EQUILIBRIUM CONFIGURATIONS FOR NONHOMOGENEOUS LINEARLY ELASTIC MATERIALS WITH SURFACE DISCONTINUITIES

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Abstract. We prove a compactness and semicontinuity result that applies to minimisation problems in nonhomogeneous linear elasticity under Dirichlet boundary conditions. This generalises a previous compactness theorem that we proved and employed to show existence of minimisers for the Dirichlet problem for the (homogeneous) Griffith energy.

Contents

1. Introduction

In this paper we study the minimisation of free discontinuity functionals describing energies for linearly elastic solids with discontinuities, under Dirichlet boundary conditions. For a solid in a given (bounded) reference configuration $\Omega \subset \mathbb{R}^n$, whose *displacement field* is $u\colon \Omega \to \mathbb{R}^n$, the minimisation of integral functionals of the form

$$E(u) := \int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u} g(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \tag{1.1}$$

accounts for the interaction of the internal elastic energy and the energy dissipated in the surface discontinuities.

The elastic properties of the solid are determined by the *elastic strain* $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$, the symmetrized gradient of u, through a function f with superlinear growth in $e(u)$ (often a quadratic form) and in general depending on the material point $x \in \Omega$. The surface term is related to dissipative phenomena such as cracks, surface tension between different elastic phases, or internal cavities, and is concentrated on the *jump set* J_u, representing the surface discontinuities of u. The jump set is such that when blowing up around any $x \in J_u$, it is approximated by a hyperplane with normal $\nu_u(x) \in \mathbb{S}^{n-1}$ and the displacement field is close to two suitable distinct values $u^+(x)$, $u^-(x) \in \mathbb{R}^n$ on the two sides of the body with respect to this hyperplane. The jump opening, denoted by $[u]$, is then $[u](x) = u^+(x) - u^-(x)$. In order to ensure that the volume and the surface term do not interact, it is usually assumed that g be greater than a positive constant, or some growth condition for small values of $[u]$ (besides the superlinear growth of f). Therefore, the functionals we consider are bounded from below through the Griffith-like energy ([32, 28])

$$G(u) := \int_\Omega |e(u)|^p\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u)\,, \qquad \text{with } p > 1\,. \tag{1.2}$$

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The first main issue in the minimisation of energies of the type (1.1) when also the control from above is only through (1.2) (in particular if g is independent of $[u]$) is how to obtain suitable compactness. This is related to the lack of good a priori integrability properties for displacements with finite energy G. In fact, a pathological situation may occur in the presence of connected components, well included in Ω, whose boundary is contained (or almost completely contained) in J_u: this allows to modify the displacement in these internal components by adding any constant, so that arbitrarily large values of u may be reached without (or slightly) modifying the energy.

Compactness results for sequences with equibounded energy (1.2) have been obtained with increasing generality. In [11] compactness w.r.t. (with respect to) strong L^1 convergence is obtained assuming a uniform L^∞ bound on the displacement field: this guarantees that the distributional symmetrized gradient $\mathrm{E}u$ is a bounded Radon measure and then u belongs to the space $SBD(\Omega)$ of *special functions of bounded deformation* [8], and in particular $u \in L^1(\Omega;\mathbb{R}^n)$. In [21], Dal Maso introduced the space of *generalised special functions of bounded deformation* $GSBD(\Omega)$ (with the smaller $GSBD^p(\Omega)$, the right energy space for (1.2), see Section 2) and proved a compactness result under a uniform mild integrability control on sequences with bounded energy, ensuring convergence in measure.

The first compactness result for (1.2) without further constraints is obtained by Friedrich [29] *in dimension two*, basing on a *piecewise Korn inequality*. This inequality permits to ensure the compactness for sequences with bounded energy, up to subtracting suitable *piecewise rigid motions*, namely functions coinciding with an *infinitesimal rigid motion* (that is an affine function with skew-symmetric gradient) on each element of a suitable Caccioppoli partition $\mathcal{P} = (P_j)_j$ of the domain (that is $\partial^*\mathcal{P} = \bigcup_j \partial^* P_j$ has finite surface measure; see [17] characterising piecewise rigid motions).

In [15] we proved in any dimension that each sequence $(u_h)_h$ with equibounded energy (1.2) converges in measure (up to subsequences) to a $GSBD^p$ function $\overline{u}$, outside an exceptional set with finite perimeter A where $|u_h| \to +\infty$. Outside the exceptional set, weak L^p convergence for the symmetrized gradients $(e(u_h))_h$ holds and $\mathcal{H}^{n-1}(J_{\overline{u}} \cup (\partial^* A \cap \Omega)) \leq \liminf_h \mathcal{H}^{n-1}(J_{u_h})$. The main ingredient for basic compactness w.r.t. the convergence in measure is the Korn-Poincaré inequality for function with small jump set proven in [14], while the semicontinuity properties are obtained through a slicing argument. In particular, this directly solves the Dirichlet minimisation problem for the energy (1.2), with volume term possibly convex with p-growth in $e(u)$, but still attaining its minimum value for $e(u) = 0$: starting from a minimising sequence $(u_h)_h$, a minimiser is given by any function equal to $\overline{u}$ in $\Omega \setminus A$ and to an infinitesimal rigid motion in A. One may argue analogously if the minimum value of f is independent of x.

However, for general nonhomogeneous materials (for instance composite materials) such that the minimum value of $f(x,\cdot)$ depends on x, this strategy does not work and a better characterisation of the limit behaviour also in the exceptional set is required. A similar issue arises when employing the compactness result by Ambrosio [2, 3, 5] in the space of *generalised functions of bounded variation* $GSBV$, and in its subspace $GSBV^p$ to the minimisation of energies

$$\int_\Omega f(x, \nabla u)\,\mathrm{d}x + \int_{J_u} g(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \tag{1.3}$$

depending on the full gradient ∇u in place of $e(u)$. For this reason a compactness result in $GSBV^p$ of different type has been derived in [30]: for any sequence with bounded energy $(u_h)_h$ (1.3) it is possible to find modifications y_h such that the energy increases at most by $\frac{1}{h}$, $\mathcal{L}^n(\{\nabla u_h \neq \nabla y_h\}) \leq \frac{1}{h}$, and $(y_h)_h$ converges in measure to some $u \in GSBV^p$. The functions y_h are indeed obtained from u_h by subtracting a piecewise constant function *up to a set of small measure*, in the same spirit of the aforementioned [29] with piecewise rigid motions replaced by piecewise constant functions.

The present work is based on a different approach: we prove that, given $(u_h)_h$ with $\sup_h(G(u_h)) < +\infty$, for suitable piecewise rigid motions a_h the sequence $(u_h - a_h)_h$ converges in measure to some $u \in GSBD^p$, such that $G(u) \leq \liminf_h G(u_h)$. Differently from

[30], we have $e(u_h) = e(u_h - a_h)$ since we subtract piecewise rigid motions (without exceptional sets of small measure); this precludes in general $u_h - a_h$ to be a minimising sequence, but nevertheless the lower semicontinuity for the surface part is obtained directly in terms of J_{u_h}. Our compactness result is the following (we use notation (2.9) for Caccioppoli partitions).

Theorem 1.1. *Let $p \in (1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ be open, bounded, and Lipschitz. For any sequence $(u_h)_h$ with $\sup_h G(u_h) < +\infty$ there exist a subsequence, not relabelled, a Caccioppoli partition $\mathcal{P} = (P_j)_j$ of Ω, a sequence of piecewise rigid motions $(a_h)_h$ of the form*

$$a_h = \sum_{j \in \mathbb{N}} a_h^j \chi_{P_j}\,, \tag{1.4}$$

with a_h^j infinitesimal rigid motions and

$$|a_h^j(x) - a_h^i(x)| \to +\infty \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \text{ for all } i \neq j\,, \tag{1.5a}$$

and a function $u \in GSBD^p(\Omega)$ such that

$$u_h - a_h \to u \quad \mathcal{L}^n\text{-a.e. in } \Omega\,, \tag{1.5b}$$

$$e(u_h) \rightharpoonup e(u) \quad \text{in } L^p(\Omega; \mathbb{M}^{n\times n}_{sym})\,, \tag{1.5c}$$

$$\mathcal{H}^{n-1}(J_u \cup (\partial^*\mathcal{P} \cap \Omega)) \leq \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h})\,. \tag{1.5d}$$

The first step of the proof consists in finding a partition $\mathcal{P}$, piecewise rigid motions a_h, and u measurable such that (1.4), (1.5a), and (1.5b) hold. In doing this, a fundamental tool is a Korn inequality for functions with small jump set, proven in two dimensions in [18, Theorem 1.2] and recently extended to any dimension in [13]. This permits, for every $\eta > 0$, to recover (1.4) and (1.5b) in a set $\Omega^\eta \subset \Omega$, such that $\mathcal{L}^n(\Omega \setminus \Omega^\eta) < \eta$. Then, the so obtained sequences of infinitesimal rigid motions are regrouped in equivalence classes for fixed η, saying that any $(a_h^i)_h$, $(a_h^j)_h$ (depending on η) are not equivalent if and only if (1.5a) holds for i, j. Finally, we pass to $\eta \to 0$ observing that this procedure is stable when η decreases: the objects found in correspondence to η coincide with those found for $\eta/2$ on $\Omega^\eta \cap \Omega^{\eta/2}$.

In the second step we prove (1.5d) through a slicing procedure. The guiding idea is that, if (1.4), (1.5a), (1.5b) hold for $n = 1$ (with Ω a real interval, a_h piecewise constant, and $(|\nabla u_h|)_h$ equibounded in L^p), then not only any jump point of u is a cluster point for $(J_{u_h})_h$ but this holds also for any point $y \in \partial^*\mathcal{P} \cap \Omega$: in fact, by (1.5a) and (1.5b), the functions u_h assume arbitrarily far values, as $h \to \infty$, in couple of points close to y but on different sides of $\Omega \setminus \{y\}$, so u_h have to jump near y for h large. We conclude by noticing that in view of (1.5d) the a_h are indeed piecewise rigid motions, so $\sup_h G(u_h - a_h) < +\infty$ and (1.5c) follows from former compactness results.

Besides compactness, we examine the semicontinuity properties of E, defined in (1.1). The lower semicontinuity of the surface term has been recently established for a large class of densities in [31], for sequences equibounded w.r.t. G (defined in (1.2)) and converging in measure (and also for functionals defined on piecewise rigid motions), providing a counterpart for the analysis of energies (1.3) in [6, 7]. We then assume that the surface part is lower semicontinuous w.r.t. the convergence in measure and move in two directions: we address the semicontinuity properties both of the volume term, and of the surface term w.r.t. the notion of convergence from Theorem 1.1. We prove the following result (see (2.8) for the definition of weak convergence in $GSBD$ and recall (2.9)).

Theorem 1.2. *Let $p \in (1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ be open, bounded, and Lipschitz. Assume that*

(f_1) *$f\colon \Omega \times \mathbb{M}^{n\times n}_{sym} \to [0, \infty)$ be a Carathéodory function;*
(f_2) *$f(x, \cdot)$ be symmetric quasi-convex for $\mathcal{L}^n$-a.e. $x \in \Omega$;*
(f_3) *for suitable $C > 0$ and $\phi \in L^1(\Omega)$, it holds*

$$\frac{1}{C}|\xi|^p \leq f(x,\xi) \leq \phi(x) + C(1 + |\xi|^p) \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{M}^{n\times n}_{sym}\,;$$

moreover assume that

(g_1) *$g\colon \Omega \times \mathbb{R}^n \times \mathbb{S}^{n-1} \to [c, +\infty)$ be measurable, with $c > 0$;*

(g_2) *$g(x,y,\nu)=g(x,-y,-\nu)$ for any x, y, ν;*

(g_3) *$g(\cdot,y,\nu)$ be continuous, uniformly w.r.t. $y\in\mathbb{R}^n$ and $\nu\in\mathbb{S}^{n-1}$;*

(g_4) *for each $x\in\Omega$ $g_x=g(x,\cdot,\cdot)$ be such that for any cube Q and any $v_h\to v$ weakly in $GSBD^p(Q)$*

$$\int_{J_v} g_x([v],\nu_v)\,\mathrm{d}\mathcal{H}^{n-1}\le\liminf_{h\to\infty}\int_{J_{v_h}} g_x([v_h],\nu_{v_h})\,\mathrm{d}\mathcal{H}^{n-1}\,;$$

(g_5) *there is $g_\infty\colon\Omega\times\mathbb{R}^n\to[0,+\infty]$ such that either $g_\infty\equiv+\infty$ or $g_\infty(x,\cdot)$ is a norm for every $x\in\Omega$, and*

$$\lim_{|y|\to+\infty} g(x,y,\nu)=g_\infty(x,\nu)\quad\textit{uniformly w.r.t. } x\in\Omega \textit{ and } \nu\in\mathbb{S}^{n-1}\,.$$

Then, for any sequence $(u_h)_h$ such that $\sup_h E(u_h)<+\infty$ there exist a subsequence (not relabelled), a Caccioppoli partition $\mathcal{P}$ of Ω, a sequence of piecewise infinitesimal rigid motions $(a_h)_h$, and $u\in GSBD^p(\Omega)$ such that (1.4), (1.5) *hold and*

$$\int_\Omega f(x,e(u))\,\mathrm{d}x+\int_{J_u\cap\mathcal{P}^{(1)}} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial^*\mathcal{P}\cap\Omega} g_\infty(x,\nu_{\mathcal{P}})\,\mathrm{d}\mathcal{H}^{n-1}\le\liminf_{h\to\infty}E(u_h)$$

if g_∞ is finite, while $\mathcal{H}^{n-1}(\partial^*\mathcal{P}\cap\Omega)=0$ and $E(u)\le\liminf_h E(u_h)$ if $g_\infty\equiv+\infty$.

The proof relies on a blow-up argument ([27]). For the bulk part we use again the result in [13]. Blowing up around a point $x_0\notin J_u$, since the density of jump vanishes in $\mathcal{H}^{n-1}$-measure, the Korn-type inequality of [13] give that the rescaled function coincides with a $W^{1,p}$ field up to a small set; we then combine this with an approximation through equi-Lipschitz functions, in the footsteps of [1, 4, 24], in order to apply Morrey's Theorem [33] in most of the blow-up ball.

As for the surface energy concentrated on $\partial^*\mathcal{P}$, we blow-up around $x_0\in\partial^*\mathcal{P}\cap\Omega$ to find that the rescaled function converges in measure, up to subtracting in the two halves of the blow-up cell two different infinitesimal rigid motions whose difference diverges as $h\to+\infty$, so that the jump has arbitrarily large amplitude near the middle of the cell. This allows to conclude through a slicing argument (anisotropic), which requires a suitable condition on g_∞, such as (g_5) (notice that this condition, which states that g_∞ is independent of the amplitude of the jump, is very restrictive, however we have currently no idea of how to treat more general cases).

Theorem 1.2 ensures existence of solutions to the class of minimisation problems

$$\min_{(u,\mathcal{P})}\left\{\int_\Omega f(x,e(u))\,\mathrm{d}x+\int_{J_u\setminus\partial^*\mathcal{P}} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial^*\mathcal{P}\cap\Omega} g_\infty(x,\nu)\,\mathrm{d}\mathcal{H}^{n-1}\right\}$$

under Dirichlet boundary condition. The further condition $\partial^*\mathcal{P}\cap\Omega\subset J_u$ may be enforced, permitting to detect the effective fractured zone by looking only at u. In this class of problems we minimise not only in u, but also on the possible partitions that may be created. If $g_\infty\equiv+\infty$, minimising sequences converge without modifications, see Proposition 4.2. The case $g(x,[u],\nu)=g_\infty(\nu)=\psi(\nu)$ corresponds to minimise an anisotropic version of (1.2) with general nonhomogeneous bulk energy (see Proposition 4.4; we refer to [20, Theorem 5.1] for an anisotropic version of (1.2) in the context of epitaxially strained materials [12]).

The paper is organised as follows: in Section 2 we recall basic notions and prove two lemmas on infinitesimal rigid motions. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2 and address the Dirichlet minimisation problems.

2. Preliminaries

In this section we fix the notation and recall the main tools employed in this work.

2.1. **Basic notation.** For every $x\in\mathbb{R}^n$ and $\varrho>0$, let $B_\varrho(x)\subset\mathbb{R}^n$ be the open ball with center x and radius ϱ, and let $Q_\varrho(x)=x+(-\varrho,\varrho)^n$, $Q^\pm_\varrho(x)=Q_\varrho(x)\cap\{x\in\mathbb{R}^n\colon\pm x_1>0\}$. For $\nu\in\mathbb{S}^{n-1}:=\{x\in\mathbb{R}^n\colon|x|=1\}$, we let also $Q^\nu_\varrho(x)$ the cube with "center" x, sidelength ϱ and with a face in a plane orthogonal to ν. We omit to write the dependence on x when $x=0$. (For x, $y\in\mathbb{R}^n$, we use the notation $x\cdot y$ for the scalar product and $|x|$ for the Euclidean norm.)

By $\mathbb{M}^{n\times n}$, $\mathbb{M}^{n\times n}_{\rm sym}$, and $\mathbb{M}^{n\times n}_{\rm skew}$ we denote the set of $n\times n$ matrices, symmetric matrices, and skew-symmetric matrices, respectively. We write χ_E for the indicator function of any $E\subset\mathbb{R}^n$, which is 1 on E and 0 otherwise. If E is a set of finite perimeter, we denote its essential boundary by ∂^*E, and by $E^{(s)}$ the set of points with density s for E, see [9, Definition 3.60]. We indicate the minimum and maximum value between $a,b\in\mathbb{R}$ by $a\wedge b$ and $a\vee b$, respectively.

We denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ the n-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively. The m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$ is indicated by γ_m for every $m\in\mathbb{N}$. For any locally compact subset $B\subset\mathbb{R}^n$, (i.e. any point in B has a neighborhood contained in a compact subset of B), the space of bounded $\mathbb{R}^m$-valued Radon measures on B [respectively, the space of $\mathbb{R}^m$-valued Radon measures on B] is denoted by $\mathcal{M}_b(B;\mathbb{R}^m)$ [resp., by $\mathcal{M}(B;\mathbb{R}^m)$]. If $m=1$, we write $\mathcal{M}_b(B)$ for $\mathcal{M}_b(B;\mathbb{R})$, $\mathcal{M}(B)$ for $\mathcal{M}(B;\mathbb{R})$, and $\mathcal{M}_b^+(B)$ for the subspace of positive measures of $\mathcal{M}_b(B)$. For every $\mu\in\mathcal{M}_b(B;\mathbb{R}^m)$, its total variation is denoted by $|\mu|(B)$. Given $\Omega\subset\mathbb{R}^n$ open, we use the notation $L^0(\Omega;\mathbb{R}^m)$ for the space of $\mathcal{L}^n$-measurable functions $v\colon\Omega\to\mathbb{R}^m$, endowed with the topology of convergence in measure.

Definition 2.1. Let $E\subset\mathbb{R}^n$, $v\in L^0(E;\mathbb{R}^m)$, and $x\in\mathbb{R}^n$ such that

$$\limsup_{\varrho\to0^+}\frac{\mathcal{L}^n(E\cap B_\varrho(x))}{\varrho^n}>0\,.$$

A vector $a\in\mathbb{R}^m$ is the *approximate limit* of v as y tends to x if for every $\varepsilon>0$ there holds

$$\lim_{\varrho\to0^+}\frac{\mathcal{L}^n(E\cap B_\varrho(x)\cap\{|v-a|>\varepsilon\})}{\varrho^n}=0\,,$$

and then we write

$$\operatorname*{ap\,lim}_{y\to x}v(y)=a\,.$$

Definition 2.2. Let $U\subset\mathbb{R}^n$ be open and $v\in L^0(U;\mathbb{R}^m)$. The *approximate jump set* J_v is the set of points $x\in U$ for which there exist a, $b\in\mathbb{R}^m$, with $a\neq b$, and $\nu\in\mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{(y-x)\cdot\nu>0,\,y\to x}v(y)=a\quad\text{and}\quad\operatorname*{ap\,lim}_{(y-x)\cdot\nu<0,\,y\to x}v(y)=b\,.$$

The triplet (a,b,ν) is uniquely determined up to a permutation of (a,b) and a change of sign of ν, and is denoted by $(v^+(x),v^-(x),\nu_v(x))$. The jump of v is the function defined by $[v](x):=v^+(x)-v^-(x)$ for every $x\in J_v$.

We note that J_v is a Borel set with $\mathcal{L}^n(J_v)=0$, and that $[v]$ is a Borel function.

2.2. ***BV* and *BD* functions.** Let $U\subset\mathbb{R}^n$ be open. We say that a function $v\in L^1(U)$ is a *function of bounded variation* on U, and we write $v\in BV(U)$, if $\mathrm{D}_iv\in\mathcal{M}_b(U)$ for $i=1,\dots,n$, where $\mathrm{D}v=(\mathrm{D}_1v,\dots,\mathrm{D}_nv)$ is its distributional derivative. A vector-valued function $v\colon U\to\mathbb{R}^m$ is in $BV(U;\mathbb{R}^m)$ if $v_j\in BV(U)$ for every $j=1,\dots,m$. The space $BV_{\rm loc}(U)$ is the space of $v\in L^1_{\rm loc}(U)$ such that $\mathrm{D}_iv\in\mathcal{M}(U)$ for $i=1,\dots,d$. If $n=1$, $v\in L^1(U)$ is a function of bounded variation if and only if its pointwise variation is finite, cf. [9, Proposition 3.6, Definition 3.26, Theorem 3.27].

A function $v\in L^1(U;\mathbb{R}^n)$ belongs to the space of *functions of bounded deformation* if the distribution $\mathrm{E}v:=\frac{1}{2}((\mathrm{D}v)^T+\mathrm{D}v)$ belongs to $\mathcal{M}_b(U;\mathbb{M}^{n\times n}_{sym})$. It is well known (see [8, 35]) that for $v\in BD(U)$, J_v is countably $(\mathcal{H}^{n-1},n{-}1)$ rectifiable, and that

$$\mathrm{E}v=\mathrm{E}^av+\mathrm{E}^cv+\mathrm{E}^jv\,,$$

where E^av is absolutely continuous w.r.t. $\mathcal{L}^n$, E^cv is singular w.r.t. $\mathcal{L}^n$ and such that $|\mathrm{E}^cv|(B)=0$ if $\mathcal{H}^{n-1}(B)<\infty$, while E^jv is concentrated on J_v. The density of E^av w.r.t. $\mathcal{L}^n$ is denoted by $e(v)$.

The space $SBD(U)$ is the subspace of all functions $v\in BD(U)$ such that $\mathrm{E}^cv=0$. For $p\in(1,\infty)$, we define $SBD^p(U):=\{v\in SBD(U)\colon e(v)\in L^p(\Omega;\mathbb{M}^{n\times n}_{sym}),\ \mathcal{H}^{n-1}(J_v)<\infty\}$. Analogous properties hold for BV, such as the countable rectifiability of the jump set and the decomposition of $\mathrm{D}v$. The spaces $SBV(U;\mathbb{R}^m)$ and $SBV^p(U;\mathbb{R}^m)$ are defined similarly, with

∇v, the density of $\mathrm{D}^a v$, in place of $e(v)$. For a complete treatment of BV, SBV functions and BD, SBD functions, we refer to [9] and to [35, 8, 11, 19], respectively.

2.3. GBD **functions.** The space GBD of *generalized functions of bounded deformation* has been introduced in [21]. We recall its definition and main properties, referring to that paper for a general treatment and more details. Since the definition of GBD is given by slicing (differently from the definition of GBV, cf. [3, 23]), we first introduce some notation. Fixed $\xi \in \mathbb{S}^{n-1}$, we let

$$\Pi^\xi := \{y \in \mathbb{R}^n \colon y \cdot \xi = 0\}, \qquad B^\xi_y := \{t \in \mathbb{R} \colon y + t\xi \in B\} \quad \text{for any } y \in \mathbb{R}^n \text{ and } B \subset \mathbb{R}^n\,, \tag{2.1}$$

and for every function $v \colon B \to \mathbb{R}^n$ and $t \in B^\xi_y$, let

$$v^\xi_y(t) := v(y + t\xi), \qquad \widehat{v}^\xi_y(t) := v^\xi_y(t) \cdot \xi\,. \tag{2.2}$$

Definition 2.3 ([21])**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $v \in L^0(\Omega;\mathbb{R}^n)$. Then $v \in GBD(\Omega)$ if there exists $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that one of the following equivalent conditions holds true for every $\xi \in \mathbb{S}^{n-1}$:

(a) for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$, the partial derivative $\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big) = \mathrm{D}\big(\tau(v\cdot\xi)\big)\cdot\xi$ belongs to $\mathcal{M}_b(\Omega)$, and for every Borel set $B \subset \Omega$
$$\big|\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big)\big|(B) \leq \lambda_v(B);$$

(b) $\widehat{v}^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset \Omega$
$$\int_{\Pi^\xi} \Big(\big|\mathrm{D}\widehat{v}^\xi_y\big|\big(B^\xi_y \setminus J^1_{\widehat{v}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\widehat{v}^\xi_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \leq \lambda_v(B)\,,$$
where $J^1_{\widehat{u}^\xi_y} := \Big\{t \in J_{\widehat{u}^\xi_y} : |[\widehat{u}^\xi_y]|(t) \geq 1\Big\}$.

The function v belongs to $GSBD(\Omega)$ if $v \in GBD(\Omega)$ and $\widehat{v}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

Every $v \in GBD(\Omega)$ has an *approximate symmetric gradient* $e(v) \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ such that for every $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ there holds

$$e(v)(y + t\xi)\xi\cdot\xi = \dot{v}^\xi_y(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^\xi_y\,; \tag{2.3}$$

where $\dot{v}^\xi_y$ denotes the density of the absolutely continuous part of the derivative $\mathrm{D}\widehat{v}^\xi_y$ of $\widehat{v}^\xi_y$, the *approximate jump set* J_v is still countably $(\mathcal{H}^{n-1}, n{-}1)$-rectifiable (*cf.* [21, Theorem 6.2]) and may be reconstructed from its slices through the identity

$$(J^\xi_v)^\xi_y = J_{\widehat{v}^\xi_y} \quad \text{and} \quad v^\pm(y + t\xi)\cdot\xi = (\widehat{v}^\xi_y)^\pm(t) \ \text{ for } t \in (J_v)^\xi_y\,, \tag{2.4}$$

where $J^\xi_v := \{x \in J_v \colon [v]\cdot\xi \neq 0\}$ (it holds that $\mathcal{H}^{n-1}(J_v \setminus J^\xi_v) = 0$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$). It follows that, if $v \in GSBD(\Omega)$ with $\mathcal{H}^{n-1}(J_v) < +\infty$, for every Borel set $B \subset \Omega$

$$\mathcal{H}^{n-1}(J_v \cap B) = (2\gamma_{n-1})^{-1} \int\limits_{\mathbb{S}^{n-1}} \Bigg(\int\limits_{\Pi^\xi} \mathcal{H}^0(J_{\widehat{v}^\xi_y} \cap B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\Bigg)\,\mathrm{d}\mathcal{H}^{n-1}(\xi) \tag{2.5}$$

and the two conditions in the definition of $GSBD$ for v hold for $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that

$$\lambda_v(B) \leq \int_B |e(v)|\,\mathrm{d}x + \mathcal{H}^{n-1}(J_v \cap B) \quad \text{for every Borel set } B \subset \Omega\,. \tag{2.6}$$

For any countably $(\mathcal{H}^{n-1}, n{-}1)$-rectifiable set $M \subset \Omega$ with unit normal $\nu \colon M \to \mathbb{S}^{n-1}$, it holds that for $\mathcal{H}^{n-1}$-a.e. $x \in M$ there exist the *traces* $v^+_M(x)$, $v^-_M(x) \in \mathbb{R}^n$ such that

$$\operatorname*{ap\,lim}_{\pm(y-x)\cdot\nu(x)>0,\, y\to x} v(y) = v^\pm_M(x) \tag{2.7}$$

and they can be reconstructed from the traces of the one-dimensional slices. This has been proven by [21, Theorem 5.2] for C^1 manifolds of dimension $n{-}1$, and may be extended to countably $(\mathcal{H}^{n-1}, n{-}1)$-rectifiable sets arguing as in [10, Proposition 4.1, Step 2].

Finally, if Ω has Lipschitz boundary, for each $v \in GBD(\Omega)$ the traces on $\partial\Omega$ are well defined in the sense that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ there exists $\mathrm{tr}(v)(x) \in \mathbb{R}^n$ such that

$$\operatorname*{ap\,lim}_{y\to x,\ y\in\Omega} v(y) = \mathrm{tr}(v)(x).$$

For $1 < p < \infty$, the space $GSBD^p(\Omega)$ is defined by

$$GSBD^p(\Omega) := \{v \in GSBD(\Omega) \colon e(v) \in L^p(\Omega; \mathbb{M}^{n\times n}_{sym}),\ \mathcal{H}^{n-1}(J_v) < \infty\}\,.$$

We say that a sequence $(v_k)_k \subset GSBD^p(\Omega)$ *converges weakly* to $v \in GSBD^p(\Omega)$ if

$$\sup_{k\in\mathbb{N}} \big(\|e(u_k)\|_{L^p(\Omega)} + \mathcal{H}^{n-1}(J_{u_k})\big) < +\infty \quad \text{and} \quad u_k \to u \text{ in } L^0(\Omega;\mathbb{R}^n)\,. \tag{2.8}$$

We say that $(v_k)_k$ *is bounded in* $GSBD^p(\Omega)$ if $\sup_k \big(\|e(u_k)\|_{L^p(\Omega)} + \mathcal{H}^{n-1}(J_{u_k})\big) < +\infty$.

We recall the following approximate Korn-type inequality for $GSBD^p$ functions with small jump set in a ball, recently proven in [13]. (We fix the case $\varepsilon = 1$ in that result.) We refer to [29] and [18] for Korn-type inequalities in $GSBD^p$ in two dimensions.

Theorem 2.4 ([13], Theorem 3.2)**.** *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $p \in (1, +\infty)$. Given $\sigma \in (0,1)$ there exist $C = C(n,p)$ and $\eta = \eta(n,p,\sigma)$, such that for every $\varrho > 0$, $v \in GSBD^p(B_\varrho)$ with $\mathcal{H}^{n-1}(J_v) \leq \eta\varrho^{n-1}$ there exist $w \in GSBD^p(B_\varrho)$ and a set of finite perimeter $\omega \subset B_\varrho$ such that $w = v$ in $B_\varrho \setminus \omega$, $\mathcal{H}^{n-1}(\partial^*\omega) < C\mathcal{H}^{n-1}(J_v)$, $w \in W^{1,p}(B_{\sigma\varrho};\mathbb{R}^n)$, and*

$$\int_{B_\varrho} |e(w)|^p \,\mathrm{d}x \leq 2\int_{B_\varrho} |e(v)|^p \,\mathrm{d}x\,, \qquad \mathcal{H}^{n-1}(J_w) \leq \mathcal{H}^{n-1}(J_v)\,.$$

Employing this result, in [13] it is proven that any function $v \in GSBD^p(\Omega)$ is approximately differentiable $\mathcal{L}^n$-a.e. in Ω, that is for $\mathcal{L}^n$-a.e. $x \in \Omega$ there exists $\nabla v(x) \in \mathbb{M}^{n\times n}$ (such that $e(v)(x) = (\nabla v(x))^{\mathrm{sym}}$ for a.e. x) for which it holds

$$\operatorname*{ap\,lim}_{y\to x} \frac{|v(y) - v(x) - \nabla v(x)(y-x)|}{|y-x|} = 0\,.$$

2.4. **Caccioppoli partitions.** A partition $\mathcal{P} = (P_j)_j$ of an open set $U \subset \mathbb{R}^n$ is said a *Caccioppoli partition* of U if $\sum_{j\in\mathbb{N}} \partial^* P_j < +\infty$. (see [9, Definition 4.16]). For Caccioppoli partitions the following structure theorem holds.

Theorem 2.5 ([9], Theorem 4.17)**.** *Let $(P_j)_j$ be a Caccioppoli partition of U. Then*

$$\bigcup_{j\in\mathbb{N}} P_j^{(1)} \cup \bigcup_{i\neq j} (\partial^* P_i \cap \partial^* P_j)$$

contains $\mathcal{H}^{n-1}$-almost all of U.

For any Caccioppoli partition $\mathcal{P} = (P_j)_j$ we set

$$\partial^*\mathcal{P} := \bigcup_{j\in\mathbb{N}} \partial^* P_j, \qquad \mathcal{P}^{(1)} := \bigcup_{j\in\mathbb{N}} P_j^{(1)}, \qquad \nu_{\mathcal{P}}(x) := \nu_{P_j}(x) \quad \text{for } x \in \partial^* P_j \setminus \bigcup_{i<j} \partial^* P_i\,. \tag{2.9}$$

2.5. **Symmetric quasi-convexity.** We recall the definition of symmetric quasi-convex functions, introduced in [25].

Definition 2.6 ([25])**.** A function $f \colon \mathbb{M}^{n\times n}_{sym} \to [0, +\infty)$ is *symmetric quasi-convex* if

$$f(\xi) \leq \frac{1}{\mathcal{L}^n(D)} \int_D f(\xi + e(\varphi)(x)) \,\mathrm{d}x$$

for every bounded open set D of $\mathbb{R}^n$, for every $\varphi \in W^{1,\infty}_0(D;\mathbb{R}^n)$, and for every $\xi \in \mathbb{M}^{n\times n}_{sym}$.

This property is related to the *quasi-convexity* in the sense of Morrey [33]; indeed f is symmetric quasi-convex if and only if $f \circ \pi$ is quasi-convex in the sense of Morrey, where π denotes the projection of $\mathbb{M}^{n\times n}$ onto $\mathbb{M}^{n\times n}_{sym}$ (see [25, Remark 2.3]).

2.6. **Two lemmas on affine functions.** We present below two lemmas that will be useful in the slicing procedure in Theorems 1.1 and 1.2. We say that $a\colon \mathbb{R}^d \to \mathbb{R}^d$ is an *infinitesimal rigid motion* if a is affine with $e(a) = \frac{1}{2}(\nabla a + (\nabla a)^{\mathrm{T}}) = 0$.

Lemma 2.7. *Let $(\mathcal{P}_j)_j$ be a Caccioppoli partition and let $(a_h)_h$ be a sequence of piecewise rigid motions such that* (1.4) *and* (1.5a) *hold. Then for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$*

$$|(a_h^j - a_h^i)(x)\cdot\xi| \to +\infty \quad \text{as } h \to +\infty \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \text{ for all } i \neq j\,, \tag{2.10}$$

Proof. For fixed $i \in \mathbb{N}$, $j \in \mathbb{N}$, with $i \neq j$, (2.10) follows from [15, Lemma 2.7] applied to $v_h = a_h^j - a_h^i$. This provides an $\mathcal{H}^{n-1}$-negligible set of ξ $N_{i,j} \subset \mathbb{S}^{n-1}$.

Then (2.10) holds for any $i \neq j$ for every $\xi \in \mathbb{S}^{n-1} \setminus N$, where $N = \bigcup_{i\neq j} N_{i,j}$ is still $\mathcal{H}^{n-1}$-negligible. □

Lemma 2.8. *Let $(a_k)_k$ be a sequence of infinitesimal rigid motions, $a_k\colon \Omega \to \mathbb{R}^n$, $a_k(x) = A_k x + b_k$, $A_k \in \mathbb{M}^{n\times n}_{\mathrm{skew}}$, $b_k \in \mathbb{R}^n$. Then, up to a subsequence, either $(a_k)_k$ converges uniformly to an infinitesimal rigid motion (with values in $\mathbb{R}^n$) or there exists an affine subspace Π of dimension at most $n-2$ such that $|a_k(x)| \to +\infty$ for every x in $\Omega \setminus \Pi$.*

Proof. If A_k and b_k are uniformly bounded on a subsequence, then we have uniform convergence to an infinitesimal rigid motion (with values in $\mathbb{R}^n$). Else, $\mu_k := |A_k| + |b_k|$ diverges. Up to a subsequence, we may assume that

$$\frac{A_k}{\mu_k} \to A \ \text{ and } \ \frac{b_k}{\mu_k} \to b \quad \text{with } |A| + |b| = 1. \tag{2.11}$$

Then, for any x,

$$Ax + b \neq 0 \quad \Rightarrow \quad \lim_{k\to+\infty} |a_k(x)| = \lim_{k\to+\infty} \mu_k |Ax + b| = +\infty$$

If $A = 0$, this is true for all x (since $b \neq 0$). Else, this is true as long as x does not belong to the affine subspace $\Pi := \{x\colon Ax + b = 0\}$, which has dimension at most $n-2$, A being a non null skew-symmetric matrix. □

3. The compactness result

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof in steps.

Step 1: Existence of $(a_h)_h$. Let $\mu_h := \mathcal{H}^{n-1} \llcorner J_{u_h} \in \mathcal{M}_b^+(\Omega)$. Since $(u_h)_h$ is bounded in $GSBD^p(\Omega)$, $\sup_h |\mu_h|(\Omega) = \mathcal{H}^{n-1}(J_{u_h}) < M$ and then, up to a (not relabelled) subsequence, $\mu_h \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_b^+(\Omega)$. We denote by

$$J := \Big\{x \in \Omega\colon \limsup_{\varrho\to 0^+} \frac{\mu(B_\varrho(x))}{\varrho^{n-1}} > 0\Big\}\,.$$

By [9, Theorem 2.56], the set J is σ-finite w.r.t. the measure $\mathcal{H}^{n-1}$, so in particular $\mathcal{L}^n(J) = 0$. Let us fix $\overline{\sigma} \in (\frac{1}{2}, 1)$ and consider $\overline{\eta} = \overline{\eta}(\overline{\sigma})$ and $C > 0$ such that the conclusion of Theorem 2.4 holds true in correspondence to $\overline{\sigma}$. (We assume n and p fixed once for all.)

Substep 1.1: Existence of $(a_h)_h$ up to a set of small measure. Let us fix $\eta \in (0, \overline{\eta})$. In the following we perform a construction in correspondence of $\overline{\sigma}$ and η; to ease the notation, we do not write explicitly the dependence on these parameters in the objects introduced in the construction. Afterwards (starting from (3.6)), we shall keep track of the dependence on η.

By definition of J, for any $x \in \Omega \setminus J$ there exists $\varrho_0 = \varrho_0(x, \eta)$ such that $\mu(B_\varrho(x)) \leq \frac{\eta}{2}\varrho^{n-1}$ for every $\varrho \in (0, \varrho_0)$. Then, in view of the weak* convergence of μ_h to μ, for every $\varrho \in (0, \varrho_0)$ such that $\mu(\partial B_\varrho(x)) = 0$ (notice that this holds for all ϱ except countable many) we have that $\lim_{h\to\infty} \mu_h(B_\varrho(x)) = \mu(B_\varrho(x))$. We denote by $T_0 = T_0(x)$ the set of $\varrho < \varrho_0$ for which $\lim_{h\to\infty} \mu_h(B_\varrho(x)) = \mu(B_\varrho(x))$. This implies that there exists $h_0 = h_0(x, \eta, \varrho)$ such that $\mu_h(B_\varrho(x)) < \eta\varrho^{n-1}$ for $\varrho \in T_0$ and $h \geq h_0$.

Applying Theorem 2.4 in correspondence to $u_h \in GSBD^p(B_\varrho(x))$ we deduce that for every $x \in \Omega \setminus J$, $\varrho \in T_0$, $h \geq h_0$, there exist $v_{h,\varrho,x} \in GSBD^p(B_\varrho(x))$ and a set of finite perimeter $\omega_{h,\varrho,x} \subset B_\varrho(x)$ such that $v_{h,\varrho,x} = u_h$ in $B_\varrho(x) \setminus \omega_{h,\varrho,x}$, $v_{h,\varrho,x} \in W^{1,p}(B_{\overline{\sigma}\varrho}(x);\mathbb{R}^n)$, and

$$\int_{B_\varrho(x)} |e(v_{h,\varrho,x})|^p \,\mathrm{d}x \leq 2 \int_{B_\varrho(x)} |e(u_h)|^p \,\mathrm{d}x\,, \tag{3.1a}$$

$$\mathcal{H}^{n-1}(\partial^* \omega_{h,\varrho,x}) \leq C\,\mathcal{H}^{n-1}(J_{u_h} \cap B_\varrho(x))\,, \tag{3.1b}$$

$$\mathcal{L}^n(\omega_{h,\varrho,x}) \leq C\,\eta^{\frac{n}{n-1}}\,\varrho^n\,, \tag{3.1c}$$

where in (3.1c) we used also the Isoperimetric Inequality. In particular, by Korn and Korn-Poincaré inequalities (see e.g. [35]) applied to $v_{h,\varrho,x}$ in $B_{\overline{\sigma}\varrho}(x)$, there exist infinitesimal rigid motions $a_{h,\varrho,x}$ such that

$$\int_{B_{\overline{\sigma}\varrho}(x)} \Big(|v_{h,\varrho,x} - a_{h,\varrho,x}|^{p^*} + |\nabla(v_{h,\varrho,x} - a_{h,\varrho,x})|^p \Big) \,\mathrm{d}x \leq C \int_{B_\varrho(x)} |e(u_h)|^p \,\mathrm{d}x\,. \tag{3.2}$$

We notice that the family

$$\mathcal{F} := \{ B_{\overline{\sigma}\varrho}(x) \colon x \in \Omega \setminus J,\ \varrho \in T_0(x) \}$$

is a fine cover of $\Omega \setminus J$ (cf. [9, Section 2.4]). Then, by Besicovitch Covering Theorem, there exists a disjoint family of balls $(B_{\overline{\sigma}\varrho(x_i)}(x_i))_i \subset \mathcal{F}$ with $\mathcal{L}^n\Big((\Omega \setminus J) \setminus \bigcup_{i\in\mathbb{N}} B_{\overline{\sigma}\varrho(x_i)}(x_i)\Big) = 0$. In particular, there exists N, depending on η, such that

$$\mathcal{L}^n\Big((\Omega \setminus J) \setminus \bigcup_{i=1}^N B_{\overline{\sigma}\varrho(x_i)}(x_i)\Big) < \eta\,. \tag{3.3}$$

Let us fix $i \in \{1, \dots, N\}$. There exist (we set $\varrho_i \equiv \varrho(x_i)$) sequences of sets of finite perimeter $(\omega_{h,\varrho_i,x_i})_h$ contained in $B_{\varrho_i}(x_i)$, of functions $(v_{h,\varrho_i,x_i})_h \subset GSBD^p(B_{\varrho_i}(x_i)) \cap W^{1,p}(B_{\overline{\sigma}\varrho_i}(x_i);\mathbb{R}^n)$ with $v_{h,\varrho_i,x_i} = u_h$ in $B_{\varrho_i}(x_i) \setminus \omega_{h,\varrho_i,x_i}$, and of infinitesimal rigid motions $(a_{h,\varrho_i,x_i})_h$ such that (3.1) and (3.2) hold for $\varrho = \varrho_i$, $x = x_i$.

Then, by (3.1b) and (3.1c), up to a subsequence (not relabelled) the characteristic functions of the sets ω_{h,ϱ_i,x_i} converge weakly* in $BV(B_{\varrho_i}(x_i))$ as $h \to +\infty$ to a set $\omega_{\varrho_i,x_i} \subset B_{\varrho_i}(x_i)$ with

$$\mathcal{L}^n(\omega_{\varrho_i,x_i}) \leq C\,\eta^{\frac{n}{n-1}}\,{\varrho_i}^n\,. \tag{3.4}$$

Moreover, again up to a (not relabelled) subsequence, by (3.2) (recall that $(e(u_h))_h$ is bounded in $L^p(\Omega;\mathbb{M}^{n\times n}_{sym})$ since $(G(u_h))_h$ is bounded) we have

$$v_{h,\varrho_i,x_i} - a_{h,\varrho_i,x_i} \rightharpoonup u_{\varrho_i,x_i} \quad \text{in } W^{1,p}(B_{\overline{\sigma}\varrho_i}(x_i);\mathbb{R}^n)\,. \tag{3.5}$$

We may assume that the convergences above hold along the same subsequence, independently on $i \in \{1, \dots, N\}$. Let us denote

$$\omega^\eta := \bigcup_{i=1}^N \big(\omega_{\varrho_i,x_i} \cap B_{\overline{\sigma}\varrho_i}(x_i)\big)\,, \quad a_h^\eta := \sum_{i=1}^N a_{h,\varrho_i,x_i}\chi_{B_{\overline{\sigma}\varrho_i}(x_i)}\,, \quad \widetilde{u}^\eta := \sum_{i=1}^N u_{\varrho_i,x_i}\chi_{B_{\overline{\sigma}\varrho_i}(x_i)}\,. \tag{3.6}$$

By (3.4) we get

$$\mathcal{L}^n(\omega^\eta) \leq C\,\eta^{\frac{n}{n-1}}\,\mathcal{L}^n(\Omega)\,, \tag{3.7a}$$

where the constant C above depends on $\overline{\sigma}$ and n. Using the fact that $u_h = v_{h,\varrho_i,x_i}$ in $B_{\overline{\sigma}\varrho_i}(x_i) \setminus \omega_{h,\varrho_i,x_i}$ and $\mathcal{L}^n(\omega_{h,\varrho_i,x_i} \triangle \omega_{\varrho_i,x_i}) \to 0$, from (3.5) we deduce that

$$u_h - a_h^\eta \to \widetilde{u}^\eta \quad \text{in } L^0(\Omega \setminus E^\eta;\mathbb{R}^n)\,, \qquad \text{for } E^\eta := \omega^\eta \cup \Big((\Omega \setminus J) \setminus \bigcup_{i=1}^N B_{\overline{\sigma}\varrho(x_i)}(x_i)\Big)\,. \tag{3.7b}$$

We may now find a partition $\mathcal{P}^\eta = (P^\eta_j)_j$ of $\Omega \setminus E^\eta$ and a function $u^\eta \in L^0(\Omega \setminus E^\eta;\mathbb{R}^n)$ such that, up to extracting a further subsequence w.r.t. h, in correspondence to any P^η_j there is a sequence $(a^\eta_{h,j})_h$ such that

$$|a^\eta_{h,j}(x) - a^\eta_{h,i}(x)| \to +\infty \text{ for a.e. } x \in \Omega \text{ whenever } i \neq j \tag{3.8a}$$

and

$$u_h - a^\eta_{h,j} \to u^\eta \quad \text{in } L^0(P^\eta_j;\mathbb{R}^n)\,. \tag{3.8b}$$

In fact, this is done as follows by regrouping the sequences of infinitesimal rigid motions in each $B_{\overline{\sigma}\varrho_i}(x_i)$ in equivalence classes, up to extracting a further subsequence.

By Lemma 2.8, denoting $a^i_h \equiv a_{h,\varrho_i,x_i}$ for every $i \in \mathbb{N}$, we may extract inductively a subsequence (not relabelled) such that for every $1 \leq i < j \leq N$ the sequence $(a^i_h - a^j_h)_h$ either converges to an infinitesimal rigid motion or diverges a.e. in Ω. We say that two sequences $(a_h)_h$ and $(b_h)_h$ of infinitesimal rigid motions are in the same equivalence class if and only if $(a_h - b_h)_h$ converges uniformly to an infinitesimal rigid motion.

We conclude (3.8) by considering the union of the $B_{\overline{\sigma}\varrho_i}(x_i) \setminus \omega^\eta$ whose sequences of infinitesimal rigid motions are in the same equivalence class. We then get a partition $\mathcal{P}^\eta = (P^\eta_j)_j$ by fixing a sequence of infinitesimal rigid motions as representative in each P^η_j.

Substep 1.2: Conclusion of Step 1. Let us now take a summable positive sequence $(\eta_k)_k$. By a diagonal argument we may assume that (3.7b) holds for the same subsequence $(u_h)_h$ for every η_k, for suitable ω^k, E^k, a^k_h, u^k, and that $\mathcal{L}^n(E^{\eta_k}) < C\eta_k^{\frac{n}{n-1}}\mathcal{L}^n(\Omega) + \eta_k$ (see (3.3) and (3.7a)). Moreover, we find partitions $\mathcal{P}^k$ such that (3.8) hold for η_k in place of η (for the notations, just replace η by k; we prefer to use the apex k in place of η_k to not overburden the notation, of course the notation in the following has nothing to do with the notation for the objects in Substep 1.1 different from those recalled just above).

Consider two sets $P^{k_1}_{j_1}$ and $P^{k_2}_{j_2}$, $k^1 \neq k^2$: if

$$\mathcal{L}^n(P^{k_1}_{j_1} \cap P^{k_2}_{j_2}) > 0\,. \tag{3.9}$$

then $a^{k_1}_{h,j_1} - a^{k_2}_{h,j_2} \to u^{k_2} - u^{k_1}$ in $L^0(P^{k_1}_{j_1} \cap P^{k_2}_{j_2};\mathbb{R}^n)$ so that $(a^{k_1}_{h,j_1} - a^{k_2}_{h,j_2})_h$ converges uniformly on bounded sets and the two sequences belong to the same equivalence class. On the other hand, for $j_2' \neq j_2$, by construction $|a^{k_2}_{h,j_2'} - a^{k_2}_{h,j_2}| \to \infty$ $\mathcal{L}^n$-a.e., so that (3.9) gives $|a^{k_2}_{h,j_2'} - a^{k_1}_{h,j_1}| \to \infty$ $\mathcal{L}^n$-a.e., which implies $\mathcal{L}^n(P^{k_1}_{j_1} \cap P^{k_2}_{j_2'}) = 0$.

Let us consider the countable set of sequences $\{(a^k_{h,j})_{h\geq 1} : k, j\}$ and $\mathcal{C} = \{\mathfrak{c}_i : i \geq 1\}$ the (countable or finite) set of its equivalence classes. For $i \geq 1$, we let:

$$P_i := \bigcup_{(k,j)\,:\,(a^k_{h,j})_h \in \mathfrak{c}_i} P^k_j\,.$$

Then if $i \neq i'$, we have that $\mathcal{L}^n(P_i \cap P_{i'}) = 0$: otherwise one could find (k_1, j_1), (k_2, j_2) as above with (3.9) and $(a^{k_1}_{h,j_1})_h \in \mathfrak{c}_i$, $(a^{k_2}_{h,j_2})_h \in \mathfrak{c}_{i'}$, and the argument after (3.9) leads to a contradiction. On the other hand, by construction, $\mathcal{L}^n(\Omega \setminus (\bigcup_k \bigcup_j P^k_j)) = 0$, hence $\mathcal{P} = \{P_i : i \geq 1\}$ is a Lebesgue partition of Ω (later on we will prove that it is a Caccioppoli partition). For $i \geq 1$, we choose a particular sequence $(a_{h,i})_h = (a^k_{h,j})_h \in \mathfrak{c}_i$ for suitable k, j and define the sequence of piecewise rigid motions $(a_h)_h$ as follows:

$$a_h = \sum_{i\geq 1} a_{h,i}\chi_{P_i}\,.$$

By construction, $u_h - a_h$ converges to some function $u \in L^0(P_i;\mathbb{R}^n)$ $\mathcal{L}^n$-a.e. in each set P_i (since it is the case in each set P^k_j with $(a^k_{h,j})_h \in \mathfrak{c}_i$) and we denote $u \in L^0(\Omega;\mathbb{R}^n)$ the function thus obtained.

Step 2: Proof of (1.5d). In this step we follow a slicing strategy, in the spirit of [15, Theorem 1.1]. In particular, the first part of the argument above is similar to that in [15, Theorem 1.1, lower semicontinuity]. We remark that we cannot directly apply that result (see Remark 3.2).

Given $\xi \in \mathbb{S}^{n-1}$ and $y \in \Pi^\xi$, we introduce

$$\mathrm{I}^\xi_y(u_h) := \int_{\Omega^\xi_y} |(\dot{u}_h)^\xi_y|^p \,\mathrm{d}t\,, \tag{3.10}$$

where $(\dot{u}_h)^\xi_y$ is the density of the absolutely continuous part of $\mathrm{D}(\widehat{u}_h)^\xi_y$, the distributional derivative of $(\widehat{u}_h)^\xi_y$ ($(\widehat{u}_h)^\xi_y \in SBV^p_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, since $u_h \in GSBD^p(\Omega)$; we denote here and in the following $\widehat{u}_h$ for $\widehat{u_h}$). Therefore, since $(u_h)_h$ is bounded in $GSBD^p(\Omega)$, by (2.3) and by Fubini-Tonelli's theorem we obtain

$$\int_{\Pi_\xi} \mathrm{I}^\xi_y(u_h)\,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_\Omega |e(u_h)(x)\xi\cdot\xi|^p \leq \int_\Omega |e(u_h)|^p\,\mathrm{d}x \leq M\,. \tag{3.11}$$

Let $u_k = u_{h_k}$ be a subsequence of u_h such that

$$\lim_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}) = \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h}) < +\infty\,, \tag{3.12}$$

so that, by (2.5), (3.11), and Fatou's lemma, we have that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$

$$\liminf_{k\to\infty} \int_{\Pi_\xi} \Big(\mathcal{H}^0\big(J_{(\widehat{u}_k)^\xi_y}\big) + \varepsilon \mathrm{I}^\xi_y(u_k)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y) < +\infty\,, \tag{3.13}$$

for a fixed $\varepsilon \in (0,1)$. Let us fix $\xi \in \mathbb{S}^{n-1}$ such that (2.10) and (3.13) hold (cf. Lemma 2.7). Then there is a subsequence $u_m = u_{k_m}$ of u_k, depending on ε and ξ, such that

$$(\widehat{u}_m - \widehat{a}_m)^\xi_y \to \widehat{u}^\xi_y \quad \text{in } L^0(\Omega^\xi_y) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi_\xi \tag{3.14}$$

where $\widehat{u}^\xi_y$ is the slicing of the function u introduced at the end of Substep 1.2, and

$$\begin{aligned}\lim_{m\to\infty} &\int_{\Pi_\xi} \Big(\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon \mathrm{I}^\xi_y(u_m)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \liminf_{k\to\infty} \int_{\Pi_\xi} \Big(\mathcal{H}^0\big(J_{(\widehat{u}_k)^\xi_y}\big) + \varepsilon \mathrm{I}^\xi_y(u_k)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y)\,.\end{aligned} \tag{3.15}$$

As for (3.14), we notice that it follows from Fubini-Tonelli's theorem and the convergence in measure of $u_h - a_h$ to u (see (1.5b)), which corresponds to $\tanh(u_h - a_h) \to \tanh(u) \in L^1(\Omega;\mathbb{R}^n)$ (with $\tanh(v) = (\tanh(v\cdot e_1), \dots, \tanh(v\cdot e_n))$ for every $v\colon \Omega \to \mathbb{R}^n$). Therefore, by (3.15) and Fatou's lemma, we have that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$

$$\liminf_{m\to\infty} \Big(\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon \mathrm{I}^\xi_y(u_m)\Big) < +\infty\,, \tag{3.16}$$

Moreover, we infer that, since ξ satisfies (2.10), then

$$\text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad |(\widehat{a}^i_h - \widehat{a}^j_h)^\xi_y(t)| = |(\widehat{a}^i_h - \widehat{a}^j_h)^\xi_y(0)| \to +\infty \quad \text{for every } t \in \Omega^\xi_y \tag{3.17}$$

for all $i \neq j$. Indeed, since $e(a^i_h) = 0$ for every i, h, then, for fixed h, $(\widehat{a}^i_h - \widehat{a}^j_h)^\xi_y$ is constant in Ω^ξ_y. Thus

$$|(a^i_h - a^j_h)\cdot\xi| \to +\infty \ \text{ in } \bigcup\{\Omega^\xi_y \colon y \in \Pi^\xi \text{ s.t. } |(a^i_h - a^j_h)(y)\cdot\xi| \to +\infty\}\,,$$

and (3.17) holds true.

Let us consider $y \in \Pi^\xi$ satisfying (3.14), (3.16), (3.17), and such that $(\widehat{u}_m)^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every m. Then we may extract a subsequence $u_j = u_{m_j}$ from u_m, depending also on y, for which

$$\lim_{j\to\infty} \Big(\mathcal{H}^0\big(J_{(\widehat{u}_j)^\xi_y}\big) + \varepsilon \mathrm{I}^\xi_y(u_j)\Big) = \liminf_{m\to\infty} \Big(\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon \mathrm{I}^\xi_y(u_m)\Big) \tag{3.18a}$$

and

$$(\widehat{u}_j - \widehat{a}_j)^\xi_y \to \widehat{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. in } \Omega^\xi_y\,, \tag{3.18b}$$

$$|(\widehat{a}^{i_1}_j - \widehat{a}^{i_2}_j)^\xi_y(t)| = |(\widehat{a}^{i_1}_j - \widehat{a}^{i_2}_j)^\xi_y(0)| \to +\infty \quad \text{for } t \in \Omega^\xi_y \text{ and } i_1 \neq i_2\,. \tag{3.18c}$$

In the following, we denote (similarly to (2.9), in dimension one)

$$\partial\mathcal{P}^\xi_y := \bigcup_{j\in\mathbb{N}} \partial(P_j)^\xi_y \cap \Omega^\xi_y \subset \Omega^\xi_y\,.$$

Since, by (3.16) and (3.18a), the number of jump points of $(\widehat{u}_j)^\xi_y$ is bounded uniformly w.r.t. j, passing to a subsequence of $((\widehat{u}_j)^\xi_y)_j$ we may assume that for every j

$$\mathcal{H}^0\big(J_{(\widehat{u}_j)^\xi_y}\big) = N_y \in \mathbb{N}\,.$$

Therefore we have $M_y \le N_y$ cluster points in the limit, denoted by

$$t_1, \dots, t_{M_y}\,.$$

Using the equiboundedness of $\mathrm{I}^\xi_y(u_j)$, which follows from (3.16) and (3.18a), we get that, for $\mathcal{L}^1$-almost any choice of $\bar{t} \in (t_l, t_{l+1})$,

$$t \mapsto (\widehat{u}_j)^\xi_y(t) - (\widehat{u}_j)^\xi_y(\bar{t}) \text{ are equibounded w.r.t. } j \text{ in } W^{1,p}_{\mathrm{loc}}(t_l, t_{l+1})\,, \tag{3.19}$$

by the Fundamental Theorem of Calculus, and then, recalling (3.18b), this sequence converges locally uniformly in (t_l, t_{l+1}), as $j \to \infty$.

Let us prove that

$$\partial\mathcal{P}^\xi_y \subset \{t_1, \dots, t_{M_y}\} \tag{3.20}$$

We argue by contradiction, assuming that there exists $l \in \{1, \dots, M_y\}$ and i_1 such that $\partial(P_{i_1})^\xi_y \cap (t_l, t_{l+1}) \neq \emptyset$. If this holds, there exist two sequences of infinitesimal rigid motions $(a^{i_1}_j)_j$, $(a^{i_2}_j)_j$ (the latter corresponds to some P_{i_2} with $i_1 \neq i_2$) such that

$$\begin{aligned} &(\widehat{u}_j - \widehat{a}^{i_1}_j)^\xi_y \to \widehat{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. in } (P_{i_1})^\xi_y \cap (t_l, t_{l+1}),\\ &(\widehat{u}_j - \widehat{a}^{i_2}_j)^\xi_y \to \widehat{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. in } (P_{i_2})^\xi_y \cap (t_l, t_{l+1})\,, \end{aligned} \tag{3.21}$$

with $\mathcal{L}^1\big((P_{i_1})^\xi_y \cap (t_l, t_{l+1})\big)$, $\mathcal{L}^1\big((P_{i_2})^\xi_y \cap (t_l, t_{l+1})\big) > 0$. But this gives, with (3.19) and since $\widehat{a}^i_j$ are infinitesimal rigid motions and $\widehat{u}^\xi_y \colon \Omega^\xi_y \to \mathbb{R}$, that $(\widehat{a}^{i_1}_j - \widehat{a}^{i_2}_j)^\xi_y$ is constant in Ω^ξ_y and uniformly bounded w.r.t. j. This is in contradiction with (3.18c). Therefore, (3.20) is proven.

Moreover, for every l there exists a unique $i \in \mathbb{N}$ such that

$$(\widehat{u}_j - \widehat{a}^i_j)^\xi_y \to \widehat{u}^\xi_y \quad \text{in } W^{1,p}_{\mathrm{loc}}(t_l, t_{l+1}) \tag{3.22}$$

and in particular the above convergence is locally uniform in (t_l, t_{l+1}). Since a^i_j are rigid motions, and hence the functions $(\widehat{a}^i_j)^\xi_y$ are constant on Ω^ξ_y, we also have that

$$\|\dot{u}^\xi_y\|_{L^p(K)} \le \liminf_{j\to\infty} \|(\dot{u}_h)^\xi_y\|_{L^p(t_l,t_{l+1})} \text{ for every compact set } K \subset (t_l, t_{l+1})\,,$$

so

$$\widehat{u}^\xi_y \in SBV^p(\Omega^\xi_y) \text{ and } J_{\widehat{u}^\xi_y} \subset \{t_1, \dots, t_{M_y}\}\,.$$

This implies, with (3.18a), that

$$\mathcal{H}^0\big(J_{\widehat{u}^\xi_y} \cup \partial\mathcal{P}^\xi_y\big) = \mathcal{H}^0\big(J_{\widehat{u}^\xi_y} \cap (\mathcal{P}^\xi_y)^{(1)}\big) + \mathcal{H}^0(\partial\mathcal{P}^\xi_y) \le \liminf_{m\to\infty}\Big(\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon\,\mathrm{I}^\xi_y(u_m)\Big)\,. \tag{3.23}$$

Notice that we have expressed $J_{\widehat{u}^\xi_y} \cup \partial\mathcal{P}^\xi_y$ as the disjoint union $\big(J_{\widehat{u}^\xi_y} \cap (\mathcal{P}^\xi_y)^{(1)}\big) \cup \partial\mathcal{P}^\xi_y$, denoting

$$(\mathcal{P}^\xi_y)^{(1)} := \bigcup_{j\in\mathbb{N}} \big((P_j)^\xi_y\big)^{(1)}$$

(recall (2.9) and that $E^{(1)}$ denotes the point where $E \subset \mathbb{R}^d$ has density 1 w.r.t. $\mathcal{L}^d$; above $d = 1$).

Integrating over $y \in \Pi^\xi$ and using Fatou's lemma with (3.15) we get

$$\begin{aligned} &\int_{\Pi^\xi} \Big[\mathcal{H}^0\big(J_{\widehat{u}^\xi_y} \cap (\mathcal{P}^\xi_y)^{(1)}\big) + \mathcal{H}^0(\partial\mathcal{P}^\xi_y)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\qquad \le \liminf_{k\to\infty} \int_{\Pi^\xi} \Big[\mathcal{H}^0\big(J_{(\widehat{u}_k)^\xi_y}\big) + \varepsilon\,\mathrm{I}^\xi_y(u_k)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y) \end{aligned} \tag{3.24}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$. In particular we deduce that each P_j has finite perimeter (*cf.* [9, Remark 3.104]) and $\sum_{j\in\mathbb{N}} \mathcal{H}^{n-1}(\partial^* P_j) < +\infty$. This confirms that $\mathcal{P}$ is a Caccioppoli partition, as claimed at the end of Step 1.

Integrating further (3.24) over $\xi \in \mathbb{S}^{n-1}$, by (2.5), (3.11), and (3.12) we get

$$\mathcal{H}^{n-1}\Big(J_u \cup \bigcup_{j\in\mathbb{N}} (\partial^* P_j \cap \Omega)\Big) \leq C\, M\varepsilon + \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h})\,, \tag{3.25}$$

for a universal constant C. By the arbitrariness of ε, (1.5d) follows.

Step 3: Proof of (1.5c) **and of** $u \in GSBD^p(\Omega)$**.** In order to prove (1.5c) it is enough to combine what we have proven so far with an abstract result on compactness and lower semicontinuity in $GSBD^p$. In fact, the sequence $(u_h - a_h)_h$ is bounded in $GSBD^p(\Omega)$: by definition of a_h^j we have that

$$e(u_h - a_h) = e(u_h)\,, \quad J_{u_h - a_h} \subset J_{u_h} \cup \bigcup_{j\in\mathbb{N}} \partial^* P_j\,,$$

and we know that $(u_h)_h$ is bounded in $GSBD^p(\Omega)$ and $\sum_j \mathcal{H}^{n-1}(\partial^* P_j) < +\infty$, by (1.5d). Since we know that $u_h - a_h \to u$ $\mathcal{L}^n$-a.e. in Ω, we are allowed to apply [15, Theorem 1.1] using that the exceptional set A therein is empty (alternatively, using the pointwise convergence one could resort to [21, Theorem 11.3]), to deduce that $u \in GSBD^p(\Omega)$ and that

$$e(u_h) = e(u_h - a_h) \rightharpoonup e(u) \quad \text{in } L^p(\Omega; \mathbb{M}^{n\times n}_{sym})\,,$$

so (1.5c) is proven and the general proof is concluded. □

Remark 3.1. With the notation of Theorem 1.1, the sequence $(u_h - a_h)_h$ is bounded in $GSBD^p(\Omega)$. This is proven in Step 3.

Remark 3.2. We cannot directly apply [15, Theorem 1.1] to $u_h - a_h^j$ for every j in Step 2. Indeed, we would obtain $\mathcal{H}^{n-1}\big((J_u \cap P_j^{(1)}) \cup \partial^* P_j \cap \Omega\big) \leq \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h})$, but the j's are countable many and we cannot localize on right-hand side, since the P_j's are not open sets.

4. Lower semicontinuity and minimisation

In this section we first prove our main lower semicontinuity result, concerning a class of free discontinuity functionals with general bulk and surface energy densities. In the second part we apply Theorem 1.2 to the minimisation of these problems.

Proof of Theorem 1.2. Up to a subsequence, we may assume that

$$\liminf_{h\to\infty} E(u_h) = \lim_{h\to\infty} E(u_h) < +\infty\,. \tag{4.1}$$

In view of the growth assumptions on f and g, we have that $(u_h)_h$ is bounded in $GSBD^p(\Omega)$. Thus we may apply Theorem 1.1 to find a subsequence (not relabelled), a Caccioppoli partition $\mathcal{P}$ of Ω, a sequence of piecewise infinitesimal rigid motions $(a_h)_h$, and $u \in GSBD^p(\Omega)$ satisfying (1.4) and (1.5).

By (4.1) we obtain that, up to a further subsequence,

$$f(x, e(u_h))\mathcal{L}^n \llcorner \Omega + g(x, [u_h], \nu_{u_h})\mathcal{H}^{n-1} \llcorner J_{u_h} =: \mu_h \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}_b^+(\Omega)$$

as $h \to \infty$. Therefore, by the Besicovitch derivation theorem and the Radon-Nikodym decomposition for μ (cf. [9, Theorem 2.2]), the result will follow from the estimates

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}(x_0) \geq f\big(x_0, e(u)(x_0)\big) \quad \text{for } \mathcal{L}^n\text{- a.e. } x_0 \in \Omega \tag{4.2}$$

and

$$\begin{aligned} &\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) \geq g\big(x_0, [u](x_0), \nu_u(x_0)\big) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in J_u \cap \mathcal{P}^{(1)}\,,\\ &\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) \geq g_\infty(x_0, \nu_{\mathcal{P}}) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in \partial^*\mathcal{P} \cap \Omega\,. \end{aligned} \tag{4.3}$$

Step 1: Proof of (4.2)**.** We divide this step into different substeps.

Substep 1.1: Choice of the blow up point x_0 and first properties. We pick x_0 in a subset of Ω of full $\mathcal{L}^n$-measure, satisfying the following five criteria. First, we notice that by the definition of Radon-Nikodym derivative and [9, Theorem 2.2] we have that for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{\varrho \to 0^+} \frac{\mu(B_\varrho(x_0))}{\gamma_n\, \varrho^n}\,. \tag{4.4}$$

Second, in [13] it is proven that every function in $GSBD^p$ is approximately differentiable $\mathcal{L}^n$-a.e., namely that for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$ there exists $\nabla u(x_0) \in \mathbb{M}^{n\times n}$ (such that $e(u)(x_0) = (\nabla u(x_0))^{\mathrm{sym}}$ for a.e. x_0) for which it holds

$$\operatorname*{ap\,lim}_{x\to x_0} \frac{|u(x) - u(x_0) - \nabla u(x_0)(x-x_0)|}{|x-x_0|} = 0\,. \tag{4.5}$$

Third, we take

$$x_0 \in \mathcal{P}^{(1)}\,, \tag{4.6}$$

which is a set of full $\mathcal{L}^n$-measure in Ω. The fourth criterion employed in the choice of x_0 is based on the properties of f. Since f is a Carathéodory function, arguing as in [24, proof of Theorem 1.2], by Scorza Dragoni Theorem (see, e.g., [26], p. 235) one deduces that there exists $F \subset \Omega$ with $\mathcal{L}^n(\Omega \setminus F) = 0$ such that for any $x_0 \in F$ there exists a compact set $K_{x_0} \subset \Omega$ (depending on x_0) such that

$$f|_{K_{x_0}\times \mathbb{M}^{n\times n}_{sym}} \quad \text{is continuous in } K_{x_0}\times \mathbb{M}^{n\times n}_{sym} \quad \text{and} \quad x_0 \in K_{x_0} \cap K^{(1)}_{x_0}\,. \tag{4.7}$$

Finally, we assume that

$$x_0 \text{ is a Lebesgue point of } \phi \text{ and } \phi(x_0) < +\infty\,, \tag{4.8}$$

where ϕ is the function that appears in (f_3). Then the set of points x_0 satisfying (4.4), (4.5), (4.6), (4.7), and (4.8) is of full $\mathcal{L}^n$-measure in Ω. Let us choose x_0 in this set.

Let us fix a sequence $(\varrho_k)_k$ converging to 0 such that $\mu(\partial B_{\varrho_k}(x_0)) = 0$ for every k (in fact this is true for any $\varrho > 0$ except at most countable many). Then, by (4.4) we have that

$$\begin{aligned}\gamma_n \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}(x_0) &= \lim_{k\to\infty}\lim_{h\to\infty} \frac{\mu_h(B_{\varrho_k}(x_0))}{\varrho_k^n}\\ &= \lim_{k\to\infty}\lim_{h\to\infty} \frac{1}{\varrho_k^n}\bigg\{\int_{B_{\varrho_k}(x_0)} f(x, e(u_h)(x))\,\mathrm{d}x + \int_{J_{u_h}\cap B_{\varrho_k}(x_0)} g(x, [u_h], \nu_{u_h})\,\mathrm{d}\mathcal{H}^{n-1}\bigg\}\,.\end{aligned} \tag{4.9}$$

Moreover, (1.5b) gives that $y \mapsto (u_h - a_h)(x_0 + \varrho_k y)$ converge pointwise in B_1 to $y \mapsto u(x_0 + \varrho_k y)$, and by (4.5), (4.6) it holds that $y \mapsto \frac{u(x_0+\varrho_k y)-u(x_0)}{\varrho_k}$ converges to $y \mapsto \nabla u(x_0)\, y$ pointwise in B_1. Therefore

$$\begin{aligned}&\lim_{k\to\infty}\lim_{h\to\infty} u^{x_0}_{k,h} \to v^{x_0} \quad \text{in } L^0(B_1;\mathbb{R}^n), \quad \text{where}\\ &\qquad v^{x_0}(y) := \nabla u(x_0) y \quad \text{and} \quad u^{x_0}_{k,h}(y) := \frac{(u_h - a_h)(x_0+\varrho_k y) - u(x_0)}{\varrho_k}\,.\end{aligned} \tag{4.10}$$

Furthermore, $\lim_{k\to\infty} \varrho_k^{-(n-1)}\mathcal{H}^{n-1}(\partial^*\mathcal{P} \cap B_{\varrho_k}(x_0)) = 0$, and so

$$\lim_{k\to\infty}\lim_{h\to\infty} \frac{\mathcal{H}^{n-1}(J_{a_h} \cap B_{\varrho_k}(x_0))}{\varrho_k^{n-1}} = 0\,. \tag{4.11}$$

Substep 1.2: Blow up argument: change of variables. We perform a blow up procedure at a point $x_0 \in \Omega$ chosen as above, in order to prove (4.2).

Let us consider the functions $u^{x_0}_{k,h}$, defined in (4.10). We notice that (4.9) and (g_1) imply that for a suitable $\widetilde{C} > 0$

$$\limsup_{k\to\infty}\limsup_{h\to\infty} \frac{1}{\varrho_k^n}\mathcal{H}^{n-1}(J_{u_h} \cap B_{\varrho_k}(x_0)) \le \widetilde{C}\,. \tag{4.12}$$

Together with (4.11), by a change of variable this gives that

$$\limsup_{k\to\infty}\limsup_{h\to\infty}\mathcal{H}^{n-1}(J_{u^{x_0}_{k,h}})=0\,. \tag{4.13}$$

Setting

$$f_k(y,\xi):=f(x_0+\varrho_k y,\xi)\,,$$

by (4.9) we obtain that

$$\gamma_n\,\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}(x_0)\geq\limsup_{k\to\infty}\limsup_{h\to\infty}\int_{B_1}f_k(y,e(u^{x_0}_{k,h})(y))\,\mathrm{d}y\,. \tag{4.14}$$

Then a diagonal argument allows to define functions $v_k:=u^{x_0}_{k,h_k}$ such that (4.10), (4.13), and (4.14) hold true for v_k (and considering the limit or the lim sup only in k).

Eventually we observe that, due to (4.7),

$$\lim_{k\to\infty}f_k(y,\xi)=f(x_0,\xi)\quad\text{for a.e. } y\in B_1,\ \text{ locally uniformly in } \mathbb{M}^{n\times n}_{sym}\,. \tag{4.15}$$

In fact, the pointwise convergence follows from the fact that $x_0\in K_{x_0}\cap K^{(1)}_{x_0}$, and the local uniform convergence in $\mathbb{M}^{n\times n}_{sym}$ by the continuity of f in $K_{x_0}\times\mathbb{M}^{n\times n}_{sym}$.

Substep 1.3: Blow up argument: lower semicontinuity. Let us fix $\overline{\sigma}\in(0,1)$. In correspondence of $\overline{\sigma}$ we find positive constants $\eta(\overline{\sigma})$ and C such that the conclusion of Theorem 2.4 holds. Fix also $\delta\in(0,\frac{\eta(\overline{\sigma})}{\widetilde{C}}\wedge 1)$, where $\widetilde{C}$ is the constant from (4.12). We notice that, up to consider a subsequence (not relabelled), we may assume that

$$\sum_{k\in\mathbb{N}}\mathcal{H}^{n-1}(J_{v_k})<\big(C^{-1}\wedge 1\big)\,\delta\,. \tag{4.16}$$

In particular, we have that $\mathcal{H}^{n-1}(J_{v_k})<\eta(\overline{\sigma})$ for every k, and we may apply Theorem 2.4 to the functions $v_k\in GSBD^p(B_1)$. This provides functions $w_k\in GSBD^p(B_1)\cap W^{1,p}(B_{\overline{\sigma}};\mathbb{R}^n)$ and sets of finite perimeter $\omega_k\subset B_1$ such that

$$w_k=v_k \text{ in } B_1\setminus\omega_k,\quad \mathcal{H}^{n-1}(\partial^*\omega_k)<C\,\mathcal{H}^{n-1}(J_{v_k}), \tag{4.17}$$

$\mathcal{H}^{n-1}(J_{w_k})\leq\mathcal{H}^{n-1}(J_{v_k})$, and

$$\int_{B_1}|e(w_k)|^p\,\mathrm{d}x\leq 2\int_{B_1}|e(v_k)|^p\,\mathrm{d}x\,. \tag{4.18}$$

By (4.16), (4.17), and the Isoperimetric Inequality $\mathcal{L}^n(\omega_k)\leq(n^n\gamma_n)^{-\frac{1}{n-1}}\mathcal{H}^{n-1}(\partial^*\omega_k)^{\frac{n}{n-1}}<\mathcal{H}^{n-1}(\partial^*\omega_k)$, we have that

$$\mathcal{L}^n(\omega_\delta)<\delta\qquad\text{for }\ \omega_\delta:=\bigcup_{k\in\mathbb{N}}\omega_k\,, \tag{4.19}$$

and

$$w_k=v_k \text{ in } B_1\setminus\omega_\delta\quad\text{for every } k\in\mathbb{N}\,. \tag{4.20}$$

Recalling the definition of v^{x_0} in (4.10), we claim that

$$w_k\rightharpoonup v^{x_0}\quad\text{in } W^{1,p}(B_{\overline{\sigma}};\mathbb{R}^n)\ :$$

first notice that $w_k\to v^{x_0}$ in $L^0(B_1;\mathbb{R}^n)$, since $v_k\to v^{x_0}$ in $L^0(B_1;\mathbb{R}^n)$ and $\mathcal{L}^n(\{w_k\neq v_k\})\to 0$ (by (4.16) and (4.17)); moreover, by Korn's inequality there are $\widetilde{a}_k$ infinitesimal rigid motions and $\widetilde{w}\in W^{1,p}(B_{\overline{\sigma}};\mathbb{R}^n)$ such that, passing to a (not relabelled) subsequence, $w_k-\widetilde{a}_k\rightharpoonup\widetilde{w}$ in $W^{1,p}(B_{\overline{\sigma}};\mathbb{R}^n)$; we conclude since then $\widetilde{a}_k$ converge pointwise and then uniformly (being affine) and by difference w_k converge weakly in $W^{1,p}(B_{\overline{\sigma}};\mathbb{R}^n)$ (to v^{x_0}).

We now perform a further approximation, through a sequence of equi-Lipschitz functions. This is done for two reasons: first, to employ (4.15) since therein the convergence holds for ξ in compact sets; second, to pass to the limit in the integral of $f(x_0,e(w_k))$ over the set $B_{\overline{\sigma}}\setminus\omega_\delta$, which is not in general open and so the semicontinuity theorem in [1] does not apply directly. Then we recall, adapting to the present case, what proven in [24, Proposition 3.1], in the spirit of [1] and [4].

In correspondence to δ, there exist a Borel set E_δ with $\mathcal{L}^n(E_\delta) < \delta$ (this replaces the sequence $(E_k)_k$ with $\mathcal{L}^n(E_k) \to 0$ as $k \to \infty$ in [24, Proposition 3.1]) and for every m and $k \in \mathbb{N}$ there exist $\widehat{w}_{k,m} \in W^{1,\infty}(B_{\overline{\sigma}};\mathbb{R}^n)$, $E_{k,m} \subset B_{\overline{\sigma}}$ Borel sets such that

$$\|\widehat{w}_{k,m}\|_{L^\infty} + \mathrm{Lip}(\widehat{w}_{k,m}) \le C(n, B_1)m, \qquad \widehat{w}_{k,m} = w_k \text{ in } B_{\overline{\sigma}} \setminus E_{k,m}\,, \tag{4.21}$$

and, up to extracting a subsequence w.r.t. k, $\widehat{w}_{k,m} \overset{*}{\rightharpoonup} \widehat{w}_m$ in $W^{1,\infty}(B_{\overline{\sigma}};\mathbb{R}^n)$ for every m, with

$$\begin{aligned} &\lim_{m\to\infty} \limsup_{k\to\infty} \int_{E_{k,m}\setminus E_\delta} \Big(1 + f_k(y, e(\widehat{w}_{k,m}))\Big)\mathrm{d}y = 0\,, \\ &\lim_{m\to\infty} m^p\, \mathcal{L}^n(A_m) = 0\,, \quad \text{ for } A_m := \{\widehat{w}_m \neq v^{x_0}\} \cap (B_{\overline{\sigma}} \setminus E_\delta)\,. \end{aligned} \tag{4.22}$$

Recalling that $f_k \ge 0$, $w_k = v_k$ in $B_1 \setminus \omega_\delta$, and $\widehat{w}_{k,m} = w_k$ in $B_{\overline{\sigma}} \setminus E_{k,m}$, it follows that

$$\begin{aligned} \int_{B_1} f_k(y, e(v_k))\,\mathrm{d}y &\ge \int_{B_{\overline{\sigma}}\setminus\omega_\delta} f_k(y, e(w_k))\,\mathrm{d}y \ge \int_{B_{\overline{\sigma}}\setminus(E_\delta\cup\omega_\delta\cup E_{k,m})} f_k(y, e(\widehat{w}_{k,m}))\,\mathrm{d}y \\ &= \int_{B_{\overline{\sigma}}\setminus(E_\delta\cup\omega_\delta)} f_k(y, e(\widehat{w}_{k,m}))\,\mathrm{d}y - \int_{E_{k,m}\setminus(E_\delta\cup\omega_\delta)} f_k(y, e(\widehat{w}_{k,m}))\,\mathrm{d}y\,. \end{aligned} \tag{4.23}$$

We now use the fact that $\widehat{w}_{k,m} \overset{*}{\rightharpoonup} \widehat{w}_m$ in $W^{1,\infty}(B_{\overline{\sigma}};\mathbb{R}^n)$ (so that $(\widehat{w}_{k,m})_k$ is a sequence of equi-Lipschitz functions and $\big(f(x_0, e(\widehat{w}_{k,m}))\big)_k$ is equi-bounded in $B_{\overline{\sigma}}$, by (f_3) and since $\phi(x_0) < +\infty$) and (4.15), to deduce that

$$\begin{aligned} \liminf_{k\to\infty} \int_{B_{\overline{\sigma}}\setminus(E_\delta\cup\omega_\delta)} f_k(y, e(\widehat{w}_{k,m}))\,\mathrm{d}y &= \liminf_{k\to\infty} \int_{B_{\overline{\sigma}}\setminus(E_\delta\cup\omega_\delta)} f(x_0, e(\widehat{w}_{k,m}))\,\mathrm{d}y\,, \\ &\ge \int_{B_{\overline{\sigma}}\setminus(E_\delta\cup\omega_\delta)} f(x_0, e(\widehat{w}_m))\,\mathrm{d}y\,. \end{aligned} \tag{4.24}$$

We observe that in the equality above we used (4.15) and the fact that x_0 is a Lebesgue point of ϕ, and to prove the latter estimate it is enough to apply Morrey's Lower Semicontinuity Theorem in an arbitrary open set containing $B_{\overline{\sigma}} \setminus (E_\delta \cup \omega_\delta)$ and observe that for any $\varepsilon > 0$, thanks to the equi-boundedness, we can find an open set $B'_\varepsilon \supset B_{\overline{\sigma}} \setminus (E_\delta \cup \omega_\delta)$ such that the integrals of $f(x_0, e(\widehat{w}_{k,m}))$ (for every k) and $f(x_0, e(\widehat{w}_m))$ over $B'_\varepsilon \setminus \big(B_{\overline{\sigma}} \setminus (E_\delta \cup \omega_\delta)\big)$ are less than ε.

The second estimate in (4.22), (4.21), and $f \ge 0$ imply that

$$\int_{B_{\overline{\sigma}}\setminus(E_\delta\cup\omega_\delta)} f(x_0, e(\widehat{w}_m))\,\mathrm{d}y \ge \mathcal{L}^n\big(B_{\overline{\sigma}} \setminus (E_\delta \cup \omega_\delta \cup A_m)\big) f(x_0, e(u)(x_0))\,. \tag{4.25}$$

Moreover, employing again $f_k \ge 0$,

$$\int_{E_{k,m}\setminus(\omega_\delta\cup E_\delta)} f_k(y, e(\widehat{w}_{k,m}))\,\mathrm{d}y \le \int_{E_{k,m}\setminus E_\delta} f_k(y, e(\widehat{w}_{k,m}))\,\mathrm{d}y\,. \tag{4.26}$$

Collecting (4.23), (4.24), (4.26), (4.25), (4.22), and passing to the lim inf in k and to the limit in m, we obtain that

$$\liminf_{k\to\infty} \int_{B_1} f_k(y, e(v_k))\,\mathrm{d}y \ge \mathcal{L}^n\big(B_{\overline{\sigma}} \setminus (E_\delta \cup \omega_\delta)\big) f(x_0, e(u)(x_0)) > (\gamma_n \overline{\sigma}^n - 2\delta) f(x_0, e(u)(x_0))\,.$$

Passing to the limit first as $\delta \to 0$ and then as $\overline{\sigma} \to 1$, by (4.14) (recall the definition $v_k = u^{x_0}_{k,h_k}$) we deduce (4.2).

Step 2: Proof of (4.3)**.** We denote $J'_u := (J_u \cap \mathcal{P}^{(1)}) \cup (\partial^*\mathcal{P} \cap \Omega)$.

Substep 2.1: Choice of the blow up point x_0 and first properties. Since J'_u is countably rectifiable and thanks to (2.7), for $\mathcal{H}^{n-1}$-a.e. $x_0 \in J'_u$ there exist $u^+(x_0)$, $u^-(x_0) \in \mathbb{R}^n$, $\nu_0 \in \mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{\substack{x\in(Q^{\nu_0}_\varrho(x_0))^\pm \\ x\to x_0}} u(x) = u^\pm(x_0)\,. \tag{4.27}$$

Notice that ν_0 denotes $\nu_u(x_0)$ if $x_0 \in J_u$ and the outer normal to P_i at x_0, if $x_0 \in \partial^*\mathcal{P} \cap \Omega$ and $x_0 \in \partial^* P_i \cap \partial^* P_j$ for $i < j$. We remark that $u^+(x_0) \neq u^-(x_0)$ for $x_0 \in J_u$.

Moreover, since μ is a positive bounded Radon measure and J'_u is countably rectifiable, there exists the Radon-Nikodym derivative of μ w.r.t. $\mathcal{H}^{n-1} \llcorner J'_u$ (which is σ-finite) and it holds (see e.g. [9, Theorems 1.28 and 2.83])

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) = \lim_{\varrho\to 0^+} \frac{\mu(Q^{\nu_0}_\varrho(x_0))}{\varrho^{n-1}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in J'_u\,. \tag{4.28}$$

We thus fix x_0 such that both (4.27) and (4.28) hold for x_0.

Recalling the pointwise convergence of $u_h - a_h$ to u, by a change of variables we obtain from (4.27) that

$$(u_h - a_h)(x_0 + \varrho_k y) \to u_0(y), \quad \text{where } u_0(y) := u^+(x_0)\chi_{Q^{\nu_0,+}_1} + u^-(x_0)\chi_{Q^{\nu_0,-}_1} \quad \text{in } L^0(Q^{\nu_0}_1;\mathbb{R}^n) \tag{4.29}$$

first as $h \to \infty$ and then as $k \to \infty$. (Recall the notation for half cubes in Section 2.) Analogously to (4.9), we fix a vanishing sequence $(\varrho_k)_k$ with $\mu(\partial Q^{\nu_0}_{\varrho_k}(x_0)) = 0$, and then

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) &= \lim_{k\to\infty}\lim_{h\to\infty} \frac{\mu_h(Q^{\nu_0}_{\varrho_k}(x_0))}{\varrho_k^{n-1}} \\ &= \lim_{k\to\infty}\lim_{h\to\infty} \frac{1}{\varrho_k^{n-1}}\bigg\{\int_{Q^{\nu_0}_{\varrho_k}(x_0)} f(x, e(u_h)(x))\,\mathrm{d}x + \int_{J_{u_h}\cap Q^{\nu_0}_{\varrho_k}(x_0)} g(x,[u_h],\nu_{u_h})\,\mathrm{d}\mathcal{H}^{n-1}\bigg\}\,.\end{aligned} \tag{4.30}$$

Substep 2.2: Blow up argument for $x_0 \in J_u \cap \mathcal{P}^{(1)}$. Given $x_0 \in J_u \cap \mathcal{P}^{(1)}$, there exists $j \in \mathbb{N}$ such that $x_0 \in P^{(1)}_j$. Then

$$\lim_{k\to\infty}\lim_{h\to\infty} |a_h(x_0+\varrho_k y) - a^j_h(x_0+\varrho_k y)| = 0 \quad \text{in } L^0(Q^{\nu_0}_1;\mathbb{R}^n)\,, \tag{4.31}$$

for a^j_h the infinitesimal rigid motion corresponding to P_j, cf. (1.4). Up to choosing a subsequence h_k in (4.29), by (4.31) we get

$$\widetilde{v}_k \to u_0 \quad \text{in } L^0(Q^{\nu_0}_1;\mathbb{R}^n), \quad \text{where } \widetilde{v}_k(y) := (u_{h_k} - a^j_{h_k})(x_0+\varrho_k y) \tag{4.32}$$

and u_0 is defined in (4.29).

By (4.30) and assumptions (g_1), (g_3) we obtain that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) = \lim_{k\to\infty}\bigg\{\varrho_k \int_{Q^{\nu_0}_1} f\Big(x_0+\varrho_k y, \frac{1}{\varrho_k} e(\widetilde{v}_k)(x)\Big)\,\mathrm{d}x + \int_{J_{\widetilde{v}_k}\cap Q^{\nu_0}_1} g(x_0,[\widetilde{v}_k],\nu_{\widetilde{v}_k})\,\mathrm{d}\mathcal{H}^{n-1}\bigg\}\,. \tag{4.33}$$

We remark that above we used that g does not depend separately on the two traces v^+ and v^- but only on $[v]$. This allowed us to infer that for any function v and infinitesimal rigid motion a the surface part evaluated on v is equal to the surface part evaluated on $v - a$.

By the growth assumptions on f and g it follows that $(\widetilde{v}_k)_k$ converges weakly in $GSBD^p(Q^{\nu_0}_1)$ to u_0.

Therefore, (g_4) and (4.33) imply that

$$\begin{aligned} g\big(x_0,[u](x_0),\nu_u(x_0)\big) &= \int_{J_{u_0}} g_{x_0}([u_0],\nu_0)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\le \liminf_{k\to\infty}\int_{J_{\widetilde{v}_k}\cap Q^{\nu_0}_1} g_{x_0}([\widetilde{v}_k],\nu_{\widetilde{v}_k})\,\mathrm{d}\mathcal{H}^{n-1} \le \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0)\,.\end{aligned}$$

Substep 2.3: Blow up argument for $x_0 \in \partial^*\mathcal{P} \cap \Omega$. Assume that $x_0 \in \partial^* P_i \cap \partial^* P_j$, for $i < j$. By Lemma 2.7 it holds that for every $k \in \mathbb{N}$ and for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$

$$\lim_{h\to\infty} |(a^i_h(x_0+\varrho_k y) - a^j_h(x_0+\varrho_k y))\cdot\xi| = +\infty \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in Q^{\nu_0}_1 . \tag{4.34}$$

Let us fix is a countable dense subset of $\mathbb{S}^{n-1}$, included into the subset of full measure for which (4.34) holds, that we call D. By a diagonal argument, for every k we may take $h_k \in \mathbb{N}$ such that the convergences in (4.29) and (4.30) hold with h_k in place of h, and moreover, setting $a^+_k(y) := a^j_{h_k}(x_0+\varrho_k y)$, $a^-_k(y) := a^i_{h_k}(x_0+\varrho_k y)$ for $y \in Q^{\nu_0}_1$, it holds

$$\lim_{k\to\infty} |(a^+_k(y) - a^-_k(y))\cdot\xi| = +\infty \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in Q^{\nu_0}_1 \text{ and } \xi \in D . \tag{4.35}$$

We notice that $e(a^+_k - a^-_k) = 0$ gives, for every $k \in \mathbb{N}$, $\xi \in \mathbb{S}^{n-1}$, $y \in \Pi^\xi$, that $(\widehat{a}^+_k - \widehat{a}^-_k)^\xi_y$ is a constant function. From (4.35) we then deduce that there are suitable $c_{k,\xi,y} > 0$ for which

$$|(\widehat{a}^+_k - \widehat{a}^-_k)^\xi_y| \equiv c_{k,\xi,y} \to +\infty \text{ as } k\to\infty, \text{ for any } \xi \in D \text{ and } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi(Q^{\nu_0}_1) . \tag{4.36}$$

Let us denote

$$v_k(y) := u_{h_k}(x_0+\varrho_k y) \quad \text{for } y \in Q^{\nu_0}_1 .$$

These functions satisfy (by a change of variables in (4.30))

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) = \lim_{k\to\infty} \left\{ \varrho_k \int_{Q^{\nu_0}_1} f\Big(x_0+\varrho_k y, \frac{1}{\varrho_k} e(\widetilde{v}_k)(x)\Big)\,\mathrm{d}x + \int_{J_{w_k}\cap Q^{\nu_0}_1} g(x,[v_k],\nu_{v_k})\,\mathrm{d}\mathcal{H}^{n-1} \right\} \tag{4.37}$$

by (4.29), $(u_{h_k} - a_{h_k})(x_0+\varrho_k\cdot) \to u_0$ in $L^0(Q^{\nu_0}_1;\mathbb{R}^n)$ as $k\to\infty$, so that

$$v_k - a^\pm_k \to u^\pm(x_0) \quad \text{in } L^0(Q^{\nu_0,\pm}_1;\mathbb{R}^n) . \tag{4.38}$$

Case $g_\infty(x_0,\nu_0) \in \mathbb{R}$. Assume that $g_\infty(x_0,\nu_0) \in \mathbb{R}$, so that g_∞ takes finite values, and fix $\eta > 0$ small. We find $\xi_0 = \xi_0(\nu_0,\eta) \in D \subset \mathbb{S}^{n-1}$ such that ξ_0 satisfies (2.10) and

$$0 \le g_\infty(x_0,\nu_0) - \frac{|\nu_0\cdot\xi_0|}{g^*_{x_0,\infty}(\xi_0)} < \eta , \tag{4.39}$$

where $g^*_{x_0,\infty}$ is the dual norm of $g_\infty(x_0,\cdot)$, given by $\phi^*(\xi) := \sup_{\phi(\nu)\le 1} |\nu\cdot\xi|$. This is done by choosing a vector $\overline{\xi}$ in $\mathbb{S}^{n-1}$ such that $g_\infty(x_0,\nu_0) = \frac{|\nu_0\cdot\overline{\xi}|}{g^*_{x_0,\infty}(\overline{\xi})}$ and by continuity, using that D is dense in $\mathbb{S}^{n-1}$.

By (g_5) there is a function $\kappa\colon [0,+\infty) \to [0,+\infty)$ with $\lim_{t\to+\infty}\kappa(t) = 0$ such that

$$g(x,y,\nu) > g_\infty(x,\nu) - \kappa(t) \quad \text{for every } x \in \Omega,\ |y| > t, \text{ and } \nu \in \mathbb{S}^{n-1},$$

and, from (g_3), $g(x,y,\nu) > g_\infty(x_0,\nu) - \kappa(t)$ in a neighbourhood of x_0. By the definition of dual norm, (g_5), (4.39), and since $|[v_k]\cdot\xi_0| \le |[v_k]|$, we get

$$g(x,[v_k],\nu_{v_k}) \ge (g_\infty(x_0,\nu_{v_k}) - \kappa(t))\chi_{\{|[v_k]|>t\}} \ge \left(\frac{|\nu_{v_k}\cdot\xi_0|}{g^*_{x_0,\infty}(\xi_0)} - \kappa(t)\right)\chi_{|\{[v_k]\cdot\xi_0|>t\}} \tag{4.40}$$

$\mathcal{H}^{n-1}$-a.e. in $J_{v_k}\cap Q^{\nu_0}_1$. We observe that by (4.37)

$$\limsup_{k\to\infty} \mathcal{H}^{n-1}(J_{v_k}\cap Q^{\nu_0}_1) =: L < +\infty , \tag{4.41}$$

since g takes values in $[c,+\infty)$. Then, using also (f_3), (4.37), and (4.40) we obtain that

$$\begin{aligned} \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) &\ge \liminf_{k\to\infty} \left\{ \int_{Q^{\nu_0}_1} \frac{|e(v_k)\xi_0\cdot\xi_0|^p}{C\varrho_k^{p-1}}\,\mathrm{d}x + \int_{J_{v_k}\cap Q^{\nu_0}_1} \left(\frac{|\nu_{v_k}\cdot\xi_0|}{g^*_{x_0,\infty}(\xi_0)}\chi_{\{|[v_k]\cdot\xi_0|>t\}} + \varepsilon \right) \mathrm{d}\mathcal{H}^{n-1} \right\} \\ &\quad - (\kappa(t)+\varepsilon)L \\ &= \liminf_{k\to+\infty} \int_{\Pi^{\xi_0}} F^{\xi_0,\varepsilon}_{y,t}((\widehat{v}_k)^{\xi_0}_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) - (\kappa(t)+\varepsilon)L \end{aligned} \tag{4.42}$$

with

$$F^{\xi_0,\varepsilon}_{y,t}(v) := \frac{1}{C}\int_{(Q^{\nu_0}_1)^{\xi_0}_y} |v'(s)|^p \,\mathrm{d}s \; + \mathcal{H}^0(\{s\colon |[v](s)| > t\})\frac{1}{g^*_{x_0,\infty}(\xi_0)} + \varepsilon\mathcal{H}^0(J_v)$$

for $v\colon (Q^{\nu_0}_1)^{\xi_0}_y \to \mathbb{R}$. We observe that the second relation in (4.42) follows from the Area Formula (cf. e.g. [34, (12.4)]) and the slicing property (2.4).

Fatou's lemma and (4.42) give that $\liminf_k F^{\xi_0,\varepsilon}_{y,t}((\widehat{v}_k)^{\xi_0}_y) < +\infty$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^{\xi_0}$, so we may find, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^{\xi_0}$, a subsequence $\widehat{v}_m = \widehat{v}_{k_m}$ (depending on y) such that

$$\lim_{m\to\infty} F^{\xi_0,\varepsilon}_{y,t}((\widehat{v}_m)^{\xi_0}_y) = \liminf_{k\to\infty} F^{\xi_0,\varepsilon}_{y,t}((\widehat{v}_k)^{\xi_0}_y)\,, \quad \mathcal{H}^0\big(J_{(\widehat{v}_m)^{\xi_0}_y}\big) = N_y \in \mathbb{N}\,. \tag{4.43}$$

Recalling (4.38), we may also choose the subsequence $(k_m)_m$ such that, denoting by $(\widehat{v}_m - \widehat{a}^\pm_m)^{\xi_0}_y$ the functions $(\widehat{v}_{k_m} - \widehat{a}^\pm_{k_m})^{\xi_0}_y$, it holds

$$(\widehat{v}_m - \widehat{a}^\pm_m)^{\xi_0}_y \to u^\pm(x_0)\cdot\xi_0 \quad \text{in } L^0\big((Q^{\nu_0,\pm}_1)^{\xi_0}_y\big)\,. \tag{4.44}$$

We now claim that given $t > 0$ there exists $\overline{m} \in \mathbb{N}$ such that

$$\Big\{s \in (Q^{\nu_0,\pm}_1)^{\xi_0}_y : |[(\widehat{v}_m)^{\xi_0}_y](s)| > t\Big\} \neq \emptyset \quad \text{for } m \geq \overline{m}\,. \tag{4.45}$$

Indeed, let us argue by contradiction assuming that (4.45) is not true. Then, by (4.43),

$$\mathrm{D}\big((\widehat{v}_m)^{\xi_0}_y\big)\big((Q^{\nu_0,\pm}_1)^{\xi_0}_y\big) \leq \int_{(Q^{\nu_0}_1)^{\xi_0}_y} \Big|\big((\widehat{v}_m)^{\xi_0}_y\big)'(s)\Big| \,\mathrm{d}s + t\,N_y \leq \widehat{C}\,, \tag{4.46}$$

for a suitable $\widehat{C} > 0$ independent of m. Therefore, for any $s^+ \in (Q^{\nu_0,+}_1)^{\xi_0}_y$, $s^- \in (Q^{\nu_0,-}_1)^{\xi_0}_y$,

$$\begin{aligned}|(\widehat{a}^+_m - \widehat{a}^-_m)^{\xi_0}_y| &\equiv |(\widehat{a}^+_m)^{\xi_0}_y(s^+) - (\widehat{a}^-_m)^{\xi_0}_y(s^-)|\\ &= \Big|(\widehat{v}_m - \widehat{a}^-_m)^{\xi_0}_y(s^-) - (\widehat{v}_m - \widehat{a}^+_m)^{\xi_0}_y(s^+) + \Big((\widehat{v}_m)^{\xi_0}_y(s^+) - (\widehat{v}_m)^{\xi_0}_y(s^-)\Big)\Big|\,.\end{aligned}$$

From the identity above we obtain a contradiction for m large enough, since the left-hand side tends to $+\infty$ by (4.36) while the right-hand side is bounded by (4.44) and (4.46). This proves (4.45).

In particular, (4.45) implies that $\lim_{m\to\infty} F^{\xi_0,\varepsilon}_{y,t}((\widehat{v}_m)^{\xi_0}_y) \geq \frac{1}{g^*_{x_0,\infty}(\xi_0)}$. Recalling (4.43) and integrating in $\Pi^{\xi_0} = \Pi^{\xi_0}(Q^{\nu_0}_1)$, by Fatou's lemma, (4.42), and the arbitrariness of t, η, ε, we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) \geq \frac{|\nu_0\cdot\xi_0|}{g^*_{x_0,\infty}(\xi_0)}\,. \tag{4.47}$$

The second estimate in (4.3) follows now by (4.39) and the arbitrariness of η.

Case $g_\infty \equiv +\infty$. We have to prove (with the usual notation $\nu_0 = \nu(x_0)$) that

$$\mathcal{H}^{n-1}(\partial^*\mathcal{P}\cap\Omega) = 0\,. \tag{4.48}$$

Assume by contradiction that there is $x_0 \in \partial^*\mathcal{P}\cap\Omega$ satisfying (4.27) and (4.28). Then, by the assumption (g_5), for any fixed large $M > 0$ there exists t_M such that

$$g(x,[v_k],\nu_{v_k}) \geq M\chi_{\{|[v_k]|>t_M\}} \geq M\,|\nu_{v_k}\cdot\xi_0|\chi_{\{|[v_k]\cdot\xi_0|>t_M\}} \tag{4.49}$$

for any $\xi_0 \in D$. Arguing as in the case $g_\infty(x_0,\nu_0) \in \mathbb{R}$, with (4.40) replaced by (4.49), we obtain that $\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0) \geq M\,|\nu_0\cdot\xi_0|$ for every $M > 0$ and $\xi_0 \in D$. Taking ξ_0 such that $|\nu_0\cdot\xi_0| > \frac{1}{2}$ we obtain a contradiction for $M > 2\,\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}}(x_0)$, so (4.48) is proven. Therefore the general proof is concluded. □

Remark 4.1. In [31] a class of functions $g\colon (\mathbb{R}^n)^3 \to [0,+\infty)$ satisfying for any bounded open set $\Omega\subset\mathbb{R}^n$

$$\int_{J_v} g(v^+,v^-,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{h\to\infty}\int_{J_{v_h}} g(v^+_h,v^-_h,\nu_{v_h})\,\mathrm{d}\mathcal{H}^{n-1} \quad \text{if } v_h \to v \text{ weakly in } GSBD^p(\Omega)$$

has been provided. This is the class of *symmetric jointly convex* functions, which are characterized by (see [31, Definition 3.1 and Theorem 5.1])

$$g(i,j,\nu) = \sup_{h\in\mathbb{N}}(f_h(i)-f_h(j))\cdot\nu \quad \text{for all } (i,j,\nu)\in(\mathbb{R}^n)^3 \text{ with } i\neq j$$

where $f_h\colon \mathbb{R}^n\to\mathbb{R}^n$ is a uniformly continuous, bounded, and conservative vector field (that is, there exists a potential $F_h\in C^1(\mathbb{R}^n)$ for which $\nabla F_h = f_h$) for every $h\in\mathbb{N}$. Any symmetric jointly convex function depending only on the difference $i-j$ provides a function satisfying (g_4). Examples of such functions are (see [31, Section 4])

$$g_1(y,\nu) = \widetilde{g}(|y|) \quad \text{for } \widetilde{g}\colon [0,+\infty)\to[0,+\infty) \text{ increasing with } \frac{\widetilde{g}(t)}{t} \text{ nonincreasing in } (0,+\infty)\,,$$

$$g_2(y,\nu) = \sup_{\{\xi_k\}_{k=1}^n \text{ orthornormal basis}} \left(\sum_{k=1}^n \theta_k\big(y\cdot\xi_k\big)^2|\nu\cdot\xi_k|^2\right)^{1/2}$$

for $\theta_k\in C(\mathbb{R};[0,+\infty))$ even and subadditive, for $k=1,\dots,n$ (this class has been introduced and studied in a BD setting in [22]), and

$$g_3(y,\nu) = \psi(\nu)$$

for ψ a norm.

From Theorem 1.2 we deduce existence for the following minimisation problems with Dirichlet conditions, in the propositions below. In the following $\Omega\subset\mathbb{R}^n$ is a bounded, open, connected and Lipschitz set. Moreover, we assume that $u_0\in W^{1,p}(\mathbb{R}^n;\mathbb{R}^n)$ and that $\partial_D\Omega\subset\partial\Omega$ be relatively open with $\partial_D\Omega = \widetilde{\Omega}\cap\partial\Omega$ for a bounded, open, connected domain $\widetilde{\Omega}\supset\Omega$.

We consider first the simpler cases corresponding to $g_\infty\equiv+\infty$ and g independent of the jump amplitude, in Propositions 4.2 and 4.4. Then we consider the case with general f, g as in Theorem 1.2, which formally includes the other two cases. We prefer to state three different results since the proofs of Propositions 4.2 and 4.4 are more direct.

Proposition 4.2. *Assume f, g as in Theorem 1.2, with $g_\infty\equiv+\infty$. Then the problem*

$$\min_{u=u_0\in\widetilde{\Omega}\setminus\overline{\Omega}}\left\{\int_\Omega f(x,e(u))\,\mathrm{d}x + \int_{J_u} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\right\} \tag{4.50}$$

admits a solution in $GSBD^p(\widetilde{\Omega})$. In particular, this holds for $g(y,\nu)=\widetilde{g}(|y|)$ with $\widetilde{g}\colon[0,+\infty)\to[0,+\infty)$ increasing, unbounded, and such that $\frac{\widetilde{g}(t)}{t}$ is nonincreasing.

Proof. Let us apply Theorem 1.2 to a minimising sequence $(u_h)_h$ for (4.50) in $GSBD^p(\widetilde{\Omega})$. Since $g_\infty\equiv+\infty$, we have that the partition $\mathcal{P}$ of $\widetilde{\Omega}$ is trivial. The fact that $u_h=u_0$ in $\widetilde{\Omega}\setminus\overline{\Omega}$ implies that one can choose a_h (which for every h reduces to a unique infinitesimal rigid motion) as $a_h=0$: in fact, if for $a_h^i=a_h$ and $a_h^j=0$ (1.5a) holds (namely, if a_h and 0 are not in the same equivalence class, as discussed in the proof of Theorem 1.1), then u_h-a_h would diverge on $\widetilde{\Omega}\setminus\overline{\Omega}$, in contrast to the pointwise convergence toward a finite valued function in (1.5b). Therefore $u_h\to u$ in $L^0(\widetilde{\Omega};\mathbb{R}^n)$. In particular $u_h=u_0$ in $\widetilde{\Omega}\setminus\overline{\Omega}$ and using again that $g_\infty\equiv+\infty$ we get the lower semicontinuity of the functional E to minimise. □

Remark 4.3. In the above assumptions, if $g(y,\nu)\geq\widetilde{c}|y|$ for some $\widetilde{c}>0$, the solutions to (4.50) belong to $SBD^p(\widetilde{\Omega})$. In fact, this follows from the fact that $[u]\in L^1(J_u;\mathbb{R}^n)$: under such condition, every $GSBD$ function is in SBD, as shown in [16, Theorem 2.9] (take $\mathbb{A}v=\mathrm{E}v$ therein, cf. [16, Remark 2.5]). For other surface densities g, such as $g(y,\nu)=c+\sqrt{|y|}$, one obtains existence for the Dirichlet problem in $GSBD^p$.

Proposition 4.4. *Assume f as in Theorem 1.2 and let $g\colon\Omega\times\mathbb{S}^{n-1}\to[c,+\infty)$ be continuous in the first variable and such that the positively one-homogeneous extension of $g(x,\cdot)$ is a norm for every $x\in\Omega$. Then the problem*

$$\min_{u=u_0\in\widetilde{\Omega}\setminus\overline{\Omega}}\left\{\int_\Omega f(x,e(u))\,\mathrm{d}x + \int_{J_u} g(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\right\} \tag{4.51}$$

admits a solution in $GSBD^p(\widetilde{\Omega})$.

Proof. Let us apply Theorem 1.2 to a minimising sequence $(u_h)_h$. By (1.5) we obtain a function $u \in GSBD^p(\widetilde{\Omega})$, with $u = u_0$ in $\widetilde{\Omega} \setminus \overline{\Omega}$. In fact, arguing as in the proof of Proposition 4.2 with (1.5a) and (1.5b), the choice $a^1_h = 0$ is possible in the set $\widetilde{\Omega} \setminus \overline{\Omega}$, which than has to be contained in a single element P_1 of the Caccioppoli partition.

Moreover, recalling Theorem 2.5 we have that

$$\begin{aligned}\int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u} g(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} &\leq \int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{(J_u \cap \mathcal{P}^{(1)}) \cup (\partial^* \mathcal{P} \cap \widetilde{\Omega})} g(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\leq \liminf_{h\to+\infty} \int_\Omega f(x, e(u_h))\,\mathrm{d}x + \int_{J_{u_h}} g(x,\nu_{u_h})\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \inf_{u=u_0 \in \widetilde{\Omega}\setminus\overline{\Omega}} \int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u} g(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned}$$

Therefore u is a solution to the problem (4.51) and this concludes the proof. We notice that by the chain of inequalities above, expressing the domain of the surface integral in the second inequality as the disjoint union $(J_u \cap \mathcal{P}^{(1)}) \cup (J_u \cap \partial^*\mathcal{P}) \cup \big((\partial^*\mathcal{P} \cap \widetilde{\Omega}) \setminus J_u\big) = J_u \cup \big((\partial^*\mathcal{P} \cap \widetilde{\Omega}) \setminus J_u\big)$, it holds

$$0 = \int_{(\partial^*\mathcal{P} \cap \widetilde{\Omega}) \setminus J_u} g(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \geq c\mathcal{H}^{n-1}\big((\partial^*\mathcal{P} \cap \widetilde{\Omega}) \setminus J_u\big)$$

and then $\partial^*\mathcal{P} \cap \widetilde{\Omega} \subset J_u$, up to a $\mathcal{H}^{n-1}$-negligible set. □

We consider now the case of general g.

Proposition 4.5. *Assume f, g as in Theorem 1.2. Then the problem*

$$\min_{\substack{u=u_0 \in \widetilde{\Omega}\setminus\overline{\Omega} \\ \mathcal{P} \text{ Caccioppoli partition of } \widetilde{\Omega},\ \partial^*\mathcal{P}\cap\widetilde{\Omega} \subset J_u}} \left\{ \int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u \setminus \partial^*\mathcal{P}} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^*\mathcal{P}\cap\widetilde{\Omega}} g_\infty(x,\nu)\,\mathrm{d}\mathcal{H}^{n-1} \right\}$$

admits a solution in $GSBD^p(\widetilde{\Omega})$.

Proof. Let us denote

$$F(u,\mathcal{P}) := \begin{cases} \displaystyle\int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u \setminus \partial^*\mathcal{P}} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^*\mathcal{P}\cap\widetilde{\Omega}} g_\infty(x,\nu)\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } \partial^*\mathcal{P} \cap \widetilde{\Omega} \subset J_u,\\ +\infty & \text{otherwise,}\end{cases}$$

and fix a minimising sequence $(u_h, \mathcal{P}_h)$ for F. To shorten the notation, in the following of the proof we write simply $\partial^*\mathcal{P}$ in place of $\partial^*\mathcal{P} \cap \widetilde{\Omega}$. We observe that, for $\mathcal{P}_h = (P_{h,j})_j$ we can assume that $\widetilde{\Omega} \setminus \overline{\Omega} \subset P_{h,1}$ (arguing as in the proofs of Propositions 4.2, 4.4, and we may choose 0 as the infinitesimal rigid motion in $P_{h,1}$) and find piecewise rigid functions $\widetilde{a}_h$ such that

$$\widetilde{a}_h = \sum_{j\in\mathbb{N}} \widetilde{a}^j_h \chi_{P_{h,j}}\,, \tag{4.52}$$

$$\widetilde{a}^1_h = 0, \quad |\widetilde{a}^j_h(x) - \widetilde{a}^i_h(x)| \to +\infty \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \text{ for all } i \neq j\,, \tag{4.53}$$

and

$$E(u_h - \widetilde{a}_h) < F(u_h, \mathcal{P}_h) + \frac{1}{h} \tag{4.54}$$

for every $h \in \mathbb{N}$. (Recall (1.1) for the definition of E). In fact, since $u_h \in GSBD^p(\widetilde{\Omega})$ it holds that $[u_h]\colon J_{u_h} \to \mathbb{R}^n$ is $\mathcal{H}^{n-1}$-measurable, and then there exists for every h a vanishing sequence $(s^h_k)_k$ such that

$$\mathcal{H}^{n-1}(J_{u_h} \cap \{|[u_h]| > k\}) < s^h_k\,. \tag{4.55}$$

Moreover, since $\mathcal{H}^{n-1}(\partial^*\mathcal{P}_h) < +\infty$, for every h, $k \in \mathbb{N}$ there exists $m^h_k \in \mathbb{N}$ such that

$$\sum_{j > m^h_k} \mathcal{H}^{n-1}(\partial^* P_{h,j}) < k^{-1}\,. \tag{4.56}$$

Then we choose, in correspondence of $(P_{h,j})_j$, a sequence $(\widetilde{a}_h^j)_j \subset \mathbb{R}^n$ (that is, each $\widetilde{a}_h^j$ is a constant function) with $\widetilde{a}_h^1 = 0$ (in view of the Dirichlet boundary conditions), such that $|\widetilde{a}_h^j - \widetilde{a}_h^i| > 2k$ for $i \neq j \leq m_k^h$. By (4.55), (4.56), and triangle inequality we find that

$$\mathcal{H}^{n-1}(\partial^*\mathcal{P}_h \cap \{|[u_h - \widetilde{a}_h]| < k\}) < s_k^h + k^{-1}\,.$$

This implies, in view of (g_5) and since g is a measurable function taking finite values, that there is $\overline{k} \in \mathbb{N}$, depending on h, large enough so that (4.54) holds true.

Let us now apply Theorem 1.2 to the sequence $(u_h - \widetilde{a}_h)_h \subset GSBD^p(\widetilde{\Omega})$ (that satisfies the assumptions of Theorem 1.2 by (4.54)): this provides a function $u \in GSBD^p(\widetilde{\Omega})$ and a sequence $(\widehat{a}_h)_h$ of piecewise rigid functions corresponding to a partition $\widehat{\mathcal{P}} = (\widehat{P}_j)_j$ (in particular, $J_{\widehat{a}_h} = \partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega}$) such that

$$u_h - \widetilde{a}_h - \widehat{a}_h \to u \quad \mathcal{L}^n\text{-a.e.}$$

and

$$\int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u \cap \widehat{\mathcal{P}}^{(1)}} g(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega}} g_\infty(x, \nu_{\widehat{\mathcal{P}}})\,\mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{h \to +\infty} E(u_h - \widetilde{a}_h)\,. \tag{4.57}$$

In particular, in view of the boundary conditions we may take $\widetilde{\Omega} \setminus \overline{\Omega} \subset \widehat{P}_1$, $\widehat{a}_h^1 = 0$ and we have that $\partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega} \subset \overline{\Omega}$ and $u = u_0$ in $\widetilde{\Omega} \setminus \overline{\Omega}$. Collecting (4.54), (4.57), and since $(u_h, \mathcal{P}_h)_h$ is a minimising sequence for F, we have that

$$\int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u \cap \widehat{\mathcal{P}}^{(1)}} g(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega}} g_\infty(x, \nu_{\widehat{\mathcal{P}}})\,\mathrm{d}\mathcal{H}^{n-1} \leq \inf_{v, \mathcal{P}} F(v, \mathcal{P})\,. \tag{4.58}$$

We notice now that we can find a piecewise rigid function $\widetilde{a}$ with $\widetilde{a} = 0$ in $\widetilde{\Omega} \setminus \overline{\Omega}$ and $J_{\widetilde{a}} \subset \partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega}$ for which $\partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega} \subset J_{u - \widetilde{a}}$. This follow from the fact that there are at most countable many $s \in \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(\partial^*\widehat{\mathcal{P}} \cap \{[u] = s\}) > 0$. Moreover, since $\partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega} \subset J_{u - \widetilde{a}}$ (and in view of the fact that g depends only on the jump amplitude, cf. below (4.33)), we have that for $\widehat{u} = u - \widetilde{a}$

$$\int_\Omega f(x, e(u))\,\mathrm{d}x + \int_{J_u \cap \widehat{\mathcal{P}}^{(1)}} g(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Omega f(x, e(\widehat{u}))\,\mathrm{d}x + \int_{J_{\widehat{u}} \cap \widehat{\mathcal{P}}^{(1)}} g(x, [\widehat{u}], \nu_{\widehat{u}})\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.59}$$

Therefore, in view of the fact that $\partial^*\widehat{\mathcal{P}} \cap \widetilde{\Omega} \subset J_{\widehat{u}}$, by (4.58) and (4.59) we get that $(\widehat{u}, \widehat{\mathcal{P}})$ is a minimiser for F. This concludes the proof. □

Remark 4.6. The minimisation problem in Proposition 4.5 formally reduces to those one in Proposition 4.2 and 4.4 noticing that $\mathcal{P} = \{\widetilde{\Omega}\}$ when $g_\infty \equiv +\infty$ (cf. Theorem 1.2) and the functional in Proposition 4.5 does not depend on $\mathcal{P}$ when g depends only on ν and coincides with g_∞.

Remark 4.7. In the minimisation problem in Proposition 4.5 the restriction $\partial^*\mathcal{P} \cap \widetilde{\Omega} \subset J_u$ may be dropped. The mechanical interpretation for including this condition is to regard $\partial^*\mathcal{P}$ as part of the discontinuity set of the displacement, where the fracture is present, since also in $\partial^*\mathcal{P}$ the material can be interpreted as fractured.

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