# Sampled-data steering of unicycles via PBC

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*Abstract*— **In this paper, on the basis of a recently proposed discrete-time port-Hamiltonian representation of sampled-data dynamics, we propose a new time-varying digital feedback for steering mobile robots. The quality of the proposed passivity-based control is validated and compared through simulations with the existing literature and the continuous-time implementation using the unicycle as a case study.**

*Index Terms*— **Sampled-data control, Nonholonomic systems, Autonomous vehicles.**

## I. Introduction

The well-known Brockett's condition [1] revealed the obstacle in stabilizing nonholonomic systems using smooth continuous control laws. As a matter of fact, this issue extends to larger classes of control problems involving nonholomic systems such as position steering, tracking or formation control in a multi-agent perspective. With this in mind, several works have been carried out with the aim of proposing design approaches for this class of systems using different tools as, for instance, time-varying [2], discontinuous [3] (both of these in a continuous-time framework) or digital multi-rate control [4] schemes. In all those works, the emphasis is on the unicycle, a benchmark example for the class of systems under investigation [5], [6]. In case of non-stationary modeling the angular velocity is described by a time-varying parameter. Thus, the overall design is performed over the obtained linear time-varying system and a proportional control is designed in order to ensure steering at a desired configuration provided that the angular component is sufficiently exciting [2]. Beside being simple and robust, this approach allows straightforward extensions to more general control problems such as position steering, tracking, rendez-vous or formation control [7]–[10].

On the other side, the digital control framework has been shown in [4], [11] to provide a natural setting for the control of nonholonomic systems, in view of the fact that these systems belong to the class of dynamics that admit, under preliminary continuous-time feedback, a finitely computable sampled-data model. In addition, when considering the unicycle, the sampled model has been proved to be fully invertible under multi-rate control (of suitable order) thereby allowing the design of digital feedback laws ensuring dead-beat (i.e., in one sampling instant) steering or tracking. However, some difficulties remain. The overall controller is piecewise continuous and consists in a fundamental continuous-time feedback component, responsible for ensuring finite discretizability. The closed loop suffers from robustness issues with respect to model uncertainties and sample-and-hold implementation due to the preliminary continuous-time loop component. Finite-time convergence in one step comes with a generally significant control effort that makes the implementation difficult in practice. Some of those issues have been partially solved in [12] embedding a Model Predictive Control (MPC) scheme. Other solutions involving sampled-data control basically rely upon emulation of the continuous-time (time-varying) control laws; i.e., through a direct implementation of the continuous-time control via sampled and hold devices without any further design [13], [14]. An exception is represented by [15] where a new sampled-data controller is designed with stability guarantees provided that the sampling period is small enough with respect to the control parameters. However, only stabilization at the origin is there considered and with the corresponding performances naturally limited by the amplitude of the sampling period.

This work is contextualized in this framework with the aim of designing a new and fully digital control law ensuring, at first, position-stabilization of a unicycle for all sampling periods and overcoming the aforementioned pathology. The contributions of the paper are detailed below. First, we prove that the sampled-data equivalent model of the kinematics admits the discrete-time port-Hamiltonian (pH) form recently proposed in [16], [17]. Thanks to this, a time-varying digital feedback on the linear velocity is defined via damping injection from the passivating output. In this way, stabilization at the desired position is guaranteed provided that the angular velocity component is set to a sufficiently exciting periodic signal. For the case under investigation, the sampled-data model that is exploited for the discrete-time design can be exactly and explicitly computed, guaranteeing that performances are enforced for all values of the sampling period. As a consequence, the effect of sampling is compensated exactly with no need of multi-rate and preliminary continuous-time feedback (that require more accurate and powerful actuation devices) while still guaranteeing good performances. In addition and contrarily to the aforementioned cases, the controller gets a simple structure as it is defined as the solution to a linear equality, so that it is easy to implement as well.

The remainder of the paper is organized as follows. In Section II, preliminaries on existing results for modeling and steering control of unicycle in continuous time are given and the problem stated. In Section III, a new sampled-data

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model for the unicycle is provided based on a discrete pH representation. Then, such a model is exploited for control design involving steering in Section IV with simulations in Section V. Section VI concludes the paper.

*Notations.* The sets $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0 respectively. The symbols $\succ$ and $\prec$ ($\succeq$ and $\preceq$) positive and negative (semi-)definite matrices. $I_n$ denotes the identity matrix of dimension $n \geq 1$ whereas 0 is the zero-matrix of suitable dimensions. Given $B \in \mathbb{R}^{n\times m}$ with $n > m$, $B^{\perp}$ be denotes the orthogonal complement verifying $B^{\perp}B = 0$. Given $m$ column vectors $g_j \in \mathbb{R}^n$ with $j = 1, \dots, m$ we denote by $\mathrm{diag}\{g_1, \dots, g_m\} \in \mathbb{R}^{mn\times m}$ the block-diagonal matrix with $g_j$ in the main diagonal whereas $\mathrm{col}\{g_1, \dots, g_m\} = (g_1^\top \ \dots g_m^\top)^\top \in \mathbb{R}^{nm}$. Given two matrices $A \in \mathbb{R}^{n_1\times n_2}$ and $B \in \mathbb{R}^{m_1\times m_2}$, the Kronecker product is denoted by $A \otimes B \in \mathbb{R}^{n_1 m_1 \times n_2 m_2}$. Given a real-valued differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, $\nabla V$ represents the gradient column-vector with $\nabla = col\{\frac{\partial}{\partial x_i}\}_{i=1,\dots,n}$ and $\nabla^2 V(\cdot)$ denotes its Hessian. For $v, w \in \mathbb{R}^n$, the discrete gradient is a vector-valued function of two variables, $\bar{\nabla} V(v)|_v^w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$\bar{\nabla} V(v)|_v^w = \int_0^1 \nabla V(v + s(w-v))ds$$

satisfying $V(w) - V(v) = (w-v)^\top \bar{\nabla} V(v)|_v^w$ with $\bar{\nabla} V(v)|_v^v = \nabla V(v)$. When $V(v) = \frac{1}{2}\nu^\top P v$ with $P = P^\top$, one gets $\bar{\nabla} V(v)|_v^w = \frac{1}{2}P(v+w)$.

## II. Problem statement and recalls

### A. Problem statement and the continuous-time solution

Consider the unicycle kinematics given by (1) as

$$\dot{z} = vr \tag{1a}$$
$$\dot{\theta} = \omega \tag{1b}$$

with $z := (x \ y)^\top \in \mathbb{R}^2$ the planar coordinates of the robot, $\theta \in \mathbb{R}$ the angle described by the chassis with respect to the horizontal axis and

$$r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad s = r^{\perp} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Assuming the input signals piecewise constant over time intervals of length $\delta > 0$ (the sampling period), the problem we address consists in designing a suitable digital control law driving the trajectories to a constant desired position in the Cartesian space. More in detail, it is formalized below.

*Problem 1 (Steering under digital control):* Consider the unicycle kinematics (1) under piecewise constant inputs over the sampling period $\delta > 0$, that is

$$\omega(t) = \omega_k, \quad v(t) = v_k \text{ for } t \in [k\delta, (k+1)\delta[. \tag{2}$$

and $z_d \in \mathbb{R}^2$ a desired plane position. Assuming tthe robot can sense its own orientation $\theta \in \mathbb{R}$ and the corresponding relative position with respect to the target point, that is

$$e_z = R(\theta)(z - z_d) \tag{3}$$

with

$$R(\theta) = \begin{pmatrix} r^\top \\ s^\top \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4}$$

The objective relies upon designing a sampled-data feedback $v_k = v^{\delta}(k, z_k, \theta_k)$ and $\omega_k = \omega^{\delta}(k, z_k, \theta_k)$ ensuring $z_k = z(k\delta) \to z_d$ as $k \to \infty$ for all initial conditions $\theta_0 \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}^n$. □

*Remark 2.1:* In the following convergence (i.e., $z_k \to z_d$ as $k \to \infty$) is enforced by making $z_d$ uniformly globally asymptotically stable (UGAS) for (1a).

In the continuous-time case, steering is ensured by the time-varying control

$$v = -\kappa r^\top (z - z_d), \quad \kappa > 0 \tag{5}$$

making (1a) UGAS at $z_d$ provided that $\omega(t)$ is persistently exciting; e.g., fixing [10]

$$\omega(t) = \cos\omega_0 t, \quad \omega_0 > 0. \tag{6}$$

Thus, the closed-loop dynamics

$$\dot{z} = -\kappa M(t)(z - z_d), \quad M(t) = rr^\top \tag{7}$$

is UGAS at $z_d \in \mathbb{R}^2$ because $M(t)$ is bounded and the signal $r(t) = r(\theta(t))$ is persistently exciting.

### B. Hamiltonian systems in discrete time

Consider now a discrete-time dynamics

$$x_{k+1} = x_k + F(x_k, u_k) \tag{8}$$

where, for simplicity, we denote by $(x_k, u_k) \in \mathbb{R}^n \times \mathbb{R}^m$, the pair of state and control variables at a generic time instant $k \geq 0$. A novel state space representation for discrete-time pH structures has been proposed in [16] through an implicit description of the drift dynamics in terms of the discrete gradient function. More precisely, denoting the free state evolution by $x^+ := x + F(x, 0)$ in the dynamics (8), a discrete-time pH system is described by

$$x^+(u) = x + (J_{\mathrm{d}}(x) - R_{\mathrm{d}}(x))\bar{\nabla} H|_x^{x^+} + g(x,u)u \tag{9a}$$
$$h(x,u) = g^\top(x,u)\bar{\nabla} H|_{x^+}^{x^+(u)} \tag{9b}$$

when setting, for simplicity, $x_{k+1} = x_k^+(u_k) = x^+(u)$, $x = x_k$, $u = u_k$, $x^+ = x + F(x, 0)$. In particular, $J_{\mathrm{d}}(x) = -J_{\mathrm{d}}{}^\top(x)$, $R_{\mathrm{d}}(x) = R_{\mathrm{d}}{}^\top(x) \succeq 0$ are the interconnection and damping matrices respectively, while $H : \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian function. As in the continuous-time case, discrete pH forms (9) verify by construction the one-step energy-balance equality so implying passivity; namely, setting $\Delta H(x) = H(x^+(u)) - H(x)$, one has

$$\Delta H(x) = \underbrace{-\bar{\nabla}^\top H|_x^{x^+} R_{\mathrm{d}}(x)\bar{\nabla} H|_{x^+}^{x^+}}_{\leq 0, \text{ one-step dissipated energy}} + \underbrace{h^\top(x,u)u}_{\text{one-step supplied energy}}.$$

It is instrumental to note that, making reference to the explicit form (8), the implicit pH representation (9) verifies the following variation equalities

$$F(x,0) = x^+ - x = (J_{\mathrm{d}}(x) - R_{\mathrm{d}}(x))\bar{\nabla} H|_x^{x^+}$$
$$F(x,u) - F(x,0) = x^+(u) - x^+ = g(x,u)u.$$

## III. The unicycle model under sampling

The evolutions of (1) under (2) at all sampling instants are described by the exact sampled equivalent model given by

$$z_{k+1} = z_k - \frac{v_k}{\omega_k}\Delta s \tag{10a}$$

$$\theta_{k+1} = \theta_k + \delta\omega_k \tag{10b}$$

with

$$\Delta s := s_{k+1} - s_k = \begin{pmatrix} -\sin\theta_{k+1} + \sin\theta_k \\ \cos\theta_{k+1} - \cos\theta_k \end{pmatrix}.$$

Considering the extended state-space $\zeta = \mathrm{col}\{z, r, s\} \in \mathbb{R}^6$ and noticing that

$$\dot r = \omega s, \quad \dot s = -\omega r$$

the sampled-data dynamics (10) is equivalently described by

$$\zeta_{k+1} = (A_0^\delta(\omega_k) \otimes I_2)\zeta_k + g^\delta(\zeta_k, \omega_k)v_k \tag{11}$$

with

$$\begin{aligned} &A_0^\delta(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & S^\delta(\omega) \end{pmatrix}, \; S^\delta(\omega) = \begin{pmatrix} \cos\delta\omega & \sin\delta\omega \\ -\sin\delta\omega & \cos\delta\omega \end{pmatrix} \\ &g^\delta(\zeta, \omega) = \frac{1}{\omega}\begin{pmatrix} -(\Delta s)^\top & 0 \end{pmatrix}^\top . \end{aligned} \tag{12}$$

The extended sampled-data model above admits a discrete pH structure as proved in the result below.

*Theorem 3.1:* The sampled-data model (11) of the unicycle kinematics (1) admits the time-varying pH form

$$\zeta^+(v) = \zeta + \frac{1}{2}\Big(\bar J^\delta(\omega) \otimes I_2\Big)\Big(\zeta^+ + \zeta\Big) + g^\delta(\zeta, \omega)v \tag{13}$$

with $\zeta = \zeta_k$, $v = v_k$, $\omega = \omega_k$, $\zeta^+(v) = \zeta^+(v_k) = \zeta_{k+1}$, $\zeta^+ = (A_0^\delta(\omega) \otimes I_2)\zeta$ and interconnection matrix

$$\bar J^\delta(\omega) = \begin{pmatrix} 0 & 0 \\ 0 & J^\delta(\omega) \end{pmatrix}, \; J^\delta(\omega) = \frac{\begin{pmatrix} 0 & \sin\omega\delta \\ -\sin\omega\delta & 0 \end{pmatrix}}{1 + \cos\delta\omega}. \tag{14}$$

The sampled-data dynamics (13) is passive with the output

$$h(\zeta, \omega, v) = \frac{1}{\delta}\big(g^\delta(\zeta, \omega)\big)^\top\Big(\zeta^+ + \frac{1}{2}g^\delta(\zeta, \omega)v\Big) \tag{15}$$

and, since $\frac{1}{2}(r^\top r + s^\top s) = 1$, quadratic storage function

$$H(\zeta) = \frac{1}{2}\zeta^\top\zeta = \frac{1}{2}z^\top z + 1. \tag{16}$$

*Proof:* Exploiting the representation (11), the proof consists in showing that (11) is equivalent to a discrete conservative pH dynamics (9) detailed as

$$\zeta^+(v) = \zeta + \tilde J^\delta(\omega)\bar\nabla H|_\zeta^{\zeta^+} + g^\delta(\zeta, \omega)v$$

for a suitable skew-symmetric matrix $\tilde J^\delta(\omega) \in \mathbb{R}^{6\times 6}$ and with the quadratic Hamiltonian (16). Because $\bar\nabla H|_\zeta^{\zeta^+} = \frac{1}{2}(\zeta + \zeta^+)$, this corresponds to solve the implicit equation

$$\zeta + \frac{1}{2}\tilde J^\delta(\omega)(\zeta^+ + \zeta) = \zeta^+.$$

Substituting $\zeta^+ = \zeta^+(0) = (A_0^\delta(\omega) \otimes I_2)\zeta$ from (11) into the equality above, one gets

$$\zeta + \frac{1}{2}\tilde J^\delta(\omega)\Big((A_0^\delta(\omega) \otimes I_2) + I_6\Big)\zeta = (A_0^\delta(\omega) \otimes I_2)\zeta.$$

that is solved for all $\zeta \in \mathbb{R}^6$ by

$$\begin{aligned} \frac{1}{2}\tilde J^\delta(\omega) &= \Big(A_0^\delta(\omega) \otimes I_2 - I_6\Big)\Big(A_0^\delta(\omega) \otimes I_2 + I_6\Big)^{-1} \\ &= \frac{1}{2}\big(\bar J^\delta(\omega) \otimes I_2\big). \end{aligned}$$

Passivity follows from the pH structure. ■

*Remark 3.1:* In the result above, we have modelled the sampled-data kinematics as a linear time-varying pH structure (13) deduced from (11) when considering $\omega \in \mathbb{R}$ a time-varying parameter and $v \in \mathbb{R}$ the control input.

*Remark 3.2:* Because $\big(\Delta s\big)^\top\Delta s = 2(1 - \cos\delta\omega)$, the passive output gets the explicit form

$$h(\zeta, \omega, v) = -\frac{1}{\delta\omega}\big(\Delta s\big)^\top z + \frac{1 - \cos\delta\omega}{\delta\omega^2}v. \tag{17}$$

*Remark 3.3:* For stabilization purposes, one can directly shift the storage function setting $V(z) = H(\zeta) - 1$ with $V(0) = 0$ and no impact on the pH representation (13).

*Remark 3.4:* The result in Theorem 3.1 provides an exact discrete-time Hamiltonian form to the sampled-data model (11) associated to the unicycle kinematics (1). This is fundamental for allowing to settle (and solve) Problem 1 in the discrete-time IDA-PBC framework [16].

## IV. A new sampled-data steering

For the sake of clarity, before providing the general solution to Problem 1, we first address the case of $z_d = (0\,0)^\top$.

*Theorem 4.1:* For all $\delta > 0$, consider the unicycle kinematics (1) with sampled-data equivalent model of the form (10). Then, Problem 1 with $z_d = 0$ is solved by the sampled-data time-varying control

$$v = \frac{\kappa}{\delta\omega + \frac{\kappa}{\omega}(1 - \cos\delta\omega)}\big(\Delta s\big)^\top z, \quad \kappa > 0 \tag{18a}$$

$$\omega = \frac{1}{\delta\omega_0}\Big(\sin\big((k+1)\omega_0\delta\big) - \sin\big(k\omega_0\delta\big)\Big) \tag{18b}$$

provided that $\omega_0 > 0$ is chosen so to satisfy, for a fixed sampling period $\delta > 0$ and $N \in \mathbb{N}$,

$$T_\delta := N\delta = \frac{2\pi}{\omega_0}, \quad N > 2. \tag{19}$$

The corresponding closed-loop sampled system is given by

$$z_{k+1} = (I_2 - \bar\kappa(\omega)M_k)z_k \tag{20a}$$

$$\theta_{k+1} = \theta_k + \frac{1}{\omega_0}\Big(\sin\big((k+1)\omega_0\delta\big) - \sin\big(k\omega_0\delta\big)\Big) \tag{20b}$$

with

$$M_k = \frac{1}{\delta\omega^2}\Delta s\big(\Delta s\big)^\top, \; \bar\kappa(\omega) = \frac{\kappa\delta\omega}{\delta\omega + \frac{\kappa}{\omega}(1 - \cos\delta\omega)}. \tag{21}$$

*Proof:* First, we note that (18a) is the damping control associated to the passive output (17) yielding

$$\begin{aligned} \Delta V(z) = \Delta H(\zeta) &= vh(\zeta, \omega, v) \\ &= -\frac{\kappa z^\top\Delta s\big(\Delta s\big)^\top z}{(\delta\omega + \frac{\kappa}{\omega}(1 - \cos\delta\omega))^2} \le 0 \end{aligned}$$

as, under feedback, the output reads

$$h(\zeta,\omega,v) = -\frac{1}{\delta\omega + \frac{\kappa}{\omega}(1-\cos\delta\omega)}\big(\Delta s\big)^\top z.$$

Accordingly, (20) is obtained substituting (18) into the sampled equivalent model (10). The closed-loop $\theta$-dynamics (20b) is periodic with period (19). Thus, Uniform Global Exponential Stability (UGES) of the closed loop follows by Lemma 1.1 (in Appendix) setting $\Phi_k = \frac{\sqrt{k}}{\omega\sqrt{\delta(1+\frac{\kappa}{\delta\omega}(1-\cos\delta\omega))}}\Delta s$ that is persistently exciting as $\Delta s$ is such under (19). ■

*Remark 4.1:* The period (19) of the sinusoidal signals in (18a) must be large enough with respect to the sampling period for exciting (20a) at all sampling instants. In other words, the sampling period and the period of the signals must be chosen in such a way that the corresponding samples, exciting the sampled-data dynamics (20a), are sufficiently rich and different.

It is worth to note that (18a) is a discrete IDA-PBC feedback [18]. As a matter of fact, it gets the form of a discrete-time damping injection over the average passivating output (17); i.e., it is the solution to the damping equality

$$v = -\frac{\kappa}{\delta}\big(g^\delta(\zeta,\omega)\big)^\top \bar\nabla H(\zeta)|_{\zeta^+}^{\zeta^+(v)}$$

$$\zeta^+ = \big(A_0^\delta(\omega)\otimes I_2\big)\zeta,\ \bar\nabla H(\zeta)|_{\zeta^+}^{\zeta^+(v)} = \frac{1}{2}(\zeta^+(v)+\zeta^+).$$

The closed-loop system (20) gets the discrete pH form

$$\zeta^+(v) = \zeta + \frac{1}{2}\Big(\bar J^\delta(\omega) - R^\delta(\omega)\Big)(\zeta^+(v)+\zeta)$$

over $\zeta = \mathrm{col}\{z,r,s\}$, with damping matrix

$$R^\delta(\omega) = \mathrm{diag}\{\kappa(\omega)M_k(I_2 - \frac{\kappa(\omega)}{2}M_k)^{-1},\ 0,\ 0\} \succeq 0. \quad (22)$$

*Remark 4.2:* Contrarily to [15] the controller in Theorem 4.1 guarantees steering for all $\delta,\kappa>0$. The gain is independent on the sampling period (and viceversa) and can be arbitrarily tuned to enforce the required performances.

At this point, to generally solve Problem 1, let us compute first the sampled-data error dynamics associated to (3) as

$$\begin{aligned} e_{z_{k+1}} &= R(\theta_{k+1})(z_{k+1} - z_d)\\ &= R(\theta_{k+1})R^\top(\theta_k)e_{z_k} - \frac{v_k}{\omega_k}\begin{pmatrix} -r_{k+1}^\top s_k \\ 1 - s_{k+1}^\top s_k\end{pmatrix}\\ &= S^\delta(\omega_k)e_{z_k} + B^\delta(\omega_k)v_k \end{aligned}$$

with $S^\delta(\omega)$ as in (12) and

$$B^\delta(\omega) = \frac{1}{\omega}\begin{pmatrix} r_{k+1}^\top s_k \\ s_{k+1}^\top s_k - 1\end{pmatrix} = \frac{1}{\omega}\begin{pmatrix}\sin\delta\omega \\ \cos\delta\omega - 1\end{pmatrix}.$$

It can be easily shown that the error dynamics above admits, once again, the conservative Hamiltonian structure

$$e^+(v) = e + \frac{1}{2}\Big(I_2\otimes J^\delta(\omega)\Big)(e+e^+) + B^\delta(\omega)v \quad (23)$$

with $e = \mathrm{col}\{e_z, r, s\}$ and interconnection matrix in (14). Passivity follows with respect to the output

$$h(e_z,\omega,v) = \frac{1}{\delta}\big(B^\delta(\omega)\big)^\top\Big(S^\delta(\omega)e_z + \frac{1}{2}B^\delta(\omega)v\Big).$$

Accordingly, the following result can be established.

*Corollary 4.1:* For all fixed $\delta>0$, consider the unicycle kinematics (1) and the sampled-data equivalent model (10). Then, Problem 1 is solved by the time-varying control

$$v = -\frac{\kappa}{\delta\omega + \frac{\kappa}{\omega}(1-\cos\omega\delta)}\begin{pmatrix}\sin\delta\omega & \cos\delta\omega - 1\end{pmatrix}e_z \quad (24a)$$

$$\omega = \frac{1}{\delta\omega_0}\Big(\sin\big((k+1)\omega_0\delta\big) - \sin\big(k\omega_0\delta\big)\Big) \quad (24b)$$

provided $\omega_0>0$ satisfies (19) for constant $\delta>0$ and $N\in\mathbb{N}$.

*Proof:* First, we apply the coordinates transformation

$$\bar e_z = R^\top(\theta)e_z = z - z_d$$

so getting the dynamics

$$\bar e_{z_{k+1}} = \bar e_{z_k} - \frac{v_k}{\omega_k}\Delta s_k.$$

In the new coordinates, the feedback (24a) reads

$$v = \frac{\kappa}{\delta\omega + \frac{\kappa}{\omega}(1-\cos\delta\omega)}\big(\Delta s\big)^\top \bar e_z$$

with corresponding closed-loop dynamics

$$\bar e_{z_{k+1}} = (I_2 - \bar\kappa(\omega)M_k)\bar e_{z_k}$$

with $M_k$ as in (21). Then, the proof follows along the lines of Theorem 4.1. ■

*Remark 4.3:* As $\delta\to 0$ the sampled-data controllers (24) naturally recover the continuous-time counterparts (5)-(6). In particular, by virtue of (3) and (4), one gets for (24a),

$$v \to -\kappa\begin{pmatrix}1 & 0\end{pmatrix}e_z = -\kappa\begin{pmatrix}1 & 0\end{pmatrix}R(\theta)(z-z_d) = -\kappa r^\top(z-z_d).$$

As in the case of $z_d = 0$, the steering feedback gets the form of a damping injection controller over the passive output; namely, of the form

$$v = -\frac{\kappa}{\delta}\Big(B^\delta(\omega)\Big)^\top \bar\nabla H(e)|_{e^+}^{e^+(v)}$$

$$\bar\nabla H(e)|_{e^+}^{e^+(v)} = \frac{1}{2}(e^+ + e^+(v)).$$

Setting for simplicity $\bar e = \mathrm{col}\{R^\top(\theta)e_z, r, s\}$, the closed-loop system gets the dissipative Hamiltonian form

$$\bar e^+(v) = \bar e + \frac{1}{2}\Big(\bar J^\delta(\omega) - R^\delta(\omega)\Big)(\bar e + \bar e^+(v)) \quad (25)$$

with interconnection and damping matrices given in (14) and (22) respectively.

## V. Simulations

The aim of the simulations presented in the sequel is twofold: validating the performances of the proposed controller with behaviors that are close to the continuous-time ones, for all sampling periods; comparing the proposed feedback with MPC, typically used in applications, and the one in [15] that is similar, in the aim, to the one we design[1]. The results are reported in Figures 1-2, with $\delta = 0.1$ and

[1] Further simulations have been performed including also the emulation-based controller but omitted for the sake of space. They can be found at https://youtu.be/Cq5WffifrCE.

$\delta = 1.8$ respectively with the continuous-time controller (5)-(6) also depicted as reference. For a fair comparison, we simulate stabilization at the origin (i.e., $z_d = (0\ 0)^\top$) under the initial condition $z_0 = (0\ 1)^\top$ and all the parameters as in [15, Sec. IV.C]. As far as the controller we propose and the continuous-time counterpart, we fix $\kappa = 1$ and $\omega_0 = 1$ to meet the requirements in Corollary 4.1. As far as MPC is concerned, we fix the prediction and control horizon at $n_p = n_c = 3$ and unitary control and state-regulation weighting matrices $Q = I_3$, $R = I_2$. In addition, for the sake of comparison, we inject to the MPC the same angular velocity persistently exciting signal component as the one in Theorem 4.1. Fig. 1 highlights that for a small sampling period (i.e. $\delta = 0.1$) the proposed controller achieves stabilization at the origin with similar performances as the continuous-time one. On the other side, the controller in [15] still achieves stabilization but with a larger transient due to the (inversely proportional) relationship among the values of the parameters and the sampling period (see Remark 4.2). This is even more evident when increasing the sampling period as depicted in Fig. 2; when $\delta = 1.8$, the controller in [15] provides better performances than the ones discussed in Fig. 1 that are still, however, overcome by the new controller we propose. Indeed, the closed-loop sampled-data system under the controller in Corollary 4.1 is only slightly affected by the increased value of the sampling period, providing behaviors that remain better than [15]. Performances of the proposed feedback are still better, yet comparable, to the ones under MPC, even when the latter one is implemented in favored conditions with respect to the typical ones (e.g., [19]). Indeed, contrarily to the common implementations, we set for the MPC $n_p = n_c = 3$ and include a persistently exciting reference on the orientation to guarantee boundedness of the closed loop under single rate control with no further terminal cost and constraints [12].

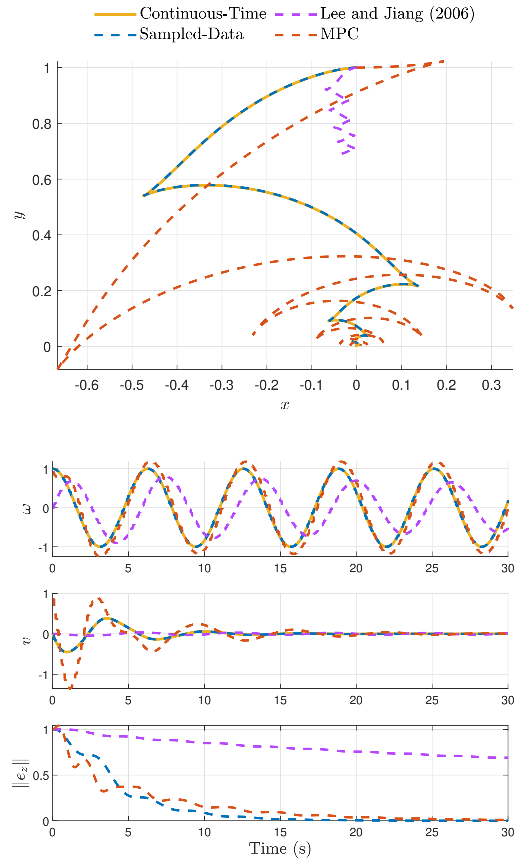


Fig. 1: Steering at the origin with $\delta = 0.1$ seconds.

## VI. Conclusions and Perspectives

A new digital control law for steering a mobile robot has been proposed. The design, based on a suitable Hamiltonian representation of the dynamics, yields a damping feedback over the linear velocity that ensures convergence to the desired position provided that the angular velocity is sufficiently persistently exciting. Current works are addressing regulation to a desired orientation and tracking at large, and extending those arguments to deal with formation control of multi-robot systems under asynchronous and aperiodic communication, measurement noise, and delay [20]–[23].

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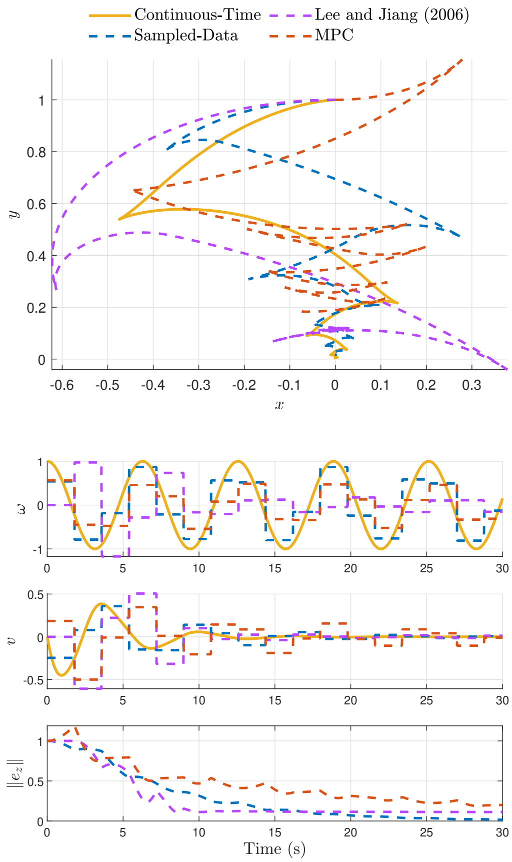


Fig. 2: Steering at the origin with $\delta = 1.8$ seconds.


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## APPENDIX

*Lemma 1.1:* Consider the linear time-varying system

$$z_{k+1} =(I_n - \Phi_k \Phi_k^\top) z_k \tag{26}$$

with $z \in \mathbb{R}^n$, $\Phi : \mathbb{N} \to \mathbb{R}^n$ and $\|\Phi_k\| \leq \phi_M$ for all $k \in \mathbb{N}$. The origin of (26) is UGES if $\Phi$ is persistently exciting; i.e., there exist positive $\mu \in \mathbb{R}_{>0}$ and $K \in \mathbb{N}$ such that

$$\sum_{j=k}^{k+K-1} \Phi_j \Phi_j^\top \succeq \mu I_n, \quad \text{for all } k \in \mathbb{N}. \tag{27}$$

*Proof:* The proof follows the lines of [24, Lemma 5] with the Lyapunov $H(z) = \frac{1}{2}\|z\|^2$ verifying $\Delta H = -\frac{1}{2}\|\Phi_k^\top z_k\|^2 \leq 0$. For $h_k := H(z_k)$ and all $k \geq k_0$, we get $h_k \leq h_{k_0} = \frac{1}{2}\|z_{k_0}\|^2$

$$\Delta^K h := h_{k+K} - h_k = -\frac{1}{2} \sum_{j=k}^{k+K-1} \|\Phi_j^\top z_j\|^2 \tag{28}$$

$$z_j = z_k - \sum_{i=k}^{j-1} \Phi_i \Phi_i^\top z_i, \quad j > k. \tag{29}$$

Substituting (29) into (28) and exploiting $\|a - b\|^2 \geq \frac{\rho}{\rho+1}\|a\|^2 - \rho\|b\|^2$ for all $\rho > 0$, one gets

$$\begin{aligned} \Delta^K h \leq & -\frac{\rho}{2(\rho+1)} \sum_{j=k}^{k+K-1} \|\Phi_j^\top z_k\|^2 \\ & + \frac{\rho}{2} \sum_{j=k}^{k+K-1} \|\Phi_j^\top \sum_{i=k}^{j-1} \Phi_i \Phi_i^\top z_i\|^2 \\ \leq & -\frac{\rho\mu}{2(\rho+1)} \sum_{j=k}^{k+K-1} \|z_k\|^2 \\ & + \frac{\rho}{2} \sum_{j=k}^{k+K-1} \|\Phi_j^\top \sum_{i=k}^{j-1} \Phi_i \Phi_i^\top z_i\|^2 \end{aligned} \tag{30}$$

by (27). At this point, because $\Phi$ is bounded and using both the triangle and Cauchy-Schwartz inequalities, one obtains

$$\sum_{j=k}^{k+K-1} \|\Phi_j^\top \sum_{i=k}^{j-1} \Phi_i \Phi_i^\top z_i\|^2 \leq \phi_M^4 \sum_{j=k}^{k+K-1} \sum_{i=k}^{j-1} \|\Phi_i^\top z_i\|^2.$$

Taking into account (30), the bound above and the fact that

$$\sum_{j=k}^{k+K-1} \sum_{i=k}^{j-1} \|\Phi_i^\top z_i\|^2 = \sum_{i=k+1}^{k+K-1} \sum_{j=i}^{k+K-1} \|\Phi_i^\top z_i\|^2$$

$$= \sum_{i=k+1}^{k+K-1} (k+K-i)\|\Phi_i^\top z_i\|^2 \leq K \sum_{i=k}^{k+K-1} \|\Phi_i^\top z_i\|^2 = 2\Delta^K h$$

one gets

$$\begin{aligned} \Delta^K h \leq & -\frac{\rho\mu}{2(\rho+1)} \sum_{j=k}^{k+K-1} \|z_k\|^2 + \rho\phi_M^4 K \Delta^K h \\ \implies & \left(1 + \rho\phi_M^4 K\right)\Delta^K h \leq -\frac{\rho\mu}{2(\rho+1)}\|z_k\|^2. \end{aligned}$$

Exploiting now that $\frac{1}{2}\|z_k\|^2 = h_k \leq h_{k_0}$, one deduces that

$$h_{k+K} \leq (1-\sigma)h_k, \quad \sigma = \frac{\rho}{\rho+1}\frac{\mu}{1+\rho\phi_M^4 K}$$

where $\rho > 0$ can be chosen to make $|1-\sigma| < 1$. From the inequality above one gets the result as

$$h_k \leq (1-\sigma)^{\frac{k-k_0}{K}} h_{k_0} \implies \|z_k\|^2 \leq (1-\sigma)^{\frac{k-k_0}{K}} \|z_{k_0}\|^2.$$

■