# A new distributed protocol for consensus of discrete-time systems

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**Abstract**

In this paper, a new distributed protocol is proposed to force consensus in a discrete-time network of scalar agents with an arbitrarily assignable convergence rate. Several simulations validate the performances and the improvements with respect to more standard protocols.

*Keywords:* Linear systems; Consensus control and estimation; Network analysis and control.

## 1. Introduction

Numerous problems in engineering and beyond can be lead to enforcing consensus in a network of discrete-time dynamics as, for instance, controlling multi-agent systems under sampled-data communication [1], modeling opinion dynamics and social networks [2, 3], non-cooperative or selfish routing in network systems [4], identification and filtering in, among many, sensor networks [5]. In all those cases, the problem consists in defining a suitable distributed interconnection protocol driving the dynamics of all agents of the

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network to a shared behavior, commonly referred to as consensus [6]. In general, such an interconnection is performed by emulating the continuous-time counterpart. Despite such a rule is simple to implement, it guarantees consensus of the network only under very restrictive conditions on the coupling strength (the gain weighting the influence of the network on each agent). As a matter of fact, even for networks of discrete-time scalar agents, the standard connection protocol guarantees convergence to an agreement state only if the coupling is small enough with respect to the network size [7, 8]. Accordingly, these results provide conservative values for the gain that are generally inversely proportional to either the smallest non zero eigenvalue of the Laplacian (which must be known to all agents) or, alternatively, the number of agents involved in the network. As a consequence the coupling gain decreases with the size of the network affecting both the convergence rate to consensus, slowed down significantly for large networks, and the amount of information the agents need to exchange before reaching an agreement. A few solutions to this problems involve the introduction of suitable weights over each edge so to make the corresponding discrete Laplacian stable. However, in that case, the consensus value is altered by the choice of the weights and might be thus significantly different from the one that one would have in continuous time. In addition, in that case it is assumed that each node can indeed separate and modify the contribution of the information coming from each neighbors, that might not be possible in practice.

In such a context, this paper aims at designing a new distributed protocol overcoming the aforementioned issues for a network of discrete-time scalar dynamical agents.

A first step toward this goal is in [9], where a new coupling control is provided for discrete-time networks forcing consensus for all the coupling strength values that can be then arbitrarily set. However, despite well-performing in the nominal cases, the proposed protocol suffered from two main issues: (*i*) it cannot be implemented in a distributed manner as the coupling protocol is implicitly defined; (*ii*) the convergence rate cannot be fixed arbitrarily as directly proportional to the coupling gain.

Starting from this result, the contribution of this paper is twofold. First, the centralized protocol in [9] is improved by allowing to fix the convergence rate to consensus arbitrary fast as directly proportional to the coupling gain. However, such a protocol is implicitly defined by a linear equality whose solution, defining the control action, cannot be instantaneously and independently computed by each node. Accordingly, as second and major contribution, we propose a distributed version of the aforementioned protocol allowing to approximately solve, in an arbitrary number of steps, the linear equation that defines the coupling input. The resulting design approach is reminiscent of a time-scale separation procedure typically employed in distributed filtering [10, 11, 12]. Roughly speaking, two consensus processes are nested over a time window of length $\gamma$, so resorting to a multi-rate consensus controller. The local control at step $t$, $u_i(t)$, is computed via an intra-consensus forward computation over a time window of length $\gamma$. At all steps $t+\tau$ (with $\tau = 1, \dots, \gamma$), each agent computes an approximate solution (say $v_i(t,\tau)$) to the equation defining the consensus protocol only based on the available (local) information and then sends it to the corresponding neighbors. At step $t+\gamma$, the actual control is deduced as the result of the approximating consensus phase after $\gamma$ steps, i.e., $u_i(t) = v_i(t,\gamma)$. The so deduced controller is ensured to enforce consensus for all values of $\gamma$ with performance of the centralized implementation recovered as $\gamma$ increases.

The rest of the paper is organized as follows. In Section 2 the problem is formulated with the centralized and decentralized solutions presented in Section 3. Simulations are given in Section 4 with concluding remarks and perspectives in Section 5.

*Notations.* $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ denote the set of complex, real and natural numbers including 0 respectively. $|\cdot| \in \mathbb{R}$ denotes, depending on the argument, either the cardinality of a set $\mathcal{S}$ or the absolute value of a complex number $\lambda \in \mathbb{C}$. $I$ and 0 denote respectively the identity and zero matrices of suitable dimensions. $I$ denotes the identity matrix of suitable dimension whereas 0 denotes zero matrix (of suitable dimension) and the zero scalar. $\mathbb{1}_c$ denotes the $c$-dimensional column vector whose elements are all ones. Given a matrix $A \in \mathbb{R}^{n\times n}$, $\sigma\{A\} \subset \mathbb{C}$ is its spectrum, $\|A\|$ its norm and $\rho(A)$ its spectral radius. A matrix is said non-negative if all its entries are non-negative. Given $x : t \mapsto x(t)$ with $t \in \mathbb{N}$ and $x(t) \in \mathbb{R}^n$, we denote for simplicity $x = x(t)$ and $x^+ = x(t+1)$. For a scalar real-valued function $H : \mathbb{R}^n \to \mathbb{R}$, $\Delta H(x) = H(x^+) - H(x)$ is the corresponding one step increment.

## 2. Problem statement and recalls

### *2.1. Recalls on graph*

We consider an unweighted directed graph (or digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors to a node $i \in \mathcal{V}$ is defined as $\mathcal{N}_i = \{j \in \mathcal{V} \text{ s.t. } (j,i) \in \mathcal{E}\}$. For all pairs of distinct nodes

$i, j \in \mathcal{V}$, a directed path from $i$ to $j$ is defined as $i \rightsquigarrow j := \{(r, r+1) \in \mathcal{E}$ s.t. $\cup_{r=0}^{\ell-1} (r, r+1) \subseteq \mathcal{E}$ with $0 = i, \ell = j$ and $\ell > 0\}$. The reachable set from a node $i \in \mathcal{V}$ is defined as $R(i) := \{i\} \cup \{j \in \mathcal{V}$ s.t. $i \rightsquigarrow j\}$. A set $\mathcal{R}$ is called a reach if it is a maximal reachable set, that is, $\mathcal{R} = R(i)$ for some $i \in \mathcal{V}$ and there is no $j \in \mathcal{V}$ such that $R(i) \subset R(j)$. Since $\mathcal{G}$ possesses a finite number of vertices, such maximal sets exist and are uniquely determined by the graph itself. Denoting by $\mathcal{R}_i$ for $i = 1, \ldots, \mu$, the reaches of $\mathcal{G}$, the exclusive part of $\mathcal{R}_i$ is defined as $\mathcal{H}_i = R_i \backslash \cup_{\ell=1, \ell \neq i}^{\mu} R_\ell$ with $h_i = |\mathcal{H}_i|$. Finally, the common part of $\mathcal{G}$ is given by $\mathcal{C} = \mathcal{V} \backslash \cup_{i=1}^{\mu} \mathcal{H}_i$ with cardinality $c = |\mathcal{C}|$.

The Laplacian matrix associated to $\mathcal{G}$ is given by $L = D - A$ with $D \in \mathbb{R}^{N \times N}$ and $A \in \mathbb{R}^{N \times N}$ being respectively the in-degree and the adjacency matrices. $L$ possesses one eigenvalue $\lambda = 0$ with both algebraic and geometric multiplicities coinciding with $\mu$, the number of reaches of $\mathcal{G}$ [13, Corollary 1]. In the following, it is assumed that $\mu = 1$, that is the network only possesses one reach.

### *2.2. Motivation and problem formulation*

Consider a multi-agent system exchanging information via a communication digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with each vertex $i \in \mathcal{V}$ being a dynamical unit of the form

$$x_i^+ = x_i + u_i \tag{1}$$

with $x_i,\ u_i \in \mathbb{R}$ and $i = 1, \ldots, N$. When coupling all agents via the standard protocol [7, 6]

$$u_i = -\kappa_l \sum_{\nu_j \in \mathcal{N}(\nu_i)} (x_i - x_j) \tag{2}$$

consensus is achieved only if the coupling gain $\kappa_l > 0$ is small enough. More precisely, denoting the agglomerate vectors

$$\begin{aligned} \boldsymbol{x} =& \text{col}\{x_i,\ i = 1, \ldots, N\} \in \mathbb{R}^N \\ \boldsymbol{u} =& \text{col}\{u_i,\ i = 1, \ldots, N\} \in \mathbb{R}^N. \end{aligned} \tag{3}$$

the network dynamics

$$\boldsymbol{x}(t+1) = (I - \kappa_l L)\boldsymbol{x}(t) \tag{4}$$

possesses dynamic matrix $I - \kappa_{\text{l}} L$ with one eigenvalue in $\lambda_d = 1$ with unitary geometric multiplicity; however, all agents converge to a suitable consensus $x_s \in \mathbb{R}$ only if all other eigenvalues lie within the open unit circle, that is when

$$\kappa_{\text{l}} \leq \frac{1}{\lambda_{\max}}, \quad \lambda_{\max} = \max_{\lambda > 0}\{\lambda \in \sigma\{L\}\}. \tag{5}$$

Accordingly, if $\kappa_{\text{l}}$ is not small enough consensus might be lost and the network might be unstable. In addition, even in the best case scenario (when consensus is preserved), because $\kappa_{\text{l}}$ should be small, the exchange of information is significantly filtered by all agents so notably affecting both the convergence rate to consensus and the amount of information that must be exchanged. Finally, $\lambda_{\max}$ might not be known by all agents and, even if upper bounds can be computed, the transient performances might not be acceptable.

In [9], a new consensus protocol solving part of the issues above has been proposed. However, the proposed solution is centralized and cannot assign an arbitrary convergence rate to consensus.

In this paper, we try to make a step farther investigating and solving the problem of designing, if any, a local and distributed control law of the form

$$u_i(t) = \kappa \varphi_i(x_i(t), \text{col}\{x_j(t), u_j(t-1) \text{ s.t. } j \in \mathcal{N}_i\})$$

making all agents asymptotically converge to some consensus $x_s \in \mathbb{R}$ for all $\kappa > 0$; namely, as $t \to \infty$

$$\boldsymbol{x}(t) \to \mathbb{1}_N x_s, \tag{6}$$

for suitably defined consensus value $x_s \in \mathbb{R}$.

## 3. Main result

### *3.1. A refined centralized consensus protocol*

First, we refine and extend the centralized algorithm proposed in [9] where it has been proved that a direct input dependence is necessary on the output that all agents exchange through the network.

**Theorem 3.1.** *Consider a network of $N$ discrete-time agents of the form* (1) *with communication digraph $\mathcal{G}$ with only one reach, i.e. the Laplacian $L$*

*has a zero eigenvalue with multiplicity* 1*. Then, for all* $i = 1, \dots, N$ *the local control law*

$$\boldsymbol{u} = -\kappa\big(I_N + \kappa g L\big)^{-1} L\boldsymbol{x} \tag{7}$$

*whose components are solutions to*

$$u_i = -\,\kappa \sum_{j\in\mathcal{N}_i} (y_i - y_j) \tag{8a}$$

$$y_i = x_i + g u_i \tag{8b}$$

*guarantee consensus for all* $\kappa > 0$ *and* $g \geq \frac{1}{2}$*; namely, for the network dynamics*

$$\boldsymbol{x}(t+1) = \Theta_c(\kappa, g)\boldsymbol{x}(t) \tag{9}$$

$$\Theta_c(\kappa, g) = \big(I_N + \kappa g L\big)^{-1}\big(I_N + \kappa(g-1)L\big) \tag{10}$$

*as* $t \to \infty$*, one gets that* (6) *holds with*

$$x_s = v_1^\top \boldsymbol{x}(0) \tag{11}$$

*with* $v_1^\top \in \mathbb{R}^N$ *such that* $v_1^\top L = 0$ *and* $v_1^\top \mathbb{1}_N = 1$*.*

*Proof:* The proof follows along the lines of [9, Thorem 4.1] noticing that for all choices of $\kappa, g \in \mathbb{R}$, the eigenvalues of $\Theta_c(\kappa, g)$ are given by

$$\lambda_d^i(\kappa, g) = \frac{1 + \kappa(g-1)\lambda^i}{1 + \kappa g \lambda^i}, \qquad \text{for all } \lambda^i \in \sigma\{L\}. \tag{12}$$

Thus, $\Theta_c$ possesses an eigenvalues in $\lambda_d = 1$ with multiplicity 1, corresponding to the eigenvalue $\lambda = 0$ of the Laplacian $L$. In addition, all other eigenvalues are in the open unit circle if and only if $\kappa > 0$ and $g > \frac{1}{2}$. As a consequence, the center subspace associated to the eigenvalue 1 yielded by

$$\mathcal{V}_c = \ker\{L\} = \operatorname{span}\{\mathbb{1}_N\}$$

is attractive and coincides with the consensus subspace. Introducing now

$$\begin{pmatrix} x_s \\ \boldsymbol{x}_r \end{pmatrix} = v_1^\top \boldsymbol{x} = \begin{pmatrix} V_0^\top \\ V_r^\top \end{pmatrix} \boldsymbol{x} \tag{13}$$

with

$$V^{-\top} = Z = \begin{pmatrix} \mathbb{1}_N & Z_r \end{pmatrix}$$
$$V^\top L Z = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^r \end{pmatrix}, \quad \Lambda^r = \mathrm{diag}\{\sigma\{L\}\backslash\{0\}\}$$

one gets that

$$V^\top \Theta_c(\kappa, g) Z = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda_d^r \end{pmatrix}$$
$$\Lambda_d^r(\kappa, g) = \mathrm{diag}\{\sigma\{\Theta_c(\kappa, g)\}\backslash\{1\}\}.$$

In detail, $x_s \in \mathbb{R}$ is the projection of the trajectories onto $\mathcal{V}_c$, whereas $\boldsymbol{x}_r \in \mathbb{R}^{N-1}$ is the orthogonal component that converges to zero, by construction of $\kappa$ and $g$. When consensus is achieved (i.e., $\boldsymbol{x} \in \mathcal{V}_c$), one gets $\boldsymbol{x}_r \equiv 0$ and

$$\boldsymbol{x} = \mathbb{1}_N x_s$$

so that the proof follows. ■

**Remark 3.1.** *The output* (8b) *that all agents exchange through the network is the one making all agents Input-Feedforward Passive [14]; i.e., fixing the storage* $S(x_i) = \frac{1}{2}x_i^2$ *one gets the dissipation inequality*

$$\Delta S(x_i) \leq u_i y_i - (g - \frac{1}{2})u_i^2. \tag{14}$$

*Accordingly, as* $g > \frac{1}{2}$, $u_i \mapsto y_i$ *is strictly passive and passive for* $g = \frac{1}{2}$.

**Remark 3.2.** *When fixing* $g = \frac{1}{2}$, *the consensus protocol proposed in [9] is recovered. However, in this case, one cannot fix the convergence rate to consensus arbitrarily small via* $\kappa > 0$. *As a matter of fact, one cannot compute the gain* $\kappa$ *to make all eigenvalues of* $\Theta_c(\kappa, \frac{1}{2})$

$$\lambda_d^i(\kappa, \frac{1}{2}) = \frac{1 - \frac{\kappa}{2}\lambda^i}{1 + \frac{\kappa}{2}\lambda^i}, \quad \textit{for all } \lambda^i \in \sigma\{L\}.$$

*arbitrarily close to* 0. *On the other side, the value of the largest eigenvalue (in module) can be minimized depending on the particular spectrum of* $L$, *which must be thus known to all nodes apriori.*

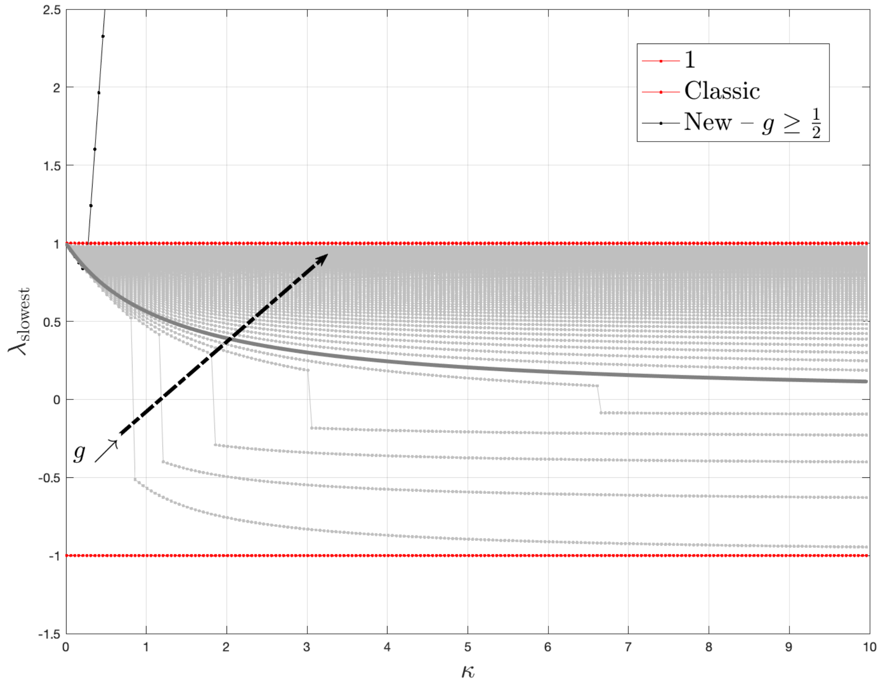


Figure 1: Plot of the farthest eigenvalue of $\Theta_c$ from 0 for increasing values of $\kappa > 0$ and $g \geq \frac{1}{2}$.

**Remark 3.3.** *When fixing $g = 1$ in the consensus protocol (8), one can modulate the convergence rate to consensus arbitrarily via $\kappa > 0$ independently on the size of the network. Precisely, the eigenvalues of $\Theta_c(\kappa, 1)$*

$$\lambda_d^i(\kappa, 1) = \frac{1}{1 + \kappa\lambda^i}, \quad \text{for all } \lambda^i \in \sigma\{L\}$$

*are as close to 0 (in module) as $\kappa$ increases; namely, trajectories of the network (9) converge to the multi-consensus with a velocity that is directly proportional to $\kappa > 0$. With this in mind, contrarily to the case [9, 6], one can assign the convergence rate arbitrarily small picking $\kappa \to \infty$, with no knowledge of the spectrum of the Laplacian. For a randomly generated graph of 10 nodes, Fig. 1 depicts the location of the slowest eigenvalue of $\Theta_c(\kappa, g)$ for increasing values of both $\kappa$ and $g$.*

The coupling rule (8) defining the consensus control (7) is implicitly defined and, in general, cannot be computed (at least statically) in a fully distributed manner. As a matter of fact, at each step $k \in \mathbb{N}$, the $i^{\text{th}}$ agent needs the input $u_j$ of all its neighbors for computing the corresponding $u_i$ so creating a bottleneck that cannot be solved locally.

The feedback (7) can be rewritten as

$$\boldsymbol{u} = -\kappa W(\kappa, g) L \boldsymbol{x} \tag{15}$$

$$W(\kappa, g) := (I_N + \kappa g L)^{-1} \tag{16}$$

with the term $W(\kappa, g)L$ being a new weighted Laplacian. This new Laplacian is associated to a new dummy graph in which new (and weighted) edges appear within connected components due to the presence of the weighting matrix $W(\kappa, g)$ which depends on the original Laplacian L. Such a matrix cannot be computed locally by each agent unless all of them possess exact knowledge on the original topology of $\mathcal{G}$.

*3.2. A distributed implementation of the new protocol*

In this section we study the problem of a distributed (dynamic) implementation of the coupling rule defined by (8). The problem is not trivial, as it is evident for general reasons. Since, as remarked in Remark 3.3, (8) allows for an arbitrary rate of convergence to consensus, it follows that its distributed implementation would yield distributed practical convergence in any finite number of steps, including 1. However, this is not possible, since the speed of the information trough a graph with discrete-time communications is obviously limited.

One possible way to overcome this conceptual limitation is to separate the time-scale of the information exchange and of the system evolution by resorting to a multi-rate controller. The idea of multiple information exchanges per time unit is not new and it has been used extensively in the field of distributed filtering [10, 11]. The coupling rule defined by (8) reads

$$u_i = -\kappa \sum_{j \in \mathcal{N}_i} (x_i - x_j) - g\kappa \sum_{j \in \mathcal{N}_i} \big(u_i - u_j\big). \tag{19}$$

By solving with respect to $u_i$ one gets, with $d_i = |\mathcal{N}_i|$,

$$u_i = -\frac{\kappa}{1 + g\kappa d_i} \sum_{j \in \mathcal{N}_i} (x_i - x_j) + \frac{g\kappa}{1 + g\kappa d_i} \sum_{j \in \mathcal{N}_i} u_j, \tag{20}$$

**Approximate $\gamma$ steps implementation of (8) at node $i$.**

1: At each time $t \geq 0$, send $x_i(t)$ to the neighbors.

2: Receive $x_j(t)$ from the neighbors, $j \in \mathcal{N}_i$, and compute, with $d_i = |\mathcal{N}_i|$,

$$v_i(t,0) = -\frac{\kappa}{1+g\kappa d_i} \sum_{j \in \mathcal{N}_{(i)}} \big(x_i(t) - x_j(t)\big) \tag{17}$$

3: For $h = 0, \ldots, \gamma - 1$ do:

3.1: Send $v_i(t,h)$ to the neighbors

3.2: Compute

$$v_i(t,h+1) = v_i(t,0) + \frac{g\kappa}{1+g\kappa d_i} \sum_{j \in \mathcal{N}_i} v_j(t,h) \tag{18}$$

4: Set $u_i(t) = v_i(t,\gamma)$.

Figure 2: Distributed approximate multi-step implementation of (8)

where the first term of the right-hand is immediately available at each time $t$, whereas the second one can be approximated by a truncated fixed-point iteration with $\gamma \in \mathbb{N}$ steps. The resulting approximate distributed implementation of (8) is summarized in Fig. 2.

**Theorem 3.2.** *If the graph $\mathcal{G}$ has exactly one reach, then at each node $i$ and time $t$ the sequence $v_i(t,\gamma)$ generated by* (18) *is such that, for all $\kappa$, $g > 0$ and $\gamma \in \mathbb{N}$,*

$$\lim_{\gamma\to\infty} v_i(t,\gamma) = u_i(t), \tag{21}$$

*with $u_i(t)$ the $i$-th component of $\boldsymbol{u} = -\kappa\big(I_N + \kappa g L\big)^{-1} L\boldsymbol{x}$.*

*Proof:* Introduce the matrices

$$\tilde{D} =(I_N + \kappa g D)^{-1} \tag{22}$$

$$G =(I_N + \kappa g D)^{-1}\kappa g A = \kappa g \tilde{D} A. \tag{23}$$

$G$ is non-negative with $\rho(G) < 1$, *i.e.* $G$ is Schur. $G$ is non-negative because $\tilde{D}$, $A$ are non-negative. Since each row $i$ of $G$ is obtained from the corresponding row of $A$ divided by $1 + \kappa g d_i$, an application of the Gerschgorin criterion [15] to the rows of $G$ allows to conclude that $\rho(G) < 1$ for any positive values of $\kappa$ and $g$. In fact, $G_{ii} = 0$ is the center of the circles and the radius is $\sum_{j\neq i} G_{ij} = \kappa g d_i/(1 + \kappa g d_i) < 1$, that implies $\rho(G) < 1$. In vector form (18) reads

$$V(t,\gamma) = -\kappa\Big(I_N + G + \cdots + G^{\gamma}\Big)\big(I + \kappa g D\big)^{-1} L\boldsymbol{x}(t). \tag{24}$$

From $L = D - A$ we obtain

$$\begin{aligned}\big(I_N + \kappa g L\big) =& I_N + \kappa g D - \kappa g A = \tilde{D}^{-1}(I_N - \kappa g \tilde{D} A)\\ =&(I_N + \kappa g D)(I_N - G)\end{aligned} \tag{25}$$

one gets

$$V(t,\gamma) = -\kappa \sum_{h=0}^{\gamma} G^h \big(I_N + \kappa g L\big)^{-1} L\boldsymbol{x}(t).$$

Since when $\rho(G) < 1$ it holds that $(I_N - G)^{-1} = \sum_{i=0}^{\infty} G^i$ one concludes, as $\gamma \to \infty$,

$$V(t,\gamma) \to -\kappa\big(I_N + \kappa g L\big)^{-1} L\boldsymbol{x}(t)$$

that is exactly the centralized control (7). ■

Theorem 3.2 shows that with a sufficient number of consensus steps for time unit, the proposed approximate coupling recovers the one defined by the equation (7) with a desired approximation order. In particular with $g = 1$ it is possible to choose any rate of convergence to the consensus value.

In practice, it is reasonable to ask if the consensus can be reached in a finite number of consensus steps $\gamma$. We therefore investigate the consensus properties of the control $u_i(t) = v_i(t, \gamma)$. The following result is instrumental to this goal.

**Lemma 3.1.** *With the control law $u_i(t) = v_i(t, \gamma)$ the network dynamics takes the form*

$$\boldsymbol{x}(t+1) = \Theta_d(\kappa, g, \gamma)\boldsymbol{x}(t) \tag{26}$$

*with*

$$\Theta_d(\kappa, g, \gamma) = I_N - \kappa(I_N - G^{\gamma+1})W(\kappa, g)L \tag{27}$$

*and where $W(\kappa, g) = (I_N + \kappa g L)^{-1}$ in (16) is non-negative.*

*Proof:* The proof follows from (24), (25) and $\sum_{h=0}^{\gamma} G^h = (I_N - G^{\gamma+1})(I_N - G)^{-1}$. We get

$$V(t, \gamma) = -\kappa \left(I - G^{\gamma+1}\right) W(\kappa, g) L\boldsymbol{x}(t) \tag{28}$$

with $G$ as in (23) and $W(\kappa, g)$ as in (16). Thus, $\boldsymbol{x}(t+1) = \boldsymbol{x}(t) + \boldsymbol{u}(t)$ can be written

$$\begin{aligned} \boldsymbol{x}(t+1) &= \left(I_N - \kappa \left(I_N - G^{\gamma+1}\right) W(\kappa, g) L\right) \boldsymbol{x}(t) \\ &= \Theta_d(\kappa, g, \gamma)\boldsymbol{x}(t). \end{aligned} \tag{29}$$

Finally, (25) implies $W = (I_N - G)^{-1}\tilde{D}$ and since both $(I_N - G)^{-1} = \sum_{h=0}^{\infty} G^h$ and $\tilde{D}$ are non-negative, $W$ is non-negative too. ■

Notice that by using (15) the matrix $\Theta_c(\kappa, g)$ in (10) can be written $\Theta_c = I_N - \kappa W L$ and, since $G$ is Schur, $\lim_{\gamma\to\infty} \Theta_d(\kappa, g, \gamma) = \Theta_c(\kappa, g)$, in accordance with Theorem 3.2.

The following result specifies conditions allowing to enforce consensus under the distributed approximation in (18) with no consensus iteration (i.e., when $\gamma = 0$).

**Lemma 3.2.** *When $\kappa g \geq 1$ and the graph contains only one reach, the control law $u_i = v_i(t, 0)$ (i.e. $\gamma = 0$) makes the agents converge to the same consensus value.*

*Proof:* From (24), $V(t, 0) = -\kappa\tilde{D}L\boldsymbol{x}(t)$ and (27) reads

$$\Theta_d(\kappa, g, 0) = I_N - \kappa\tilde{D}L. \tag{30}$$

$\Theta_d(\kappa, g, 0)$ is row-stochastic, i.e. $\Theta_d(\kappa, g, 0)\mathbb{1}_N = \mathbb{1}_N$. Moreover, if $\kappa g \geq 1$ then $\Theta_d(\kappa, g, 0)$ is non-negative. An application of the Gerschgorin circles to the rows of $\Theta_d(\kappa, g, 0)$ yields $\rho(\Theta_d(\kappa, g, 0)) \leq 1$. Since it is immediate to verify that $\lambda_1 = 1 \in \sigma\{\Theta_d(\kappa, g, 0)\}$ with right eigenvector $u_1 = \mathbb{1}_N$ we have $\rho(\Theta_d(\kappa, g, 0)) = 1$. When the graph is strongly connected, $\Theta_d(\kappa, g, 0)$ is irreducible and, by the presence of at least a non zero element on the diagonal, primitive [15]. Consequently, $\lambda_1 = 1$ is the only eigenvalue on the unit circle and it has algebraic and geometric multiplicity 1 by the Frobenius-Perron theorem. Thus, $\boldsymbol{x}(t)$ tends to the eigenspace of $\lambda_1$, that is, to a vector of the form $\mathbb{1}_N x_s$, $x_s = v_1^\top \boldsymbol{x}(0) \in \mathbb{R}$, where $v_1$ is the corresponding left eigenvector of $\Theta_d(\kappa, g, 0)$ for $\lambda_1 = 1$ satisfying $v_1^\top \mathbb{1}_N = 1$. When the graph is weakly connected and with one reach only, the multiplicity of $\lambda_1 = 1$ is still 1 (see [16, Theorem 3.2]). In this situation, the set of roots of the graph form a strongly connected sub-graph and therefore they reach consensus. The consensus of the remaining nodes of the graph follows from a simple partioning of $L$ between root and non-root nodes (see [17, Section 2]). ■

Theorem 3.1 and Lemma 3.2 show that the distributed control (18) on weakly connected digraphs with only one reach (*i.e.* $\mu = 1$) guarantees consensus for, respectively, $\gamma = \infty$ and $\gamma = 0$ consensus iterations. Our main result, here below, states that this property holds for *any* number of consensus iterations and for all $\gamma \geq 0$, $\kappa > 0$, $g \geq 1$.

**Theorem 3.3.** *If the graph $\mathcal{G}$ has exactly one reach, then the control $u_i(t) = v_i(t, \gamma)$ generated by (18) makes the agents converge to the same consensus value $x_s \in \mathbb{R}$ for all $\gamma \geq 0$, $\kappa > 0$, $g \geq 1$.*

*Proof:* We know from Lemma 3.1 that the collective dynamics of the nodes is $\boldsymbol{x}(t+1) = \Theta_d(\kappa, g, \gamma)\boldsymbol{x}(t)$ with $\Theta_d(\kappa, g, \gamma)$ given in (27), that can be rewritten as

$$\begin{aligned}\Theta_d(\kappa, g, \gamma) =& I_N - \kappa W(\kappa, g)L + \kappa G^{\gamma+1} W(\kappa, g)L \\ =& \Theta_c(\kappa, g) + \kappa G^{\gamma+1} W(\kappa, g)L.\end{aligned} \tag{31}$$

Let $W = W(\kappa, g)$ for concision. We now prove that $W = I_N - \kappa g W L$. In fact,

$$I_N - \kappa g W L = W(W^{-1} - \kappa g L) = W \cdot I_N = W.$$

Consequently, $\kappa W L = (I_N - W)/g$ and

$$\Theta_c(\kappa, g) = I_N - \kappa W L = I_N - \frac{1}{g}\left(I_N - W\right)$$

Replacing into (31) yields

$$\Theta_d(\kappa, g) = I_N - \frac{1}{g} I_N + \frac{1}{g} G^{\gamma+1} + \frac{1}{g}\left(I_N - G^{\gamma+1}\right) W. \tag{32}$$

We notice that, for $g \geq 1$, $I_N - \frac{1}{g} I_N > 0$ and $G^{\gamma+1} > 0$ because $G$ is positive. Finally,

$$\left(I_N - G^{\gamma+1}\right) W = \sum_{h=0}^{\gamma} G^h (I_N - G) W = \sum_{h=0}^{\gamma} G^h \tilde{D} \geq 0, \tag{33}$$

where we have used $(I_N - G)W = \tilde{D}$ that descends from (25), and $\tilde{D} \geq 0$. We can now conclude $\Theta(\kappa, g, \gamma) \geq 0$ whenever $g \geq 1$, $\kappa > 0$, $\gamma \geq 0$. Since $\Theta(\kappa, g, \gamma)\mathbb{1}_N = \mathbb{1}_N$ we can repeat the same steps as in Lemma 3.2 to conclude that $\rho(\Theta(\kappa, g, \gamma)) = 1$, with exactly one eigenvalue $\lambda_1 = 1$ on the unit circle, and this guarantees consensus. ■

## 4. Simulations

In this section, we report the results of the proposed algorithm (in both the centralized and distributed implementation) when applied to different networks. Performances are compared with respect to the standard discrete-time protocol (2) when fixing the coupling gain as the largest value guaranteeing that (5) holds. For evaluating performances, we use the following parameters: the $M\%$-consensus settling time $t_{\mathrm{s}}^M$ defined as the minimum amount of steps that is required for the trajectories of the network to reach $M\%$ of the corresponding consensus value. For (2), and the proposed centralized algorithm in Proposition 3.1, $t_{\mathrm{s}}^M$ provides an estimate of the minimum number of iterations that are required for consensus to be achieved; for the distributed implementation in Theorem 3.3, such a quantity is given by $(\gamma + 1)t_{\mathrm{s}}^M$, with $\gamma \in \mathbb{N}$ as in (28) and $M = 10^{-1}$.

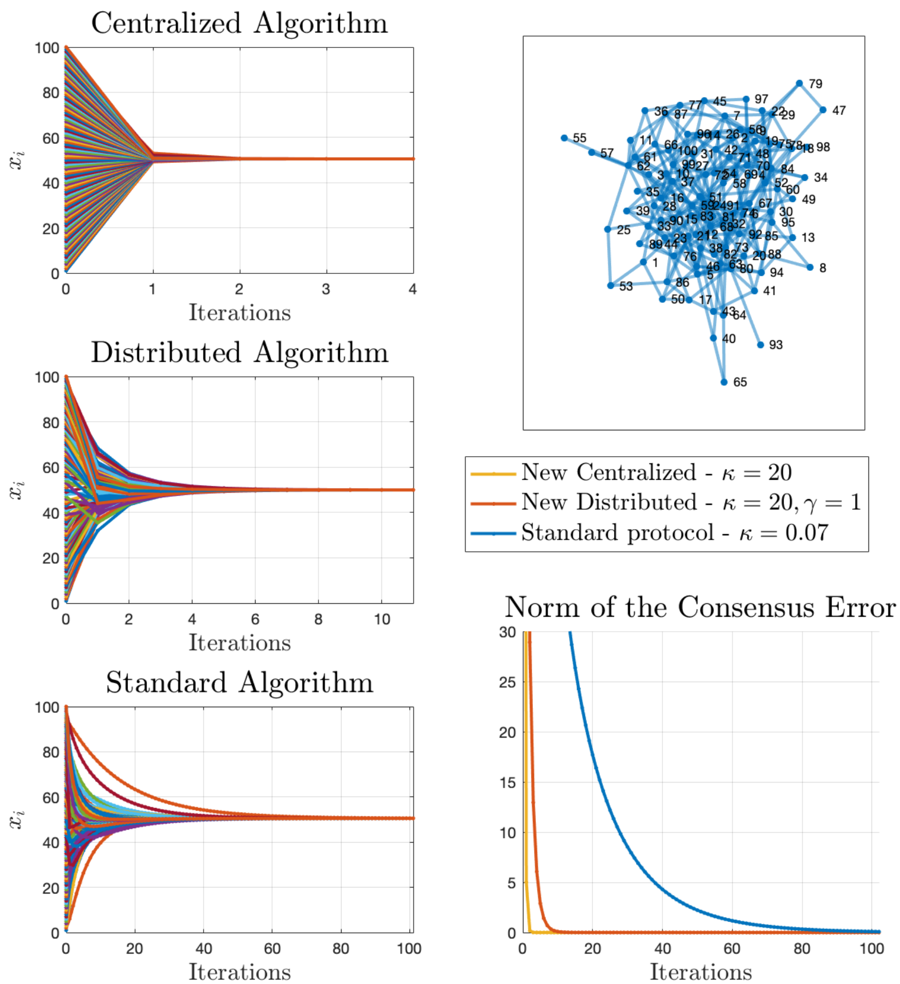


Figure 3: Network of $N = 100$ scalar agents over an undirected graph.

In Fig. 3, an undirect network of $N = 100$ is simulated when fixing, for the proposed algorithm, $\kappa = 10$ and $\gamma = 1$. Despite the amount of agents, the network controlled via the centralized algorithm converges to consensus in three iterations (and $t_{\mathrm{s}}^{10^{-1}} = 3$); the distributed implementation yields $t_{\mathrm{s}}^{10^{-1}} = 10$ so getting convergence with good performances even when $\gamma = 1$ and, in this case, in exactly 20 time steps. Those performances are

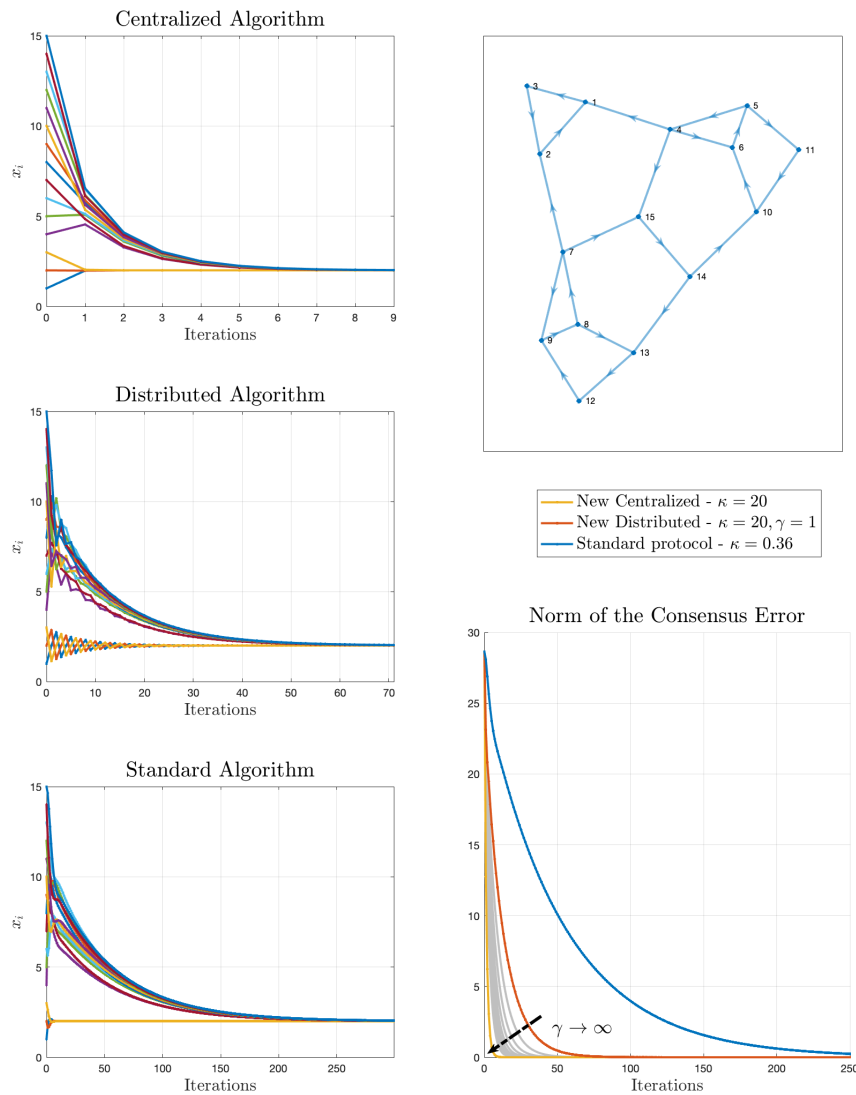


Figure 4: Network of $N = 15$ scalar agents over a digraph.

far better than the ones exhibited by the standard consensus algorithm for which $t_{\mathrm{s}}^{10^{-1}} = 100$.

In Fig. 4, the simulations are performed for a directed network of $N = 15$ agents. Similar considerations as in the previous case hold. In particular, fixing $\kappa = 1$ consensus is achieved in 14 iterations in exactly 20 time steps; the corresponding distributed implementation with $\gamma = 1$ requires 148 iterations to converge to consensus (with $t_{\mathrm{s}}^{10^{-1}} = 74$). Such a result is notably better than the one provided by the standard algorithm for which approximately $t_{\mathrm{s}}^{10^{-1}} = 298$. We underline that, as highlighted in Fig. 4, performances of the distributed algorithm improve (over the big-consensus steps) as $\gamma$ increases. In that case, the distributed algorithm approaches the nominal performances (i.e., the ones under (7)) for finite and reasonable values of $\gamma$ at the price of larger computational delays introduced by the consensus steps. Summarizing, performances achieved by the distributed algorithm are always far better than the usual one even when $\gamma$ is small and fixed to 1, the worst case scenario.

**Remark 4.1.** $\kappa_{\mathrm{l}}$ *is in general fixed smaller and inversely proportional to the size of the network [7] . However, in our specific case, we have fixed it as the minimal value guaranteeing that network dynamics* (2) *is critically stable with, thus, an attractive consensus. This choice has been performed to compare the solution we propose with the most favorable implementation of the algorithm in the literature.*

## 5. Conclusions and perspectives

The centralized consensus protocol proposed in [9] has been generalized to assign the convergence rate to consensus arbitrarily fast, independently on the size of the network. Then, a distributed implementation of such an algorithm is proposed. It is based on a multi-rate forward computation over an arbitrary number of consensus steps. Future works include the extension of this protocol to deal with multi-consensus of heterogeneous networks in discrete time in presence of delays too [18]. Work is progressing to allow each agent to fix a different value of the weighting parameter in (8a) possibly adopting adaptive control strategies.

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