# A STOCHASTIC PARTICLE SYSTEM APPROXIMATING THE BGK EQUATION 

Paolo Buttà and Mario Pulvirenti<br>Dipartimento di Matematica, Sapienza Università di Roma

P.le Aldo Moro 5, 00185 Roma, Italy


#### Abstract

We consider a stochastic $N$-particle system on a torus in which each particle moving freely can instantaneously thermalize according to the particle configuration at that instant. Following [2], we show that the propagation of chaos does hold and that the one-particle distribution converges to the solution of the BGK equation. The improvement with respect to [2] consists in the fact that here, as suggested by physical considerations, the thermalizing transition is driven only by the restriction of the particle configuration in a small neighborhood of the jumping particle. In other words, the Maxwellian distribution of the outgoing particle is computed via the empirical hydrodynamical fields associated to the fraction of particles sufficiently close to the test particle and not, as in [2], via the whole particle configuration.


## 1. Introduction

Consider the following kinetic equation for the one-particle distribution function $f(x, v, t)$,

$$
\begin{equation*}
\left(\partial_{t} f+v \cdot \nabla_{x} f\right)(x, v, t)=\lambda(\varrho)\left(\varrho(x, t) M_{f}(x, v, t)-f(x, v, t)\right), \tag{1.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& M_{f}(x, v, t)=\frac{1}{(2 \pi T(x, t))^{3 / 2}} \exp \left(-\frac{|v-u(x, t)|^{2}}{2 T(x, t)}\right), \\
& \varrho(x, t)=\int \mathrm{d} v f(x, v, t), \quad \varrho u(x, t)=\int \mathrm{d} v f(x, v, t) v, \\
& \varrho\left(|u|^{2}+3 T\right)(x, t)=\int \mathrm{d} v f(x, v, t)|v|^{2},
\end{aligned}
$$

and $\lambda$ is a suitable positive function of the spatial density. Here, $(x, v)$ denotes position and velocity of the particle, respectively, and $t>0$ is the time.

The evolution equation (1.1) describes the behavior of a particle moving freely. In addition, the particle thermalizes instantaneously at a random time of intensity $\lambda>0$. The Maxwellian $M_{f}$ has mean velocity and temperature given by $f$ itself.

Such a model was introduced by P.L. Bhatnagar, E.P. Gross, and M. Krook in [1] to deal with situations when the mean free path of the particle system is very small, but the hydrodynamical regime is not yet appropriate.

In the original paper [1] the authors consider $\lambda(\varrho)=\varrho$. This choice, even if natural, presents serious problems from the mathematical side. Indeed, up to now there exists a constructive existence and uniqueness result only when $\lambda=1[7]$,

[^0]which is the case treated here, or when $\lambda$ is a smooth bounded function, case in which the analysis of [7] can be easily extended.

Clearly, the BGK model must preserve mass, momentum, energy, and satisfy the $H$-Theorem. It also exhibits the usual hydrodynamic behavior whenever $\lambda \rightarrow \infty$. The interest of the model is related to the fact that the instantaneous thermalization described by (1.1) is much easier to compute compared to a huge amount of collisions which however produces the same effect at the end.

We now try to present an heuristic derivation of the BGK equation, which is indeed not a simple arbitrary toy model but it is based on reasonable physical arguments.

The starting point is the usual Boltzmann equation,

$$
\begin{equation*}
\left(\partial_{t}+v \cdot \nabla_{x}\right) f=\frac{1}{\varepsilon} Q(f, f) \tag{1.2}
\end{equation*}
$$

where $Q$ is the collision operator, which we do not make explicit here, and $\varepsilon$ is a very small scale parameter. Fixed $t>0$, we represent the solution $f(t)$ to (1.2) with initial datum $f_{0}$ in terms of the Trotter product formula: letting $n=\lfloor t / \tau\rfloor$ with $\tau>0$ very small,

$$
\begin{equation*}
f(t) \approx f(n \tau) \approx\left(S_{0}(\tau) S_{h}(\tau)\right)^{n} f_{0} \tag{1.3}
\end{equation*}
$$

where $S_{0}(t) f(x, v)=f(x-v t, v)$ is the free stream operator and $S_{h}(t) f_{0}$ is the solution to the homogeneous Boltzmann equation

$$
\varepsilon \partial_{t} f=Q(f, f)
$$

with initial datum $f_{0}$.
By virtue of the well known properties of the homogeneous Boltzmann equation,

$$
\lim _{\varepsilon \rightarrow 0} S_{0}(\tau) f=\varrho M_{f},
$$

so that we can replace in Eq. (1.3) the term $S_{h}(\tau)$ by the transition probability $P$ such that $f \rightarrow \varrho M_{f}$ with probability $\tau<1$ and $f \rightarrow f$ with probability $1-\tau$. Thus

$$
f(n \tau) \approx\left(S_{0}(\tau) P\right)^{n} f_{0}
$$

i.e.,

$$
\begin{align*}
f(n \tau) & \approx S_{0}(\tau)\left(S_{0}(\tau) P\right)^{n-1} f_{0}+S_{0}(\tau)(P-1)\left(S_{0}(\tau) P\right)^{n-1} f_{0} \\
& =\cdots \cdots \cdots \\
& =S_{0}(n \tau) f_{0}+\sum_{k=1}^{n} S_{0}(k \tau)(P-1)\left(S_{0}(\tau) P\right)^{n-k} f_{0} \\
& \approx S_{0}(n \tau) f_{0}+\sum_{k=1}^{n} S_{0}(k \tau)(P-1) f[(n-k) \tau] \tag{1.4}
\end{align*}
$$

But

$$
\frac{1}{\tau}(P-1) f=\varrho M_{f}-f
$$

thus in the limit $\tau \rightarrow 0$ Eq. (1.4) reads

$$
f(t)=S_{0}(t) f_{0}+\int_{0}^{t} \mathrm{~d} s S_{0}(s)\left(\varrho M_{f}-f\right)(t-s)
$$

or, equivalently, (1.1) with $\lambda=1$.

We assume $\lambda=1$ however, when the density is high, the transition probability of the event $f \rightarrow \varrho M_{f}$ should increase so that we can recover, by the above heuristic argument, the BGK equation with $\lambda=\varrho$ as well, as originally proposed in [1].

The heuristic argument leading to the BGK picture starts from the Boltzmann equation and applies in situations close to the hydrodynamical regime, but still with a finite mean free path. Thus, this equation does not seem to be consequence of a scaling limit for which one obtains either the Boltzmann equation in the low density (or Boltzmann-Grad) limit, or the hydrodynamical equations in a mere space-time scaling.

In the present paper we introduce a stochastic system of $N$ interacting particles on a $d$-dimensional torus $(d=2,3)$, and prove the convergence of the one-particle law to the corresponding solution to the BGK equation in the limit $N \rightarrow \infty$. In doing this, we follow a previous similar approach developed in [2]. The improvements we present here are related to the following aspect. In the particle model we introduce a function $x \rightarrow \varphi(x)$ ( $x$ is the space variable) which we use to compute the local empirical hydrodynamical fields,

$$
\begin{aligned}
& \varrho(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-x_{j}\right), \quad \varrho u(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-x_{j}\right) v_{j} \\
& \varrho\left(|u|^{2}+3 T\right)(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-x_{j}\right) v_{j}^{2}
\end{aligned}
$$

where $\left(x_{1} \cdots x_{N} ; v_{1} \cdots v_{N}\right)$ are the sequence of positions and velocities of the particle configuration. In [2] the function $\varphi$ was assumed to be strictly positive to avoid divergences due to the possibility of a local vacuum. This is physically not very reasonable because long-range effects should not play any role in the jump mechanism of a tagged particle. Here, we remove such an hypothesis allowing $\varphi$ to be compactly supported. In other words, $\varphi$ can be thought as a smoothed version of the characteristic function of a small ball.

We remark that the analysis of the present paper is, in a sense, equivalent to the introduction of the stochastic particle system which is the inhomogeneous version of the well known Kac's model [8, 3], and it is the conceptual basis of the usual Monte Carlo Direct Simulation Method (Bird's scheme), see, e.g., [4], to approximate the solutions of the usual Boltzmann equation.

Here, we work in a canonical context, i.e., the number of particles $N$ is fixed. In [5] the authors derive a linear version of the homogeneous BGK equation, starting from a suitable two species particle system in the microcanonical setting. Namely, the energy of the system they consider is also fixed. This is more related to the spirit of the original Kac's model.

The plan of the paper is the following. We first fix the cutoff function $\varphi$ and a regularized version of the BGK equation (see Eq. (2.14) below) in which the hydrodynamical fields are smeared. After fixing notation and establishing the main result in Section 2, we introduce in Section 3 a coupling between the particle system and $N$ independent copies of a one-particle stochastic process associated to the regularized BGK equation, which is the basic tool for the proof of convergence. Sections 4 and 5 contain the preliminary lemmas and the key result on the proximity of the particle system to the regularized equation. In Section 6 we remove the cutoff as in [2]. But here we choose a different method, by working not on the equations but on the processes, again with a coupling technique. As matter of facts, the
convergence of the laws is obtained in the 2-Wasserstein distance, hence weaker with respect to the result in [2] holding in a weighted $L^{1}$ space, but in addition here we also prove the convergence of the processes, and this is closer to the spirit of the present analysis. The final result follows by a diagonal limit.

## 2. Notation and results

For $d=2,3$ we let

- $\mathbb{T}^{d}=\left(\mathbb{R} /\left(\frac{1}{2}+\mathbb{Z}\right)\right)^{d}$ be the $d$-dimensional torus of side length one.
- $\varphi \in C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}_{+}\right)$be a smearing function such that ${ }^{1}$

$$
\begin{align*}
& \varphi_{0}:=\varphi(0)=\max \varphi \geq 1, \quad \varphi(x)=0 \text { for }|x| \geq \frac{1}{2} \\
& \varphi(x)=\varphi(-x), \quad \int \mathrm{d} y \varphi(y)=1 \tag{2.1}
\end{align*}
$$

where for $x \in \mathbb{T}^{d}$ we denote by $|x|$ its distance from 0 (on the torus).

- $M_{u, T}=M_{u, T}(v), v \in \mathbb{R}^{d}$, be the normalized Maxwellian density of mean velocity $u \in \mathbb{R}^{d}$ and temperature $T$, i.e.,

$$
\begin{equation*}
M_{u, T}(v)=\frac{1}{(2 \pi T)^{d / 2}} \exp \left(-\frac{|v-u|^{2}}{2 T}\right) . \tag{2.2}
\end{equation*}
$$

Note that

$$
u=\int \mathrm{d} v M_{u, T}(v) v, \quad T=\frac{1}{d} \int \mathrm{~d} v M_{u, T}(v)|v-u|^{2} .
$$

2.1. The BGK equation. We denote by $f=f(t)=f(x, v, t)$, where $(x, v) \in$ $\mathbb{T}^{d} \times \mathbb{R}^{d}$ and $t \in \mathbb{R}_{+}$is the time, the solution to the BGK equation,

$$
\begin{equation*}
\partial_{t} f+v \cdot \nabla_{x} f=\varrho_{f} M_{f}-f \tag{2.3}
\end{equation*}
$$

where $\varrho_{f}=\varrho_{f}(x, t)$ is the local density defined by

$$
\begin{equation*}
\varrho_{f}(x, t)=\int \mathrm{d} v f(x, v, t) \tag{2.4}
\end{equation*}
$$

while $M_{f}=M_{f}(x, v, t)$ is the (local) Maxwellian given by

$$
\begin{equation*}
M_{f}(x, v, t)=M_{u_{f}(x, t), T_{f}(x, t)}(v), \tag{2.5}
\end{equation*}
$$

with $u_{f}=u_{f}(x, t)$ and $T_{f}=T_{f}(x, t)$ the local velocity and temperature,

$$
\begin{align*}
\varrho_{f}(x, t) u_{f}(x, t) & =\int \mathrm{d} v f(x, v, t) v  \tag{2.6}\\
\varrho_{f}(x, t) T_{f}(x, t) & =\frac{1}{d} \int \mathrm{~d} v f(x, v, t)\left|v-u_{f}(x, t)\right|^{2} \tag{2.7}
\end{align*}
$$

Well-posedness of the BGK equation together with $L^{\infty}$ estimates for the hydrodynamical fields can be found in [7]. In particular, we consider as initial condition a probability density $f_{0}$ on $\mathbb{T}^{d} \times \mathbb{R}^{d}$ such that there are a function $a \in C\left(\mathbb{R}^{d}\right)$ and positive constants $C_{1}, \alpha>0$ such that

$$
\begin{align*}
& a(v) \leq f_{0}(x, v) \leq C_{1} \mathrm{e}^{-\alpha|v|^{2}} \quad \forall(x, v) \in \mathbb{T}^{d} \times \mathbb{R}^{d} \\
& a \geq 0, \quad C_{2}:=\int \mathrm{d} v a(v)>0 \tag{2.8}
\end{align*}
$$

[^1]Therefore, from [7, Theorem 3.1] the following proposition follows.
Proposition 2.1. There exists a mild solution $f=f(t)=f(x, v, t)$ to Eq. (2.3) with initial condition $f(x, v, 0)=f_{0}(x, v)$ satisfying Eq. (2.8). ${ }^{2}$ Moreover, there are a non-decreasing finite function $t \mapsto K_{q, t}=K_{q, t}\left(f_{0}\right), q \in \mathbb{N}$, and a non-increasing positive function $t \mapsto A_{t}=A_{t}\left(f_{0}\right)$ such that, for any $(x, t) \in \mathbb{T}^{d} \times \mathbb{R}_{+}$,

$$
\begin{align*}
& \left|u_{f}(x, t)\right|+T_{f}(x, t)+\mathcal{N}_{q}(f(t)) \leq K_{q, t}  \tag{2.9}\\
& \varrho_{f}(x, t) \geq C_{2} \mathrm{e}^{-t}, \quad T_{f}(x, t) \geq A_{t} \tag{2.10}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{N}_{q}(f):=\sup _{(x, v) \in \mathbb{T}^{d} \times \mathbb{R}^{d}} f(x, v)\left(1+|v|^{q}\right) \tag{2.11}
\end{equation*}
$$

Finally, the above solution is unique in the class of functions $f=f(t)=f(x, v, t)$ such that, for some $q>d+2$, $\sup _{t \leq \tau} \mathcal{N}_{q}(f(t))<+\infty$ for any $\tau>0$.
2.2. The stochastic particle system. We consider a system of $N$ particles confined in the torus $\mathbb{T}^{d}$. We denote by $Z_{N}=\left(X_{N}, V_{N}\right)$ the state of the system, where $X_{N} \in\left(\mathbb{T}^{d}\right)^{N}$ and $V_{N} \in\left(\mathbb{R}^{d}\right)^{N}$ are the positions and velocities of the particles, respectively. The particles move randomly, governed by the stochastic dynamics defined as below.

Setting $X_{N}=\left(x_{1}, \ldots, x_{N}\right)$ and $V_{N}=\left(v_{1}, \ldots, v_{N}\right)$, we introduce the smeared empirical hydrodynamical fields $\varrho_{N}^{\varphi}$, $u_{N}^{\varphi}$, and $T_{N}^{\varphi}$ (depending on $Z_{N}$ ) defined by

$$
\begin{align*}
& \varrho_{N}^{\varphi}(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-x_{j}\right), \quad \varrho_{N}^{\varphi} u_{N}^{\varphi}(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-x_{j}\right) v_{j}  \tag{2.12}\\
& \varrho_{N}^{\varphi} T_{N}^{\varphi}(x)=\frac{1}{N d} \sum_{j=1}^{N} \varphi\left(x-x_{j}\right)\left|v_{j}-u_{N}^{\varphi}(x)\right|^{2}
\end{align*}
$$

with $\varphi$ as in Eq. (2.1). The particle system evolves according to the Markovian stochastic dynamics whose generator $\mathcal{L}_{N}$ is defined as

$$
\begin{align*}
\mathcal{L}_{N} G\left(Z_{N}\right)= & {\left[\left(V_{N} \cdot \nabla_{X_{N}}\right) G\right]\left(Z_{N}\right)+\sum_{i=1}^{N} \int \mathrm{~d} \tilde{x}_{i} \varphi\left(\tilde{x}_{i}-x_{i}\right) } \\
& \times\left[\int \mathrm{d} \tilde{v}_{i} M_{Z_{N}}^{\varphi}\left(\tilde{x}_{i}, \tilde{v}_{i}\right) G\left(Z_{N}^{i,\left(\tilde{x}_{i}, \tilde{v}_{i}\right)}\right)-G\left(Z_{N}\right)\right] \tag{2.13}
\end{align*}
$$

Above (for given $\left.(y, w) \in \mathbb{T}^{d} \times \mathbb{R}^{d}\right), Z_{N}^{i,(y, w)}=\left(X_{N}^{i, y}, V_{N}^{i, w}\right)$ is the state obtained from $Z_{N}=\left(X_{N}, V_{N}\right)$ by replacing the position $x_{i}$ and velocity $v_{i}$ of the $i$-th particle by $y$ and $w$ respectively; $G$ is a test function on the state space, and $M_{Z_{N}}^{\varphi}(x, v)$ is the Maxwellian associated to the empirical fields, i.e.,

$$
M_{Z_{N}}^{\varphi}(x, v)=M_{u_{N}(x), T_{N}^{\varphi}(x)}(v)
$$

Otherwise stated, the evolution $Z_{N}(t)=\left(X_{N}(t), V_{N}(t)\right)$ is the Markov process in which at a random exponential time of intensity one the $i$-th particle performs a

[^2]which formally derives from Eq. (2.3) via Duhamel formula.
jump from its state $\left(x_{i}, v_{i}\right)$ to a new one $\left(\tilde{x}_{i}, \tilde{v}_{i}\right)$, extracted according to the distribution $\varphi\left(\cdot-x_{i}\right)$ for the position and then to the empirical Maxwellian $M_{Z_{N}}^{\varphi}\left(\tilde{x}_{i}, \cdot\right)$ for the velocity.

The stochastic evolution is well posed since $\varrho_{N}^{\varphi}\left(\tilde{x}_{i}\right) \geq N^{-1} \varphi\left(\tilde{x}_{i}-x_{i}\right)$, so that the Maxwellian $M_{Z_{N}}^{\varphi}\left(\tilde{x}_{i}, \tilde{v}_{i}\right)$ is well defined when $\varphi\left(\tilde{x}_{i}-x_{i}\right)>0$, and the integration in the right-hand side of Eq. (2.13) makes sense. On the other hand, the smeared hydrodynamical temperature $T_{N}^{\varphi}\left(\tilde{x}_{i}\right)$ may vanish, in which case we replace the Maxwellian $M_{Z_{N}}^{\varphi}\left(\tilde{x}_{i}, \tilde{v}_{i}\right)$ by a Dirac mass in $u_{N}^{\varphi}\left(\tilde{x}_{i}\right)$. In particular, this happens in the special case $\varrho_{N}^{\varphi}\left(\tilde{x}_{i}\right)=N^{-1} \varphi\left(\tilde{x}_{i}-x_{i}\right)$, which implies $u_{N}^{\varphi}\left(\tilde{x}_{i}\right)=v_{i}$, so that $M_{Z_{N}}^{\varphi}\left(\tilde{x}_{i}, \tilde{v}_{i}\right)=\delta\left(\tilde{v}_{i}-v_{i}\right)$ (the velocity does not jump).

In the sequel, we will denote by $F_{N}(t)=F_{N}\left(Z_{N}, t\right)$ the density of the law of $Z_{N}(t)$ (but we will often refer to it as simply the law of the process).
2.3. The regularized BGK equation. The kinetic limit of the particle system introduced in Section 2.2 will be shown to be governed by the following regularized version of Eq. (2.3),

$$
\begin{equation*}
\partial_{t} g+v \cdot \nabla_{x} g=\varrho_{g}^{\varphi} M_{g}^{\varphi}-g \tag{2.14}
\end{equation*}
$$

for the unknown distribution function $g=g(t)=g(x, v, t)$, where $M_{g}^{\varphi}$ is the Maxwellian

$$
\begin{equation*}
M_{g}^{\varphi}(x, v, t)=M_{u_{g}^{\varphi}(x, t), T_{g}^{\varphi}(x, t)}(v), \tag{2.15}
\end{equation*}
$$

and the fields $\varrho_{g}^{\varphi}=\varrho_{g}^{\varphi}(x, t), u_{g}^{\varphi}=u_{g}^{\varphi}(x, t)$, and $T_{g}^{\varphi}=T_{g}^{\varphi}(x, t)$ are given by

$$
\begin{align*}
\varrho_{g}^{\varphi}(x, t) & =\left(\varphi * \varrho_{g}\right)(x, t)=\int \mathrm{d} y \varphi(x-y) \varrho_{g}(y, t)  \tag{2.16}\\
\varrho_{g}^{\varphi}(x, t) u_{g}^{\varphi}(x, t) & =\int \mathrm{d} y \mathrm{~d} v \varphi(x-y) g(y, v, t) v  \tag{2.17}\\
\varrho_{g}^{\varphi}(x, t) T_{g}^{\varphi}(x, t) & =\frac{1}{d} \int \mathrm{~d} y \mathrm{~d} v \varphi(x-y) g(y, v, t)\left|v-u_{g}^{\varphi}(x, t)\right|^{2}, \tag{2.18}
\end{align*}
$$

with

$$
\begin{equation*}
\varrho_{g}(x, t)=\int \mathrm{d} v g(x, v, t) \tag{2.19}
\end{equation*}
$$

The content of Proposition 2.1 extends to the regularized BGK equation, in particular the $L^{\infty}$ estimates do not depend on the smearing function $\varphi$. This is the matter of [2, Proposition 2.2] - which we report below for the convenience of the reader, noticing that it applies also in the present context since the proof does not depend on the assumption (done in [2]) that $\varphi$ is strictly positive.

Proposition 2.2. Let $g=g(t)=g(x, v, t)$ be the solution to Eq. (2.14) with initial condition $g(x, v, 0)=f_{0}(x, v), f_{0}$ as in Proposition 2.1, i.e., satisfying Eq. (2.8). Then, similar estimates hold for the corresponding hydrodynamical fields, namely,

$$
\begin{align*}
& \left|u_{g}^{\varphi}(x, t)\right|+T_{g}^{\varphi}(x, t)+\mathcal{N}_{q}(g(t)) \leq K_{q, t}  \tag{2.20}\\
& \varrho_{g}(x, t) \geq C_{2} \mathrm{e}^{-t}, \quad \varrho_{g}^{\varphi}(x, t) \geq C_{2} \mathrm{e}^{-t}  \tag{2.21}\\
& T_{g}^{\varphi}(x, t) \geq A_{t} \tag{2.22}
\end{align*}
$$

(with $t \mapsto K_{q, t}=K_{q, t}\left(f_{0}\right), q \in \mathbb{N}$ non-decreasing and $t \mapsto A_{t}=A_{t}\left(f_{0}\right)$ nonincreasing, both positive and independent of $\varphi$ ).
2.4. Kinetic limit. We can now state the key result of the paper, concerning the kinetic limit of the stochastic particle system.

Theorem 2.3. Suppose that the law of $Z_{N}(0)$ has density $F_{N}(0)=f_{0}^{\otimes N}$, where $f_{0}$ satisfies the assumptions detailed in Eq. (2.8), and let $g=g(t)=g(x, v, t)$ be the solution to Eq. (2.14) with initial condition $g(0)=f_{0}$.

Let $f_{j}^{N}(t), j \in\{1, \ldots, N\}$, be the $j$-particle marginal distribution function of the (symmetric) law $F_{N}(t)$,i.e.,

$$
f_{j}^{N}\left(x_{1}, \ldots, x_{j}, v_{1}, \ldots, v_{j}, t\right)=\int \mathrm{d} x_{j+1} \cdots \mathrm{~d} x_{N} \mathrm{~d} v_{j+1} \cdots \mathrm{~d} v_{N} F_{N}\left(X_{N}, V_{N}, t\right)
$$

Then, the 2-Wasserstein distance $\mathcal{W}_{2}\left(f_{j}^{N}(t), g(t)^{\otimes j}\right)$ vanishes as $N \rightarrow+\infty$ for any $j \in \mathbb{N}$ and $t \geq 0$. More precisely, for each $T>0$ there exists $L_{T}=L_{T}\left(f_{0}\right)$ such that, for any $j \in\{1, \ldots, N\}$,

$$
\begin{equation*}
\mathcal{W}_{2}\left(f_{j}^{N}(t), g(t)^{\otimes j}\right)^{2} \leq \frac{j L_{T}}{N^{1 / 4}} \exp \left(L_{T} \Gamma_{\varphi}\right) \quad \forall t \in[0, T] \quad \forall N>N_{\varphi} \tag{2.23}
\end{equation*}
$$

where $\Gamma_{\varphi}$ and $N_{\varphi}$ are explicitly computable positive numbers depending solely on the smearing function $\varphi$ (see Eq. (5.25) below). In particular, the one particle marginal distribution function $f_{1}^{N}(t)$ converges weakly to $g(t)$ as $N \rightarrow+\infty$ for any $t \geq 0$.

Remark 2.4. We recall that if $\mu$ and $\nu$ are two probability measures on a metric space $(M, d)$ with finite second moment, the 2 -Wasserstein distance between $\mu$ and $\nu$ is defined as

$$
\mathcal{W}_{2}(\mu, \nu)=\left(\inf _{\gamma \in \mathcal{P}(\mu, \nu)} \int_{M \times M} \mathrm{~d} \gamma\left(x, x^{\prime}\right) d\left(x, x^{\prime}\right)^{2}\right)^{1 / 2}
$$

where $\mathcal{P}(\mu, \nu)$ denotes the collection of all the probability measures on $M \times M$ with marginals $\mu$ and $\nu$. In Theorem 2.3, $M=\left(\mathbb{T}^{d}\right)^{j} \times\left(\mathbb{R}^{d}\right)^{j}$ and $\mathcal{W}_{2}\left(f_{j}^{N}(t), g(t)^{\otimes j}\right)$ denotes the 2-Wasserstein distance between the probability measures with densities $f_{j}^{N}(t)$ and $g(t)^{\otimes j}$, respectively.

The proof of Theorem 2.4 will be presented in Section 5 after some preliminaries in Sections 3 and 4.

The convergence of the particle system to the true BGK equation Eq. (2.3) is now obtained through a rescaling of the smearing function $\varphi$, by setting

$$
\begin{equation*}
\varphi(x)=\varphi^{(\varepsilon)}(x):=\frac{1}{\varepsilon^{d}} \bar{\varphi}\left(\frac{x}{\varepsilon}\right) \tag{2.24}
\end{equation*}
$$

where $\bar{\varphi}$ is fixed (it varies on the scale of order one) and satisfies (2.1). Clearly, in this case $\varphi_{0} \approx \varepsilon^{-d}$ and $\|\nabla \varphi\|_{\infty} \approx \varepsilon^{-d-1}$. We have the following result.

Theorem 2.5. Let $\varphi=\varphi^{(\varepsilon)}$ be as in Eq. (2.24), suppose $f_{0}$ satisfies Eq. (2.8) and in addition that, for some $q>d+2$,

$$
\begin{equation*}
\mathcal{N}_{q}\left(\left|\nabla_{x} f_{0}\right|\right)<+\infty \tag{2.25}
\end{equation*}
$$

Then, for each $T>0$ there exists $C_{T}=C_{T}\left(f_{0}\right)$ such that,

$$
\begin{equation*}
\mathcal{W}_{2}(f(t), g(t))^{2} \leq C_{T} \varepsilon \quad \forall t \in[0, T] \tag{2.26}
\end{equation*}
$$

where $f(t)$ and $g(t)$ are the solutions to Eq. (2.3) and Eq. (2.14) respectively, with same initial condition $f_{0}$.

We are now in position to formulate the main result.

Theorem 2.6. Under the hypotheses of Theorem 2.3 and Theorem 2.5, suppose that $\varepsilon$ vanishes gently when $N$ diverges (for instance $\varepsilon=(\log N)^{-\mu}$ with $\mu$ sufficiently small). Then, for all integer positive $j$,

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \mathcal{W}_{2}\left(f_{j}^{N}(t), f(t)^{\otimes j}\right)=0 \quad \forall t \in[0, T] \tag{2.27}
\end{equation*}
$$

Some comments are in order. Theorem 2.6 is actually a corollary of Theorem 2.3 and Theorem 2.5 via a triangular inequality. The short proof will be presented at the end of Section 6 .

The convergence expressed in Theorem 2.6 is very slow. In particular, choosing $\varepsilon=(\log N)^{-\mu}$ with $\mu$ sufficiently small we obtain

$$
\mathcal{W}_{2}\left(f_{j}^{N}(t), f(t)^{\otimes j}\right)^{2} \leq \mathrm{const} j(\log N)^{-\mu}
$$

The condition on $\varepsilon$ is due to the fact that (when $\left.\varphi=\varphi^{(\varepsilon)}\right)$ Eq. (2.23) holds with $\Gamma_{\varphi} \approx$ $\varepsilon^{-a}$ and $N_{\varphi} \approx \varepsilon^{-b}$ for suitable $a, b>1$, so that the condition $N>N_{\varphi}$ is satisfied and the divergence $\exp \left(C \varepsilon^{-a}\right)$ appearing in the right-hand side is compensated by the term $N^{-1 / 4}$. We did not tried to optimize further our estimates since an effort in this direction would not improve so much the result. A similar feature is also present in [2], where the physically reasonable scaling is discussed in Section 5.

## 3. Reformulation of the problem

Following the strategy developed in [2], we prove Theorem 2.3 by showing that the stochastic particle system is close (as $N \rightarrow+\infty$ ) to an auxiliary process, whose asymptotic as $N \rightarrow \infty$ is obvious.
3.1. Coupling with an independent process. The auxiliary process is denoted by $\Sigma_{N}(t)=\left(Y_{N}(t), W_{N}(t)\right) \in\left(\mathbb{T}^{d}\right)^{N} \times\left(\mathbb{R}^{d}\right)^{N}$ and it is defined according to the following construction.

Let $g=g(t)=g(x, v, t)$ be as in Proposition 2.2. Denote by $(x(t), v(t)) \in \mathbb{T}^{d} \times \mathbb{R}^{d}$ the one-particle jump process whose generator is given by

$$
\begin{equation*}
\mathcal{L}_{1}^{g} \psi(x, v)=\left[\left(v \cdot \nabla_{x}\right) \psi\right](x, v)+\int \mathrm{d} \tilde{x} \varphi(\tilde{x}-x)\left[\int \mathrm{d} \tilde{v} M_{g}^{\varphi}(\tilde{x}, \tilde{v}) \psi(\tilde{x}, \tilde{v})-\psi(x, v)\right] \tag{3.1}
\end{equation*}
$$

where $\psi$ is a test function and $M_{g}^{\varphi}$ is defined in Eq. (2.15). In particular, if the initial distribution has a density, the same holds at any positive time and the probability density of $(x(t), v(t))$ solves the regularized BGK equation (2.14). This kind of process is usually called non-linear since its generator is implicitly defined through the law of the process itself.

The process $\Sigma_{N}(t)$ is then defined by $N$ independent copies of the above process, i.e., as the Markov process on $\left(\mathbb{T}^{d}\right)^{N} \times\left(\mathbb{R}^{d}\right)^{N}$ with generator

$$
\begin{align*}
\mathcal{L}_{N}^{g} G\left(Z_{N}\right)= & {\left[\left(V_{N} \cdot \nabla_{X_{N}}\right) G\right]\left(Z_{N}\right)+\sum_{i=1}^{N} \int \mathrm{~d} \tilde{x}_{i} \varphi\left(\tilde{x}_{i}-x_{i}\right) } \\
& \times\left[\int \mathrm{d} \tilde{v}_{i} M_{g}^{\varphi}\left(\tilde{x}_{i}, \tilde{v}_{i}\right) G\left(Z_{N}^{i,\left(\tilde{x}_{i}, \tilde{v}_{i}\right)}\right)-G\left(Z_{N}\right)\right] . \tag{3.2}
\end{align*}
$$

Note that the only difference with respect to Eq. (2.13) is the replacement of $M_{Z_{N}}^{\varphi}$ by $M_{g}^{\varphi}$.

As in [2], the closeness of $Z_{N}(t)$ and $\Sigma_{N}(t)$ is proved by introducing a suitable coupled process $Q_{N}(t)=\left(Z_{N}(t), \Sigma_{N}(t)\right)$. More precisely, the coupled process is
the Markov process whose generator $\mathcal{L}_{Q}$ is defined in the following way. We let $Z_{N}=\left(X_{N}, V_{N}\right), \Sigma_{N}=\left(Y_{N}, W_{N}\right)$, with $X_{N}=\left(x_{1}, \ldots, x_{N}\right), V_{N}=\left(v_{1}, \ldots, v_{N}\right)$, $Y_{N}=\left(y_{1}, \ldots, y_{N}\right)$, and $W_{N}=\left(w_{1}, \ldots, w_{N}\right)$. Then, for any test function $G=$ $G\left(Z_{N}, \Sigma_{N}\right)$,

$$
\begin{align*}
& \mathcal{L}_{Q} G\left(Z_{N}, \Sigma_{N}\right)=\left[\left(V_{N} \cdot \nabla_{X_{N}}+W_{N} \cdot \nabla_{Y_{N}}\right) G\right]\left(Z_{N}, \Sigma_{N}\right)+\int \mathrm{d} \tilde{x}_{i} \mathrm{~d} \tilde{y}_{i} \Phi_{x_{i}, y_{i}}\left(\tilde{x}_{i}, \tilde{y}_{i}\right) \\
& \quad \times\left[\int \mathrm{d} \tilde{v}_{i} \mathrm{~d} \tilde{w}_{i} \mathcal{M}^{\varphi}\left(\tilde{x}_{i}, \tilde{v}_{i} ; \tilde{y}_{i}, \tilde{w}_{i}\right) G\left(Z_{N}^{i,\left(\tilde{x}_{i}, \tilde{v}_{i}\right)}, \Sigma_{N}^{i,\left(\tilde{y}_{i}, \tilde{w}_{i}\right)}\right)-G\left(Z_{N}, \Sigma_{N}\right)\right] \tag{3.3}
\end{align*}
$$

In Eq. (3.3), for given $\tilde{x}, \tilde{y} \in \mathbb{T}^{d}, \mathcal{M}^{\varphi}(\tilde{x}, v ; \tilde{y}, w)$ is the joint representation of the Maxwellians $M_{Z_{N}}^{\varphi}(\tilde{x}, v)$ and $M_{g}^{\varphi}(\tilde{y}, w)$ that realizes the 2-Wasserstein distance between the marginals, whose square is given by (see, e.g., [6])

$$
\begin{equation*}
\mathcal{W}_{2}\left(M_{Z_{N}}^{\varphi}(x, \cdot), M_{g}^{\varphi}(y, \cdot)\right)^{2}=\left|u_{N}^{\varphi}(x)-u_{g}^{\varphi}(y)\right|^{2}+d\left(\sqrt{T_{N}^{\varphi}(x)}-\sqrt{T_{g}^{\varphi}(y)}\right)^{2} \tag{3.4}
\end{equation*}
$$

While, for given $x, y \in \mathbb{T}^{d}, \Phi_{x, y}(\tilde{x}, \tilde{y})$ is the joint representation of the probability densities $\varphi_{x}(\tilde{x})=\varphi(\tilde{x}-x)$ and $\varphi_{y}(\tilde{y})=\varphi(\tilde{y}-y)$ defined as

$$
\begin{equation*}
\Phi_{x, y}(\tilde{x}, \tilde{y})=\varphi_{x}(\tilde{x}) \delta(\tilde{x}-x-\tilde{y}+y) \tag{3.5}
\end{equation*}
$$

where $\delta(z)$ denotes the Dirac measure on $\mathbb{T}^{d}$ centered in $z=0$. In particular, for any integrable function $J$ on $\mathbb{T}^{d}$,

$$
\begin{equation*}
\int \mathrm{d} \tilde{x} \mathrm{~d} \tilde{y} \Phi_{x, y}(\tilde{x}, \tilde{y}) J(\tilde{x}-\tilde{y})=J(x-y) \tag{3.6}
\end{equation*}
$$

In words, the coupling is given by the Markov process in which at a random exponential time of intensity one, the $i$-th pair of particles makes the jump from $\left(x_{i}, v_{i}, y_{i}, w_{i}\right)$ to $\left(\tilde{x}_{i}, \tilde{v}_{i}, \tilde{y}_{i}, \tilde{w}_{i}\right)=\left(x_{i}+\xi, \tilde{v}_{i}, y_{i}+\xi, \tilde{w}_{i}\right)$, where $\xi$ is distributed according to $\varphi$, and ( $\tilde{v}_{i}, \tilde{w}_{i}$ ) according to the prescribed joint representation of $M_{Z_{N}}^{\varphi}\left(\tilde{x}_{i}, \cdot\right)$ and $M_{g}^{\varphi}\left(\tilde{y}_{i}, \cdot\right)$. We denote by $\mathrm{d} R_{N}(t)=\mathrm{d} R_{N}\left(Z_{N}, \Sigma_{N}, t\right)$ the law of $Q_{N}(t)$ and assume that, initially,

$$
\mathrm{d} R_{N}(0)=\delta\left(X_{N}-Y_{N}\right) \delta\left(V_{N}-W_{N}\right) f_{0}^{\otimes N}\left(X_{N}, V_{N}\right) \mathrm{d} Z_{N} \mathrm{~d} \Sigma_{N}
$$

In particular, recalling the notation introduced in Remark 2.4,

$$
\mathrm{d} R_{N}(t) \in \mathcal{P}\left(F_{N}(t) \mathrm{d} Z_{N}, g(t)^{\otimes N} \mathrm{~d} \Sigma_{N}\right)
$$

3.2. Estimating the distance between the processes. We adopt the same strategy of [2] and introduce the quantity

$$
I_{N}(t):=\int \mathrm{d} R_{N}(t)\left(\left|x_{1}-y_{1}\right|^{2}+\left|v_{1}-w_{1}\right|^{2}\right)
$$

As $\mathrm{d} R_{N}(t)$ is symmetric with respect to particle permutations we have

$$
I_{N}(t)=\frac{1}{j} \int \mathrm{~d} R_{N}(t) \sum_{i=1}^{j}\left(\left|x_{i}-y_{i}\right|^{2}+\left|v_{i}-w_{i}\right|^{2}\right) \quad \forall j \in\{1, \ldots, N\}
$$

so that

$$
\mathcal{W}_{2}\left(f_{j}^{N}(t), g(t)^{\otimes j}\right) \leq \sqrt{j I_{N}(t)} \quad \forall j \in\{1, \ldots, N\}
$$

by the definition of the 2-Wasserstein distance. Therefore, the proof of Theorem 2.3 reduces to show that for each $T>0$ there exists $L_{T}=L_{T}\left(f_{0}\right)$ such that,

$$
\begin{equation*}
I_{N}(t) \leq \frac{L_{T}}{N^{1 / 4}} \exp \left(L_{T} \Gamma_{\varphi}\right) \quad \forall t \in[0, T] \quad \forall N>N_{\varphi} \tag{3.7}
\end{equation*}
$$

for suitable $\Gamma_{\varphi}$ and $N_{\varphi}$. To this end, we compute

$$
\begin{aligned}
\dot{I}_{N}(t)= & \int \mathrm{d} R_{N}(t) \mathcal{L}_{Q}\left(\left|x_{1}-y_{1}\right|^{2}+\left|v_{1}-w_{1}\right|^{2}\right) \\
= & \int \mathrm{d} R_{N}(t)\left(v_{1} \cdot \nabla_{x_{1}}+w_{1} \cdot \nabla_{y_{1}}\right)\left|x_{1}-y_{1}\right|^{2} \\
& -N \int \mathrm{~d} R_{N}(t)\left(\left|x_{1}-y_{1}\right|^{2}+\left|v_{1}-w_{1}\right|^{2}\right) \\
& +\sum_{i=2}^{N} \int \mathrm{~d} R_{N}(t)\left(\left|x_{1}-y_{1}\right|^{2}+\left|v_{1}-w_{1}\right|^{2}\right)+\int \mathrm{d} R_{N}(t)\left|x_{1}-y_{1}\right|^{2} \\
& +\int \mathrm{d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) \int \mathrm{d} \tilde{v}_{1} \mathrm{~d} \tilde{w}_{1} \mathcal{M}^{\varphi}\left(x_{1}+\xi, \tilde{v}_{1} ; y_{1}+\xi, \tilde{w}_{1}\right)\left|\tilde{v}_{1}-\tilde{w}_{1}\right|^{2}
\end{aligned}
$$

where the first two terms in the right-hand side arise from the stream part ( $V_{N}$. $\left.\nabla_{X_{N}} G+W_{N} \cdot \nabla_{Y_{N}} G\right)$ and the loss part $(-N G)$ of the generator $\mathcal{L}_{Q}$, respectively. We note that the loss term is partially compensated by the third term, while the stream part is equal to

$$
2 \int \mathrm{~d} R_{N}(t)\left(v_{1}-w_{1}\right) \cdot\left(x_{1}-y_{1}\right) \leq \int \mathrm{d} R_{N}(t)\left(\left|x_{1}-y_{1}\right|^{2}+\left|v_{1}-w_{1}\right|^{2}\right)
$$

where, with an abuse of notation, in the left-hand side we denote by $\left(x_{1}-y_{1}\right)$ a vector $\eta \in \mathbb{R}^{d}$ in the equivalence class defined by $x_{1}-y_{1} \in \mathbb{T}^{d}=\left(\mathbb{R} /\left(\frac{1}{2}+\mathbb{Z}\right)\right)^{d}$ with $|\eta|=\left|x_{1}-y_{1}\right|$ and, when not uniquely determined by these conditions, with the minimum value of $\left(v_{1}-w_{1}\right) \cdot \eta$ (however, this is an event of vanishing measure and will not play any role in the sequel). Finally, the last term is given by Eq. (3.4). Therefore

$$
\begin{equation*}
\dot{I}_{N}(t) \leq I_{N}(t)+\int \mathrm{d} R_{N}(t) D\left(Z_{N}, \Sigma_{N}\right) \tag{3.8}
\end{equation*}
$$

with

$$
\begin{align*}
D\left(Z_{N}, \Sigma_{N}\right)= & \int \mathrm{d} \xi \varphi(\xi)\left|u_{N}^{\varphi}\left(x_{1}+\xi\right)-u_{g}^{\varphi}\left(y_{1}+\xi\right)\right|^{2} \\
& +\int \mathrm{d} \xi \varphi(\xi) d\left(\sqrt{T_{N}^{\varphi}\left(x_{1}+\xi\right)}-\sqrt{T_{g}^{\varphi}\left(y_{1}+\xi\right)}\right)^{2} \tag{3.9}
\end{align*}
$$

Our goal is to estimate $\int \mathrm{d} R_{N}(t) D\left(Z_{N}, \Sigma_{N}\right)$ from above with a constant (independent of $N$ ) multiple of $I_{N}(t)$ plus a small term of order $1 / N^{1 / 4}$, so that Eq. (3.7) follows from Grönwall's inequality.

As noticed in [2], in estimating $D\left(Z_{N}, \Sigma_{N}\right)$ it is useful to replace $\varrho_{g}^{\varphi}, u_{g}^{\varphi}, T_{g}^{\varphi}$ by

$$
\begin{align*}
& \tilde{\varrho}_{N}^{\varphi}(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-y_{j}\right), \quad \tilde{\varrho}_{N}^{\varphi} \tilde{u}_{N}^{\varphi}(x)=\frac{1}{N} \sum_{j=1}^{N} \varphi\left(x-y_{j}\right) w_{j} \\
& \tilde{\varrho}_{N}^{\varphi} \tilde{T}_{N}^{\varphi}(x)=\frac{1}{N d} \sum_{j=1}^{N} \varphi\left(x-y_{j}\right)\left|w_{j}-\tilde{u}_{N}^{\varphi}(x)\right|^{2} \tag{3.10}
\end{align*}
$$

i.e., the empirical fields constructed via the variables $Y_{N}=\left(y_{1}, \ldots, y_{N}\right)$ and $W_{N}=$ $\left(w_{1}, \ldots, w_{N}\right)$, distributed independently according to $g(t)^{\otimes N}$. By the law of large numbers, this replacement is expected to be small for large $N$.

In the present case, the function $\varphi$ has compact support, so that there are particle configurations for which the smeared empirical densities defined in Eqs. (2.12) and
(3.10) assume very small values (order $1 / N$ ). This makes impossible to obtain (as in [2]) a point-wise estimate of $D\left(Z_{N}, \Sigma_{N}\right)$. To overcome this difficulty, we decompose the phase space as the union of a "good set" $\mathcal{G}$, which will be defined in the next section, and its complement, the "bad set" $\mathcal{G}^{\complement}$. Roughly speaking, in the set $\mathcal{G}$, $D\left(Z_{N}, \Sigma_{N}\right)$ can be controlled similarly to what done in [2], while the contribution to $\dot{I}_{N}(t)$ coming from the integration on $\mathcal{G}^{c}$ will be treated by suitable probability estimates (actually, the decomposition of $\dot{I}_{N}(t)$ is more involved, as explained at the beginning of Section 5).
A notation warning. In what follows, we shall denote by $C$ a generic positive constant whose numerical value may change from line to line and it may possibly depend on the fixed time $T$ and the initial condition $f_{0}$.

Furthermore, we will use both the notations $\mathbb{1}_{B}$ and $\mathbb{I}(B)$ to denote the characteristic function of the set $B$. We shall also use the shorten notation $\mathrm{d} g(t)^{\otimes N}$ to denote integration with respect to $\mathrm{d} \Sigma_{N} g(t)^{\otimes N}$.

## 4. Preliminary estimates

Recalling the assumptions Eq. (2.1) on $\varphi$, we fix $r \in\left(0, \frac{1}{10}\right)$, with $r^{-1} \in \mathbb{N}$ and such that

$$
\begin{equation*}
\varphi(x)>\frac{\varphi_{0}}{2} \quad \forall x \in[-5 r, 5 r]^{d} \tag{4.1}
\end{equation*}
$$

Denote by $\{\Delta\}$ a partition of $\mathbb{T}^{d}$ into square boxes of side $r$. As a consequence, we have the following lower bound on the empirical densities,

$$
\begin{equation*}
N \tilde{\varrho}_{N}^{\varphi}(x) \geq \frac{\varphi_{0}}{2} N_{\Delta}^{Y} \quad \text { if } x \in \Delta \tag{4.2}
\end{equation*}
$$

where $N_{\Delta}^{Y}$ is the number of particles of the configuration $Y_{N}$ contained in $\Delta$.
Lemma 4.1. Given $T>0$ there is $A>0$ (depending only on $T$ and initial condition $f_{0}$ ) such that if

$$
\begin{equation*}
\mathcal{B}_{A}:=\left\{\left(Z_{N}, \Sigma_{N}\right): \tilde{\varrho}_{N}^{\varphi}(x)>A r^{d} \varphi_{0} \forall x \in \mathbb{T}^{d}\right\} \tag{4.3}
\end{equation*}
$$

then

$$
\begin{equation*}
\int \mathrm{d} R_{N}(t) \mathbb{I}_{\mathcal{B}_{A}^{\complement}} \leq \frac{C}{r^{3 d} N} \quad \forall t \in[0, T] . \tag{4.4}
\end{equation*}
$$

Proof. By Eq. (4.2),

$$
\begin{aligned}
\int \mathrm{d} R_{N}(t) \mathbb{I}_{\mathcal{B}_{A}^{0}} & =\int \mathrm{d} g(t)^{\otimes N} \mathbb{I}\left(\left\{Y_{N}: \exists x \in \mathbb{T}^{d} \text { s.t. } \tilde{\varrho}_{N}^{\varphi}(x) \leq A r^{d} \varphi_{0}\right\}\right) \\
& \leq \int \mathrm{d} g(t)^{\otimes N} \mathbb{I}\left(\left\{Y_{N}: \exists \Delta \text { s.t. } N_{\Delta}^{Y} \leq 2 A r^{d} N\right\}\right) \\
& \leq \sum_{\{\Delta\}} \int \mathrm{d} g(t)^{\otimes N} \mathbb{I}_{N_{\Delta}^{Y} \leq 2 A r^{d} N} \leq \frac{1}{r^{d}} \max _{\Delta} \int \mathrm{d} g(t)^{\otimes N} \mathbb{I}_{N_{\Delta}^{Y} \leq 2 A r^{d} N} .
\end{aligned}
$$

We observe that $N_{\Delta}^{Y}=N \xi_{N}$ with $\xi_{N}=N^{-1} \sum_{j=1}^{N} \mathbb{I}_{y_{j} \in \Delta}$ the arithmetic mean of $N$ i.i.d. random variables whose common expected value is

$$
\mathbb{E} \xi_{N}=\mathbb{E} \mathbb{I}_{y_{1} \in \Delta}=\int_{\Delta} \mathrm{d} y \varrho_{g}(x, t) \geq C_{2} \mathrm{e}^{-T} r^{d} \quad \forall t \in[0, T]
$$

having used Eq. (2.21) in the last inequality. We then choose $A=C_{2} \mathrm{e}^{-T} / 4$, whence

$$
\mathbb{I}_{N_{\Delta}^{Y} \leq 2 A r^{d} N}=\mathbb{1}_{\xi_{N} \leq 2 A r^{d}} \leq \mathbb{1}_{\left|\xi_{N}-\mathbb{E} \xi_{N}\right| \geq \mathbb{E} \xi_{N} / 2} \leq \mathbb{I}_{\left|\xi_{N}-\mathbb{E} \xi_{N}\right| \geq C_{2} \mathrm{e}^{-T_{r} r^{d} N / 4}}
$$

Therefore, by Chebyshev's inequality,

$$
\int \mathrm{d} \Sigma_{N} g(t)^{\otimes N} \mathbb{I}_{N_{\Delta}^{Y} \leq 2 A r^{d} N} \leq \frac{16 \mathrm{e}^{2 T}}{C_{2}^{2} r^{2 d} N} \mathbb{E}\left(\mathbb{I}_{y_{1} \in \Delta}-\mathbb{E} \mathbb{I}_{y_{1} \in \Delta}\right)^{2} \leq \frac{C}{r^{2 d} N}
$$

Eq. (4.4) is thus proved.
Lemma 4.2 (The good set). Given $A>0$ as in Lemma 4.1, we let

$$
\begin{equation*}
A_{\varphi}=\frac{A r^{d} \varphi_{0}}{2\|\nabla \varphi\|_{\infty}} \tag{4.5}
\end{equation*}
$$

and define

$$
\begin{equation*}
\mathcal{G}:=\mathcal{G}_{1} \cap \mathcal{B}_{A} \text { with } \mathcal{G}_{1}:=\left\{\left(Z_{N}, \Sigma_{N}\right): \frac{1}{N} \sum_{j=1}^{N}\left|x_{j}-y_{j}\right| \leq A_{\varphi}\right\} \tag{4.6}
\end{equation*}
$$

Then

$$
\begin{equation*}
\varrho_{N}^{\varphi}(x)>\frac{A r^{d} \varphi_{0}}{2}, \quad \tilde{\varrho}_{N}^{\varphi}(x)>A r^{d} \varphi_{0} \quad \forall x \in \mathbb{T}^{d} \quad \text { in the set } \mathcal{G} . \tag{4.7}
\end{equation*}
$$

Proof. The lower bound on $\tilde{\varrho}_{N}^{\varphi}(x)$ follows trivially from the definition of $\mathcal{B}_{A}$. Concerning the other bound, in the set $\mathcal{G}$ we have

$$
\begin{aligned}
\varrho_{N}^{\varphi}(x) & \geq \tilde{\varrho}_{N}^{\varphi}(x)-\left|\varrho_{N}^{\varphi}(x)-\tilde{\varrho}_{N}^{\varphi}(x)\right| \geq A r^{d} \varphi_{0}-\frac{\|\nabla \varphi\|_{\infty}}{N} \sum_{j=1}^{N}\left|x_{j}-y_{j}\right| \\
& \geq A r^{d} \varphi_{0}-\|\nabla \varphi\|_{\infty} A_{\varphi}=\frac{A r^{d} \varphi_{0}}{2}
\end{aligned}
$$

and the lemma is proved.
Lemma 4.3. Define

$$
\begin{align*}
& p_{i, j}=p_{i, j}(\xi):=\frac{\varphi\left(x_{i}+\xi-x_{j}\right)}{\sum_{k} \varphi\left(x_{i}+\xi-x_{k}\right)}=\frac{\varphi\left(x_{i}+\xi-x_{j}\right)}{N \varrho_{N}^{\varphi}\left(x_{i}+\xi\right)}  \tag{4.8}\\
& q_{i, j}=q_{i, j}(\xi):=\frac{\varphi\left(y_{i}+\xi-y_{j}\right)}{\sum_{k} \varphi\left(y_{i}+\xi-y_{k}\right)}=\frac{\tilde{\varphi}\left(y_{i}+\xi-y_{j}\right)}{N \varrho_{N}^{\varphi}\left(y_{i}+\xi\right)} \tag{4.9}
\end{align*}
$$

Then, recalling $\varphi_{0}=\max \varphi$,

$$
\begin{align*}
& \sum_{j=1}^{N} p_{i, j}=1, \quad \int \mathrm{~d} \xi \varphi(\xi) \sum_{i=1}^{N} p_{i, j} \leq \varphi_{0}  \tag{4.10}\\
& \sum_{j=1}^{N} q_{i, j}=1, \quad \int \mathrm{~d} \xi \varphi(\xi) \sum_{i=1}^{N} q_{i, j} \leq \varphi_{0} \tag{4.11}
\end{align*}
$$

Proof. The proofs of Eq. (4.10) and (4.11) are the same, let us consider the first one. The normalization property $\sum_{j=1}^{N} p_{i, j}=1$ is obvious, while (with the change of variable $\xi^{\prime}=x_{i}+\xi$ )

$$
\begin{aligned}
\int \mathrm{d} \xi \varphi(\xi) \sum_{i=1}^{N} p_{i, j} & \leq \varphi_{0} \int \mathrm{~d} \xi \varphi(\xi) \sum_{i=1}^{N} \frac{1}{N \varrho_{N}^{\varphi}\left(x_{i}+\xi\right)} \\
& =\varphi_{0} \sum_{i=1}^{N} \int \mathrm{~d} \xi^{\prime} \frac{\varphi\left(\xi^{\prime}-x_{i}\right)}{N \varrho_{N}^{\varphi}\left(\xi^{\prime}\right)}=\varphi_{0} \int \mathrm{~d} \xi^{\prime} \frac{N \varrho_{N}^{\varphi}\left(\xi^{\prime}\right)}{N \varrho_{N}^{\varphi}\left(\xi^{\prime}\right)}=\varphi_{0}
\end{aligned}
$$

(recall the volume of the torus $\mathbb{T}^{d}$ is one).

Lemma 4.4. Given $T>0$, for each $p \in \mathbb{N}$ there is $M$ (depending only on $T, p$, and initial condition $f_{0}$ ) such that the following holds.
(1) For any $j=1, \ldots, N$ we have

$$
\begin{equation*}
\int \mathrm{d} R_{N}(t)\left|w_{j}\right|^{p}=\int \mathrm{d} g^{\otimes N}(t)\left|w_{j}\right|^{p}=\int \mathrm{d} g(t)|w|^{p} \leq \frac{M}{2} \quad \forall t \in[0, T] \tag{4.12}
\end{equation*}
$$

(2) If

$$
\begin{equation*}
\mathcal{G}_{M, p}:=\left\{\left(Z_{N}, \Sigma_{N}\right): \frac{1}{N} \sum_{j=1}^{N}\left|w_{j}\right|^{p} \leq M\right\} \tag{4.13}
\end{equation*}
$$

then

$$
\begin{equation*}
\int \mathrm{d} R_{N}(t) \mathbb{I}_{\mathcal{G}_{M, p}^{\mathrm{C}}} \leq \frac{C}{N} \quad \forall t \in[0, T] \tag{4.14}
\end{equation*}
$$

Proof. From the estimate on $\mathcal{N}_{q}(g)$ in Eq. (2.20), there is $M=M\left(T, p, f_{0}\right)$ such that $\int \mathrm{d} y \mathrm{~d} w g(y, w, t)|w|^{p} \leq M / 2$ for any $t \in[0, T]$, which proves Eq. (4.12). Moreover, letting $\xi_{N}=\frac{1}{N} \sum_{j}\left|w_{j}\right|^{p}$ and $\mathbb{E}\left(\xi_{N}\right)=\int \mathrm{d} g(t)^{\otimes N} \xi_{N}$, we have

$$
\int \mathrm{d} R_{N}(t) \mathbb{I}_{\mathcal{G}_{M, p}^{\mathrm{c}}}=\int_{\sum_{j}\left|w_{j}\right|^{p}>M N} \mathrm{~d} g(t)^{\otimes N} \leq \int_{\left|\xi_{N}-\mathbb{E}\left(\xi_{N}\right)\right|>M / 2} \mathrm{~d} g(t)^{\otimes N}
$$

whence Eq. (4.14) follows from the law of large numbers (i.e., Chebyshev's inequality).

## 5. Proofs

We deduce an upper bound for the quantity $D\left(Z_{N}, \Sigma_{N}\right)$ introduced in Eq. (3.9), which is the sum of several terms. To estimate the expectation of some of them, a separate analysis on the good set and its complement will be necessary. To this purpose, we first introduce the "mixed temperature"

$$
\bar{T}_{N}^{\varphi}\left(x_{1}+\xi, y_{1}+\xi\right)=\frac{1}{d} \sum_{j=1}^{N} p_{1, j}\left|w_{j}-\tilde{u}_{N}^{\varphi}\left(y_{1}+\xi\right)\right|^{2}
$$

To simplify the notation, in what follows we will omit sometimes the explicit dependence on $x_{1}+\xi$ and $y_{1}+\xi$. By virtue of Eq. (3.9) we have

$$
\begin{align*}
D\left(Z_{N}, \Sigma_{N}\right) \leq & \int \mathrm{d} \xi \varphi(\xi)\left(2\left|u_{N}^{\varphi}-\tilde{u}_{N}^{\varphi}\right|^{2}+2\left|\tilde{u}_{N}^{\varphi}-u_{g}^{\varphi}\right|^{2}\right) \\
& +\int \mathrm{d} \xi \varphi(\xi)\left[2 d\left(\sqrt{T_{N}^{\varphi}}-\sqrt{\bar{T}_{N}^{\varphi}}\right)^{2}+2 d\left(\sqrt{\bar{T}_{N}^{\varphi}}-\sqrt{T_{g}^{\varphi}}\right)^{2}\right] \tag{5.1}
\end{align*}
$$

where $\tilde{u}_{N}^{\varphi}$ is defined in Eq. (3.10). Recalling the definitions Eqs. (4.8) and (4.9), from Cauchy-Schwarz inequality,

$$
\begin{equation*}
\left|u_{N}^{\varphi}-\tilde{u}_{N}^{\varphi}\right|^{2} \leq 2 \mathcal{V}+2\left(\sum_{j=1}^{N}\left|p_{1, j}-q_{1, j}\right|\left|w_{j}\right|\right)^{2} \tag{5.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{V}=\mathcal{V}\left(\xi, Z_{N}, \Sigma_{N}\right):=\sum_{j=1}^{N} p_{1, j}\left|v_{j}-w_{j}\right|^{2} \tag{5.3}
\end{equation*}
$$

Concerning the difference between the empirical and mixed temperature, we observe that

$$
\begin{aligned}
\left|T_{N}^{\varphi}-\bar{T}_{N}^{\varphi}\right| \leq & \left.\frac{1}{d} \sum_{j=1}^{N} p_{1, j}| | v_{j}-\left.u_{N}^{\varphi}\right|^{2}-\left|w_{j}-\tilde{u}_{N}^{\varphi}\right|^{2} \right\rvert\, \\
= & \sum_{j=1}^{N} p_{1, j}\left|\left(v_{j}-u_{N}^{\varphi}-w_{j}+\tilde{u}_{N}^{\varphi}\right) \cdot\left(v_{j}-u_{N}^{\varphi}+w_{j}-\tilde{u}_{N}^{\varphi}\right)\right| \\
\leq & \frac{1}{d} \sum_{j=1}^{N} p_{1, j}\left(\left|v_{j}-w_{j}\right|+\left|u_{N}^{\varphi}-\tilde{u}_{N}^{\varphi}\right|\right)\left|v_{j}-u_{N}^{\varphi}\right| \\
& +\frac{1}{d} \sum_{j=1}^{N} p_{1, j}\left(\left|v_{j}-w_{j}\right|+\left|u_{N}^{\varphi}-\tilde{u}_{N}^{\varphi}\right|\right)\left|w_{j}-\tilde{u}_{N}^{\varphi}\right| \\
\leq & \frac{1}{\sqrt{d}}\left(\sqrt{\mathcal{V}}+\left|u_{N}^{\varphi}-\tilde{u}_{N}^{\varphi}\right|\right)\left(\sqrt{T_{N}^{\varphi}}+\sqrt{\bar{T}_{N}^{\varphi}}\right)
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality in the last passage. Therefore, by Eq. (5.2),

$$
\begin{align*}
d\left(\sqrt{T_{N}^{\varphi}}-\sqrt{\bar{T}_{N}^{\varphi}}\right)^{2} & =d\left(\frac{T_{N}^{\varphi}-\bar{T}_{N}^{\varphi}}{\sqrt{T_{N}^{\varphi}}+\sqrt{\bar{T}_{N}^{\varphi}}}\right)^{2} \leq 2 \mathcal{V}+2\left|u_{N}^{\varphi}-\tilde{u}_{N}^{\varphi}\right|^{2} \\
& \leq 6 \mathcal{V}+4\left(\sum_{j=1}^{N}\left|p_{1, j}-q_{1, j}\right|\left|w_{j}\right|\right)^{2} \tag{5.4}
\end{align*}
$$

On the other hand, from Eq. (2.22),

$$
\begin{equation*}
d\left(\sqrt{\bar{T}_{N}^{\varphi}}-\sqrt{T_{g}^{\varphi}}\right)^{2}=d\left(\frac{\bar{T}_{N}^{\varphi}-T_{g}^{\varphi}}{\sqrt{T_{N}^{\varphi}}+\sqrt{T_{g}^{\varphi}}}\right)^{2} \leq \frac{2 d\left(\bar{T}_{N}^{\varphi}-\tilde{T}_{N}^{\varphi}\right)^{2}}{A_{t}}+\frac{2 d\left(\tilde{T}_{N}^{\varphi}-T_{g}^{\varphi}\right)^{2}}{A_{t}}, \tag{5.5}
\end{equation*}
$$

where $\tilde{T}_{N}^{\varphi}$ is defined in Eq. (3.10). Hence, by Eqs. (5.1), (5.2), (5.4) and (5.5),

$$
\begin{equation*}
D\left(Z_{N}, \Sigma_{N}\right) \leq D_{1}\left(Z_{N}, \Sigma_{N}\right)+D_{2}\left(Z_{N}, \Sigma_{N}\right)+\mathcal{E}\left(\Sigma_{N}\right) \tag{5.6}
\end{equation*}
$$

with

$$
\begin{align*}
D_{1}\left(Z_{N}, \Sigma_{N}\right) & =\int \mathrm{d} \xi \varphi(\xi) 16 \mathcal{V}\left(\xi, Z_{N}, \Sigma_{N}\right)  \tag{5.7}\\
D_{2}\left(Z_{N}, \Sigma_{N}\right) & =\int \mathrm{d} \xi \varphi(\xi)\left[12\left(\sum_{j=1}^{N}\left|p_{1, j}-q_{1, j}\right|\left|w_{j}\right|\right)^{2}+\frac{4 d\left(\bar{T}_{N}^{\varphi}-\tilde{T}_{N}^{\varphi}\right)^{2}}{A_{t}}\right]  \tag{5.8}\\
\mathcal{E}\left(\Sigma_{N}\right) & =\int \mathrm{d} \xi \varphi(\xi) 2\left|\tilde{u}_{N}^{\varphi}-u_{g}^{\varphi}\right|^{2}+\int \mathrm{d} \xi \varphi(\xi) \frac{4 d\left(\tilde{T}_{N}^{\varphi}-T_{g}^{\varphi}\right)^{2}}{A_{t}} \tag{5.9}
\end{align*}
$$

From Eqs. (3.8), (5.6), and recalling the definition Eq. (4.6) of the good set, we arrive at the following estimate on the derivative of $I_{N}(t)$,

$$
\begin{equation*}
\dot{I}_{N}(t) \leq I_{N}(t)+\mathcal{D}_{a}(t)+\mathcal{D}_{b}(t)+\mathcal{D}_{c}(t)+\mathcal{D}_{d}(t) \tag{5.10}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\mathcal{D}_{a}(t)=\int \mathrm{d} R_{N}(t) D_{1}\left(Z_{N}, \Sigma_{N}\right), & \mathcal{D}_{b}(t)=\int \mathrm{d} R_{N}(t) D_{2}\left(Z_{N}, \Sigma_{N}\right) \mathbb{I}_{\mathcal{G}^{\mathrm{c}}} \\
\mathcal{D}_{c}(t)=\int \mathrm{d} R_{N}(t) D_{2}\left(Z_{N}, \Sigma_{N}\right) \mathbb{I}_{\mathcal{G}}, & \mathcal{D}_{d}(t)=\int \mathrm{d} g(t)^{\otimes N} \mathcal{E}\left(\Sigma_{N}\right)
\end{array}
$$

5.1. Upper bound on $\mathcal{D}_{a}(t)$. Since $\mathrm{d} R_{N}(t)$ is symmetric with respect to particle permutations,

$$
\begin{align*}
\mathcal{D}_{a}(t) & =\int \mathrm{d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) 16 \mathcal{V}\left(\xi, Z_{N}, \Sigma_{N}\right) \\
& =\frac{16}{N} \sum_{i=1}^{N} \int \mathrm{~d} R_{N}(t) \int \mathrm{d} \xi \sum_{j=1}^{N} p_{i, j}\left|v_{j}-w_{j}\right|^{2} \\
& =\frac{16}{N} \int \mathrm{~d} R_{N}(t) \sum_{j=1}^{N}\left(\int \mathrm{~d} \xi \varphi(\xi) \sum_{i=1}^{N} p_{i, j}\right)\left|v_{j}-w_{j}\right|^{2} \\
& \leq 16 \varphi_{0} I_{N}(t) \tag{5.11}
\end{align*}
$$

where we used the upper bound of Eq. (4.10) in the last estimate.
5.2. Upper bound on $\mathcal{D}_{b}(t)$. By repeatedly applying the Cauchy-Schwarz inequality we have,

$$
\begin{gathered}
\left(\sum_{j=1}^{N}\left|p_{1, j}-q_{1, j}\right|\left|w_{j}\right|\right)^{2} \leq\left(\sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left|w_{j}\right|\right)^{2} \leq 2 \sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left|w_{j}\right|^{2} \\
\left(\bar{T}_{N}^{\varphi}-\tilde{T}_{N}^{\varphi}\right)^{2} \leq\left(\sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left|w_{j}-\tilde{u}_{N}^{\varphi}\right|^{2}\right)^{2} \leq C \sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left|w_{j}\right|^{4}+C\left|\tilde{u}_{N}^{\varphi}\right|^{4},
\end{gathered}
$$

and

$$
\left|\tilde{u}_{N}^{\varphi}\right|^{4} \leq\left(\sum_{j=1}^{N} q_{1, j}\left|w_{j}\right|\right)^{4} \leq \sum_{j=1}^{N} q_{1, j}\left|w_{j}\right|^{4}
$$

Therefore, by Eq. (5.8) and recalling the definition Eq. (4.6) of $\mathcal{G}$,

$$
\begin{align*}
\mathcal{D}_{b}(t) & \leq C \int \mathrm{~d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) \sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}^{\mathbb{C}}} \\
& \leq C\left(\mathcal{R}_{1}+\mathcal{R}_{2}\right) \tag{5.12}
\end{align*}
$$

where

$$
\begin{aligned}
& \mathcal{R}_{1}=\int \mathrm{d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) \sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}}, \\
& \mathcal{R}_{2}=\int \mathrm{d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) \sum_{j=1}^{N}\left(p_{1, j}+q_{1, j}\right)\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}_{1}^{\mathrm{C}}} .
\end{aligned}
$$

Since $\mathrm{d} R_{N}(t)$ and $\mathbb{I}_{\mathcal{B}_{A}^{\text {© }}}$ are symmetric with respect to particle permutations,

$$
\begin{aligned}
\mathcal{R}_{1} & =\frac{1}{N} \sum_{i=1}^{N} \int \mathrm{~d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) \sum_{j=1}^{N}\left(p_{i, j}+q_{i, j}\right)\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}} \\
& =\int \mathrm{d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(\int \mathrm{~d} \xi \varphi(\xi) \sum_{i=1}^{N}\left(p_{i, j}+q_{i, j}\right)\right)\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}} \\
& \leq 2 \varphi_{0} \int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}}
\end{aligned}
$$

where we used the upper bounds of Eqs. (4.10) and (4.11) in the last inequality. Therefore, from the Cauchy-Schwarz inequality and Eqs. (4.4) and (4.12),

$$
\begin{align*}
\mathcal{R}_{1} & \leq 2 \varphi_{0}\left(\int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right)^{2}\right)^{1 / 2}\left(\int \mathrm{~d} R_{N}(t) \mathbb{I}_{\mathcal{B}_{A}^{0}}\right)^{1 / 2} \\
& \leq \frac{C \varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}}\left(\int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(\left|w_{j}\right|^{4}+\left|w_{j}\right|^{8}\right)\right)^{1 / 2} \leq \frac{C \varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}} \tag{5.13}
\end{align*}
$$

Analogously, since $\mathrm{d} R_{N}(t)$ and $\mathbb{1}_{\mathcal{G}_{1}^{\text {c }}}$ are symmetric with respect to particle permutations, by applying the upper bound of Eq. (4.10) and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\mathcal{R}_{2} & =\frac{1}{N} \sum_{i=1}^{N} \int \mathrm{~d} R_{N}(t) \int \mathrm{d} \xi \varphi(\xi) \sum_{j=1}^{N}\left(p_{i, j}+q_{i, j}\right)\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}_{1}^{\mathrm{C}}} \\
& \leq 2 \varphi_{0} \int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(\left|w_{j}\right|^{2}+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}_{1}^{\mathrm{C}}} \\
& \leq C \varphi_{0} \int \mathrm{~d} R_{N}(t)\left(1+\frac{1}{N} \sum_{j=1}^{N}\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}_{1}^{\mathrm{C}}}
\end{aligned}
$$

Recalling Eq. (4.13), we estimate $\mathbb{1}_{\mathcal{G}_{1}^{\mathrm{C}}} \leq \mathbb{1}_{\mathcal{G}_{1}^{\complement} \cap \mathcal{G}_{M, 4}}+\mathbb{1}_{\mathcal{G}_{M, 4}^{\mathrm{C}}}$ so that

$$
\begin{align*}
\mathcal{R}_{2} \leq & C(1+M) \varphi_{0} \int \mathrm{~d} R_{N}(t) \mathbb{I}_{\mathcal{G}_{1}^{\mathrm{C}}}+C \varphi_{0} \int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(1+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}_{M, 4}^{\mathrm{C}}} \\
\leq & \frac{C(1+M) \varphi_{0}}{A_{\varphi}^{2}} \int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{i=1}^{N}\left|x_{i}-y_{i}\right|^{2} \\
& +C \varphi_{0}\left[\int \mathrm{~d} R_{N}(t)\left(\frac{1}{N} \sum_{j=1}^{N}\left(1+\left|w_{j}\right|^{4}\right)\right)\right]^{1 / 2}\left(\int \mathrm{~d} R_{N}(t) \mathbb{1}_{\mathcal{G}_{M, 4}^{\mathrm{c}}}\right)^{1 / 2} \\
\leq & \frac{C(1+M) \varphi_{0}}{A_{\varphi}^{2}} I_{N}(t)+\frac{C \varphi_{0}}{N^{1 / 2}} \tag{5.14}
\end{align*}
$$

where we used Chebyshev's inequality, Cauchy-Schwarz inequality twice, and finally Eqs. (4.12) and (4.14).

From Eqs. (5.12), (5.13), and (5.14), and by Eq. (4.5), we finally obtain

$$
\begin{equation*}
\mathcal{D}_{b}(t) \leq C \frac{\|\nabla \varphi\|_{\infty}^{2}}{r^{2 d} \varphi_{0}} I_{N}(t)+\frac{C \varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}} \tag{5.15}
\end{equation*}
$$

5.3. Upper bound on $\mathcal{D}_{c}(t)$. As

$$
\begin{aligned}
p_{1, j}-q_{1, j}= & \frac{\varphi\left(x_{1}+\xi-x_{j}\right)-\varphi\left(y_{1}+\xi-y_{j}\right)}{N \varrho_{N}^{\varphi}\left(x_{1}+\xi\right)} \\
& +\varphi\left(y_{1}+\xi-y_{j}\right) \frac{\sum_{k}\left[\varphi\left(y_{1}+\xi-y_{k}\right)-\varphi\left(x_{1}+\xi-x_{k}\right)\right]}{N^{2} \varrho_{N}^{\varphi}\left(x_{1}+\xi\right) \varrho_{N}^{\varphi}\left(y_{1}+\xi\right)}
\end{aligned}
$$

from Eq. (4.7) we have that

$$
\begin{align*}
\mid p_{1, j} & -q_{1, j} \left\lvert\, \leq \frac{2\|\nabla \varphi\|_{\infty}}{A r^{d} \varphi_{0} N}\left(\left|x_{1}-y_{1}\right|+\left|x_{j}-y_{j}\right|\right)\right. \\
& +\frac{2 \varphi_{0}\|\nabla \varphi\|_{\infty}}{N^{2}\left(A r^{d} \varphi_{0}\right)^{2}} \sum_{k=1}^{N}\left(\left|x_{1}-y_{1}\right|+\left|x_{k}-y_{k}\right|\right) \\
\quad \leq & \frac{C\|\nabla \varphi\|_{\infty}}{r^{2 d} \varphi_{0} N}\left(\left|x_{1}-y_{1}\right|+\left|x_{j}-y_{j}\right|+\frac{1}{N} \sum_{k=1}^{N}\left|x_{k}-y_{k}\right|\right) \quad \text { in the set } \mathcal{G} . \tag{5.16}
\end{align*}
$$

Therefore, from Cauchy-Schwarz inequality, in the set $\mathcal{G}$,

$$
\left(\sum_{j=1}^{N}\left|p_{1, j}-q_{1, j} \| w_{j}\right|\right)^{2} \leq \frac{C\|\nabla \varphi\|_{\infty}^{2}}{r^{4 d} \varphi_{0}^{2}}\left(\left|x_{1}-y_{1}\right|^{2}+\frac{1}{N} \sum_{k=1}^{N}\left|x_{k}-y_{k}\right|^{2}\right) \frac{1}{N} \sum_{j=1}^{N}\left|w_{j}\right|^{2}
$$

Analogously, still in $\mathcal{G}$,

$$
\begin{aligned}
\left(\bar{T}_{N}^{\varphi}-\tilde{T}_{N}^{\varphi}\right)^{2} \leq & \left(\sum_{j=1}^{N}\left|p_{1, j}-q_{1, j} \| w_{j}-\tilde{u}_{N}^{\varphi}\right|^{2}\right)^{2} \\
& \leq \frac{C\|\nabla \varphi\|_{\infty}^{2}}{r^{4 d} \varphi_{0}^{2}}\left(\left|x_{1}-y_{1}\right|^{2}+\frac{1}{N} \sum_{k=1}^{N}\left|x_{k}-y_{k}\right|^{2}\right) \frac{1}{N} \sum_{j=1}^{N}\left|w_{j}-\tilde{u}_{N}^{\varphi}\right|^{4} \\
& \leq \frac{C\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}}\left(\left|x_{1}-y_{1}\right|^{2}+\frac{1}{N} \sum_{k=1}^{N}\left|x_{k}-y_{k}\right|^{2}\right) \frac{1}{N} \sum_{j=1}^{N}\left|w_{j}\right|^{4}
\end{aligned}
$$

where in the last inequality, we used that, because of Eq. (4.7),

$$
\left|\tilde{u}_{N}^{\varphi}\right|^{4} \leq\left(\sum_{j=1}^{N} q_{1, j}\left|w_{j}\right|\right)^{4} \leq \sum_{j=1}^{N} q_{1, j}\left|w_{j}\right|^{4} \leq \frac{1}{A r^{d} N} \sum_{j=1}^{N}\left|w_{j}\right|^{4} \quad \text { in the set } \mathcal{G}
$$

Recalling Eq. (5.8), the above estimates allow to control $D_{2}\left(Z_{N}, \Sigma_{N}\right)$ in the set $\mathcal{G}$,

$$
\mathcal{D}_{c}(t) \leq \frac{C\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}} \int \mathrm{~d} R_{N}(t)\left(\left|x_{1}-y_{1}\right|^{2}+\frac{1}{N} \sum_{k=1}^{N}\left|x_{k}-y_{k}\right|^{2}\right) \frac{1}{N} \sum_{j=1}^{N}\left(1+\left|w_{j}\right|^{4}\right)
$$

We now argue analogously to what done to get Eq. (5.14): recalling Eq. (4.13) and inserting $1=\mathbb{I}_{\mathcal{G}_{M, 4}}+\mathbb{I}_{\mathcal{G}_{M, 4}^{\mathcal{C}}}$ in the right-hand side we have that

$$
\begin{align*}
\mathcal{D}_{c}(t) & \leq \frac{2 C(1+M)\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}} I_{N}(t)+\frac{C\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}} \int \mathrm{~d} R_{N}(t) \frac{1}{N} \sum_{j=1}^{N}\left(1+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{G}_{M, 4}^{\mathrm{c}}} \\
& \leq \frac{2 C(1+M)\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}} I_{N}(t)+\frac{C\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2} N^{1 / 2}} \tag{5.17}
\end{align*}
$$

where in estimating the integrand in $\mathcal{G}_{M, 4}^{\complement}$ we used that the mutual distance among the particles is not greater than one.

### 5.4. Upper bound on $\mathcal{D}_{d}(t)$. We decompose

$$
\mathcal{D}_{d}(t)=\mathcal{D}_{d}^{(1)}(t)+\mathcal{D}_{d}^{(2)}(t)
$$

where

$$
\begin{aligned}
D_{d}^{(1)}(t) & :=\int \mathrm{d} g(t)^{\otimes N} \int \mathrm{~d} \xi \varphi(\xi) 2\left|\tilde{u}_{N}^{\varphi}-u_{g}^{\varphi}\right|^{2} \\
\mathcal{D}_{d}^{(2)}(t) & :=\int \mathrm{d} g(t)^{\otimes N} \int \mathrm{~d} \xi \varphi(\xi) \frac{4 d\left(\tilde{T}_{N}^{\varphi}-T_{g}^{\varphi}\right)^{2}}{A_{t}}
\end{aligned}
$$

and analyze the two terms separately.
Upper bound on $\mathcal{D}_{d}^{(1)}(t)$. After introducing the random variables

$$
\mathcal{U}_{j}(\xi):=\varrho_{g}^{\varphi}(\xi, t) \varphi\left(\xi-y_{j}\right) w_{j}-\varphi\left(\xi-y_{j}\right)\left(\varrho_{g}^{\varphi} u_{g}^{\varphi}\right)(\xi, t)
$$

we observe that

$$
\begin{aligned}
& \int \mathrm{d} \xi \varphi(\xi)\left|\tilde{u}_{N}^{\varphi}-u_{g}^{\varphi}\right|^{2} \leq \int \mathrm{d} \xi \varphi(\xi)\left[\left|\tilde{u}_{N}^{\varphi}-u_{g}^{\varphi}\right|^{2} \mathbb{I}_{\mathcal{B}_{A}}+\left(\left|\tilde{u}_{N}^{\varphi}\right|^{2}+\left|u_{g}^{\varphi}\right|^{2}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}}\right] \\
& \quad=\int \mathrm{d} \xi \frac{\varphi\left(\xi-y_{1}\right)}{\varrho_{N}^{\varphi}(\xi)^{2} \varrho_{g}^{\varphi}(\xi)^{2}}\left|\frac{1}{N} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{2} \mathbb{1}_{\mathcal{B}_{A}}+\int \mathrm{d} \xi \varphi(\xi)\left(\left|\tilde{u}_{N}^{\varphi}\right|^{2}+\left|u_{g}^{\varphi}\right|^{2}\right) \mathbb{1}_{\mathcal{B}_{A}^{C}} \\
& \quad \leq \frac{C}{\varphi_{0}^{2} r^{2 d}} \int \mathrm{~d} \xi \varphi\left(\xi-y_{1}\right)\left|\frac{1}{N} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{2}+C \int \mathrm{~d} \xi \varphi(\xi) \sum_{j=1}^{N} q_{1, j}\left(1+\left|w_{j}\right|^{2}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}}
\end{aligned}
$$

Above, we applied (after the change of variables $\xi \rightarrow \xi+y_{1}$ ) the definition Eq. (4.3) and the lower bound Eq. (2.21) to estimate the first term in the right-hand side, the Cauchy-Schwarz inequality together with Eq. (2.20) to estimate the second term.

We notice that the variables $\mathcal{U}_{j}(\xi)$ are i.i.d. and satisfy

$$
\begin{align*}
& \int \mathrm{d} g(t)^{\otimes N} \mathcal{U}_{j}(\xi)=0 \\
& \int \mathrm{~d} g(t)^{\otimes N}\left|\mathcal{U}_{j}(\xi)\right|^{2} \leq C \varphi_{0} \int \mathrm{~d} y \mathrm{~d} w \varphi(\xi-y) g(y, w, t)|w|^{2} \leq C \varphi_{0} \tag{5.18}
\end{align*}
$$

where we used the upper bound on $\mathcal{N}_{q}(g(t))$ given in Eq. (2.20) with $q>2+d$. Therefore,

$$
\begin{align*}
\int \mathrm{d} g(t)^{\otimes N} \int \mathrm{~d} \xi \varphi\left(\xi-y_{1}\right)\left|\frac{1}{N} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{2} & \leq \varphi_{0} \int \mathrm{~d} \xi \int \mathrm{~d} g(t)^{\otimes N}\left|\frac{1}{N} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{2} \\
& \leq \frac{C \varphi_{0}^{2}}{N} \tag{5.19}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
\int \mathrm{d} g(t)^{\otimes N} & \int \mathrm{~d} \xi \varphi(\xi) \sum_{j=1}^{N} q_{1, j}\left(1+\left|w_{j}\right|^{2}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}} \\
& =\frac{1}{N} \sum_{i=1}^{N} \int \mathrm{~d} g(t)^{\otimes N} \int \mathrm{~d} \xi \varphi(\xi) \sum_{j=1}^{N} q_{i, j}\left(1+\left|w_{j}\right|^{2}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}} \\
& \leq C \varphi_{0} \int \mathrm{~d} g(t)^{\otimes N} \frac{1}{N} \sum_{j=1}^{N}\left(1+\left|w_{j}\right|^{2}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}} \leq \frac{C \varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}} \tag{5.20}
\end{align*}
$$

where we used that $\mathrm{d} g(t)^{\otimes N}$ and $\mathbb{1}_{\mathcal{B}_{A}^{\text {C }}}$ are symmetric with respect to particle permutations, the upper bound of Eqs. (4.11), and finally, as done in Eq. (5.13), the Cauchy-Schwarz inequality and Eqs. (4.4) and (4.12).

Putting the above together, we obtain

$$
\begin{equation*}
\mathcal{D}_{d}^{(1)}(t) \leq C\left(\frac{1}{r^{2 d} N}+\frac{\varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}}\right) \tag{5.21}
\end{equation*}
$$

Upper bound on $\mathcal{D}_{d}^{(2)}(t)$. We argue similarly to the previous case. Since

$$
d\left(\tilde{T}_{N}^{\varphi}-T_{g}^{\varphi}\right)=\sum_{j=1}^{N} q_{1, j}\left|w_{j}\right|^{2}-\left(d T_{g}^{\varphi}+\left|u_{g}^{\varphi}\right|^{2}\right)+\left|u_{g}^{\varphi}\right|^{2}-\left|\tilde{u}_{N}^{\varphi}\right|^{2}
$$

after introducing the random variables

$$
\mathcal{T}_{j}(\xi)=\varrho_{g}^{\varphi}(\xi, t) \varphi\left(\xi-y_{j}\right)\left|w_{j}\right|^{2}-\varphi\left(\xi-y_{j}\right)\left(\varrho_{g}^{\varphi} d T_{g}^{\varphi}+\left|u_{g}^{\varphi}\right|^{2}\right)(\xi, t)
$$

and using that $\left(\left|u_{g}^{\varphi}\right|^{2}-\left|\tilde{u}_{N}^{\varphi}\right|^{2}\right)^{2} \leq\left|\tilde{u}_{N}^{\varphi}-u_{g}^{\varphi}\right|^{4}$, we have

$$
\begin{align*}
& \int \mathrm{d} \xi \varphi(\xi)\left(\tilde{T}_{N}^{\varphi}-T_{g}^{\varphi}\right)^{2} \leq \int \mathrm{d} \xi \varphi(\xi)\left[\left(\tilde{T}_{N}^{\varphi}-T_{g}^{\varphi}\right)^{2} \mathbb{I}_{\mathcal{B}_{A}}+\left(\left(\tilde{T}_{N}^{\varphi}\right)^{2}+\left(T_{g}^{\varphi}\right)^{2}\right) \mathbb{I}_{\mathcal{B}_{A}^{\mathrm{C}}}\right] \\
& =\int \mathrm{d} \xi \frac{\varphi\left(\xi-y_{1}\right)}{\tilde{\varrho}_{N}^{\varphi}(\xi)^{2} \varrho_{g}^{\varphi}(\xi)^{2}}\left\{\left(\frac{1}{N d} \sum_{j=1}^{N} \mathcal{T}_{j}(\xi)\right)^{2}+\frac{1}{\tilde{\varrho}_{N}^{\varphi}(\xi)^{2} \varrho_{g}^{\varphi}(\xi)^{2}}\left|\frac{1}{N d} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{4}\right\} \mathbb{B}_{\mathcal{B}_{A}} \\
& \quad+\int \mathrm{d} \xi \varphi(\xi)\left[\left(\tilde{T}_{N}^{\varphi}\right)^{2}+\left(T_{g}^{\varphi}\right)^{2}\right] \mathbb{1}_{\mathcal{B}_{A}^{\text {© }}} \\
& \leq \\
& \quad \frac{C}{\varphi_{0}^{2} r^{2 d}} \int \mathrm{~d} \xi \varphi\left(\xi-y_{1}\right)\left\{\left(\frac{1}{N} \sum_{j=1}^{N} \mathcal{T}_{j}(\xi)\right)^{2}+\frac{1}{\varphi_{0}^{2} r^{2 d}}\left|\frac{1}{N} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{4}\right\}  \tag{5.22}\\
& \quad+C \int \mathrm{~d} \xi \varphi(\xi) \sum_{j=1}^{N} q_{1, j}\left(1+\left|w_{j}\right|^{4}\right) \mathbb{I}_{\mathcal{B}_{A}^{\text {a }}} .
\end{align*}
$$

Clearly, the expectation with respect to $\mathrm{d} g(t)^{\otimes N}$ of the last term in the right-hand side of Eq. (5.22) can be bounded as in Eq. (5.20), while, by Eq. (5.18),

$$
\int \mathrm{d} g(t)^{\otimes N} \int \mathrm{~d} \xi \varphi\left(\xi-y_{1}\right)\left|\frac{1}{N} \sum_{j=1}^{N} \mathcal{U}_{j}(\xi)\right|^{4} \leq \frac{C \varphi_{0}^{3}}{N^{2}}
$$

Finally, concerning the expectation with respect to $\mathrm{d} g(t)^{\otimes N}$ of the first term in the right-hand side of Eq. (5.22), we observe that also the variables $\mathcal{T}_{j}(\xi)$ are i.i.d. and
satisfy

$$
\begin{aligned}
& \int \mathrm{d} g(t)^{\otimes N} \mathcal{T}_{j}(\xi)=0, \\
& \int \mathrm{~d} g(t)^{\otimes N}\left|\mathcal{T}_{j}(\xi)\right|^{2} \leq C \varphi_{0} \int \mathrm{~d} y \mathrm{~d} w \varphi(\xi-y) g(y, w)|w|^{4} \leq C \varphi_{0},
\end{aligned}
$$

where we used the upper bound on $\mathcal{N}_{q}(g(t))$ given in Eq. (2.20) with $q>4+d$. Therefore, analogously to Eq. (5.19),

$$
\int \mathrm{d} g(t)^{\otimes N} \int \mathrm{~d} \xi \varphi\left(\xi-y_{1}\right)\left(\frac{1}{N} \sum_{j=1}^{N} \mathcal{T}_{j}(\xi)\right)^{2} \leq \frac{C \varphi_{0}^{2}}{N}
$$

Collecting together the above bounds, we obtain

$$
\begin{equation*}
\mathcal{D}_{d}^{(2)}(t) \leq C\left(\frac{1}{r^{2 d} N}+\frac{1}{r^{4 d} \varphi_{0} N^{2}}+\frac{\varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}}\right) \tag{5.23}
\end{equation*}
$$

From Eqs. (5.21) and (5.23) we conclude that

$$
\begin{equation*}
\mathcal{D}_{d}(t) \leq C\left(\frac{1}{r^{2 d} N}+\frac{1}{r^{4 d} \varphi_{0} N^{2}}+\frac{\varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}}\right) \tag{5.24}
\end{equation*}
$$

5.5. Proof of Eq. (3.7). From Eqs. (5.10), (5.11), (5.15), (5.17), and (5.24) we get

$$
\begin{aligned}
\dot{I}_{N}(t) \leq & C\left(\varphi_{0}+\frac{\|\nabla \varphi\|_{\infty}}{r^{2 d} \varphi_{0}}+\frac{\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}}\right) I_{N}(t) \\
& +\left(\frac{1}{r^{2 d} N}+\frac{1}{r^{4 d} \varphi_{0} N^{2}}+\frac{\varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}}+\frac{\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2} N^{1 / 2}}\right)
\end{aligned}
$$

We then choose

$$
\begin{equation*}
\Gamma_{\varphi}=\frac{\varphi_{0}^{3}+\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}}, \quad N_{\varphi}=\frac{\varphi_{0}^{4}}{r^{6 d}}+\frac{\|\nabla \varphi\|_{\infty}^{8}}{\left(r^{5 d} \varphi_{0}^{2}\right)^{4}} \tag{5.25}
\end{equation*}
$$

so that, recalling $\varphi_{0} \geq 1$,

$$
\left(\varphi_{0}+\frac{\|\nabla \varphi\|_{\infty}}{r^{2 d} \varphi_{0}}+\frac{\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2}}\right) \leq 2 \Gamma_{\varphi}
$$

and

$$
\left(\frac{1}{r^{2 d} N}+\frac{1}{r^{4 d} \varphi_{0} N^{2}}+\frac{\varphi_{0}}{\left(r^{3 d} N\right)^{1 / 2}}+\frac{\|\nabla \varphi\|_{\infty}^{2}}{r^{5 d} \varphi_{0}^{2} N^{1 / 2}}\right) \leq \frac{4}{N^{1 / 4}} \quad \forall N>N_{\varphi} .
$$

Therefore, $\dot{I}_{N}(t) \leq C \Gamma_{\varphi} I_{N}(t)+C / N^{1 / 4}$ for any $N>N_{\varphi}$, from which Eq. (3.7) follows by Grönwall's inequality and hence the proof of Theorem 2.3 is achieved.

## 6. Removing the cutoff and conclusion

The regularized BGK equation Eq. (2.14) reduces to the usual one Eq. (1.1) when the cutoff function $\varphi$ converges to the $\delta$-function, at least formally. In this section, $\varphi=\varphi^{(\varepsilon)}$ is the rescaled function given by (2.24) and we assume $t \in[0, T]$ for fixed $T>0$. We recall that $C$ denotes a generic positive constant, possibly depending only on $T$ and $f_{0}$, hence independent of $\varepsilon$.

Proof of Theorem 2.5. The limit $\varepsilon \rightarrow 0$ was investigated in [2], where the convergence $g \rightarrow f$ is proven in a weighted $L^{1}$ space, see [2, Theorem 2.4]. Here, given $f$ and $g$, we rather study the processes $(x(t), v(t))$ and $(y(t), w(t))$ whose generators are given by ( $\psi$ denotes a test function)

$$
\begin{equation*}
\mathcal{L}_{1} \psi(x, v)=\left(v \cdot \nabla_{x}-1\right) \psi(x, v)+\int \mathrm{d} \tilde{v} M_{f}(x, \tilde{v}) \psi(x, \tilde{v}) \tag{6.1}
\end{equation*}
$$

and $\mathcal{L}_{2}=\mathcal{L}_{1}^{g}$ as in Eq. (3.1), i.e.,

$$
\begin{equation*}
\mathcal{L}_{2} \psi(y, w)=\left(w \cdot \nabla_{y}-1\right) \psi(y, w)+\int \mathrm{d} \tilde{y} \varphi(\tilde{y}-y) \int \mathrm{d} \tilde{w} M_{g}^{\varphi}(\tilde{y}, \tilde{w}) \psi(\tilde{y}, \tilde{w}) \tag{6.2}
\end{equation*}
$$

respectively.
We fix $f_{0}$ as common initial datum for Eqs. (2.3) and (2.14), and assume that it satisfies the hypotheses Eqs. (2.8) and (2.25). Note that, in particular, the hypothesis of Propositions 2.1 and 2.2 are satisfied.

We couple the two processes by defining in the product space the process whose generator is given by ( $G$ denotes a test function)

$$
\begin{aligned}
\mathcal{L}_{Q} G(x, v, y, w)= & \left(v \cdot \nabla_{x}+w \cdot \nabla_{y}-1\right) G(x, v, y, w) \\
& +\int \mathrm{d} \tilde{y} \int \mathrm{~d} \tilde{v} \int \mathrm{~d} \tilde{w} \mathcal{M}(x, \tilde{v} ; \tilde{y}, \tilde{w}) \varphi(\tilde{y}-y) G(x, \tilde{v}, \tilde{y}, \tilde{w})
\end{aligned}
$$

where, analogously to what done in $\operatorname{Sec} .3 .1, \mathcal{M}(x, \tilde{v} ; \tilde{y}, \tilde{w})$ is the joint representation of the Maxwellians $M_{f}$ and $M_{g}^{\varphi}$ that realizes the 2-Wasserstein distance between the marginals. We denote by $\mathrm{d} R(t)=\mathrm{d} R(x, v ; y, w ; t)$ the law of this process assuming that initially

$$
\mathrm{d} R(x, v ; y, w ; 0)=f_{0}(x, v) \delta(x-y) \delta(v-w)
$$

Clearly,

$$
\begin{equation*}
\mathcal{W}_{2}(f(t), g(t))^{2} \leq I(t):=\int \mathrm{d} R(t)\left(|x-y|^{2}+|v-w|^{2}\right) \tag{6.3}
\end{equation*}
$$

To obtain an upper bound for $I(t)$ we compute,

$$
\begin{aligned}
\dot{I}(t)= & 2 \int \mathrm{~d} R(t)(x-y) \cdot(v-w)-\int \mathrm{d} R(t)|x-y|^{2} \\
& +\int \mathrm{d} R(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y})|x-\tilde{y}|^{2}+\frac{\mathrm{d}}{\mathrm{~d} t} \int \mathrm{~d} R(t)|v-w|^{2} \\
\leq & \int \mathrm{d} R(t)|v-w|^{2}+2 \int \mathrm{~d} R(t)|x-y|^{2}+E_{1}+\frac{\mathrm{d}}{\mathrm{~d} t} \int \mathrm{~d} R(t)|v-w|^{2}
\end{aligned}
$$

where

$$
E_{1}=2 \int \mathrm{~d} R(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y})|y-\tilde{y}|^{2} \leq C \varepsilon^{2}
$$

while, by virtue of the choice of $\mathcal{M}(x, \tilde{v} ; \tilde{y}, \tilde{w})$,

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \int \mathrm{~d} R(t)|v-w|^{2}=S_{1}+S_{2}-\int \mathrm{d} R(t)|v-w|^{2}
$$

with

$$
S_{1}=\int \mathrm{d} R(t)\left|u_{f}(x, t)-u_{g}^{\varphi}(y, t)\right|^{2}, \quad S_{2}=\int \mathrm{d} R(t) d\left(\sqrt{T_{f}(x, t)}-\sqrt{T_{g}^{\varphi}(y, t)}\right)^{2}
$$

Therefore,

$$
\begin{equation*}
\dot{I}(t) \leq 2 I(t)+S_{1}+S_{2}+C \varepsilon^{2} \tag{6.4}
\end{equation*}
$$

and it remains to estimate $S_{1}$ and $S_{2}$.

Upper bound on $S_{1}$. Setting

$$
\begin{aligned}
\mathrm{d} r(x, y ; t) & =\int_{v, w} \mathrm{~d} R(x, v ; y, w ; t) \\
\mathrm{d} R^{\varphi}(x, v ; y, w ; t) & =\int \mathrm{d} \tilde{y} \varphi(y-\tilde{y}) \mathrm{d} R(x, v ; \tilde{y}, w ; t) \\
\mathrm{d} r^{\varphi}(x, y ; t) & =\int \mathrm{d} \tilde{y} \varphi(y-\tilde{y}) \mathrm{d} r(x, \tilde{y} ; t)
\end{aligned}
$$

we have

$$
S_{1}=\bar{S}_{1}+E_{2}
$$

where

$$
\bar{S}_{1}=\int \mathrm{d} r^{\varphi}(t)\left|u_{f}(x, t)-u_{g}^{\varphi}(y, t)\right|^{2}, \quad E_{2}=\int\left(\mathrm{d} r-\mathrm{d} r^{\varphi}\right)(t)\left|u_{f}(x, t)-u_{g}^{\varphi}(y, t)\right|^{2}
$$

We start by estimating $\bar{S}_{1}$, noticing that if

$$
g^{\varphi}(y, w, t):=\int \mathrm{d} \tilde{y} \varphi(y-\tilde{y}) g(\tilde{y}, w, t)
$$

then

$$
\begin{align*}
\bar{S}_{1} & =\int \mathrm{d} r^{\varphi}(t)\left|\frac{\int \mathrm{d} v f(x, v, t) v}{\varrho_{f}(x, t)}-\frac{\int \mathrm{d} w g^{\varphi}(y, w, t) w}{\varrho_{g}^{\varphi}(y, t)}\right|^{2} \\
& =\int \mathrm{d} r^{\varphi}(t)\left|\int \mathrm{d} \Lambda_{x, y}(v, w)(v-w)\right|^{2} \leq \int \mathrm{d} r^{\varphi}(t) \int \mathrm{d} \Lambda_{x, y}(v, w)|v-w|^{2} \tag{6.5}
\end{align*}
$$

with (remarking also that $\varrho_{g}^{\varphi}=\varrho_{g}$ )

$$
\begin{equation*}
\mathrm{d} \Lambda_{x, y}(v, w) \in \mathcal{P}\left(\frac{f(x, v, t) \mathrm{d} v}{\varrho_{f}(x, t)}, \frac{g^{\varphi}(y, w, t) \mathrm{d} w}{\varrho_{g}^{\varphi}(y, t)}\right) \tag{6.6}
\end{equation*}
$$

to be fixed later (recall $\mathcal{P}(\mu, \nu)$ denotes the collection of all joint probability measures in the product space with marginals $\mu$ and $\nu$, see Remark 2.4).

Looking at the coupled stochastic processes $(x(t), y(t))$, we realize it is of the form

$$
\begin{aligned}
x(t) & =x_{0}+v_{0} t_{1}+v_{1}\left(t_{2}-t_{1}\right)+v_{2}\left(t_{3}-t_{2}\right) \cdots, \\
y(t) & =x_{0}+v_{0} t_{1}+\xi_{1}+w_{1}\left(t_{2}-t_{1}\right)+\xi_{2}+w_{2}\left(t_{3}-t_{2}\right) \cdots
\end{aligned}
$$

where $t_{1}<t_{2}<\cdots<t_{k}$ are exponential times in which the jumps in velocity are simultaneously performed. The outgoing velocities are Maxwellian computed via the hydrodynamical fields depending on $x\left(t_{k}\right)$ and $y\left(t_{k}\right)+\xi_{k}$. Finally, the extra displacements $\xi_{k}$ are i.i.d. distributed according to $\varphi$.

Now, let $R(\mathrm{~d} v \mathrm{~d} w \mid x, y ; t)$, resp. $R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t)$, be the conditional probability of $\mathrm{d} R(x, v ; y, w ; t)$, resp. $\mathrm{d} R^{\varphi}(x, v ; y, w ; t)$, conditioned to the values $x, y$ at time $t$. For what just noticed on the structure of the coupled process, the conditional probability has the property that $\int R(\mathrm{~d} v \mathrm{~d} w \mid x, y ; t) \psi(v)$ and $\int R(\mathrm{~d} v \mathrm{~d} w \mid x, y ; t) \psi(w)$ are independent of $y$ and $x$, respectively. Similarly, also $\int R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t) \psi(v)$ and $\int R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t) \psi(w)$ are independent of $y$ and $x$, respectively.

Remark 6.1. Note that the measures $\mathrm{d} R(x, v ; y, w ; t)$ and $\mathrm{d} r(x, y ; t)$ are not absolutely continuous with respect to the Lebesgue measure. Indeed, the contribution due to the event with zero jumps transports the initial delta function in position
and velocity. Incidentally, this event does not give any contribution in the evaluation of $I(t)$. On the other hand, due to the convolution with $\varphi, \mathrm{d} r^{\varphi}$ is absolutely continuous with respect to the Lebesgue measure, and we denote by $r^{\varphi}$ its density.

We now observe that

$$
R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t) \in \mathcal{P}\left(\frac{f(x, v, t) \mathrm{d} v}{\varrho_{f}(x, t)}, \frac{g^{\varphi}(y, w, t) \mathrm{d} w}{\varrho_{g}^{\varphi}(y, t)}\right)
$$

Indeed, since $\varrho_{g}^{\varphi}(y, t)^{-1} \int \mathrm{~d} x r^{\varphi}(x, y ; t)=1$, for any test function $\psi$,

$$
\begin{gathered}
\int \mathrm{d} y \int R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t) \psi(y, w)=\int \mathrm{d} x \int \mathrm{~d} y \frac{r^{\varphi}(x, y ; t)}{\varrho_{g}^{\varphi}(y, t)} \int R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t) \psi(y, w) \\
=\int \mathrm{d} R^{\varphi}(x, v ; y, w ; t) \frac{\psi(y, w)}{\varrho_{g}^{\varphi}(y, t)}=\int \mathrm{d} y \int \frac{g^{\varphi}(y, w, t) \mathrm{d} w}{\varrho_{g}^{\varphi}(y, t)} \psi(y, w)
\end{gathered}
$$

where, in the first step, we used the independence of $x$ of the left-hand side. The same argument can be used to prove that

$$
\int \mathrm{d} x \int R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t) \psi(x, v)=\int \mathrm{d} x \int \frac{f(x, v, t) \mathrm{d} v}{\varrho_{f}(x, t)} \psi(x, v)
$$

By choosing $\mathrm{d} \Lambda_{x, y}(v, w)=R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t)$ in Eq. (6.5) we obtain

$$
\begin{align*}
\bar{S}_{1} & \leq \int \mathrm{d} r^{\varphi}(t) \int R^{\varphi}(\mathrm{d} v \mathrm{~d} w \mid x, y ; t)|v-w|^{2} \\
& =\int \mathrm{d} R^{\varphi}(t)|v-w|^{2}=\int \mathrm{d} R(t)|v-w|^{2} \tag{6.7}
\end{align*}
$$

Concerning the error term $E_{2}$, since $\int\left(\mathrm{d} r-\mathrm{d} r^{\varphi}\right)(t)\left|u_{f}(x)\right|^{2}=0$ we have

$$
E_{2}=\int\left(\mathrm{d} r-\mathrm{d} r^{\varphi}\right)(t)\left(\left|u_{g}^{\varphi}(y, t)\right|^{2}-2 u_{f}(x, t) \cdot u_{g}^{\varphi}(y, t)\right)=E_{2}^{(1)}+E_{2}^{(2)}
$$

with

$$
\begin{aligned}
& E_{2}^{(1)}=\int \mathrm{d} r(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y})\left(\left|u_{g}^{\varphi}(y, t)\right|^{2}-\left|u_{g}^{\varphi}(\tilde{y}, t)\right|^{2}\right) \\
& E_{2}^{(2)}=-2 \int \mathrm{~d} r(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y}) u_{f}(x, t) \cdot\left(u_{g}^{\varphi}(y, t)-u_{g}^{\varphi}(\tilde{y}, t)\right)
\end{aligned}
$$

The assumptions Eqs. (2.8) and (2.25) are the hypotheses of [2, Lemma 4.1], which in particular states

$$
\mathcal{N}_{q}\left(\left|\nabla_{x} g(t)\right|\right) \leq C
$$

This implies (see the proof of the same lemma)

$$
\begin{equation*}
\left|\nabla_{x} \varrho_{g}\right|+\left|D_{x} u_{g}\right|+\left|\nabla_{x} T_{g}\right| \leq C \tag{6.8}
\end{equation*}
$$

Therefore, using also Eqs. (2.9) and (2.20),

$$
\left|E_{2}^{(1)}\right|+\left|E_{2}^{(2)}\right| \leq C \int \mathrm{~d} r(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y})|y-\tilde{y}| \leq C \varepsilon
$$

In conclusion, from Eq. (6.7) and the above estimate,

$$
\begin{equation*}
S_{1} \leq C \int \mathrm{~d} R(t)|v-w|^{2}+C \varepsilon \leq I(t)+C \varepsilon \tag{6.9}
\end{equation*}
$$

Upper bound on $S_{2}$. We proceed analogously by setting

$$
S_{2}=\bar{S}_{2}+E_{3}
$$

where

$$
\bar{S}_{2}=\int \mathrm{d} r^{\varphi}(t) d\left(\sqrt{T_{f}(x, t)}-\sqrt{T_{g}^{\varphi}(y, t)}\right)^{2}
$$

and, by arguing as done for $E_{2}$,

$$
\begin{aligned}
E_{3}= & \int\left(\mathrm{d} r-\mathrm{d} r^{\varphi}\right)(t)\left(T_{g}^{\varphi}(y, t)-2 \sqrt{T_{f}(x, t)} \sqrt{T_{g}^{\varphi}(y, t)}\right) \\
= & \int \mathrm{d} r(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y})\left(T_{g}^{\varphi}(y, t)-T_{g}^{\varphi}(\tilde{y}, t)\right) \\
& -2 \int \mathrm{~d} r(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y}) \sqrt{T_{f}(x, t)}\left(\sqrt{T_{g}^{\varphi}(y, t)}-\sqrt{T_{g}^{\varphi}(\tilde{y}, t)}\right)
\end{aligned}
$$

From Eqs. (6.8) and (2.22) the error term $E_{3}$ satisfies

$$
\begin{equation*}
\left|E_{3}\right| \leq C \int \mathrm{~d} r(t) \int \mathrm{d} \tilde{y} \varphi(y-\tilde{y})|y-\tilde{y}| \leq C \varepsilon \tag{6.10}
\end{equation*}
$$

Concerning $\bar{S}_{2}$, we observe that (omitting for brevity the dependence on $(x, t)$ in $u_{f}(x, t), T_{f}(x, t)$ and on $(y, t)$ in $\left.u_{g}^{\varphi}(y, t), T_{g}^{\varphi}(y, t)\right)$

$$
\begin{aligned}
\left|T_{f}-T_{g}^{\varphi}\right|= & \frac{1}{d}\left|\int \mathrm{~d} \Lambda_{x, y}(v, w)\left(\left|v-u_{f}\right|^{2}-\left|w-u_{g}^{\varphi}\right|^{2}\right)\right| \\
= & \frac{1}{d}\left|\int \mathrm{~d} \Lambda_{x, y}(v, w)\left(v-w+u_{g}^{\varphi}-u_{f}\right) \cdot\left(v-u_{f}+w-u_{g}^{\varphi}\right)\right| \\
\leq & \frac{2}{d}\left[\int \mathrm{~d} \Lambda_{x, y}(v, w)\left(|v-w|^{2}+\left|u_{g}^{\varphi}-u_{f}\right|^{2}\right)\right]^{1 / 2} \\
& \times\left[\int \mathrm{d} \Lambda_{x, y}(v, w)\left(\left|v-u_{f}\right|^{2}+\left|w-u_{g}^{\varphi}\right|^{2}\right)\right]^{1 / 2} \\
\leq & C\left[\int \mathrm{~d} \Lambda_{x, y}(v, w)\left(|v-w|^{2}+\left|u_{g}^{\varphi}-u_{f}\right|^{2}\right)\right]^{1 / 2}\left(\sqrt{T_{f}}+\sqrt{T_{g}^{\varphi}}\right) .
\end{aligned}
$$

where in the first and last step we used (6.6). Therefore,

$$
\begin{aligned}
\bar{S}_{2} & =\int \mathrm{d} r^{\varphi}(t) d\left(\frac{T_{f}(x, t)-T_{g}^{\varphi}(y, t)}{\sqrt{T_{f}(x, t)}+\sqrt{T_{g}^{\varphi}(y, t)}}\right)^{2} \\
& \leq C \int \mathrm{~d} r^{\varphi}(t) \int \mathrm{d} \Lambda_{x, y}(v, w)\left(|v-w|^{2}+\left|u_{g}^{\varphi}(y, t)-u_{f}(x, t)\right|^{2}\right) \\
& =C \int \mathrm{~d} R(t)|v-w|^{2}+C \int \mathrm{~d} R(t)\left|u_{g}^{\varphi}(y, t)-u_{f}(x, t)\right|^{2} \\
& =C \int \mathrm{~d} R(t)|v-w|^{2}+C S_{1}
\end{aligned}
$$

Therefore, from Eq. (6.10) and the above estimate,

$$
\begin{equation*}
S_{2} \leq C I(t)+C S_{1}+C \varepsilon \tag{6.11}
\end{equation*}
$$

From Eqs. (6.3), (6.4), (6.9), (6.11), and Grönwall's inequality, Eq. (2.26) follows and Theorem 2.5 is proved.

Proof of Theorem 2.6. We have,

$$
\begin{aligned}
\mathcal{W}_{2}\left(f_{j}^{N}(t), f(t)^{\otimes j}\right) & \leq \mathcal{W}_{2}\left(f_{j}^{N}(t), g(t)^{\otimes j}\right)+\mathcal{W}_{2}\left(g(t)^{\otimes j}, f(t)^{\otimes j}\right) \\
& \leq C \sqrt{j}\left(\frac{1}{N^{1 / 8}} \exp \frac{C\left(\varphi_{0}^{3}+\|\nabla \varphi\|_{\infty}^{2}\right)}{r^{5 d} \varphi_{0}^{2}}+\sqrt{\varepsilon}\right),
\end{aligned}
$$

after using Theorem 2.5, Theorem 2.3, and inserting the expression of $\Gamma_{\varphi}$ given in Eq. (5.25).

Since $\|\nabla \varphi\|_{\infty} \leq C \varepsilon^{-(d+1)}, \varphi_{0} \leq C \varepsilon^{-d}, r>C \varepsilon$ (in order to satisfy Eq. (4.1)), and $N>N_{\varphi}$, the limit Eq. (2.27) follows provided $\varepsilon$ is vanishing slowly when $N$ diverges (e.g., $\varepsilon=(\log N)^{-\mu}$ with $\mu$ sufficiently small). The theorem is thus proved.

## References

[1] P.L. Bhatnagar, E.P. Gross and M. Krook, A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems, Phys. Rev., 94 (1954), 511-525.
[2] P. Buttà, M. Hauray and M. Pulvirenti, Particle approximation of the BGK equation, Arch. Ration. Mech. Anal., 240 (2021), 785-808.
[3] S. Caprino, M. Pulvirenti and W. Wagner, Stationary particle systems approximating stationary solutions to the Boltzmann equation, SIAM J. Math. Anal., 4 (1998), 913-934.
[4] C. Cercignani, R. Illner and M. Pulvirenti, The Mathematical Theory of Dilute Gases, Applied Mathematical Sciences 106 Springer-Verlag, New York, 1994.
[5] D. Mustafa and B. Wennberg, The BGK equation as the limit of an $N$ particle system, $J$. Stat. Phys., 181 (2020), 715-737.
[6] I. Olkin and F. Pukelsheim, The distance between two random vectors with given dispersion matrices, Linear Algebra Appl., 48 (1982), 25-263.
[7] B. Perthame and M. Pulvirenti, Weighted $L^{\infty}$ bounds and uniqueness for the Boltzmann BGK model, Arch. Ration. Mech. Anal., 125 (1993), 289-295.
[8] M. Pulvirenti, W. Wagner and M.B. Zavelani Rossi, Convergence of particle schemes for the Boltzmann equation, Eur. J. Mech. B/Fluids, 13 (1994), 339-351.
Email address: butta@mat.uniroma1.it
Email address: pulviren@mat.uniroma1.it


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[^1]:    ${ }^{1}$ The assumption $\varphi_{0} \geq 1$ in unnecessary but it makes cleaner some estimates. On the other hand, it is not restrictive as we are interested in the case when $\varphi$ converges to the $\delta$-function.

[^2]:    ${ }^{2}$ This means that $f$ solves the integral equation,

    $$
    f(x, v, t)=f_{0}(x-v t, v)+\int_{0}^{t} \mathrm{~d} s\left(\varrho_{f} M_{f}-f\right)(x-v(t-s), v, s),
    $$

