begin{aligned}
-\lambda \exp \left(2 f_{\xi_{n}}(q) / C_{t}\right) & \geq-\xi_{n} s^{t}(\omega)(q)-\lambda\left(1-\xi_{n}\right) \geq \xi_{n}\left(-s^{t}(\omega)(q)+\lambda\right)-\lambda \\
& \geq \min \left\{\min _{M}\left(-s^{t}(\omega)\right),-\lambda\right\}>0 .
\end{aligned}
$$

The above estimates provide the claimed uniform constant $K_{0}$. The same argument holds also for $C_{t}<0$, indeed, in this case, at a maximum point $p \in M$ for $f_{\xi_{n}}$, we have

$$
\begin{aligned}
\lambda \exp \left(2 f_{\xi_{n}}(p) / C_{t}\right) & \geq \xi_{n} s^{t}(\omega)(p)+\lambda\left(1-\xi_{n}\right) \geq \xi_{n}\left(s^{t}(\omega)(p)-\lambda\right)+\lambda \\
& \geq \min \left\{\min _{M}\left(s^{t}(\omega)\right), \lambda\right\}
\end{aligned}
$$

while at a minimum point $q \in M$ for $f_{\xi_{n}}$, there holds

$$
\lambda \exp \left(2 f_{\xi_{n}}(q) / C_{t}\right) \leq \xi_{n} s^{t}(\omega)(q)+\lambda\left(1-\xi_{n}\right) \leq \max _{M} s^{t}(\omega)+\lambda
$$

Hence the lemma is proved.
Now it remains to prove the uniform equicontinuity of the functions $\left\{f_{\xi_{n}}\right\}$ in $\mathcal{C}^{2, \alpha}(M ; \mathbb{R})$ in order to apply the Ascoli-Arzelà theorem. We define the elliptic operators

$$
L_{n} f:=\Delta_{\omega}^{C h} f+\xi_{n} s^{t}(\omega)+\lambda\left(1-\xi_{n}\right) .
$$

For the functions $f_{\xi_{n}}$ we get the equalities

$$
L_{n} f_{\xi_{n}}=\lambda \exp \left(2 f_{\xi_{n}} / C_{t}\right)
$$

The estimate of Lemma 3.3 .1 gives a uniform $L^{\infty}$ control of the right-hand side $\lambda \exp \left(2 f_{\xi_{n}} / C_{t}\right)$ of the equation and hence a uniform $L^{p}$ control of $L_{n} f_{\xi_{n}}$ for any $p \in(1, \infty)$. Then, by the Calderon-Zygmund inequality we can control the $p$-norm of the second-order derivatives by the $p$-norms of the function and its Laplacian; hence, iterating it twice, we get that $f_{\xi_{n}} \in W^{4, p}(M ; \mathbb{R})$ with uniform bound on the norms. Finally, we can use the Sobolev embedding taking $p$ large enough so that we find an a priori $\mathcal{C}^{3}$ uniform bound on the solutions. Finally, we can apply the Ascoli-Arzelà theorem to get a subsequence (which we still call $\left.\left\{f_{\xi_{n}}\right\}\right)$ converging in $\mathcal{C}^{2, \alpha}(M ; \mathbb{R})$ to a function $f_{\infty}$. We can take the limit in the equation (3.6) ; in this way we see that $f_{\infty}$ is a solution $\mathrm{GaYa}\left(\xi_{\infty}, f_{\infty}\right)=0$ as needed.

So far we achieved the existence of a $\mathcal{C}^{2, \alpha}$ solution $f$ to the Gauduchon-Yamabe equation, $\operatorname{GaYa}(1, f)=0$. Hence we have $f \in \mathcal{C}^{2, \alpha}$ such that

$$
\Delta_{\omega}^{C h} f=\lambda e^{2 f / C_{t}}-s^{t}(\omega)
$$

Notice that the right-hand side has the same regularity of $f$, hence the smooth regularity of the solution follows by the usual bootstrap argument via Schauder's estimates for elliptic operators.

Now we have a smooth function $f$ solving $\Delta_{\omega}^{C h} f=\lambda e^{2 f / C_{t}}-s^{t}(\omega)$ and we want to prove its uniqueness. Notice that by Proposition 3.1.1. since we have $\lambda=\Gamma_{M}^{t}(\{\omega\}), e^{f} \omega$ must be in $\{\omega\}_{1}$; moreover, any other metric in $\{\omega\}_{1}$ with constant $\nabla^{t}$-scalar curvature must solve the same equation. Thus suppose we have two conformal metrics $\omega_{1}=\exp \left(2 f_{1} / C_{t}\right) \omega$ and $\omega_{2}=\exp \left(2 f_{2} / C_{t}\right) \omega$ in $\{\omega\}_{1}$ with constant $\nabla^{t}$-scalar curvature equal to $\lambda$. Hence we have the equations

$$
\Delta_{\omega}^{C h} f_{1}+s^{t}(\omega)=\lambda \exp \left(2 f_{1} / C_{t}\right), \quad \text { and } \quad \Delta_{\omega}^{C h} f_{2}+s^{t}(\omega)=\lambda \exp \left(2 f_{2} / C_{t}\right)
$$

Taking the difference between these, we get the equation

$$
\Delta_{\omega}^{C h}\left(f_{1}-f_{2}\right)=\lambda\left(\exp \left(2 f_{1} / C_{t}\right)-\exp \left(2 f_{2} / C_{t}\right)\right)
$$

At a first glance, we should distinguish the cases $C_{t}>0$ or $C_{t}<0$ for which we respectively have $\lambda<0$ and $\lambda>0$; however, in both cases at a maximum point $p \in M$ for $f_{1}-f_{2}$, we find $f_{1}(p)-f_{2}(p) \leq 0$ while at a minimum point $q \in M$, we have $f_{1}(q)-f_{2}(q) \geq 0$, proving that $f_{1}$ and $f_{2}$ coincide.

Remark 3.3.1. In case $C_{t} \Gamma_{M}^{t}(\{\omega\})>0$ the maximum principle does not apply and the Gauduchon-Yamabe equation loses its good analytic properties. For the Chern connection, this case corresponds to having a positive Gauduchon degree, which in turn implies Kodaira dimension $-\infty$, by the Gauduchon Plurigenera Theorem [135]. It is investigated in [207, Section 6] in the almost-Hermitian case, and in [14, Section 5] in the Hermitian case. In the latter, some sufficient criteria for the existence of positive constant Chern scalar curvature metrics are found. Moreover, non-homogeneous examples of Hermitian metrics of positive constant Chern scalar curvature have been constructed in [196] and [19]. Similarly, it would be interesting to find new explicit examples of constant Bismut scalar curvature metrics as well as some sufficient (and, possibly, necessary) conditions which ensure the existence of metrics with negative constant scalar curvature for the Bismut connection.

For the Bismut connection, the constant $C_{-1}=2-n$ is always negative unless $\operatorname{dim}_{\mathbb{R}} M=4$ when it vanishes. We thus have the following result.

Corollary 3.3.1. Let $M$ be a compact manifold with dimension greater or equal than 6 and almost-Hermitian structure $(\omega, J)$. If $\Gamma_{M}^{-1}(\{\omega\}) \geq 0$, then there exists a unique $\widetilde{\omega} \in\{\omega\}_{1}$ with constant Bismut scalar curvature. Moreover, $s^{B}(\widetilde{\omega})=\Gamma_{M}^{-1}(\{\omega\})$.

Remark 3.3.2. As proved in [6, Proposition 3.1], if a compact Hermitian manifold ( $M, J, g$ ) satisfies $\Gamma_{M}^{-1}(\{\omega\}) \geq 0$ then for all the plurigenera $p_{m}$ with $m>0$ it holds $p_{m} \leq 1$, and thus the Kodaira dimension of $M$ is either 0 or $-\infty$. Furthermore, the plurigenera vanish if $\Gamma_{M}^{-1}(\{\omega\})>0$, whence the Kodaira dimension is $-\infty$.

## Chapter 4

## The Ricci curvature of the Chern connection on almost-Hermitian 4-manifolds

In this chapter, we study an Einstein equation that involves the second Ricci curvature tensor of the Chern connection and is stated as follows. Given an almost-complex manifold ( $M, J$ ), an almost-Hermitian metric $g$ is said to be second-Chern-Einstein if it satisfies

$$
R i c^{C h, 2}(g)=\lambda g, \quad \text { for some } \lambda \in \mathcal{C}^{\infty}(M ; \mathbb{R}) .
$$

Taking the trace in the above equation, one sees that the function $\lambda$ is equal to $\frac{1}{n} s^{C h}(g)$. Hence the second-Chern-Einstein equation becomes

$$
\operatorname{Ric}^{C h, 2}(g)=\frac{s^{C h}(g)}{n} g
$$

We investigate the existence of second-Chern-Einstein metrics on 4-dimensional almostHermitian manifolds and collect some general results about the geometry of these manifolds. In Section 4.5, we give some new explicit examples of compact 4-dimensional second-Chern-Einstein almost-Hermitian manifolds. In some of these examples, the second-Chern-Einstein metric has positive Chern scalar curvature, while in some others it has zero Chern scalar curvature. We remark that in the integrable case second-Chern-Einstein Hermitian non-Kähler metrics with negative Chern scalar curvature are still missing (see [19]). We also observe that the second-Chern-Einstein problem on a complex surface is equivalent to an Einstein condition for the Bismut connection. Even if this equivalence is no longer true for almost-Hermitian structures, some crucial relations persist, see for example Theorem 4.1.1 and Proposition 4.6.1 Finally, using the relation between the second Chern-Ricci tensor $R i c^{C h}, 2$ and the Bismut-Ricci tensor $R i c^{B}$ given in Proposition 4.6.1, in Section 4.6, we give a classification of 4-dimensional unimodular almost-Abelian Lie algebras equipped with left-invariant almost-Hermitian second-Chern-Einstein metrics.

The original results of this chapter have been obtained in a joint work with Mehdi Lejmi [36].

### 4.1 Second-Chern-Einstein and Weyl-Einstein metrics

Thanks to (3.1), which describes the variation of the second Chern-Ricci tensor under conformal change of the metric, the second-Chern-Einstein equation is conformally invariant, meaning that if an almost-Hermitian metric is second-Chern-Einstein then any other metric in its conformal class is so. Since the canonical Weyl connection $\nabla^{W}$ is invariant under conformal change of the metric, the classical approach to attack the second-Chern-Einstein problem consists in exploring
the relation between second-Chern-Einstein metrics and Weyl-Einstein metrics (as in [143). Given a complex manifold $(M, J)$, a Hermitian metric $g$ is said to be Weyl-Einstein if

$$
\operatorname{Ric}^{W}(g)=\lambda g, \quad \text { for some } \lambda \in \mathcal{C}^{\infty}(M ; \mathbb{R}) .
$$

The Weyl-Einstein problem was studied, for example, in [37, 71, 140, 177, 242, 252, 256]. In particular, in 143 Gauduchon and Ivanov proved that the only non-Kähler Weyl-Einstein manifold of complex dimension 2 is the Hopf surface with its standard Hermitian structure. Moreover, in 252 Pedersen, Poon, and Swann studied Weyl-Einstein manifolds such that $\nabla^{W} J=0$. They proved the following

Theorem (Proposition 3.2 in [252]). Let ( $M, J, g$ ) be a compact Hermitian manifold of complex dimension $n \geq 3$, such that $\nabla^{W} J=0$ and $g$ is a Weyl-Einstein metric. Then, $\theta^{\sharp}$ and $J \theta^{\sharp}$ are commuting real holomorphic Killing vector fields; moreover $\nabla^{L C} \theta=0$.

With some extra assumption on the regularity of the leaves induced by $\theta^{\sharp}$ and $J \theta^{\sharp}$ it is also possible to prove [252] that the Weyl-Einstein manifold must be the total space of a toric fibration over a Kähler-Einstein manifold of positive curvature (emulating the structure of the Hopf surface).

Let us now briefly focus on the condition $\nabla^{W} J=0$. Since the Weyl connection $\nabla^{W}$ is torsion-free, $\nabla^{W} J=0$ implies that $J$ is integrable. Moreover, in real dimension 4 it holds (for more details see (181)

$$
\begin{equation*}
g\left(\left(\nabla_{z}^{W} J\right) x, J y\right)=-2 g\left(N_{J}(x, y), z\right), \tag{4.1}
\end{equation*}
$$

hence $J$ is integrable if and only if $\nabla^{W} J=0$. On the other hand, in real dimension greater than $4, \nabla^{W}$ preserves $J$ if and only if the manifold is locally conformally Kähler. In details,

Theorem (Proposition 2.1 in [252] or Theorem 2.2 in [312]). Any almost-Hermitian manifold $(M, J, g)$ of real dimension at least 6 such that $\nabla^{W} J=0$ is locally conformally Kähler. Conversely, a locally conformally Kähler manifold of dimension at least 4 satisfies $\nabla^{W} J=0$.

It turns out that under the condition $\nabla^{W} J=0$, the second-Chern-Einstein problem and the Weyl-Einstein problem are equivalent. Indeed, in the integrable case of complex dimension 2, the second-Chern-Einstein condition is equivalent to the Weyl-Einstein condition by Theorem 1 of [143]; while the equivalence between these two problems is preserved in higher dimensions under the extra assumption of locally conformally Kähler metric, see the Remark 4.1.1. Somehow, the crucial condition which leads to the equivalence of the two problems is that in complex dimension 2 or for LCK manifolds it holds

$$
d \omega=\theta \wedge \omega
$$

Henceforth, the second-Chern-Einstein problem in the integrable case is reasonably wellunderstood through its relation with the Weyl-Einstein problem. Thus, we now show how this equivalence generalizes to the non-integrable case.

Let $(M, J, g)$ be an almost-Hermitian manifold of real dimension 4. Set $\theta$ the associated Lee form. It follows from $d \omega=\theta \wedge \omega$ that the Chern connection is related to the Levi-Civita connection by

$$
\begin{equation*}
\nabla_{x}^{C h} y=\nabla_{x}^{L C} y-\frac{1}{2} \theta(J x) J y-\frac{1}{2} \theta(y) x+\frac{1}{2} g(x, y) \theta^{\sharp}+N_{J}(x, y) . \tag{4.2}
\end{equation*}
$$

Moreover, the canonical Weyl connection is related to the Levi-Civita connection by

$$
\begin{equation*}
\nabla_{x}^{W} y=\nabla_{x}^{L C} y-\frac{1}{2} \theta(x) y-\frac{1}{2} \theta(y) x+\frac{1}{2} g(x, y) \theta^{\sharp} . \tag{4.3}
\end{equation*}
$$

Hence, by (4.3) and (4.2) we obtain the relation between the Chern connection and the canonical Weyl connection:

$$
\nabla_{x}^{C h} y=\nabla_{x}^{W} y+\frac{1}{2} \theta(x) y-\frac{1}{2} \theta(J x) J y+N(x, y) .
$$

Notice that on any 4-dimensional almost-Hermitian manifold, it holds [141, Proposition 1]

$$
\begin{equation*}
g\left(N_{J}(x, y), z\right)+g\left(N_{J}(y, z), x\right)+g\left(N_{J}(z, x), y\right)=0 \tag{4.4}
\end{equation*}
$$

Therefore, from (4.4) and (4.1), we get that

$$
\nabla_{x}^{C h} y=\nabla_{x}^{W} y+\frac{1}{2} \theta(x) y-\frac{1}{2} \theta(J x) J y+\frac{1}{2}\left(\nabla_{x}^{W} J\right) J y
$$

Finally, we can compute the relations between the curvatures $R^{C h}$ and $R^{W}$ as

$$
\begin{align*}
R_{x, y}^{C h} z=R_{x, y}^{W} z & -\frac{1}{2}(d J \theta)(x, y) J z-\frac{1}{2}(d \theta)(x, y) z \\
& -\frac{1}{2}\left(\nabla_{x}^{W}\left(\nabla_{y}^{W} J\right)-\nabla_{y}^{W}\left(\nabla_{x}^{W} J\right)-\nabla_{[x, y]}^{W} J\right) J z  \tag{4.5}\\
& +\frac{1}{4}\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right)-\left(\nabla_{y}^{W} J\right)\left(\nabla_{x}^{W} J\right)\right) z
\end{align*}
$$

We remark that when $J$ is integrable (or the manifold is higher-dimensional but LCK), the above relation reduces to (see 143])

$$
R^{C h}=R^{W}-\frac{1}{2}(d J \theta) \otimes J-\frac{1}{2}(d \theta) \otimes I d
$$

We also remark that the part of $R_{x, y}^{W}$ that anti-commutes with $J$ is given precisely by (see for example [181, Equation (2.15)])

$$
\begin{equation*}
\left(R_{x, y}^{W}\right)^{J,-}=\frac{1}{2}\left(\nabla_{x}^{W}\left(\nabla_{y}^{W} J\right)-\nabla_{y}^{W}\left(\nabla_{x}^{W} J\right)-\nabla_{[x, y]}^{W} J\right) \tag{4.6}
\end{equation*}
$$

We thus can substitute 4.6 in 4.5 obtaining

$$
\begin{align*}
R_{x, y}^{C h} z=\left(R_{x, y}^{W}\right)^{J,+} z & -\frac{1}{2}(d J \theta)(x, y) J z-\frac{1}{2}(d \theta)(x, y) z \\
& +\frac{1}{4}\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right)-\left(\nabla_{y}^{W} J\right)\left(\nabla_{x}^{W} J\right)\right) z \tag{4.7}
\end{align*}
$$

It follows from 4.7) that we can relate the second Chern-Ricci curvature with Ric ${ }^{W, J}$ extending the relation in 143, Theorem 1] to the non-integrable case.
Theorem 4.1.1 (Corollary 3 in $[36])$. Let $(M, J, g)$ be an almost-Hermitian 4-dimensional manifold. Then,

$$
R i c^{C h, 2}(\omega)=\left(\operatorname{Ric}^{W, J}(\omega)\right)^{J,+}+\frac{1}{2}\left(d^{*} \theta+\|\theta\|_{g}^{2}\right) \omega-\frac{1}{4}\left\|N_{J}\right\|_{g}^{2} \omega
$$

In particular, $g$ is second-Chern-Einstein if and only if $\left(\operatorname{Ric}^{W, J}(\omega)\right)^{J,+}$ is a multiple of $\omega$.
Proof. First of all, notice that $g(d J \theta, \omega)=-d^{*} \theta-\|\theta\|_{g}^{2}$, and $g(d \theta, \omega)=0$. Then, consider a $J$-adapted $g$-orthonormal frame of the tangent bundle $\left\{e_{1}, e_{2}=J e_{1}, e_{3}, e_{4}=J e_{3}\right\}$. Using (4.1), we see that

$$
\begin{aligned}
& \sum_{i=1}^{4} g\left(\left(\nabla_{e_{i}}^{W} J\right)\left(\nabla_{J e_{i}}^{W} J\right)-\left(\nabla_{J e_{i}}^{W} J\right)\left(\nabla_{e_{i}}^{W} J\right) z, V\right) \\
& \quad=-2 \sum_{i=1}^{4} g\left(\left(\nabla_{e_{i}}^{W} J\right)\left(\nabla_{e_{i}}^{W} J\right) z, J v\right)=2 \sum_{i=1}^{4} g\left(\left(\nabla_{e_{i}}^{W} J\right) z,\left(\nabla_{e_{i}}^{W} J\right) J v\right) \\
& \quad=4 \sum_{i=1}^{4} g\left(N_{J}\left(z,\left(\nabla_{e_{i}}^{W} J\right) J v\right), J e_{i}\right)=4 \sum_{i, j=1}^{4} g\left(N_{J}\left(z, e_{j}\right), J e_{i}\right) g\left(\left(\nabla_{e_{i}}^{W} J\right) J v, e_{j}\right) \\
& \quad=8 \sum_{i, j=1}^{4} g\left(N_{J}\left(z, e_{j}\right), J e_{i}\right) g\left(N_{J}\left(J v, e_{j}\right), J e_{i}\right)=-8 \sum_{i, j=1}^{4} g\left(N_{J}\left(J z, e_{j}\right), e_{i}\right) g\left(N_{J}\left(v, e_{j}\right), e_{i}\right)
\end{aligned}
$$

Now, we use the fact that in dimension 4 it holds $\left\|N_{J}\right\|_{g}^{2}=8\left\|N_{J}\left(e_{1}, e_{3}\right)\right\|_{g}^{2}$. Then

$$
\begin{aligned}
& \sum_{i, j=1}^{4} g\left(N_{J}\left(J z, e_{j}\right), e_{i}\right) g\left(N_{J}\left(v, e_{j}\right), e_{i}\right) \\
& \quad=\sum_{i, j, k, l=1}^{4} g\left(J z, e_{k}\right) g\left(v, e_{l}\right) g\left(N_{J}\left(e_{k}, e_{j}\right), e_{i}\right) g\left(N_{J}\left(e_{l}, e_{j}\right), e_{i}\right) \\
& \quad=\sum_{j, k, l=1}^{4} g\left(J z, e_{k}\right) g\left(v, e_{l}\right) g\left(N_{J}\left(e_{k}, e_{j}\right), N_{J}\left(e_{l}, e_{j}\right)\right) \\
& \quad=\sum_{j, k=1}^{4} g\left(J z, e_{k}\right) g\left(v, e_{k}\right) g\left(N_{J}\left(e_{k}, e_{j}\right), N_{J}\left(e_{k}, e_{j}\right)\right)=\frac{1}{4}\left\|N_{J}\right\|_{g}^{2} g(J z, v)
\end{aligned}
$$

and the result then follows.
Theorem 4.1.1 shows that the metric $g$ is second-Chern-Einstein if and only if $\left(R i c^{W, J}\right)^{J,+}=$ $\lambda \omega$ for some function $\lambda$. This extends Theorem 1 of 143 because when $J$ is integrable, $R_{x, y}^{W}$ commutes with $J$ and then $\operatorname{Ric}^{W}(J x, y)=\operatorname{Ric}^{W, J}(x, y)$.

Remark 4.1.1. A simplified version of the above proof shows that on a complex manifold $(M, J)$ with an LCK metric $g$, it holds

$$
\operatorname{Ric}^{C h, 2}(\omega)=\operatorname{Ric}^{W}(\omega)+\frac{1}{2}\left(d^{*} \theta+\|\theta\|_{g}^{2}\right) \omega
$$

Therefore, the second-Chern-Einstein and the Weyl-Einstein problems are equivalent for an LCK manifold, while it still is an open question if for higher dimensional LCaK manifolds a relation similar to that in Theorem 4.1.1 holds.

### 4.2 Lee form of a second-Chern-Einstein metric

We have seen (Remark 4.1.1) that for an LCK manifold, the second-Chern-Einstein problem is equivalent to the Weyl-Einstein problem. Moreover, given a locally conformally Kähler manifold $(M, J, g)$ whose Hermitian metric $g$ is Weyl-Einstein, then $\nabla^{L C} \theta=0$ and both $\theta^{\sharp}$ and $J \theta^{\sharp}$ are real holomorphic Killing vector fields [140, 252]. In this section, we study under which hypothesis on almost-Hermitian second-Chern-Einstein manifold $(M, J, g)$ of real dimension 4 the vector fields $\theta^{\sharp}$ or $J \theta^{\sharp}$ are real holomorphic Killing vector fields.

First of all, when $M$ is compact, the condition that $J \theta^{\sharp}$ is a Killing vector field implies that $(M, J, g)$ is LCaK. Indeed, applying the Lie derivative $\mathcal{L}_{J \theta^{\sharp}}$ to the relation $\omega=g(J \cdot, \cdot)$ we get

$$
d \theta=-2\left(\nabla^{L C} J \theta\right)^{\text {sym }}(J \cdot, \cdot)-g\left(\mathcal{L}_{J \theta^{\sharp}} J \cdot, \cdot\right) .
$$

Hence, if $\left(\nabla^{L C} J \theta\right)^{s y m}=0$ then $d \theta$ is $J$-anti-invariant. Thus, $d \theta$ is a self-dual $d$-exact 2 -form on a compact manifold so $d \theta=0$. However, the converse is not true in general: if $d \theta=0$, then only the symmetric $J$-invariant part of $\nabla^{L C} J \theta$ vanishes, that is $\left(\nabla^{L C} J \theta\right)^{s y m, J,+}=0$; therefore, $J \theta^{\sharp}$ is not necessarily a Killing vector field.

On the other hand, we have the following proposition regarding the condition of $\theta^{\sharp}$ being a real holomorphic vector field (when the metric is Weyl-Einstein see 296 and also [140]).

Proposition 4.2.1 (Lemma 6 in $[36])$. Let $(M, J, g)$ be a compact 4-dimensional almostHermitian manifold, and let $g$ be a second-Chern-Einstein metric. Suppose that the unit-volume Gauduchon metric $\eta$ in the conformal class $\{g\}$ satisfies $\left(\nabla^{L C} \theta\right)^{\text {sym,J,- }}=0$, then $\theta^{\sharp}$ is a Killing vector field of $\eta$.

Proof. First of all, from equation (4.3) we obtain

$$
\begin{equation*}
\left(\operatorname{Ric}^{W, J}(g)\right)^{J,+}(\cdot, J \cdot)=\left(\rho^{\star}\right)^{\text {sym }}+\left(\nabla^{L C} \theta\right)^{\text {sym,J,+ }}-\frac{1}{4}\|\theta\|_{g}^{2} g+\frac{1}{2}(\theta \otimes \theta)^{J,+} \tag{4.8}
\end{equation*}
$$

Then, combining Theorem 4.1.1 with (4.8) and using 1.12 and 1.13 , we deduce the following relation between the second Chern-Ricci form and the Ricci tensor of the Levi-Civita connection:

$$
\begin{equation*}
R i c^{C h, 2}(g)(\cdot, J \cdot)=\left(\operatorname{Ric}^{L C}(g)\right)^{J,+}+\frac{1}{4}\left\|N_{J}\right\|_{g}^{2} g+\left(\nabla^{L C} \theta\right)^{s y m, J,+}+\frac{1}{2}(\theta \otimes \theta)^{J,+} \tag{4.9}
\end{equation*}
$$

Since the second-Chern-Einstein condition is conformally invariant, we can specialize (4.9) to the unit-volume Gauduchon metric $\eta$ (from now on $\theta$ refers to the Lee form of $\eta$ ) obtaining

$$
\frac{s^{C h}(\eta)}{4} \eta=\left(\operatorname{Ric}^{L C}(\eta)\right)^{J,+}+\frac{1}{4}\left\|N_{J}\right\|_{\eta}^{2} \eta+\left(\nabla^{L C} \theta\right)^{s y m, J,+}+\frac{1}{2}(\theta \otimes \theta)^{J,+}
$$

We recall that $d^{*} \theta=-\eta\left(\nabla^{L C} \theta, \eta\right)$. Taking the inner product with $\left(\nabla^{L C} \theta\right)^{\text {sym }}=\left(\nabla^{L C} \theta\right)^{\text {sym, } J,+}$ and integrating we have

$$
\begin{aligned}
-\int_{M} \frac{s^{C h}(\eta)}{4} d^{*} \theta \operatorname{Vol}_{\eta} & =\int_{M} \eta\left(\left(\operatorname{Ric}^{L C}(\eta)\right)^{J,+},\left(\nabla^{L C} \theta\right)^{s y m}\right) \\
& -\frac{1}{4}\left\|N_{J}\right\|_{\eta}^{2} d^{*} \theta+\left\|\left(\nabla^{L C} \theta\right)^{s y m}\right\|_{\eta}^{2}+\frac{1}{2} \eta\left((\theta \otimes \theta)^{J,+},\left(\nabla^{L C} \theta\right)^{s y m}\right) \mathrm{Vol}_{\eta}
\end{aligned}
$$

Since the metric is Gauduchon, i. e. $d^{*} \theta=0$, we obtain

$$
\begin{aligned}
\left|\left(\nabla^{L C} \theta\right)^{s y m}\right|_{\eta}^{2} & =-\int_{M} \eta\left(\left(\operatorname{Ric}^{L C}(\eta)\right)^{J,+}+\frac{1}{2}(\theta \otimes \theta)^{J,+},\left(\nabla^{L C} \theta\right)^{s y m}\right) \operatorname{Vol}_{\eta} \\
& =-\int_{M} \eta\left(\operatorname{Ric}^{L C}(\eta), \nabla^{L C} \theta\right)+\frac{1}{2} \eta\left(\theta \otimes \theta, \nabla^{L C} \theta\right) \mathrm{Vol}_{\eta} \\
& =-\int_{M} \eta\left(d^{*} \operatorname{Ric}^{L C}(\eta), \theta\right)+\frac{1}{2} \eta\left(d^{*}(\theta \otimes \theta), \theta\right) \mathrm{Vol}_{\eta} \\
& =-\int_{M} \eta\left(-\frac{1}{2} d s^{L C}(\eta), \theta\right)+\frac{1}{2}\left(d^{*} \theta\|\theta\|_{\eta}^{2}-\eta\left(\nabla_{\theta}^{L C} \theta, \theta\right)\right) \operatorname{Vol}_{\eta} \\
& =-\int_{M}-\frac{1}{2} s^{L C}(\eta) d^{*} \theta+\frac{1}{2}\left(-\frac{1}{2} \eta\left(d\|\theta\|_{\eta}^{2}, \theta\right)\right) \operatorname{Vol}_{\eta}=\frac{1}{4} \int_{M}\|\theta\|_{\eta}^{2} d^{*} \theta \operatorname{Vol}_{\eta}=0
\end{aligned}
$$

where we used the contracted Bianchi identity $d_{\eta}^{*} R i c^{L C}(\eta)=-\frac{1}{2} d s^{L C}(\eta)$. The result follows.
For an almost-Hermitian manifold of real dimension 4 , the vanishing of the symmetric $J$-antiinvariant component of $\nabla^{L C} \theta$ is necessary to obtain $\mathcal{L}_{\theta^{\sharp}} \eta=0$, indeed there are counterexamples as shown in Section 4.5. Moreover, we remark that $\left(\nabla^{L C} \theta\right)^{s y m, J,-}=0$ is equivalent to $\left(\mathcal{L}_{\theta^{\sharp}} J\right)^{s y m}=0$. Hence, the flow of the vector field $\theta^{\sharp}$ does not necessarily preserve $J$, i. e. $\theta^{\sharp}$ is not necessarily a real holomorphic vector field.

We end this section highlighting the relation between the second-Chern-Einstein and the Weyl-Einstein equations under the extra assumption that $\left(\nabla^{L C} \theta\right)^{s y m, J,-}=0$. Namely, we prove the following result, see also Corollary 4.4.1.

Proposition 4.2.2 (Corollary 11 in $[36]$ ). Suppose that $(M, J, g)$ is a compact 4-dimensional almost-Hermitian manifold where $g$ is a second-Chern-Einstein metric. Suppose that the unitvolume Gauduchon metric $\eta$ in the conformal class $\{g\}$ satisfies $\left(\nabla^{L C} \theta\right)^{\text {sym,J,- }}=0$, then

$$
\operatorname{Ric}^{W}(\eta)=\operatorname{Ric}^{L C}(\eta)-\frac{1}{2}\left(\|\theta\|_{\eta}^{2} \eta-\theta \otimes \theta\right)=\frac{s^{W}(\eta)}{4} \eta+\left(\operatorname{Ric}^{L C}(\eta)\right)^{J,-}+\frac{1}{2}(\theta \otimes \theta)^{J,-}
$$

In particular, the metric $\eta$ is Weyl-Einstein if and only if

$$
\left(R i c^{L C}(\eta)\right)^{J,-}=-\frac{1}{2}(\theta \otimes \theta)^{J,-}
$$

Proof. Thanks to equation (4.9) and Proposition 4.2.1, we get that

$$
\begin{equation*}
\left(R i c^{L C}\right)^{J,+}=\frac{s^{C h}}{4} \eta-\frac{1}{4}\left\|N_{J}\right\|_{\eta}^{2} \eta-\frac{1}{2}(\theta \otimes \theta)^{J,+} . \tag{4.10}
\end{equation*}
$$

Then, the statement follows by applying again Proposition 4.2.1 to 4.3 and 4.10 .

### 4.3 Second-Chern-Einstein metrics with constant scalar curvature

In the literature, the second-Chern-Einstein metrics are called strong or weak depending on whether their Chern scalar curvature is constant or not. Since the second-Chern-Einstein problem is conformally invariant, by using [207, Corollary 5.10] or equivalently Theorem 3.3 .2 , one can prove that in the case of non-positive Gauduchon degree, the existence of a weak solution implies the existence of a strong solution. In particular, Theorem B of [15] can be generalized to the almost-Hermitian setting (see also [14, 70, 138, 166, 206]).

Theorem 4.3.1 (Theorem 8 in $[36])$. Let $(M, J, g)$ be a compact almost-Hermitian manifold and suppose that $g$ is a weak second-Chern-Einstein metric. Then, there is a representative in the conformal class $\{g\}$ such that its Hermitian scalar curvature has the same sign as $\Gamma_{M}(\{g\})$. Moreover, if $\Gamma_{M}(\{g\}) \leq 0$, then there is a strong second-Chern-Einstein representative in $\{g\}$.

In the Hermitian 4-dimensional case, weak solutions of the second-Chern-Einstein equation are always strong solutions up to conformal change [143, Theorem 2]. On the other hand, for almost-Hermitian manifolds, it is possible to prove that the conformal scalar curvature $s^{W}$ is constant under the extra conditions that $\theta^{\sharp}$ and $J \theta^{\sharp}$ are Killing vector fields. More precisely, we prove the following result.

Proposition 4.3.1 (Proposition 10 in $|36|)$. Let $(M, J, g)$ be a compact 4-dimensional almostHermitian manifold and $g$ be a second-Chern-Einstein metric. Suppose that the unit-volume Gauduchon metric $\eta$ in the conformal class $\{g\}$ satisfies $\left(\nabla^{L C} \theta\right)^{\text {sym,J,- }}=0$, and that Ric ${ }^{L C}$ is $J$-invariant, and $J \theta^{\sharp}$ is a Killing vector field. Then the conformal scalar curvature $s^{W}$ of $\eta$ is constant.

Proof. Applying the codifferential $d^{*}$ to (4.10), and using the contracted Bianchi identity $d_{\eta}^{*} \operatorname{Ric}^{L C}(\eta)=-\frac{1}{2} d s^{L C}(\eta)$ we obtain

$$
\begin{aligned}
-\frac{1}{2} d s^{L C}(\eta) & =-\frac{1}{4} d s^{C h}(\eta)+\frac{1}{4} d\left(\left\|N_{J}\right\|_{\eta}^{2}\right)-\frac{1}{4} d^{*}(\theta \otimes \theta)-\frac{1}{4} d^{*}(J \theta \otimes J \theta) \\
& =-\frac{1}{4} d s^{C h}(\eta)+\frac{1}{4} d\left(\left\|N_{J}\right\|_{\eta}^{2}\right)-\frac{1}{4}\left(d^{*} \theta \otimes \theta-\nabla_{\theta^{\sharp}}^{L C} \theta\right)-\frac{1}{4}\left(d^{*} J \theta \otimes \theta-\nabla_{J \theta^{\sharp}}^{L C} J \theta\right) \\
& =-\frac{1}{4} d s^{C h}(\eta)+\frac{1}{4} d\left(\left\|N_{J}\right\|_{\eta}^{2}\right)+\frac{1}{4} \nabla_{\theta^{\sharp}}^{L C} \theta+\frac{1}{4} \nabla_{J \theta^{\sharp}}^{L C} J \theta .
\end{aligned}
$$

Hence,

$$
d s^{L C}(\eta)=\frac{1}{2} d s^{C h}(\eta)-\frac{1}{2} d\left(\left\|N_{J}\right\|_{\eta}^{2}\right)-\frac{1}{2} \nabla_{\theta^{\sharp}}^{L C} \theta-\frac{1}{2} \nabla_{J \theta^{\sharp}}^{L C} J \theta .
$$

On the other hand, from the trace of 4.10 , we have that

$$
s^{L C}(\eta)=s^{C h}(\eta)-\left\|N_{J}\right\|_{\eta}^{2}-\frac{1}{2}\|\theta\|_{\eta}^{2}
$$

Thus,

$$
d\left(s^{C h}(\eta)-\left\|N_{J}\right\|_{\eta}^{2}\right)=d\left(\|\theta\|_{\eta}^{2}\right)-\nabla_{\theta^{\sharp}}^{L C} \theta-\nabla_{J \theta^{\sharp}}^{L C} J \theta .
$$

Applying again the codifferential $d^{*}$, we have

$$
\begin{equation*}
\Delta_{\eta}\left(s^{C h}(\eta)-\left\|N_{J}\right\|_{\eta}^{2}\right)=\Delta_{\eta}\left(\|\theta\|_{\eta}^{2}\right)-d^{*} \nabla_{\theta^{\sharp}}^{L C} \theta-d^{*} \nabla_{J \theta^{\sharp}}^{L C} J \theta . \tag{4.11}
\end{equation*}
$$

Then thanks to Proposition 4.2.1, $\theta^{\sharp}$ is Killing. Thus, since $\mathcal{L}_{\theta^{\sharp}} \theta=0$ using the Cartan formula we get

$$
d^{*} \nabla_{\theta^{\sharp}}^{L C} \theta=\frac{1}{2} d^{*}\left(d \theta\left(\theta^{\sharp}, \cdot\right)\right)=-\frac{1}{2} d^{*} d\left(\|\theta\|_{\eta}^{2}\right)=-\frac{1}{2} \Delta_{\eta}\left(\|\theta\|_{\eta}^{2}\right) .
$$

Similarly, since $J \theta^{\sharp}$ is a Killing vector field, we have that

$$
d^{*} \nabla_{J \theta^{\sharp}}^{L C} J \theta=-\frac{1}{2} \Delta_{\eta}\left(\|\theta\|_{\eta}^{2}\right) .
$$

Finally, from 4.11, we obtain

$$
\Delta_{\eta}\left(s^{C h}(\eta)-\left\|N_{J}\right\|_{\eta}^{2}-2\|\theta\|_{\eta}^{2}\right)=0
$$

Thus, since $M$ is compact from (1.11) we get that

$$
s^{W}(\eta)=s^{L C}(\eta)-\frac{3}{2}\|\theta\|_{\eta}^{2}=s^{C h}(\eta)-\left\|N_{J}\right\|_{\eta}^{2}-2\|\theta\|_{\eta}^{2}
$$

is a constant.

### 4.4 Geometry of almost-Hermitian second-Chern-Einstein manifolds

In this section, we collect our main results on the geometry of the almost-Hermitian second-Chern-Einstein manifolds. In particular, in Theorems 4.4.1 and 4.4.2 we characterize these manifolds under the additional assumption of $\rho^{\star}\left(\theta^{\sharp}, \theta^{\sharp}\right)=0$. As a matter of fact, the vanishing of this quantity leads to a condition on the almost-Hermitian structure when the metric is second-Chern-Einstein. Indeed, combining equations (1.12) and (1.13) we obtain

$$
\left(\rho^{\star}(g)\right)^{s y m}-\left(R i c^{L C}(g)\right)^{J,+}=-\frac{1}{4}\left(2 d^{*} \theta+\|\theta\|_{g}^{2}-2\left\|N_{J}\right\|_{g}^{2}\right) g .
$$

Moreover, in 4.10 we already computed $\left(\operatorname{Ric}^{L C}(g)\right)^{J,+}$ for a Gauduchon second-Chern-Einstein metric. Hence, with these assumptions, we have that

$$
\begin{equation*}
\left(\rho^{\star}(g)\right)^{s y m}=\frac{1}{4}\left(s^{C h}(g)-\|\theta\|_{g}^{2}+\left\|N_{J}\right\|_{g}^{2}\right) g-\frac{1}{2}(\theta \otimes \theta)^{J,+} . \tag{4.12}
\end{equation*}
$$

We start computing $\rho^{\star}(g)\left(\theta^{\sharp}, \cdot\right)$ in the following lemma.
Lemma 4.4.1 (Lemma 12 in [36). Let $(M, g, J)$ be a 4-dimensional compact almost-Hermitian manifold. Suppose that $\theta^{\sharp}$ is a Killing vector field. Then,

$$
\rho^{\star}(g)\left(\theta^{\sharp}, x\right)=-\frac{1}{2}(d \theta)^{J,-}\left(\theta^{\sharp}, x\right)+g\left(d \theta, N_{x}\right),
$$

for any vector field $x$, where $N_{x}(y, z)=g\left(N_{J}(y, z), x\right)$.
Proof. Consider a $J$-adapted $g$-orthonormal frame of the tangent bundle $\left\{e_{1}, e_{2}=J e_{1}, e_{3}, e_{4}=\right.$
$\left.J e_{3}\right\}$. Let $\alpha$ be a 1-form and $x$ be a vector field. Then

$$
\begin{aligned}
\left(d^{*}\right. & \left.\left(\nabla^{L C} \alpha\right)^{J,+}-d^{*}\left(\nabla^{L C} \alpha\right)^{J,-}\right)(x)=-\sum_{i=1}^{4}\left(\nabla_{e_{i}}^{L C}\left(\left(\nabla^{L C} \alpha\right)^{J,+}-\left(\nabla^{L C} \alpha\right)^{J,-}\right)\right)\left(e_{i}, x\right) \\
= & \sum_{i=1}^{4}-\nabla_{e_{i}}^{L C}\left(\nabla_{J e_{i}}^{L C} \alpha(J x)\right)+\nabla^{L C} \alpha\left(J \nabla_{e_{i}}^{L C} e_{i}, J x\right)+\nabla^{L C} \alpha\left(J e_{i}, J \nabla_{e_{i}}^{L C} x\right) \\
= & \sum_{i=1}^{4}-\nabla_{e_{i}}^{L C}\left(\nabla_{J e_{i}}^{L C} \alpha(J x)\right)+\nabla^{L C} \alpha\left(\nabla_{e_{i}}^{L C}\left(J e_{i}\right), J x\right)+\nabla^{L C} \alpha\left(J e_{i}, \nabla_{e_{i}}^{L C}(J x)\right) \\
& -\nabla^{L C} \alpha\left(\left(\nabla_{e_{i}}^{L C} J\right) e_{i}, J x\right)-\nabla^{L C} \alpha\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right) \\
= & \sum_{i=1}^{4}-\left(\nabla_{e_{i}}^{L C}\left(\nabla_{J e_{i}}^{L C} \alpha\right)\right)(J x)+\nabla^{L C} \alpha\left(\nabla_{e_{i}}^{L C}\left(J e_{i}\right), J x\right)-\nabla^{L C} \alpha\left(\left(\nabla_{e_{i}}^{L C} J\right) e_{i}, J x\right) \\
& -\nabla^{L C} \alpha\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right) \\
= & \sum_{i=1}^{4} \frac{1}{2} g\left(R_{e_{i}, J e_{i}}^{L C} \alpha^{\sharp}, J x\right)-\nabla^{L C} \alpha\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right)-\nabla^{L C} \alpha\left(J \theta^{\sharp}, J x\right) \\
= & -\nabla^{L C} \alpha\left(J \theta^{\sharp}, J x\right)-\sum_{i=1}^{4} \nabla^{L C} \alpha\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right)+\rho^{\star}\left(\alpha^{\sharp}, x\right) .
\end{aligned}
$$

On the other hand, substituting $\theta$ to $\alpha$ and using that $g(d \theta, \omega)=0$, we have

$$
\begin{aligned}
d^{*}\left(\nabla^{L C} \theta\right)^{J,+}-d^{*}\left(\nabla^{L C} \theta\right)^{J,-} & =\frac{1}{2} d^{*}\left((d \theta)^{J,+}-(d \theta)^{J,-}\right)=\frac{1}{2} d^{*}\left((d \theta)^{-}-(d \theta)^{+}\right) \\
& =-\frac{1}{2} \star_{g} d \star_{g}\left((d \theta)^{-}-(d \theta)^{+}\right)=\frac{1}{2} \star_{g} d\left((d \theta)^{-}+(d \theta)^{+}\right) \\
& =\frac{1}{2} \star_{g} d(d \theta)=0 .
\end{aligned}
$$

We then deduce that

$$
\begin{equation*}
\rho^{\star}(g)\left(\theta^{\sharp}, x\right)=\nabla^{L C} \theta\left(J \theta^{\sharp}, J x\right)+\sum_{i=1}^{4} \nabla^{L C} \theta\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right) . \tag{4.13}
\end{equation*}
$$

Now, we would like to compute the second term on the right-hand side of (4.13). We first recall that

$$
\left(\nabla_{x}^{L C} J\right) y=\frac{1}{2} g(x, y) J \theta^{\sharp}+\frac{1}{2} \theta(J y) x+\frac{1}{2} g(J x, y) \theta^{\sharp}-\frac{1}{2} \theta(y) J x+2\left(g\left(N_{J}(y, \cdot), J x\right)\right)^{\sharp},
$$

which can be easily deduced from (4.1) and (4.3). Hence,

$$
\begin{align*}
\sum_{i=1}^{4} \nabla^{L C} \theta\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right)= & \sum_{i=1}^{4}\left[\nabla_{J e_{i}}^{L C} \theta\left(\frac{1}{2} g\left(e_{i}, x\right) J \theta^{\sharp}\right)+\nabla_{J e_{i}}^{L C} \theta\left(\frac{1}{2} \theta(J x) e_{i}\right)\right. \\
& +\nabla_{J e_{i}}^{L C} \theta\left(\frac{1}{2} g\left(J e_{i}, x\right) \theta^{\sharp}\right)-\nabla_{J e_{i}}^{L C} \theta\left(\frac{1}{2} \theta(x) J e_{i}\right) \\
& \left.+2 g\left(N_{J}\left(x, \nabla_{J e_{i}}^{L C} \theta^{\sharp}\right), J e_{i}\right)\right] \\
= & \frac{1}{2} g\left(\nabla_{J x}^{L C} \theta, J \theta\right)+\frac{1}{2} g\left(\nabla_{x}^{L C} \theta, \theta\right)+2 \sum_{i=1}^{4} g\left(N_{J}\left(x, \nabla_{J e_{i}}^{L C} \theta^{\sharp}\right), J e_{i}\right) . \tag{4.14}
\end{align*}
$$

Moreover, using (4.4), we also compute the third term in (4.14) as

$$
\begin{aligned}
& \sum_{i=1}^{4} g\left(N_{J}\left(x, \nabla_{J e_{i}}^{L C} \theta^{\sharp}\right), J e_{i}\right)=\sum_{i, j=1}^{4} g\left(\nabla_{J e_{i}}^{L C} \theta^{\sharp}, e_{j}\right) g\left(N_{J}\left(x, e_{j}\right) J e_{i}\right) \\
& \quad=\sum_{i, j=1}^{4}\left[-g\left(\nabla_{J e_{i}}^{L C} \theta^{\sharp}, e_{j}\right) g\left(N_{J}\left(e_{j}, J e_{i}\right), x\right)-g\left(\nabla_{J e_{i}}^{L C} \theta^{\sharp}, e_{j}\right) g\left(N_{J}\left(J e_{i}, x\right), e_{j}\right)\right] \\
& \quad=\sum_{i, j=1}^{4}\left[-g\left(\nabla_{J e_{i}}^{L C} \theta^{\sharp}, e_{j}\right) g\left(N_{J}\left(e_{j}, J e_{i}\right), x\right)-g\left(\nabla_{e_{j}}^{L C} \theta^{\sharp}, J e_{i}\right) g\left(N_{J}\left(x, J e_{i}\right), e_{j}\right)\right] .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{i=1}^{4} g\left(N_{J}\left(x, \nabla_{J e_{i}}^{L C} \theta^{\sharp}\right), J e_{i}\right) & =-\frac{1}{2} \sum_{i, j=1}^{4} g\left(\nabla_{J e_{i}}^{L C} \theta^{\sharp}, e_{j}\right) g\left(N_{J}\left(e_{j}, J e_{i}\right), x\right) \\
& =\frac{1}{2} \sum_{i, j=1}^{4} g\left(\nabla_{e_{i}}^{L C} \theta^{\sharp}, e_{j}\right) g\left(N_{J}\left(e_{i}, e_{j}\right), x\right)=\frac{1}{2} g\left(d \theta, N_{x}\right) .
\end{aligned}
$$

From (4.14), we deduce that

$$
\sum_{i=1}^{4} \nabla^{L C} \theta\left(J e_{i},\left(\nabla_{e_{i}}^{L C} J\right) x\right)=\frac{1}{2} g\left(\nabla_{J x}^{L C} \theta, J \theta\right)+\frac{1}{2} g\left(\nabla_{x}^{L C} \theta, \theta\right)+g\left(d \theta, N_{x}\right)
$$

Finally, from (4.13) and (4.14), we conclude that

$$
\begin{aligned}
\rho^{\star}(g)\left(\theta^{\sharp}, x\right) & =\nabla^{L C} \theta\left(J \theta^{\sharp}, J x\right)+\frac{1}{2} g\left(\nabla_{J x}^{L C} \theta, J \theta\right)+\frac{1}{2} g\left(\nabla_{x}^{L C} \theta, \theta\right)+g\left(d \theta, N_{x}\right) \\
& =\frac{1}{2} \nabla^{L C} \theta\left(J \theta^{\sharp}, J x\right)+\frac{1}{2} g\left(\nabla_{x}^{L C} \theta, \theta\right)+g\left(d \theta, N_{x}\right) \\
& =-\frac{1}{2}(d \theta)^{J,-}\left(\theta^{\sharp}, x\right)+g\left(d \theta, N_{x}\right) .
\end{aligned}
$$

Then the result follows.
As consequences of Lemma 4.4.1, we see that if the almost-Hermitian metric $g$ is locally conformally almost-Kähler, then, $\rho^{\star}\left(\theta^{\sharp}, x\right)=0$ for any vector field $x$. Moreover, from Lemma 4.4.1 we also deduce that

$$
\begin{equation*}
\rho^{\star}(g)\left(\theta^{\sharp}, \theta^{\sharp}\right)=g\left(d \theta, N_{\theta^{\sharp}}\right), \tag{4.15}
\end{equation*}
$$

extending Lemma 2 in 143 to the almost-Hermitian setting. Notice that when $J$ is integrable $\rho^{\star}\left(\theta^{\sharp}, \theta^{\sharp}\right)=0$. As a matter of fact, this condition simplifies in a natural way the problem, thus we will now prove our main theorems with the assumption that $d \theta=0$ (Theorem 4.4.1) or the weaker $N_{\theta^{\sharp}}=0$ (Theorem 4.4.2).

### 4.4.1 Second-Chern-Einstein locally conformally almost-Kähler metrics

We start by describing the geometry of the manifolds equipped with a second-Chern-Einstein locally conformally almost-Kähler metric. We have seen that in the Hermitian context, the LCK condition naturally arises by studying the second-Chern-Einstein and Weyl-Einstein problems. Furthermore, it turns out that in the (4-dimensional) almost-Hermitian case, the LCaK condition is a natural substitute for the integrability.
Theorem 4.4.1 (Theorem 15 in [36]). Let ( $M, J, g$ ) be a 4-dimensional compact locally conformally almost-Kähler manifold, and $g$ be a second-Chern-Einstein metric. Suppose that the unit-volume Gauduchon metric $\eta$ in the conformal class $\{g\}$ satisfies $\left(\nabla^{L C} \theta\right)^{\text {sym, } J,-}=0$. Then, either

1. $(M, J, g)$ is a second-Chern-Einstein globally conformally almost-Kähler manifold, or
2. $\theta$ is $\nabla^{L C}$-parallel and the conformal scalar curvature $s^{W}(\eta)$ is non-positive and the $\star$-scalar curvature $s^{\star}(\eta)$ is a positive constant. Moreover, $s^{W}(\eta)$ is identically zero if and only if $J$ is integrable, in which case $(M, J)$ is a Hopf surface. Furthermore, if $s^{W}(\eta)$ is nowhere zero then $\chi=\sigma=0$, where $\chi$ and $\sigma$ are the Euler class and signature of $M$ respectively.

Proof. From Proposition 4.2.1, $\theta^{\sharp}$ is a Killing vector field. Since $d \theta=0$, it follows that $\theta$ is $\nabla^{L C}$-parallel and so $\theta^{\sharp}$ has constant length. Combining (4.12) and (4.15), we obtain

$$
\left(s^{C h}(\eta)+\left\|N_{J}\right\|_{\eta}^{2}-2\|\theta\|_{\eta}^{2}\right)\|\theta\|_{\eta}^{2}=0 .
$$

Hence either

$$
\theta=0, \quad \text { or } \quad s^{C h}(\eta)=2\|\theta\|_{\eta}^{2}-\left\|N_{J}\right\|_{\eta}^{2} .
$$

Now, if the second holds, then

$$
s^{L C}(\eta)=\frac{3}{2}\|\theta\|_{\eta}^{2}-2\left\|N_{J}\right\|_{\eta}^{2} .
$$

Hence, thanks to (1.11), we get that

$$
s^{W}(\eta)=-2\left\|N_{J}\right\|_{\eta}^{2} .
$$

Thus $s^{W}(\eta) \equiv 0$ if and only if $J$ is integrable. Moreover, if $s^{W}(\eta)$ is nowhere zero then $5 \chi+6 \sigma=0$ using [22, Lemma 3]. The existence of a non-trivial Killing vector field of constant length implies $\chi=0$ by Hopf theorem 169 hence $\chi=\sigma=0$. Furthermore, using (1.13) we have that

$$
s^{\star}(\eta)=s^{L C}(\eta)-\|\theta\|_{\eta}^{2}+2\left\|N_{J}\right\|_{\eta}^{2}=\frac{3}{2}\|\theta\|_{\eta}^{2}-2\left\|N_{J}\right\|_{\eta}^{2}-\|\theta\|_{\eta}^{2}+2\left\|N_{J}\right\|_{\eta}^{2}=\frac{1}{2}\|\theta\|_{\eta}^{2},
$$

and so $s^{\star}(\eta)$ is constant.

Remark 4.4.1. We recall that there are many restrictions to the existence of a non-zero Killing vector field of constant length on a Riemannian manifold, see for example [43, 238]. Moreover, when $J$ is integrable, if $\theta$ is $\nabla^{L C}$-parallel (the metric is called Vaisman) then $\theta^{\sharp}$ and $J \theta^{\sharp}$ are both real holomorphic Killing vector fields [108]. However, when $J$ is not integrable that is not necessarily true; see for instance the first example in Section 4.5.

We can then deduce the following result characterizing the almost-Hermitian metrics which are both second-Chern-Einstein and Weyl-Einstein.

Corollary 4.4.1 (Theorem 3 in [140], and Corollary 4.2 in [181]). Let ( $M, J, g$ ) be a 4-dimensional compact locally conformally almost-Kähler manifold. Suppose that the unit-volume Gauduchon metric $\eta \in\{g\}$ is a second-Chern-Einstein and a Weyl-Einstein metric. Then $(M, J, \eta)$ is either an almost-Kähler Riemannian Einstein manifold or a Hopf surface.

Proof. It follows from Theorem 4.4.1 that either $\eta$ is almost-Kähler or $s^{W}$ is non-positive. Moreover, in Theorem 4.4.1 we also proved that if $s^{W}$ is identically zero then the manifold is the Hopf surface. On the other hand, in [140, Théorème 2] (in dimension 3 see also [256]) Gauduchon proved that if the conformal scalar curvature $s^{W}$ of a compact Weyl-Einstein manifold is non-positive but not identically zero then the Gauduchon metric is an almost-Kähler Riemannian Einstein metric.

### 4.4.2 Second-Chern-Einstein metrics with vanishing $N_{\theta^{\sharp}}$

From equation 4.15), we know that $N_{\theta^{\sharp}}=0$ also gives $\rho^{\star}\left(\theta^{\sharp}, \theta^{\sharp}\right)=0$. The vanishing of $N_{\theta^{\sharp}}$ means that $\theta^{\sharp}$ is $g$-orthogonal to $\operatorname{span}\left(N_{J}\right)$, which is the distribution spanned by all the vector fields $N_{J}(x, y)$. Moreover, if $N_{\theta^{\sharp}}=0$ then $N_{J \theta^{\sharp}}=0$. In fact, in real dimension 4, at each point the dimension of $\operatorname{span}\left(N_{J}\right)$ is equal to 0 or 2 [68] (see also 234 for more details). Then, a proof similar to that of Theorem4.4.1 gives the following result.

Theorem 4.4.2 (Theorem 19 in (36]). Let ( $M, J, g$ ) be a 4-dimensional compact almost-Hermitian manifold, and $g$ be a second-Chern-Einstein metric. Suppose that the Gauduchon metric $\eta$ in the conformal class $\{g\}$ satisfies $\left(\nabla^{L C} \theta\right)^{\text {sym, } J,-}=0, \theta^{\sharp}$ has constant length, and $N_{\theta^{\sharp}}=0$. Then, either

1. $(M, J, g)$ is a second-Chern-Einstein globally conformally almost Kähler manifold, or
2. $\theta^{\sharp}$ is a non-zero Killing vector field and the conformal scalar curvature $s^{W}(\eta)$ is non-positive and the $\star$-scalar curvature $s^{\star}(\eta)$ is a positive constant. Moreover, $s^{W}(\eta)$ is identically zero if and only if $J$ is integrable, in which case $(M, J)$ is a Hopf surface. In addition, if $s^{W}(\eta)$ is nowhere zero then $\chi=\sigma=0$.

Now, we would like to investigate the condition $N_{\theta^{\sharp}}=0$ and see if it can be implied by the $J$-invariance of different Ricci forms. We first prove the following preliminary result.

Lemma 4.4.2. Let $(M, J, g)$ be a 4-dimensional almost-Hermitian manifold. Then

$$
\operatorname{Ric}^{C h, 1}(g)=\operatorname{Ric}^{W, J}(g)-d J \theta-(d \theta)^{J,-}(J \cdot, \cdot)-\frac{1}{2} \sum_{i=1}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b} .
$$

In particular

$$
\left(\operatorname{Ric}^{C h, 1}(g)\right)^{J,-}=\left(\operatorname{Ric}^{W, J}(g)\right)^{J,-}-(d J \theta)^{J,-}-(d \theta)^{J,-}(J \cdot, \cdot)
$$

Proof. First of all, we fix a $J$-adapted $g$-orthonormal frame of the tangent bundle $\left\{e_{1}, e_{2}=\right.$ $\left.J e_{1}, e_{3}, e_{4}=J e_{3}\right\}$. Then, from equation (4.5), we have that

$$
\begin{aligned}
& 2 R i c^{C h, 1}(g)(x, y)=\sum_{i=1}^{4} g\left(R_{x, y}^{\nabla} e_{i}, J e_{i}\right) \\
& \qquad=\sum_{i=1}^{4}\left[g\left(R_{x, y}^{W} e_{i}, J e_{i}\right)-\frac{1}{2}(d J \theta)(x, y) g\left(J e_{i}, J e_{i}\right)-\frac{1}{2}(d \theta)(x, y) g\left(e_{i}, J e_{i}\right)\right. \\
& \quad-\frac{1}{2} g\left(\left(\nabla_{x}^{W}\left(\nabla_{y}^{W} J\right)-\nabla_{y}^{W}\left(\nabla_{x}^{W} J\right)-\nabla_{[x, y]}^{W} J\right) J e_{i}, J e_{i}\right) \\
& \left.\quad+\frac{1}{4} g\left(\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right)-\left(\nabla_{y}^{W} J\right)\left(\nabla_{x}^{W} J\right)\right) e_{i}, J e_{i}\right)\right] \\
& = \\
& \sum_{i=1}^{4}\left[g\left(R_{x, y}^{W} e_{i}, J e_{i}\right)+\frac{1}{4} g\left(\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right)-\left(\nabla_{y}^{W} J\right)\left(\nabla_{x}^{W} J\right)\right) e_{i}, J e_{i}\right)\right]-2(d J \theta)(x, y) .
\end{aligned}
$$

Since $\nabla^{W}$ is torsion-free, by using the relation

$$
g\left(R_{x, y}^{W} z, w\right)+g\left(R_{x, y}^{W} w, z\right)=(d \theta)(x, y) g(z, w),
$$

we can deduce the following relation for any vector fields $x, y, z, w$

$$
\begin{aligned}
& 2 g\left(R_{x, y}^{W} z, w\right)=2 g\left(R_{z, w}^{W} x, y\right)-d \theta(z, w) g(x, y)+d \theta(x, y) g(z, w)-d \theta(x, w) g(y, z) \\
&-d \theta(y, z) g(x, w)+d \theta(y, w) g(x, z)+d \theta(x, z) g(y, w) .
\end{aligned}
$$

In particular,

$$
\sum_{i=1}^{4} g\left(R_{e_{i}, J e_{i}}^{W} x, y\right)=\sum_{i=1}^{4} g\left(R_{x, y}^{W} e_{i}, J e_{i}\right)+d \theta(J x, y)+d \theta(x, J y) .
$$

Hence,

$$
\begin{align*}
& 2 R i c^{C h, 1}(g)(x, y)=2 R i c^{W, J}(g)(x, y)-d \theta(J x, y)-d \theta(x, J y)-2(d J \theta)(x, y) \\
&+\frac{1}{4} \sum_{i=1}^{4} g\left(\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right)-\left(\nabla_{y}^{W} J\right)\left(\nabla_{x}^{W} J\right)\right) e_{i}, J e_{i}\right) . \tag{4.16}
\end{align*}
$$

Moreover, using 4.1) we also have

$$
\begin{aligned}
\sum_{i=1}^{4} g\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right) e_{i}, J e_{i}\right) & =-2 \sum_{i=1}^{4} g\left(N_{J}\left(\left(\nabla_{y}^{W} J\right) e_{i}, e_{i}\right), x\right) \\
& =-2 \sum_{i, j=1}^{4} g\left(N_{J}\left(e_{j}, e_{i}\right), x\right) g\left(\left(\nabla_{y}^{W} J\right) e_{i}, e_{j}\right) \\
& =-4 \sum_{i, j=1}^{4} g\left(N_{J}\left(e_{i}, e_{j}\right), x\right) g\left(J N\left(e_{i}, e_{j}\right), y\right) .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \sum_{i=1}^{4} g\left(\left(\left(\nabla_{x}^{W} J\right)\left(\nabla_{y}^{W} J\right)-\left(\nabla_{y}^{W} J\right)\left(\nabla_{x}^{W} J\right)\right) e_{i}, J e_{i}\right) \\
& \quad=-4 \sum_{i, j=1}^{4} g\left(N_{J}\left(e_{i}, e_{j}\right), x\right) g\left(J N_{J}\left(e_{i}, e_{j}\right), y\right)+4 \sum_{i, j=1}^{4} g\left(N_{J}\left(e_{i}, e_{j}\right), y\right) g\left(J N_{J}\left(e_{i}, e_{j}\right), x\right) \\
& \quad=-4 \sum_{i, j=1}^{4}\left(\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b}\right)(x, y) .
\end{aligned}
$$

Finally, from (4.16) we deduce that

$$
\begin{aligned}
\operatorname{Ric}^{C h, 1}(g)(x, y)=\operatorname{Ric} c^{W, J}(g)(x, y)-\frac{1}{2}(d \theta(J x, y) & +d \theta(x, J y))-(d J \theta)(x, y) \\
& -\frac{1}{2} \sum_{i, j=1}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b}(x, y) .
\end{aligned}
$$

Remark 4.4.2. We can compute the $J$-anti-invariant part of $R i c^{W, J}$ and it is given by

$$
\left(\operatorname{Ric}^{W, J}(g)\right)^{J,-}=-\sum_{i=1}^{4} g\left(\left(\nabla_{e_{i}}^{L C} N_{J}\right)(\cdot, \cdot), e_{i}\right)+\frac{3}{2} N_{\theta^{\sharp}} .
$$

Proposition 4.4.1. Let $(M, J, g)$ be a 4-dimensional almost-Hermitian manifold. Suppose that Ric ${ }^{C h, 1}$ and Ric ${ }^{W, J}$ are J-invariant. Then $N_{\theta^{\sharp}}=0$.

Proof. First of all, from Lemma 4.4.2, we obtain that

$$
\begin{equation*}
(d J \theta)^{J,-}=-(d \theta)_{J,,}^{J,-} . \tag{4.17}
\end{equation*}
$$

Then, by applying the Lie derivative $\mathcal{L}_{\theta^{\sharp}}$ to the relation $\omega=g(J \cdot, \cdot)$ and using the Cartan formula we obtain

$$
d J \theta=-\|\theta\|_{g}^{2} \omega+\theta \wedge J \theta+2\left(\nabla^{L C} \theta\right)^{s y m}(J \cdot, \cdot)+g\left(\mathcal{L}_{\theta^{\sharp}} J \cdot, \cdot\right) .
$$

In particular

$$
\begin{equation*}
(d J \theta)^{J,-}=\left(\mathcal{L}_{\theta^{\sharp}} J\right)^{a n t i-s y m} . \tag{4.18}
\end{equation*}
$$

Similarly, we have that

$$
d \theta=-2\left(\nabla^{L C} J \theta\right)^{s y m}(J \cdot, \cdot)-g\left(\mathcal{L}_{J \theta^{\sharp}} J \cdot, \cdot\right),
$$

and thus,

$$
\begin{equation*}
(d \theta)^{J,-}(J \cdot, \cdot)=g\left(J\left(\mathcal{L}_{J \theta^{\sharp}} J\right)^{\text {anti-sym }} \cdot, \cdot\right) \tag{4.19}
\end{equation*}
$$

Now combining (4.17), 4.18), and 4.19 we deduce that

$$
\begin{equation*}
\left(\mathcal{L}_{J \theta^{\sharp}} J\right)^{\text {anti-sym }}=J\left(\mathcal{L}_{\theta^{\sharp}} J\right)^{a n t i-s y m} . \tag{4.20}
\end{equation*}
$$

Moreover, for any almost-Hermitian manifold, we have that

$$
\mathcal{L}_{J \theta^{\sharp}} J-J\left(\mathcal{L}_{\theta^{\sharp}} J\right)=4 N_{J}\left(\theta^{\sharp}, \cdot\right) .
$$

On the other hand, from (4.4) we see that

$$
\left(g\left(\left(N_{J}\left(\theta^{\sharp}, \cdot\right), \cdot\right)\right)^{a n t i-s y m}=-2 N_{\theta^{\sharp}}\right.
$$

so that

$$
g\left(\left(\mathcal{L}_{J \theta^{\sharp}} J\right)^{\text {anti-sym }}-J\left(\mathcal{L}_{\theta^{\sharp}} J\right)^{\text {anti-sym }} \cdot, \cdot\right)=-2 N_{\theta^{\sharp}} .
$$

Finally, from 4.20, we deduce that $N_{\theta^{\sharp}}=0$.

### 4.5 Examples of almost-Hermitian second-Chern-Einstein metrics on compact 4 -manifolds

In this section, we collect some new and explicit examples of almost-Hermitian non-integrable second-Chern-Einstein manifolds. The first three are locally conformally almost-Kähler while the last one is not. Notice that the second and third examples show that for a second-Chern-Einstein almost-Hermitian manifold, the condition $\left(\nabla^{L C} \theta\right)^{s y m, J,-}=0$ is necessary for $\theta^{\sharp}$ to be a Killing vector field. We provide our examples on compact solvmanifolds, and we use the same notation for Lie algebras as 251.

1. Lie algebra $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}$ : the structure of the Lie algebra is

$$
\left[e_{1}, e_{3}\right]=-e_{2}, \quad\left[e_{2}, e_{3}\right]=e_{1}
$$

where $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ is a basis of $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}$. The simply connected group associated to the Lie algebra $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}$ admits lattices, see for example $10,13,53$ (in the notation of 10 $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}$ corresponds to $\left.\mathfrak{r}_{3,0}^{\prime} \times \mathbb{R}\right)$. We consider the almost-complex structure

$$
J e_{1}=e_{3}, \quad J e_{2}=e_{4}
$$

The almost-complex structure $J$ is non-integrable because $N_{J}\left(e_{1}, e_{2}\right)=\frac{1}{4} e_{3}$.
We consider the following $J$-compatible metric $g$

$$
g=(\sqrt{5}-1)\left(e^{1} \otimes e^{1}+e^{3} \otimes e^{3}\right)+e^{2} \otimes e^{2}+e^{4} \otimes e^{4}
$$

where $\left\{e^{1}, e^{2}, e^{3}, e^{4}\right\}$ is the dual basis. Then the pair $(J, g)$ induces the fundamental form

$$
\omega=(\sqrt{5}-1) e^{13}+e^{24}
$$

where $e^{13}=e^{1} \wedge e^{3}$ etc. It holds $d \omega=e^{134}$. Moreover, the Lee form is given by

$$
\theta=\frac{1}{(\sqrt{5}-1)} e^{4}
$$

Hence $d \theta=0$. Moreover, $\left(\nabla^{L C} \theta\right)^{s y m, J,-}=0$. On the other hand, the second Chern-Ricci form is given by

$$
R i c^{C h, 2}(\omega)=\frac{1}{4} e^{13}+\frac{1}{4(\sqrt{5}-1)} e^{24}
$$

so the metric $g$ is a second-Chern-Einstein metric with a positive Hermitian scalar curvature $s^{C h}(g)=\frac{1}{\sqrt{5}-1}$. Thus, $\theta$ is $\nabla^{L C}$-parallel and $\mathcal{L}_{\theta^{\sharp}} J=0$ but $\mathcal{L}_{J \theta^{\sharp}} J \neq 0$. We also remark that $N_{\theta^{\sharp}}=0$ and the first Chern-Ricci form $\operatorname{Ric}{ }^{C h, 1}(\omega)=\frac{1}{2} e^{13}$ is $J$-invariant.
2. Lie algebra $\mathcal{A}_{4,1}$ : the structure of the Lie algebra is

$$
\left[e_{2}, e_{4}\right]=e_{1}, \quad\left[e_{3}, e_{4}\right]=e_{2}
$$

where $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ is a basis of $\mathcal{A}_{4,1}$. The simply connected group associated to the Lie algebra $\mathcal{A}_{4,1}$ admits lattices, see for example $\sqrt{10}, 13,53$ (in the notation of $\sqrt{10} \mathcal{A}_{4,1}$ corresponds to $\mathfrak{n}_{4}$ ).

We consider the almost-complex structure

$$
J e_{1}=e_{3}, \quad J e_{2}=e_{4}
$$

which is non-integrable because $N_{J}\left(e_{1}, e_{2}\right)=\frac{1}{4} e_{2}$. We also consider the following $J$ compatible metric $g$

$$
g=\frac{1}{2}\left(e^{1} \otimes e^{1}+e^{3} \otimes e^{3}\right)+e^{2} \otimes e^{2}+e^{4} \otimes e^{4}
$$

where $\left\{e^{1}, e^{2}, e^{3}, e^{4}\right\}$ is the dual basis. Thus the pair $(J, g)$ induces the fundamental form

$$
\omega=\frac{1}{2} e^{13}+e^{24}
$$

and it holds $d \omega=\frac{1}{2} e^{234}$. Moreover, the Lee form is given by

$$
\theta=-\frac{1}{2} e^{3}
$$

Hence $d \theta=0$. However in this example, $\left(\nabla^{L C} \theta\right)^{s y m, J,-}$ does not vanish and $\theta^{\sharp}$ is not a Killing vector field. Explicitly,

$$
\left(\nabla^{L C} \theta\right)^{s y m, J,-}=\frac{1}{2}\left(e^{2} \otimes e^{4}+e^{4} \otimes e^{2}\right)
$$

On the other hand, the second Chern-Ricci form vanishes, so the metric $g$ is a second-Chern-Einstein metric with vanishing Hermitian scalar curvature. We also remark that $N_{\theta \sharp}=0$ and the first Chern-Ricci form vanishes.
3. Lie algebra $\mathcal{A}_{4,8}$ : the structure of the Lie algebra is

$$
\left[e_{2}, e_{3}\right]=e_{1}, \quad\left[e_{2}, e_{4}\right]=e_{2}, \quad\left[e_{3}, e_{4}\right]=-e_{3}
$$

where $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ is a basis of $\mathcal{A}_{4,8}$. The simply connected group associated to the Lie algebra $\mathcal{A}_{4,8}$ admits lattices, see for example [13, 53] (in the notation of $13 \mathcal{A}_{4,8}$ corresponds to $\mathfrak{d}_{4}$ ). We consider the almost-complex structure

$$
J e_{1}=e_{4}, \quad J e_{2}=e_{3}
$$

which is non-integrable because $N_{J}\left(e_{1}, e_{2}\right)=\frac{1}{2} e_{3}$. We also consider the following $J_{-}$ compatible metric $g$

$$
g=\sum_{i=1}^{4} e^{i} \otimes e^{i}
$$

where $\left\{e^{1}, e^{2}, e^{3}, e^{4}\right\}$ is the dual basis. Then the pair $(J, g)$ induces the fundamental form

$$
\omega=e^{14}+e^{23}
$$

which satisfies $d \omega=-e^{234}$. Moreover, the Lee form is given by

$$
\theta=-e^{4}
$$

Hence $d \theta=0$. However in this example, $\left(\nabla^{L C} \theta\right)^{s y m, J,-}$ does not vanish and $\theta^{\sharp}$ is not a Killing vector field. Explicitly,

$$
\left(\nabla^{L C} \theta\right)^{s y m, J,-}=e^{3} \otimes e^{3}-e^{2} \otimes e^{2}
$$

On the other hand, the second Chern-Ricci form vanishes, so the metric $g$ is a second-Chern-Einstein metric with vanishing Hermitian scalar curvature. We also remark that $N_{\theta^{\sharp}}=0$ and the first Chern-Ricci form vanishes.
4. Lie algebra $\mathcal{A}_{4,10}$ : the structure of the Lie algebra is

$$
\left[e_{2}, e_{3}\right]=e_{1}, \quad\left[e_{2}, e_{4}\right]=-e_{3}, \quad\left[e_{3}, e_{4}\right]=e_{2}
$$

where $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ is a basis of $\mathcal{A}_{4,10}$. The simply connected group associated to the Lie algebra $\mathcal{A}_{4,8}$ admits lattices, see for example 13 , 53 (in the notation of [13] $\mathcal{A}_{4,10}$ corresponds to $\mathfrak{d}_{4,0}^{\prime}$ ). We consider the almost-complex structure

$$
J e_{1}=e_{3}, \quad J e_{2}=e_{4}
$$

which is non-integrable because $N_{J}\left(e_{1}, e_{2}\right)=\frac{1}{4} e_{2}+\frac{1}{4} e_{3}$. We also consider the following $J$-compatible metric $g$

$$
g=\frac{1+\sqrt{17}}{8}\left(e^{1} \otimes e^{1}+e^{3} \otimes e^{3}\right)+\left(e^{2} \otimes e^{2}+e^{4} \otimes e^{4}\right)
$$

where $\left\{e^{1}, e^{2}, e^{3}, e^{4}\right\}$ is the dual basis. Then the pair $(J, g)$ induces the fundamental form

$$
\omega=\frac{1+\sqrt{17}}{8} e^{13}+e^{24}
$$

which satisfies $d \omega=-\frac{1+\sqrt{17}}{8} e^{124}$. Moreover, the Lee form is given by

$$
\theta=-\frac{1+\sqrt{17}}{8} e^{1}
$$

Hence $d \theta \neq 0$. In this example, $\left(\nabla^{L C} \theta\right)^{\text {sym,J,- }}$ vanishes. On the other hand, the second Chern-Ricci form is

$$
R i c^{C h, 2}(\omega)=\frac{1+\sqrt{17}}{32} e^{13}+\frac{1}{4} e^{24}
$$

so the metric $g$ is a second-Chern-Einstein metric with positive Hermitian scalar curvature $s^{C h}(g)=1$. Henceforth, $\theta^{\sharp}$ is a Killing vector field but not $\nabla^{L C}$-parallel. We also remark that in this example $N_{\theta^{\sharp}} \neq 0$ and the first Chern-Ricci form Ric $^{C h}, 1(\omega)=\frac{1}{2} e^{24}-\frac{1}{2} e^{34}$ is not $J$-invariant.

### 4.6 Second-Chern-Einstein metrics on almost-Abelian Lie algebras

An almost-Abelian Lie group $G$ is a Lie group whose Lie algebra $\mathfrak{g}$ has a codimension-one Abelian ideal $\mathfrak{n} \subset \mathfrak{g}$. Since the Lie algebra is almost entirely commutative, the geometry of these spaces can be encoded in a few parameters. Henceforth, they represent a perfect environment where to produce explicit examples.

In this section, we classify the second-Chern-Einstein metrics on four-dimensional almostAbelian Lie groups equipped with left-invariant almost-Hermitian structures. We do it by explicitly computing the second Chern-Ricci tensor in terms of the structure constant characterizing the almost-Abelian Lie algebras. The first step is to relate the second Chern-Ricci tensor and the Bismut-Ricci curvature tensor. Then an explicit formulation can be deduced using the results in 317 .

### 4.6.1 Second Chern-Ricci and Bismut-Ricci tensors

Thanks to Theorem 4.1.1 and Lemma 4.4 .2 both the first Chern-Ricci form Ric ${ }^{C h, 1}$ and the second Chern-Ricci form Ric ${ }^{C h, 2}$ can be expressed in terms of Ric ${ }^{W, J}$. Hence, we get the following result.

Proposition 4.6.1 (Proposition 24 in $[36])$. Let $(M, J, g)$ be a 4-dimensional almost-Hermitian manifold. Then,

$$
\begin{aligned}
R i c^{C h, 2}(\omega)=\left(R i c^{C h, 1}(\omega)+d J \theta\right)^{J,+}+ & \frac{1}{4}\left(2 d^{*} \theta+2\|\theta\|_{g}^{2}-\left\|N_{J}\right\|_{g}^{2}\right) \omega \\
& +\frac{1}{2} \sum_{i, j}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b} .
\end{aligned}
$$

In particular, thanks to 1.8),

$$
\begin{aligned}
& R i c^{C h, 2}(\omega)=\left(R i c^{B}(\omega)\right)^{J,+}+\frac{1}{4}\left(2 d^{*} \theta+2\|\theta\|_{g}^{2}-\left\|N_{J}\right\|_{g}^{2}\right) \omega \\
&+\frac{1}{2} \sum_{i, j}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b} .
\end{aligned}
$$

Therefore, $g$ is second-Chern-Einstein if and only if

$$
\left(\operatorname{Ric}^{B}(\omega)\right)^{J,+}+\frac{1}{2} \sum_{i, j}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b}
$$

is proportional to $\omega$.
From Lemma 4.4 .2 and 1.8 it also follows that on a 4 -dimensional Hermitian manifold

$$
R i c^{W, J}=\left(R i c^{B}\right)^{J,+} .
$$

Therefore, on a complex surface, a Hermitian metric $\omega$ is second-Chern-Einstein if and only if the $J$-invariant part of the Bismut-Ricci form is proportional to the metric, that is

$$
\left(\operatorname{Ric}^{B}(\omega)\right)^{1,1}=\lambda \omega, \quad \text { for some } \lambda \in \mathcal{C}^{\infty}(M ; \mathbb{R}) .
$$

Remark 4.6.1. One can check that the crucial property of 4-dimensional manifolds that leads to these relations between the second Chern-Ricci form, the first Bismut-Ricci form, and Ric ${ }^{W, J}$ is $d \omega=\theta \wedge \omega$. As a consequence, in higher dimensions $2 n>4$, if we assume the Hermitian structure to be locally conformally Kähler we obtain similar relations (see [6, Lemma 4.4]). For example, computations analogous to the one we did here can show that on an LCK manifold

$$
R i c^{W, J}=R i c^{\frac{1}{1-n}} .
$$

### 4.6.2 Almost-Abelian Lie groups

Given an almost-Hermitian left-invariant structure ( $J, g$ ) on a $2 n$-dimensional almost-Abelian Lie group $G$, define

$$
\mathfrak{n}_{1}:=\mathfrak{n} \cap J \mathfrak{n}, \quad \text { and } \quad J_{1}:=J_{\mid \mathfrak{n}_{1}} .
$$

Then we can choose an orthonormal basis $\left\{e_{1}, \ldots, e_{2 n}\right\}$ for $\mathfrak{g}$ such that

$$
\mathfrak{n}=\operatorname{span}_{\mathbb{R}}\left\langle e_{1}, \ldots, e_{2 n-1}\right\rangle, \quad \text { and } \quad J e_{i}=e_{2 n-i+1} \text { for } i=1, \ldots, n .
$$

Hence, the fundamental form $\omega(\cdot, \cdot):=g(J \cdot, \cdot)$ associated to the almost-Hermitian structure $(J, g)$ is

$$
\omega=e^{1} \wedge e^{2 n}+e^{2} \wedge e^{2 n-1}+\cdots+e^{n} \wedge e^{n+1}
$$

given in terms of the dual left-invariant frame $\left\{e^{1}, \ldots, e^{2 n}\right\}$.
Since $\mathfrak{n}$ has codimension one, the algebra structure of $\mathfrak{g}$ is completely described by the adjoint map

$$
\operatorname{ad}_{e_{2 n}}: \mathfrak{g} \rightarrow \mathfrak{g}: x \mapsto\left[e_{2 n}, x\right] .
$$

This restricts to an endomorphism of $\mathfrak{n}$ whose associated matrix is

$$
\left.\operatorname{ad}_{\left.e_{2 n}\right|_{\mathfrak{n}}}=\left(\begin{array}{cc}
a & b  \tag{4.21}\\
v & A
\end{array}\right), \quad a \in \mathbb{R}, b, v \in \mathfrak{n}_{1}, A \in \mathfrak{g l (} \mathfrak{n}_{1}\right)
$$

The data $(a, b, v, A)$ completely characterizes the almost-Hermitian structure $(J, g)$. For example, the integrability of $J$ can be expressed in terms of ( $a, b, v, A$ ) asking that $b=0$ and $A \in$ $\mathfrak{g l}\left(n_{1}, J_{1}\right)$, where $\mathfrak{g l}\left(n_{1}, J_{1}\right)$ denotes endomorphisms of $\mathfrak{n}_{1}$ commuting with $J_{1}$, see 23, Lemma 4.1]. Henceforth, from now on, we indicate with the quadruple $\left(G,[\cdot, \cdot]_{(a, b, v, A)}, J, g\right)$ an almostAbelian almost-Hermitian Lie group. The Lee form of $\left(G,[\cdot, \cdot]_{(a, b, v, A)}, J, g\right)$ is given by

$$
\theta=J d^{*} \omega=(J v)^{b}-(\operatorname{tr} A) e^{2 n}
$$

with respect to the adapted orthonormal basis $\left\{e_{1}, \ldots, e_{2 n}\right\}$, see for example [118, Lemma 2.1].
The first Ricci form of the canonical connections on a Lie group ( $G, \mathfrak{g}$ ) equipped with an almost-Hermitian structure ( $J, g$ ) were computed in 317 . In particular, for any parameter $t \in \mathbb{R}$ these are given by

$$
\operatorname{Ric}^{t}(x, y)=-\frac{1}{2}\left\{\operatorname{tr}\left(\operatorname{ad}_{[x, y]} \circ J\right)-t \operatorname{tr} \operatorname{ad}_{J[x, y]}+(t-1) g\left(\omega, d[x, y]^{b}\right)\right\}
$$

Then, a direct computation leads to the following result.
Proposition 4.6.2 (Lemma 26 in 36 ). Let $\left(G,[\cdot, \cdot]_{(a, b, v, A)}, J, g\right)$ be an almost-Abelian almostHermitian Lie group, endowed with an adapted unitary basis $\left\{e_{1}, \ldots, e_{2 n}\right\}$, determining the algebraic data $(a, b, v, A)$ by 4.21). Then, the first Ricci form associated to the canonical connection $\nabla^{t}$ is

$$
\operatorname{Ric}^{t}=\frac{1}{2}\left\{\left((t-1)|v|^{2}-2 a^{2}-t a \operatorname{tr} A-b \cdot v\right) e^{1}+\left(A^{t}((t-1) v-b)-(2 a+t \operatorname{tr} A) b\right)^{b}\right\} \wedge e^{2 n}
$$

In particular,

$$
R i c^{B}=-\frac{1}{2}\left\{\left(2 a^{2}-a \operatorname{tr} A+2|v|^{2}+b \cdot v\right) e^{1} \wedge e^{2 n}+\left((2 a-\operatorname{tr} A) b+A^{t} b+2 A^{t} v\right)^{b} \wedge e^{2 n}\right\} .
$$

### 4.6.3 Classification of second-Chern-Einstein almost-Abelian Lie algebras

Now we can classify the second-Chern-Einstein metrics on compact almost-Abelian almostHermitian Lie groups $\left(G,[\cdot, \cdot]_{(a, b, v, A)}, J, g\right)$ of real dimension 4. Thanks to Proposition 4.6.1, in the integrable case, the second-Chern-Einstein problem reduces to an Einstein-type equation for the Bismut connection. On the other hand, in the non-integrable case, a factor depending on the Nijenhuis tensor pops up:

$$
\begin{equation*}
\frac{1}{2} \sum_{i, j}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b} . \tag{4.22}
\end{equation*}
$$

We choose an adapted unitary basis $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ for $\mathfrak{g}$, determining the algebraic data $(a, b, v, A)$ by (4.21). Then (4.22) can be written as

$$
\begin{align*}
& \frac{1}{2} \sum_{i, j}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b}=|b|^{2} e^{1} \wedge e^{4}+\left(\left(A_{1}^{2}+A_{2}^{1}\right)^{2}+\left(A_{1}^{1}-A_{2}^{2}\right)^{2}\right) e^{2} \wedge e^{3} \\
& \quad+\left(b_{2}\left(A_{1}^{1}-A_{2}^{2}\right)-b_{1}\left(A_{2}^{1}+A_{1}^{2}\right)\right) e^{1} \wedge e^{2}+\left(b_{2}\left(A_{1}^{2}+A_{2}^{1}\right)+b_{1}\left(A_{1}^{1}-A_{2}^{2}\right)\right) e^{1} \wedge e^{3} \\
& \quad+\left(b_{1}\left(A_{1}^{1}-A_{2}^{2}\right)+b_{2}\left(A_{2}^{1}+A_{1}^{2}\right)\right) e^{2} \wedge e^{4}+\left(b_{1}\left(A_{1}^{2}+A_{2}^{1}\right)-b_{2}\left(A_{1}^{1}-A_{2}^{2}\right)\right) e^{3} \wedge e^{4}, \tag{4.23}
\end{align*}
$$

where $b=\left(b_{1}, b_{2}\right)$ and $A_{j}^{i}$ is the $(i, j)$-th element of $A$.
Theorem 4.6.1 (Theorem 32 in $[36]$ ). Let $\mathfrak{g}$ be a 4-dimensional unimodular almost-Abelian Lie algebra equipped with a left-invariant almost-Hermitian non-Hermitian structure ( $J, g$ ) such that the Lee form $\theta$ is $\nabla^{L C}$-parallel and non-zero. Suppose that $(J, g)$ is a solution to the second-Chern-Einstein problem. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras

1. $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}:\left[e_{1}, e_{3}\right]=-e_{2}, \quad\left[e_{2}, e_{3}\right]=e_{1}$;
2. $\mathcal{A}_{3,4} \oplus \mathcal{A}_{1}:\left[e_{1}, e_{3}\right]=e_{1}, \quad\left[e_{2}, e_{3}\right]=-e_{2}$;
which, in the notation of [10], correspond respectively to $\mathfrak{r}_{3,0}^{\prime} \times \mathbb{R}$, and $\mathfrak{r}_{3,-1} \times \mathbb{R}$. Both the Lie algebras admit compact quotients.
Proof. First of all, we choose an adapted unitary basis $\left\{e_{1}, e_{2}, e_{3}, e_{4}\right\}$ for $\mathfrak{g}$, determining the algebraic data $(a, b, v, A)$ by (4.21). By hypothesis, the Lie algebra $\mathfrak{g}$ is unimodular, hence it holds $a=-\operatorname{tr} A$. Thanks to proposition 4.6.1 we have to verify that

$$
\left(\operatorname{Ric}^{B}(\omega)\right)^{J,+}+\frac{1}{2} \sum_{i, j}^{4}\left(N_{J}\left(e_{i}, e_{j}\right)\right)^{b} \wedge\left(J N_{J}\left(e_{i}, e_{j}\right)\right)^{b}
$$

is proportional to $\omega$. We computed these terms on a generic almost-Hermitian almost-Abelian Lie group $\left(G,[\cdot, \cdot]_{(a, b, v, A)}, J, g\right)$ of real dimension 4 in Proposition 4.6.2 and equation 4.23). In particular, we get that $g$ is second-Chern-Einstein if and only if the structure constants $(a, b, v, A)$ satisfy the following system of equations:

$$
\left\{\begin{array}{l}
2|b|^{2}-3 a^{2}-b \cdot v-2|v|^{2}=2\left(A_{1}^{1}-A_{2}^{2}\right)^{2}+2\left(A_{2}^{1}+A_{1}^{2}\right)^{2}, \\
3 a b_{1}+A_{1}^{1} b_{1}+A_{1}^{2} b_{2}+2 A_{1}^{1} v_{1}+2 A_{1}^{2} v_{2}=4 b_{1}\left(A_{1}^{1}-A_{2}^{2}\right)+4 b_{2}\left(A_{2}^{1}+A_{1}^{2}\right), \\
3 a b_{2}+A_{2}^{1} b_{1}+A_{2}^{2} b_{2}+2 A_{2}^{1} v_{1}+2 A_{2}^{2} v_{2}=4 b_{1}\left(A_{2}^{1}+A_{1}^{2}\right)-4 b_{2}\left(A_{1}^{1}-A_{2}^{2}\right) .
\end{array}\right.
$$

Since the Lee form is given by $\theta=v_{1} e^{3}-v_{2} e^{2}+a e^{4}$, the condition $\nabla^{L C} \theta=0$ implies that

$$
\begin{aligned}
a & =0, \\
b_{1} v_{2} & =b_{2} v_{1}, \\
v_{1}\left(A_{2}^{1}-A_{1}^{2}\right) & =0, \\
v_{2}\left(A_{2}^{1}-A_{1}^{2}\right) & =0, \\
v_{1}\left(A_{2}^{1}+A_{1}^{2}\right) & =2 A_{1}^{1} v_{2}, \\
v_{2}\left(A_{2}^{1}+A_{1}^{2}\right) & =2 A_{2}^{2} v_{1} .
\end{aligned}
$$

Now, suppose that $v_{1} \neq 0$. Then the above equations imply that $A_{1}^{1}=A_{2}^{2}=A_{2}^{1}=A_{1}^{2}=0$ and

$$
\left\{\begin{array}{l}
2 b_{1}^{2}-b_{1} v_{1}-2 v_{1}^{2}=0 \\
2 b_{2}^{2}-b_{2} v_{2}-2 v_{2}^{2}=0
\end{array}\right.
$$

We remark that $b \cdot v \neq 0$. Moreover, the isomorphism classes of almost-Abelian Lie algebras can be described using Jordan forms of $\operatorname{ad}_{e_{2 n} \mid \mathfrak{n}}$ up to scaling (see [10, Lemma 2.1] and $[9]$ ). We thus have two cases:

1. $b \cdot v>0$ : the canonical Jordan form of $\operatorname{ad}_{e_{4} \mid n}$ up to scaling is $\left(\begin{array}{llc}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right)$, which corresponds to $\mathcal{A}_{3,4} \oplus \mathcal{A}_{1}$.
2. $b \cdot v<0$ : the canonical Jordan form of $\operatorname{ad}_{e_{4} \mid \mathfrak{n}}$ up to scaling is $\left(\begin{array}{ccc}0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0\end{array}\right)$, which corresponds to $\mathcal{A}_{3,6} \oplus \mathcal{A}_{1}$.

On the other hand, if $v_{1}=0$, then $\theta=v_{2} e^{2} \neq 0$ implies that $v_{2} \neq 0$. We can then deduce that $b_{1}=A_{1}^{1}=A_{2}^{2}=A_{2}^{1}=A_{1}^{2}=0$. Since $J$ is not integrable, $b_{2}$ must vanish. Therefore, $2 b_{2}^{2}-b_{2} v_{2}-2 v_{2}^{2}=0$, with $b_{2} v_{2} \neq 0$. We then obtain the same canonical Jordan forms as above.

Finally, notice that these Lie algebras admit compact quotients since [13, Proposition 5.1].

## Chapter 5

## The Ricci curvature of the Bismut connection

In this chapter, we study Calabi-Yau with torsion metrics (CYT in short), which are Hermitian metrics $g$ with vanishing Bismut-Ricci form,

$$
\operatorname{Ric}^{B}(g) \equiv 0 .
$$

Since the peculiar geometry of these manifolds, we observe, in Section 5.1, that explicit examples may be naturally constructed on toric bundles over Hermitian manifolds. Therefore, in Section 5.2, by emulating the computations in [151, we describe the CYT condition for metrics of submersion type on the total spaces of $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over Hermitian manifolds. In Proposition 5.2.1, we characterize the submersion metrics which give a CYT structure in terms of the metric on the base-space and the characteristic class of the principal bundle. From this analysis, in Section 5.3, we derive results of existence and uniqueness of CYT structures over class $\mathcal{C}$ manifolds in the sense of [263]. These are constructed as the total spaces of homogeneous $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over the product of two compact irreducible Hermitian symmetric spaces. As a particular case, we apply these results to the Calabi-Eckmann manifolds, showing that their standard homogeneous metrics are the unique (among the invariant ones) Calabi-Yau with torsion metrics.

The original results of this chapter have been obtained in 34 .

### 5.1 Calabi-Yau with torsion manifolds

For an $n$-dimensional CYT manifold, the restricted holonomy of the Bismut connection satisfies $131,145,210,288$ (see also 260 for a survey)

$$
\operatorname{Hol}_{\nabla^{B}} \subset \mathrm{SU}(n),
$$

while in general $\operatorname{Hol}_{\nabla^{B}} \subset \mathrm{U}(n)$ since $\nabla^{B}$ is a Hermitian connection. Hence the CYT geometry is related to Yau's Problem 87 in 333 on compact Hermitian manifolds with holonomy reduced to a subgroup of $\mathrm{U}(n)$. Furthermore, thanks to (1.8), CYT manifolds have vanishing first Chern class, which means that $\nabla^{B}$ gives rise to a flat unitary connection on the canonical line bundle. We now show that given a complex manifold, it is possible to construct a toric bundle over it whose total space has vanishing first Chern class. Henceforth, toric bundles over Hermitian manifolds are natural spaces where looking for CYT structures.

Let $(X, J)$ be a complex manifold. Given a lattice $\Lambda \subset \mathbb{C}$, denote by $\mathbb{T}:=\mathbb{C} / \Lambda \cong \mathbb{S}^{1} \times \mathbb{S}^{1}$ the complex torus. Let $(X, J)$ be a complex manifold, then the local sections of the constant sheaves

$$
0 \rightarrow \Lambda \rightarrow \mathbb{C} \rightarrow \mathbb{T} \rightarrow 0
$$

give an exact sequence of sheaves on $X$

$$
0 \rightarrow \Lambda \rightarrow \mathcal{O}_{X} \rightarrow \mathcal{O}_{X}(\mathbb{T}) \rightarrow 0
$$

Notice that $H^{1}\left(\mathcal{O}_{X}(\mathbb{T})\right)$ is the space of $\mathbb{T}$-principal bundles over $X$, and the above exact sequence gives a boundary map in cohomology

$$
c: H^{1}\left(\mathcal{O}_{X}(\mathbb{T})\right) \rightarrow H^{2}(X, \Lambda) \cong H^{2}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda .
$$

Given a $\mathbb{T}$-principal bundle $M \xrightarrow{\pi} X, c(M)$ is its characteristic class. If the characteristic class is of type $(1,1)$ then $M$ inherits an integrable complex structure so that the projection map $\pi: M \rightarrow X$ is holomorphic, see [151, Lemma 1] for details. Then we get the following exact sequence of holomorphic vector bundles

$$
0 \rightarrow \mathbb{C} \rightarrow T^{1,0} M \rightarrow \pi^{*} T^{1,0} X \rightarrow 0,
$$

where the first non-zero term represents the trivial holomorphic bundle on $M$ with fiber $\mathbb{C}$. It follows that the pull-back map $\pi^{*}$ induces a holomorphic isomorphism between the canonical bundles $K_{M}$ and $K_{X}$, i. e. $\pi^{*} K_{X}=K_{M}$. Consequently, it also induces equality at the level of Chern classes, $c_{1}(M)=\pi^{*}\left(c_{1}(X)\right)$. Furthermore, the exact sequence induced by the fiber bundle structure holds in integral cohomology [168, Proposition 5.6]

$$
\cdots \rightarrow H^{1}(\mathbb{T}, \mathbb{Z}) \xrightarrow{\delta} H^{2}(X, \mathbb{Z}) \xrightarrow{\pi^{*}} H^{2}(M, \mathbb{Z}) \rightarrow \cdots .
$$

Hence, $c_{1}(M)=0$ if and only if $c_{1}(X) \in \operatorname{Im} \delta$. Finally, after the identification $H^{1}(\mathbb{T}, \mathbb{Z}) \cong$ $\operatorname{Hom}(\Lambda, \mathbb{Z})$, the map $\delta$ can be seen as an element in $H^{2}(X, \Lambda)$, and it actually coincides with $c(M)$ [168, Theorem 6.1]. To summarize, given a complex manifold $(X, J)$, there is a standard procedure to construct a $\mathbb{T}$-principal bundle $M \rightarrow X$ whose total space satisfies $c_{1}(M)=0$.

Proposition 5.1.1. Let $(X, J)$ be a complex manifold. Consider two forms $\omega_{i} \in H^{1,1}(X, \mathbb{Z})$ such that

$$
\lambda_{1}\left[\omega_{1}\right]+\lambda_{2}\left[\omega_{2}\right]=c_{1}(X), \quad \text { for } \lambda_{i} \in \mathbb{Z} .
$$

Then the total space of the $\mathbb{T}$-principal bundle $M \xrightarrow{\pi} X$ with characteristic class given by the pair $\left(\omega_{1}, \omega_{2}\right)$ inherits a complex structure such that the projection map $\pi$ is holomorphic and $c_{1}(M)=0$.

The Hopf manifolds and in general the Calabi-Eckmann manifolds are $\mathbb{T}$-principal bundles over the product of complex projective spaces. Since Kähler-Ricci-flat metrics, as well as Bismut flat metrics, are trivial examples of Calabi-Yau with torsion metrics, then the standard Hermitian structures on the torus $\mathbb{S}^{1} \times \mathbb{S}^{1}$, the Hopf surface $\mathbb{S}^{1} \times \mathbb{S}^{3}$ and the Calabi-Eckmann threefold $\mathbb{S}^{3} \times \mathbb{S}^{3}$ are actually CYT, see Sections 2.2 and 2.3 . Moreover, thanks to equation (2.7) the metrics $g\left(1, \frac{2-n}{2 n-2}\right)$ (and their multiples) are the unique homogeneous metrics on the $n$-dimensional Hopf manifolds that have vanishing Bismut-Ricci form. Therefore, these give explicit homogeneous examples of CYT metrics on all the Hopf manifolds. Moreover, in the following section, we will explicitly construct CYT metrics on all the Calabi-Eckmann manifolds, which will be unique among the homogeneous ones. As a matter of fact, Theorem 3 in 152 ensures that any compact homogeneous manifold with vanishing first Chern class admits a CYT Hermitian structure. In details,

Theorem 5.1.1 (Theorem 3 in $\sqrt{152]}$ ). Let $G$ be a compact Lie group and $H$ a closed subgroup, such that the homogeneous space $M=G / H$ admits a $G$-invariant complex structure $J_{0}$ with vanishing first Chern class. Then there is a 1-parameter family of invariant complex structures which connects $J_{0}$ and a complex structure $J_{1}$ admitting a compatible CYT metric.

If the compact Lie group $G$ is even dimensional, it can be equipped with a left-invariant complex structure such that the Killing metric on $G$ is Bismut flat, see Chapter 6. In general, by taking the quotient with $H$, the restriction of the Killing metric is not Bismut flat anymore but the theorem above shows that it still has a vanishing Bismut-Ricci form. It was conjectured 157 , Conjecture 1] that any compact complex manifold with vanishing first Chern class should admit a Hermitian CYT metric. However, in [115] the authors rejected this conjecture by providing counterexamples on compact quotients of nilpotent Lie groups. Finally, non-homogeneous examples of Calabi-Yau with torsion manifolds were produced in [151] where the authors studied $\mathbb{T}$-principal bundles over the blown-up $\mathbb{C P}^{2}$. They showed that the threefolds resulting as total spaces are diffeomorphic to $k\left(\mathbb{S}^{2} \times \mathbb{S}^{4}\right) \#(k+1)\left(\mathbb{S}^{3} \times \mathbb{S}^{3}\right)$ for $k \in \mathbb{N}$ and they can be equipped with both a CYT metric and a (possibly different) pluriclosed metric (which coincide when $k=0)$.

### 5.2 Calabi-Yau with torsion metrics on toric bundles over Hermitian manifolds

We have seen that toric bundles over complex manifolds represent a natural environment where to construct examples of Calabi-Yau with torsion metrics. Henceforth, in this section, we describe the CYT condition on the total spaces of $\mathbb{T}$-principal bundles over Hermitian manifolds equipped with metrics of submersion type.

Given a Hermitian manifold $\left(X, J, \omega_{X}\right)$, consider a principal toric bundle

$$
\mathbb{S}^{1} \times \mathbb{S}^{1} \hookrightarrow M \xrightarrow{\pi} X
$$

with characteristic classes of type $(1,1)$. It has a connection one-form with values in the Lie algebra of $\mathbb{S}^{1} \times \mathbb{S}^{1}$ given by $\left(\theta_{1}, \theta_{2}\right)$, and such that $d \theta_{i}=\pi^{*} \omega_{i}$, with $\omega_{i}(1,1)$-forms on $X$. Thanks to Proposition 5.1.1, $c_{1}(M)=0$ if and only if there exist constants $\lambda_{1}, \lambda_{2} \in \mathbb{Z}$ such that

$$
\lambda_{1}\left[\omega_{1}\right]+\lambda_{2}\left[\omega_{2}\right]=c_{1}(X)=\left[\frac{\mathrm{i}}{2 \pi} \operatorname{Ric}^{C h}\left(\omega_{X}\right)\right]
$$

in de Rham cohomology. We then consider the Hermitian metrics on $M$ for which $\pi$ becomes a Riemannian submersion. These are all of the following form:

$$
\omega=\pi^{*}\left(\omega_{X}\right)+f \theta_{1} \wedge \theta_{2} .
$$

Here, $f$ is a positive function on $M$, which is constant along the fibers. Namely, $f=\pi^{*} f^{\prime}$ for some positive function $f^{\prime}$ on $X$ (for the sake of simplicity, we will always identify $f$ and $f^{\prime}$ ). The Bismut-Ricci form of $\omega$ is given by

$$
\begin{equation*}
\operatorname{Ric}^{B}(\omega)=\pi^{*}\left(\operatorname{Ric}^{B}\left(\omega_{X}\right)\right)-d d^{*}\left(f \theta_{1} \wedge \theta_{2}\right) . \tag{5.1}
\end{equation*}
$$

This follows from the relation between the Bismut-Ricci form and the Chern-Ricci form given in 1.8), indeed

$$
\begin{aligned}
\operatorname{Ric}^{B}(\omega) & =\operatorname{Ric}^{C h}(\omega)-d d^{*} \omega=\pi^{*} \operatorname{Ric}^{C h}\left(\omega_{X}\right)-d d^{*} \omega \\
& =\pi^{*}\left(\operatorname{Ric}^{B}\left(\omega_{X}\right)+d d^{*} \omega_{X}\right)-d d^{*} \omega=\pi^{*}\left(\operatorname{Ric}^{B}\left(\omega_{X}\right)\right)+d d^{*}\left(\pi^{*} \omega_{X}\right)-d d^{*} \omega \\
& =\pi^{*} \operatorname{Ric}^{B}\left(\omega_{X}\right)-d d^{*}\left(f \theta_{1} \wedge \theta_{2}\right)
\end{aligned}
$$

where the second equality comes from the following lemma.
Lemma (Lemma 3 of 151$]$. Let $\operatorname{Ric}^{C h}(\omega)$ and $\operatorname{RiC}^{C h}\left(\omega_{X}\right)$ be the Ricci forms of the Chern connections on $(M, \omega)$ and $\left(X, \omega_{X}\right)$ respectively. Then $\operatorname{Ric}^{C h}(\omega)=\pi^{*}\left(\operatorname{Ric}^{C h}\left(\omega_{X}\right)\right)$.

We want to explicitly compute $d d^{*} \widehat{\omega}$ where $\widehat{\omega}:=f \theta_{1} \wedge \theta_{2}$. Therefore, we work on a Hermitian frame $\left\{e_{1}, \ldots, e_{2 n}, t_{1}, t_{2}\right\}$ on an open subset of $M$ which comes from a local Hermitian frame $\left\{e_{1}, \ldots, e_{2 n}\right\}$ on an open subset of $X$ extended so that the vector fields $t_{1}, t_{2}$ are dual to the 1 -forms $\theta_{1}, \theta_{2}$.

Lemma 5.2.1. The following equations hold:

- $\left[t_{i}, e_{j}\right]=0$ for any $i=1,2$ and $j=1, \ldots, 2 n$;
- $\theta_{i}\left(\sum_{j}\left[e_{2 j-1}, e_{2 j}\right]\right)=-\operatorname{tr}_{\omega_{X}}\left(\omega_{i}\right)$ for $i=1,2$.

Proof. We derive these equations from the conditions $d \theta_{i}=\pi^{*} \omega_{i}(i=1,2)$.
First of all, since the Lie brackets are $\pi$-related, i.e. $\pi_{*}[u, v]=\left[\pi_{*} u, \pi_{*} v\right]$ for any smooth vector fields $u, v$, we have that $\left[t_{i}, e_{j}\right]$ must be vertical. However,

$$
\theta_{k}\left(\left[t_{i}, e_{j}\right]\right)=-d \theta_{k}\left(t_{i}, e_{j}\right)=-\pi^{*} \omega_{k}\left(t_{i}, e_{j}\right)=0
$$

We similarly obtain the second equation, indeed

$$
\theta_{i}\left(\left[e_{2 j-1}, e_{2 j}\right]\right)=-d \theta_{i}\left(e_{2 j-1}, e_{2 j}\right)=-\omega_{i}\left(e_{2 j-1}, e_{2 j}\right)
$$

thus

$$
\theta_{i}\left(\sum_{j}\left[e_{2 j-1}, e_{2 j}\right]\right)=-\sum_{j} \omega_{i}\left(e_{2 j-1}, e_{2 j}\right)=-\operatorname{tr}_{\omega_{X}} \omega_{i}
$$

Recall that the co-differential of a form could be expressed in terms of the contraction of Levi-Civita connection as

$$
d^{*} \widehat{\omega}=-\sum_{j=1}^{2 n} \nabla_{e_{j}}^{L C} \widehat{\omega}\left(e_{j}, \cdot\right)-\sum_{i=1,2} \nabla_{t_{i}}^{L C} \widehat{\omega}\left(t_{i}, \cdot\right)
$$

Moreover, for any smooth vector fields $u, v, w$ on a Hermitian manifold it holds

$$
-2\left(\nabla_{u}^{L C} \widehat{\omega}(v, w)\right)=d \widehat{\omega}(u, J v, J w)-d \widehat{\omega}(u, v, w)
$$

Hence, we have

$$
\begin{align*}
d d^{*} \widehat{\omega}= & d\left(\sum_{j} d \widehat{\omega}\left(J \cdot, e_{2 j-1}, e_{2 j}\right)+d \widehat{\omega}\left(J \cdot, t_{1}, t_{2}\right)\right) \\
= & d\left(\sum_{j} d \widehat{\omega}\left(t_{2}, e_{2 j-1}, e_{2 j}\right) \theta_{1}-\sum_{j} d \widehat{\omega}\left(t_{1}, e_{2 j-1}, e_{2 j}\right) \theta_{2}+d \widehat{\omega}\left(J \cdot, t_{1}, t_{2}\right)\right) \\
= & d\left(-\sum_{j} \widehat{\omega}\left(\left[e_{2 j-1}, e_{2 j}\right], t_{2}\right) \theta_{1}+\sum_{j} \widehat{\omega}\left(\left[e_{2 j-1}, e_{2 j}\right], t_{1}\right) \theta_{2}+d \widehat{\omega}\left(J \cdot, t_{1}, t_{2}\right)\right) \\
= & d\left(-f\left(\theta_{1}\left(\sum_{j}\left[e_{2 j-1}, e_{2 j}\right]\right) \theta_{1}+\theta_{2}\left(\sum_{j}\left[e_{2 j-1}, e_{2 j}\right]\right) \theta_{2}\right)+\left(\left(\left(J e_{j}\right)\right) f\right) e^{j}\right) \\
= & d\left(f\left(\operatorname{tr} \omega_{1} \theta_{1}+\operatorname{tr} \omega_{2} \theta_{2}\right)+\left(\left(J e_{j}\right) f\right) e^{j}\right) \\
= & d f \wedge\left(\operatorname{tr} \omega_{1} \theta_{1}+\operatorname{tr} \omega_{2} \theta_{2}\right)+f d\left(\operatorname{tr} \omega_{1} \theta_{1}+\operatorname{tr} \omega_{2} \theta_{2}\right)+d\left(\left(\left(J e_{j}\right) f\right) e^{j}\right) \\
= & \left(e_{i}\left(J e_{j}\right) f-e_{j}\left(J e_{i}\right) f\right) e^{i} \wedge e^{j}+f\left(\operatorname{tr} \omega_{1} \pi^{*} \omega_{1}+\operatorname{tr} \omega_{2} \pi^{*} \omega_{2}\right)  \tag{5.2}\\
& +\left(e_{j}\left(f \operatorname{tr} \omega_{i}\right)-t_{i}\left(J e_{j}\right) f\right) e^{j} \wedge \theta^{i}+\left(t_{1}\left(f \operatorname{tr} \omega_{2}\right)-t_{2}\left(f \operatorname{tr} \omega_{1}\right)\right) \theta_{1} \wedge \theta_{2}
\end{align*}
$$

where we used the Einstein notation for repeated indices and dropped the subscript $\omega_{X}$ on the traces $\operatorname{tr}_{\omega_{X}} \omega_{i}$ for convenience. Since $f$ is constant along the fibers we obtain

$$
\begin{equation*}
d d^{*} \widehat{\omega}=\pi^{*} d d^{c} f+f\left(\operatorname{tr} \omega_{1} \pi^{*} \omega_{1}+\operatorname{tr} \omega_{2} \pi^{*} \omega_{2}\right)+\left(e_{j}\left(f \operatorname{tr} \omega_{i}\right)\right) e^{j} \wedge \theta^{i} \tag{5.3}
\end{equation*}
$$

From this identity and equation (5.1) we get the following result.
Proposition 5.2.1. On the total space $M$ of a toric bundle $\mathbb{S}^{1} \times \mathbb{S}^{1} \hookrightarrow M \xrightarrow{\pi} X$ over a Hermitian manifold $\left(X, \omega_{X}\right)$ with connection one-forms $\left(\theta_{1}, \theta_{2}\right)$, the metric $\omega=\pi^{*}\left(\omega_{X}\right)+f \theta_{1} \wedge \theta_{2}$ defines a Calabi-Yau with torsion structure if there are constants $c_{1}, c_{2}$ such that

$$
\left\{\begin{array}{l}
\operatorname{Ric}^{B}\left(\omega_{X}\right)=d d^{c} f+\left(c_{1} \omega_{1}+c_{2} \omega_{2}\right), \\
f \operatorname{tr}_{\omega_{X}} \omega_{i}=c_{i} .
\end{array}\right.
$$

Notice that if $\operatorname{tr}_{\omega_{X}} \omega_{i}$ vanishes at some point, then it must vanish everywhere since $f \operatorname{tr}_{\omega_{X}} \omega_{i}$ is supposed to be a constant function and $f>0$. We now analyze a special case in which $\operatorname{tr}_{\omega_{X}} \omega_{1}$ and $\operatorname{tr}_{\omega_{X}} \omega_{2}$ are simultaneously zero. First, we exploit the equation on the variation of the Bismut-Ricci tensor under conformal change of the metric to prove the following Lemma.
Lemma 5.2.2 (Lemma 6 of [151]). Let $\left(M, J, g_{M}\right)$ be a Hermitian manifold of complex dimension greater than two. Suppose that the Bismut-Ricci form is $\partial \bar{\partial}$-exact. Then the metric $g_{M}$ is conformally a CYT metric. In other words, there exists a conformal change of $g_{M}$ such that the Ricci form of the induced Bismut connection vanishes.
Proof. We recall the argument of the proof in [151. The result directly comes from the formula for the conformal change of the Ricci curvature form of the Gauduchon connections. Indeed, setting $t=-1$ in (3.2) we obtain

$$
\left(\operatorname{Ric}^{B}\left(e^{f} \omega\right)\right)^{1,1}=\left(\operatorname{Ric}^{B}(\omega)\right)^{1,1}+(n-2) d d^{c} f .
$$

Then it is sufficient to notice that the $(2,0)$ and the $(0,2)$ components of the Bismut-Ricci tensor are invariant for conformal changes and are zero by hypothesis.

When the Bismut-Ricci form is $\partial \bar{\partial}$-exact there is also another construction that can be performed to obtain a CYT structure. Specifically, we can consider a trivial $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundle over $\left(X, \omega_{X}\right)$. In this case, we have the following result.
Proposition 5.2.2. Given a compact Hermitian manifold $\left(X, J, \omega_{X}\right)$ such that the Bismut-Ricci curvature is $\partial \bar{\partial}$-exact, then the product manifold $M:=\mathbb{S}^{1} \times \mathbb{S}^{1} \times X$, equipped with the induced complex structure, admits a CYT metric.
Proof. By hypothesis $R i c^{B}\left(\omega_{X}\right)=\sqrt{-1} \partial \bar{\partial} f$. We can suppose that $f$ is positive since it is defined up to an additive constant on a compact manifold. Thus a submersion-type metric on $M$ can be defined as

$$
\omega=\pi^{*}\left(\omega_{X}\right)+f \theta_{1} \wedge \theta_{2},
$$

where $\pi$ is the natural projection of $M$ onto $X$ and $\theta_{1}, \theta_{2}$ are dual to the vector fields $t_{1}, t_{2}$ on the fibers. We hence have that

$$
\operatorname{Ric}^{B}\left(\pi^{*}\left(\omega_{X}\right)+f \theta_{1} \wedge \theta_{2}\right)=\pi^{*}\left(\operatorname{Ric}^{B}\left(\omega_{X}\right)\right)-d d^{*}\left(f \theta_{1} \wedge \theta_{2}\right)=0
$$

since from (5.3) we get $d d^{*}\left(f \theta_{1} \wedge \theta_{2}\right)=\pi^{*}\left(d d^{c} f\right)$.
Thanks to this result, and by reversing the argument in Lemma 5.2 .2 we obtain the following corollary.
Corollary 5.2.1. Given a complex manifold $(X, J)$ with a CYT metric $g_{X}$, for any positive function $f>0$, the submersion metric $\omega=\pi^{*}\left(e^{f} \omega_{X}\right)+(n-2) f \theta_{1} \wedge \theta_{2}$ is a CYT metric on the product manifold $\mathbb{S}^{1} \times \mathbb{S}^{1} \times X$ equipped with the induced complex structure.

Corollary 5.2 .1 shows that in general on a complex manifold, the Calabi-Yau with torsion metrics are not unique.

### 5.3 Calabi-Yau with torsion metrics on class $\mathcal{C}$ manifolds

In this section, we specialize the results of Proposition 5.2.1 to class $\mathcal{C}$ manifolds. In particular, in Theorems 5.3.1 and 5.3.2 we explicitly construct Calabi-Yau with torsion metrics on class $\mathcal{C}$ manifolds and we prove that, other than some particular cases, they are the unique homogeneous CYT metrics. These results partially answer to the general problem of finding canonical metrics on the homogeneous non-Kähler manifolds. We remark that there are other canonical metrics on the class $\mathcal{C}$ manifolds since they can also be equipped with invariant second Chern--Einstein metrics 263]. However, we will see that the natural homogeneous metrics are Calabi-Yau with torsion.

In [263] the author defined a class $\mathcal{C}$ manifold as a homogeneous manifold $M=G / L$, where $G=G_{1} \times G_{2}$ for compact simply-connected simple Lie groups $G_{1}, G_{2}$, and $L$ is a connected closed subgroup of $G$. We also assume that there exist two irreducible compact Hermitian symmetric spaces $G_{1} / H_{1}, G_{2} / H_{2}$ so that the subgroups $H_{i}$ are of the form $H_{i}=\left\langle Z_{i}\right\rangle \cdot L_{i}$ for $i=1,2$ and $L=L_{1} \times L_{2}$. Therefore, we have the following diagram

$$
\mathbb{S}^{1} \times \mathbb{S}^{1} \longrightarrow\left(G_{1} / L_{1}\right) \times\left(G_{2} / L_{2}\right) \xrightarrow{\phi_{1} \times \phi_{2}}\left(G_{1} / H_{1}\right) \times(G_{2} / \underbrace{\left.H_{2}\right)}_{\pi_{2}}
$$

where the $\phi_{i}$ are the Tits fibrations given by

$$
\phi_{i}: G_{i} / L_{i} \rightarrow G_{i} / H_{i}: g \cdot L_{i} \mapsto g \cdot H_{i} .
$$

Then $M$ is the product of two manifolds, $M=\left(G_{1} / L_{1}\right) \times\left(G_{2} / L_{2}\right)$, where $G_{1} / L_{1}$ and $G_{2} / L_{2}$ are M-manifolds as defined in [263], meaning that $L_{1}$ and $L_{2}$ are the semisimple part of the centralizer of some torus, $H_{i}=C_{G_{i}}\left(\left\langle Z_{i}\right\rangle\right)$. By Theorem C in [319] the manifolds $G_{i} / H_{i}$ are also simply-connected, hence they are generalized flag manifolds. It is known since [226 that any generalized flag manifold can be endowed with an invariant Kähler-Einstein Fano metric which is unique (up to homothety) once the invariant complex structure is fixed. Moreover, the left-invariant complex structures on $M$ are all given by choosing left-invariant complex structures on the symmetric spaces $G_{1} / H_{1}, G_{2} / H_{2}$ and on the torus $\mathbb{S}^{1} \times \mathbb{S}^{1}$, see Samelson's construction in Section 6.3.2. Among these, the standard complex structures on $M$ are that given by choosing $I\left(Z_{1}\right)=Z_{2}$ on the torus and a complex structure on the symmetric spaces. By exploiting the structure of class $\mathcal{C}$ manifolds we can explicitly construct submersion metrics on them which are CYT, namely, we prove the following theorem.

Theorem 5.3.1. Let $M=M_{1} \times M_{2}$ be a class $\mathcal{C}$ manifold, and equip it with a standard complex structure. $M_{1}$ and $M_{2}$ are $M$-manifolds which fiber through the Tits fibrations $\phi_{i}$ over two generalized flag manifolds $X_{1}=G_{1} / H_{1}$ and $X_{2}=G_{2} / H_{2}$ with $\mathbb{S}^{1}$-fibers. Denote $\theta_{1}$ and $\theta_{2}$ the connection one-forms on the fiber bundles such that $d \theta_{i}=\phi_{i}^{*} \omega_{i}$, for $\omega_{i}$ the unique invariant Kähler-Einstein metrics on $X_{i}$ with Einstein constants $n_{i}=\operatorname{dim}\left(X_{i}\right)$. Then the metric on $M$ given by

$$
\omega=\phi_{1}^{*}\left(\omega_{1}\right)+\phi_{2}^{*}\left(\omega_{2}\right)+\theta_{1} \wedge \theta_{2}
$$

defines a CYT structure on $M$.
Proof. The metric on the base space $X=X_{1} \times X_{2}$ is $\omega_{X}=\omega_{1}+\omega_{2}$. Then the metric $\omega$ satisfies

$$
\left\{\begin{array}{l}
\operatorname{Ric}^{B}\left(\omega_{X}\right)=d d^{c}(1)+n_{1} \omega_{1}+n_{2} \omega_{2}, \\
\operatorname{tr}_{\omega_{X}} \omega_{i}=n_{i}, \quad \text { for } i=1,2,
\end{array}\right.
$$

and hence it is Bismut-Ricci flat by Proposition 5.2.1. Thus, we only need to check that the Tits fibrations represent the $U(1)$-principal bundles over $G_{i} / H_{i}$ with curvature $\omega_{i} \in c_{1}\left(G_{i} / H_{i}\right)$ chosen to be the unique Kähler-Einstein metrics on $X_{i}$. We know that the isomorphism classes of principal $U(1)$-bundles over a manifold $X$ are parametrized by its cohomology group $H^{2}(X)$; moreover, we can extract the following piece from the exact sequence in the cohomology of the Tits fibration:

$$
\mathbb{R} \cong H^{1}\left(\mathbb{S}^{1}\right) \xrightarrow{\delta} H^{2}(G / H) \xrightarrow{\phi^{*}} H^{2}(G / L)=0,
$$

where the last term vanishes since the M-manifolds have zero second Betti number 319, Theorem D]. Thus, on the $U(1)$-principal bundles on the $G_{i} / H_{i}$ given by the Tits fibrations we can always find connection one-forms $\theta_{i}$ with curvature in $c_{1}\left(G_{i} / H_{i}\right)$.

The existence of CYT structures on class $\mathcal{C}$ manifolds can also be derived by Theorem 5.1.1. Indeed, the metric $-B(\cdot, \cdot)$ given by the negative of the Killing form of $G$ is Hermitian with respect to the standard complex structures on $M$. To see this, consider the decomposition of the Lie algebra $\mathfrak{g}$ of $G$ as

$$
\mathfrak{g}=\mathfrak{g}_{1}+\mathfrak{g}_{2}=\left(\mathfrak{m}_{1}+\mathfrak{l}_{1}\right)+\left(\mathfrak{m}_{2}+\mathfrak{l}_{2}\right)=\left(\mathfrak{n}_{1}+\mathbb{R} Z_{1}+\mathfrak{l}_{1}\right)+\left(\mathfrak{n}_{2}+\mathbb{R} Z_{2}+\mathfrak{l}_{2}\right) .
$$

Here, $\mathfrak{g}_{1}$ and $\mathfrak{g}_{2}$ are the Lie algebras of $G_{1}$ and $G_{2}$ respectively, while the $\mathfrak{l}_{i}$ 's are the Lie algebras of the $L_{i}$ 's; moreover, the Lie algebras $\mathfrak{h}_{i}$ 's of the $H_{i}$ 's satisfy $\mathfrak{h}_{i}=\mathfrak{l}_{i}+\mathbb{R} Z_{i}$ for $i=1,2$. The Killing form $B$ is Hermitian on $\mathfrak{n}:=\mathfrak{n}_{1}+\mathfrak{n}_{2}$, moreover, the tori $\left\langle Z_{1}\right\rangle$ and $\left\langle Z_{2}\right\rangle$ are orthogonal to the $\mathfrak{n}_{i}$ 's as well as one to each other. It only remains to verify that $B$ is Hermitian on $\mathfrak{t}$, that is $B\left(Z_{1}, Z_{1}\right)=B\left(Z_{2}, Z_{2}\right)$.

The CYT metrics constructed above can be characterized as the unique CYT metrics among the homogeneous ones. Namely, we prove the following result.

Theorem 5.3.2. Take a class $\mathcal{C}$ manifold $M$ as in Theorem 5.3.1. Suppose that none of the $X_{i}$ 's is $S O(k+2) / S O(2) \times S O(k)$ for $k \geq 3$, then the metric

$$
\omega=\phi_{1}^{*}\left(\omega_{1}\right)+\phi_{2}^{*}\left(\omega_{2}\right)+\theta_{1} \wedge \theta_{2},
$$

constructed in Theorem 5.3.1, is the unique (up to homothety) homogeneous CYT metric on $M$.
Proof. First of all, we verify that the homogeneous metrics on $M$ make $\phi_{1} \times \phi_{2}$ a Riemannian submersion. Indeed, with the same notations as above, a $G$-invariant Hermitian metric $g^{\prime}$ on $M$, can be seen as an ad $\left(\mathfrak{l}_{1}+\mathfrak{l}_{2}\right)$-invariant Hermitian inner product on $\mathfrak{m}_{1}+\mathfrak{m}_{2}$. As the $\mathfrak{l}_{i}$ 's are not trivial, $\mathfrak{l}=\mathfrak{l}_{1}+\mathfrak{l}_{2}$ acts non-trivially on $\mathfrak{n}=\mathfrak{n}_{1}+\mathfrak{n}_{2}$ and trivially on $\mathfrak{t}$, therefore $g^{\prime}(\mathfrak{t}, \mathfrak{n})=0$. Moreover, the ad $(\mathfrak{l})$-modules $\mathfrak{n}_{i}$ are mutually non-equivalent, hence $g^{\prime}\left(\mathfrak{n}_{1}, \mathfrak{n}_{2}\right)=0$. Since we are avoiding the special case of $\mathfrak{g}_{i}=\mathfrak{s o}(n+2)$ and $\mathfrak{h}_{i}=\mathfrak{s o}(2)+\mathfrak{s o}(n)$, for $n \geq 3$ the $\mathfrak{n}_{i}$ 's are $\mathfrak{l}_{i}$-irreducible. Hence, the Schur Lemma implies that $g^{\prime}$ on $\mathfrak{n}_{i} \times \mathfrak{n}_{i}$ restricts to a multiple ( $\lambda_{i} \in \mathbb{R}_{+}$) of the Killing form $B_{i}$ on $G_{i}$, i.e.

$$
g_{\mid \mathbf{n}_{i} \times \mathrm{n}_{i}}^{\prime}=-\lambda_{i}\left(B_{i}\right)_{\mid} .
$$

In other words, the homogeneous metrics on $M$ are all of the types

$$
\omega^{\prime}=\lambda_{1} \phi_{1}^{*}\left(\omega_{1}\right)+\lambda_{2} \phi_{2}^{*}\left(\omega_{2}\right)+\lambda \theta_{1} \wedge \theta_{2}
$$

and by Proposition 5.2.1 any homogeneous CYT metric $g^{\prime}$ have to satisfy

$$
\left\{\begin{array}{l}
\operatorname{Ric}^{B}\left(\lambda_{1} \omega_{1}+\lambda_{2} \omega_{2}\right)=n_{1} \omega_{1}+n_{2} \omega_{2}, \\
\lambda \operatorname{tr}_{\left(\lambda_{1} \omega_{1}+\lambda_{2} \omega_{2}\right)} \omega_{i}=n_{i}, \quad \text { for } i=1,2 .
\end{array}\right.
$$

However, for $i=1,2$,

$$
n_{i}=\lambda \operatorname{tr}_{\left(\lambda_{1} \omega_{1}+\lambda_{2} \omega_{2}\right)} \omega_{i}=\frac{\lambda}{\lambda_{i}} \operatorname{tr}_{\left(\omega_{1}+\omega_{2}\right)} \omega_{i}=\frac{\lambda}{\lambda_{i}} n_{i},
$$

proving that $\lambda=\lambda_{1}=\lambda_{2}$, and hence $g^{\prime}$ is a positive multiple of $g$.

## Calabi-Yau with torsion metrics on Calabi-Eckmann manifolds

Theorem 5.3.1 and Theorem 5.3.2 apply to give unique homogeneous CYT metrics on the Calabi-Eckmann manifolds when they are equipped with their standard complex structures. Indeed, as class $\mathcal{C}$ manifolds, they are given by taking $G_{i}=\mathrm{SU}\left(n_{i}+1\right), L_{i}=\mathrm{SU}\left(n_{i}\right)$, and $H_{i}=\mathrm{SU}\left(n_{i}\right) \times \mathrm{U}(1)$, see Section 2.3 . In particular, the Tits fibrations agree with the Hopf fibrations

$$
\mathbb{S}^{1} \hookrightarrow \mathbb{S}^{2 n_{i}+1} \xrightarrow{\phi_{i}} \mathbb{C} \mathbb{P}^{n_{i}} .
$$

Then we have the following corollary.
Corollary 5.3.1. Let $M_{n_{1}, n_{2}}$ be a Calabi-Eckmann manifold, and equip it with its standard complex structure. Consider the Fubini-Study metrics $\omega_{i}$ 's on the complex projective spaces $\mathbb{C P}^{n_{i}}$ 's with Einstein constants $n_{i}$ 's, and set $\theta_{1}$ and $\theta_{2}$ the connection one-forms on the fiber bundles such that $d \theta_{i}=\phi_{i}^{*} \omega_{i}$ for $i=1,2$. Then, the metric

$$
\omega=\phi_{1}^{*}\left(\omega_{1}\right)+\phi_{2}^{*}\left(\omega_{2}\right)+\theta_{1} \wedge \theta_{2}
$$

is a CYT metric on $M_{n_{1}, n_{2}}$; moreover, it is the unique homogeneous $C Y T$ metric on it up to homothety.

## Chapter 6

## The curvature of the Bismut connection

In this chapter, we describe the geometry of Bismut flat manifolds. Specifically, given a Hermitian manifold ( $M, J, g$ ), it is Bismut flat if the curvature tensor of its Bismut connection vanishes identically, that is

$$
R^{B} \equiv 0
$$

The compact Bismut flat manifolds are characterized by [321]: up to taking the universal cover, they are connected and simply-connected Lie groups with a left-invariant Hermitian structure whose metric is also right-invariant. Therefore they can be distinguished by means of the dimension of the maximal torus in the universal cover Lie group. We then focus on Bismut flat manifolds with maximal torus of dimension 2 and compute their cohomologies. In particular, we prove that the ( 1,1 )-Aeppli cohomology of this special class of manifolds is of dimension one in Theorem 6.4.1. This, combined with the analysis on the long-time behavior of the pluriclosed flow in 132, yields a result on the global stability of the pluriclosed flow on this class of manifolds, as stated in Theorem 7.3.3 (see Chapter 7).

The original results of this chapter have been obtained in [32].

### 6.1 Flat connections with torsion

The Bismut connection is a particular connection with skew-symmetric torsion chosen to adapt to the Hermitian context (see Section 1.2 .2 for details). Since flat connections with skew-symmetric torsion have been studied in [3, 82, 83, 325, 326, we start by briefly recalling some results in this more general setting.

Given a Riemannian manifold ( $M, g$ ), the metric connections on it are uniquely determined by their torsion, see for instance the proof of Proposition 1.2.4. In particular, the torsion tensor $T$ lies in $\mathcal{A}^{2}(M ; T M)$, which decomposes under $\mathrm{SO}(n)$ action (for $n \geq 3$ ) in

$$
\mathcal{A}^{2}(M ; T M)=\mathcal{C}^{\infty}(M ; T M) \oplus \mathcal{A}^{3}(M) \oplus \mathcal{T} .
$$

This is known as the Cartan decomposition of metric connections [80]. In the above formula, $\mathcal{T}$ is an algebraic term without a geometric interpretation, while the connections with torsion in $\mathcal{A}^{3}(M)$ (respectively $\mathcal{C}^{\infty}(M ; T M)$ ) are called connections with skew (respectively, vectorial) torsion. The connections with skew torsion are by far the richest and the best-understood class among the three Cartan classes. In fact, such connections are always geodesically complete and correspond precisely to those metric connections that have the same geodesics as the Levi-Civita connection. Furthermore, they represent a natural replacement for the Levi-Civita connection on many geometries. For example, any simple Lie group carries two flat connections, usually
called $\pm$-connection (compare with Definition 1.2.3), with torsion $\pm[\cdot, \cdot]$ (83] (see also 195, pages 198-199]). Then, if one chooses a bi-invariant metric, these connections are metric and the torsion becomes a 3 -form.

Example 6.1.1. Consider a compact Lie group $G$ with bi-invariant metric $g$. This condition is equivalent to asking that

$$
\begin{equation*}
g([z, x], y)+g(x,[z, y])=0 \tag{6.1}
\end{equation*}
$$

for $x, y, z \in \mathfrak{g}$. Here, the Levi-Civita connection is $\nabla_{x}^{L C} y=\frac{1}{2}[x, y]$, while connections with torsion equal to $\pm[\cdot, \cdot]$ can be easily defined following (1.2) as

$$
\nabla_{x}^{ \pm} y=\frac{1 \pm 1}{2}[x, y]
$$

Finally, the torsion 3-tensor of $\nabla^{ \pm}$is $g\left(T^{ \pm}(x, y), z\right)= \pm g([x, y], z)$, which is a three form thanks to (6.1). Moreover, using the Jacobi identity, it can be shown that the Levi-Civita connection has curvature tensor equal to

$$
R_{x, y}^{L C} z=\frac{1}{4}[z,[x, y]]
$$

while the $\pm$-connections have flat curvature tensors.
Here one replaces the torsion-free condition of the Levi-Civita connection with the assumption of flat curvature tensor to obtain a connection which reflects the geometry of the manifold. The question of whether there are any further examples of flat metric connections with skewsymmetric torsion beside products of Lie groups was firstly answered by Cartan and Schouten in 82, and then their result was reproved by other means in 325, 326, and more recently in 3]. We state it here.

Theorem 6.1.1 ([3, 82]). Let $(M, g)$ be a simply-connected, complete, and irreducible Riemannian manifold equipped with a flat metric connection $\nabla$ with skew-symmetric torsion $T \neq 0$. Then, $M$ is either isometric to a compact simple Lie group or isometric to $S^{7}$.

### 6.2 Flat Bismut connection

Thanks to Theorem 6.1.1, a compact simply-connected Hermitian non-Kähler manifold ( $M, J, g$ ) whose Bismut connection is flat must be isometric to a product of compact simple Lie groups. Hence, as Riemannian manifolds, the structure of such spaces is well-understood. It then remains to describe the complex structures compatible with these Riemannian metrics. In general, it is a challenging task to find the set of all possible complex structures compatible with a given Riemannian metric. However, by an explicit construction (see Section 6.3), Samelson showed 271 that any even-dimensional compact Lie group admits a left-invariant complex structure compatible with the bi-invariant metric coming from the Killing form. Moreover, Alexandrov and Ivanov [6] proved that any even dimensional connected Lie group equipped with a bi-invariant metric $g$ and a left-invariant complex structure which is compatible with $g$ is Bismut flat. Afterward, Wang, Yang, and Zheng [321] showed that up to taking the universal cover, these are the only existing compact Bismut flat manifolds. In other words, simply-connected compact Bismut flat manifolds have been characterized as Samelson spaces, whose definition is as follows.

Definition 6.2.1 ( $(\boxed{321})$. A Samelson space is a Hermitian manifold $(G, g, J)$, where $G$ is a connected and simply-connected, even-dimensional Lie group, $g$ a bi-invariant metric on $G$, and $J$ a left-invariant complex structure on $G$ that is compatible with $g$.

By Milnor's Lemma [230, Lemma 7.5], a simply-connected Lie group $G^{\prime}$ with a bi-invariant metric must be the product of a compact semisimple Lie group with an additive vector group.

Lemma 6.2.1 (Lemma 7.5 of 230 ). Let $G$ be a simply-connected Lie group with a bi-invariant metric $\langle\cdot, \cdot\rangle$. Then $G$ is isomorphic and isometric to the product $G_{1} \times \cdots \times G_{r} \times \mathbb{R}^{k}$ where each $G_{i}$ is a simply-connected compact simple Lie group and $\mathbb{R}^{k}$ is the additive vector group with the flat metric.

Taking quotients of these manifolds one obtains the local Samelson spaces (definition below). Such Hermitian manifolds are Bismut flat since their universal cover is so.
Definition 6.2.2 ( $\sqrt[321]{ })$. Let $\left(G^{\prime}, g, J\right)$ be a Samelson space, where $G^{\prime}=G \times \mathbb{R}^{k}$ with $G$ semisimple. Let $\rho: \mathbb{Z}^{k} \rightarrow I(G)$ be a homomorphism into the isometry group of $G$. Then $\Gamma_{\rho} \sim \mathbb{Z}^{k}$ acts on $G \times \mathbb{R}^{k}$ by $\gamma(x, y)=(\rho(\gamma)(x), y+\gamma)$ as isometries, and it acts freely and properly discontinuously, so one gets a compact quotient $M_{\rho}=\left(G \times \mathbb{R}^{k}\right) / \Gamma_{\rho}$. If the complex structure of $G^{\prime}$ is preserved by $\Gamma_{\rho}$, then it descends down to $M_{\rho}$ and makes it a complex manifold. In this case, the compact Hermitian manifold $M_{\rho}$ is called local Samelson space.

Theorem 6.2.1 (Theorem 1 in [321]). Let $(M, J, g)$ be a compact Hermitian manifold whose Bismut connection is flat. Then there exists a finite cover $M^{\prime}$ of $M$ such that $M^{\prime}$ is a local Samelson space $M_{\rho}$ defined as above. Also, $M_{\rho}$ is diffeomorphic to $G \times\left(\mathbb{S}^{1}\right)^{k}$.

It is possible to go even further in the classification of the Bismut flat manifolds. Indeed, the compact simply-connected simple Lie groups are fully classified $76,77,109,110,318,322,324$; they are:

$$
\begin{aligned}
& A_{k}=\mathrm{SU}(k+1), k \geq 1, \\
& B_{k}=\operatorname{Spin}(2 k+1), k \geq 2, \\
& C_{k}=\operatorname{Sp}(2 k), k \geq 3 \\
& D_{k}=\operatorname{Spin}(2 k), k \geq 4, \\
& E_{6} \\
& E_{7}, \\
& E_{8} \\
& F_{4} \\
& G_{2}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{dim}\left(A_{k}\right) & =k(k+2) \\
\operatorname{dim}\left(B_{k}\right) & =k(2 k+1) \\
\operatorname{dim}\left(C_{k}\right) & =k(2 k+1) \\
\operatorname{dim}\left(D_{k}\right) & =k(2 k-1) \\
\operatorname{dim}\left(E_{6}\right) & =78 \\
\operatorname{dim}\left(E_{7}\right) & =133 \\
\operatorname{dim}\left(E_{8}\right) & =248 \\
\operatorname{dim}\left(F_{4}\right) & =52 \\
\operatorname{dim}\left(G_{2}\right) & =14
\end{aligned}
$$

$$
\begin{aligned}
& \operatorname{rank}\left(A_{k}\right)=k \\
& \operatorname{rank}\left(B_{k}\right)=k \\
& \operatorname{rank}\left(C_{k}\right)=k \\
& \operatorname{rank}\left(D_{k}\right)=k \\
& \operatorname{rank}\left(E_{6}\right)=6 \\
& \operatorname{rank}\left(E_{7}\right)=7 \\
& \operatorname{rank}\left(E_{8}\right)=8 \\
& \operatorname{rank}\left(F_{4}\right)=4 \\
& \operatorname{rank}\left(G_{2}\right)=2
\end{aligned}
$$

Here the rank of a group coincides with the dimension of its maximal torus, which is the maximal compact, connected, abelian Lie subgroup. The above classification follows from the classification of Dynkin diagrams. As a matter of fact, semisimple Lie algebras over algebraically closed fields can be classified via their root system, which in turn can be represented by a Dynkin diagram. In particular, $A_{n}, B_{n}, C_{n}, D_{n}$ correspond to the Lie algebras associated with classical groups over the complex numbers, while $E_{6}, E_{7}, E_{8}, F_{4}, G_{2}$ are called exceptional because they do not have a classical geometric interpretation, and the names for the exceptional groups coincide with the associated Dynkin diagrams. For details on the classification of simple Lie algebras, we refer to 191. Chapter II].
Remark 6.2.1. $A_{2}=\mathrm{SU}(3), B_{2}=\mathrm{Spin}(5)$ and $\mathrm{G}_{2}$ are the only compact simply-connected simple Lie groups of rank two. Together with the complex torus $\mathbb{T}$, the Hopf surface $\mathbb{S}^{1} \times \mathrm{SU}(2)$, and the Calabi-Eckmann 3 -fold $\mathrm{SU}(2) \times \mathrm{SU}(2)$ these give the only compact Bismut flat manifolds whose maximal torus is $\mathbb{T}$.

### 6.3 Complex structures on Bismut flat manifolds

In [261] Pittie gave a complete description of the moduli of left-invariant, integrable complex structures on even-dimensional compact Lie groups, proving that they all come from Samelson's construction in [271], namely, from a choice of a maximal torus, a complex structure on the Lie algebra of the torus, and a choice of positive roots for the Cartan decomposition. Let us now recall this construction in more detail.

### 6.3.1 Cartan decomposition

Let $G$ be an even-dimensional connected Lie group and denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathfrak{g}^{\mathbb{C}}$ its complexification. We recall that the Killing form of $G$ is given by

$$
B(X, Y):=\operatorname{tr}\left(\operatorname{ad}_{X} \circ \operatorname{ad}_{Y}\right),
$$

where $\operatorname{ad}_{X}(Y)=[X, Y]$ for $X, Y \in \mathfrak{g}$. This is a symmetric bilinear form, and thanks to the Cartan criterion 76, Théorème 1 in Chapitre IV] it is non-degenerate if and only if $\mathfrak{g}$ is semisimple. If this is the case, its opposite is a metric which we indicate with $\langle\cdot, \cdot\rangle:=-B(\cdot, \cdot)$. Moreover, if $G$ is a simple Lie group then any invariant symmetric bilinear form on it is a scalar multiple of the Killing form.

Given $K$ a maximal torus of $G$, we denote by $\mathfrak{k}$ its Lie algebra. The action of ad $\mathfrak{k}$ on $\mathfrak{g}$ is simultaneously diagonalizable [191, Corollary 2.23], and the eigenvalues of ad $\mathfrak{k}$ on $\mathfrak{g}$ are called roots of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Then one has the ad $(K)$-invariant roots decomposition

$$
\mathfrak{g}^{\mathbb{C}}=\mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_{\alpha},
$$

where $\mathfrak{k}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{k}, R$ is the roots space and

$$
\mathfrak{g}_{\alpha}:=\left\{v \in \mathfrak{g}^{\mathbb{C}} \mid[H, v]=\alpha(H) v \forall H \in \mathfrak{k}\right\} .
$$

The algebra $\mathfrak{k}^{\mathbb{C}}$ is also known as Cartan subalgebra and it is unique, up to conjugation by an automorphism of $\mathfrak{g}$ (see [191, Theorem 2.15]). Moreover, the root spaces $\mathfrak{g}_{\alpha}$ are one dimensional, and it holds 191, Proposition 2.17]

$$
\begin{equation*}
\left\langle\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}\right\rangle=0 \quad \text { if } \alpha+\beta \neq 0 . \tag{6.2}
\end{equation*}
$$

The bracket relations between root spaces can be easily computed and are

$$
\left[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}\right] \begin{cases}=\mathfrak{g}_{\alpha+\beta} & \text { if } \alpha+\beta \text { is a non-zero-root; }  \tag{6.3}\\ =0 & \text { if } \alpha+\beta \text { is not a root; } \\ \subset \mathfrak{k} & \text { if } \alpha+\beta=0\end{cases}
$$

Furthermore, if $\alpha$ is a root, then also $-\alpha$ is a root, and $\mathfrak{g}_{-\alpha}=\overline{\mathfrak{g}_{\alpha}}$. Therefore, one fix an element $H \in \mathfrak{k}$ such that $\alpha(H) \neq 0$ for every $\alpha \in R$ (which exists since $R$ is finite), and says that a root $\alpha$ is positive if $\mathrm{i} \alpha(H)>0$. A choice of the positive root system leads to the Cartan decomposition, namely $\mathfrak{g}^{\mathbb{C}}$ decomposes as

$$
\begin{equation*}
\mathfrak{g}^{\mathbb{C}}=\mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in R^{+}} \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}, \tag{6.4}
\end{equation*}
$$

where $R^{+}$is the space of positive roots.

### 6.3.2 Samelson construction

The left-invariant almost-complex structures $J$ on $G$ are uniquely determined by their restriction to $\mathfrak{g}$, which we still indicate with the same symbol. Hence we look at them as linear maps $J: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $J^{2}=-\mathrm{id}_{\mathfrak{g}}$. Equivalently, an almost-complex structure on $G$ is determined by the subspace $\mathfrak{s} \subset \mathfrak{g}^{\mathbb{C}}$ of ( 1,0 )-vectors, which clearly satisfies $\mathfrak{s} \cap \mathfrak{g}=0$, and $\mathfrak{s} \oplus \overline{\mathfrak{s}}=\mathfrak{g}^{\mathbb{C}}$. Finally, the integrability condition becomes $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$, and thus, the complex structures on $G$ are in one-to-one correspondence with the complex Lie subalgebras $\mathfrak{s} \subset \mathfrak{g}^{\mathbb{C}}$, such that

$$
\mathfrak{s} \cap \mathfrak{g}=0, \quad \text { and } \quad \mathfrak{s} \oplus \overline{\mathfrak{s}}=\mathfrak{g}^{\mathbb{C}} .
$$

Such subspaces are called Samelson subalgebras of $\mathfrak{g}^{\mathbb{C}}$ 261].

Samelson [271] first constructed examples of left-invariant complex structures on compact Lie groups as follows. Consider the Cartan decomposition (6.4). Since $\operatorname{dim}(G)$ is even, the abelian Lie algebra $\mathfrak{k}$ is even-dimensional as well. Thus it is possible to choose a complex structure on $\mathfrak{k}$. As before, this is equivalent to choosing a complex subalgebra $\mathfrak{a} \subset \mathfrak{k}^{\mathbb{C}}$ such that

$$
\mathfrak{a} \cap \mathfrak{k}=0, \quad \text { and } \quad \mathfrak{a} \oplus \overline{\mathfrak{a}}=\mathfrak{k}^{\mathbb{C}} .
$$

Now one could simply take

$$
\mathfrak{s}=\mathfrak{a} \oplus \sum_{\alpha \in R^{+}} \mathfrak{g}_{\alpha}
$$

to be a Samelson subalgebra of $\mathfrak{g}$. Thanks to the relation in 6.2, the positive root spaces are orthogonal with respect to $\langle\cdot, \cdot\rangle$. Therefore, if we choose a Hermitian metric on the torus, then the above complex structure is Hermitian together with the Killing metric.

Pittie 261 proved that any left-invariant complex structure on $G$ is obtained as above. Moreover, he described the moduli-space of left-invariant complex structures on $G$. Specifically, he proved that the space $m_{2}(G)$ of left-invariant complex structures on $G$ up to automorphisms of $G$ is given by

$$
\begin{equation*}
m_{2}(G)=(G L(2 k, \mathbb{R}) / G L(k, \mathbb{C})) / F \tag{6.5}
\end{equation*}
$$

where $2 k$ is the rank of $G$ and $F$ is a discrete group generated by the automorphisms of the abelian factor of $G$, the automorphisms of the Dynkin diagrams of the simple factors, and permutations among isomorphic simple factors.

### 6.3.3 Isotropic complex structures on Bismut flat manifolds of rank two

As shown in Theorem 6.2.1, the simply-connected Bismut flat manifolds are isomorphic and isometric to even-dimensional Lie groups equipped with a bi-invariant metric and a compatible left-invariant structure. For this reason, here, we are interested in the left-invariant complex structures which give Hermitian structures with respect to the Killing metric. In the terminology of Pittie [261], these are called isotropic left-invariant complex structures, and he showed that the Dolbeault cohomology of the compact simply-connected simple Lie groups of rank 2 equipped with a left-invariant complex structure only depends on whether this is isotropic or not.

Since the moduli-space of left-invariant complex structures on $G$ up to automorphisms is computed as in (6.5), the isotropic ones are given by the quotient by $F$ of $O(2 k) / U(k)$. In fact, as explained above, the isotropic left-invariant complex structures are precisely the ones which are $B$-orthogonal on the Cartan subalgebra, or equivalently, the ones for which the maximal torus is a complex submanifold of $G$. In particular, for compact simply-connected simple Lie groups $G$ of rank 2, the moduli-space of left-invariant complex structures is

$$
m_{2}(G)=\left(\mathcal{H}_{+} \cup \mathcal{H}_{-}\right) / F,
$$

where $\mathcal{H}_{ \pm}$represent respectively the upper and lower half-planes in $\mathbb{C}$, see the example in [261, page 123]. Indeed, in this case $G L(2, \mathbb{R}) / G L(1, \mathbb{C}) \simeq \mathcal{H}_{+} \cup \mathcal{H}_{-}$. Moreover, the space of the isotropic complex structures is $O(2) / U(1) \simeq\{ \pm i\}$ as a subset of $\mathcal{H}_{+} \cup \mathcal{H}_{-}$.

In the rest of this section, we follow the Samelson construction to describe the left-invariant complex structures on compact simply-connected simple Lie groups of rank two. We are going to focus on the complex structures which are Hermitian with respect to the Killing metric. In particular, since the only compact simply-connected simple Lie groups of rank two are $\mathrm{SU}(3)$, $\operatorname{Spin}(5)$ and $\mathrm{G}_{2}$ (see Remark 6.2.1) we proceed case by case (be aware of the fact that we use the same symbols for objects which refer to different groups). Notice that for $\operatorname{SU}(3)$ we have that $F=\mathbb{Z} / 2$ [261, page 123], hence there is a unique isotropic left-invariant complex structure on it up to automorphisms. On the other hand, $\operatorname{Spin}(5)$ and $\mathrm{G}_{2}$ have two isotropic left-invariant complex structures up to automorphisms, since $F$ is trivial for them.

SU(3)
The group $\operatorname{SU}(3)$ is the group of $3 \times 3$ unitary matrices with unit determinant,

$$
\operatorname{SU}(3):=\left\{U \in M(3 \times 3 ; \mathbb{C}) \mid U U^{\dagger}=I d \wedge \operatorname{det} U=1\right\} .
$$

By differentiating these two conditions we get that its Lie algebra $\mathfrak{s u}(3)$ is made by the skewHermitian $(3 \times 3)$-matrices with zero trace. We thus can take i-times the Gell-Mann matrices [147, Table I] as a basis of $\mathfrak{s u}(3)$ :

$$
\begin{array}{lll}
e^{1}=\left(\begin{array}{lll}
0 & \mathrm{i} & 0 \\
\mathrm{i} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), & e^{2}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), & e^{3}=\left(\begin{array}{ccc}
\mathrm{i} & 0 & 0 \\
0 & -\mathrm{i} & 0 \\
0 & 0 & 0
\end{array}\right), \\
e^{4}=\left(\begin{array}{lll}
0 & 0 & \mathrm{i} \\
0 & 0 & 0 \\
\mathrm{i} & 0 & 0
\end{array}\right), & e^{5}=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{array}\right), & e^{8}=\frac{1}{\sqrt{3}}\left(\begin{array}{ccc}
\mathrm{i} & 0 & 0 \\
0 & \mathrm{i} & 0 \\
0 & 0 & -2 \mathrm{i}
\end{array}\right) .
\end{array}
$$

In this way, we obtain the global left-invariant frame $\left\{e^{1}, e^{2}, e^{3}, e^{4}, e^{4}, e^{5}, e^{6}, e^{7}, e^{8}\right\}$ on $\mathrm{SU}(3)$ with structure constants

$$
\left[e^{i}, e^{j}\right]=2 \sum_{k=1}^{8} \lambda^{i j k} e^{k}
$$

given by

$$
\begin{array}{l|ccccccccc}
\mathrm{ijk} & 123 & 147 & 156 & 246 & 257 & 345 & 367 & 458 & 678 \\
\hline \lambda^{i j k} & -1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}
\end{array}
$$

Here $\lambda^{i j k}=(-1)^{|\sigma|} \lambda^{\sigma(i, j, k)}$ for any permutation $\sigma$. With this notation

$$
\left\langle e^{i}, e^{j}\right\rangle=4 \sum_{p, q=1}^{8} \lambda^{i p q} \lambda^{j p q} .
$$

Thus all the $e^{i}$ have the same norm equal to $\sqrt{6}$.
The maximal torus in $\operatorname{SU}(3)$ is given by the diagonal matrices and its Lie algebra $\mathfrak{k} \subset \mathfrak{s u}(3)$ is generated by $e^{3}$ and $e^{8}$. The remaining six generators, outside the Cartan subalgebra, could be arranged into six roots. In particular, $e^{1} \pm \mathrm{i} e^{2}, e^{4} \pm \mathrm{i} e^{5}$ and $e^{6} \pm \mathrm{i} e^{7}$ give the following relations:

$$
\begin{array}{ll}
{\left[e^{3}, e^{1} \pm \mathrm{i} e^{2}\right]= \pm 2 \mathrm{i}\left(e^{1} \pm \mathrm{i} e^{2}\right),} & \\
{\left[e^{8}, e^{1} \pm \mathrm{i} e^{2}\right]=0,} \\
{\left[e^{3}, e^{4} \pm \mathrm{i} e^{5}\right]= \pm \mathrm{i}\left(e^{4} \pm \mathrm{i} e^{5}\right),} & \\
{\left[e^{3}, e^{6} \pm \mathrm{i} e^{7}\right]=\mp \mathrm{i}\left(e^{6} \pm e^{4} \pm \mathrm{i} e^{7}\right),} & \\
\left.e^{5}\right]= \pm \sqrt{3} \mathrm{i}\left(e^{4} \pm \mathrm{i} e^{5}\right), \\
& {\left[e^{8}, e^{6} \pm \mathrm{i} e^{7}\right]= \pm \sqrt{3} \mathrm{i}\left(e^{6} \pm \mathrm{i} e^{7}\right) .}
\end{array}
$$

Therefore, three positive roots are $(2,0),(1, \sqrt{3})$, and $(1,-\sqrt{3})$ with associated eigenspaces generated respectively by $e^{1}+\mathrm{i} e^{2}, e^{4}+\mathrm{i} e^{5}$ and $e^{6}-\mathrm{i} e^{7}$. We shall define

$$
\begin{array}{ll}
\varphi^{1}=e^{1}+\mathrm{i} e^{2}, & \varphi^{3}=e^{6}-\mathrm{i} e^{7}, \\
\varphi^{2}=e^{4}+\mathrm{i} e^{5}, & \varphi^{4}=(1-a \mathrm{i}) e^{3}-b \mathrm{i} e^{8} \quad \text { with } a+\mathrm{i} b \in \mathcal{H}_{-},
\end{array}
$$

so that $\left\langle\varphi^{1}, \varphi^{2}, \varphi^{3}, \varphi^{4}\right\rangle$, generate the Samelson subalgebras on $\operatorname{SU}(3)$. Among these, the only isotropic Samelson subalgebra is detected by the choice $J e^{3}=e^{8}$ since $\left\langle e^{3}, e^{8}\right\rangle=0$ and they have the same norm. It corresponds to $a+\mathrm{i} b=-\mathrm{i}$ and we indicate it with $J_{0,-1}$.

Spin(5)
The group $\operatorname{Spin}(5)$ is the universal cover of the special orthogonal group $\mathrm{SO}(5)$. Namely,

$$
0 \rightarrow \mathbb{Z}_{2} \rightarrow \operatorname{Spin}(5) \rightarrow \mathrm{SO}(5) \rightarrow 0
$$

where

$$
\mathrm{SO}(5):=\left\{Q \in M(5 \times 5 ; \mathbb{R}) \mid Q Q^{T}=\mathrm{id}=Q^{T} Q\right\}
$$

Consequently, they share the same Lie algebra, which we indicate with $\mathfrak{s p i n}(5)$. It is given by differentiating the two conditions above, and thus it is the 10 -dimensional algebra of $5 \times 5$ skew-symmetric matrices. We take the following generators as basis of the Spin(5) Lie algebra $\mathfrak{s p i n}(5)$ :

$$
\begin{array}{llll}
e^{1}=A_{1,2}, & e^{2}=A_{1,3}, & e^{3}=A_{2,3}, & e^{4}=A_{1,4}, \\
e^{6}=A_{3,4}, & e^{7}=A_{1,5}, & e^{8}=A_{2,5}, & e^{9}=A_{3,5},
\end{array}
$$

where $A_{i, j}$ represents the $5 \times 5$ skew-symmetric matrix with 1 in the $(i, j)$-position; more precisely, $\left(A_{i, j}\right)_{p, q}=\delta_{i, p} \delta_{j, q}-\delta_{i, q} \delta_{j, p}$. Using this notation we describe the structure constants of the global left-invariant frame $\left\{e^{1}, e^{2}, e^{3}, e^{4}, e^{4}, e^{5}, e^{6}, e^{7}, e^{8}, e^{9}, e^{10}\right\}$ on $\operatorname{Spin}(5)$ as

$$
\left[A_{i, j}, A_{m, n}\right]=\delta_{m j} A_{i n}-\delta_{n j} A_{i m}-\delta_{m i} A_{j n}+\delta_{n i} A_{j m} .
$$

A maximal torus in $\mathrm{SO}(5)$ is given by the block-diagonal matrices of the form

$$
\left(\begin{array}{ccc}
B_{1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & B_{2}
\end{array}\right)
$$

where $B_{1}, B_{2} \in \mathrm{SO}(2)$. Thus its Lie algebra $\mathfrak{k} \subset \mathfrak{s p i n}(5)$ is generated by $e^{1}$ and $e^{10}$. The remaining eight generators, outside the Cartan subalgebra, could be rearranged into eight roots. Consider the following relations:

$$
\begin{array}{ll}
{\left[e^{1}, e^{2} \pm \mathrm{i} e^{3}\right]=\mp \mathrm{i}\left(e^{2} \pm \mathrm{i} e^{3}\right),} & {\left[e^{10}, e^{2} \pm \mathrm{i} e^{3}\right]=0} \\
{\left[e^{1}, e^{6} \pm \mathrm{i} e^{9}\right]=0,} & {\left[e^{10}, e^{6} \pm \mathrm{i} e^{9}\right]=\mp \mathrm{i}\left(e^{6} \pm \mathrm{i} e^{9}\right),} \\
{\left[e^{1},\left(e^{4}-e^{8}\right) \pm \mathrm{i}\left(e^{7}+e^{5}\right)\right]=\mp \mathrm{i}\left(\left(e^{4}-e^{8}\right) \pm \mathrm{i}\left(e^{7}+e^{5}\right)\right),} \\
{\left[e^{1},\left(e^{4}+e^{8}\right) \pm \mathrm{i}\left(e^{7}-e^{5}\right)\right]= \pm \mathrm{i}\left(\left(e^{4}+e^{8}\right) \pm \mathrm{i}\left(e^{7}-e^{5}\right)\right),} \\
{\left[e^{10},\left(e^{4}-e^{8}\right) \pm \mathrm{i}\left(e^{7}+e^{5}\right)\right]=\mp \mathrm{i}\left(\left(e^{4}-e^{8}\right) \pm \mathrm{i}\left(e^{7}+e^{5}\right)\right),} \\
{\left[e^{10},\left(e^{4}+e^{8}\right) \pm \mathrm{i}\left(e^{7}-e^{5}\right)\right]=\mp \mathrm{i}\left(\left(e^{4}+e^{8}\right) \pm \mathrm{i}\left(e^{7}-e^{5}\right)\right) .}
\end{array}
$$

It follows that the eight roots are $\pm(\mathrm{i}, 0), \pm(0, \mathrm{i}), \pm(\mathrm{i}, \mathrm{i}), \pm(\mathrm{i},-\mathrm{i})$. Now choose $e^{1}+2 e^{10} \in \mathfrak{k}$ as an element on which none of the roots vanishes. It defines $(i, 0),(0, i),(i, i),(-i, i)$ as positive roots. We thus obtain the Samelson subalgebras of $\operatorname{Spin}(5)$ with generators $\left\langle\varphi^{1}, \varphi^{2}, \varphi^{3}, \varphi^{4}, \varphi^{5}\right\rangle$ given by

$$
\begin{array}{lll}
\varphi^{1}=(1-a \mathrm{i}) e^{1}-b \mathrm{i} e^{10} \text { with } a+\mathrm{i} b \in \mathcal{H}_{+} \cup \mathcal{H}_{-}, & \varphi^{2}=e^{2}+\mathrm{i} e^{3}, & \varphi^{3}=e^{4}+\mathrm{i} e^{5}, \\
& \varphi^{4}=e^{7}+\mathrm{i} e^{8}, & \varphi^{5}=e^{6}+\mathrm{i} e^{9} .
\end{array}
$$

The two isotropic Samelson subalgebras are detected by the choices $a+\mathrm{i} b= \pm \mathrm{i}$ and we indicate the corresponding complex structures with $J_{ \pm}$. Indeed, it can be verified that $\left\langle e^{1}, e^{10}\right\rangle=0$ and they have the same norm.
$\mathrm{G}_{2}$
The group $G_{2}$ is the simple exceptional Lie group of rank two. Its Lie algebra $\mathfrak{g}_{2}$ is described by the Dynkin diagram. Hence, we can fix a system of simple roots given by $\left\{\alpha_{1}, \alpha_{2}\right\}$ generating all the positive roots as (see [191, Chapter II] for details)

$$
R^{+}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{1}+\alpha_{2}, 2 \alpha_{1}+\alpha_{2}, 3 \alpha_{1}+\alpha_{2}, 3 \alpha_{1}+2 \alpha_{2}\right\} .
$$

We now construct a basis of $\mathfrak{g}_{2}^{\mathbb{C}}$ adapted to the roots system as follows. Take $\varphi_{1} \in \mathfrak{g}_{2, \alpha_{1}}, \varphi_{2} \in \mathfrak{g}_{2, \alpha_{2}}$ and $\bar{\varphi}_{1} \in \mathfrak{g}_{2,-\alpha_{1}}, \bar{\varphi}_{2} \in \mathfrak{g}_{2,-\alpha_{2}}$. Thanks to the relations in (6.3), we define the eigenvectors of the other roots starting from $\varphi_{1}, \bar{\varphi}_{1}, \varphi_{2}, \bar{\varphi}_{2}$. Namely,

$$
\begin{array}{llll}
\varphi_{3}=\left[\varphi_{1}, \varphi_{2}\right], & \varphi_{4} & =\left[\varphi_{1}, \varphi_{3}\right], & \varphi_{5} \\
\bar{\varphi}_{3}=\left[\varphi_{1}, \varphi_{4}\right], & \left.\bar{\varphi}_{1}, \bar{\varphi}_{2}\right], & \bar{\varphi}_{4}=\left[\bar{\varphi}_{1}, \bar{\varphi}_{3}\right], & \bar{\varphi}_{5}=\left[\bar{\varphi}_{1}, \bar{\varphi}_{4}\right],
\end{array}
$$

while, the generators of the torus $\mathfrak{k} \in \mathfrak{g}_{2}$ can be chosen as $h_{i}=\left[\varphi_{i}, \bar{\varphi}_{i}\right]$ for $i=1,2$. Furthermore, we know that $\left[\mathfrak{g}_{2, \alpha}, \mathfrak{g}_{2, \beta}\right]=0$ if $\alpha+\beta$ is not a root (see again (6.3). Thus using the Jacobi identity we can compute all the non-vanishing products, which we summarize in the following Table 6.1

| $[\cdot, \cdot]$ | $\varphi_{1}$ | $\bar{\varphi}_{1}$ | $\varphi_{2}$ | $\bar{\varphi}_{2}$ | $\varphi_{3}$ | $\bar{\varphi}_{3}$ | $\varphi_{4}$ | $\bar{\varphi}_{4}$ | $\varphi_{5}$ | $\bar{\varphi}_{5}$ | $\varphi_{6}$ | $\bar{\varphi}_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h_{1}$ | $2 \varphi_{1}$ | $-2 \bar{\varphi}_{1}$ | $-3 \varphi_{2}$ | $3 \bar{\varphi}_{2}$ | $-\varphi_{3}$ | $\bar{\varphi}_{3}$ | $\varphi_{4}$ | $-\bar{\varphi}_{4}$ | $3 \varphi_{5}$ | $-3 \bar{\varphi}_{5}$ | 0 | 0 |
| $h_{2}$ | $-\varphi_{1}$ |  | $2 \varphi_{2}$ | $-2 \bar{\varphi}_{2}$ | $\varphi_{3}$ | $-\bar{\varphi}_{3}$ | 0 | - | $-\varphi_{5}$ | $\bar{\varphi}_{5}$ | $\varphi_{6}$ | $-\bar{\varphi}_{6}$ |
| $\varphi_{1}$ |  | $h_{1}$ | $\varphi_{3}$ | 0 | $\varphi_{4}$ | $3 \bar{\varphi}_{2}$ | $\varphi_{5}$ | $4 \bar{\varphi}_{3}$ | 0 | $3 \bar{\varphi}_{4}$ | 0 |  |
| $\bar{\varphi}_{1}$ |  |  | 0 | $\bar{\varphi}_{3}$ | $3 \varphi_{2}$ | $\bar{\varphi}_{4}$ | $4 \varphi_{3}$ | $\bar{\varphi}_{5}$ | $3 \varphi_{4}$ | 0 | 0 | 0 |
| $\varphi_{2}$ |  |  |  | $h_{2}$ | - | $-\bar{\varphi}_{1}$ | 0 | 0 | $\varphi_{6}$ | 0 | 0 | $\bar{\varphi}_{5}$ |
| $\bar{\varphi}_{2}$ |  |  |  |  | $-\varphi_{1}$ | 0 | 0 | 0 | 0 | $\bar{\varphi}_{6}$ | $\varphi_{5}$ | 0 |
| $\varphi_{3}$ |  |  |  |  |  | $\begin{aligned} & -h_{1} \\ & -3 h_{2} \end{aligned}$ | $-\varphi_{6}$ | $4 \bar{\varphi}_{1}$ | - | 0 | , | $3 \bar{\varphi}_{4}$ |
| $\bar{\varphi}_{3}$ |  |  |  |  |  |  | $4 \varphi_{1}$ | $-\bar{\varphi}_{6}$ | 0 | 0 | $3 \varphi_{4}$ | 0 |
| $\varphi_{4}$ |  |  |  |  |  |  |  | $\begin{aligned} & 8 h_{1} \\ & +12 h_{2} \end{aligned}$ | 0 | $-12 \bar{\varphi}_{1}$ | 0 | $12 \bar{\varphi}_{3}$ |
| $\bar{\varphi}_{4}$ |  |  |  |  |  |  |  |  | $-12 \varphi_{1}$ | 0 | $12 \varphi_{3}$ | 0 |
| $\varphi_{5}$ |  |  |  |  |  |  |  |  |  | $\begin{aligned} & -36 h_{1} \\ & -36 h_{2} \end{aligned}$ | 0 | $36 \bar{\varphi}_{2}$ |
| $\bar{\varphi}_{5}$ |  |  |  |  |  |  |  |  |  |  | $36 \varphi_{2}$ |  |
| $\varphi_{6}$ |  |  |  |  |  |  |  |  |  |  |  | $36 h_{1}$ $+72 h_{2}$ |

Table 6.1. Algebra structure of $\mathfrak{g}_{2}$

Finally, it remains to assign a complex structure on the torus to describe the Samelson subalgebras of $\mathfrak{g}_{2}$. Thus we define $\varphi_{7}:=(1-a \mathrm{i}) h_{1}-b \mathrm{i} h_{2}$ with $a+\mathrm{i} b \in \mathcal{H}_{+} \cup \mathcal{H}_{-}$, and then the Samelson subalgebras of $\mathfrak{g}_{2}$ are generated by $\left\langle\varphi^{1}, \varphi^{2}, \varphi^{3}, \varphi^{4}, \varphi^{5}, \varphi^{6}, \varphi^{7}\right\rangle$, where $\varphi_{7}=$ $(1-a \mathrm{i}) h_{1}-b \mathrm{i} h_{2}$ with $a+\mathrm{i} b \in \mathcal{H}_{+} \cup \mathcal{H}_{-}$. Since, $\left\|\alpha_{1}\right\|^{2}=3\left\|\alpha_{2}\right\|^{2}$ and $\left\langle h_{1}, h_{2}\right\rangle=-\frac{1}{2}\left\|h_{1}\right\|^{2}$ the two isotropic complex structures are given by $J h_{1}= \pm \sqrt{3}\left(h_{1}+2 h_{2}\right)$, that is

$$
\varphi_{7}=(1 \mp \sqrt{3} \mathrm{i}) h_{1} \mp 2 \sqrt{3} \mathrm{i} h_{2} .
$$

We indicate these complex structures with $J_{ \pm}$.

### 6.4 Bott-Chern cohomology of Bismut flat manifolds of rank two

The Dolbeault cohomology of compact simply-connected simple Lie groups of rank 2 equipped with left-invariant complex structures is computed in 261. In particular, given $G$ a $2 n$ dimensional compact simply-connected simple Lie group of rank 2 and $J$ a left-invariant complex
structure on it, a model of the cohomology ring $H_{\bar{\partial}}^{\bullet \bullet \bullet}(G, J)$ is given, depending on whether $J$ is isotropic or not. Namely [261, Proposition 4.5],

$$
H_{\bar{\partial}}^{\bullet, \bullet}(G, J) \cong \begin{cases}\mathbb{C}\left[y_{1,1}\right] /\left(\left(y_{1,1}\right)^{n-1}\right) \otimes \wedge^{\bullet \bullet}\left(\mathbb{C}\left\langle\left[u_{2,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle\right) & \text { if } J \text { is isotropic }  \tag{6.6}\\ \wedge^{\bullet, \bullet}\left(\mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[y_{1,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[u_{n, n-1}\right]\right\rangle\right) & \text { if } J \text { is not isotropic }\end{cases}
$$

where subscripts denote the bi-degree of the generators $x, y, u$.
Since we are interested in Bismut flat manifolds, in this section we compute the Bott-Chern and the Aeppli cohomology of compact simply-connected simple Lie groups of rank 2 when they are equipped with an isotropic complex structure. We do it in two different ways. In the first place (Section 6.4.1), we recover the structure of the double complexes from the knowledge of the Dolbeault and de Rham cohomologies of these spaces. This leads automatically to the understanding of all the quantitative cohomological properties. In particular, we are able to specify the Hodge diamonds of the Bott-Chern and Aeppli cohomologies. We remark that this argument is independent on the classification of compact simply-connected simple Lie groups. On the other hand, in Section 6.4 .2 we study case by case $\left(\mathrm{SU}(3), J_{0,-1}\right)$, ( $\left.\operatorname{Spin}(5), J_{ \pm}\right)$, and $\left(\mathrm{G}_{2}, J_{ \pm}\right)$, using Theorem 1.3 .1 to prove that the cohomologies of these spaces arise just from the left-invariant classes. Consequently, direct computations can be performed to give the harmonic representatives of the Bott-Chern and Aeppli cohomologies. This last strategy has been performed in 32 for $\mathrm{SU}(3)$ and $\operatorname{Spin}(5)$, while a greater computational power was needed for $G_{2}$.

For the sake of simplicity, we use the following notation: $\varphi^{i \bar{j}}=\varphi^{i} \wedge \bar{\varphi}^{j}$, and similarly for higher order.

### 6.4.1 Structure of the double complexes

We perform here the computation of the (1,1)-Aeppli cohomology of compact simply-connected simple Lie groups of rank 2 equipped with left-invariant isotropic complex structures. We do it using the relations between the Aeppli, Dolbeault, and de Rham cohomologies which reflect the structure of the double complex. This will lead (in Theorem 7.3.2) to a proof of the global stability of the pluriclosed flow. Afterward, we reconstruct the double complexes of $\left(\mathrm{G}_{2}, J_{ \pm}\right)$(up to squares), and thus the associated Bott-Chern diamonds. This covers the missing cases in 32 .

The de Rham cohomology of the compact Lie groups has been computed in $64,81,94,96$, 99, 170, 264, 267, 270, 334. In particular, Hopf [170] and Samelson [270] showed that for a compact Lie group the Poincaré polynomial, whose coefficients are the Betti numbers, is of the form

$$
P(t)=\prod_{i=1}^{k}\left(1+t^{p_{i}}\right)
$$

where $k$ is the rank of the group and the $p_{i}$ 's are odd integers. Then, for compact simple Lie groups, the $p_{i}$ 's were computed in [96, Page 354], and are

$$
\begin{array}{ll}
A_{k}: 3,5,7,9, \ldots, 2 k+1, & E_{6}: 3,9,11,15,17,23 \\
B_{k}: 3,7,11,15, \ldots, 4 k-1, & E_{7}: 3,11,15,19,23,27,35 \\
C_{k}: 3,7,11,15, \ldots, 4 k-1, & E_{8}: 3,15,23,27,35,39,47,59 \\
D_{k}: 3,7,11, \ldots, 4 k-5,2 k-1, & F_{4}: 3,11,15,23 \\
& G_{2}: 3,11
\end{array}
$$

In particular, for the three compact simple Lie groups of rank 2 the non-zero Betti numbers are

$$
\begin{array}{ll}
A_{2}=\mathrm{SU}(3): & b_{0}=b_{3}=b_{5}=b_{8}=1 \\
B_{2}=\operatorname{Spin}(5): & b_{0}=b_{3}=b_{7}=b_{10}=1 \\
G_{2}: & b_{0}=b_{3}=b_{11}=b_{14}=1
\end{array}
$$

Fix a compact simply-connected simple Lie group $G$ of rank 2, and equip it with a leftinvariant isotropic complex structure $J$. Thanks to 6.6 the lower bi-degrees of the Dolbeault diamond of $(G, J)$ are as in Figure 6.1.


Figure 6.1. Lower bi-degrees of the Dolbeault diamond of $(G, J)$.

Henceforth, the following implications hold.

- In bi-degree $(0,0)$ there must be a dot, because $b_{0}=1=h_{\bar{\partial}}^{0,0}$;
- In bi-degree $(0,1)$ there is a zig-zag starting in the $\partial$-direction since $h_{\bar{\partial}}^{0,1}=1$. This can not have length 1 or 3 otherwise it would give a non-zero class in $H_{d R}^{1}(G)$, while $b_{1}=0$. We exclude also length 4 because it would give a non-zero class in the ( 2,0 )-Dolbeault cohomology. Hence, it must be of length 2, and by symmetry, there is another length- 2 zig-zag connecting the bi-degrees $(1,1)$ and $(1,0)$. Notice that these zig-zags also give $h_{\bar{\partial}}^{1,1}=1$.
- There is a zig-zag starting or ending in bi-degree $(2,1)$, because $h_{\bar{\partial}}^{2,1}=1$. This can not be a dot, otherwise, by symmetry, it would be $b_{3}=2$. Suppose it starts at bi-degree $(2,1)$, meaning that it moves forward in the $\partial$-direction; then it can be neither of length 2 , because $h_{\bar{\partial}}^{3,1}=0$, nor of length 3 , because $h_{\bar{\partial}}^{0,3}=0$ (use symmetry); a contradiction. Thus it must end in bi-degree (2,1). Length 2, 4, and 5 are not admissible because $h_{\bar{\partial}}^{1,1} \neq 2, h_{\bar{\partial}}^{0,2}=0$, and $h_{\bar{\partial}}^{3,0}=0$ respectively. Therefore, it has length 3.
It then turns out that just by knowing the dimensions of the de Rham and Dolbeault cohomology groups we are able to reconstruct (the lower bi-degrees of) the double complex of $(G, J)$. We picture it in Figure 6.2.

We indicate with $\omega_{B F}$ the fundamental (1,1)-form associated to the Killing metric. Thanks to Theorem 2.1.1 it is a $\partial \bar{\partial}$-closed form. Furthermore, by integrating on the maximal torus one sees that it is not $(\partial+\bar{\partial})$-exact (see the proof of Theorem 7.3 .2 for the precise argument). Thus [ $\omega_{B F}$ ] is a non-zero element in $H_{A}^{1,1}(G, J)$. The theorem below follows immediately.
Theorem 6.4.1. Let $G$ be a compact simply-connected simple Lie group of rank 2. Consider the Hermitian structure given by the Killing metric and a compatible let-invariant complex structure $J$. Then the Aeppli cohomology of $(G, J)$ in bi-degree $(1,1)$ is one-dimensional, i.e.

$$
H_{A}^{1,1}(G, J) \cong \mathbb{C}
$$

It is actually possible to reconstruct the whole double complex of $(G, J)$ by performing the above arguments. We picture in Figure 6.3 the double complex of $\left(\mathrm{G}_{2}, J_{ \pm}\right)$(the ones of $\left(\mathrm{SU}(3), J_{0,-1}\right)$ and $\left(\operatorname{Spin}(5), J_{ \pm}\right)$are similar, see Figure 0.1).


Figure 6.2. Lower bi-degrees of the double complex of $(G, J)$, up to squares.


Figure 6.3. Double complex of $\left(\mathrm{G}_{2}, J_{ \pm}\right)$, up to squares.

Consequently, the Bott-Chern diamond of $\left(\mathrm{G}_{2}, J_{+}\right)$and $\left(\mathrm{G}_{2}, J_{-}\right)$are both equal to


### 6.4.2 Harmonic Bott-Chern representatives

On homogeneous spaces, it is rather uncommon that the double complex of left-invariant forms computes all the cohomologies (see [98, 120] and the references therein for analysis on nilmanifolds and solvmanifolds). Nonetheless, this was proved true, for example, for the Hopf surface $\mathbb{S}^{1} \times \mathrm{SU}(2)$ and the Calabi-Eckmann threefold $\mathrm{SU}(2) \times \mathrm{SU}(2)$ by Angella and Tomassini [21]. We show here that the double complex of left-invariant forms computes all the cohomologies of compact simply-connected simple Lie groups of rank 2 when they are equipped with left-invariant isotropic complex structures. As a consequence of [17, Theorem 1.1] it must be true also for small deformations of the isotropic complex structures, and we expect it to be also true in general in the non-isotropic case. The argument combines the knowledge of the Dolbeault cohomology (thanks to 261 ) and Theorem 1.3.1. It goes as follows. Fix a compact simply-connected simple Lie group $G$ of rank 2 , and equip it with a left-invariant isotropic complex structure $J$. We consider the sub-double complex of left-invariant forms

$$
\iota: \mathcal{L I}^{\bullet, \bullet} \hookrightarrow \mathcal{A}^{\bullet \bullet} .
$$

Notice that the double complex of left-invariant forms is always a direct summand in the double complex of all forms $\mathcal{A}^{\bullet \bullet}$. Indeed, since $G$ is compact averaging out a form is a map of double complexes that gives a one-sided inverse to the inclusion $\iota$ (see also [97, Lemma 7]). Therefore, left-invariant cohomology is always a direct summand in any cohomology. In particular, $H_{\bar{\partial}}(\iota)$ is injective. We then show that $\iota$ is also surjective in Dolbeault cohomology by checking that the dimensions of the Dolbeault left-invariant cohomology groups equal the ones given by (6.6). Namely, $\operatorname{dim} H_{\bar{\partial}}^{p, q}(\mathcal{L I})=h_{\bar{\partial}}^{p, q}(G, J)$. We remark that, since the Dolbeault cohomology is generated by $x_{0,1}, y_{1,1}$, and $u_{2,1}$, it is sufficient to check that $H_{\bar{\partial}}(\iota)$ is surjective in bi-degrees $(0,1),(1,1)$, and $(2,1)$. Finally, Theorem 1.3.1 ensures that the map $\iota$ also induces isomorphisms in Bott-Chern and Aeppli cohomologies. As a consequence, given $p, q \in\{1, \ldots, n\}$, the 4 -th order differential operator $\Delta_{B C}^{g}$ reduces to an endomorphism of the $\binom{n}{p}\binom{n}{q}$-dimensional vector space $\mathcal{L I}^{p, q}$. Therefore, with the help of the symbolic computation software Sage [293], we are able to explicitly compute

$$
\operatorname{ker}\left(\left(\Delta_{B C}^{g}\right)_{\mid}: \mathcal{L I}^{p, q} \rightarrow \mathcal{L I}^{p, q}\right)
$$

We perform this computation to find the harmonic representative of the Bott-Chern cohomologies of $\left(\mathrm{SU}(3), J_{0,-1}\right)$ and $\left(\operatorname{Spin}(5), J_{ \pm}\right)$, while for $\left(\mathrm{G}_{2}, J_{ \pm}\right)$they require more powerful computational tools than the one we have.

## SU(3)

We consider the Bismut flat manifold $\left(\mathrm{SU}(3), J_{0,-1}, \omega_{B F}\right)$ where $J_{0,-1}$ is the left-invariant isotropic complex structure given in Section 6.3 .3 and $\omega_{B F}$ represents the Hermitian metric coming from the Killing form; more precisely it is

$$
\omega_{B F}:=\frac{\mathrm{i}}{2} \sum_{k=1}^{4} \varphi^{k} \wedge \bar{\varphi}^{k}
$$

By computing the complex structure equations, we obtain

$$
\left\{\begin{array} { l } 
{ \partial \varphi ^ { 1 } = - \mathrm { i } \varphi ^ { 1 4 } + \mathrm { i } \varphi ^ { 2 3 } , } \\
{ \partial \varphi ^ { 2 } = - \frac { 1 } { 2 } ( \sqrt { 3 } + \mathrm { i } ) \varphi ^ { 2 4 } , } \\
{ \partial \varphi ^ { 3 } = \frac { 1 } { 2 } ( \sqrt { 3 } - \mathrm { i } ) \varphi ^ { 3 4 } , } \\
{ \partial \varphi ^ { 4 } = 0 , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
\bar{\partial} \varphi^{1}=-\mathrm{i} \varphi^{1 \overline{4}}, \\
\bar{\partial} \varphi^{2} \\
\bar{\partial} \varphi^{3} \\
\bar{\partial} \varphi^{4}
\end{array}=-\mathrm{i} \varphi^{1 \overline{3}}+\frac{1}{2}(\sqrt{3}-\mathrm{i}) \varphi^{1 \overline{2}}-\frac{1}{2}(\sqrt{3}+\mathrm{i}) \varphi^{1 \overline{4}}+\frac{1}{2}(-\sqrt{3}+\mathrm{i}) \varphi^{2 \overline{4}}+\frac{1}{2}(\sqrt{3}+\mathrm{i}) \varphi^{3 \overline{3}} .\right.\right.
$$

Thanks to (6.6), when $\mathrm{SU}(3)$ is equipped with its isotropic left-invariant complex structure it holds

$$
H_{\bar{\partial}}^{\bullet \bullet \bullet}(\mathrm{SU}(3)) \simeq \mathbb{C}\left[y_{1,1}\right] /\left(\left(y_{1,1}\right)^{3}\right) \otimes \wedge^{\bullet \bullet}\left(\mathbb{C}\left\langle\left[u_{2,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle\right)
$$

We thus recover the Hodge numbers of $\left(\operatorname{SU}(3), J_{0,-1}\right)$, which are


Consider the sub-complex of left-invariant forms

$$
\iota: \bigwedge\left\langle\varphi^{1}, \varphi^{2}, \varphi^{3}, \varphi^{4}, \bar{\varphi}^{1}, \bar{\varphi}^{2}, \bar{\varphi}^{3}, \varphi^{4}\right\rangle \hookrightarrow \wedge^{\bullet \bullet} \operatorname{SU}(3)
$$

A direct computation leads to the following conditions

$$
\begin{aligned}
& H_{\bar{\partial}}^{0,1}(\mathrm{SU}(3))_{\text {inv }}=\mathbb{C}\left\langle\left[\bar{\varphi}^{4}\right]\right\rangle ; \\
& H_{\bar{\partial}}^{1,1}(\mathrm{SU}(3))_{\text {inv }}=\mathbb{C}\left\langle\left[\varphi^{1 \overline{1}}+\varphi^{2 \overline{2}}\right]\right\rangle ; \\
& H_{\bar{\partial}}^{2,1}(\mathrm{SU}(3))_{\text {inv }}=\mathbb{C}\left\langle\left[2 \varphi^{14 \overline{1}}-2 \varphi^{23 \overline{1}}+(1-\sqrt{3} \mathrm{i}) \varphi^{24 \overline{2}}+(1+\sqrt{3} \mathrm{i}) \varphi^{34 \overline{3}}\right]\right\rangle,
\end{aligned}
$$

where the subscript "inv" indicates that those are the invariant cohomology groups. Therefore, $H_{\bar{\partial}}(\iota)$ is an isomorphism and the formal representative $x_{0,1}, y_{1,1}$ and $u_{2,1}$ of Pittie's model are respectively in the left-invariant classes

$$
\left[\bar{\varphi}^{4}\right],\left[\varphi^{1 \overline{1}}+\varphi^{2 \overline{2}}\right] \text { and }\left[2 \varphi^{14 \overline{1}}-2 \varphi^{23 \overline{1}}+(1-\sqrt{3} \mathrm{i}) \varphi^{24 \overline{2}}+(1+\sqrt{3} \mathrm{i}) \varphi^{34 \overline{3}}\right]
$$

In particular, we can compute the harmonic representatives of the Dolbeault cohomology. They are

$$
\begin{aligned}
& H_{\bar{\partial}}^{\bullet} \bullet \\
&(\mathrm{SU}(3))= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{\overline{4}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[2 \varphi^{1 \overline{1}}+(1-\sqrt{3} \mathrm{i}) \varphi^{2 \overline{2}}+(1+\sqrt{3} \mathrm{i}) \varphi^{3 \overline{3}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{14 \overline{1}}-2 \varphi^{23 \overline{1}}+(1-\sqrt{3} \mathrm{i}) \varphi^{24 \overline{2}}+(1+\sqrt{3} \mathrm{i}) \varphi^{34 \overline{3}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{1 \overline{14}}+(1-\sqrt{3} \mathrm{i}) \varphi^{2 \overline{44}}+(1+\sqrt{3} \mathrm{i}) \varphi^{3 \overline{34}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{14 \overline{14}}-2 \varphi^{23 \overline{14}}+(1-\sqrt{3} \mathrm{i}) \varphi^{24 \overline{24}}+(1+\sqrt{3} \mathrm{i}) \varphi^{34 \overline{34}}\right],\right. \\
& {\left.\left[2 \varphi^{12 \overline{12}}-(1-\sqrt{3} \mathrm{i}) \varphi^{13 \overline{13}}-(1+\sqrt{3} \mathrm{i}) \varphi^{14 \overline{23}}+(1+\sqrt{3} \mathrm{i}) \varphi^{23 \overline{3 \overline{3}}}\right]\right\rangle } \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{124 \overline{12}}-(1-\sqrt{3} \mathrm{i}) \varphi^{134 \overline{13}}+(1+\sqrt{3} \mathrm{i}) \varphi^{234 \overline{33}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{12 \overline{124}}-(1-\sqrt{3} \mathrm{i}) \varphi^{13 \overline{34}}-(1+\sqrt{3} \mathrm{i}) \varphi^{14 \overline{234}}+(1+\sqrt{3} \mathrm{i}) \varphi^{23 \overline{334}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{124 \overline{124}}-(1-\sqrt{3} \mathrm{i}) \varphi^{134134}+(1+\sqrt{3} \mathrm{i}) \varphi^{2342 \overline{234}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{1234 \overline{123}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{1234 \overline{1234}}\right]\right\rangle .
\end{aligned}
$$

By Theorem 1.3.1 (also Theorem 1.3 and Proposition 2.2 in 16 would work properly), we have that also $H_{B C}(\iota)$ is an isomorphism. In particular, the Bott-Chern cohomology arises from just the left-invariant forms, and we can directly compute the harmonic representatives for
$H_{B C}^{\bullet \bullet \bullet}\left(\mathrm{SU}(3), J_{0,-1}\right)$. We list them here.

$$
\begin{aligned}
H_{B C}^{\bullet \bullet}(\mathrm{SU}(3))= & \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{1 \overline{1}}+\varphi^{2 \overline{2}}\right],\left[\varphi^{2 \overline{2}}-\varphi^{3 \overline{3}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{14 \overline{1}}-2 \varphi^{23 \overline{1}}+(1-\sqrt{3} \mathrm{i}) \varphi^{24 \overline{2}}+(1+\sqrt{3} \mathrm{i}) \varphi^{34 \overline{3}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{1 \overline{14}}-2 \varphi^{1 \overline{23}}+(1+\sqrt{3} \mathrm{i}) \varphi^{2 \overline{24}}+(1-\sqrt{3} \mathrm{i}) \varphi^{3 \overline{34}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[8 \varphi^{12 \overline{12}}-(3-\sqrt{3} \mathrm{i}) \varphi^{23 \overline{14}}+8 \varphi^{23 \overline{3}}\right],\right. \\
& {\left.\left[8 \varphi^{13 \overline{313}}-(3+\sqrt{3} \mathrm{i}) \varphi^{23 \overline{14}}+8 \varphi^{23 \overline{23}}\right]\right\rangle } \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{124 \overline{12}}-(1-\sqrt{3} \mathrm{i}) \varphi^{134 \overline{13}}+(1+\sqrt{3} \mathrm{i}) \varphi^{234 \overline{23}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[2 \varphi^{12 \overline{124}}-(1+\sqrt{3} \mathrm{i}) \varphi^{13 \overline{134}}+(1-\sqrt{3} \mathrm{i}) \varphi^{23 \overline{344}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[3 \varphi^{123 \overline{23}}+\varphi^{124 \overline{124}}+\varphi^{134 \overline{134}}-\varphi^{234 \overline{234}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{1234 \overline{123}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{123 \overline{1234}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{1234 \overline{1234}}\right]\right\rangle .
\end{aligned}
$$

Finally, the Bott-Chern numbers are


Spin(5)
We now consider the Bismut flat manifolds ( $\left.\operatorname{Spin}(5), J_{ \pm}, \omega_{B F}\right)$ where $J_{ \pm}$are the only two leftinvariant isotropic complex structures on $\operatorname{Spin}(5)$ as described in Section 6.3 .3 and $\omega_{B F}$ represents the Hermitian metric coming from the Killing form, which is

$$
\omega_{B F}:=\frac{\mathrm{i}}{2} \sum_{k=1}^{5} \varphi^{k} \wedge \bar{\varphi}^{k}
$$

By computing the complex structure equations, we obtain

$$
\left\{\begin{array} { l } 
{ \partial \varphi ^ { 1 } = 0 , } \\
{ \partial \varphi ^ { 2 } = \mathrm { i } \varphi ^ { 1 2 } - \varphi ^ { 3 5 } - \mathrm { i } \varphi ^ { 4 5 } , } \\
{ \partial \varphi ^ { 3 } = \mathrm { i } \varphi ^ { 1 3 } \pm \mathrm { i } \varphi ^ { 1 4 } + \varphi ^ { 2 5 } , } \\
{ \partial \varphi ^ { 4 } = \mp \mathrm { i } \varphi ^ { 1 3 } + \mathrm { i } \varphi ^ { 1 4 } + \mathrm { i } \varphi ^ { 2 5 } , } \\
{ \partial \varphi ^ { 5 } = \mp \varphi ^ { 1 5 } , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{rl}
\bar{\partial} \varphi^{1} & =\mathrm{i} \varphi^{2 \overline{2}}+\mathrm{i} \varphi^{3 \overline{3}} \pm \mathrm{i} \varphi^{3 \overline{4}} \mp \mathrm{i} \varphi^{4 \overline{3}}+\mathrm{i} \varphi^{4 \overline{4}} \pm \varphi^{5 \overline{5}}, \\
\bar{\partial} \varphi^{2} & =-\mathrm{i} \varphi^{2 \overline{1}}-\varphi^{3 \overline{5}}+\mathrm{i} \varphi^{4 \overline{5}}, \\
\bar{\partial} \varphi^{3} & =\varphi^{2 \overline{5}}-\mathrm{i} \varphi^{3 \overline{1}} \pm \mathrm{i} \varphi^{4 \overline{1}}, \\
\bar{\partial} \varphi^{4} & =-\mathrm{i} \varphi^{2 \overline{5}} \mp \mathrm{i} \varphi^{3 \overline{1}}-\mathrm{i} \varphi^{4 \overline{1}}, \\
\bar{\partial} \varphi^{5} & =-\varphi^{2 \overline{3}}+\mathrm{i} \varphi^{2 \overline{4}}+\varphi^{3 \overline{2}}-\mathrm{i} \varphi^{4 \overline{2}} \mp \varphi^{5 \overline{1}}
\end{array}\right.\right.
$$

where $\pm$ depend on the choice of the complex structure $J_{+}$or $J_{-}$.
When $\operatorname{Spin}(5)$ is equipped with a isotropic left-invariant complex structure, (6.6) gives

$$
H_{\bar{\partial}}^{\bullet, \bullet}(\operatorname{Spin}(5)) \simeq \mathbb{C}\left[y_{1,1}\right] /\left(\left(y_{1,1}\right)^{4}\right) \otimes \wedge^{\bullet, \bullet}\left(\mathbb{C}\left\langle\left[u_{2,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle\right)
$$

and we recover the Hodge diamond


We consider the sub-complex of left-invariant forms

$$
\iota: \bigwedge\left\langle\varphi^{1}, \varphi^{2}, \varphi^{3}, \varphi^{4}, \varphi^{5}, \bar{\varphi}^{1}, \bar{\varphi}^{2}, \bar{\varphi}^{3}, \bar{\varphi}^{4}, \bar{\varphi}^{5}\right\rangle \hookrightarrow \wedge^{\bullet \bullet} \operatorname{Spin}(5),
$$

and we check that $H_{\bar{\partial}}(\iota)$ is surjective in bi-degree $(0,1),(1,1)$ and $(2,1)$ for both $J_{+}$and $J_{-}$. More precisely, we verify that the sub-complex has cohomologies $H_{\bar{\partial}}^{0,1}(\operatorname{Spin}(5))_{i n v}, H_{\bar{\partial}}^{1,1}(\operatorname{Spin}(5))_{\text {inv }}$ and $H_{\bar{\partial}}^{2,1}(\operatorname{Spin}(5))_{i n v}$ of dimension one generated respectively by

$$
\left[\bar{\varphi}^{1}\right],\left[\varphi^{2 \overline{2}}+\varphi^{3 \overline{3}}+\varphi^{4 \overline{4}}\right]
$$

and

$$
\left[\varphi^{12 \overline{2}}+\varphi^{13 \overline{3}} \mp \varphi^{13 \overline{4}} \pm \varphi^{14 \overline{3}}+\varphi^{14 \overline{4}} \pm \mathrm{i} \varphi^{15 \overline{5}}-\mathrm{i} \varphi^{25 \overline{3}}+\varphi^{25 \overline{4}}+\mathrm{i} \varphi^{35 \overline{2}}-\varphi^{45 \overline{2}}\right]
$$

Therefore, $H_{\bar{\partial}}(\iota)$ is an isomorphism and we can directly compute the harmonic representatives of the Dolbeault cohomology ring. They are

$$
\begin{aligned}
& H \stackrel{\rightharpoonup}{\bullet} \cdot(\operatorname{Spin}(5))=\mathbb{C}\langle 1\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{\overline{1}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{2 \overline{2}}+\varphi^{3 \overline{3}} \mp \varphi^{3 \overline{4}} \pm \varphi^{4 \overline{3}}+\varphi^{4 \overline{4}} \pm \mathrm{i} \varphi^{5 \overline{5}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{2 \overline{12}}+\varphi^{3 \overline{13}} \mp \varphi^{3 \overline{14}} \pm \varphi^{4 \overline{13}}+\varphi^{4 \overline{14}} \pm \mathrm{i} \varphi^{5 \overline{15}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{12 \overline{2}}+\varphi^{13 \overline{3}} \mp \varphi^{13 \overline{4}} \pm \varphi^{14 \overline{3}}+\varphi^{14 \overline{4}} \pm \mathrm{i} \varphi^{15 \overline{5}}-\mathrm{i} \varphi^{25 \overline{3}}+\varphi^{25 \overline{4}}+\mathrm{i} \varphi^{35 \overline{2}}-\varphi^{45 \overline{2}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{12 \overline{12}}+\varphi^{13 \overline{13}} \mp \varphi^{13 \overline{14}} \pm \varphi^{14 \overline{13}}+\varphi^{14 \overline{14}} \pm \mathrm{i} \varphi^{15 \overline{15}}-\mathrm{i} \varphi^{25 \overline{13}}+\varphi^{25 \overline{14}}+\mathrm{i} \varphi^{35 \overline{12}}-\varphi^{45 \overline{12}}\right],\right. \\
& {\left[\varphi^{12 \overline{35}}-\mathrm{i} \varphi^{12 \overline{45}} \mp \mathrm{i} \varphi^{13 \overline{25}} \mp \varphi^{14 \overline{25}}+(\mathrm{i} \pm 1) \varphi^{23 \overline{23}}-(1 \pm \mathrm{i}) \varphi^{23 \overline{24}}+(1 \pm \mathrm{i}) \varphi^{24 \overline{23}}+(\mathrm{i} \pm 1) \varphi^{24 \overline{24}}\right.} \\
& \left.\left.+2(\mathrm{i} \mp 1) \varphi^{25 \overline{25}}+(\mathrm{i} \pm 1) \varphi^{34 \overline{34}}+(\mathrm{i} \mp 1) \varphi^{35 \overline{35}}+(1 \mp \mathrm{i}) \varphi^{35 \overline{45}}-(1 \mp \mathrm{i}) \varphi^{45 \overline{35}}+(\mathrm{i} \mp 1) \varphi^{45 \overline{45}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 2 \varphi^{123 \overline{23}} \mp 2 \varphi^{123 \overline{24}} \pm 2 \varphi^{124 \overline{23}}+2 \varphi^{124 \overline{24}} \pm 2 \mathrm{i} \varphi^{125 \overline{25}}+4 \varphi^{134 \overline{34}} \pm 2 \mathrm{i} \varphi^{135 \overline{35}}-2 \mathrm{i} \varphi^{135 \overline{45}}\right.\right. \\
& \left.\left.+2 \mathrm{i} \varphi^{145 \overline{35}} \pm 2 \mathrm{i} \varphi^{145 \overline{45}}+(1 \mp \mathrm{i}) \varphi^{235 \overline{34}}+(\mathrm{i} \pm 1) \varphi^{245 \overline{34}}-(1 \pm \mathrm{i}) \varphi^{345 \overline{23}}-(\mathrm{i} \mp 1) \varphi^{345 \overline{24}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{12 \overline{135}}-\mathrm{i} \varphi^{12 \overline{145}} \mp \mathrm{i} \varphi^{13 \overline{125}} \mp \varphi^{14 \overline{125}}+(\mathrm{i} \pm 1) \varphi^{23 \overline{123}}-(1 \pm \mathrm{i}) \varphi^{23 \overline{124}}+(1 \pm \mathrm{i}) \varphi^{24 \overline{123}}+(\mathrm{i} \pm 1) \varphi^{24 \overline{124}}\right.\right. \\
& \left.\left.+2(\mathrm{i} \mp 1) \varphi^{25 \overline{125}}+(\mathrm{i} \pm 1) \varphi^{34 \overline{134}}+(\mathrm{i} \mp 1) \varphi^{35 \overline{135}}+(1 \mp \mathrm{i}) \varphi^{35 \overline{145}}-(1 \mp \mathrm{i}) \varphi^{45 \overline{135}}+(\mathrm{i} \mp 1) \varphi^{45 \overline{145}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 2 \varphi^{123 \overline{123}} \mp 2 \varphi^{123 \overline{124}} \pm 2 \varphi^{124 \overline{123}}+2 \varphi^{124 \overline{124}} \pm 2 \mathrm{i} \varphi^{125 \overline{125}}+4 \varphi^{134 \overline{134}} \pm 2 \mathrm{i} \varphi^{135 \overline{135}}-2 \mathrm{i} \varphi^{135 \overline{145}}\right.\right. \\
& \left.+2 \mathrm{i} \varphi^{145 \overline{135}} \pm 2 \mathrm{i} \varphi^{145 \overline{145}}+(1 \mp \mathrm{i}) \varphi^{235 \overline{134}}+(\mathrm{i} \pm 1) \varphi^{245 \overline{134}}-(1 \pm \mathrm{i}) \varphi^{345 \overline{123}}-(\mathrm{i} \mp 1) \varphi^{345 \overline{124}}\right], \\
& {\left[\varphi^{123 \overline{345}}-\mathrm{i} \varphi^{124 \overline{345}}-\varphi^{134 \overline{235}}+\mathrm{i} \varphi^{134 \overline{245}} \pm \mathrm{i} \varphi^{234 \overline{234}}-\varphi^{235 \overline{235}} \pm \varphi^{235 \overline{245}} \mp \varphi^{245 \overline{235}}-\varphi^{245 \overline{245}}\right.} \\
& \left.\left.-\varphi^{345 \overline{345}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{1234 \overline{234}} \pm \mathrm{i} \varphi^{1235 \overline{235}}-\mathrm{i} \varphi^{1235 \overline{245}}+\mathrm{i} \varphi^{1245 \overline{235}} \pm \mathrm{i} \varphi^{1245 \overline{245}} \pm \mathrm{i} \varphi^{1345 \overline{345}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{123 \overline{1345}}-\mathrm{i} \varphi^{124 \overline{1345}}-\varphi^{134 \overline{1235}}+\mathrm{i} \varphi^{134 \overline{1245}} \pm \mathrm{i} \varphi^{234 \overline{1234}}-\varphi^{235 \overline{1235}} \pm \varphi^{235 \overline{1245}} \mp \varphi^{245 \overline{1235}}\right.\right. \\
& \left.\left.-\varphi^{245 \overline{1245}}-\varphi^{345 \overline{1345}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{12345 \overline{2345}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{12345 \overline{12345}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{1234 \overline{1234}} \pm \mathrm{i} \varphi^{\left.\left.1235 \overline{1235}-\mathrm{i} \varphi^{1235 \overline{1245}}+\mathrm{i} \varphi^{1245 \overline{1235}} \pm \mathrm{i} \varphi^{1245 \overline{1245}} \pm \mathrm{i} \varphi^{1345 \overline{1345}}\right]\right\rangle,}\right.\right.
\end{aligned}
$$

As before, applying Theorem 1.3.1 we obtain that $H_{B C}(\iota)$ is an isomorphism. In particular, the Bott-Chern cohomology arises from just the left-invariant forms as well as the Aeppli cohomology. We thus compute the harmonic representatives of the Bott-Chern cohomology, which are

$$
\begin{aligned}
& H_{B C}^{\bullet \bullet \bullet}(\operatorname{Spin}(5))=\mathbb{C}\langle 1\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{2 \overline{2}}+\varphi^{3 \overline{3}}+\varphi^{4 \overline{4}}\right],\left[\varphi^{3 \overline{4}}-\varphi^{4 \overline{3}}-\mathrm{i} \varphi^{5 \overline{5}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{12 \overline{2}}+\varphi^{13 \overline{3}} \mp \varphi^{13 \overline{4}} \pm \varphi^{14 \overline{3}}+\varphi^{14 \overline{4}} \pm \mathrm{i} \varphi^{15 \overline{5}}-\mathrm{i} \varphi^{25 \overline{3}}+\varphi^{25 \overline{4}}+\mathrm{i} \varphi^{35 \overline{2}}-\varphi^{45 \overline{2}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[\varphi^{2 \overline{12}}+\varphi^{3 \overline{13}} \mp \varphi^{4 \overline{13}} \pm \varphi^{3 \overline{14}}+\varphi^{4 \overline{14}} \mp \mathrm{i} \varphi^{5 \overline{15}}+\mathrm{i} \varphi^{3 \overline{25}}+\varphi^{4 \overline{25}}-\mathrm{i} \varphi^{2 \overline{35}}-\varphi^{2 \overline{45}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 9 \varphi^{23 \overline{23}}+9 \varphi^{24 \overline{24}}+(\mathrm{i} \mp 1) \varphi^{25 \overline{13}}-(1 \pm \mathrm{i}) \varphi^{25 \overline{14}}+16 \varphi^{34 \overline{34}}+(\mathrm{i} \pm 1) \varphi^{35 \overline{12}}-9 \mathrm{i} \varphi^{35 \overline{45}}-(1 \mp \mathrm{i}) \varphi^{45 \overline{12}}\right.\right. \\
& \left.+9 \mathrm{i} \varphi^{45 \overline{35}}\right],\left[3 \varphi^{23 \overline{24}}-3 \varphi^{24 \overline{23}}-(1 \mp \mathrm{i}) \varphi^{25 \overline{13}}-6 \mathrm{i} \varphi^{25 \overline{25}}-(\mathrm{i} \pm 1) \varphi^{25 \overline{14}} \mp 2 \varphi^{34 \overline{34}}+(1 \pm \mathrm{i}) \varphi^{35 \overline{12}}\right. \\
& \left.\left.-3 \mathrm{i} \varphi^{35 \overline{35}}+(\mathrm{i} \mp 1) \varphi^{45 \overline{12}}-3 \mathrm{i} \varphi^{45 \overline{45}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 316 \varphi^{123 \overline{23}} \mp 332 \varphi^{123 \overline{24}} \pm 332 \varphi^{124 \overline{23}}+316 \varphi^{124 \overline{24}}-(42 \mathrm{i} \mp 50) \varphi^{125 \overline{13}}+(42 \pm 50 \mathrm{i}) \varphi^{125 \overline{14}}\right.\right. \\
& \pm 342 \mathrm{i} \varphi^{125 \overline{25}}+788 \varphi^{134 \overline{34}}-(42 \mathrm{i} \pm 50) \varphi^{135 \overline{12}} \pm 324 \mathrm{i} \varphi^{135 \overline{35}}-380 \mathrm{i} \varphi^{135 \overline{45}}+(42 \mp 50 \mathrm{i}) \varphi^{145 \overline{12}} \\
& +(189 \mp 125 \mathrm{i}) \varphi^{235 \overline{34}}+(189 \mathrm{i} \pm 125) \varphi^{245 \overline{34}}+380 \mathrm{i} \varphi^{145 \overline{35}} \pm 324 \mathrm{i} \varphi^{145 \overline{45}}-(189 \pm 125 \mathrm{i}) \varphi^{345 \overline{23}} \\
& \left.\left.-(189 \mathrm{i} \mp 125) \varphi^{345 \overline{24}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 316 \varphi^{23 \overline{123}} \mp 332 \varphi^{24 \overline{123}} \pm 332 \varphi^{23 \overline{124}}+316 \varphi^{24 \overline{124}}+(42 \mathrm{i} \pm 50) \varphi^{13 \overline{125}}+(42 \mp 50 \mathrm{i}) \varphi^{14 \overline{125}}\right.\right. \\
& \mp 342 \mathrm{i} \varphi^{25 \overline{125}}+788 \varphi^{34 \overline{134}}+(42 \mathrm{i} \mp 50) \varphi^{12 \overline{135}} \mp 324 \mathrm{i} \varphi^{35 \overline{135}}+380 \mathrm{i} \varphi^{45 \overline{135}}+(42 \pm 50 \mathrm{i}) \varphi^{12 \overline{145}} \\
& +(189 \pm 125 \mathrm{i}) \varphi^{34235}-(189 \mathrm{i} \mp 125) \varphi^{34245}-380 \mathrm{i} \varphi^{35 \overline{45}} \mp 324 \mathrm{i} \varphi^{45 \overline{45}}-(189 \mp 125 \mathrm{i}) \varphi^{23 \overline{45}} \\
& \left.\left.+(189 \mathrm{i} \pm 125) \varphi^{24 \overline{345}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 22 \varphi^{123 \overline{123}}-11(1 \pm \mathrm{i}) \varphi^{123 \overline{345}}+22 \varphi^{124 \overline{124}}+11(\mathrm{i} \mp 1) \varphi^{124 \overline{345}}+22 \mathrm{i} \varphi^{135 \overline{145}}+11(1 \mp \mathrm{i}) \varphi^{134 \overline{235}}\right.\right. \\
& -11(\mathrm{i} \pm 1) \varphi^{134 \overline{245}}-22 \mathrm{i} \varphi^{145 \overline{135}} \pm 208 \mathrm{i} \varphi^{234 \overline{234}}+276 \varphi^{235 \overline{235}}+(209 \pm \mathrm{i}) \varphi^{235 \overline{134}} \pm 208 \varphi^{235 \overline{245}} \\
& \left.+(209 \mathrm{i} \mp 1) \varphi^{245 \overline{134}} \mp 208 \varphi^{245 \overline{235}}+276 \varphi^{245 \overline{245}}-(209 \mp \mathrm{i}) \varphi^{345 \overline{123}}-(209 \mathrm{i} \pm 1) \varphi^{345 \overline{124}}+276 \varphi^{345 \overline{345}}\right] \text {, } \\
& {\left[22 \varphi^{123 \overline{124}}-22 \varphi^{124 \overline{123}}-11(\mathrm{i} \mp 1) \varphi^{123 \overline{345}}-11(1 \pm \mathrm{i}) \varphi^{124 \overline{345}}-11(\mathrm{i} \pm 1) \varphi^{134 \overline{235}}-11(1 \mp \mathrm{i}) \varphi^{134 \overline{245}}\right.} \\
& +44 \mathrm{i} \varphi^{125 \overline{125}}-22 \mathrm{i} \varphi^{135 \overline{135}}-22 \mathrm{i} \varphi^{145 \overline{145}}+452 \mathrm{i} \varphi^{234 \overline{234}}-(\mathrm{i} \mp 341) \varphi^{235 \overline{134}} \pm 340 \varphi^{235 \overline{235}} \\
& +452 \varphi^{235 \overline{245}}+(1 \pm 341 \mathrm{i}) \varphi^{245 \overline{134}}-452 \varphi^{245 \overline{235}} \pm 340 \varphi^{245 \overline{245}}-(\mathrm{i} \pm 341) \varphi^{345 \overline{123}} \pm 340 \varphi^{345 \overline{345}} \\
& \left.\left.+(1 \mp 341 \mathrm{i}) \varphi^{345 \overline{124}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[2 \varphi^{1234 \overline{1234}}+\mathrm{i} \varphi^{1235 \overline{1245}}-\mathrm{i} \varphi^{1245 \overline{1235}}+3 \varphi^{2345 \overline{2345}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 7 \varphi^{1234 \overline{234}}+2 \varphi^{1235 \overline{134}} \pm 7 \mathrm{i} \varphi^{1235 \overline{235}}-7 \mathrm{i} \varphi^{1235 \overline{245}}+2 \mathrm{i} \varphi^{1245 \overline{134}}+7 \mathrm{i} \varphi^{1245 \overline{235}} \pm 7 \mathrm{i} \varphi^{1245 \overline{245}}\right.\right. \\
& \left.\left.+2 \varphi^{1345 \overline{123}}+2 \mathrm{i} \varphi^{1345 \overline{124}} \pm 7 \mathrm{i} \varphi^{1345 \overline{345}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{12345 \overline{2345}}\right]\right\rangle \\
& \oplus \mathbb{C}\left\langle\left[ 7 \varphi^{234 \overline{1234}}+2 \varphi^{134 \overline{1235}} \mp 7 \mathrm{i} \varphi^{235 \overline{1235}}+7 \mathrm{i} \varphi^{245 \overline{1235}}-2 \mathrm{i} \varphi^{134 \overline{1245}}-7 \mathrm{i} \varphi^{235 \overline{1245}} \mp 7 \mathrm{i} \varphi^{245 \overline{1245}}\right.\right. \\
& \left.\left.+2 \varphi^{123 \overline{1345}}-2 \mathrm{i} \varphi^{124 \overline{1345}} \mp 7 \mathrm{i} \varphi^{345 \overline{1345}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{2345 \overline{12345}}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[\varphi^{12345 \overline{12345}}\right]\right\rangle .
\end{aligned}
$$

Consequently, the Bott-Chern numbers are

$\mathrm{G}_{2}$
Finally, we consider the Bismut flat manifolds $\left(\mathrm{G}_{2}, J_{ \pm}, \omega_{B F}\right)$ where $J_{ \pm}$are the only two leftinvariant isotropic complex structures on $\mathrm{G}_{2}$ as described in the Section 6.3.3 and $\omega_{B F}$ represents the Hermitian metric coming from the Killing form, which is

$$
\omega_{B F}:=\mathrm{i}\left(3 \varphi^{1 \overline{1}}+\varphi^{2 \overline{2}}-3 \varphi^{3 \overline{3}}+12 \varphi^{4 \overline{4}}-36 \varphi^{5 \overline{5}}+36 \varphi^{6 \overline{6}}-12 \varphi^{7 \overline{7}}\right) .
$$

By computing the complex structure equations, we obtain

$$
\left\{\begin{aligned}
\partial \varphi^{1} & =-2 \varphi^{17} \\
\partial \varphi^{2} & =( \pm \mathrm{i} \sqrt{3}+3) \varphi^{27} \\
\partial \varphi^{3} & =\varphi^{12}+( \pm \mathrm{i} \sqrt{3}+1) \varphi^{37} \\
\partial \varphi^{4} & =\varphi^{13}+( \pm \mathrm{i} \sqrt{3}-1) \varphi^{47} \\
\partial \varphi^{5} & =\varphi^{14}+( \pm \mathrm{i} \sqrt{3}-3) \varphi^{57} \\
\partial \varphi^{6} & =\varphi^{25}-\varphi^{34} \pm 2 \mathrm{i} \sqrt{3} \varphi^{67} \\
\partial \varphi^{7} & =0
\end{aligned}\right.
$$

$$
\text { and }\left\{\begin{aligned}
\bar{\partial} \varphi^{1}= & -\varphi^{2 \overline{3}}+4 \varphi^{3 \overline{4}}-12 \varphi^{4 \overline{5}}-2 \varphi^{7 \overline{1}} \\
\bar{\partial} \varphi^{2}= & ( \pm \mathrm{i} \sqrt{3}-3) \varphi^{2 \overline{7}}-3 \varphi^{3 \overline{1}}-36 \varphi^{6 \overline{5}} \\
\bar{\partial} \varphi^{3}= & ( \pm \mathrm{i} \sqrt{3}-1) \varphi^{3 \overline{7}}-4 \varphi^{4 \overline{1}}-12 \varphi^{6 \overline{4}} \\
\bar{\partial} \varphi^{4}= & ( \pm \mathrm{i} \sqrt{3}+1) \varphi^{4 \overline{7}}-3 \varphi^{5 \overline{1}}-3 \varphi^{6 \overline{3}}, \\
\bar{\partial} \varphi^{5}= & ( \pm \mathrm{i} \sqrt{3}+3) \varphi^{5 \overline{7}}-\varphi^{6 \overline{2}} \\
\bar{\partial} \varphi^{6}= & \pm 2 \mathrm{i} \sqrt{3} \varphi^{6 \overline{7}} \\
\bar{\partial} \varphi^{7}= & \frac{1}{2} \varphi^{\overline{1} \overline{1}} \pm\left(\frac{1}{12} \mathrm{i} \sqrt{3} \mp \frac{1}{4}\right) \varphi^{2 \overline{2}} \\
& \mp\left(\frac{1}{4} \mathrm{i} \sqrt{3} \mp \frac{1}{4}\right) \varphi^{3 \overline{3}} \pm(\mathrm{i} \sqrt{3} \pm 1) \varphi^{4 \overline{4}} \\
& \mp(3 \mathrm{i} \sqrt{3} \pm 9) \varphi^{5 \overline{5}} \pm 6 \mathrm{i} \sqrt{3} \varphi^{6 \overline{6}},
\end{aligned}\right.
$$

As for the previous cases, we specialize (6.6) to $\left(\mathrm{G}_{2}, J_{ \pm}\right)$obtaining

$$
H_{\bar{\partial}}^{\bullet \bullet \bullet}\left(\mathrm{G}_{2}\right) \simeq \mathbb{C}\left[y_{1,1}\right] /\left(\left(y_{1,1}\right)^{6}\right) \otimes \wedge^{\bullet \bullet \bullet}\left(\mathbb{C}\left\langle\left[u_{2,1}\right]\right\rangle \oplus \mathbb{C}\left\langle\left[x_{0,1}\right]\right\rangle\right),
$$

and we recover the Hodge numbers


We consider the sub-complex of left-invariant forms

$$
\iota: \bigwedge\left\langle\varphi^{1}, \varphi^{2}, \varphi^{3}, \varphi^{4}, \varphi^{5}, \varphi^{6}, \varphi^{7}, \bar{\varphi}^{1}, \bar{\varphi}^{2}, \bar{\varphi}^{3}, \bar{\varphi}^{4}, \bar{\varphi}^{5}, \bar{\varphi}^{6}, \bar{\varphi}^{7}\right\rangle \hookrightarrow \wedge^{\bullet \bullet} \mathrm{G}_{2}
$$

As before, we check that, for both the complex structures $J_{+}$and $J_{-}$, the sub-complex has cohomologies $H_{\bar{\partial}}^{0,1}\left(\mathrm{G}_{2}\right)_{\text {inv }}, H_{\bar{\partial}}^{1,1}\left(\mathrm{G}_{2}\right)_{\text {inv }}$ and $H_{\bar{\partial}}^{2,1}\left(\mathrm{G}_{2}\right)_{\text {inv }}$ of dimension one. Therefore, $H_{\bar{\partial}}(\iota)$ is an isomorphism and Theorem 1.3.1 applies giving that also $H_{B C}(\iota)$ and $H_{A}(\iota)$ are isomorphisms. In other words, we conclude that the Dolbeault, Bott-Chern, and Aeppli cohomologies arise from just the left-invariant forms. Consequently, one could explicitly compute the harmonic representatives of these cohomologies. We uploaded our sage script in [33.

## Chapter 7

## The Pluriclosed flow and the stability of the Bismut flat metrics


#### Abstract

In this chapter, we introduce a class of flows of Hermitian metrics known as Hermitian Curvature Flows following [286]. Among these, we focus on the pluriclosed flow, firstly presented and studied in [285]. In particular, in Theorem 7.3.3 we prove that on Bismut flat manifolds of rank 2 , the Bismut flat metrics are globally stable for the pluriclosed flow. This follows from combining the recent developments in the understanding of the long-time behavior of the pluriclosed flow, achieved in [132], and the knowledge of the cohomologies of this special class of manifolds, that we obtained in the previous chapter. We finally study an explicit case in Example 7.4.1 to highlight the difficulties of extending our arguments to the higher rank cases.


The original results of this chapter have been obtained in [32].

### 7.1 Hermitian curvature flows

The Ricci flow on a Riemannian manifold ( $M, g_{0}$ ) was first introduced by Hamilton [161], and it evolves a Riemannian metric in the direction of its Riemannian Ricci curvature:

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g=-2 R i c^{L C}(g), \\
g_{\mid t=0}=g_{0} .
\end{array}\right.
$$

Perelman's landmark resolution of Thurston's Geometrization Conjecture for 3-manifolds using the Ricci flow $774,75,190,257,259$ showed the effectiveness of this flow, which now occupies a central position as one of the key tools in geometry. His revolutionary work sparked interest in the study of geometric flows also on complex manifolds. In particular, if the Ricci flow starts from a Kähler metric on a complex manifold, the evolving metrics will remain Kähler, and the resulting PDE is called the Kähler-Ricci flow. It has been demonstrated to be useful in facing various problems in Kähler geometry, see [63] and the references therein. For example, Cao initiated the study of Kähler-Ricci flow [73], using it to reprove the Calabi-Yau and Aubin-Yau Theorems [25, 331, 332], explicitly constructing Kähler-Einstein metrics on manifolds with $c_{1}=0, c_{1}<0$ respectively. Moreover, Chen, Sun, and Tian [91] used the Kähler-Ricci flow to obtain an alternative proof of the Frankel conjecture avoiding the Siu-Yau and Mori result. However, it is usually the case that the Riemannian Ricci tensor of a Hermitian (non-Kähler) metric is not $(1,1)$, and thus in general, the Hermitian condition is not preserved by the Ricci flow. Moreover, even if $R i c^{L C}$ is $J$-invariant it may not be enough for the Ricci flow to preserve this symmetry (it happens, for instance, if $\nabla^{L C}$ is Kähler-like 18, Theorem 33]). Thus, in general, the Ricci flow can not be used for studying complex geometry which is not Kähler. Then, since the success of Ricci flow for Riemannian manifolds and Kähler-Ricci flow for Kähler geometry, it is natural to try also in non-Kähler geometry to associate special metrics to complex manifolds via a geometric flow construction. It is indeed a general purpose in geometry to use properties of
the resulting metrics to provide further insights which capture aspects of the underlying complex structure, see for instance the works of Ustinovskiy 309 311. The Chern-Ricci flow represents one of the first attempts in this direction. It has been first investigated by Gill [149] in the context of non-Kähler Calabi-Yau manifolds, and then by Tosatti and Weinkove (see [299] and the references therein), who studied it in general on Hermitian non-Kähler manifolds. It evolves the metric in the direction of its Chern-Ricci form:

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g=-R i c^{C h, 1}(g), \\
g_{\left.\right|_{t=0}}=g_{0}
\end{array}\right.
$$

In 286 Streets and Tian suggest new curvature evolution equations on Hermitian manifolds. These are called Hermitian Curvature flows, HCFs in short. The evolution equation of these flows is based on the second Chern-Ricci curvature $R i c^{C h, 2}$, which naturally is a $(1,1)$ curvature tensor associated to a Hermitian metric. More precisely, fixed $Q$ a quadratic term in the Chern torsion, a Hermitian curvature flow evolves the metric as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g=- \text { Ric }^{C h, 2}(g)+Q(g) \\
g_{\mid t=0}=g_{0}
\end{array}\right.
$$

In the original formulation, $Q$ is chosen as a linear combination of the following real symmetric $(1,1)$ tensors $Q^{i}$ :

$$
\begin{array}{ll}
Q_{i \bar{j}}^{1}(g):=g^{k \bar{l}} g^{m \bar{n}} T_{i k \bar{n}}^{C h} T_{\overline{j l m}}^{C h}, & Q_{i \bar{j}}^{2}(g):=g^{k \bar{l}} g^{m \bar{n}} T_{k m \bar{j}}^{C h} T_{\overline{l n} i}^{C h}, \\
Q_{i \bar{j}}^{3}(g):=g^{k \bar{l}} g^{m \bar{n}} T_{i k \bar{l}}^{C h} T_{\overline{j n} m}^{C h}, & Q_{i \bar{j}}^{4}(g):=\frac{1}{2} g^{k \bar{l}} g^{m \bar{n}}\left(T_{m k \bar{l}}^{C h} T_{n j i}^{C h}+T_{m i \bar{j}}^{C h} T_{n l k}^{C h}\right) ;
\end{array}
$$

however, we remark that in general, $Q$ might be any quadratic polynomial in $T^{C h}$.
Definition 7.1.1. Let $(M, J)$ be a complex manifold with Hermitian metric $g_{0}$. A one-parameter family of Hermitian metrics $g(t)$ is a solution to the Hermitian curvature flow with quadratic term $Q$ and initial condition $g_{0}$ if

$$
g(0)=g_{0}, \quad \text { and } \quad \frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t))+Q(g(t))
$$

The operator $g \mapsto R i c^{C h, 2}(g)$ is strictly elliptic [286, Proposition 4.1], giving to the HCFs good existence properties. More in detail, the map

$$
R i c^{C h, 2}-Q: \operatorname{Sym}_{\mathbb{R}}^{1,1} \mathrm{~T}^{*} \mathrm{M} \rightarrow \operatorname{Sym}_{\mathbb{R}}^{1,1} \mathrm{~T}^{*} \mathrm{M}
$$

from the space $\operatorname{Sym}_{\mathbb{R}}^{1,1} \mathrm{~T}^{*} \mathrm{M}$ of real symmetric (1,1)-tensors to itself is a nonlinear second-order strictly elliptic operator, since

$$
\left(R i c^{C h, 2}-Q\right)(g)_{i \bar{j}}=-g^{k \bar{l}} \partial_{k} \bar{\partial}_{l} g_{i \bar{j}}+\mathcal{O}(\partial g, \bar{\partial} g)
$$

Henceforth, the HCF equation is strictly parabolic, and thus short-time existence and uniqueness follow from standard theory. As a consequence of the uniqueness of the solution, starting from a Kähler metric the evolution of the HCFs must coincide with that of the Kähler-Ricci flow. The effectiveness of this family of flows lies in the fact that the quadratic term $Q$ can be chosen in order to adapt to different problems. Indeed, given the rich diversity of Hermitian geometry, it is natural to expect that different Hermitian curvature flows, i.e. different choices of quadratic term $Q$, would be needed to address different situations. Some of them have proved to be powerful tools in revealing information about the complex geometry of the manifolds, see for example the works of Ustinovskiy $\sqrt[309]{311}$, or the results of Streets and Tian on the pluriclosed flow $\sqrt[284]{ }$ 285. We look at some examples:

- Gradient flow: In 286 the authors focus on a particular choice of quadratic term,

$$
Q=\frac{1}{2} Q^{1}-\frac{1}{4} Q^{2}-\frac{1}{2} Q^{3}+Q^{4},
$$

that was identified as corresponding to the Euler equation of a certain Hilbert-type functional in Hermitian geometry

$$
\mathbb{F}=\int_{M} s^{C h}-\frac{1}{4}\left\|T^{C h}\right\|^{2}-\frac{1}{2}\|\theta\|^{2}
$$

- Ustinovskiy flow: In [310], Ustinovskiy showed that there exists a quadratic term $Q$, such that the flow preserves various curvature positivity conditions. In particular, he chose $Q$ as

$$
Q=-\frac{1}{2} Q^{2} .
$$

By exploiting the properties of this flow, Ustinovskiy 310 proved an extension of the classical Frankel conjecture to non-Kähler geometry. We refer to Section 8.1.1 for more details.

We finally focus on the pluriclosed flow in the next section.

### 7.2 Pluriclosed flow

The pluriclosed flow was introduced in [285] as an evolution equation for Hermitian metrics which preserves the pluriclosed condition. Indeed, it evolves a pluriclosed metric in the direction of the ( 1,1 )-component of its Bismut-Ricci form, which is $d d^{c}$-closed by (1.8). More precisely, given a complex manifold $(M, J)$ together with a pluriclosed metric $\omega_{0}$ the pluriclosed flow evolves as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} \omega_{t}=-\left(\operatorname{Ric}^{B}\left(\omega_{t}\right)\right)^{1,1}, \\
\omega_{\mid t=0}=\omega_{0} .
\end{array}\right.
$$

As a Hermitian curvature flow, it is prescribed by the choice of quadratic term $Q=Q^{1}$.
A crucial distinction between this flow and the Chern-Ricci flow is that for the latter the Bismut torsion remains fixed along the flow, i.e. $d \omega_{t}=d \omega_{0}$, whereas for pluriclosed flow the Bismut torsion tensor satisfies a parabolic PDE:

$$
\begin{equation*}
\frac{\partial}{\partial t} \partial \omega_{t}=\partial \overline{\partial \partial}^{*} \omega_{t} \tag{7.1}
\end{equation*}
$$

or equivalently, along the flow, $\partial \omega_{t}=\partial \omega_{0}+\bar{\partial} \beta(t)$, where $\beta$ evolves as

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} \beta=-\left(\operatorname{Ric}^{B}\left(\omega_{t}\right)\right)^{2,0} \\
\beta_{\mid t=0}=0
\end{array}\right.
$$

As a consequence, this prevents the Chern-Ricci flow from converging to a Kähler metric when starting from a non-Kähler one. For instance, given a generic Hermitian metric on the torus, the Chern-Ricci flow will exist globally and converge to a Chern-Ricci flat, but not necessarily flat, metric [149]. This is related to the fact that there is an infinite-dimensional moduli space of pluriclosed Chern-Ricci flat metrics on the torus, obtained by perturbing the flat metric via $\partial \bar{\alpha}+\bar{\partial} \alpha$. Alternatively, the pluriclosed flow on the torus, with arbitrary initial pluriclosed metric, exists globally and converges to a flat metric [282, Theorem 1.1] see also Remark 7.3.1.

In complex dimension two, the pluriclosed condition agrees with the Gauduchon condition. Then, thanks to Theorem 1.1.1, on any compact Hermitian surface there exist pluriclosed metrics, which are suitable starting points for the pluriclosed flow. As a matter of fact, the pluriclosed
flow has been introduced as an analytic tool to understand the topology and geometry of compact complex surfaces. In particular, there is a strong link between the existence of curves on complex surfaces and the evolution of the pluriclosed flow; we now recall it.

It is well known that the class VII surfaces are the last family in the Kodaira classification of complex surfaces which has not been completely classified yet. These are compact complex non-Kähler surfaces with Kodaira dimension $-\infty$. The class VII surfaces with second Betti number $b_{2}=0$ have been classified $[54,55,211,212,291$, and are either Hopf (if there are elliptic curves) or Inoue-Bombieri surfaces (if there are no curves). On the other hand, those with $b_{2}=1$ were classified by Nakamura in [236 under an additional assumption that the surface has a curve, which was later proved by Teleman in 292 . Those with $b_{2}>1$ are not classified in general yet. A construction method for class VII surfaces, and thus a large class of examples, have been introduced by Kato [184], and at present, these are the only known minimal surfaces of class VII with positive $b_{2}$. They are known as Kato surfaces and give examples of minimal class VII surfaces for any $b_{2} \in \mathbb{N}$. The Global Spherical Shell Conjecture (see 235 and the references therein) claims that these are all the existent class VII surfaces. Moreover, the dedicated effort of many authors, culminating in the theorem of Dloussky-Oeljeklaus-Toma 104, has reduced the problem of completing the classification of class VII surfaces to finding $b_{2}$ rational curves in the minimal model of the surface. Indeed, this would imply that they are Kato surfaces.

Eyssidieux, Guedj, Song, Tian, Weinkove, and Zeriahi among others studied the analytic minimal model program [113, 276, 277, 279, 280], seeking to attack the classification of surfaces through the singularities of the Kähler-Ricci flow. In 279 and 278 an analytic version of the Castelnuovo criterion was obtained. Namely, the Kähler-Ricci flow on a compact Kähler surface contracts ( -1 )-curves in the sense of Gromov-Hausdorff and converges smoothly outside of the curves. Indeed, thanks to the Nakai-Moishezon criterion of [67] and [201] it can be proved that, if the maximal existence time of the flow is finite, then either the volume of the surface goes to zero, or the volume of a curve of negative self-intersection goes to zero. The same behavior was proved to occur also for the Chern-Ricci flow on compact non-Kähler surfaces with the extra assumption that the starting metric is pluriclosed $[298,300]$. It is conjectured that the pluriclosed flow should also have this behaviour [287, Section 5], meaning that the pluriclosed flow should exist until either the volume collapses or it becomes singular on an effective divisor with negative self-intersection.
Conjecture 7.2.1 (Conjecture 5.9 of 287$]$ ). Let $(M, J)$ be a compact complex surface with pluriclosed metric $\omega_{0}$. Let $\omega(t)$ be the solution to the pluriclosed flow with initial condition $\omega_{0}$, and suppose $\omega(t)$ exists on $[0, T)$ and that

- $\lim _{t \rightarrow T} \int_{M} \omega(t) \wedge \omega(t)>0$,
- there exists $A>0$ such that $\frac{1}{A}<\lim _{t \rightarrow T} \int_{D} \omega(t)<A$ for every effective divisor $D$ with negative self intersection.

Then there exists a uniform bound on the curvature of $\omega(t)$ depending on $A$.
The proof of this conjecture would lead to an analytic proof of the global spherical shell conjecture for $b_{2}=1$. As a matter of fact, if this conjecture holds, any class VII surface with $b_{2}>0$ should contain an irreducible effective divisor of non-positive self-intersection.

Theorem (Theorem 7.1 of [287]). Suppose Conjecture 7.2.1 holds true. Then any Class VII surface with positive $b_{2}$ contains an irreducible effective divisor of non-positive self-intersection.

Then, by general theory $(\| 236$, Lemma 2.2$])$ there would be only two possible cases: the curve is either a rational curve or an elliptic curve. If the curve is elliptic, the geometry of the manifold is well-understood by $112, \sqrt{236}$; in the other case, the surface would contain exactly $b_{2}=1$ rational curves. In 284 further analysis of the conjectural behavior of the pluriclosed flow on class VII surfaces has been made.

Up to now, there is no definitive understanding of the singularity that the pluriclosed flow may encounter. A complete description of the long-time evolution of this flow has been reached in the locally homogeneous setting for complex surfaces [56], and for metric of nonpositive, flat, or negative holomorphic bisectional curvature [282]. In [23] and [130] it has been developed an analysis of the behavior of the pluriclosed flow acting on invariant metrics on almost-abelian Lie algebras, and Oeljeklaus-Toma manifolds respectively. Recently, long-time existence and convergence of the pluriclosed flow have been achieved on Bismut flat manifolds, under a natural cohomological ansatz 132 (see Theorem 7.3.1). This represents a step forward in proving that static points of the flow are attractive in the sense of Theorem 7.3.1. As a matter of fact, as for the Kähler-Ricci flow and the Chern-Ricci flow $[73,149$, it is expected that also the pluriclosed flow should converge to its static points or to its solitons (if any). The static points of the pluriclosed flow are pluriclosed metrics $g$ which satisfy

$$
\begin{equation*}
\left(\operatorname{Ric}^{B}(g)\right)^{1,1}=\lambda g, \quad \lambda \in \mathbb{R} \tag{7.2}
\end{equation*}
$$

indeed, the pluriclosed flow acts by homothety on them when $\lambda \neq 0$ and keeps them fixed when $\lambda=0$. On the other hand, a pluriclosed soliton is a pair $(g, f)$ of a pluriclosed metric and a function satisfying

$$
\left(\operatorname{Ric}^{B}(g)\right)^{1,1}-\lambda g=\mathcal{L}_{\nabla f} g, \quad \lambda \in \mathbb{R}
$$

for $\mathcal{L}$ the Lie derivative; it is called steady, shrinking or expanding depending on $\lambda=0, \lambda>0$ or $\lambda<0$. It is expected that there are no non-Kähler examples of of static points for the pluriclosed flow with $\lambda \neq 0$ while there are non-Kähler metrics which are fixed points of the pluriclosed flow.

## Static metrics with $\lambda=0$

The basic examples of non-Kähler pluriclosed metrics which satisfy $\left(\operatorname{Ric}^{B}(g)\right)^{1,1}=0$ are given by the standard Calabi-Eckmann Hermitian structures on the Hopf surface $\mathbb{S}^{1} \times \mathbb{S}^{3}$ and the Calabi-Eckmann threefold $\mathbb{S}^{3} \times \mathbb{S}^{3}$. In 133 , the authors asked if the higher-dimensional CalabiEckmann manifolds also admit such special Hermitian structures. The answer is negative and it comes from the fact that, for cohomological reasons, $\mathbb{S}^{1} \times \mathbb{S}^{1}, \mathbb{S}^{1} \times \mathbb{S}^{3}$ and $\mathbb{S}^{3} \times \mathbb{S}^{3}$ are the only Calabi-Eckmann manifolds which can admit a pluriclosed structure, see Theorem 2.3.1. On the other hand, when equipped with the standard Calabi-Eckmann complex structure, these manifolds admit metrics such that $\left(R i c^{B} \omega\right)^{1,1}$ vanishes. Hence, the following picture appears:

- $\mathbb{S}^{1} \times \mathbb{S}^{1}$ has a flat Kähler metric;
- $\mathbb{S}^{3} \times \mathbb{S}^{1}$ has a Bismut flat (hence Bismut Hermitian-Einstein) metric;
- $\mathbb{S}^{3} \times \mathbb{S}^{3}$ has a Bismut flat (hence Bismut Hermitian-Einstein) metric;
- $\mathbb{S}^{2 n+1} \times \mathbb{S}^{2 m+1}$ with $n \geq 2, m \geq 0$ have Bismut-Ricci flat metrics (by Corollary 5.3.1) which are not pluriclosed (by Theorem 2.3.1).


## Staic metrics with $\lambda \neq 0$

As outlined in [133, page 172], when $\lambda \neq 0$ there are restrictions that suggest that the equation (7.2) should imply that the Hermitian structure is Kähler. Some of these can be found in 283, where the author classifies solitons of the pluriclosed flow.
Proposition (Proposition 3.5 of [283]). Let $\left(M^{2 n}, J\right)$ be a compact Kähler manifold, and suppose $(g, f)$ is a pluriclosed steady or shrinking soliton on M. Then $(g, f)$ is a Kähler-Ricci soliton.
Example 7.2.1. The non-Kähler cases of $\mathbb{S}^{1} \times \mathbb{S}^{1}$-principal bundles over Hermitian manifolds are not covered by the above proposition. However, one can easily verify that even on these manifolds there are no metrics of submersion type that satisfy (7.2) with $\lambda \neq 0$ as stated in the proposition below.

Proposition 7.2.1 (Proposition 4.9 in 34 ). Let $\mathbb{S}^{1} \times \mathbb{S}^{1} \hookrightarrow M \xrightarrow{\pi} X$ be a principal toric bundle over a Hermitian manifold $\left(X, \omega_{X}\right)$, with connection one-forms $\theta_{1}, \theta_{2}$ such that $d \theta_{i}=\pi^{*} \omega_{i}$ for $(1,1)$-forms $\omega_{i}$. Then there are no Hermitian metrics of submersion type which satisfy the equation

$$
\left(\operatorname{Ric}^{B}(\omega)\right)^{1,1}=\lambda \omega \quad \text { for } \lambda \neq 0
$$

Proof. We first recall that the submersion metrics are all of the type

$$
\omega=\pi^{*}\left(\omega_{X}\right)+f \theta_{1} \wedge \theta_{2}
$$

where $f$ is a positive function on $M$. Then, thanks to the analysis of the Bismut-Ricci form associated to a submersion metric given in $(\sqrt[5.2]{2}$, the Einstein-type problem $\sqrt{7.2}$ in this setting reduces to solve

$$
\left\{\begin{array}{l}
\pi^{*}\left(\operatorname{Ric}^{B}\left(\omega_{X}\right)\right)^{1,1}=\lambda \pi^{*}\left(\omega_{X}\right)+\left(e_{i}\left(J e_{j}\right) f-e_{j}\left(J e_{i}\right) f\right) e^{i} \wedge e^{j}+f\left(t r_{\omega_{X}} \omega_{1} \pi^{*} \omega_{1}+t r_{\omega_{X}} \omega_{2} \pi^{*} \omega_{2}\right), \\
t r_{\omega_{X}} \omega_{1} t_{2} f-t r_{\omega_{X}} \omega_{2} t_{1} f=\lambda f, \\
\left(\left(e_{j}\left(f t r_{\omega_{X}} \omega_{i}\right)-t_{i}\left(J e_{j}\right) f\right) \theta^{i} \wedge e^{j}\right)^{1,1}=0
\end{array}\right.
$$

In particular, $f$ has to verify

$$
\operatorname{tr}_{\omega_{X}} \omega_{1} t_{2} f-\operatorname{tr}_{\omega_{X}} \omega_{2} t_{1} f=\lambda f
$$

Since the fibers are compact, if we fix one of them, there should be a critical point for $f$ on it. At this point, both $t_{1} f$ and $t_{2} f$ vanish giving a contradiction with the above equality, since $f>0$.

### 7.3 Stability of the Bismut flat metrics for the pluriclosed flow

Up to now, the only known static metrics for the pluriclosed flow are the Bismut flat metrics and (trivially) the Kähler Einstein metrics. In [286, Theorem 1.2], dynamic stability of the HCFs near Kähler-Einstein metrics with negative or zero first Chern class is achieved. In particular, if the pluriclosed flow starts "close enough" to a Kähler-Einstein metric with non-positive Einstein constant then it evolves converging in infinite time to a Kähler-Einstein metric. On the other hand, the Bismut flat metrics are attractive (in the sense of Theorem 7.3.1) for the pluriclosed flow in their torsion classes, which, for a generic metric $\omega$, is the class $[\partial \omega] \in H_{\bar{\partial}}^{2,1}$. Indeed, in 132 the authors implemented beautiful machinery based on Generalized Geometry to compare metrics with the same torsion class. Thanks to it, they proved the first result showing that a natural class of non-Kähler metrics is attractive for the pluriclosed flow in the following sense.

Theorem 7.3.1 (Theorem 1.2 of 132 ). Let $\left(M, J, \omega_{B F}\right)$ be a compact Bismut flat manifold. Given $\omega_{0}$ a pluriclosed metric such that $\left[\partial \omega_{0}\right]=\left[\partial \omega_{B F}\right] \in H_{\bar{\partial}}^{2,1}(M, J)$, the solution of the pluriclosed flow with initial data $\omega_{0}$ exists on $[0, \infty)$ and converges to a Bismut flat metric $\omega_{\infty}$.

Remark. In general, the endpoint of the flow $\omega_{\infty}$ needs not to be equal to the background metric $\omega_{B F}$. However, it actually happens in some specific cases (see Theorem 7.3.2).

Some of the ideas behind Theorem 7.3.1 were already in the proof of Theorem 1.1 in 282 where the author shows the global stability of the flat metrics on the torus for the pluriclosed flow. There, the condition on the torsion class appears in order to guarantee that it is possible to choose proper background data; similarly, in the proof of Theorem 7.3.1, the cohomological condition is needed to compare the evolving metrics with the background metric on the same holomorphic Courant algebroid. Furthermore, the hypothesis $\left[\partial \omega_{0}\right]=\left[\partial \omega_{B F}\right] \in H_{\bar{\partial}}^{2,1}(M, J)$ is natural, for along the pluriclosed flow, the class of the torsion $\partial \omega(t)$ is fixed in the Bott-Chern cohomology by (7.1). Finally, Theorem 7.3.1 reduces the problem of understanding the long-time behavior and global stability of the pluriclosed flow to check this cohomological condition. As a
consequence, Theorem 7.3.1 together with the knowledge of the ( 1,1 )-Aeppli cohomology of the manifold may lead to global stability results. For example, with Theorem 7.3.1 at hand, the stability of the flat metrics on the torus for the pluriclosed flow can be achieved by checking the torsion classes.

Example 7.3.1. Let $(M, J)$ be a complex manifold biholomorphic to a torus. Then, given a pluriclosed metric $\omega$, thanks to the $\partial \bar{\partial}$-lemma, $\partial \omega$ is $\partial \bar{\partial}$-exact hence the torsion class of $\omega$ vanishes in $H_{\bar{\partial}}^{2,1}(M, J)$. Consequently, all the pluriclosed metrics, including the flat Kähler metrics, have the same torsion class, i. e. zero. Thus the solution of the pluriclosed flow with initial data $\omega$ exists in $[0, \infty)$ and converges to a Bismut flat metric. Finally, notice that the Bismut flat metrics on $M$ are precisely the flat Kähler metrics.

It is evident that Theorem 7.3 .1 is extremely powerful when combined with the knowledge of the cohomology of the manifold. Moreover, in some particular cases, the knowledge of the dimension of the ( 1,1 )-Aeppli cohomology group is enough to derive the global stability of the Bismut flat metrics. This happens, for example, for the Hopf surface and the Calabi-Eckmann threefold.

Example 7.3.2 (Examples 2.7 and 2.8 in 132 ). Consider the Hopf surface $\mathbb{S}^{1} \times \mathrm{SU}(2)$ and the Calabi-Eckmann threefold $\mathrm{SU}(2) \times \mathrm{SU}(2)$ equipped with their standard complex structures and bi-invariant metrics $\omega_{B F}$. As seen in Sections 2.2 and 2.3, they are Bismut flat, and the $(1,1)$-Aeppli cohomology is one-dimensional and generated by the class of $\omega_{B F}$. Furthermore, they have the fiber $\mathbb{S}^{1} \times \mathbb{S}^{1}$ as a complex submanifold. Thus, by integrating over it, $[\omega] \neq 0$ in $H_{A}^{1,1}(M, J)$. Henceforth there exists a positive $\lambda$ such that $[\omega]=\lambda\left[\omega_{B F}\right]$ in $H_{A}^{1,1}(M, J)$. Using now that $[\partial \omega] \in H_{\bar{\partial}}^{2,1}(M, J)$ is the image of the natural map

$$
H_{A}^{1,1}(M, J) \xrightarrow{\partial} H_{\bar{\partial}}^{2,1}(M, J):[\omega] \mapsto[\partial \omega],
$$

we have that $[\partial \omega]=\left[\partial\left(\lambda \omega_{B F}\right)\right]$ in $H_{\bar{\partial}}^{2,1}(M, J)$, where $\lambda \omega_{B F}$ is again Bismut flat. Hence, Theorem 7.3.1 applies to give long-time existence and convergence of the pluriclosed flow to a Bismut flat metric for any initial pluriclosed data.

The same ideas of the above example also apply to compact simply-connected simple Lie groups of rank two (described in the previous chapter), proving the following result.

Theorem 7.3.2 (Theorems 4.1, 4.2, and 4.3 of [32]). Given a compact simply-connected simple Lie group $M$ of rank 2 (which are $\mathrm{SU}(3)$, $\operatorname{Spin}(5)$ and $\mathrm{G}_{2}$ ), consider a Bismut flat Hermitian structure ( $J, \omega_{B F}$ ) coming from the Killing form (as in Section 6.3). Then for any pluriclosed metric $\omega_{0}$ on $(M, J)$ there exists a positive $\lambda$ such that the solution to the pluriclosed flow with initial data $\omega_{0}$ exists on $[0, \infty)$ and converges to $\lambda \omega_{B F}$ up to diffeomorphism.
Proof. First of all, the complex structure $J$ is such that the maximal torus $\mathbb{S}^{1} \times \mathbb{S}^{1}$ in $M$ is a complex submanifold. Indeed, following Samelson's construction (presented in Section 6.3), $J$ has been defined firstly by choosing a complex structure on the maximal torus $\mathfrak{t}^{\mathbb{C}}$ and then completing it to a complex structure of $\mathfrak{g}^{\mathbb{C}}$ by choosing a system of positive roots.

Now, given any pluriclosed metric $\omega_{0}$ on $(M, J)$, by integrating over the maximal torus we see that $\left[\omega_{0}\right] \neq 0$ in $H_{A}^{1,1}(M)$. As shown in Theorem 6.4.1. the ( 1,1 )-Aeppli cohomology of $(M, J)$ is one dimensional, thus there exists a constant $\lambda$ such that $\left[\omega_{0}\right]=\lambda\left[\omega_{B F}\right]$ in $H_{A}^{1,1}(M)$, which must be positive because the integrals of $\omega_{0}$ and $\omega_{B F}$ on $\mathbb{S}^{1} \times \mathbb{S}^{1}$ are both positive. Then, Theorem 7.3.1 applies to ensure the long-time existence of the pluriclosed flow with initial data $\omega_{0}$ and convergence to a Bismut flat metric $\omega_{\infty} \in \lambda\left[\omega_{B F}\right]$.

Thanks to the classification in Section 6.2, Bismut flat metrics are bi-invariant with respect to some Lie group structure, and it is well known that any invariant symmetric bi-linear form on a simple Lie group must be a multiple of the Killing form. A priori there might be different Lie group structures on $M$, and hence different Killing forms. However, the Milnor result (Lemma
6.2.1) ensures that there is only one Lie group structure on $\mathrm{SU}(3)$, $\operatorname{Spin}(5)$, or $\mathrm{G}_{2}$ (as manifolds) which may admit a bi-invariant metric, up to diffeomorphisms of the underling manifold. In fact, thanks to Lemma 6.2.1, a compact simply-connected Lie group $G$ with a bi-invariant metric must be isomorphic and isometric to a product of simply-connected compact simple Lie groups. In particular, the only compact simply-connected Lie group of complex dimension 4 with a bi-invariant metric is $\mathrm{SU}(3)$, and the same holds for $\operatorname{Spin}(5)$ in complex dimension 5 . For $\mathrm{G}_{2}$ one has to verify that it can not be isomorphic to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, which follows for example by cohomological reasons. Consequently, there is a unique Killing form on $M$ up to diffeomorphisms, and $\omega_{\infty}$ and $\omega_{B F}$ are bi-invariant with respect to two isomorphic Lie group structures on $M$. In particular, $\omega_{\infty}$ must be a positive multiple of $\phi^{*} \omega_{B F}$ for some $\phi \in \operatorname{Diff}(M)$, and hence finally $\omega_{\infty}=\lambda \phi^{*} \omega_{B F}$.

Theorem 7.3.1 only applies to compact Bismut flat manifolds. Thanks to the classification in Section 6.2, the only compact Bismut flat manifolds with maximal torus of dimension 2 are $\mathbb{S}^{1} \times \mathbb{S}^{1}, \mathbb{S}^{1} \times \mathrm{SU}(2), \mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(3), \mathrm{Spin}(5)$, and $\mathrm{G}_{2}$ equipped with the Bismut flat Hermitian structures as described in Section 6.3. Henceforth, Theorem 7.3 .2 together with Examples 7.3.1 and 7.3.2 show the global stability for the pluriclosed flow on these manifolds.

Theorem 7.3.3. Given a compact connected Bismut flat manifold $M$ with maximal torus $\mathbb{S}^{1} \times \mathbb{S}^{1}$, consider a Bismut flat Hermitian structure $\left(J, \omega_{B F}\right)$. Then for any pluriclosed metric $\omega_{0}$ on $(M, J)$ there exists a positive $\lambda$ such that the solution to the pluriclosed flow with initial data $\omega_{0}$ exists on $[0, \infty)$ and converges to a Bismut flat metric $\omega_{\infty} \in \lambda\left[\omega_{B F}\right] \in H_{A}^{1,1}(M, J)$. In particular, for $\mathrm{SU}(3)$, $\operatorname{Spin}(5)$, and $\mathrm{G}_{2}$ it holds $\omega_{\infty}=\lambda \omega_{B F}$ up to diffeomorphism.

### 7.4 Further analysis on higher rank cases

A natural question is whether the global stability of the Bismut flat metrics can also be proved on Bismut flat manifolds of higher rank. On the one hand, the first step needed to generalize the argument presented in this thesis would be to understand the Dolbeault cohomology, and then the Aeppli cohomology of these manifolds. However, this is still unclear on compact simply-connected simple Lie groups of rank 4 and higher. On the other hand, one could try to find a counterexample by restricting to invariant forms. However, the smallest compact simply-connected simple Lie group of rank 4 is $\mathrm{SU}(5)$, which has real dimension 24 ; hence explicit examples became already hard to be investigated. Semisimple Lie groups can be considered in order to work with groups of higher rank but small dimensions. For these manifolds, the $(1,1)$-Aeppli cohomology grows according to the dimension of the maximal torus following a Künneth-type formula. Thus, to apply the same argument of Theorem 7.3 .2 one should check that the subspace of $H_{A}^{1,1}$ generated by the classes of the Bismut flat metrics is still filling the whole $H_{A}^{1,1}$. In the following, we give an example of this behavior in the simplest non-trivial case, namely on the semisimple Lie group $\mathrm{SU}(3) \times \mathbb{S}^{1} \times \mathbb{S}^{1}$, which is of rank 4 .

Example 7.4.1. Consider the Bismut flat manifold $\left(\mathrm{SU}(3), J_{0,-1}, \omega_{B F}\right)$ and the complex torus with the standard complex structure $\left(\mathbb{S}^{1} \times \mathbb{S}^{1}, J_{S t}\right)$. Define $J:=J_{0,-1} \times J_{S t}$ the product complex structure on $M:=\mathrm{SU}(3) \times \mathbb{S}^{1} \times \mathbb{S}^{1}$. The double complex associated to $(M, J)$ is given by the product of the double complex of $\left(\mathrm{SU}(3), J_{0,-1}\right)$, (a) in Figure 7.1 (see Section 6.4.2 for details), and $\left(\mathbb{S}^{1} \times \mathbb{S}^{1}, J_{S t}\right)$, (b) in Figure 7.1.

More precisely, one obtains the picture in Figure 7.1 repeated four times: in its position, shifted of one unit on the right, shifted of one unit on the top, and shifted of one unit on the right and one on the top; which correspond to wedging with $1, \psi, \bar{\psi}, \psi \wedge \bar{\psi}$ respectively, for $\psi$ the $(1,0)$-form generating the cohomology of the torus. Focusing just on the lower bi-degrees, the double complex of $(M, J)$ looks like in Figure 7.2.

In particular, the $(1,1)$-Aeppli cohomology of $(M, J)$ is of dimension 4 generated by the classes of $\omega_{B F}, \psi \wedge \bar{\psi}, \varphi^{4} \wedge \bar{\psi}, \psi \wedge \bar{\varphi}^{4}$. Therefore, given a generic pluriclosed metric $\omega$, its class

(a)

(b)

Figure 7.1. Diagram (a) represents the double complex of ( $\left.\mathrm{SU}(3), J_{0,-1}\right)$, up to squares. Diagram (b) represents the double complex of $\left(\mathbb{S}^{1} \times \mathbb{S}^{1}, J_{S t}\right)$, up to squares.


Figure 7.2. Lower bi-degrees of the double complex of $(M, J)$, up to squares.
in the $(1,1)$ Aeppli cohomology group $H_{A}^{1,1}(M, J)$ is

$$
\begin{equation*}
[\omega]=\alpha\left[\omega_{B F}\right]+\beta \frac{\mathrm{i}}{2}[\psi \wedge \bar{\psi}]+u\left[\varphi^{4} \wedge \bar{\psi}\right]-\bar{u}\left[\psi \wedge \bar{\varphi}^{4}\right] \tag{7.3}
\end{equation*}
$$

for coefficients $\alpha, \beta \in \mathbb{R}$, and $u \in \mathbb{C}$ such that

$$
\left\{\begin{array}{l}
\alpha>0, \quad \beta>0  \tag{7.4}\\
\alpha \beta>4|u|^{2}
\end{array}\right.
$$

The above equation (7.3) defines the subset of $H_{A}^{1,1}(M, J)$ generated by the classes of the pluriclosed metrics on $(M, J)$. Henceforth, to apply the argument in Theorem 7.3.2 one should verify that it
is generated just by the classes of the Bismut flat metrics. We thus define a family of left-invariant pluriclosed metrics as

$$
\omega_{\alpha, \beta, u}:=\alpha \omega_{B F}+\beta \frac{\mathrm{i}}{2} \psi \wedge \bar{\psi}+u \varphi^{4} \wedge \bar{\psi}-\bar{u} \psi \wedge \bar{\varphi}^{4}
$$

with coefficients $\alpha, \beta$ and $u$ as in (7.4). By straightforward computations, all these metrics are Bismut flat, hence the pluriclosed flow is globally stable on $(M, J)$.

Note that $\mathrm{SU}(3)$ acts trivially on $\mathbb{S}^{1} \times \mathbb{S}^{1}$, and $\mathbb{S}^{1} \times \mathbb{S}^{1}$ acts isometrically on $\left(\mathrm{SU}(3), \omega_{B F}\right)$. Thus $\omega_{B F}$ and $\psi \wedge \bar{\psi}$ are both bi-invariant forms on $M$, since they come from the Killing forms on $\mathrm{SU}(3)$ and $\mathbb{S}^{1} \times \mathbb{S}^{1}$ respectively. Now, $\varphi^{4}=e^{3}+\mathrm{i} e^{8}$ is left-invariant but not right-invariant on $\mathrm{SU}(3)$ because the matrix

$$
e^{3}+\mathrm{i} e^{8}=\left(\begin{array}{ccc}
\mathrm{i}-\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & -\mathrm{i}-\frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{2}{\sqrt{3}}
\end{array}\right)
$$

does not commute with all the unitary $3 \times 3$-matrices. Therefore, on $\mathrm{SU}(3) \times \mathbb{S}^{1} \times \mathbb{S}^{1}$ there exist Lie groups structure different from the product one such that the metrics $\omega_{\alpha, \beta, u}$ with $u \neq 0$ are bi-invariant with respect to them. This corresponds with taking a non-trivial action $\rho: \mathbb{Z}^{2} \rightarrow I(\mathrm{SU}(3))$ when performing the quotient of $\mathrm{SU}(3) \times \mathbb{R}^{2}$, see Definition 6.2.2.

We wonder whether this cohomological behavior is a peculiarity of all the Bismut flat manifolds. Namely, if in general on a Bismut flat manifold, any pluriclosed metric is in the same $(1,1)$-Aeppli cohomology class of a Bismut flat metric.

## Chapter 8

## Bismut positivity along Hermitian Curvature Flows


#### Abstract

In this chapter, we introduce and study the Bismut positivity notion. It involves the curvature of the Bismut connection, naturally emulating the Griffiths positivity of the holomorphic tangent bundle, which is the holomorphic bisectional curvature associated to the Chern connection. We perform our analysis on two concrete classes of examples, namely, Hopf manifolds and 6-dimensional Calabi-Yau solvmanifolds with holomorphically-trivial canonical bundle. The latter are compact quotients of solvable Lie groups endowed with an invariant complex structure and having holomorphically-trivial canonical bundle. In particular, in Sections 8.1.3 and 8.1.4 we characterize the homogeneous metrics on both these classes of manifolds which are Bismut-positive or Bismut-non-negative. We then investigate the behavior of this positivity notion under the evolution of the Hermitian curvature flows. In particular, in Section 8.2, we prove a global stability result for the Hermitian curvature flows over the Hopf manifolds. This result completely describes the evolution of the Hermitian curvature flows on the Hopf manifolds. Then, in Section 8.3 we use it to characterize those HCFs which preserve or do not preserve Bismut non-negativity when evolving homogeneous metrics on the Hopf manifolds.


The original results of this chapter have been obtained in [31]. We remark that in that article we named our positivity notion as Bismut-Griffiths-positivity.

### 8.1 Positivity notions for Hermitian manifolds

In this section, we introduce the notion of Bismut positivity and we motivate it by comparing it with the Griffiths positivity of the holomorphic tangent bundle. We then test this condition on Hopf manifolds and Calabi-Yau solvmanifolds with holomorphically-trivial canonical bundle. Let us start by recalling some basic notions about the Griffiths positivity.

### 8.1.1 Griffiths positivity

Let $(E, h)$ be a holomorphic vector bundle. It is said Griffiths-positive (respectively Griffiths-non-negative), if its Chern curvature tensor

$$
\left(\nabla^{C h}\right)^{2}=: \Omega \in \mathcal{A}^{1,1}(M) \otimes E^{*} \otimes E \cong \mathcal{A}^{1,1}(M) \otimes \operatorname{End}(E)
$$

is positive (respectively non-negative) on all non-zero tensors $x \otimes \bar{x} \otimes v \otimes \bar{v}$, with $x \in T^{1,0} M$ and $v \in E$. Griffiths positivity implies ampleness of the bundle $E$, and conjecturally any ample bundle $E$ admits Griffiths-positive Hermitian metric $h$, see [103, Problem 11.14]. Given a Hermitian manifold $(M, J, g)$, set $(E, h)=\left(T^{1,0} M, g\right)$. Then, the tensor $\Omega$ coincides with $R^{C h}$. Moreover, $g$ has Griffiths-positive curvature if and only if the holomorphic tangent bundle $T^{1,0} M$ is positive in the sense of Griffiths, see 103 .

We remark that if a Hermitian manifold $(M, J, g)$ is Kähler, then the curvature tensor $R^{C h}(x, J x, y, J y)$ coincides with the holomorphic bisectional curvature by definition. In particular, in this case, $g$ has Griffiths-positive curvature if and only if the holomorphic bisectional curvature of $g$ is positive.

It is known that the induced metric on a quotient bundle of a Griffiths-non-negative Hermitian bundle $(E, h)$ is Griffiths-non-negative, $\sqrt{103}$, Proposition 6.10$]$. Hence, if $T^{1,0} M$ is globally generated, then the Hermitian metric induced on it by the natural projection from the trivial bundle $H^{0}\left(M, T^{1,0} M\right) \cong \mathbb{C}^{m}$ with a flat metric is Griffiths-non-negative. As a consequence, any complex homogeneous space has a metric that is Griffiths-non-negative. On the other hand, the Frankel conjecture proved by Mori [232, Siu-Yau [275], and then extended by Mok [231], in its differential-geometric formulation, states that a compact Griffiths positive Kähler manifold must be biholomorphic to the complex projective space. Bando 29], and Mok 231 proved that the positivity of the bisectional holomorphic curvature is preserved under the evolution of the Kähler-Ricci flow. Then emulating their arguments, Ustinovskiy showed 310 that there exists a quadratic term in the torsion of the Chern connection such that the associated Hermitian curvature flow preserves various curvature positivity conditions. In the notations of Section 7.1, Ustinovskiy's quadratic term is given by

$$
Q_{i \bar{j}}=-\frac{1}{2} g^{k \bar{l}} g^{m \bar{n}} T_{k m \bar{j}}^{C h} T_{\bar{l} \bar{n} i}^{C h}=-\frac{1}{2} Q_{i \bar{j}}^{2}
$$

In particular, he showed that it preserves Griffiths positivity and non-negativity, and it evolves a metric which has Griffiths-non-negative curvature everywhere and positive at least in one point to a metric with Griffiths-positive curvature everywhere. These regularization properties of Ustinovskiy's flow allow us to prove a uniformization theorem which extends the classical Frankel conjecture to non-Kähler geometry. Namely, also using the result of Mori, Ustinovskiy proved that a compact Hermitian manifold of complex dimension $n$ such that its curvature is Griffiths-non-negative everywhere and strictly positive somewhere must be biholomorphic to the projective space $\mathbb{C P}^{n}$ 310, Proposition 0.3].

### 8.1.2 Bismut positivity

Given a Hermitian manifold $(M, J, g)$, we define the notion of Bismut positivity by evaluating its holomorphic Bismut bisectional curvature. In details,
Definition. Let $(M, J, g)$ be a Hermitian manifold. It has Bismut-positive (respectively Bismut-non-negative) curvature if its Bismut curvature tensor $R^{B}$ satisfies ( Cplx and for any non-zero $x, y \in T^{1,0} M$,

$$
R^{B}(x, \bar{x}, y, \bar{y})>0 \quad(\text { respectively } \geq 0)
$$

The request for the (Cplx condition to be satisfied is motivated by the fact that the holomorphic bisectional curvature only describes the geometry of the ( 1,1 ) part of $R^{B}$ ignoring the $(2,0)$ and $(0,2)$ components. Hence, we ask the $(1,1)$ component to be the whole tensor.

Given a complex manifold $(M, J)$ equipped with a pluriclosed Hermitian metric $g$, using the formula for the Gauduchon curvature tensors given in (1.5), the Bismut curvature tensor (in holomorphic coordinates) becomes:

$$
\begin{equation*}
R_{i \bar{j} k \bar{l}}^{B}=R_{k \bar{l} \bar{j}}^{C h}-g^{p \bar{q}} T_{k p \bar{j}}^{C h} \overline{T_{l q \bar{i}}^{C h}} \tag{8.1}
\end{equation*}
$$

Notice that the trace of the second term in the left-hand side is

$$
\begin{equation*}
g^{k \bar{l}} g^{i \bar{j}} g^{p \bar{q}} T_{k p \bar{j}}^{C h} \overline{T_{l q \bar{i}}^{C h}}=\left\|T^{C h}\right\|_{g}^{2} \tag{8.2}
\end{equation*}
$$

hence for pluriclosed metrics, the holomorphic Bismut bisectional curvature is less positive than the holomorphic Chern bisectional curvature. Symmetrically, Bismut positivity (non-negativity) is stronger than Griffiths positivity (non-negativity).
Remark. In 297 Tong showed that the tensor $R_{k \bar{l} \bar{j}}^{C h}-g^{p \bar{q}} T_{k p \bar{j}}^{C h} \overline{T_{l q \bar{i}}^{C h}}$ arises in a Bochner-Kodairatype formula for closed $(1,1)$-forms. Then he studied its positivity.

### 8.1.3 Bismut positivity on 6-dimensional Calabi-Yau solvmanifolds

In this section, we analyze the symmetries of $(\mathrm{Cplx})$ and the notion of Bismut positivity on 6 -dimensional Calabi-Yau solvmanifolds, which is a class of complex non-Kähler manifolds with holomorphically-trivial canonical bundle. More precisely, by solvmanifold, we mean a compact quotient of a connected simply-connected solvable Lie group by a co-compact discrete subgroup. We consider those that can be endowed with a Hermitian structure $(g, J)$ which is invariant under left translations when lifted to the universal cover; moreover, we ask these solvmanifolds to be Calabi-Yau, that is, the complex structure $J$ is such that the canonical bundle is holomorphically-trivial. This means that there exists a holomorphic 3-form which is nowhere zero, and we assume it to be invariant. We remark that any even-dimensional nilmanifold endowed with an invariant complex structure has holomorphically-trivial canonical bundle 269 .

We refer to the classification (up to linear equivalence) of the invariant complex structures on 6-dimensional nilmanifolds (Table 8.1) and solvmanifolds non-nilmanifolds with holomorphicallytrivial canonical bundle (Table 8.2) as outlined in the works of Andrada, Barberis, Ceballos, Dotti, Fino, Otal, Salamon, Ugarte and Villacampa [8, 86, 117, 246, 269, 305, 306].

| Name | Complex structure | Lie algebra (Notation of [269]) |
| :---: | :---: | :---: |
| ( Np ) | $d \varphi^{1}=d \varphi^{2}=0, d \varphi^{3}=\rho \varphi^{12}, \quad$ where $\rho \in\{0,1\}$ | $\rho=0: \mathfrak{h}_{1}=(0,0,0,0,0,0)$ |
|  |  | $\rho=1: \mathfrak{h}_{5}=(0,0,0,0,13+42,14+23)$ |
| (Ni) | $\begin{gathered} d \varphi^{1}=d \varphi^{2}=0 \\ d \varphi^{3}=\rho \varphi^{12}+\varphi^{1 \overline{1}}+\lambda \varphi^{1 \overline{2}}+D \varphi^{2 \overline{2}} \end{gathered}$ <br> where $\rho \in\{0,1\}, \lambda \in \mathbb{R}^{\geq 0}, D \in \mathbb{C}$ with $\Im D \geq 0$ | $\mathfrak{h}_{2}=(0,0,0,0,12,34)$ |
|  |  | $\mathfrak{h}_{3}=(0,0,0,0,0,12+34)$ |
|  |  | $\mathfrak{h}_{4}=(0,0,0,0,12,14+23)$ |
|  |  | $\mathfrak{h}_{5}=(0,0,0,0,13+42,14+23)$ |
|  |  | $\mathfrak{h}_{6}=(0,0,0,0,12,13)$ |
|  |  | $\mathfrak{h}_{8}=(0,0,0,0,0,12)$ |
| (Nii) | $d \varphi^{1}=0, \quad d \varphi^{2}=\varphi^{1 \overline{1}}$ $d \varphi^{3}=\rho \varphi^{12}+B \varphi^{1 \overline{2}}+c \varphi^{2 \overline{1}},$ <br> where $\rho \in\{0,1\}, B \in \mathbb{C}, c \in \mathbb{R}^{\geq 0}$, with $(\rho, B, c) \neq(0,0,0)$ | $\mathfrak{h}_{7}=(0,0,0,12,13,23)$ |
|  |  | $\mathfrak{h}_{9}=(0,0,0,0,12,14+25)$ |
|  |  | $\mathfrak{h}_{10}=(0,0,0,12,13,14)$ |
|  |  | $\mathfrak{h}_{11}=(0,0,0,12,13,14+23)$ |
|  |  | $\mathfrak{h}_{12}=(0,0,0,12,13,24)$ |
|  |  | $\mathfrak{h}_{13}=(0,0,0,12,13+14,24)$ |
|  |  | $\mathfrak{h}_{14}=(0,0,0,12,14,13+42)$ |
|  |  | $\mathfrak{h}_{15}=(0,0,0,12,13+42,14+23)$ |
|  |  | $\mathfrak{h}_{16}=(0,0,0,12,14,24)$ |
| (Niii) | $\begin{gathered} d \varphi^{1}=0, \quad d \varphi^{2}=\varphi^{13}+\varphi^{1 \overline{3}}, \\ d \varphi^{3}=\mathrm{i} \rho \varphi^{1 \overline{1}} \pm \mathrm{i}\left(\varphi^{1 \overline{2}}-\varphi^{2 \overline{1}}\right), \quad \text { where } \rho \in\{0,1\} \end{gathered}$ | $\mathfrak{h}_{19}^{-}=(0,0,0,12,23,14-35)$ |
|  |  | $\mathfrak{h}_{26}^{+}=(0,0,12,13,23,14+25)$ |

Table 8.1. Invariant complex structures on 6 -dimensional nilmanifolds up to linear equivalence, see [8], [86, 306.

In the formulas in Tables 8.1 and 8.2 the authors refer to a co-frame $\left(\varphi^{1}, \varphi^{2}, \varphi^{3}, \bar{\varphi}^{1}, \bar{\varphi}^{2}, \bar{\varphi}^{3}\right)$ where $\left(\varphi^{1}, \varphi^{2}, \varphi^{3}\right)$ is an invariant co-frame of $(1,0)$-forms with respect to $J$. The generic invariant

| Name | Complex structure | Lie algebra (Notation of 269]) |
| :---: | :---: | :---: |
|  |  | $\mathfrak{g}_{1}=(15,-25,-35,45,0,0)$ when $\theta=0$ |
| (Si) | $\begin{aligned} & d \varphi^{2}=-A \varphi^{23}-A \varphi^{2 \overline{3}}, \quad d \varphi^{3}=0 \\ & \text { where } A=\cos \theta+\mathrm{i} \sin \theta, \theta \in[0, \pi) \end{aligned}$ | $\mathfrak{g}_{2}^{\alpha}=(\alpha \times 15+25,-15+\alpha \times 25,-\alpha \times 35+45,-35-\alpha \times 45,0,0)$ <br> with $\alpha=\frac{\cos \theta}{\sin \theta} \geq 0$, when $\theta \neq 0$ |
| (Sii) | $\begin{gathered} d \varphi^{1}=0, \quad d \varphi^{2}=-\frac{1}{2} \varphi^{13}-\left(\frac{1}{2}+\mathrm{i} x\right) \varphi^{1 \overline{3}}+\mathrm{i} x \varphi^{3 \overline{1}}, \\ d \varphi^{3}=\frac{1}{2} \varphi^{12}+\left(\frac{1}{2}-\frac{\mathrm{i}}{4 x}\right) \varphi^{1 \overline{2}}+\frac{\mathrm{i}}{4 x} \varphi^{2 \overline{1}}, \\ \text { where } x \in \mathbb{R}^{>0} \end{gathered}$ | $\mathfrak{g}_{3}=(0,-13,12,0,-46,-45)$ |
| (Siii1) | $\begin{gathered} d \varphi^{1}=\mathrm{i} \varphi^{13}+\mathrm{i} \varphi^{1 \overline{3}} \\ d \varphi^{2}=-\mathrm{i} \varphi^{23}-\mathrm{i} \varphi^{2 \overline{3}} \\ d \varphi^{3}= \pm \varphi^{1 \overline{1}} \end{gathered}$ | $\mathfrak{g}_{4}=(23,-36,26,-56,46,0)$ |
| (Siii2) | $\begin{gathered} d \varphi^{1}=\varphi^{13}+\varphi^{1 \overline{3}} \\ d \varphi^{2}=-\varphi^{23}-\varphi^{2 \overline{3}} \\ d \varphi^{3}=\varphi^{1 \overline{2}}+\varphi^{2 \overline{1}} \end{gathered}$ | $\mathfrak{g}_{5}=(24+35,26,36,-46,-56,0)$ |
| (Siii3) | $\begin{gathered} d \varphi^{1}=\mathrm{i} \varphi^{13}+\mathrm{i} \varphi^{1 \overline{3}} \\ d \varphi^{2}=-\mathrm{i} \varphi^{23}-\mathrm{i} \varphi^{2 \overline{3}} \\ d \varphi^{3}=\varphi^{1 \overline{1}}+\varphi^{2 \overline{2}} \end{gathered}$ | $\mathfrak{g}_{6}=(24+35,-36,26,-56,46,0)$ |
| (Siii4) | $\begin{gathered} d \varphi^{1}=\mathrm{i} \varphi^{13}+\mathrm{i} \varphi^{1 \overline{3}} \\ d \varphi^{2}=-\mathrm{i} \varphi^{23}-\mathrm{i} \varphi^{2 \overline{3}} \\ d \varphi^{3}= \pm\left(\varphi^{1 \overline{1}}-\varphi^{2 \overline{2}}\right) \end{gathered}$ | $\mathfrak{g}_{7}=(24+35,46,56,-26,-36,0)$ |
| (Siv1) | $d \varphi^{1}=-\varphi^{13}, \quad d \varphi^{2}=\varphi^{23}, \quad d \varphi^{3}=0$ |  |
| (Siv2) | $\begin{aligned} d \varphi^{1} & =2 \mathrm{i} \varphi^{13}+\varphi^{3 \overline{3}}, \quad x \in\{0,1\} \\ d \varphi^{2} & =-2 \mathrm{i} \varphi^{23}+x \varphi^{3 \overline{3}}, \quad d \varphi^{3}=0 \end{aligned}$ | $\mathfrak{g}_{8}=(16-25,15+26,-36+45,-35-46,0,0)$ |
| (Siv3) | $\begin{gathered} d \varphi^{1}=A \varphi^{13}-\varphi^{1 \overline{3}} \\ d \varphi^{2}=-A \varphi^{23}+\varphi^{2 \overline{3}}, \quad d \varphi^{3}=0 \\ A \in \mathbb{C} \text { with }\|A\| \neq 1 \end{gathered}$ | $\mathfrak{g}_{8}=(16-25,15+26,-36+45,-35-46,0$, |
| (Sv) | $\begin{gathered} d \varphi^{1}=-\varphi^{3 \overline{3}} \\ d \varphi^{2}=\frac{\mathrm{i}}{2} \varphi^{12}+\frac{1}{2} \varphi^{1 \overline{3}}-\frac{\mathrm{i}}{2} \varphi^{2 \overline{1}} \\ d \varphi^{3}=-\frac{\mathrm{i}}{2} \varphi^{13}+\frac{\mathrm{i}}{2} \varphi^{3 \overline{1}} \end{gathered}$ | $\mathfrak{g}_{9}=(45,15+36,14-26+56,-56,46,0)$ |

Table 8.2. Invariant complex structures on 6 -dimensional solvmanifolds non-nilmanifolds with holomorphically-trivial canonical bundle up to linear equivalence, see [246, [117].

Hermitian structure $\omega=g(J \cdot, \cdot)$ is given by

$$
\begin{equation*}
2 \omega=\mathrm{i}\left(r^{2} \varphi^{1 \overline{1}}+s^{2} \varphi^{2 \overline{2}}+t^{2} \varphi^{3 \overline{3}}\right)+u \varphi^{1 \overline{2}}-\bar{u} \varphi^{2 \overline{1}}+v \varphi^{2 \overline{3}}-\bar{v} \varphi^{3 \overline{2}}+z \varphi^{1 \overline{3}}-\bar{z} \varphi^{3 \overline{1}} \tag{8.3}
\end{equation*}
$$

where $\varphi^{i \bar{j}}:=\varphi^{i} \wedge \bar{\varphi}^{j}$ and the coefficients satisfy the following inequalities coming from the fact that $g$ is positive definite:

$$
\begin{gathered}
r^{2}>0, \quad s^{2}>0, \quad t^{2}>0 \\
r^{2} s^{2}>|u|^{2}, \quad r^{2} t^{2}>|z|^{2}, \quad s^{2} t^{2}>|v|^{2} \\
\text { 8i det } \Xi=r^{2} s^{2} t^{2}+2 \Re(\mathrm{i} \overline{u v} z)-\left(r^{2}|v|^{2}+t^{2}|u|^{2}+s^{2}|z|^{2}\right)>0
\end{gathered}
$$

where, $\Xi$ denotes the Hermitian matrix associated to the Hermitian structure, i.e.

$$
\Xi=\left(\begin{array}{ccc}
\mathrm{i} \frac{r^{2}}{2} & \frac{u}{2} & \frac{z}{2} \\
-\frac{\bar{u}}{2} & \mathrm{i} \frac{s^{2}}{2} & \frac{v}{2} \\
-\frac{\bar{z}}{2} & -\frac{\bar{v}}{2} & \mathrm{i} \frac{t^{2}}{2}
\end{array}\right) .
$$

Following the classification in Tables 8.1 and 8.2 we analyze case by case the families of nilmanifolds and solvmanifolds. We thus obtain the following results, whose proofs are collected in Section 8.4

Theorem 8.1.1. Let $M$ be a 6 -dimensional solvmanifold endowed with an invariant Hermitian structure ( $J, g$ ) with $\omega$ as in (8.3) and $J$ such that the canonical bundle is holomorphically-trivial. The Bismut curvature tensor satisfies (Cplx precisely in the cases (Np), (Ni), (Nii), (Si), (Siiii1), (Siv1), and (Siv3) when the conditions on the invariant structures of Table 8.3 are satisfied.

| Name | Cplx condition | Bismut non-negativity |
| :---: | :---: | :---: |
| ( Np ) | Always satisfied | $\rho=0$ : flat <br> $\rho=1$ : nowhere non-negative nor non-positive |
| (Ni) | $\rho=0$ | $\mathfrak{h}_{2}$ : non-negative if $u=0$ <br> $\mathfrak{h}_{3}, D=1$ : non-negative <br> $\mathfrak{h}_{3}, D=-1$ : nowhere non-negative nor non-positive <br> $\mathfrak{h}_{4}$ : nowhere non-negative nor non-positive <br> $\mathfrak{h}_{5}$ : nowhere non-negative nor non-positive <br> $\mathfrak{h}_{8}$ : non-negative |
| (Nii) | $c=B=0, \rho=1, v=0$ | nowhere non-negative nor non-positive |
| (Si) | $u=v=z=0$ | $A=\mathrm{i} \text { : flat }$ <br> $A \neq \mathrm{i}$ : nowhere non-negative nor non-positive |
| (Siii1) | $u=v=z=0$ | non-negative |
| (Siv1) | Always satisfied | nowhere non-negative nor non-positive |
| (Siv3) | $\begin{gathered} u=v=z=0 \\ A=0, v=z=0 \end{gathered}$ | nowhere non-negative nor non-positive in both cases |

Table 8.3. Conditions on the underlying complex structure, invariant Hermitian metric and Lie algebra.

Remark. In [18] the authors studied the existence of Gauduchon Kähler-like connections on 6-dimensional Calabi-Yau solvmanifolds. Comparing Theorem 13 in [18] with our result it is evident that for the Bismut connection the Kähler-like condition is strictly stronger than (Cplx. See for example the cases (Nii), (Siv1), and (Siv3), and the subcases in the other families.

In the cases where (Cplx) is satisfied we look at the holomorphic Bismut bisectional curvature.
Theorem 8.1.2. Let $M$ be a 6 -dimensional solvmanifold endowed with an invariant Hermitian structure ( $J, g$ ) with $\omega$ as in 8.3) and $J$ such that the canonical bundle is holomorphically-trivial. If $J$ is in the families (Siiii1) or (Ni) with Lie algebra $\mathfrak{h}_{2}, \mathfrak{h}_{8}$ and $\mathfrak{h}_{3}$ (with $D=1$ ) then the Bismut curvature tensor is non-negative. If $J$ is in the family (Si) with Lie algebra $\mathfrak{g}_{2}^{0}$ and diagonal metric the manifold is Bismut-flat. In all the other cases where (Cplx) is satisfied the invariant metrics are neither non-positive nor non-negative. See Table 8.3.

Lemmas 8.4.1, 8.4.2, 8.4.3, 8.4.4 and 8.4.5 collected in Section 8.4 lead to the following result.
Theorem 8.1.3. Let $M$ be a 6 -dimensional solvmanifold endowed with an invariant Hermitian structure ( $J, g$ ) with $\omega$ as in (8.3) and $J$ such that the canonical bundle is holomorphically-trivial. Then the symmetries of $(\overline{\mathrm{Cplx})}$ are preserved by any Hermitian curvature flow. Moreover, the Hermitian curvature flows preserve Bismut non-negativity and Bismut flatness.

All the computations on 6-dimensional Calabi-Yau solvmanifolds within the proofs of the above statements are contained in Section 8.4.

### 8.1.4 Bismut positivity on Hopf manifolds

In this section, we analyze the notion of Bismut positivity on diagonal Hopf manifolds. In particular, we focus on the homogeneous metrics $g(\alpha, \beta)$ for parameters $\alpha, \beta \in \mathbb{R}$ with $\beta>-\alpha$ and $\alpha>0$; see the precise definition (2.3) in Section 2.2

The family of metrics $g(\alpha, \beta)$ naturally arises from studying the evolution of HCFs on linear Hopf manifolds (see also 298 for the Chern-Ricci flow on Hopf manifolds). As a matter of fact, the standard metric $g_{H}$ belongs to this family being $g_{H}=g(1,0)$. Moreover, since the Hermitian curvature flows preserve the $S^{1} \times U(n)$-invariance of the metrics, the $g(\alpha, \beta)$ family is closed by their action.

As we saw in Section 2.2, the Bismut curvature tensor associated to a metric $g(\alpha, \beta)$ on a Hopf manifold satisfies various symmetries, including ( Cplx ). The 2-dimensional case is special because the standard metric $g_{H}$ on the Hopf surface is Bismut flat. Henceforth, thanks to the equations (2.4) and 2.8 , a metric $g(\alpha, \beta)$ on the Hopf surface is Bismut-non-negative if and only if $\beta \leq 0$. In general, since the metrics $g(\alpha, \beta)$ are described up to homotheties by the ratio $\gamma=\frac{\beta}{\alpha}$, the non-negativity of their Bismut curvature only depends on $\gamma$. In detail, we prove the following result.

Proposition 8.1.1. Let $(M, J)$ be a diagonal Hopf manifold of complex dimension $n$. Consider $\alpha, \gamma \in \mathbb{R}$ with $\alpha>0$ and $\gamma>-1$. Then the metric $g(\alpha, \gamma \alpha)$ is Bismut-non-negative if and only if $\gamma \leq 0$ for $n=2$, and $\gamma \leq-\frac{1}{2}$ for $n \geq 3$.
Proof. First of all, recall that from (2.4) the Bismut curvature tensor of a metric $g(\alpha, \beta)$ is composed by two terms, namely

$$
R^{B}(g(\alpha, \beta))=\alpha U^{\alpha}+2 \beta U^{\beta}
$$

where $U^{\alpha}$ equals the Bismut curvature of the standard metric $g_{H}$, and $U^{\beta}$ is non-positive since (2.8.

The Hopf surface is Bismut flat when equipped with the metric $g_{H}$, hence the metric $g(\alpha, \beta)$ on the Hopf surface is Bismut-non-negative if and only if $\beta \leq 0$.
Now suppose that $n \geq 3$. For any $\alpha>0$, the metric $g\left(\alpha,-\frac{1}{2} \alpha\right)$ is Bismut-non-negative. In particular, given any $x, y \in T^{1,0} M$, the following equation holds,

$$
R^{B}(g(\alpha, \beta))(x, \bar{x}, y, \bar{y})_{z}=\left.\frac{\alpha}{|z|^{8}}|(x \cdot y)| z\right|^{2}-\left.(x \cdot z)(z \cdot y)\right|^{2} \geq 0
$$

Moreover, for any $\varepsilon>0$ the metric $g(\alpha,(-1 / 2+\varepsilon) \alpha$ ) has Bismut curvature tensor given by

$$
R^{B}(g(\alpha,(-1 / 2+\varepsilon) \alpha))(x, \bar{x}, y, \bar{y})_{z}=\left.\frac{\alpha}{|z|^{8}}|(x \cdot y)| z\right|^{2}-\left.(x \cdot z)(z \cdot y)\right|^{2}+2 \varepsilon \alpha U_{\beta}(x, \bar{x}, y, \bar{y})
$$

On a point $z \in M$ with two zero coordinates (say $k$ and $l$ ), by equation 2.8 we get

$$
R^{B}(g(\alpha,(-1 / 2+\varepsilon) \alpha))\left(\partial_{k}, \bar{\partial}_{k}, \partial_{l}, \bar{\partial}_{l}\right)_{z}=2 \varepsilon \alpha U^{\beta}\left(\partial_{k}, \bar{\partial}_{k}, \partial_{l}, \bar{\partial}_{l}\right)_{z}=-\frac{2 \varepsilon \alpha}{|z|^{4}}<0
$$

The result follows.
To simplify the exposition we collect the critical values detected in the above proposition as

$$
\gamma_{n}= \begin{cases}0 & \text { if } n=2  \tag{8.4}\\ -\frac{1}{2} & \text { otherwise }\end{cases}
$$

Remark. Notice that $R^{B}(g(\alpha, \beta))(z)$ is nowhere positive because in any point $z \in M$ both the terms $U^{\alpha}(x, \bar{x}, y, \bar{y})_{z}$ and $U^{\beta}(x, \bar{x}, y, \bar{y})_{z}$ vanish if $x=\lambda z$ or $y=\lambda z$ for $\lambda \in \mathbb{C}$.

### 8.2 Evolution of the Hermitian curvature flows on Hopf manifolds

In this section, we describe the evolution of the Hermitian curvature flows on the homogeneous metrics on the Hopf manifolds. In particular, in Theorem 8.2.1 we obtain a global stability result for the HCFs on the homogeneous metrics $g(\alpha, \beta)$.

First of all, we compute the terms $R i c^{C h, 2}$ and $Q$ of the Hermitian curvature flows in the explicit case of a diagonal Hopf manifold equipped with a metric $g(\alpha, \beta)$. In local holomorphic coordinates, the Christoffel symbols of the Chern connection are

$$
\left(\Gamma^{C h}\right)_{i j}^{k}=\frac{1}{|z|^{2}}\left(\frac{\beta}{\alpha} \delta_{i}^{k} \bar{z}_{j}-\delta_{j}^{k} \bar{z}_{i}\right)-\frac{\beta}{\alpha} \frac{\bar{z}_{i} \bar{z}_{j} z^{k}}{|z|^{4}}
$$

and the Chern curvatures are

$$
\begin{aligned}
R^{C h}(g(\alpha, \beta))_{i \bar{j} k \bar{l}}=\frac{1}{|z|^{2}}\left[\delta_{k l}\left(\delta_{i j}-\frac{\bar{z}_{i} z_{j}}{|z|^{2}}\right)-\frac{\beta}{\alpha} \delta_{i l}\left(\delta_{j k}-\right.\right. & \left.\frac{\bar{z}_{k} z_{j}}{|z|^{2}}\right) \\
& \left.+\frac{\beta}{\alpha} \frac{\left(\delta_{j k} \bar{z}_{i}+\delta_{i j} \bar{z}_{k}\right)|z|^{2}-2 \bar{z}_{i} z_{j} \bar{z}_{k}}{|z|^{4}} z_{l}\right]
\end{aligned}
$$

The Chern torsion is

$$
\left(T^{C h}\right)_{i j}^{k}=\frac{1}{|z|^{2}}\left(\frac{\beta}{\alpha}+1\right)\left(\delta_{i}^{k} \bar{z}_{j}-\delta_{j}^{k} \bar{z}_{i}\right)
$$

and, referring to the notations in Section 7.1. we have the following quadratic terms in $T^{C h}$ :

$$
\begin{aligned}
& Q^{1}(g(\alpha, \beta))_{i \bar{j}}=\frac{1}{|z|^{2}}\left(\frac{\beta}{\alpha}+1\right)^{2}\left[\frac{\alpha}{\alpha+\beta} \delta_{i j}+\left(n-2+\frac{\beta}{\alpha+\beta}\right) \frac{\bar{z}_{i} z_{j}}{|z|^{2}}\right] \\
& Q^{2}(g(\alpha, \beta))_{i \bar{j}}=\frac{2}{|z|^{2}}\left(\frac{\beta}{\alpha}+1\right)^{2} \frac{\alpha}{\alpha+\beta}\left[\delta_{i j}-\frac{\bar{z}_{i} z_{j}}{|z|^{2}}\right] \\
& Q^{3}(g(\alpha, \beta))_{i \bar{j}}=(n-1)^{2}\left(\frac{\beta}{\alpha}+1\right)^{2} \frac{\bar{z}_{i} z_{j}}{|z|^{4}} \\
& Q^{4}(g(\alpha, \beta))_{i \bar{j}}=\frac{1}{|z|^{2}}\left(\frac{\beta}{\alpha}+1\right)^{2} \frac{\alpha}{\alpha+\beta}(n-1)\left[\delta_{i j}-\frac{\bar{z}_{i} z_{j}}{|z|^{2}}\right]
\end{aligned}
$$

Since the solution for the Hermitian curvature flows equations are unique, the following statement holds.

Proposition 8.2.1. Let $(M, J)$ be a diagonal Hopf manifold equipped with the Hermitian metric $g\left(\alpha_{0}, \beta_{0}\right)$. Then, given $a, b, c, d \in \mathbb{R}$, the generic Hermitian curvature flow

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t))+a Q^{1}(g(t))+b Q^{2}(g(t))+c Q^{3}(g(t))+d Q^{4}(g(t)) \\
g(0)=g\left(\alpha_{0}, \beta_{0}\right)
\end{array}\right.
$$

evolves the metric as

$$
g(t)_{i \bar{j}}=g(\alpha(t), \beta(t))=\alpha(t) \frac{\delta_{i j}}{|z|^{2}}+\beta(t) \frac{\bar{z}_{i} z_{j}}{|z|^{4}} \quad \text { for } t \geq 0
$$

where $\alpha(t)$ and $\beta(t)$ satisfy the $O D E$ system

$$
\left\{\begin{array}{l}
\alpha(0)=\alpha_{0}, \quad \beta(0)=\beta_{0}  \tag{8.5}\\
\dot{\alpha}(t)=\frac{\beta}{\alpha}+1-n+\left(\frac{\beta}{\alpha}+1\right)(a+2 b+(n-1) d) \\
\dot{\beta}(t)=-n \frac{\beta}{\alpha}+\left(\frac{\beta}{\alpha}+1\right)^{2}(n-1)(a+(n-1) c-1)-\left(\frac{\beta}{\alpha}+1\right)(a+2 b+(n-1)(d-1))
\end{array}\right.
$$

We set $\gamma:=\frac{\beta}{\alpha}$, and we rewrite the ODE system (8.5) as

$$
\left\{\begin{array}{l}
\alpha(0)=\alpha_{0}, \quad \beta(0)=\beta_{0} \\
\dot{\alpha}(t)=\gamma+1-n+(\gamma+1)(a+2 b+(n-1) d), \\
\dot{\beta}(t)=-n \gamma+(\gamma+1)^{2}(n-1)(a+(n-1) c-1)-(\gamma+1)(a+2 b+(n-1)(d-1))
\end{array}\right.
$$

Consequently, $\gamma$ evolves as

$$
\left\{\begin{array}{l}
\gamma(0)=\frac{\beta_{0}}{\alpha_{0}},  \tag{8.6}\\
\dot{\gamma}(t)=\frac{1}{\alpha}(\gamma+1)[(F-n) \gamma+F], \\
\dot{\alpha}(t)=(\gamma+1) L-n,
\end{array}\right.
$$

where

$$
\begin{aligned}
F(a, b, c, d, n) & :=(n-2) a-2 b+(n-1)^{2} c-(n-1) d, \quad \text { and } \\
L(a, b, c, d) & :=1+a+2 b+(n-1) d .
\end{aligned}
$$

Recall that $\beta>-\alpha$, hence $\frac{\beta}{\alpha}>-1$. Therefore, a metric $g(\alpha, \beta)$ can have ratio $\gamma=\frac{F}{n-F}$ only if $F<n$. Moreover, thanks to Proposition 8.2.1 and equation 8.6), the metrics $g(\alpha, \beta)$ with $\gamma=\frac{F}{n-F}$ are static for the HCF, meaning that it evolves them by homotheties. The following result shows that these static metrics are globally stable for the HCFs among the $g(\alpha, \beta)$ metrics.

Theorem 8.2.1. Consider an n-dimensional diagonal Hopf manifold equipped with a metric $g_{0}=g\left(\alpha_{0}, \beta_{0}\right)$, and the Hermitian curvature flow

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t))+a Q^{1}(g(t))+b Q^{2}(g(t))+c Q^{3}(g(t))+d Q^{4}(g(t)), \\
g(0)=g_{0} .
\end{array}\right.
$$

Suppose that the coefficients $(a, b, c, d)$ are such that $F(a, b, c, d, n)<n$. Then the metric $g\left(1, \frac{F}{n-F}\right)$, as well as any of its positive multiples, is static for the flow, and the metric $g_{0}$ evolves along the flow so that the ratio $\gamma$ converges to $\frac{F}{n-F}$.
Proof. Suppose that the starting metric $g\left(\alpha_{0}, \beta_{0}\right)$ has ratio $\gamma_{0}<\frac{F}{n-F}$. By the evolution equation (8.6) for $\gamma$ we know that $\gamma$ is strictly increasing along the flow, moreover, it is bounded above from $\frac{F}{n-F}$. We now distinguish two cases, depending on $L$ : when $\alpha$ is decreasing along the flow, meaning that $\left(\frac{F}{n-F}+1\right) L-n \leq 0$, and when it is not.
In the first case, suppose that $\gamma$ does not converge to $\frac{F}{n-F}$, then it needs to converge to some $\gamma_{\infty}$ with $\gamma_{0}<\gamma_{\infty}<\frac{F}{n-F}$. Hence,

$$
\dot{\alpha}<\left(\gamma_{\infty}+1\right) L-n<\left(\frac{F}{n-F}+1\right) L-n \leq 0,
$$

and thus $\dot{\alpha}$ is uniformly strictly negative and so $\alpha$ will get to zero in finite time, say $T$; at the same time $T, \gamma$ will be increasing with infinite slope (by equation (8.6)), which is a contradiction to the convergence $\gamma \rightarrow \gamma_{\infty}$.
In the second case, namely $\left(\frac{F}{n-F}+1\right) L-n>0$, we can suppose without loss of generality that $\left(\gamma_{0}+1\right) L-n>0$. Moreover, since the term $(\gamma+1)$ in the equation (8.6) for $\dot{\gamma}$ is positive and increasing we can suppress it and prove the convergence of $\gamma$ to $\frac{F}{n-F}$ with evolution equation

$$
\dot{\gamma}=\frac{1}{\alpha}[(F-n) \gamma+F] .
$$

Since $\gamma$ is bounded from above, then also $\dot{\alpha}$ is so. This means that we can bound $\alpha$ above with a straight line with positive slope $\alpha \leq \alpha(0)+A t$. Thus finally we get

$$
(\alpha(0)+A t) \dot{\gamma}(t) \geq(F-n) \gamma(t)+F .
$$

We have an explicit solution for this ODE, namely

$$
\gamma(t) \geq C(A t+\alpha(0))^{\frac{F-n}{A}}+\frac{F}{n-F}
$$

where the constant $C$ depends on the initial value $\gamma_{0}$. Since the exponent $\frac{F-n}{A}$ is negative we get the convergence to $\frac{F}{n-F}$ for $t \rightarrow \infty$.
A similar argument holds true also in the opposite case, namely if $\gamma_{0}>\frac{F}{n-F}$.
Remark. Theorem 8.2.1 represents evidence of (global) stability for the Hermitian curvature flows in the non-Kähler setting, compare with [286, Theorem 1.2].

### 8.3 Hermitian curvature flows preserving Bismut positivity

In this section, we detect a subset of the Hermitian curvature flows which preserve the Bismut non-negativity on diagonal Hopf manifolds equipped with metrics of type $g(\alpha, \beta)$. This subfamily is prescribed by inequalities of the coefficients ( $a, b, c, d$ ) characterizing the flows which depend on the complex dimension $n$ of the Hopf manifold.

Consider the coefficients $\gamma_{n}$ obtained as defined in (8.4).
Theorem 8.3.1. Let $(M, J)$ be a Hopf manifold of complex dimension $n$. Suppose that $a, b, c, d \in$ $\mathbb{R}$ are such that

$$
(n-2) a-2 b+(n-1)^{2} c-(n-1) d \leq n \frac{\gamma_{n}}{\gamma_{n}+1}
$$

Then if the metric $g_{0}=g\left(\alpha_{0}, \beta_{0}\right)$ is Bismut-non-negative, the Hermitian curvature flow

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t))+a Q^{1}(g(t))+b Q^{2}(g(t))+c Q^{3}(g(t))+d Q^{4}(g(t)), \\
g(0)=g_{0}
\end{array}\right.
$$

preserves the Bismut non-negativity.
Proof. Notice that since the metric $g\left(\alpha_{0}, \beta_{0}\right)$ is Bismut-non-negative, the initial ratio $\gamma_{0}=\frac{\alpha_{0}}{\beta_{0}}$ must satisfy $\gamma_{0} \leq \gamma_{n}$. Moreover, we have that

$$
F(a, b, c, d, n)=(n-2) a-2 b+(n-1)^{2} c-(n-1) d \leq n \frac{\gamma_{n}}{\gamma_{n}+1} \leq 0<n
$$

Therefore, thanks to Theorem 8.2.1, the ratio $\gamma$ will evolve along the flow converging to a value $\gamma_{\infty}=\frac{F}{n-F} \leq \gamma_{n}$. This means that the metric will remain Bismut-non-negative along the flow.

On the other hand, when the inequality in Theorem 8.3.1 is not satisfied, the flow does not preserve Bismut non-negativity. More precisely

Proposition 8.3.1. Let $(M, J)$ be a Hopf manifold of complex dimension n. The metric $g_{0}=g\left(1, \gamma_{n}\right)$ is Bismut-non-negative, and the Hermitian curvature flows

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} g(t)=-\operatorname{Ric}^{C h, 2}(g(t))+a Q^{1}(g(t))+b Q^{2}(g(t))+c Q^{3}(g(t))+d Q^{4}(g(t)), \\
g(0)=g_{0}
\end{array}\right.
$$

with coefficients ( $a, b, c, d$ ) such that

$$
(n-2) a-2 b+(n-1)^{2} c-(n-1) d>n \frac{\gamma_{n}}{\gamma_{n}+1}
$$

evolve it into a metric that is no more non-negative.

Proof. Consider the metric $g_{0}=g\left(1, \gamma_{n}\right)$ as starting point for a Hermitian curvature flow with coefficients $(a, b, c, d)$ as in the statement. The metric $g_{0}$ is Bismut-non-negative and $\gamma_{n}$ is the largest ratio for which this happens. Therefore, to prove the statement it is enough to verify that the HCF evolves the metric so that it holds $\dot{\gamma}(0)>0$. Thanks to (8.6),

$$
\dot{\gamma}(0)=\frac{1}{\alpha_{0}}\left(\gamma_{n}+1\right)\left[(F-n) \gamma_{n}+F\right],
$$

and $(F-n) \gamma_{n}+F>0$ by hypothesis.
Remark. Proposition 8.3.1 shows that the inequalities of Theorem 8.3.1 are sharp, meaning that they detect the largest possible set of HCFs which preserve Bismut non-negativity on diagonal Hopf manifolds equipped with metrics of type $g(\alpha, \beta)$. Moreover, taking the limit on $n$ in the inequality of Proposition 8.3 .1 one sees, for example, that the Hermitian curvature flows with coefficient $c<0$ do not preserve Bismut non-negativity.

## Bismut positivity and the pluriclosed flow

We end this section by testing the above results on the pluriclosed flow. We start by noticing that the quadratic term $Q$ associated to the pluriclosed flow is identified by the coefficients $a=1$ and $b=c=d=0$. With these values, we get that $F=n-2$. Therefore, for $n>2$, the pluriclosed flow evolving on homogeneous metrics on the Hopf manifolds does not preserve the Bismut non-negativity. On the other hand, the pluriclosed flow performed on the Hopf surfaces with metrics of type $g(\alpha, \beta)$ preserves the Bismut non-negativity. We notice that a metric $g(\alpha, \beta)$ on a diagonal Hopf manifold is pluriclosed if and only if the dimension is $n=2$. It is then remarkable that the pluriclosed flow behaves well with the Bismut non-negativity only in dimension two. However, the Bismut positivity condition becomes uninteresting under the assumption of pluriclosed metric. Indeed, thanks to the equations (8.1) and (8.2), a pluriclosed Bismut-positive metric must be Griffiths-positive, and hence equal to the Fubini-Study metric on some complex projective space.

### 8.4 Computations on 6-dimensional Calabi-Yau solvmanifolds

We collect here the computations on 6 -dimensional Calabi-Yau solvmanifolds that lead to Theorems 8.1.1, 8.1 .2 and 8.1.3. Some of them were performed with the help of the symbolic computation software Sage [293].

### 8.4.1 Nilmanifolds

## Holomorphically-parallelizable nilmanifolds in Family (Np)

Consider 6-dimensional holomorphically-parallelizable nilmanifolds, i.e. nilmanifolds with holomorphically-trivial tangent bundle. On these nilmanifolds, the complex structure equations are

$$
d \varphi^{1}=d \varphi^{2}=0, \quad d \varphi^{3}=\rho \varphi^{12} ; \quad \rho=0,1 .
$$

The case $\rho=0$ refers to the torus, which is Kähler and flat; thus we consider $\rho=1$ which corresponds to the Iwasawa manifold. A direct calculation shows that the Bismut curvature tensor satisfies (Cplx). Then, we compute the following determinant and coefficient:

$$
\begin{gathered}
R_{1 \overline{1} \overline{3} \overline{3}}^{B} R_{2 \overline{2} 3 \overline{3}}^{B}-R_{1 \overline{2} 3 \overline{3}}^{B} R_{2 \overline{1} 3 \overline{3}}^{B}=-\frac{t^{10}}{32 \mathrm{i} \operatorname{det} \Xi}, \\
R_{1 \overline{1} 3 \overline{3}}^{B}=\frac{t^{4}\left(r^{2} t^{2}-|z|^{2}\right)}{16 \mathrm{idet} \Xi},
\end{gathered}
$$

showing that the Bismut curvature tensor is neither non-positive nor non-negative.

## Nilmanifolds in Family (Ni)

Consider the generic Hermitian structure of this family

$$
d \varphi^{1}=d \varphi^{2}=0 ; \quad d \varphi^{3}=\rho \varphi^{12}+\varphi^{1 \overline{1}}+\lambda \varphi^{1 \overline{2}}+D \varphi^{2 \overline{2}},
$$

where $\rho \in\{0,1\}, \lambda \geq 0$, and $\operatorname{Im} D \geq 0$. Then, according to [307, (2.4) and (2.5)], up to linear biholomorphism we can take $v=z=0$ and $r^{2}=1$ in the generic expression (8.3):

$$
2 \omega=\mathrm{i}\left(\varphi^{1 \overline{1}}+s^{2} \varphi^{2 \overline{2}}+t^{2} \varphi^{3 \overline{3}}\right)+u \varphi^{1 \overline{2}}-\bar{u} \varphi^{2 \overline{1}} .
$$

The element $R_{231 \overline{3}}^{B}=\frac{\rho s^{2} t^{2}}{16 \overline{i d e t}^{2} \Xi}$ vanishes if and only if $\rho=0$. Therefore, taking into account the classification of complex structures up to equivalence in [86], we set the coefficients $\rho, \lambda$ and $D$ (and the Lie algebras) as follows:

- $(\rho, \lambda, D)=(0,0, \mathbf{i})$, Lie algebra $\mathfrak{h}_{2}$;
- $(\rho, \lambda, D)=(0,0, \pm 1)$, Lie algebra $\mathfrak{h}_{3} ;$
- $(\rho, \lambda, D)=\left(0,1, \frac{1}{4}\right)$, Lie algebra $\mathfrak{h}_{4}$;
- $(\rho, \lambda)=(0,1)$ and $D \in\left[0, \frac{1}{4}\right)$, Lie algebra $\mathfrak{h}_{5} ;$
- $(\rho, \lambda, D)=(0,0,0)$, Lie algebra $\mathfrak{h}_{8}$.

In any of these cases, direct computations of the Bismut curvature tensor yield that it satisfies (Cplx). Suppose now that $\lambda=0$ (thus Lie algebras $\mathfrak{h}_{2}, \mathfrak{h}_{3}$ and $\mathfrak{h}_{8}$ ). We have the following elements of the Bismut curvature tensor

$$
\begin{aligned}
& R_{1 \overline{1} 1 \overline{1}}^{B}=t^{2}, \\
& R_{2 \overline{1} 1 \overline{1}}^{B}=\Re(D) t^{2},
\end{aligned}
$$

$$
R_{3 \overline{3} 1 \overline{2}}^{B}=-\frac{\Re(\mathrm{i} D)}{\left(s^{2}-|u|^{2}\right)} t^{4} u, \quad \quad R_{3 \overline{3} 2 \overline{1}}^{B}=-\frac{\Re(\mathrm{i} D)}{\left(s^{2}-|u|^{2}\right)} t^{4} \bar{u}
$$

Thus if $D=-1$ (which corresponds to $\mathfrak{h}_{3}$ ), then $R_{112 \overline{2}}^{B}<0<R_{1 \overline{1} 1 \overline{1}}^{B}$ and the curvature tensor is neither non-negative nor non-positive. On the other hand, if $D=\mathrm{i}$ (which corresponds to $\mathfrak{h}_{2}$ ), then the determinant

$$
R_{3 \overline{3} 1 \overline{1}}^{B} R_{3 \overline{3} 2 \overline{2}}^{B}-R_{3 \overline{1} 1 \overline{2}}^{B} R_{3 \overline{2} 2 \overline{1}}^{B}=-\frac{t^{8}|u|^{2}}{\left(s^{2}-|u|^{2}\right)^{2}} \leq 0 .
$$

Thus the Bismut curvature tensor is non-negative if and only if $u=0$. Finally, for $D=1$ or $D=0$ (which correspond to $\mathfrak{h}_{3}$ or $\mathfrak{h}_{8}$ respectively) we have Bismut non-negativity.
Lemma 8.4.1. Suppose we are in case of Lie algebra $\mathfrak{h}_{2}$, that is $(\rho, \lambda, D)=(0,0, \mathrm{i})$. Suppose also that $u=0$ (i.e. the metric $g$ is diagonal, since we are supposing $v=z=0$ ) then also Ric ${ }^{C h, 2}$ and $Q$ are diagonal. This means that any Hermitian curvature flow preserves the condition $u=v=z=0$.

Now we turn to the Lie algebras $\mathfrak{h}_{4}$ and $\mathfrak{h}_{5}$, for which we compute the following element and the determinant of the curvature tensor:

$$
\begin{gathered}
R_{1 \overline{1} 1 \overline{1}}^{B}=t^{2}>0 \\
R_{1 \overline{1} 1 \overline{1}}^{B} R_{1 \overline{1} 2 \overline{2}}^{B}-R_{1 \overline{1} 1 \overline{2}}^{B} R_{1 \overline{1} 2 \overline{1}}^{B}=t^{4}\left(D-\frac{1}{4}\right) .
\end{gathered}
$$

Thus, in the case of Lie algebra $\mathfrak{h}_{5}$ (i.e. $D<\frac{1}{4}$ ) the Bismut curvature tensor is neither nonpositive nor non-negative. While, if $D=\frac{1}{4}$, the coefficients of second Bismut-Ricci tensor (which is $\left.R i c_{i \bar{j}}^{B, 2}:=g^{k \bar{l}} R_{k \bar{l} \bar{j}}^{B}\right)$ give

$$
R i c_{1 \overline{1}}^{B, 2} R i c_{2 \overline{2}}^{B, 2}-R i c_{1 \overline{2}}^{B, 2} R i c_{2 \overline{1}}^{B, 2}=-\frac{\left|4 s^{2}-4 \mathrm{i} \bar{u}+1\right|^{2}}{16\left(s^{2}-|u|^{2}\right)^{2}}<0 .
$$

## Nilmanifolds in Family (Nii)

Consider the complex structure equations

$$
d \varphi^{1}=0, \quad d \varphi^{2}=\varphi^{1 \overline{1}}, \quad d \varphi^{3}=\rho \varphi^{12}+B \varphi^{1 \overline{2}}+c \varphi^{2 \overline{1}}
$$

where $\rho \in\{0,1\}, c \geq 0, B \in \mathbb{C}$ satisfy $(\rho, B, c) \neq(0,0,0)$. If $\rho=0$ we have the coefficients

$$
R_{231 \overline{2}}^{B}=\frac{c\left(s^{2} t^{2}-|v|^{2}\right)^{2}}{16 \mathrm{i} \operatorname{det} \Xi}, \quad \quad R_{232 \overline{1}}^{B}=\frac{-\bar{B}\left(s^{2} t^{2}-|v|^{2}\right)^{2}}{16 \mathrm{i} \operatorname{det} \Xi}
$$

These vanish only if $c=B=0$ which is in contradiction with the hypothesis. Thus we take $\rho=1$ and compute

$$
R_{232 \overline{3}}^{B}=\frac{s^{2} t^{2}-|v|^{2}}{16 \mathrm{idet} \Xi} t^{4} \bar{B}
$$

Therefore, $B=0$ is necessary for (Cplx).
Now we prove that $c=v=0$. First of all, if $c=0$, we have

$$
R_{231 \overline{2}}^{B}=\frac{s^{2} t^{2}-|v|^{2}}{16 \mathrm{i} \operatorname{det} \Xi} \bar{v}^{2}
$$

which implies $v=0$; on the other hand, if $v=0$, we have

$$
R_{131 \overline{3}}^{B}=c t^{4} \frac{r^{2} t^{2}-|z|^{2}}{16 \mathrm{i} \operatorname{det} \Xi}
$$

which implies $c=0$. Thus $c=0$ if and only if $v=0$. Suppose $c \neq 0$ (hence $v \neq 0$ ), then we compute the following elements of the Bismut curvature tensor:

$$
\begin{aligned}
& R_{131 \overline{1}}^{B}=\frac{1}{16 \mathrm{idet} \Xi}\left[\mathrm{i} c t^{2}\left(r^{2} t^{2} \bar{z}+\mathrm{i} t^{2}|u|^{2}+\overline{u v} z-u v \bar{z}-\bar{z}|z|^{2}\right)-\mathrm{i} t^{2} u \overline{v z}-(c v+\bar{v})|z|^{2} \bar{v}\right] \\
& R_{131 \overline{2}}^{B}=\frac{1}{16 \mathrm{i} \operatorname{det} \Xi}\left[\mathrm{i} c t^{2}\left(s^{2} t^{2} u-u|v|^{2}-\bar{v}|z|^{2}+r^{2} t^{2} \bar{v}-\mathrm{i} s^{2} \bar{v} z\right)-\mathrm{i} t^{2} u \bar{v}^{2}-(c v+\bar{v}) \bar{v}^{2} z\right] \\
& R_{121 \overline{3}}^{B}=\frac{1}{8 \operatorname{idet} \Xi}\left[c t^{2}\left(-8 \mathrm{i} \operatorname{det} \Xi+r^{2}|v|^{2}-\mathrm{i} s^{4} z+s^{2} u v-\mathrm{i} \overline{u v} z\right)+\left(\mathrm{i} s^{2} \bar{v} z-u|v|^{2}\right)(c v+\bar{v})\right] \\
& R_{121 \overline{2}}^{B}=\frac{1}{8 \mathrm{i} \operatorname{det} \Xi}\left[t^{2}\left(\mathrm{i} u|v|^{2}+s^{2} \bar{v} z-c t^{2} \bar{u} z-\mathrm{i} c r^{2} t^{2} v\right)\right]
\end{aligned}
$$

From $R_{131 \overline{1}}^{B}=R_{131 \overline{1}}^{B}=0$ we get:

$$
\begin{aligned}
& {\left[\mathrm{i} c t^{2}\left(r^{2} t^{2}-|z|^{2}\right)-\left(\mathrm{i} t^{2} u+\overline{v z}\right)(c v+\bar{v})\right] \bar{z}=c t^{2} \bar{u}\left(t^{2} u-\mathrm{i} \bar{v} z\right)} \\
& {\left[\mathrm{i} c t^{2}\left(r^{2} t^{2}-|z|^{2}\right)-\left(\mathrm{i} t^{2} u+\overline{v z}\right)(c v+\bar{v})\right] \bar{v}=-c t^{2} s^{2}\left(\mathrm{i} t^{2} u+\bar{v} z\right)}
\end{aligned}
$$

Hence

$$
\left(\mathrm{i} t^{2} u+\bar{v} z\right)\left(\overline{u v}+\mathrm{i} s^{2} \bar{z}\right)=0
$$

Notice that from this equation we also get that $u=0$ if and only if $z=0$; however they can not vanish or we would get $c v=0$ from $R_{131 \overline{2}}^{B}=0$. Thus $u, z$ (as well as $v, c$ ) are different from zero and we distinguish two cases: $\mathrm{i} t^{2} u+\bar{v} z=0$ and $\overline{u v}+\mathrm{i} s^{2} \bar{z}=0$. In the first case, we have

$$
0=\mathrm{i} c t^{2}\left(r^{2} t^{2}-|z|^{2}\right)-\left(\mathrm{i} t^{2} u+\overline{v z}\right)(c v+\bar{v})=\mathrm{i} c t^{2}\left(r^{2} t^{2}-|z|^{2}\right)
$$

thus $c=0$, which is a contradiction. In the second case, $R_{121 \overline{2}}^{B}=0$ and $\overline{u v}+\mathrm{i} s^{2} \bar{z}=0$ imply

$$
0=\mathrm{i} u|v|^{2}+s^{2} \bar{v} z=c t^{2}\left(\bar{u} z+\mathrm{i} r^{2} v\right)
$$

Multiplying by $\bar{v}$ and using again $\overline{u v}+\mathrm{i} s^{2} \bar{z}=0$ we obtain $s^{2}|z|^{2}=r^{2}|v|^{2}$. Finally, with these equations $R_{121 \overline{3}}^{B}$ becomes

$$
R_{121 \overline{3}}^{B}=-c t^{2} \neq 0
$$

This shows that $v=c=0$ is needed to satisfy (Cplx).
With the above choice of parameters (namely, $\rho=1, c=B=0$ ) and $v=0,(\mathrm{Cplx})$ is satisfied and we get the following element and determinant of the Bismut curvature tensor:

$$
R_{2 \overline{2} 3 \overline{3}}^{B}=\frac{s^{2} t^{6}}{16 \mathrm{idet} \Xi} \quad R_{2 \overline{2} 3 \overline{3}}^{B} R_{1 \overline{1} 3 \overline{3}}^{B}-R_{12 \overline{3} \overline{3}}^{B} R_{2 \overline{1} 3 \overline{3}}^{B}=-\frac{t^{10}}{32 \mathrm{idet} \Xi}
$$

showing that the curvature tensor is neither non-negative nor non-positive.
Lemma 8.4.2. With parameters ( $\rho=1 ; c=B=0$ ) the condition $v=0$ (and hence (Cplx)) is preserved by any Hermitian curvature flow in the family HCF.

## Nilmanifolds in Family (Niii)

Consider the complex structure equations

$$
d \varphi^{1}=0, \quad d \varphi^{2}=\varphi^{13}+\varphi^{1 \overline{3}}, \quad d \varphi^{3}=\mathrm{i} \rho \varphi^{1 \overline{1}}+\delta \mathrm{i}\left(\varphi^{1 \overline{2}}-\varphi^{2 \overline{1}}\right),
$$

where $\rho \in\{0,1\}$ and $\delta= \pm 1$. From a direct computation, we get the following elements of the Bismut curvature tensor:

$$
\begin{aligned}
& R_{122 \overline{3}}^{B}=-\frac{\left(\mathrm{i} u v+s^{2} z\right) v^{2}}{16 \mathrm{i} \operatorname{det} \Xi}, \\
& R_{121 \overline{2}}^{B}=-\frac{\left(\mathrm{i} \delta \rho s^{2} z-\mathrm{i} r^{2} v-\delta \rho u v-\bar{u} z\right) s^{2} v}{16 \mathrm{i} \operatorname{det} \Xi}, \\
& R_{132 \bar{\Omega}}^{B}=\frac{\left(\mathrm{i} t^{2} u+z \bar{v}\right) s^{2} v}{16 \mathrm{i} \operatorname{det} \Xi}, \\
& R_{122 \overline{1}}^{B}=\frac{\mathrm{i} s^{4} z^{2}-2 s^{2} u v z+s^{2} \bar{u} v z-\mathrm{i} u^{2} v^{2}+\mathrm{i} v^{2}|u|^{2}}{16 \mathrm{i} \operatorname{det} \Xi} .
\end{aligned}
$$

First of all, we prove that for (Cplx to hold true $u, v$ and $z$ must vanish: suppose $v \neq 0$, then imposing $R_{122 \overline{3}}^{B}=0$ we get $s^{2} z=-\mathrm{i} u v$. Now $R_{121 \overline{2}}^{B}=0$ implies $r^{2} v=\mathrm{i} \bar{u} z$, and $R_{132 \overline{2}}^{B}=0$ implies $t^{2} u=\mathrm{i} \bar{v} z$. These three equations together would imply that $\operatorname{det} \Xi=0$ which is a contradiction, thus $v$ must vanish. Moreover, if $v=0$ from $R_{122 \overline{1}}^{B}=0$ we get also $z=0$. Finally, $R_{133 \overline{2}}^{B}$ with $v=z=0$ is

$$
R_{133 \overline{2}}^{B}=-\frac{s^{2} t^{2}\left(\mathrm{i} s^{2}-t^{2}\right) u}{16 \operatorname{idet} \Xi} .
$$

Thus, also $u$ must vanish. However, for $u=v=z=0$ we have $R_{133 \overline{1}}^{B}=\frac{1}{2}\left(\delta i t^{2}-s^{2}\right) \neq 0$, showing that (Cplx) is never satisfied.

### 8.4.2 Solvmanifolds

## Solvmanifolds in Family ( $\mathbf{S i}$ )

Consider the generic Hermitian structure of this family

$$
d \varphi^{1}=A\left(\varphi^{13}+\varphi^{1 \overline{3}}\right), \quad d \varphi^{2}=-A\left(\varphi^{23}+\varphi^{2 \overline{3}}\right), \quad d \varphi^{3}=0,
$$

where $A=\cos \theta+\mathrm{i} \sin \theta$ and $\theta \in[0, \pi)$. We directly compute

$$
R_{123 \overline{3}}^{B}=-\mathrm{i}|A|^{2} \frac{r^{2} u v^{2}+s^{2} z^{2} \bar{u}}{8 \mathrm{i} \operatorname{det} \Xi} .
$$

This vanishes only if $r^{2} u v^{2}+s^{2} z^{2} \bar{u}=0$ since $A \neq 0$. We then compute the following coefficients of the Bismut curvature tensor:

$$
\begin{aligned}
R_{133 \overline{1}}^{B}=-\frac{\bar{A}}{16 \mathrm{i} \operatorname{det} \Xi}\left[4 A r^{2} t^{2}|u|^{2}+(A+\bar{A}) r^{4}|v|^{2}-(\bar{A}\right. & +3 A) \mathrm{i} r^{2} z \overline{u v} \\
& \left.+(A+\bar{A}) \mathrm{i} r^{2} u v \bar{z}-(A-\bar{A})|u|^{2}|z|^{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
R_{133 \overline{2}}^{B}=-\frac{\bar{A}}{16 \mathrm{idet} \Xi}\left[-4 \mathrm{i} A r^{2} s^{2} t^{2} u-(A+\bar{A}) r^{2} s^{2} z \bar{v}\right. & +(A-\bar{A}) \mathrm{i} r^{2}|v|^{2} u \\
& \left.+(3 A-\bar{A}) \mathrm{i} s^{2}|z|^{2} u-(A-\bar{A}) u^{2} v \bar{z}\right], \\
R_{233 \overline{1}}^{B}=\frac{\bar{A}}{16 \mathrm{idet} \Xi}\left[-4 \mathrm{i} A r^{2} s^{2} t^{2} \bar{u}+(A+\bar{A}) r^{2} s^{2} v \bar{z}+\right. & (3 A-\bar{A}) \mathrm{i} r^{2}|v|^{2} \bar{u} \\
& \left.+(A-\bar{A}) \mathrm{i} s^{2}|z|^{2} \bar{u}+(A-\bar{A}) z \bar{u}^{2} \bar{v}\right], \\
R_{233 \overline{2}}^{B}=-\frac{\bar{A}}{16 \mathrm{idet} \Xi}\left[4 A s^{2} t^{2}|u|^{2}+(A+\bar{A}) s^{4}|z|^{2}-(A\right. & +\bar{A}) \mathrm{i} s^{2} z \overline{u v} \\
& \left.+(3 A+\bar{A}) \mathrm{i} s^{2} u v \bar{z}-(A-\bar{A})|u|^{2}|v|^{2}\right] .
\end{aligned}
$$

The system of equations generated by the vanishing of these four coefficients has $u=v=z=0$ as a unique solution. The computations follow exactly the same structure as for solvmanifolds in the family (Siv3), see Section 8.4.2. Moreover, a direct computation shows that with this hypothesis (Cplx) is satisfied.

Lemma 8.4.3. The invariant metric $g$ with $u=v=z=0$ is Chern-flat. Moreover, with these parameters also $Q$ is diagonal; hence (Cplx) is preserved by any HCF.

We computed the following elements of the Bismut curvature tensor:

$$
R_{1 \overline{1} \overline{1} \overline{1}}^{B}=2 \Re(A)^{2} \frac{r^{4}}{t^{2}}, \quad \quad R_{1 \overline{1} \overline{3} \overline{3}}^{B}=-2 r^{2} \Re(A)^{2}
$$

Then if $A \neq \mathrm{i}$ the Bismut curvature tensor is neither non-negative nor non-positive. On the other hand, for parameter $A=\mathrm{i}$, corresponding to the Lie algebra $\mathfrak{g}_{2}^{0}$, the diagonal metrics are Kähler, hence, Kähler-flat. By [57, the complex solvmanifold is in fact biholomorphic to a holomorphically-parallelizable manifold.

## Solvmanifolds in Family (Sii)

Consider the complex structure equations (with $x \in \mathbb{R}_{>0}$ )

$$
\begin{aligned}
& d \varphi^{1}=0 \\
& d \varphi^{2}=-\frac{1}{2} \varphi^{13}-\left(\frac{1}{2}+\mathrm{i} x\right) \varphi^{1 \overline{3}}+\mathrm{i} x \varphi^{3 \overline{1}}, \\
& d \varphi^{3}=\frac{1}{2} \varphi^{12}+\left(\frac{1}{2}-\frac{\mathrm{i}}{4 x}\right) \varphi^{1 \overline{2}}+\frac{\mathrm{i}}{4 x} \varphi^{2 \overline{1}} .
\end{aligned}
$$

Working on the elements $R_{232 \overline{3}}^{B}$ and $R_{233 \overline{3}}^{B}$ (which we set equal to zero) we get $s^{2}=t^{2}$, see 18 , Section 3.2.2] for details. Then

$$
R_{121 \overline{2}}^{B}=\frac{t^{2}(2 x-\mathrm{i})}{16 x} \neq 0,
$$

and so (Cplx) is never satisfied.
Solvmanifolds in Families (Siii1), (Siii3), (Siii4)
Recall that the Lie algebras underlying (Siii1), (Siii3), and (Siii4) are, respectively, $\mathfrak{g}_{4}, \mathfrak{g}_{6}$, and $\mathfrak{g}_{7}$. In order to give a unified argument, we will gather the complex structure equations as follows:

$$
d \varphi^{1}=\mathrm{i}\left(\varphi^{13}+\varphi^{1 \overline{3}}\right), \quad d \varphi^{2}=-\mathrm{i}\left(\varphi^{23}+\varphi^{2 \overline{3}}\right), \quad d \varphi^{3}=x \varphi^{1 \overline{1}}+y \varphi^{2 \overline{2}},
$$

where $(x, y)=( \pm 1,0)$ for $\mathfrak{g}_{4},(x, y)=(1,1)$ for $\mathfrak{g}_{6}$ and $(x, y=-x)=( \pm 1, \mp 1)$ for $\mathfrak{g}_{7}$. In particular $x \neq 0$. Imposing the symmetries (Cplx) on the Bismut curvature tensor, we get that $y$ must be zero (meaning that the underlying Lie algebra is $\mathfrak{g}_{4}$ ) and the metric described by the equation (8.3) needs to satisfy $u=v=z=0$; see [18, Section 3.2.3] for details. With these condition (Cplx) is satisfied and the only non-zero coefficients of type $R_{i \bar{j} k \bar{l}}^{B}$ of the curvature tensor is $R_{1 \overline{1} 1 \overline{1}}^{B}=t^{2}$.
Lemma 8.4.4. If $u=v=z=0$ (i.e. the metric $g$ is diagonal) then also Ric ${ }^{C h, 2}$ and $Q$ are diagonal. Consequently, any HCF preserves the condition $u=v=z=0$.

## Solvmanifolds in Family (Siii2)

The complex structure equations for this family are the following:

$$
d \varphi^{1}=\varphi^{13}+\varphi^{1 \overline{3}}, \quad d \varphi^{2}=-\varphi^{23}-\varphi^{2 \overline{3}}, \quad d \varphi^{3}=\varphi^{1 \overline{2}}+\varphi^{2 \overline{1}} .
$$

Imposing the symmetries (Cplx) on the Bismut curvature tensor, we get that the metric described by the equation (8.3) needs to satisfy $v=z=0$; see [18, Section 3.2.4] for details. From a direct computation, we get

$$
R_{133 \overline{1}}^{B}=\mathrm{i} r^{2} t^{2} u \frac{t^{2}+2 \mathrm{i} \bar{u}}{8 \operatorname{idet} \Xi}, \quad \quad R_{133 \overline{2}}^{B}=r^{2} s^{2} t^{2} \frac{t^{2}+2 \mathrm{i} u}{8 \operatorname{idet} \Xi}
$$

Then $R_{133 \overline{2}}^{B}=0$ implies $t^{2}+2 \mathrm{i} u=0$, and then $R_{133 \overline{1}}^{B}=0$ leads to $t^{2}+2 \mathrm{i} \bar{u}$. These two equations together imply that $u$ is real which is in contradiction with both of them. This shows that (Cplx) is never satisfied.

## Solvmanifolds in Families (Siv1)

Consider the complex structure equations for this family:

$$
d \varphi^{1}=-\varphi^{13}, \quad d \varphi^{2}=\varphi^{23}, \quad d \varphi^{3}=0 .
$$

A direct computation shows that the Bismut curvature tensor satisfies (Cplx). Moreover, we have the following coefficient and determinant of the curvature tensor:

$$
R_{1 \overline{1} 1 \overline{1}}^{B}=\frac{r^{2} s^{2}-|u|^{2}}{16 \mathrm{i} \operatorname{det} \Xi} r^{4} \geq 0, \quad R_{1 \overline{1} 1 \overline{1}}^{B} R_{1 \overline{1} 3 \overline{3}}^{B}-R_{1 \overline{1} 1 \overline{3}}^{B} R_{1 \overline{1} 3 \overline{1}}^{B}=-\frac{r^{2} s^{2}-|u|^{2}}{32 \mathrm{i} \operatorname{det} \Xi} r^{6} \leq 0 .
$$

## Solvmanifolds in Families (Siv2)

Recall the complex structure equations for this family:

$$
d \varphi^{1}=2 \mathrm{i} \varphi^{13}+\varphi^{3 \overline{3}}, \quad d \varphi^{2}=-2 \mathrm{i} \varphi^{23}-x \varphi^{3 \overline{3}}, \quad d \varphi^{3}=0 .
$$

where $x=0,1$. Consider the terms $R_{123 \overline{1}}^{B}$ and $R_{123 \overline{2}}^{B}$ :

$$
\begin{aligned}
& R_{123 \overline{1}}^{B}=-\frac{\left(r^{2} s^{2}-|u|^{2}\right)\left(x r^{2} s^{2}+x|u|^{2}+2 \mathrm{i} r^{2} \bar{u}\right)}{8 \operatorname{det} \Xi} \\
& R_{123 \overline{2}}^{B}=-\frac{\left(r^{2} s^{2}-|u|^{2}\right)\left(r^{2} s^{2}+|u|^{2}-2 \mathrm{i} x s^{2} u\right)}{8 \operatorname{det} \Xi}
\end{aligned}
$$

Notice that, $R_{123 \overline{1}}^{B}=R_{123 \overline{2}}^{B}=0$ if and only if

$$
x r^{2} s^{2}+x|u|^{2}+2 \mathrm{i} r^{2} \bar{u}=r^{2} s^{2}+|u|^{2}-2 \mathrm{i} x s^{2} u=0 .
$$

If $x=1$, these equations imply $\Re(u)=0, \Im(u)=-r^{2}$ and $r^{2}=s^{2}$, which is a contradiction to the positive definiteness of the metric. Hence, $x=0$ and $R_{123 \overline{2}}^{B}$ is always different from zero.

## Solvmanifolds in Families (Siv3)

The complex structure equations for this family are the following (with $A \in \mathbb{C} \backslash \mathbb{S}^{1}$ ):

$$
d \varphi^{1}=A \varphi^{13}-\varphi^{1 \overline{3}}, \quad d \varphi^{2}=-A \varphi^{23}+\varphi^{2 \overline{3}}, \quad d \varphi^{3}=0
$$

We directly compute

$$
R_{123 \overline{3}}^{B}=\frac{\mathrm{i}\left(r^{2} u v^{2}+s^{2} z^{2} \bar{u}\right) A}{8 \operatorname{idet} \Xi}
$$

This vanishes if $A=0$ or $r^{2} u v^{2}+s^{2} z^{2} \bar{u}=0$. We start analyzing the case $A \neq 0$. We compute the following coefficients of the Bismut curvature tensor:

$$
\begin{aligned}
& R_{133 \overline{1}}^{B}=\frac{1}{16 \mathrm{i} \operatorname{det} \Xi}\left[4 A r^{2} t^{2}|u|^{2}+(A-1) r^{4}|v|^{2}+(1-3 A) \mathrm{i} r^{2} z \overline{u v}\right. \\
& \left.+(A-1) \mathrm{i} r^{2} u v \bar{z}-(A+1)|u|^{2}|z|^{2}\right], \\
& R_{133 \overline{2}}^{B}=\frac{1}{16 \mathrm{idet} \Xi}\left[-4 \mathrm{i} A r^{2} s^{2} t^{2} u+(3 A+1) \mathrm{i} s^{2} u|z|^{2}+(1+A) \mathrm{i} r^{2} u|v|^{2}\right. \\
& \left.+(1-A) r^{2} s^{2} z \bar{v}-(A+1) u^{2} v \bar{z}\right], \\
& R_{233 \overline{1}}^{B}=\frac{1}{16 \mathrm{i} \operatorname{det} \Xi}\left[4 \mathrm{i} A r^{2} s^{2} t^{2} \bar{u}+(1-A) r^{2} s^{2} v \bar{z}-(3 A+1) \mathrm{i} r^{2}|v|^{2} \bar{u}\right. \\
& \left.-(A+1) \mathrm{i} s^{2}|z|^{2} \bar{u}-(A+1) z \bar{u}^{2} \bar{v}\right], \\
& R_{233 \overline{2}}^{B}=\frac{1}{16 \mathrm{idet} \Xi}\left[4 A s^{2} t^{2}|u|^{2}+(A-1) s^{4}|z|^{2}+(1-A) \mathrm{i} s^{2} z \overline{u v}\right. \\
& \left.+(3 A-1) \mathrm{i} s^{2} u v \bar{z}-(A+1)|u|^{2}|v|^{2}\right] .
\end{aligned}
$$

Notice that $A-1 \neq 0$ by hypothesis, thus if $u=0$ we get also $v=0$ and $z=0$ from $R_{133 \overline{1}}^{B}=0$ and $R_{233 \overline{2}}^{B}=0$ respectively. On the other hand, if $u \neq 0$ then $v$ vanishes if and only if $z$ vanishes (from $r^{2} u v^{2}+s^{2} z^{2} \bar{u}=0$ ), and they can not vanish together otherwise $u$ should also be 0 (from $\left.R_{233 \overline{2}}^{B}=0\right)$. Now suppose $u, v, z \neq 0$ and consider the following equations:

$$
\begin{align*}
R_{133 \overline{2}}^{B}-\overline{R_{233 \overline{1}}^{B}} & =\frac{s^{2}|z|^{2}-r^{2}|v|^{2}}{8 \mathrm{i} \operatorname{det} \Xi} A \mathrm{i}=0, \\
R_{133 \overline{1}}^{B}|v|^{2}-R_{233 \overline{2}}^{B}|z|^{2} & =-\frac{\frac{u v}{u v}+u v \bar{z}}{8 \mathrm{i} \operatorname{det} \Xi} A \mathrm{i} r^{2}|v|^{2}=0, \\
R_{233 \overline{1}}^{B} u v-R_{233 \overline{2}}^{B} \bar{u} z & =A|u|^{2} \frac{2^{2} t^{2}\left(\mathrm{i} r^{2} v-\bar{u} z\right)-2 \mathrm{i} r^{2}|v|^{2} v}{4 \mathrm{idet} \Xi}=0,  \tag{8.7}\\
R_{133 \overline{1}}^{B} u v-R_{133 \overline{2}}^{B} \bar{u} z & =A|u|^{2} \frac{r^{2} t^{2}\left(u v-\mathrm{i} s^{2} z\right)-2 \mathrm{i} r^{2}|v|^{2} z}{4 \mathrm{i} \operatorname{det} \Xi}=0, \tag{8.8}
\end{align*}
$$

where we used the first one to get the second and the first two to get the last two. Finally from (8.7) $\cdot z-8.8 \cdot v=0$ we get $v z=0$ which is a contradiction. This shows that $u, v$ and $z$ must be zero and a direct computation shows that with this hypothesis (Cplx) is satisfied.

In case $A=0, R_{123 \overline{1}}^{B}$ and $R_{123 \overline{2}}^{B}$ become

$$
R_{123 \overline{1}}^{B}=\frac{\left(r^{2} s^{2}-|u|^{2}\right)\left(\mathrm{i} r^{2} v+z \bar{u}\right)}{16 \mathrm{i} \operatorname{det} \Xi}, \quad R_{123 \overline{2}}^{B}=\frac{\left(r^{2} s^{2}-|u|^{2}\right)\left(u v-\mathrm{i} s^{2} z\right)}{16 \mathrm{i} \operatorname{det} \Xi} .
$$

The equations $\mathrm{i} r^{2} v+z \bar{u}=0$ and $\mathrm{i} s^{2} z-u v=0$ implies that $v$ vanishes if and only if $z$ vanishes. Moreover, if they are both different from zero, we can multiply the first one by $\bar{v}$ and the second one by $\bar{z}$; this leads to $\mathrm{i} r^{2}|v|^{2}+\overline{u v} z=0=\mathrm{i} s^{2}|z|^{2}-u v \bar{z}$ which is impossible. Hence $v$ and $z$ must be zero and with this hypothesis (Cplx) is satisfied.

Lemma 8.4.5. Both the conditions $u=v=z=0 \neq A$ and $v=z=A=0$ are preserved by any Hermitian curvature flow in the family HCF. Consequently, any Hermitian curvature flow preserves (Cplx) for the family (Siv3).

Now, setting $v=z=0$, we get the following elements of the curvature tensor:

$$
\begin{aligned}
& R_{1 \overline{1} 1 \overline{1}}^{B}=\frac{1}{2} \frac{r^{4}}{t^{2}}(A-1)(\bar{A}-1), \\
& R_{1 \overline{1} 3 \overline{3} \overline{3}}^{B}=-\frac{1}{2} \frac{(A-1)(\bar{A}-1) r^{4} s^{2}-((A-1) \bar{A}-A-3) r^{2}|u|^{2}}{r^{2} s^{2}-|u|^{2}},
\end{aligned}
$$

showing that in both cases $u=0$ and $A=0$ the Bismut curvature tensor is neither non-negative nor non-positive.

## Solvmanifolds in Families (Sv)

Recall the complex structure equations for this family:

$$
d \varphi^{1}=-\varphi^{3 \overline{3}}, \quad d \varphi^{2}=\frac{\mathrm{i}}{2} \varphi^{12}+\frac{1}{2} \varphi^{1 \overline{3}}-\frac{\mathrm{i}}{2} \varphi^{2 \overline{1}}, \quad d \varphi^{3}=-\frac{\mathrm{i}}{2} \varphi^{13}+\frac{\mathrm{i}}{2} \varphi^{3 \overline{1}} .
$$

Consider the terms $R_{233 \overline{3}}^{B}$ and $R_{123 \overline{1}}^{B}$ : if we set

$$
R_{233 \overline{3}}^{B}=\frac{s^{4}|z|^{2}}{32 \mathrm{idet} \Xi}=0,
$$

we get $z=0$, but then $R_{123 \overline{1}}^{B}=-\frac{r^{2} s^{2}-|u|^{2}}{4 t^{2}} \neq 0$; thus (Cplx) is never satisfied.

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