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The Boltzmann–Grad limit of a hard sphere system: analysis of the correlation error

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Abstract We present a quantitative analysis of the Boltzmann–Grad (low-density) limit of a hard sphere system. We introduce and study a set of functions, the *correlation errors*, measuring the deviations in time from the statistical independence of particles (propagation of chaos). In the context of the BBGKY hierarchy, a correlation error of order k measures the event where k particles are connected by a chain of interactions preventing the factorization. We show that, provided $k < \varepsilon^{-\alpha}$, such an error flows to zero with the average density ε, for short times, as $\varepsilon^{\gamma k}$, for some positive $\alpha, \gamma \in (0, 1)$. This provides an information on the *size of chaos*, namely j different particles behave as dictated by the Boltzmann equation even when j diverges as a negative power of ε. The result requires a rearrangement of Lanford perturbative series into a cumulant type expansion, and an analysis of many-recollision events.

In memory of Oscar Erasmus Lanford III.

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List of symbols

$(\boldsymbol{\alpha}, \boldsymbol{\beta})$	Table of recollisions
B	Boltzmann collision kernel
B^ε	BBGKY collision kernel
$\mathcal{C}^\varepsilon_{j+1}$	BBGKY collision operator
$\mathcal{C}^{\mathscr{E}}_{j+1}$	Enskog hierarchical collision operator
$\mathcal{C}_{j+1}$	Boltzmann hierarchical collision operator
$\chi_{H,K}$, $\bar{\chi}_{H,K}$	Generic constraints on graphs
$\chi^{ov}_{H,K}$	Overlap constraint
$\chi^{rec}_{L_0}$	Recollision constraint
Δ	Axis of $\mathcal{T}^\varepsilon_\xi$
$d\Lambda$	Integration measure in the tree expansion
$\mathbb{E}^{\mathcal{B}}$	Expectation with respect to the Boltzmann density
$\boldsymbol{\eta}^\varepsilon(\cdot)$	Velocities in the IBF
E_j	Correlation error of order j
$E^{\mathcal{B}}_j$	Boltzmann error term
$E^{\mathscr{E}}_j$	Enskog error term
$E^0_{\mathcal{K}}$	Time-zero correlation error associated to the partition in the clusters $\mathcal{K}$
$\bar{E}^0_{\mathcal{K}}$	Extension of $E^0_{\mathcal{K}}$ to the whole space
$\bar{E}_{\mathcal{K}}$	Extension of $E_{\mathcal{K}}$ to the whole space
F_i	Observable in the particle system, associated to the test function φ_i
f	Solution to the Boltzmann equation
f_j	j-particle function solving the Boltzmann hierarchy
f^ε_j	Rescaled correlation function (r.c.f.) of order j
F_{θ_3}	A cutoffed function of the energy
g^ε	Solution to the Enskog equation
g^ε_j	j-particle function solving the Enskog hierarchy
Γ_i	Tree generated by particle i
$\Gamma(j,n)$	n-collision, j-particle tree
$\mathcal{H}_K$	Energy of the trees in K
J	Set of indices of particles $\{1, 2, \dots, j\}$
$\mathcal{J}$	Set of indices of clusters $\{1, 2, \dots, j\}$
$\mathcal{M}$	Grand-canonical phase space
$\mathcal{M}_n$	Canonical n-particle phase space
$\mathcal{M}^x_n(\delta)$	Position space of n particles with mutual distance larger than δ
n_Δ	Fraction of particles in the region $\Delta \subset \mathbb{R}^3 \times \mathbb{R}^3$
ρ^ε_j	Correlation function of order j

$S(i)$	Set of particles belonging to the tree Γ_i
$\mathcal{S}_j^\varepsilon$	j-particle interacting flow operator
$\mathcal{S}_j$	j-particle free flow operator
$\mathbf{t}_n$	Times of scattering (creation) in backwards flow
(t_i, ω_i, v_{j+i})	Triple describing a scattering (creation) in backwards flow
$\mathcal{T}_\xi^\varepsilon$	"Tube" of external recollision
T_n^ε	n-particle hard sphere flow
v_i	Velocity of particle i
$\mathbf{W}^\varepsilon$	State of the hard sphere system: a collection of measures $\{W_{0,n}^\varepsilon\}_{n\geq 0}$
$\boldsymbol{\xi}^\varepsilon(\cdot)$	Positions in the IBF
x_i	Position of particle i
$\boldsymbol{\zeta}^\varepsilon(\cdot)$	Interacting backwards flow
$\boldsymbol{\zeta}^\mathcal{B}(\cdot)$	Boltzmann backwards flow
$\boldsymbol{\zeta}^\mathscr{E}(\cdot)$	Enskog backwards flow
$\tilde{\boldsymbol{\zeta}}^\varepsilon(\cdot)$	Uncorrelated interacting backwards flow
$\bar{\zeta}^i(\cdot)$	Virtual trajectory of particle i in the flow $\bar{\zeta}$
z_i	State (position x_i, velocity v_i) of particle i
$\mathbf{z}_j$	Vector $(z_1, \ldots, z_j)$
$\mathbf{z}_{j,n}$	Vector $(z_{j+1}, \ldots, z_{j+n})$

Contents

1 Introduction

1.1 Lanford's theorem and beyond

In 1975 O. E. Lanford III presented his celebrated proof of the mathematical validity of the Boltzmann equation, in a time interval small enough [25]. To recall his result, let us consider a system of identical hard spheres of diameter ε moving in the space $\mathbb{R}^3$ with collisions governed by the laws of elastic reflection.

The initial data are random, including the number of particles $n \in \mathbb{N}$. The state of the system is specified by an absolutely continuous probability on the grand canonical phase space $\cup_{n\geq 0}(\mathbb{R}^3 \times \mathbb{R}^3)^n$. Restricted to n particles, it is given by a function $W^\varepsilon_{0,n}$ such that

$$\frac{1}{n!} W^\varepsilon_{0,n}(z_1, \ldots, z_n) dz_1 \cdots dz_n$$

is the probability of finding exactly n particles in $dz_1 \cdots dz_n$ at the initial time, say $t = 0$. Here $z_i = (x_i, v_i) \in \mathbb{R}^3 \times \mathbb{R}^3$ stands for the position and the velocity of the i-th particle. $W^\varepsilon_{0,n}$ is symmetric by permutation of the particle labels. If $p_n = (1/n!) \int W^\varepsilon_{0,n}$, then $\sum_{k\geq 0} p_k = 1$ and the average number of particles is

$$\langle n \rangle := \sum_{k\geq 0} k\, p_k.$$

We analyze a low-density limit, the *Boltzmann–Grad limit* [18,19] defined by

$$\langle n \rangle \to \infty, \quad \varepsilon \to 0 \quad \text{and} \quad \langle n \rangle \varepsilon^2 \to \lambda^{-1} > 0, \tag{1.1}$$

at times of order 1, where λ is a fixed constant proportional to the mean free path. The fraction of volume occupied by the particles $\sim\langle n \rangle \varepsilon^3$ goes to zero.

It is not conceivable to follow in time the positions and the velocities of the entire particle system. We are rather interested in the amount of single particles, couples of particles, triples etc., in given configurations. We define the vector of *correlation functions* $\{\rho^{\varepsilon}_{0,j}\}_{j\geq 0}$ by

$$\rho^{\varepsilon}_{0,j}(z_1,\ldots,z_j)=\sum_{k=0}^{\infty}\frac{1}{k!}\int dz_{j+1}\cdots dz_{j+k}\, W^{\varepsilon}_{0,j+k}(z_1,\ldots,z_{j+k}).$$

Under a boundedness assumption, the map $\{W^{\varepsilon}_{0,n}\}_{n\geq 0} \rightarrow \{\rho^{\varepsilon}_{0,j}\}_{j\geq 0}$ is invertible (with $W^{\varepsilon}_{0,n}=\sum_{k=0}^{\infty}\left((-1)^k/k!\right)\int dz_{n+1}\cdots dz_{n+k}\,\rho^{\varepsilon}_{0,n+k}$) and the correlation functions are an alternative way to encode the statistical properties of the particle system.

Since $\int \rho^{\varepsilon}_{0,j}(z_1,\ldots,z_j)\,dz_1\cdots dz_j=\langle n(n-1)\cdots(n-j+1)\rangle$, the scaling (1.1) requires that $\int \rho^{\varepsilon}_{0,j}\sim\varepsilon^{-2j}$ as $\varepsilon\rightarrow 0$. This leads to introduce *rescaled correlation functions* (r.c.f.)

$$f^{\varepsilon}_{0,j}=\varepsilon^{2j}\rho^{\varepsilon}_{0,j},$$

expected to be finite when $\varepsilon\rightarrow 0$. The r.c.f. differ from the *marginals* of the measure only by proper normalization factors.

We focus on the quantities $f^{\varepsilon}_j(t)$, namely the r.c.f. of the system at time $t>0$, evolved deterministically (starting from $f^{\varepsilon}_{0,j}$) according to the hard sphere dynamics.

Lanford proves that, if the initial state factorizes in the limit, meaning that

$$\lim_{\varepsilon\rightarrow 0} f^{\varepsilon}_{0,j}=f_0^{\otimes j}, \tag{1.2}$$

then there exists $\bar{t}>0$ such that

$$\lim_{\varepsilon\rightarrow 0} f^{\varepsilon}_j(t)=f(t)^{\otimes j}\quad \text{for } 0\leq t<\bar{t}. \tag{1.3}$$

Here f_0 is a given one-particle probability density and $f(t)$ is a solution of the Boltzmann equation with initial datum f_0. The convergence fails on certain exceptional sets as will be discussed in detail. Initially, j particles are "almost independent" by (1.2) and (1.3) shows that this property propagates, at least for short times.

Viewed probabilistically, the result is a law of large numbers, that is with probability close to 1, the fraction of particles in the volume element Δ at time t is approximated by $\int_{\Delta} dz f(z,t)$. Moreover, it has been shown that, looked on the scale of the interparticle distance $\sim\varepsilon^{2/3}$, the distribution of particles is close to a homogeneous Poisson law with density $\int dv f(x,v,t)$ at (x,t) [39].

Note that we found convenient to recall the theorem of Lanford as stated in [23] (or also in [3,35]), namely with random total number of particles n. The advantage of this formulation in our context will be discussed later on.

We write here the *Boltzmann equation* for the density $f = f(x, v, t)$, with hard sphere kernel and mean free path λ [8],

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = \lambda^{-1} \int_{\mathbb{R}^3} dv_1 \int_{S^2_+} d\omega (v - v_1) \cdot \omega \times \left\{ f(x, v'_1, t) f(x, v', t) - f(x, v_1, t) f(x, v, t) \right\} \tag{1.4}$$

where $S^2_+ = \{\omega \in S^2 | \, (v - v_1) \cdot \omega \geq 0\}$, S^2 is the unit sphere in $\mathbb{R}^3$ (with surface measure $d\omega$), (v, v_1) is a pair of velocities in incoming collision configuration and (v', v'_1) is the corresponding pair of outgoing velocities defined by the elastic reflection rules

$$\begin{cases} v' = v - \omega[\omega \cdot (v - v_1)] \\ v'_1 = v_1 + \omega[\omega \cdot (v - v_1)]. \end{cases} \tag{1.5}$$

This equation is well understood in the case of spatially homogeneous solutions [29] and of perturbations of the equilibrium [41,42]; we refer to [44] for a review on the Cauchy problem. In the general inhomogeneous setting, global existence and stability have been proved based on the notion of *renormalized solution* [14].

Lanford's result also implies, of course, local in time existence under proper hypotheses on the initial data. More precisely, the initial datum is bounded from above by a Maxwellian. The time $\bar{t}$ in (1.3) is a fraction of the mean free flight time. Nevertheless, it is enough to show unambiguously that there is no contradiction between the reversible microscopic dynamics and the irreversible behaviour described by the Boltzmann equation. We shall adopt the restriction $0 \leq t < \bar{t}$ in the present work.

So far, an extension of Lanford's theorem to arbitrary times has been achieved in the special situation of a rare cloud of gas expanding in the infinite space [20,21]. This relies on a smallness assumption on the initial datum ("perturbation of the vacuum") for which the Boltzmann equation becomes close to the free transport equation. More recently, it has been shown that the Boltzmann–Grad limit can be controlled up to very long times for a tracer particle in a hard sphere fluid at equilibrium [4]. Brownian motion is then derived in a hydrodynamic regime, using the *linear* Boltzmann equation as intermediate step.

The proof in [25], and in the subsequent works on the subject, is carried out by assuming suitable uniform estimates on the family of r.c.f. at time zero.

The available estimates deteriorate in time in such a way that, at time $\bar{t}$, any possibility of a uniform control is lost. Indeed the strategy of Lanford is based on an expansion of $f_j^\varepsilon(t)$, for given $j > 0$, in terms of a series (involving only the initial data $f_{0,j}^\varepsilon$) which is absolutely convergent, uniformly in ε, only for $0 \leq t < \bar{t}$. To complete the proof it is enough to exploit the term by term convergence. Further discussions can be found in [13,16,37,39,40,43].

The original argument of Lanford is qualitative, in the sense that (1.3) is shown without an explicit rate of convergence. Recently, a rate of convergence has been obtained in [16]. It is of the form

$$|f_j^\varepsilon(t) - f(t)^{\otimes j}| \leq C^j \varepsilon^\gamma \quad \text{for } 0 \leq t < \bar{t}, \tag{1.6}$$

for some $C, \gamma > 0$, outside a subset of $(\mathbb{R}^3 \times \mathbb{R}^3)^j$ which measure goes to zero as $\varepsilon \to 0$ (see also [30] for a different class of potentials).

The purpose of the present paper is to introduce a notion of *correlation error* for the many-particle system, providing explicit estimates.

Before giving precise definitions, let us explain our result informally. From now on, we assume $0 \leq t < \bar{t}$.

1.2 The size of chaos

The result (1.3) has been proved by Lanford for any fixed $j > 0$. This cannot be uniform in j since, for very large j e.g. $j \simeq \varepsilon^{-2}$, $f_j^\varepsilon(t)$ is far from a tensor product. It is however natural to ask whether there exists a notion of convergence holding for $j = j(\varepsilon)$ suitably diverging with ε.

A first answer is given by the quantitative analysis in [16,30]. From (1.6) it follows that the convergence of r.c.f. holds for $j \leq C_0|\log \varepsilon|$, for some positive C_0. One would expect, instead, a power–law divergence, at least along the following heuristic argument.

The proof of the asymptotic behavior (1.3) is intimately connected with the problem of *propagation of chaos*, i.e. the conservation in time of the statistical independence of particles (provided that it holds at time zero). Given a group of j particles, consider, for any $i = 1, 2, \ldots, j$, the set B_i of particles "really influencing" the dynamics of particle i up to the time t. We assume that the cardinality of the sets B_i is finite to have a correct kinetic behaviour in the limit. For the propagation of chaos to hold, the groups B_i must be disjoint. Therefore, the probability that two given particles in the group $\{1, 2, \ldots, j\}$ are dynamically correlated will be $O(1/\langle n \rangle)$. Correspondingly, the probability that the j particles do *not* behave as mutually independent will be $O(j^2/\langle n \rangle)$, which is small for $j \ll \varepsilon^{-1}$. (We refer to [2] for related considerations.)

Our goal here is to analyze the "size of chaos", i.e., how large can be a cluster of asymptotically independent particles.

In the effort of going beyond the logarithmic scale we immediately realize that there is no hope to improve estimate (1.6). Even ignoring the correlations and assuming that $f_j^\varepsilon(t) \approx (f_1^\varepsilon(t))^{\otimes j}$, one cannot do better than expanding $(f_1^\varepsilon(t))^{\otimes j} - (f(t))^{\otimes j}$. On the other hand, trivially, $(f_1^\varepsilon(t) - f(t))^{\otimes j} = O(\varepsilon^{\gamma j})$ uniformly in j. We are led then to give the following notion of error. Let $\mathbf{z}_n = (z_1, \ldots, z_n)$ be a configuration of the particle system and let $\mathbf{z}_n(t) = (z_1(t), \ldots, z_n(t))$ be the corresponding time-evolved configuration. Given a sequence of test functions over $\mathbb{R}^3 \times \mathbb{R}^3$, denoted $\varphi_1, \varphi_2, \ldots$, we consider the one-particle observables $F_1 = F_1(t)$, $F_2 = F_2(t), \ldots$ defined by

$$F_i(t)(\mathbf{z}_n) = \varepsilon^2 \sum_{j=1}^{n} \varphi_i(z_j(t)). \tag{1.7}$$

When φ_i is the characteristic function of the set Δ, then $F_i(t)$ is the fraction of particles in Δ at time t. In terms of law of large numbers, the validity of the Boltzmann equation can be rephrased by saying that the error $(F_i(t) - \mathbb{E}^{\mathcal{B}}[\varphi_i(t)]) \approx 0$ for ε small, where $\mathbb{E}^{\mathcal{B}}[\varphi(t)] = \int dx dv\, \varphi(x, v)\, f(x, v, t)$. We look now at the product of j such simultaneous deviations and compute its expected value $\mathbb{E}^\varepsilon$ in the state of the particle system. We will prove that, in the Boltzmann–Grad limit, there exists a constant $\alpha \in (0, 1)$ such that, if $0 \le t < \bar{t}$,

$$\lim_{\varepsilon \to 0} \sup_{j < \varepsilon^{-\alpha}} \left| \mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^{\mathcal{B}}[\varphi_i(t)] \right) \right] \right| = 0. \tag{1.8}$$

Roughly speaking, with respect to (1.3), we replace the difference of products with the product of differences, which is expected to be much smaller (but more difficult to control).

By (1.8) groups of up to $\varepsilon^{-\alpha}$ particles become statistically independent and simultaneous deviations of the particles behaviour from the Boltzmann behaviour are negligible in the limit.

1.3 Result on correlation errors

The notion of error in (1.8) is closely related to what is known, in statistical physics, as *fluctuation* around the average value. Usually one focuses on the particle system and ignores the convergence error $\mathbb{E}^{\mathcal{B}}[\varphi_i(t)] - \mathbb{E}^\varepsilon[F_i(t)]$. The quantity

$$\mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^\varepsilon[F_i(t)] \right) \right] \approx 0 \tag{1.9}$$

gives the j-th order moment of the fluctuation field and is formally seen to be $O(\varepsilon^j)$ for any fixed j. For previous results on the fluctuations in the Boltzmann–Grad limit, see [3,35,36].

To be more concrete, let us choose a collection of disjoint sets $\Delta_1, \ldots, \Delta_j$ in $\mathbb{R}^3 \times \mathbb{R}^3$ and, as φ_i, the indicator function of the set Δ_i. $F_i(t) = n_{\Delta_i}$ is the fraction of particles in the region. We have

$$\mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(n_{\Delta_i} - \mathbb{E}^\varepsilon [n_{\Delta_i}] \right) \right] = \int_{\Delta_1 \times \cdots \times \Delta_j} dz_1 \cdots dz_j \, E_j(z_1, \ldots, z_j, t), \tag{1.10}$$

which introduces a new sequence of (ε-dependent) functions $E_j = E_j(\mathbf{z}_j, t)$, $j = 1, 2, \ldots$. Here we call $E_j(t)$ the *correlation error* of order j. Its size is a measure of the statistical dependence of j distinct particles in different regions. In our work, these will be the fundamental objects.

Technically, E_j is connected to the r.c.f. by a cumulant type expansion. Explicitly,

$$\begin{aligned} E_1(z_1) &= 0, \\ E_2(z_1, z_2) &= f_2^\varepsilon(z_1, z_2) - f_1^\varepsilon(z_1) f_1^\varepsilon(z_2), \\ E_3(z_1, z_2, z_3) &= f_3^\varepsilon(z_1, z_2, z_3) - f_2^\varepsilon(z_1, z_2) f_1^\varepsilon(z_3) - f_2^\varepsilon(z_1, z_3) f_1^\varepsilon(z_2) \\ &\quad - f_2^\varepsilon(z_2, z_3) f_1^\varepsilon(z_1) + 2 f_1^\varepsilon(z_1) f_1^\varepsilon(z_2) f_1^\varepsilon(z_3), \end{aligned} \tag{1.11}$$

etc., and for generic j

$$E_J(t) = \sum_{K \subset J} (-1)^k (f_1^\varepsilon(t))^{\otimes K} f_{J \setminus K}^\varepsilon(t), \tag{1.12}$$

where $J = \{1, 2, \ldots, j\}$, K is a subset of indices in J ($\emptyset$ and J are included in the sum, with the convention $f_\emptyset^\varepsilon = 1 = (f_1^\varepsilon)^{\otimes \emptyset}$), $k = |K|$ is the cardinality of the set K and, if $Q = \{i_1, \ldots, i_q\}$, one denotes

$$\begin{aligned} & f_Q^\varepsilon(t) = f_Q^\varepsilon(\mathbf{z}_Q, t) = f_q^\varepsilon(z_{i_1}, \ldots, z_{i_q}, t), \\ & E_Q(t) = E_Q(\mathbf{z}_Q, t) = E_q(z_{i_1}, \ldots, z_{i_q}, t), \\ & (f_1^\varepsilon(t))^{\otimes Q} = f_1^\varepsilon(z_{i_1}, t) f_1^\varepsilon(z_{i_2}, t) \cdots f_1^\varepsilon(z_{i_q}, t). \end{aligned} \tag{1.13}$$

Equation (1.12) can be inverted to give

$$f_J^\varepsilon(t) = \sum_{K \subset J} (f_1^\varepsilon(t))^{\otimes K} E_{J \setminus K}(t), \tag{1.14}$$

having a physical interpretation as a sum over subgroups of uncorrelated particles. In this paper we shall adopt (1.14) as definition of correlation error. The connection with fluctuations and cumulant expansions will be further discussed in Sects. 4.7 and 4.8.1.

We stress again that the above definitions involve the particle system only and do not refer to any kinetic equation. The propagation of chaos amounts to say that, for any given j, as $\varepsilon \to 0$

$$E_j(0) \to 0 \quad \text{for all } j \geq 1 \Longrightarrow \quad E_j(t) \to 0 \quad \text{for all } j \geq 1 \tag{1.15}$$

for $t > 0$. The correlation errors identify and strictly isolate those dynamical events which destroy propagation of chaos.

Our main result will be the following. *There exist constants* $\alpha, \gamma \in (0, 1)$ *such that, in the Boltzmann–Grad limit, if* $0 \leq t < \bar{t}$ *and provided* $j < \varepsilon^{-\alpha}$,

$$\int dv_1 \cdots dv_j |E_j(z_1, \ldots, z_j, t)| \leq \varepsilon^{\gamma j} \tag{1.16}$$

for any given configuration $x_1, \ldots, x_j$ *of distinct points*. The available estimates are such that a linear combination of α, γ with positive coefficients is bounded by $c < 1$. Moreover, $c \to 0$ as $x_1, \ldots, x_j$ approach a small scale distance. Of course an estimate similar to (1.16) has to be assumed at time zero, together with uniform estimates on the family of r.c.f. (as in Lanford's theorem). We shall construct natural examples of initial states satisfying such hypotheses. The convergence result (1.8) will be a consequence of the main estimate (1.16).

1.4 Strategy: hierarchical particle flows

Let us comment briefly on the difficulties. The only known strategy to rigorously derive estimates on the particle system goes through a reformulation of the problem in terms of characteristics of a set of hierarchical equations. Such flows share all the features of the interacting dynamics of finite groups of particles and their control is a delicate task.

The breakdown of the statistical independence is indeed due to *mechanical* effects. First of all, one should keep in mind that any given state of the system (in particular, whatever choice of the time-zero state) cannot be exactly factorized, simply because of the hard core exclusion. This is a static feature. Secondly, and most importantly, correlations between particles are generated by the dynamics itself. In the context of [25] and of the subsequent literature, the events responsible for these dynamical correlations are called *recollisions*. Their effective control is quite complicated since they generally depend on the full particle dynamics.

To be more precise, one looks at the BBGKY hierarchy, namely the set of coupled evolution equations for $\{f_j^\varepsilon\}_{j\geq 1}$.[1] The iteration of the hierarchy gives an expression of $f_j^\varepsilon(t)$ in terms of a series expansion depending only on the initial data. Each term of this expansion is in one-to-one correspondence with a special trajectory of clusters of particles flowing backwards in time. It is looking at these flows that we single out precisely the (re-)collisions that generate correlations.

Roughly, formulas (1.14), (1.16) will be constructed starting from the BBGKY expansion for f_j^ε, by systematically replacing such collision-events with "free overlap-events" where the two considered particles ignore and cross each other freely, and estimating the consequent errors (see Sect. 4.2 for a simple example). In addition, one has to extract the correlation error of the initial state due to the exclusion. The main technical part of the work shall consist of (1) a suitable cluster expansion (needed to control the total number of produced terms) and (2) geometric estimates for trajectories of j particles showing up many recollisions.

The net result expresses $f_j^\varepsilon(t)$ as a sum of contributions (1.14). The first, $O(1)$, is just the product state. Then, we sum over all possible ways of choosing two correlated particles, the remaining $j-2$ particles being uncorrelated. This events are $O(\varepsilon^{2\gamma})$ (actually $O(\varepsilon^2)$). Then we pass to the events in which three particles are correlated, which give a contribution $O(\varepsilon^{3\gamma})$, and so on.

Note that we derive the bound on E_j, as roughly explained above, exploiting the series expansion for f_j^ε. Another possibility is to use (1.12) and the evolution equations for f_j^ε and f_1^ε. However a closed evolution equation for correlation errors seems to be more difficult to handle.

1.5 The Enskog error

Although the main technical effort in the present work will concern the correlation error E_j, it is important to observe that this function describes a part, but not *all*, of the total correlation between particles. The remainder is encoded in the one-point function f_1^ε.

Working again in terms of backwards flows, one may extract from the definition of f_1^ε a second (and last) class of recollision-events. This operation leads to define another sequence of quantities $E_j^{\mathscr{E}}(\mathbf{z}_j, t),\ j = 1, 2, \ldots$, given by

$$f_J^\varepsilon(t) = \sum_{K\subset J} (g^\varepsilon(t))^{\otimes K} E_{J\setminus K}^{\mathscr{E}}(t) \tag{1.17}$$

[1] Originally written for smooth potentials by Bogolyubov, Born, Green, Kirkwood and Yvon [6] and, later on, by Cercignani for the hard sphere system [12].

(where we extend the notations of (1.13)) or by

$$E_J^{\mathscr{E}}(t) = \sum_{K \subset J} (-1)^k (g^\varepsilon(t))^{\otimes K} f_{J\setminus K}^\varepsilon(t), \tag{1.18}$$

where $g^\varepsilon(t)$ is defined by an explicit expression that does *not* involve any more correlations among particles. Namely, $g^\varepsilon(t)$ is the series solution to the *Enskog equation* (more properly, the Boltzmann–Enskog equation [7]), which we recall:

$$\begin{aligned}(\partial_t + v \cdot \nabla_x) g^\varepsilon(x, v, t) = \lambda^{-1} \int_{\mathbb{R}^3} dv_1 \int_{S_+^2} d\omega\, (v - v_1) \cdot \omega \\ \times \left\{ g^\varepsilon(x - \omega\varepsilon, v_1', t) g^\varepsilon(x, v', t) - g^\varepsilon(x + \omega\varepsilon, v_1, t) g^\varepsilon(x, v, t) \right\}. \end{aligned} \tag{1.19}$$

Here we used the notations introduced next to (1.4). We shall refer to this second class of correlation errors $E_j^{\mathscr{E}}(\mathbf{z}_j, t)$, $j = 1, 2, \ldots$, as the *Enskog error* terms.

Note that if $f_j^\varepsilon(t)$ factorizes strictly, i.e. $f_J^\varepsilon(t) = (f_1^\varepsilon(t))^{\otimes J}$, then

$$E_J^{\mathscr{E}}(t) = \left((f_1^\varepsilon - g^\varepsilon)(t)\right)^{\otimes J}.$$

More properly, $E_J^{\mathscr{E}}(t)$ measures both the breakdown of propagation of chaos and the error in the convergence of f_1^ε to g^ε. We will show that $E_j^{\mathscr{E}}(t)$ can be bounded as $E_j(t)$, i.e.

$$\int dv_1 \cdots dv_j |E_j^{\mathscr{E}}(t)| \leq \varepsilon^{\gamma j} \tag{1.20}$$

for $0 \leq t < \bar{t}$ and $j < \varepsilon^{-\alpha}$, as soon as $f_1^\varepsilon(0)$ is assumed to converge uniformly as a power of ε to the initial datum for the Enskog equation.

In our framework, the Enskog equation appears as a natural *bridge* between the hard sphere dynamics and the Boltzmann equation. In particular, to obtain the representation (1.17)–(1.20), no regularity property needs to be assumed for the state of the system. The Enskog picture is what emerges from the mechanical system once we eliminate all the sources of correlation, including both the dynamical correlations and the static correlations of the time-zero state.

1.6 The Boltzmann error

Finally, the only difference between the Enskog system described by $g^\varepsilon(t)$ and the Boltzmann system described by $f(t)$ and (1.4) (with same initial datum), is that the interactions occur at distance ε instead of zero. In other words, microscopic translations of the Enskog flow lead to the Boltzmann flow. A simple continuity property (assumed for the initial data) implies now

$$\int dv_1 \cdots dv_j |E_j^{\mathcal{B}}(t)| \leq \varepsilon^{\gamma j} \tag{1.21}$$

for $0 \leq t < \bar{t}$ and $j < \varepsilon^{-\alpha}$, where the *Boltzmann error* term $E_j^{\mathcal{B}}$, $j = 1, 2, \ldots$ is defined by

$$f_J^{\varepsilon}(t) = \sum_{K \subset J} (f(t))^{\otimes K} E_{J \setminus K}^{\mathcal{B}}(t) \tag{1.22}$$

or by

$$E_J^{\mathcal{B}}(t) = \sum_{K \subset J} (-1)^k (f(t))^{\otimes K} f_{J \setminus K}^{\varepsilon}(t). \tag{1.23}$$

Equations (1.21)–(1.22) reformulate Lanford's result, together with an explicit representation of the error. The restriction to short times is also the same. However if the Boltzmann equation were globally valid, the statistical independence could not fail to hold and, in this case, we believe that our estimations would be also globally valid.

The quantities $E_j^{\mathcal{B}}(t)$, under the name "v-functions", were previously introduced in [9–11] in dealing with kinetic limits of stochastic particle systems.

Equation (1.8) follows from a further estimate of contraction terms due to the fact that, for generic observables, the same particle may appear simultaneously in the computation of F_i and F_j, $i \neq j$.

2 Assumptions and main results

In this section we describe precisely our setting, fix the notation and state the main results.

2.1 The hard sphere system

We consider a *system of hard spheres* of unit mass and of diameter $\varepsilon > 0$ moving in the whole space $\mathbb{R}^3$. We will denote

$$z_i = (x_i, v_i) \in \mathbb{R}^3 \times \mathbb{R}^3$$

the state (position, velocity) of the i-th particle, $i = 1, 2, \ldots$. For groups of particles we shall use the notation

$$\mathbf{z}_j = z_1, \ldots, z_j, \quad \mathbf{z}_{j,n} = z_{j+1}, \ldots, z_{j+n},$$

and call "particle i" a particle whose configuration is labelled by the index i.

We will work in the grand-canonical *phase space*

$$\mathcal{M}(\varepsilon) = \cup_{n\geq 0} \mathcal{M}_n(\varepsilon), \tag{2.1}$$

where

$$\mathcal{M}_n(\varepsilon) = \left\{ \mathbf{z}_n \in \mathbb{R}^{6n}, \ \ |x_i - x_j| > \varepsilon, \ i \neq j \right\}, \quad \mathcal{M}_0(\varepsilon) = \emptyset. \tag{2.2}$$

Unless necessary we omit, for simplicity, the dependence of the spaces on ε.

Notice that $\mathcal{M}_N$, with $N \sim \varepsilon^{-2}$, is the canonical N-particle phase space used in [25] and in most of the subsequent literature on the Boltzmann–Grad limit. In this paper we find convenient to consider a more general class of measures where the exact number of particles n is not necessarily fixed. The advantage of this picture will be discussed in Sect. 2.4.1, Remark 6.

The equations of motion for the n-particle system are defined as follows. Between collisions each particle moves on a straight line with constant velocity. When two hard spheres collide with positions x_i, x_j (at distance ε), impact direction

$$\omega = (x_i - x_j)/|x_i - x_j| = (x_i - x_j)/\varepsilon \in S^2$$

and incoming velocities v_i, v_j (that means $(v_i - v_j) \cdot \omega < 0$), these are instantaneously transformed to outgoing velocities v_i', v_j' (with $(v_i' - v_j') \cdot \omega > 0$) through the relations

$$\begin{aligned} v_i' &= v_i - \omega[\omega \cdot (v_i - v_j)], \\ v_j' &= v_j + \omega[\omega \cdot (v_i - v_j)]. \end{aligned} \tag{2.3}$$

The collision transformation is invertible and preserves the Lebesgue measure on $\mathbb{R}^6$.

The above prescription defines the *flow of the n-particle dynamics*, $t \mapsto \mathrm{T}_n^\varepsilon(t)\mathbf{z}_n$. Observe that these rules do not cover all possible situations, e.g. triple collisions are excluded. Nevertheless, as proved by Alexander in [1], there exists a full-measure subset of $\mathcal{M}_n$, over which $\mathrm{T}_n^\varepsilon(t)$ is uniquely defined for all t (see also [13,28]). Thus $\mathrm{T}_n^\varepsilon(t)$ can be defined as a one-parameter group of Borel maps on $\mathcal{M}_n$, leaving the Lebesgue measure invariant.

Notice that the flow $\mathrm{T}_n^\varepsilon(t)$ is piecewise continuous in t (we do not identify outgoing and incoming configurations). If necessary, we may distinguish the limit from the future $(+)$ and the limit from the past $(-)$ by writing $\mathrm{T}_n^\varepsilon(t^\pm)\mathbf{z}_n = \lim_{\varepsilon\to 0^+} \mathrm{T}_n^\varepsilon(t \pm \varepsilon)\mathbf{z}_n$. Moreover, we shall fix the convention of right-continuity of the flow, $\mathrm{T}_n^\varepsilon(t)\mathbf{z}_n = \mathrm{T}_n^\varepsilon(t^+)\mathbf{z}_n$.

2.2 Statistical states and kinetic limit

Let us turn now to a statistical description. We adopt a general formulation, in the spirit of classical statistical mechanics [32].

We introduce the set of density functions over $\mathcal{M}$, denoted $\mathbf{W}_0^\varepsilon = \{W_{0,n}^\varepsilon\}_{n\geq 0}$, where $W_{0,n}^\varepsilon : \mathcal{M}_n \to \mathbb{R}^+$ is a positive Borel function symmetric in the particle labels. The quantity $(1/n!)W_{0,n}^\varepsilon(\mathbf{z}_n)$ gives the probability density of finding exactly n particles in $z_1, \dots, z_n$. We refer to $\mathbf{W}_0^\varepsilon$ as the *state of the particle system.*

Note that n, the total number of particles, is a random variable, and $(1/n!)\int W_{0,n}^\varepsilon$ is its distribution. The normalization condition reads

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int W_{0,n}^\varepsilon = 1. \tag{2.4}$$

Given an initial measure over $\mathcal{M}$ with density specified by $\mathbf{W}_0^\varepsilon$, its evolution at time t is given by the *Liouville equation*

$$W_n^\varepsilon(\mathbf{z}_n, t) = W_{0,n}^\varepsilon(\mathrm{T}_n^\varepsilon(-t)\mathbf{z}_n), \tag{2.5}$$

to be valid almost everywhere in $\mathcal{M}_n$. This defines $\mathbf{W}^\varepsilon(t)$, the state at time t.

For notational convenience, we shall sometimes extend the definition of the state to the whole space as

$$W_n^\varepsilon(\mathbf{z}_n, t) = 0 \quad \text{if} \quad |x_i - x_k| < \varepsilon \tag{2.6}$$

for some $i \neq k$.

We define next the vector of *correlation functions* over $\mathcal{M}$ as $\boldsymbol{\rho}^\varepsilon(t) = \{\rho_j^\varepsilon(t)\}_{j\geq 0}$, $t \geq 0$, by

$$\rho_j^\varepsilon(\mathbf{z}_j, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathcal{M}_k} dz_{j+1} \cdots dz_{j+k} W_{j+k}^\varepsilon(\mathbf{z}_{j+k}, t). \tag{2.7}$$

A state admits correlation functions when the series in the right hand side of (2.7) is convergent, together with the series in the inverse formula

$$W_j^\varepsilon(\mathbf{z}_j, t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathcal{M}_k} dz_{j+1} \cdots dz_{j+k} \rho_{j+k}^\varepsilon(\mathbf{z}_{j+k}, t). \tag{2.8}$$

In this case, the set of functions $\boldsymbol{\rho}^\varepsilon(t)$ describes all the properties of the system. Later on, we will assume explicit estimates ensuring the convergence of the series for any finite ε.

The normalization condition for the correlation functions is

$$\int_{\mathcal{M}_j} \rho_j^\varepsilon(\mathbf{z}_j, t) d\mathbf{z}_j = \mathbb{E}_t(n(n-1)\cdots(n-j+1)) = \mathbb{E}_0(n(n-1)\cdots(n-j+1)), \tag{2.9}$$

where n is the total number of particles, and the expectation $\mathbb{E}_t$ is done with respect to the state $\mathbf{W}^\varepsilon(t)$.

In this setting, the *Boltzmann–Grad scaling* is given by the following condition: the *average* number of particles diverges as ε^{-2}, that is

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_{\mathbb{R}^6} \rho_1^\varepsilon(z_1, t) = \lambda^{-1}, \tag{2.10}$$

where $\lambda > 0$ is proportional to the mean free path. From now on, we shall fix

$$\lambda = 1$$

for simplicity.

The central object of our study becomes the collection of *rescaled correlation functions* (r.c.f.) defined by

$$f_j^\varepsilon(\mathbf{z}_j, t) = \varepsilon^{2j} \rho_j^\varepsilon(\mathbf{z}_j, t). \tag{2.11}$$

These are expected to be $O(1)$ as $\varepsilon \to 0$.

2.3 Assumptions on the initial state

2.3.1 Initial data for the particle system

The state of the hard sphere system at time zero, $\mathbf{W}_0^\varepsilon$, admits, as rescaled correlation functions, the collection $f_{0,j}^\varepsilon : \mathcal{M}_j \to \mathbb{R}^+$, $j \geq 0$, which are by definition Borel functions, symmetric for permutation of particle labels.

We assume:

Hypothesis 2.1 There exist constants $z, \beta > 0$ and a function $h \in L^1(\mathbb{R}^3; \mathbb{R}^+)$ with $\operatorname{ess\,sup}_x h(x) = z$, such that the rescaled functions at time zero, $f_j^\varepsilon(\cdot, 0) \equiv f_{0,j}^\varepsilon$, satisfy the bound

$$f_{0,j}^\varepsilon(\mathbf{z}_j) \leq h(x_1)\cdots h(x_j)\, e^{-(\beta/2)\sum_{i=1}^j v_i^2} \leq z^j\, e^{-(\beta/2)\sum_{i=1}^j v_i^2}. \tag{2.12}$$

Hypothesis 2.2 There exist two positive constants α_0, γ_0 such that the initial r.c.f. admit the following representation:

$$f_{0,J}^\varepsilon = \sum_{H\subset J} (f_{0,1}^\varepsilon)^{\otimes H} E_{J\setminus H}^0 \tag{2.13}$$

with $E^0_\emptyset = 1$, $E^0_k : \mathcal{M}_k \to \mathbb{R}$ and, for ε small enough,

$$|E^0_K| \le \varepsilon^{\gamma_0 k} z^k \, e^{-(\beta/2)\sum_{i\in K} v_i^2} \quad \forall\, k < \varepsilon^{-\alpha_0}. \tag{2.14}$$

The bound (2.14) holds almost everywhere in $\mathcal{M}_k(\varepsilon)$. Observe that $E^0_1 = 0$.

Here we are using the same notation introduced in Sect. 1.3 which we recall now and that will be adopted throughout all the paper. We use capital latin letters $(J, H, K, \ldots)$ for subsets of indices of $\{1, 2, 3, \ldots\}$ and corresponding small letters for the cardinality of the same sets $(j = |J|, k = |K|, \ldots)$, namely, in (2.13), $J = \{1, 2 \cdots j\}$ and $H = \{i_1, i_2 \cdots i_h\} \subset J$. In addition, $\mathbf{z}_H = (z_{i_1}, z_{i_2}, \ldots, z_{i_h})$ and, for given functions f_h, f, we abbreviate $f_H = f_h(\mathbf{z}_H)$ and $f^{\otimes H} = \prod_{i\in H} f(z_i)$. Finally, the conventions $f_\emptyset = f^{\otimes\emptyset} = 1$ are used.

Notice that, with respect to the hypotheses of Lanford's theorem, we are requiring the additional explicit information (2.13)–(2.14). Recall that the hard core exclusion, $|x_i - x_\ell| > \varepsilon$, prevents the full factorization of the state. A class of measures which are "maximally factorized", in the sense that the correlations are only those arising from the exclusion, is constructed in Appendix A. Such a class of states fulfills the hypotheses above.

2.3.2 *Initial data for the kinetic equation*

The initial datum for the Boltzmann and Enskog equations $f_0 = f_0(x, v)$ is a probability density over $\mathbb{R}^3 \times \mathbb{R}^3$ $(\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0 = 1)$.

As regards the error bound involving the kinetic equation, we postulate:

Hypothesis 2.3 There exists a positive constant γ_0 such that, for ε small enough,

$$|(f^\varepsilon_{0,1} - f_0)(x, v)| \le \varepsilon^{\gamma_0}\, z\, e^{-(\beta/2)v^2}. \tag{2.15}$$

In particular, condition (2.10) is satisfied with $\lambda = 1$. Here the constant γ_0 has been chosen equal to the one in Hypothesis 2.2 for notational simplicity.

Putting together the Hypotheses 2.2 and 2.3, it follows that the r.c.f. of the hard sphere system admit as well the following representation in terms of f_0:

$$f^\varepsilon_{0,J} = \sum_{H\subset J} f_0^{\otimes H} E^{\mathcal{B},0}_{J\setminus H}, \tag{2.16}$$

with $E^{\mathcal{B},0}_k : \mathcal{M}_k \to \mathbb{R}$ satisfying

$$|E^{\mathcal{B},0}_K| \le \varepsilon^{\gamma_0' k} z^k \, e^{-(\beta/2)\sum_{i\in K} v_i^2} \quad \forall\, k < \varepsilon^{-\alpha_0}, \tag{2.17}$$

for some $\gamma_0' > 0$ and ε small enough.

2.4 Results

We are now in a position to formulate our main results, summarized in the following theorem. Let

$$\mathcal{M}_n^x(\delta) = \left\{ \mathbf{x}_n \in \mathbb{R}^{3n}, \ \ |x_i - x_j| > \delta, \ i \neq j \right\} \tag{2.18}$$

where

$$\delta = \varepsilon^\theta \tag{2.19}$$

and $\theta \in (0, 1]$.

Theorem 2.4 *Let $\mathbf{W}_0^\varepsilon$ be a state of the hard sphere system with rescaled correlation functions $f_{0,j}^\varepsilon$ satisfying Hypotheses 2.1 and 2.2. Let $\mathbf{W}^\varepsilon(t)$ be the state evolved at time $t > 0$, with r.c.f. $f_j^\varepsilon(t)$. There exist positive constants θ, α, γ, a time $t^* > 0$ and $\varepsilon_0 > 0$ such that*

$$f_J^\varepsilon(t) = \sum_{H \subset J} (f_1^\varepsilon(t))^{\otimes H} E_{J \setminus H}(t) \tag{2.20}$$

and

$$\int_{\mathbb{R}^{3k}} d\boldsymbol{v}_k |E_K(t)| \leq \varepsilon^{\gamma k} \quad \forall k < \varepsilon^{-\alpha}, \tag{2.21}$$

for any $t < t^*$, $\varepsilon < \varepsilon_0$ and $\boldsymbol{x}_k \in \mathcal{M}_k^x(\delta)$.

Moreover, let $g^\varepsilon(t)$, $f(t)$ be the solutions to the Enskog and the Boltzmann equation respectively, with f_0 the common initial datum.

If f_0 satisfies Hypothesis 2.3, then for any $t < t^*$, $\varepsilon < \varepsilon_0$ and $\boldsymbol{x}_k \in \mathcal{M}_k^x(\delta)$,

$$f_J^\varepsilon(t) = \sum_{H \subset J} (g^\varepsilon(t))^{\otimes H} E_{J \setminus H}^{\mathscr{E}}(t), \tag{2.22}$$

$$\int_{\mathbb{R}^{3k}} d\boldsymbol{v}_k |E_K^{\mathscr{E}}(t)| \leq \varepsilon^{\gamma k} \quad \forall k < \varepsilon^{-\alpha}. \tag{2.23}$$

If, additionally, f_0 is Lipschitz continuous with respect to the space variables, with Lipschitz constant $Le^{-(\beta/2)v^2}$, $L > 0$, then for any $t < t^*$, $\varepsilon < \varepsilon_0$ and $\boldsymbol{x}_k \in \mathcal{M}_k^x(\delta)$,

$$f_J^\varepsilon(t) = \sum_{H \subset J} (f(t))^{\otimes H} E_{J \setminus H}^{\mathcal{B}}(t), \tag{2.24}$$

$$\int_{\mathbb{R}^{3k}} d\boldsymbol{v}_k |E_K^{\mathcal{B}}(t)| \leq \varepsilon^{\gamma k} \quad \forall k < \varepsilon^{-\alpha}. \tag{2.25}$$

The main result (2.21) is proved in Sect. 4.3.5 and the geometric estimates on recolliding trajectories are postponed to Sects. 4.4, 4.5. Finally, (2.23) and (2.25) are proved in Sect. 4.6.

As we shall see in the course of the proof, the solutions $g^\varepsilon(t)$ and $f(t)$ are local in time and constructed by means of a series expansion.

Equations (2.20)–(2.21) are an expression for the propagation of chaos, with an explicit representation of the error, while Eqs. (2.22)–(2.23) and (2.24)–(2.25) express in addition the asymptotic equivalence of the r.c.f. with the solution of the Enskog and the Boltzmann equation.

The convergence to the Boltzmann equation can be also expressed in terms of deviation from average values of observables. To this purpose, let us denote $\mathbb{E}^\varepsilon$, $\mathbb{E}^\mathcal{B}$ the average values with respect to the hard sphere state and the Boltzmann density respectively. Then the following result holds:

Theorem 2.5 *Let $\varphi_i \in C_c(\mathbb{R}^6;\mathbb{R})$, $i = 1, 2, \dots$ be a sequence of test functions with $\max(\|\varphi_i\|_{L^\infty}, \|\varphi_i\|_{L^1_x(L^\infty_v)}) \leq G$ for some $G > 0$. Moreover, let $F_i = F_i(t) : \mathcal{M} \to \mathbb{R}$ be the associated sequence of observables defined by (1.7). Then, if Hypotheses 2.1, 2.2 and 2.3 hold, there exists a positive constant α' such that, for any $t < t^*$,*

$$\lim_{\varepsilon\to 0} \sup_{j<\varepsilon^{-\alpha'}} \left| \mathbb{E}^\varepsilon\left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^\mathcal{B}[\varphi_i(t)] \right) \right] \right| = 0. \tag{2.26}$$

The theorem is proved in Sect. 4.7.

2.4.1 Comments on the result

1. The constants α and γ can be computed explicitly. Upper bounds will be given in Sect. 4.3.3.a, as a byproduct of the proof. They are the result of a balance between the combinatorial factors and the geometric recollision estimates. These bounds are certainly not optimal (for instance, (A.15) in appendix shows that there is no limitation on α for maximally factorized initial data). In this paper we are not concerned with optimal bounds on rates of convergence nor with the optimality of the coefficient α. Improvements in this direction would complicate the proof considerably. An exception is the geometrical estimate of internal recollisions (see Lemma 4.12), which can be shown to be ε^{γ_1} for arbitrary $\gamma_1 < 1$ by following the proof in Appendix D.
2. The limiting time t^* is obtained by imposing the absolute (uniform in ε) convergence of the series expansions appearing in the proof, and is determined only by z, β (see Hypothesis 2.1).

3. The use of the L^1-norm in the velocity variables is essential in the proof of the estimate of many-recollision events (Lemmas 4.10 and 4.12 below), to obtain a k-dependent rate of convergence such as (2.21). Chebyshev's inequality implies then that $|E_K(t)| \leq \varepsilon^{\bar{\gamma}k}$ for some $\bar{\gamma} > 0$ (and similar estimates for $E^{\mathscr{E}}$ and $E^{\mathcal{B}}$), outside a subset of $\mathcal{M}_k$ of measure smaller than $\varepsilon^{(\gamma-\bar{\gamma})k}$.
4. In particular, the comparison with the uniform estimate in Hypothesis 2.2 shows that the set where the convergence takes place deteriorates in time. This is a feature of the Boltzmann–Grad limit. In fact, it will be clear from the proof that, due to recollisions, the propagation of chaos $f_J^\varepsilon(t) \to (f_1^\varepsilon(t))^{\otimes J}$ necessarily fails over the time-dependent set

$$\left\{\mathbf{z}_J \;\middle|\; \min_{i,k\in J} \min_{s\in(0,t)} [(x_i - x_k) - (v_i - v_k)\, s] = 0\right\}.$$

   Actually it can be proved that, over compacts outside this null-measure set, $E_K(t) = O(\varepsilon^\eta)$ for some $\eta > 0$ and ε small enough (e.g. [30], where this is done for the more difficult case of smoothly interacting particles).
5. The above discussion does not prevent, however, the following integrated result.

Proposition 2.6 *Let φ_i be test funtions as in Theorem 2.5. If Hypotheses 2.1 and 2.2 hold, there exists $\alpha' > 0$ such that, for $t < t^*$ and ε small enough,*

$$\left|\int_{\mathbb{R}^{6k}} dz_k\, \varphi(z_1)\cdots\varphi(z_k)\, E_K(z_k, t)\right| \leq \varepsilon^{\gamma k} \quad \forall\, k < \varepsilon^{-\alpha'}. \tag{2.27}$$

Observe that, by definition (remind (2.6)), since $f_J^\varepsilon(\mathbf{z}_J, t) = 0$ when two particles in $\mathbf{z}_J$ are at distance smaller than ε, inside the "excluded" region $\mathbb{R}^{6k}\backslash\mathcal{M}_k$ (of small measure) the correlation errors will generally satisfy the bad estimate $|E_K(t)| \leq (const.)^k$ (by (1.12)). Equation (2.27) will be derived in Sect. 4.7.

We conclude with some comments on the choice of the setting.

6. We are working with rescaled correlation functions in a grand canonical formalism (no fixed total number of particles N) in place of the more usual formulation in terms of marginals in the canonical setting (fixed $N = \varepsilon^{-2}$). This choice is convenient when dealing with fluctuations and truncation formulas of cumulant type, see e.g. [3,35,36]. The reason is that the mere facts of (1) fixing the number of particles N, and (2) labelling the particles from 1 to N (implied in the definition of marginal) are itself a source of correlation. Consequently, even though the r.c.f. are asymptotically equivalent to the marginals of the canonical setting, here additional error terms are

produced which should be expanded and estimated in order to get a quantitative result like (2.21). We do not deal with this problem in the present paper. More details on this point will be provided in Sect. 4.8.3.

7. Another simplification comes from the choice of the unbounded spatial domain $\mathbb{R}^3$. Since we do not use dispersive properties, our analysis in the whole space can be transferred to the case of a bounded box (assuming periodic or reflecting boundary conditions) with minor modifications (see Sect. 4.8.4). One faces here two extra difficulties. The first one arises from the fact that the recollisions are more likely. This has been discussed in [4]. The second one is that, as in the canonical formalism, the total number of particles cannot exceed a given integer, that is the close-packing number N_{cp}. However $N_{cp} = O(\varepsilon^{-3})$ is much larger than the average density and the corresponding error of correlation is very small and easily tractable.

2.4.2 Further remarks on the initial states

In the proof of the main result we shall find more convenient to use a representation of the initial data different from the one given by Hypothesis 2.2. We illustrate it in this section.

Let $S = \{1, \ldots, s\}$ be a set of indices (particles) and $\{S_1, S_2, \ldots, S_j\}$ a partition of S into nontrivial clusters, i.e. $\cup_{i=1}^{j} S_i = S$, $S_i \cap S_k = \emptyset$ for $i \neq k$, $|S_i| > 0$.

Denote by $\mathcal{J} = \{1, \ldots, j\}$ the set of indices of the clusters $\{S_i\}$. We introduce an expansion on products of higher order, not only 1-point, rescaled correlation functions. We use a calligraphic capital letter for the subsets of $\mathcal{J}$.

Property 1 *There exist two positive constants α_0, γ_0 such that the initial r.c.f. admit the following collection of representations*:

$$f^{\varepsilon}_{0,S} = \sum_{\mathcal{H} \subset \mathcal{J}} \left(\prod_{i \in \mathcal{H}} f^{\varepsilon}_{0,S_i} \right) E^0_{\mathcal{J} \backslash \mathcal{H}} \tag{2.28}$$

for any partition of the set S, where $E^0_{\emptyset} = 1$, $E^0_{\mathcal{K}} : \mathcal{M}_k \to \mathbb{R}$ and, for ε small enough,

$$|E^0_{\mathcal{K}}| \leq \varepsilon^{\gamma_0 |\mathcal{K}|} z^k \, e^{-(\beta/2) \sum_{i \in K} v_i^2} \quad \forall \, k < \varepsilon^{-\alpha_0}, \tag{2.29}$$

with $|\mathcal{K}|$ = total number of elements (clusters) in $\mathcal{K}$, and k = total number of indices (particles) in $K = \cup_{i \in \mathcal{K}} S_i$.

Note that $E^0_{\mathcal{K}} = 0$ for $|\mathcal{K}| = 1$. We stress that $E^0_{\mathcal{K}}$ and E^0_K denote different quantities (unless all the clusters in $\mathcal{K}$ are singletons).

Property 1 is actually equivalent to Hypothesis 2.2. For the proof, we refer to Appendix A.

Observe that, again, $E^0_{\mathcal{K}}$ will be order 1 (in ε) outside $\mathcal{M}_k(\varepsilon)$. As already mentioned, this is due to the hard sphere exclusion which is a first obvious source of correlation, namely the fact that the r.c.f. $f^\varepsilon_j(t)$ is naturally defined on $\mathcal{M}_j$ and extended to zero outside. On the other hand, we will need to compare the r.c.f. with $(f^\varepsilon_1(t))^{\otimes j}$ which is defined in the extended phase space $\mathbb{R}^{6j}$. In particular it will be necessary, in the course of the proof, to embed (2.28) in the whole space $\mathbb{R}^{6s}$ as follows:

$$f^\varepsilon_{0,S} = \bar{\chi}^0_S \sum_{\mathcal{H}\subset\mathcal{J}} \left(\prod_{i\in\mathcal{H}} f^\varepsilon_{0,S_i}\right) E^0_{\mathcal{J}\setminus\mathcal{H}}, \tag{2.30}$$

where

$$\bar{\chi}^0_S = \prod_{\substack{i,k\in S\\ i\neq k}} \bar{\chi}^0_{i,k}$$

and $\bar{\chi}^0_{i,k}$ is the indicator function of the set $\{|x_i - x_k| > \varepsilon\}$.

In Sect. 4.3 we develop a technique which allows a useful expansion of $\bar{\chi}^0_S$, for which we can prove (see Appendix A):

Property 2 *Equation* (2.28) *can be extended in* $\mathbb{R}^{6s}$ *according to*

$$f^\varepsilon_{0,S} = \sum_{\mathcal{H}\subset\mathcal{J}} \left(\prod_{i\in\mathcal{H}} \bar{\chi}^0_{S_i} f^\varepsilon_{0,S_i}\right) \bar{E}^0_{\mathcal{J}\setminus\mathcal{H}}, \tag{2.31}$$

with

$$|\bar{E}^0_{\mathcal{K}}| \leq \sum_{\substack{\mathcal{H}_1,\mathcal{H}_2\\ \mathcal{H}_1\cup\mathcal{H}_2=\mathcal{K}\\ \mathcal{H}_1\cap\mathcal{H}_2=\emptyset}} \left(C^{|\mathcal{H}_1|}\, |\mathcal{H}_1|!\, \chi^0_{\mathcal{H}_1,\mathcal{K}} \prod_{i\in\mathcal{H}_1} \bar{\chi}^0_{S_i} f^\varepsilon_{0,S_i}\right) \left(\bar{\chi}^0_{H_2} |E^0_{\mathcal{H}_2}|\right), \tag{2.32}$$

where:

(i) $\chi^0_{\mathcal{H}_1,\mathcal{K}} = 1$ *if and only if any cluster* S_i, *with* $i \in \mathcal{H}_1$, *has, at least, one particle "overlapping" with another particle in* S_j *with* $j \in \mathcal{K}$, $j \neq i$;
(ii) $H_2 = \cup_{i\in\mathcal{H}_2} S_i$.

By *overlap* of two particles we mean that their relative distance is smaller than ε.

Note that $\bar{\chi}^0_{H_2}$ allows to insert (2.29) into (2.32), while the particles contained in $\mathcal{H}_1$ are constrained to lie in a small set. Explicitly,

$$|\bar{E}^0_{\mathcal{K}}| \le z^k\, e^{-(\beta/2)\sum_{i\in K} v_i^2} \sum_{\substack{\mathcal{H}_1,\mathcal{H}_2\\ \mathcal{H}_1\cup\mathcal{H}_2=\mathcal{K}\\ \mathcal{H}_1\cap\mathcal{H}_2=\emptyset}} C^{|\mathcal{H}_1|}\, |\mathcal{H}_1|!\, \chi^0_{\mathcal{H}_1,\mathcal{K}}\, \varepsilon^{\gamma_0|\mathcal{H}_2|}, \tag{2.33}$$

for all $k < \varepsilon^{-\alpha_0}$.

3 Hierarchies

In this section we introduce the standard description of the evolution of a statistical state of particles, namely the chain of BBGKY hierarchy equations (Sect. 3.1). We also introduce the analogue hierarchies at the kinetic level, which can be obtained by formally taking the limit $\varepsilon \to 0$ (Sects. 3.2, 3.3). An explicit representation of the solution to the BBGKY can be given in terms of a tree expansion and of a class of special flows of particles evolving backwards in time. An analogous description is also possible for the Boltzmann (or the Enskog) evolution equation. These well known expressions, which will be our basic tool, are introduced in Sects. 3.4 and 3.5, together with some new expansions that will have the role of intermediate object in the transition towards the kinetic limit. We conclude the section with a summary of the main tools introduced.

3.1 BBGKY hierarchy

We describe here the time evolution of the hard sphere system for any fixed $\varepsilon > 0$. The evolution equations for the considered quantities were first derived formally by Cercignani in [12].

Assuming some explicit bound and sufficient smoothness, he deduced the hard sphere version of the *BBGKY hierarchy* of equations [6], which for the rescaled correlation functions takes the form

$$\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}\right) f^\varepsilon_j(\mathbf{z}_j, t) = \sum_{i=1}^{j} \int_{S^2\times\mathbb{R}^3} d\omega\, dv_{j+1}\, B^\varepsilon(\omega; v_{j+1}-v_i) \times f^\varepsilon_{j+1}(\mathbf{z}_j, x_i+\varepsilon\omega, v_{j+1}, t), \tag{3.1}$$

where

$$B^\varepsilon(\omega; v_{j+1}-v_i) = \omega\cdot(v_{j+1}-v_i)\, \mathbb{1}_{\{\min_{\ell=1,\dots,j;\ell\neq i}|x_i+\omega\varepsilon-x_\ell|>\varepsilon\}}(\omega). \tag{3.2}$$

Here $\mathbb{1}_A$ denotes the indicator function of the set A.

Notice that the difference of this formula with respect to the hierarchy written for marginals in the canonical setting (as for instance in [12]), is that the factor $\varepsilon^2(N-j)$ is absent in the right hand side (N = fixed total number of particles). This is a small notational advantage in using correlation functions.

The *series solution* of the hierarchy (obtained from integration and repeated iteration of the above formula) is

$$f_j^\varepsilon(t) = \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\mathcal{S}_j^\varepsilon(t-t_1)\mathcal{C}_{j+1}^\varepsilon \mathcal{S}_{j+1}^\varepsilon(t_1-t_2)\cdots \mathcal{C}_{j+n}^\varepsilon \mathcal{S}_{j+n}^\varepsilon(t_n) f_{j+n}^\varepsilon(0), \tag{3.3}$$

where we used the definitions of *interacting flow operator* $\mathcal{S}_j^\varepsilon(t)$ and *BBGKY collision operator* $\mathcal{C}_{j+1}^\varepsilon$, i.e. respectively

$$\mathcal{S}_j^\varepsilon(t) f_j^\varepsilon(\mathbf{z}_j, \cdot) = f_j^\varepsilon(\mathrm{T}_j^\varepsilon(-t)\mathbf{z}_j, \cdot) \tag{3.4}$$

and

$$\mathcal{C}_{j+1}^\varepsilon = \sum_{k=1}^{j} \mathcal{C}_{k,j+1}^\varepsilon$$
$$\mathcal{C}_{k,j+1}^\varepsilon f_{j+1}^\varepsilon(\mathbf{z}_j, \cdot) = \int_{S^2\times\mathbb{R}^3} d\omega dv_{j+1} B^\varepsilon(\omega; v_{j+1}-v_k)$$
$$f_{j+1}^\varepsilon(\mathbf{z}_j, x_k + \omega\varepsilon, v_{j+1}, \cdot). \tag{3.5}$$

Rigorous derivations of the hard sphere hierarchy, under rather weak assumptions on the initial measure, have been discussed later on, e.g. [22,34,38].[2] The latter references focus mainly on the validity of the series expansion (3.3).

Let us formulate the result in a form useful for our analysis.

Proposition 3.1 (BBGKY series expansion) *Let $\mathbf{W}_0^\varepsilon$ be a state of the hard sphere system satisfying Hypothesis 2.1. Then the measure at any time $t > 0$ has rescaled correlation functions $f_j^\varepsilon(t)$ given by (3.3), for almost all points in $\mathcal{M}_j$.*

For a complete proof of the validity result as formulated above, we refer to [34].[3]

Proposition 3.1 is the starting point of our analysis. All the formulas involving the r.c.f. at positive times will be valid only almost everywhere.

[2] See also [31], appeared before revision of the present paper.

[3] Note that the quoted result of [34] (Corollary 2) is stated for a system of particles in a finite box. Given the explicit assumption on the spatial decay (2.12), the result can be easily established on the full space along the same lines.

3.2 Boltzmann hierarchy

We want to give a picture of the Boltzmann equation which can be conveniently compared to (3.3).

Suppose that f is a solution to the Boltzmann equation (1.4) (with $\lambda = 1$). Consider the products

$$f_j(\mathbf{z}_j, t) = f(t)^{\otimes j}(\mathbf{z}_j) = f(z_1, t) f(z_2, t) \cdots f(z_j, t). \tag{3.6}$$

The family of f_j solves the hierarchy of equations (*Boltzmann hierarchy*):

$$\left(\partial_t + \sum_{i=1}^{j} v_i \cdot \nabla_{x_i}\right) f_j = \mathcal{C}_{j+1} f_{j+1},$$

where

$$\begin{aligned}
&\mathcal{C}_{j+1} = \sum_{k=1}^{j} \mathcal{C}_{k,j+1} \\
&\mathcal{C}_{k,j+1} = \mathcal{C}^{+}_{k,j+1} - \mathcal{C}^{-}_{k,j+1} \\
&\mathcal{C}^{+}_{k,j+1} f_{j+1}(\mathbf{z}_j, \cdot) = \int_{\mathbb{R}^3} dv_{j+1} \int_{S^2_+} d\omega (v_k - v_{j+1}) \\
&\quad \cdot \omega f_{j+1}(z_1, \ldots, x_k, v'_k, \ldots, z_j, x_k, v'_{j+1}, \cdot) \\
&\mathcal{C}^{-}_{k,j+1} f_{j+1}(\mathbf{z}_j, \cdot) = \int_{\mathbb{R}^3} dv_{j+1} \int_{S^2_+} d\omega (v_k - v_{j+1}) \\
&\quad \cdot \omega f_{j+1}(z_1, \ldots, x_k, v_k, \ldots, z_j, x_k, v_{j+1}, \cdot),
\end{aligned} \tag{3.7}$$

with

$$\begin{cases} v'_k = v_k - \omega[\omega \cdot (v_k - v_{j+1})] \\ v'_{j+1} = v_{j+1} + \omega[\omega \cdot (v_k - v_{j+1})] \end{cases} \tag{3.8}$$

and

$$S^2_+ = \{\omega \mid (v_k - v_{j+1}) \cdot \omega \geq 0\}. \tag{3.9}$$

The corresponding series solution reads

$$\begin{aligned}
f_j(t) = \sum_{n \geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\
\times \mathcal{S}_j(t - t_1) \mathcal{C}_{j+1} \mathcal{S}_{j+1}(t_1 - t_2) \cdots \mathcal{C}_{j+n} \mathcal{S}_{j+n}(t_n) f_{0,j+n},
\end{aligned} \tag{3.10}$$

where now $\mathcal{S}_j(t)$ is the *free flow operator*, defined as

$$\mathcal{S}_j(t) f_j(\mathbf{z}_j, \cdot) = f_j(x_1 - v_1 t, v_1, \ldots, x_j - v_j t, v_j, \cdot), \tag{3.11}$$

and

$$f_{0,j} = f_0^{\otimes j} \tag{3.12}$$

are the initial data.

The absolute convergence of this formula has been discussed in [25] and holds (over all $\mathbb{R}^{6j}$) only for a sufficiently small time. We shall give a proof, for completeness, in Sect. 4.1 (Proposition 4.1). This implies, in particular, local existence and uniqueness of the solution to the time-integrated version of the Boltzmann hierarchy in the class of continuous functions such that $|f_j(t)| \leq c^j e^{-c' \sum_{i=1}^j v_i^2}$ for some $c, c' > 0$. Moreover, in the case of initial product states, factorization is propagated in time, each factor being the local solution to the time-integrated Boltzmann equation (see formula (3.52) below).

The similarity of (3.10) and (3.3) follows from the decomposition of the BBGKY collision operator into its positive and negative part.

3.3 Enskog hierarchy

We provide here an intermediate item between the BBGKY hierarchy and the Boltzmann hierarchy, that is the so called *Enskog hierarchy*.

Let g^ε be a solution to the *Enskog Equation* (1.19) (with $\lambda = 1$). Proceeding as above, the products

$$g_j^\varepsilon(\mathbf{z}_j, t) = g^\varepsilon(t)^{\otimes j}(\mathbf{z}_j) \tag{3.13}$$

satisfy

$$\left(\partial_t + \sum_{i=1}^j v_i \cdot \nabla_{x_i} \right) g_j^\varepsilon = \mathcal{C}_{j+1}^{\mathscr{E}} g_{j+1}^\varepsilon, \tag{3.14}$$

where the definition of $\mathcal{C}_{j+1}^{\mathscr{E}}$ is induced by that of the collision operator on the right hand side of (1.19) (the symbol $\mathscr{E}$ stands for "Enskog", while we drop the dependence on ε), i.e.

$$\begin{aligned} \mathcal{C}_{j+1}^{\mathscr{E}} g_{j+1}^\varepsilon(\mathbf{z}_j, \cdot) = \sum_{k=1}^j \Bigg\{ & \int_{\mathbb{R}^3} dv_{j+1} \int_{S_+^2} d\omega (v_k - v_{j+1}) \\ & \cdot \omega \, g_{j+1}^\varepsilon(z_1, \ldots, x_k, v_k', \ldots, z_j, x_k - \omega\varepsilon, v_{j+1}', \cdot) \\ & - \int_{\mathbb{R}^3} dv_{j+1} \int_{S_+^2} d\omega (v_k - v_{j+1}) \\ & \cdot \omega \, g_{j+1}^\varepsilon(z_1, \ldots, x_k, v_k, \ldots, z_j, x_k + \omega\varepsilon, v_{j+1}, \cdot) \Bigg\}. \end{aligned} \tag{3.15}$$

We derive the corresponding series solution:

$$g_j^\varepsilon(t) = \sum_{n\geq 0} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\times \mathcal{S}_j(t-t_1)\mathcal{C}_{j+1}^{\mathscr{E}}\mathcal{S}_{j+1}(t_1-t_2)\cdots\mathcal{C}_{j+n}^{\mathscr{E}}\mathcal{S}_{j+n}(t_n)g_{0,j+n}^\varepsilon, \quad (3.16)$$

where

$$g_{0,j}^\varepsilon = f_0^{\otimes j} \quad (3.17)$$

are the initial data (which in this paper will be assumed, for simplicity, equal to the initial data for the Boltzmann hierarchy).

Notice that the operator $\mathcal{C}_{j+1}^{\mathscr{E}}$ is identical to $\mathcal{C}_{j+1}$ introduced in (3.7), except for the fact that the particle $j+1$ has position $x_k - \omega\varepsilon$ in the gain and $x_k + \omega\varepsilon$ in the loss.

Local existence, uniqueness and propagation of chaos are discussed exactly as for the Boltzmann hierarchy (see the comment after (3.12), and formula (3.46) below).

3.4 The tree expansion

In this section we shall follow mainly [30] Sec. 6, adapting discussions and notation therein to the simpler case of hard spheres. Our purpose is to rewrite formulas (3.3) and (in the next section) (3.10) in a convenient and more explicit way.

We start from (3.3), which we write as

$$f_j^\varepsilon(t) = \sum_{n\geq 0}\sum_{\mathbf{k}_n}^{*} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\times \mathcal{S}_j^\varepsilon(t-t_1)\mathcal{C}_{k_1,j+1}^\varepsilon\mathcal{S}_{j+1}^\varepsilon(t_1-t_2)\cdots\mathcal{C}_{k_n,j+n}^\varepsilon\mathcal{S}_{j+n}^\varepsilon(t_n)f_{0,j+n}^\varepsilon, \quad (3.18)$$

where

$$\sum_{\mathbf{k}_n}^{*} = \sum_{k_1=1}^{j}\sum_{k_2=1}^{j+1}\cdots\sum_{k_n=1}^{j+n-1}. \quad (3.19)$$

We introduce the ***n*-collision**, ***j*-particle tree**, denoted $\Gamma(j,n)$, as the collection of integers $k_1,\ldots,k_n$ that are present in the sum (3.19), i.e.

$$k_1 \in I_j, k_2 \in I_{j+1}, \ldots, k_n \in I_{j+n-1}, \quad \text{with} \quad I_s = \{1,2,\ldots,s\}, \quad (3.20)$$

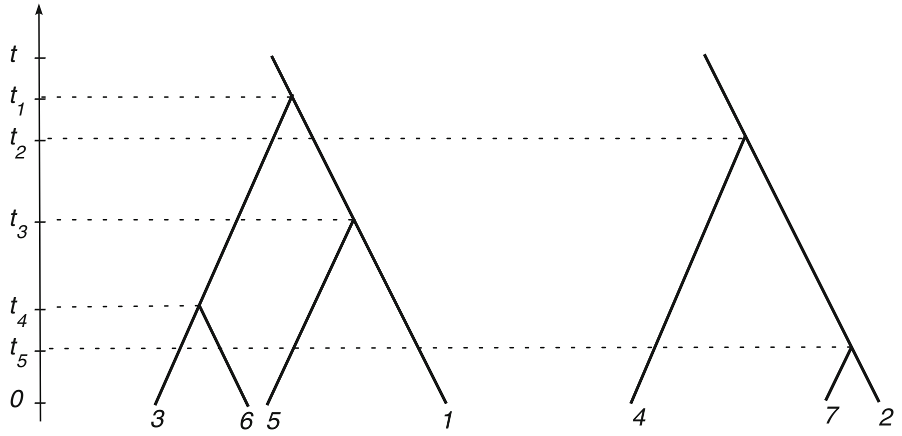


Fig. 1 The two-particle tree $\Gamma(2,5) = (1,2,1,3,2)$. The tree associated to 1 is $\Gamma_1 = (1,1,3)$, while $\Gamma_2 = (2,2)$

so that

$$\sum_{\mathbf{k}_n}^{*} = \sum_{\Gamma(j,n)} . \tag{3.21}$$

The name "tree" is justified by its natural graphical representation, which we explain by means of an example: see Fig. 1 corresponding to $\Gamma(2,5)$ given by $1,2,1,3,2$. In the figure, we have also drawn a time arrow in order to associate times to the nodes of the trees: at time t_i the line $j+i$ is "created". Lines 1 and 2 of the example, existing for all times, are called "root lines".

3.4.1 The interacting backwards flow (IBF)

Given a j-particle tree $\Gamma(j,n)$ and fixed a value of all the integration variables in the expansion (3.18) (times, unit vectors, velocities), we associate to them a special (ε-dependent) trajectory of particles, which we call *interacting backwards flow* (IBF in the following), since it will be naturally defined by going backwards in time. The rules for the construction of this evolution are explained in what follows.

First, we introduce a notation for the configuration of particles in the IBF, by making use of Greek alphabet, i.e. $\boldsymbol{\zeta}^{\varepsilon}(s)$, where $s \in [0,t]$ is the time. Note that there is no label specifying the number of particles. This number depends indeed on the time. If $s \in (t_{r+1}, t_r)$ (with the convention $t_0 = t$, $t_{n+1} = 0$), there are exactly $j+r$ particles:

$$\boldsymbol{\zeta}^{\varepsilon}(s) = (\zeta_1^{\varepsilon}(s), \ldots, \zeta_{j+r}^{\varepsilon}(s)) \in \mathcal{M}_{j+r} \quad \text{for } s \in (t_{r+1}, t_r), \tag{3.22}$$

with

$$\zeta_i^\varepsilon(s) = (\xi_i^\varepsilon(s), \eta_i^\varepsilon(s)), \tag{3.23}$$

the positions and velocities of the particles being respectively

$$\begin{aligned} \boldsymbol{\xi}^\varepsilon(s) &= (\xi_1^\varepsilon(s), \dots, \xi_{j+r}^\varepsilon(s)), \\ \boldsymbol{\eta}^\varepsilon(s) &= (\eta_1^\varepsilon(s), \dots, \eta_{j+r}^\varepsilon(s)). \end{aligned} \tag{3.24}$$

Our final goal is to write (3.18) in terms of the IBF (to be defined below), i.e.:

$$f_j^\varepsilon(\mathbf{z}_j, t) = \sum_{n\geq 0} \sum_{\Gamma(j,n)} \mathcal{T}^\varepsilon(\mathbf{z}_j, t) \tag{3.25}$$

where $\mathcal{T}^\varepsilon(\mathbf{z}_j, t)$ is the *value of the tree* $\Gamma(j, n)$ *with configuration* $\mathbf{z}_j$ *at time* t, *for the interacting flow*,

$$\mathcal{T}^\varepsilon(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\omega_i; v_{j+i} - \eta_{k_i}^\varepsilon(t_i)) f_{0,j+n}^\varepsilon(\boldsymbol{\zeta}^\varepsilon(0)), \tag{3.26}$$

$$\begin{aligned} \mathbf{t}_n &= t_1, \dots, t_n, \\ \boldsymbol{\omega}_n &= \omega_1, \dots, \omega_n, \\ \mathbf{v}_{j,n} &= v_{j+1}, \dots, v_{j+n}, \end{aligned} \tag{3.27}$$

$d\Lambda$ is the measure on $\mathbb{R}^n \times S^{2n} \times \mathbb{R}^{3n}$

$$d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) = \mathbb{1}_{\{t>t_1>t_2\dots>t_n>0\}} dt_1 \cdots dt_n d\omega_1 \cdots d\omega_n dv_{j+1} \cdots dv_{j+n}, \tag{3.28}$$

and we use the shorthand notation

$$B^\varepsilon(\omega_i; v_{j+i} - \eta_{k_i}^\varepsilon(t_i)) = \omega_i \cdot (v_{j+i} - \eta_{k_i}^\varepsilon(t_i)) \mathbb{1}_{\{|\xi_{j+i}^\varepsilon(t_i) - \xi_k^\varepsilon(t_i)| > \varepsilon \ \forall k \neq k_i\}}. \tag{3.29}$$

In other words, in the generic term $\mathcal{T}^\varepsilon(\mathbf{z}_j, t)$, the initial datum $f_{0,j+n}^\varepsilon$ is integrated, with the suitable weight, over all the possible time-zero states of the IBF associated to $\Gamma(j, n)$.

In formula (3.26), the triple (t_i, ω_i, v_{j+i}) may be thought as associated to the node of $\Gamma(j, n)$ where line $j + i$ is created (see Fig. 1). In the rest of the paper, we shall abbreviate further

$$\int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod B^\varepsilon = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod_{i=1}^n B^\varepsilon(\omega_i; v_{j+i} - \eta_{k_i}^\varepsilon(t_i)), \tag{3.30}$$

where the $\eta^\varepsilon_{k_i}(t_i)$ in the factors B^ε have to be computed through the rules specified below, starting from the set of variables $(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$, the corresponding j-particle tree (whose nodes are labeled by $(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$), together with the associated value of $\mathbf{z}_j, t$.

Let us construct $\boldsymbol{\zeta}^\varepsilon(s)$ for a fixed collection of variables $\Gamma(j,n), \mathbf{z}_j, \mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}$, with

$$t \equiv t_0 > t_1 > t_2 > \cdots > t_n > t_{n+1} \equiv 0, \tag{3.31}$$

and $\boldsymbol{\omega}_n$ satisfying a further constraint that will be specified soon. The root lines of the j-particle tree are associated to the first j particles, with configuration $\zeta^\varepsilon_1, \ldots, \zeta^\varepsilon_j$. Each branch $j+\ell$ ($\ell = 1, \ldots, n$) represents a new particle with the same label, and state $\zeta^\varepsilon_{j+\ell}$. This new particle appears, going backwards in time, at time t_ℓ in a collision configuration with a previous particle (branch) $k_\ell \in \{1, \ldots, j+\ell-1\}$, with either incoming or outgoing velocity.

More precisely, in the time interval (t_r, t_{r-1}) particles $1, \ldots, j+r-1$ flow according to the interacting dynamics $\mathtt{T}^\varepsilon_{j+r-1}$. This defines $\boldsymbol{\zeta}^\varepsilon_{j+r-1}(s)$ starting from $\boldsymbol{\zeta}^\varepsilon_{j+r-1}(t_{r-1})$. At time t_r the particle $j+r$ is "*created*" by particle k_r in the position

$$\xi^\varepsilon_{j+r}(t_r) = \xi^\varepsilon_{k_r}(t_r) + \omega_r\,\varepsilon \tag{3.32}$$

and with velocity v_{j+r}. This defines $\boldsymbol{\zeta}^\varepsilon(t_r) = (\zeta^\varepsilon_1(t_r), \ldots, \zeta^\varepsilon_{j+r}(t_r))$.

The characteristic function in the collision operator (3.2)–(3.5) (or the characteristic function in (3.29)), is a constraint on ω_r ensuring that two hard spheres cannot be at distance smaller than ε.

Next, the evolution in (t_{r+1}, t_r) is contructed applying to this configuration the dynamics $\mathtt{T}^\varepsilon_{j+r}$ (with negative times). We have two cases. If $\omega_r \cdot (v_{j+r} - \eta^\varepsilon_{k_r}(t_r)) \le 0$, then the velocities are incoming and no scattering occurs, namely for times $s < t_r$ the pair of particles moves backwards freely with velocities $\eta^\varepsilon_{k_r}(t_r)$ and v_{j+r}. If instead $\omega_r \cdot (v_{j+r} - \eta^\varepsilon_{k_r}(t_r)) \ge 0$, the pair is post-collisional. Then the presence of the interaction in the flow $\mathtt{T}^\varepsilon_{j+r}$ forces the pair to perform a (backwards) instantaneous collision. The two situations are depicted in Fig. 2.

Proceeding inductively, the IBF is constructed for all times $s \in [0, t]$.

3.4.2 Recollisions and factorization

Observe that between two creation times t_{r+1}, t_r any pair of particles among the existing $j+r$ can possibly interact. These interactions are called *recollisions*, because they may involve particles that have already interacted at some creation time (in the future) with another particle of the IBF. In our language, recollisions are the "interactions different from creations".

Let us focus now in more detail on the structure of the backwards flow and on the mechanisms of correlation.

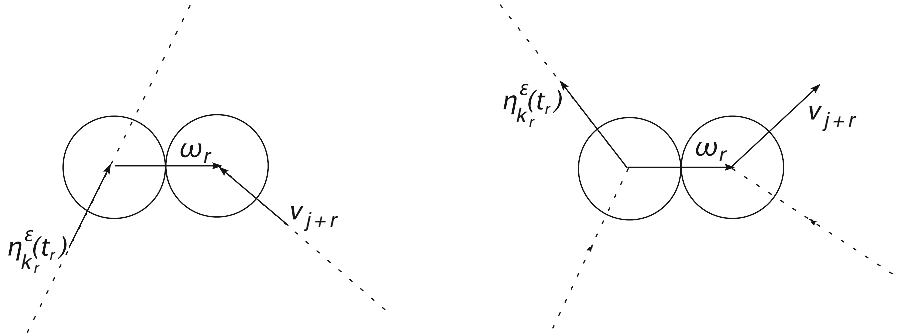


Fig. 2 At time t_r, particle $j+r$ is *created* by particle k_r, either in incoming ($\omega_r \cdot (v_{j+r} - \eta^{\varepsilon}_{k_r}(t_r)) \le 0$) or in outgoing ($\omega_r \cdot (v_{j+r} - \eta^{\varepsilon}_{k_r}(t_r)) \ge 0$) collision configuration. Particle k_r is called the *progenitor* of particle $j+r$

We observe preliminarily that the graphical representation of a n-collision, j-particle tree $\Gamma(j,n) = (k_1, \dots, k_n)$ consists of j connected components. Each of these components is associated to a root line $i \in \{1, 2, \dots, j\}$ and collects n_i nodes $i_1, i_2, \dots, i_{n_i}$. In particular, we have the following map:

$$\begin{aligned} &\Gamma(j,n) \longrightarrow \mathbf{\Gamma}_j = \Gamma_1, \dots, \Gamma_j, \\ &\Gamma_i = (k^i_1, \dots, k^i_{n_i}), \quad k^i_h = k_{i_h}. \end{aligned} \tag{3.33}$$

*In the sequel we will call simply **tree** (generated by i) the collection of integers* Γ_i. In the example of Fig. 1 one has $\Gamma_1 = (1,1,3)$, $\Gamma_2 = (2,2)$.

Note that the map (3.33) is not invertible, since the collection $\mathbf{\Gamma}_j$ does *not* specify the ordering of particles belonging to different trees. A one-to-one correspondence is instead the following:

$$n,\ \Gamma(j,n),\ \mathbf{t}_n \longleftrightarrow n_1, \Gamma_1, \mathbf{t}^1_{n_1}, \dots, n_j, \Gamma_j, \mathbf{t}^j_{n_j}, \tag{3.34}$$

where

$$\mathbf{t}^i_{n_i} = t^i_1, \dots, t^i_{n_i}, \quad t^i_h = t_{i_h}.$$

Clearly $n = \sum_i n_i$.

For a given sequence of trees $\mathbf{\Gamma}_j$, there are several j-particle trees $\Gamma(j,n)$ having $\mathbf{\Gamma}_j$ as image of the map (3.33). However summing the time–ordered product over such trees $\Gamma(j,n)$ is equivalent to a free time integration leaving only the partial ordering dictated by the sequence $\mathbf{\Gamma}_j$. Namely it holds:

$$\sum_{\Gamma(j,n)} \int \mathbb{1}_{\{t>t_1>t_2\cdots>t_n>0\}} dt_1 \cdots dt_n\, F = \prod_{i=1}^{j} \sum_{\Gamma_i} \int \mathbb{1}_{\{t>t^i_1>t^i_2\cdots>t^i_{n_i}>0\}} dt^i_1 \cdots dt^i_{n_i}\, F \tag{3.35}$$

where $F = F(\mathbf{\Gamma}_j, \mathbf{t}_n)$.

Applying this property to the expansion (3.25), we obtain the following factorization result:

$$
\begin{aligned}
f_j^\varepsilon(\mathbf{z}_j,t) &= \sum_{n\geq 0}\sum_{\Gamma(j,n)}\int d\Lambda(\mathbf{t}_n,\boldsymbol{\omega}_n,\mathbf{v}_{j,n})\prod B^\varepsilon f^\varepsilon_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)) \\
&= \prod_{i=1}^{j}\left(\sum_{n_i,\Gamma_i}\int d\Lambda(\mathbf{t}^i_{n_i},\boldsymbol{\omega}^i_{n_i},\mathbf{v}^i_{1,n_i})\right)\prod B^\varepsilon f^\varepsilon_{0,j+n}(\boldsymbol{\zeta}^\varepsilon(0)). \qquad (3.36)
\end{aligned}
$$

In (3.36), the triples in $(\mathbf{t}^i_{n_i},\boldsymbol{\omega}^i_{n_i},\mathbf{v}^i_{1,n_i})$ are associated to the nodes of the tree Γ_i, while the IBF (hence the integrand $\prod B^\varepsilon f^\varepsilon_{0,j+n}$) has to be computed with the rules specified in the previous subsection.

With the notations introduced above (see in particular Fig. 1), it should be clear that each particle of the IBF "belongs" to exactly one tree Γ_i. Therefore we may distinguish two types of recollisions. The *internal recollisions*, occurring among particles of the same tree and the *external recollisions*, occurring between particles belonging to different trees. Because of the external recollisions, we say that different trees are *correlated*, in the sense that their interacting backwards flows are not pairwise independent.

Remark Formula (3.36) shows a *partial factorization*: a full factorization is prevented by the correlations of the initial datum $f^\varepsilon_{0,j+n}$, the forbidden (external) overlaps of created particles at the creation times (written in B^ε) and, more importantly, the external recollisions in the IBF. If we simply ignore these effects and replace $f^\varepsilon_{0,j+n}$ with a tensor product, then (3.36) becomes a completely factorized expression.

From now on, in handling formula (3.36) and similar ones established in the sequel, we will use intensively the notations

$$\Gamma_i = \text{tree generated by particle } i \in \{1,\ldots,j\}, \qquad (3.37)$$

which is a (n_i-collision) tree with associated configuration z_i at time t,

$$(\mathbf{t}^i_{n_i},\boldsymbol{\omega}^i_{n_i},\mathbf{v}^i_{1,n_i}) = \text{collection of triples associated to the nodes of } \Gamma_i, \qquad (3.38)$$

and

$$S(i) = \text{set of particles associated to } \Gamma_i. \qquad (3.39)$$

Moreover,

$$S(K) = \cup_i S(i), \qquad (3.40)$$

where K is any subset of $\{1,\ldots,j\}$.

3.5 Factorized expansions

3.5.1 The uncorrelated *IBF*

Using the symmetry of the state, we could change notation in the integrals (3.36), by substituting $\boldsymbol{\zeta}^\varepsilon(0)$ with $(\boldsymbol{\zeta}^\varepsilon_{S(1)}(0), \dots, \boldsymbol{\zeta}^\varepsilon_{S(j)}(0))$, where $\boldsymbol{\zeta}^\varepsilon_{S(i)} = \{\zeta^\varepsilon_k \ ; \ k \in S(i)\}$. As already pointed out, however, configurations $\boldsymbol{\zeta}^\varepsilon_{S(i)}$ with different values of i are correlated through the external recollisions.

Let us introduce a different notion of backwards flow, in which the correlations among different groups $\boldsymbol{\zeta}^\varepsilon_{S(i)}$ are ignored. Suppose that we want the tree Γ_i to be "uncorrelated". Then, for all $k \in S(i)$, we substitute the IBF $\zeta^\varepsilon_k(s)$ with the evolution

$$\tilde{\zeta}^\varepsilon_k(s), \tag{3.41}$$

to be constructed as $\zeta^\varepsilon_k(s)$ with the additional prescription that *its external recollisions are ignored* (see Fig. 3). The constraint excluding overlaps of created particles in Γ_i with particles of different trees at the moment of creation, has to be also ignored. Notice that (3.41) is a function of the only $z_i, \Gamma_i, \mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}$.

For instance if we want *all* the trees in the expansions (3.36) to be uncorrelated in the above sense, we shall replace $\boldsymbol{\zeta}^\varepsilon(s) \to \tilde{\boldsymbol{\zeta}}^\varepsilon(s)$ inside the formula and require that:

- factors B^ε associated to different trees become completely independent;
- the initial data are evaluated in the time-zero configuration $\tilde{\boldsymbol{\zeta}}^\varepsilon(0) = (\tilde{\boldsymbol{\zeta}}^\varepsilon_{S(1)}(0), \dots, \tilde{\boldsymbol{\zeta}}^\varepsilon_{S(j)}(0))$ (with $\tilde{\boldsymbol{\zeta}}^\varepsilon_{S(i)} = \{\tilde{\zeta}^\varepsilon_k \ ; \ k \in S(i)\}$), that is a collection of j independent objects. The resulting quantity would differ from the tensorized product $f^\varepsilon_1(t)^{\otimes j}(\mathbf{z}_j)$ *just* because of the correlations assumed for the initial r.c.f. $f^\varepsilon_{0,j+n}$.

3.5.2 The Enskog backwards flow (EBF)

Even after replacing the IBF with the uncorrelated flow in (3.36), there is still a nontrivial correlation among particles of the *same* tree. This is due to the internal recollisions in $\tilde{\boldsymbol{\zeta}}^\varepsilon$, among particles of each set $S(i)$. To get rid of them, one has to introduce the completely uncorrelated backwards flow

$$\zeta^{\mathscr{E}}_k(s) \tag{3.42}$$

(where $\mathscr{E}$ stands for "Enskog") for all $k \in S(i)$, to be constructed as $\tilde{\zeta}^\varepsilon_k(s)$ with the additional prescription that *its internal recollisions are ignored*, together with the constraint excluding overlaps of created particles at the moment of creation.

The evolution $\boldsymbol{\zeta}^{\mathscr{E}}$ will be called *Enskog backwards flow* (EBF). In this flow, particles are created at distance ε (from their progenitor), but they may reach a distance smaller than ε during the evolution (in particular, its time-zero state $\boldsymbol{\zeta}^{\mathscr{E}}(0)$ varies in $\mathbb{R}^{6(j+\sum_i n_i)}$).

Alternatively, we may say that the EBF is constructed as the IBF, with the following differences:

– except for the scattering at the creation times, the interacting dynamics $\mathtt{T}^{\varepsilon}$ is replaced by the simple free dynamics;
– there is no constraint on ω_r.

The name "Enskog" is due to the obvious connection with the Enskog equation. Indeed (3.15)–(3.16) can be rewritten explicitly

$$g_j^{\varepsilon}(\mathbf{z}_j, t) = \sum_{n=0}^{\infty} \sum_{\Gamma(j,n)} \mathcal{T}^{\mathscr{E}}(\mathbf{z}_j, t) \tag{3.43}$$

where $\mathcal{T}^{\mathscr{E}}(\mathbf{z}_j, t)$ is the *value of the tree* $\Gamma(j, n)$ *with configuration* $\mathbf{z}_j$ *at time* t, *for the Enskog flow*

$$\mathcal{T}^{\mathscr{E}}(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod B^{\mathscr{E}} g_{0,j+n}^{\varepsilon}(\boldsymbol{\zeta}^{\mathscr{E}}(0)), \tag{3.44}$$

with $\prod B^{\mathscr{E}} = \prod_{i=1}^{n} B(\omega_i; v_{j+i} - \eta_{k_i}^{\mathscr{E}}(t_i))$,

$$B(\omega_i; v_{j+i} - \eta_{k_i}^{\mathscr{E}}(t_i)) = \omega_i \cdot (v_{j+i} - \eta_{k_i}^{\mathscr{E}}(t_i)). \tag{3.45}$$

Note that the EBF allows a complete factorization, whenever the initial datum does. Namely if $g_{0,j}^{\varepsilon} = (f_0)^{\otimes j}$ for all j, the expansion above gives immediately

$$g_j^{\varepsilon}(\mathbf{z}_j, t) = \prod_{i=1}^{j} \left(\sum_{n_i, \Gamma_i} \int d\Lambda(\mathbf{t}_{n_i}^i, \boldsymbol{\omega}_{n_i}^i, \mathbf{v}_{1,n_i}^i) \prod B^{\mathscr{E}} g_{0,1+n_i}^{\varepsilon}(\boldsymbol{\zeta}_{S(i)}^{\mathscr{E}}(0)) \right), \tag{3.46}$$

where $\boldsymbol{\zeta}_{S(i)}^{\mathscr{E}} = \{\zeta_k^{\mathscr{E}}\ ;\ k \in S(i)\}$.

3.5.3 The Boltzmann backwards flow (BBF)

The previous discussion can be repeated, with minor changes, for the case of the Boltzmann series (3.10). The interacting backwards flow is now substituted by the *Boltzmann backwards flow* (BBF) $\boldsymbol{\zeta}(s)$. For it, we use the same notations of (3.22)–(3.24) with the superscript ε omitted.

Since the collision operator (3.7) is splitted into a gain and a loss term, then, together with the sum over $\Gamma(j,n)$, we have an additional $\sum_{\boldsymbol{\sigma}_n}$ with $\boldsymbol{\sigma}_n = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i = \pm$. To have a compact expression, we change variables $\omega \to -\omega$ inside the positive part of the collision operators. As a result, in each term of the expansion, σ_i fixes the sign of the product $\omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i))$ (where the relative velocity at the moment of creation appears). Note that the same procedure has to be followed when rewriting (3.16) in the form (3.43)–(3.45).

The BBF turns out to be defined exactly as the IBF, except for the following differences:

- the interacting dynamics T^ε is replaced by the simple free dynamics;
- in the right hand side of (3.32) the second term is missing, i.e. the created particle appears at the same position of its progenitor: $\xi_{j+r}(t_r) = \xi_{k_r}(t_r)$;
- there is no constraint on ω_r other than the sign (implied by the value of σ_r);
- if $\sigma_r = +$, to determine the configuration of particles in (t_{r+1}, t_r), *before* applying free evolution we have to change velocities according to $(\eta_{k_r}(t_r^+), v_{j+r}) \to (\eta_{k_r}(t_r^-), \eta_{j+r}(t_r^-))$, where $\to$ denotes the elastic scattering rule with scattering vector ω_r. We recall that, in our conventions, $\eta_{k_r}(t_r) \equiv \eta_{k_r}(t_r^+)$ (which indicates the limit from the future, while $\eta_{k_r}(t_r^-)$ indicates the limit from the past).

Equation (3.10) can then be rewritten:

$$f_j(\mathbf{z}_j, t) = \sum_{n=0}^{\infty} \sum_{\Gamma(j,n)} \mathcal{T}(\mathbf{z}_j, t), \tag{3.47}$$

where $\mathcal{T}(\mathbf{z}_j, t)$ is the *value of the tree* $\Gamma(j,n)$ *with configuration* $\mathbf{z}_j$ *at time* t, *for the Boltzmann flow*,

$$\mathcal{T}(\mathbf{z}_j, t) = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \prod B \; f_{0,j+n}(\boldsymbol{\zeta}(0)), \tag{3.48}$$

with $\prod B = \prod_{i=1}^n B(\omega_i; v_{j+i} - \eta_{k_i}(t_i))$ and

$$\begin{aligned} B(\omega_i; v_{j+i} - \eta_{k_i}(t_i)) &= \sum_{\sigma_i} \sigma_i |\omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i))| \mathbb{1}_{\{\sigma_i \omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i)) \geq 0\}} \\ &= \omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i)). \end{aligned} \tag{3.49}$$

Note that, in the final formula, the difference between gain and loss collision operators is hidden inside the *rule* for the construction of the BBF, which depends, as explained above, on the sign of each product $\omega_i \cdot (v_{j+i} - \eta_{k_i}(t_i))$.

Note also that

$$\boldsymbol{\eta} = \boldsymbol{\eta}^{\mathscr{E}} \tag{3.50}$$

and

$$B = B^{\mathscr{E}}, \tag{3.51}$$

the only difference between the BBF and the EBF being due to the position in space of created particles.

As before, (3.47) can be immediately written in the form

$$f_j(\mathbf{z}_j, t) = \prod_{i=1}^{j} \left(\sum_{n_i, \Gamma_i} \int d\Lambda(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i}) \prod B \right) f_{0,j+n}(\boldsymbol{\zeta}(0)), \tag{3.52}$$

which shows a complete factorization in the case of factorized initial data.

3.6 Summary

We have introduced:

(i) the tree expansion for the evolution of the hard sphere system, solution to the BBGKY hierarchy of equations: see (3.25)–(3.26) (equivalently, (3.36));
(ii) an "uncorrelated" tree expansion described in Sect. 3.5.1;
(iii) the tree expansion for the Enskog equation, solution to the Enskog hierarchy: (3.43)–(3.44);
(iv) the tree expansion for the Boltzman equation, solution to the Boltzmann hierarchy: (3.47)–(3.48);

and, correspondingly:

(i') the interacting backwards flow $\boldsymbol{\zeta}^{\varepsilon}(s)$, expressing the evolution of the rescaled correlation functions of the hard sphere system;
(ii') the partially uncorrelated flow $\tilde{\boldsymbol{\zeta}}^{\varepsilon}(s)$, obtained from the IBF by ignoring the external recollisions;
(iii') the Enskog backwards flow $\boldsymbol{\zeta}^{\mathscr{E}}(s)$, obtained from the IBF by ignoring all the recollisions;
(iv') the Boltzmann backwards flow $\boldsymbol{\zeta}(s)$, describing the evolution of functions obeying the Boltzmann hierarchy, and obtained from the EBF by making the particles interact at distance zero instead of ε.

See Fig. 3 below.

The flows in (iii') and (iv') will be used to prove convergence of the hard sphere system to the Enskog and the Boltzmann equation, while (ii') will be enough for the proof of propagation of chaos.

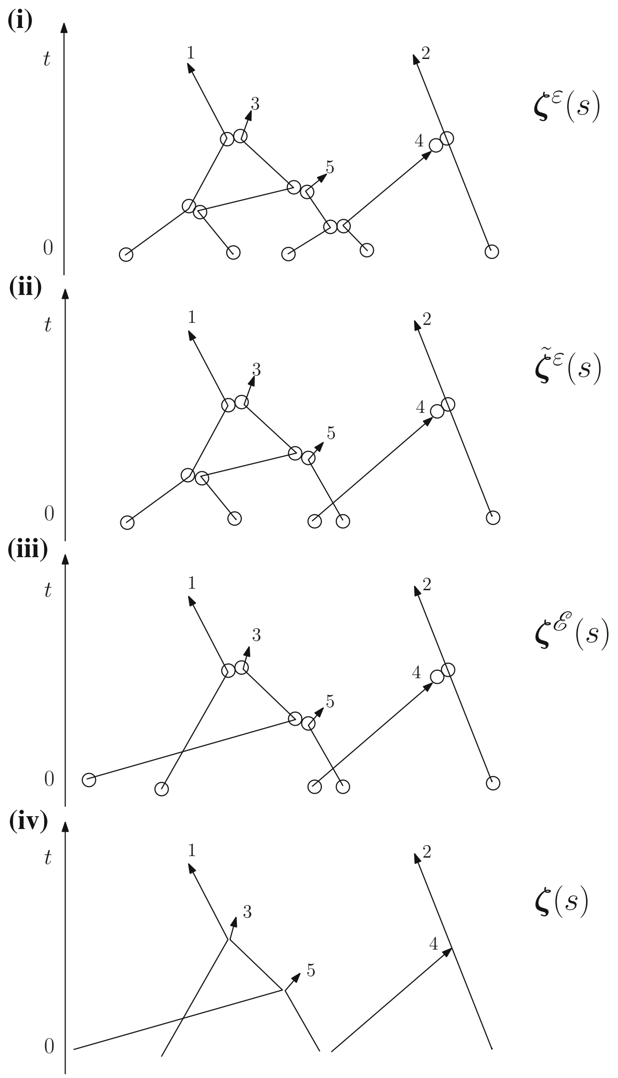


Fig. 3 An example of trajectories drawn by the four types of backwards flow introduced, in the case of the two-particle tree $\Gamma(2,3) = (1,2,3)$, for fixed values of the variables (here $\mathbf{z}_2, \mathbf{t}_3, \boldsymbol{\omega}_3, \mathbf{v}_{2,3}$). The hard spheres of diameter ε are pictured at the creation times and at the recollision times

4 Proof

4.1 Basic estimates

In this section we recall the basic estimate given by Lanford, in a form well suited for our purposes.

The following convergence property of the BBGKY, Enskog and Boltzmann series expansions introduced in Sect. 3, is preliminary to our work.

Proposition 4.1 (Short time estimates) *If the initial data* $f^{\varepsilon}_{0,j}$ *and* $f_{0,j}$ *are bounded as in* (2.12), *then the absolute convergence of the expansions* (3.25), (3.43) (*uniformly in* ε) *and* (3.47) *holds, for any* $t < \bar{t} = \bar{t}(z, \beta)$.

The proof, reported here for completeness, reduces immediately to the bound given by the following lemma, which is stated in a somewhat general form.

Lemma 4.2 *Let* $a = 1, 2$. *There exist constants* $\bar{t}, \bar{C} > 0$ (*depending on* z, β) *such that, for any* $t < \bar{t}$, *the following estimate holds*:

$$\sum_{n\geq 0} z^{j+n} \sum_{\Gamma(j,n)} \int d\Lambda(\boldsymbol{t}_n, \boldsymbol{\omega}_n, \boldsymbol{v}_{j,n}) \left(\prod |B^{\varepsilon}|\right)^a \; e^{-(\beta/2)\sum_{i\in S(J)}(\eta^{\varepsilon}_i(0))^2} \leq \bar{C}^j e^{-(\beta/4)\sum_{i\in J} v_i^2}. \tag{4.1}$$

The same result holds when $B^{\varepsilon}, \boldsymbol{\zeta}^{\varepsilon}$ *are replaced by* $B^{\mathscr{E}}, \boldsymbol{\zeta}^{\mathscr{E}}$ *(Enskog flow) or* $B, \boldsymbol{\zeta}$ *(Boltzmann flow).*

Remind that $J = \{1, \ldots, j\}$ and, by the notation (3.40), $S(J) = \{1, 2, \ldots, j+n\}$.

In the case $a = 1$, this shows that the expansions of Proposition 4.1 are absolutely convergent in the norm $\operatorname{ess\,sup}_{\mathbf{x}_j} \int d\mathbf{v}_j$. The case $a = 2$ will be used as technical tool in Appendix C (Lemma C.1).

Proof of Lemma 4.2 The conservation of energy at collisions implies

$$\sum_{i\in S(J)} (\eta^{\varepsilon}_i(0))^2 = \sum_{i=1}^{j+n} v_i^2. \tag{4.2}$$

In particular $\sum_{k_i=1}^{j+i-1}(\eta^{\varepsilon}_{k_i}(t_i))^2 \leq \sum_{i=1}^{j+n} v_i^2$. Using the expression of B^{ε} (3.29) and (3.19), we find

$$\sum_{\Gamma(j,n)} \left(\prod |B^{\varepsilon}|\right)^a \leq a^n \prod_{i=1}^{n} \left[(j+n)|v_{j+i}|^a + (j+n)^{\frac{2-a}{2}} \left(\sum_{l=1}^{j+n} v_l^2\right)^{\frac{a}{2}} \right]. \tag{4.3}$$

Moreover,

$$\left(\sum_{l=1}^{j+n} v_l^2\right)^{\frac{1}{2}} e^{-\frac{\beta}{4n}\sum_{i=1}^{j+n} v_i^2} \leq \sqrt{\frac{2n}{e\beta}}, \quad \left(\sum_{l=1}^{j+n} v_l^2\right) e^{-\frac{\beta}{4n}\sum_{i=1}^{j+n} v_i^2} \leq \frac{4n}{e\beta}. \tag{4.4}$$

Inserting these estimates in the l.h.s. of (4.1), it follows that we can bound it by

$$e^{-(\beta/4)\sum_{i\in J} v_i^2} \sum_{n\geq 0} 2^n z^{j+n} \int d\Lambda \prod_{i=1}^{n} e^{-\frac{\beta}{4} v_{j+i}^2} \times \left((j+n)|v_{j+i}|^a + \frac{\sqrt{2n(j+n)}}{\sqrt{e\beta}} + \frac{4n}{e\beta} \right). \tag{4.5}$$

The integral on the velocities factorizes so that

$$(4.5) \leq e^{-(\beta/4)\sum_{i\in J} v_i^2} \sum_{n\geq 0} C(z,\beta)^{j+n} \frac{t^n}{n!} (j+n)^n \tag{4.6}$$

for a suitable constant $C(z,\beta) > 0$ (explicitly computable in terms of gaussian integrals). Since

$$\frac{(j+n)^n}{n!} \leq \frac{(j+n)^{j+n}}{(j+n)!} \leq e^{j+n}, \tag{4.7}$$

we have that (4.6) is bounded by a geometric series. Hence choosing

$$\bar{t} < \frac{1}{C(z,\beta)e}, \tag{4.8}$$

we obtain (4.1).

The cases of the Enskog and of the Boltzmann flow are treated in the same way. □

Remark Lemma 4.2 and Proposition 4.1 imply of course that the correlation errors E_K, $E_K^{\mathcal{E}}$, $E_K^{\mathcal{B}}$ introduced in Theorem 2.4 are bounded by $(const.)^k$, uniformly in ε, for all $t < \bar{t}$. Note that this is also true (and optimal) in the regions $\mathbb{R}^{6k}\backslash\mathcal{M}_k$, as soon as the definition of the r.c.f. is extended there as $f_J^\varepsilon(\mathbf{z}_J, t) = 0$.

4.2 Plan of the proof

In this section we outline the main technical difficulties in proving our result and give some intuitive explanation of the strategies and sketch of the arguments developed to overcome them. We concentrate on estimate (2.21) which is the main result of the paper. There are three main issues:

Step 1: Combinatorics;
Step 2: Ordering of recollisions;
Step 3: Estimate of the single recollision event.

In Step 1 we construct a perturbative expression for the correlation error E_K and control the number of terms. Steps 2, 3 deal with the estimate of such an expression. All these steps present peculiar difficulties and we shall discuss them separately.

General strategy. We start from the explicit formula for the evolution of rescaled correlation functions f_j^ε, that is the tree expansion described in Sect. 3.4, see (3.36). It is important to keep in mind the structure of this formula: (a) a sum over j binary tree graphs; (b) an integral over characteristic flows of type $\boldsymbol{\zeta}^\varepsilon$ associated to the trees (see Fig. 3-i above).

Our first purpose is to manipulate directly the formula and *reorganize* it into the cumulant type expansion (1.14).

Reconstructing formula (1.14) from the tree expansion means to *extract*, among the j trees, different *subsets* of independent one-particle trees. "Independent" here has a precise meaning, namely the *value* of the tree does *not* depend on particles external to the tree. Conversely, "correlated" trees are not independent trees. By working with formula (3.36) we are in a favourable position, since the sources of correlation become totally explicit (remind the discussion in Sect. 3.5.1).

There are two different effects: the propagation in time of the initial correlations due to the hard sphere exclusion, and the dynamical correlations induced by the external recollisions. As an example, consider the two-particle function f_2^ε. An associated trajectory is Fig. 3-i where the trees Γ_1, Γ_2 are correlated through the external recollision. To get independent trees we would need: (1) to replace $\boldsymbol{\zeta}^\varepsilon$ by $\tilde{\boldsymbol{\zeta}}^\varepsilon$, where the particles of different trees ignore and cross each other freely (*overlap*); (2) to replace the time-zero distribution $f_{0,5}^\varepsilon(z_1, \dots, z_5)$ with $f_{0,3}^\varepsilon(z_1, z_3, z_5) f_{0,2}^\varepsilon(z_2, z_4)$. Point (1) is achieved by the following elementary addition/subtraction procedure: for any integrable function of the flow F (here $F = F(\mathbf{z}_2, \mathbf{t}_3, \boldsymbol{\omega}_3, \mathbf{v}_{2,3})$), it holds that (here $\int = \int d\Lambda(\mathbf{t}_3, \boldsymbol{\omega}_3, \mathbf{v}_{2,3})$)

$$\begin{aligned}\int F(\boldsymbol{\zeta}^\varepsilon) &= \int F\chi_{1,2}^{rec}(\boldsymbol{\zeta}^\varepsilon) + \int F(1-\chi_{1,2}^{rec})(\boldsymbol{\zeta}^\varepsilon)\\ &= \int F(\tilde{\boldsymbol{\zeta}}^\varepsilon) + \int \Big[F\chi_{1,2}^{rec}(\boldsymbol{\zeta}^\varepsilon) - F\chi_{1,2}^{ov}(\tilde{\boldsymbol{\zeta}}^\varepsilon)\Big],\end{aligned}$$

where $\chi_{1,2}^{rec}$ ($\chi_{1,2}^{ov}$) is the indicator function of the recollision (overlap) condition between the two trees. The last integral, which will be $O(\varepsilon^{2\gamma})$ for some $\gamma > 0$, is part of the correlation error E_2, and to have the complete expression it suffices now to add the error term in point (2).

We stress that the flows in Fig. 3 above, as well as the notions of recollision and overlap do not have direct physical interpretation. Technically, the estimation of recollisions and overlaps will be treated in the same way in Sect. 4.4 ($\boldsymbol{\zeta}^\varepsilon$ has external recollisions *iff* $\tilde{\boldsymbol{\zeta}}^\varepsilon$ has overlaps). But to give an explicit expression for E_K, we need to distinguish between the two.

The extension of the procedure to the case of $k > 2$ particles leads to a combinatorial problem. One could write $1 = \prod_{i<\ell}[\chi^{rec}_{i,\ell} + (1 - \chi^{rec}_{i,\ell})]$ and proceed as above. For any pair of recolliding trees we hope to gain a factor $O(\varepsilon^{2\gamma})$. But for k recolliding trees one would get $\sim 2^{\frac{k(k-1)}{2}}$ terms, therefore nothing better than $E_K \sim 2^{\frac{k(k-1)}{2}} \varepsilon^{\gamma k}$ (yielding $k \leq O(\log \varepsilon^{-1})$). To reach $k \sim O(\varepsilon^{-\alpha})$, we use a simple graph expansion procedure (Lemma 4.4 below) allowing to improve the bad counting factor to $k!$. In Sect. 4.3 we introduce this technique (in a sense reminiscent of the cluster expansion in equilibrium statistical mechanics) in an abstract form, and then apply it to both the dynamical and the initial correlation.

Let us explain here the method in a few words.

Step 1. Suppose to have a set J of trees. Some of them recollide externally, say those in $L_0 \subset J$, while the other trees $L = J \backslash L_0$ do not. We indicate these conditions with the characteristic functions of the flow: $\chi^{rec}_{L_0}$, $\bar{\chi}^{rec}_{L,J}$. Obviously the second function makes the trees in L *not* independent. To make them independent we need to substitute $\boldsymbol{\zeta}^\varepsilon \to \tilde{\boldsymbol{\zeta}}^\varepsilon$: we produce an error

$$F\,\bar{\chi}^{rec}_{L,J} = F - \sum_{\emptyset \neq L_1 \subset L} F\,\chi^{ov}_{L_1, L_0 \cup L_1}\,\bar{\chi}^{rec}_{L \backslash L_1, J}$$

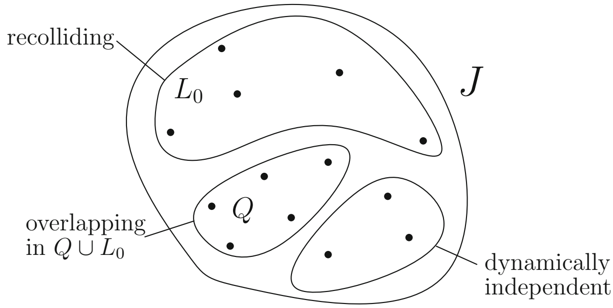
where everything is evaluated in $\tilde{\boldsymbol{\zeta}}^\varepsilon$, and $\chi^{ov}_{L_1, L_0 \cup L_1}$ means that *all* the trees in L_1 are constrained to overlap with some other tree in $L_0 \cup L_1$. From the $|L_1|$ overlapping condition, we want to gain $O(\varepsilon^{\gamma |L_1|})$ (steps 2, 3 below). The trees $L \backslash L_1$ are still correlated through $\bar{\chi}^{rec}_{L \backslash L_1, J}$, but here we just iterate the formula. It follows that

$$1 = \sum_{L_0 \subset J} \chi^{rec}_{L_0}\,\bar{\chi}^{rec}_{L,J} = \sum_{L_0 \subset J} \chi^{rec}_{L_0} \sum_{Q \subset J \backslash L_0} R(Q, L_0) \tag{4.9}$$

where $R(Q, L_0)$ contains q overlapping conditions and a number of terms growing as the number of partitions of a set with cardinality $q = |Q|$, i.e.

$$\sum_{Q \subset J \backslash L_0} R(Q, L_0) \approx C^q q!\,\chi^{ov}_{Q, Q \cup L_0} \tag{4.10}$$

for some constant $C > 0$ (see Appendix B for the exact expression). The result is summarized in the picture.



The trees in $J\backslash(L_0 \cup Q)$ are *free* of dynamical conditions and evaluated in the uncorrelated flow $\tilde{\boldsymbol{\zeta}}^{\varepsilon}$, therefore they are correlated only through the initial data. Such a correlation, which is due to the hard sphere exclusion at time zero, can be treated by the same method, with the "recollision condition" replaced by the "overlap of spheres at time zero" (Sect. 2.4.2, Property 2). This makes the final expression for the correlation error E_K slighty more complicated than what can be guessed from (4.9) (see (4.27) below).

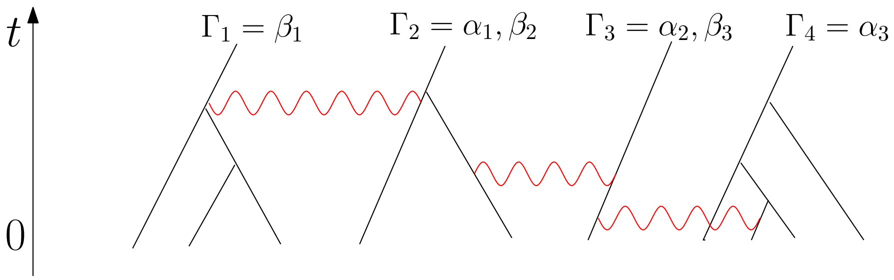
Step 1.1 Once derived the expression for E_K it becomes clear that we need to face an estimate of events with many external recollisions and overlaps, like $\int \chi^{rec}_{L_0} \chi^{ov}_{Q,Q\cup L_0}(\cdots)$. This will be the object of steps 2 and 3. A preliminary, crucial operation is the substitution of the integrand $(\cdots)$ with a simplified expression. This is based on estimates known from previous papers. In Sect. 4.3.3 we will show that the Hypotheses on the initial data and the introduction of several cutoffs allow to replace the integrands in the expansion for E_K with a positive, bounded, compactly supported function of the *energy of trees*.

Step 2. Let $F = F(K)$ be a nice function of the energy of the trees in K. Consider, for simplicity ($Q = \emptyset$ above), the estimate of K recolliding trees with n created particles: $\int \chi^{rec}_K F(K)$, from which we want to gain a factor $\varepsilon^{\gamma k}$. This is a delicate point because, while it is understood how to estimate a single (internal or external) recollision (see [16,30] and Appendix D of the present paper) it is not obvious at all that, in case of k recolliding trees—implying at least $k/2$ external recollisions—one can gain the $\varepsilon^{\gamma k}$ by means of a sufficiently large number of integrations.

The $k/2$ recollision conditions are, of course, *not independent*. Therefore, we need to proceed carefully. First, we *order* the recollisions in time. Secondly, for any possible sequence, we estimate the recollisions one by one *iteratively*, following the time order.

To clarify this better, let us consider the following possible ordering. Going backwards in time, the first two trees to collide are Γ_2 and Γ_1. Going further backwards, the first collision involving a *new* tree is between Γ_3 and

Γ_2, the second is between Γ_4 and Γ_3 and so on up to the last collision of the chain, between Γ_k and Γ_{k-1}. It is natural to say that the trees α and β are in a relation "bullet-target" when the *first* external recollision of α going backwards in time occurs with β. In the case considered, we have a sequence $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_{k-1}, \beta_{k-1})$, where $(\alpha_i, \beta_i) = (\Gamma_{i+1}, \Gamma_i)$ and the first external recollision of α_i occurs in the future with respect to the first external recollision of α_{i+1}. For instance for $k = 4$:

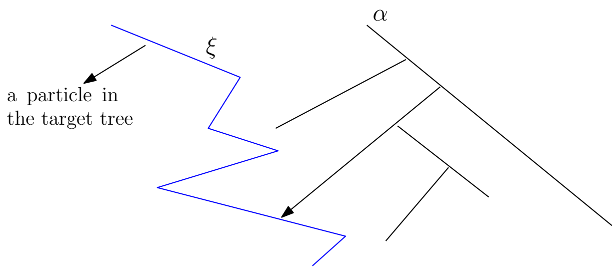


where we represented with wavy lines the recollisions characterizing the bullets.

The velocities at time t of the particles generating the trees are denoted by $v_1, v_2, \dots, v_k$ respectively. Now *fix* all the integration variables but those relative to the last tree Γ_k. Then we can integrate with respect to the latter variables (including v_k) with *no interference* with the other constraint, thanks to the fact that the recollision $\Gamma_k - \Gamma_{k-1}$ is the last one in backward order. We gain a small factor ε^{γ_1} by this integration (Lemma 4.10 below) and we iterate the procedure to obtain the desired result.

More generally, to handle with $\int \chi_K^{rec} F(K)$ we shall introduce, in Sect. 4.4, a "table of recollisions" $\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_\ell, \beta_\ell)\}$, characterizing one of all possible choices of the couples bullet-target and of the orderings of bullets in time. Whatever is the sequence, we know that $\ell \geq k/2$, and we reduce to an ordered integral ((4.46) below) which allows to estimate the constraints one by one (integrate out the α_ℓ-variables first, then the $\alpha_{\ell-1}$-variables, and so on up to α_1).

Step 3. The price we pay for the approach in Step 2 is that we need now to control the single overlap of a given bullet against one target tree whose particles perform a very complicated trajectory, possibly due to several recollisions. In fact, even if we ignore the internal recollisions of the bullet tree (as can be actually done producing a small error, see Sect. 4.5.1), we are still left with the following kind of challenge



namely the *geometry* of the constraint is more complex with respect to the one studied in [16,30]. In the latter references, targets move freely, while here they may have an uncontrolled number of recollisions.

The key ingredients to deal with such an estimate are: (1) a parametrization of the constraint in terms of the *relative velocities* of the bullet tree at the creation times (Sect. 4.5.3); (2) to exploit the variable v_α, namely the velocity of the *root* of the bullet tree (Sect. 4.5.5). Indeed, thanks to (1), the constraint may be rewritten as $\mathbb{1}(v_\alpha \in \mathcal{T}^\varepsilon_\xi)$, where $\mathcal{T}^\varepsilon_\xi$ is a *thin tube* around a curve of parametric equation given by ξ (which is frozen) and by all the variables (but v_α) spanning the flow of the bullet ((4.77) below).

No scattering transform is required in this procedure. On the other hand, it is crucial to integrate in the velocities $\int d\mathbf{v}_k$ as stated in the main theorem. For this reason, we cannot treat the internal recollisions with the same method.

4.3 Step 1: Combinatorics

In this section we develop the graph expansion technique, Lemma 4.4 below, which we use to reconstruct the representation (2.20) with an explicit expression for $E_K(t)$. We find convenient to discuss this method of expansion in an abstract formulation first (i.e. for generic graphs) since it will be used twice in the sequel, that is it will be applied to the BBGKY series (3.36) (in Sect. 4.3.1) and to the initial data (proof of Property 2 in Appendix A, applied in Sect. 4.3.2). Next, the discussion in Sects. 4.3.3–4.3.5 will reduce the proof of the theorem to an estimate of many-recollision events.

Let us start with some classical definitions.

Definition 4.3 (i) A graph over a set $\mathcal{I} = \{a_1 \cdots a_n\}$ of vertices, is a collection of edges (links) $\{\ell_{i,j}\}_{i \neq j}$, where $\ell_{i,j}$ takes the values $1, 0$ if the vertices a_i and a_j are connected or not respectively (e.g. Fig. 4).

(ii) $\mathcal{G}$ is the family of all graphs over $\mathcal{I}$.

(iii) We introduce the following characteristic functions on $\mathcal{G}$:

$$\chi_{i,K} = 1$$

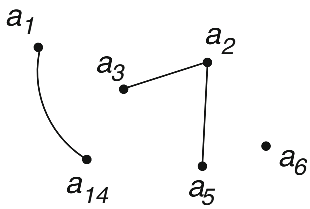


Fig. 4 Graph $\ell_{1,4} = \ell_{2,3} = \ell_{2,5} = 1$, and different $\ell_{i,j} = 0$

if and only if the vertex a_i is connected with some vertex in $K \subset \mathcal{I}$;

$$\bar{\chi}_{i,K} = 1 - \chi_{i,K};$$

and, for $H \subset \mathcal{I}$,

$$\chi_{H,K} = \prod_{i \in H} \chi_{i,K},$$
$$\bar{\chi}_{H,K} = \prod_{i \in H} \bar{\chi}_{i,K}.$$

$\chi_{H,K} = 1$ if and only if any vertex of H is connected with some vertex in K, and $\bar{\chi}_{H,K} = 1$ if and only if any vertex in H is not connected with any vertex in K. Note that a vertex cannot be self connected, i.e. $\chi_{i,i} = 0$ and $\bar{\chi}_{i,i} = 1$.

The following Lemma is the abstract, rigorous formulation of the ideas explained in 'Sect. 4.2—General strategy and Step 1' (where $\mathcal{I} = J = L \cup L_0$ is a set of trees).

Lemma 4.4 *Let $L \subset \mathcal{I}$ and $L_0 = \mathcal{I} \backslash L$. Then,*

$$\bar{\chi}_{L,L\cup L_0} = \sum_{Q \subset L} R(Q, L_0) \tag{4.11}$$

where, for some pure constant $C > 0$,

$$|R(Q, L_0)| \leq C^q \, q! \, \chi_{Q,Q\cup L_0}. \tag{4.12}$$

We remind the notation $q = |Q|$ and the convention $\bar{\chi}_{\emptyset,\cdot} = \chi_{\emptyset,\cdot} = 1$.

Note that each term of the expansion (4.11) does not depend on $L \backslash Q$, i.e. there is no condition on the vertices of this set (they are "free" vertices).

The proof of the Lemma is a simple algebraic computation and can be found in Appendix B (see (B.5) for the exact expression of R).

4.3.1 Expanding the dynamical constraints

We start by rewriting the formula, introduced in Sect. 3.4, yielding the reduced correlation functions at time t, namely

$$f_J^\varepsilon(\mathbf{z}_J, t) = \sum_{n\geq 0} \sum_{\Gamma(j,n)} \int d\Lambda \prod B^\varepsilon f_{0,S(J)}^\varepsilon, \tag{4.13}$$

where we abbreviate

$$\int d\Lambda = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}), \tag{4.14}$$

$$f_{0,S(J)}^\varepsilon = f_{0,|S(J)|}^\varepsilon(\boldsymbol{\zeta}_{S(J)}^\varepsilon(0)), \tag{4.15}$$

$\Gamma(j, n)$ denotes the set of j-particle trees with n created particles, $(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$ are the collections of node variables in the tree and $S(J)$ denotes the set of indices of the particles created in the backwards flow $\boldsymbol{\zeta}^\varepsilon$ at time 0. Clearly, $|S(J)| = j + n$.

4.3.1.a Selection of the recolliding set Let us focus on the external recollisions.

Consider the map (3.33). We say that two trees, say Γ_i and Γ_k (or, briefly, i and k) *recollide* if there is a particle in $S(i)$ which collides with a particle in $S(k)$.

We introduce the characteristic function $\chi_{i,K}^{rec}$ defined by:

$$\chi_{i,K}^{rec} = 1$$

if and only if the tree i recollides with some tree in $K \subset J$. This is a function of the IBF $\boldsymbol{\zeta}^\varepsilon$. Also, we introduce

$$\chi_K^{rec} = \prod_{i\in K} \chi_{i,K}^{rec},$$

so that $\chi_K^{rec} = 1$ if and only if all the trees in K recollide with some other tree in K. Finally,

$$\bar{\chi}_{i,K}^{rec} = 1 - \chi_{i,K}^{rec}$$

and, for $H \subset J$,

$$\bar{\chi}_{H,K}^{rec} = \prod_{i\in H} \bar{\chi}_{i,K}^{rec}.$$

That is $\bar{\chi}_{H,K}^{rec} = 1$ if and only if the trees in H do *not* recollide with any tree in K.

With these definitions, one has

$$1 = \sum_{L_0\subset J} \chi_{L_0}^{rec} \bar{\chi}_{J\setminus L_0,J}^{rec}. \tag{4.16}$$

Observe that, if $L_0 \neq \emptyset$, $l_0 = |L_0| \geq 2$. Inserting now this partition of unity into (4.13), we find

$$f_J^\varepsilon(\mathbf{z}_J, t) = \sum_{n\geq 0} \sum_{\Gamma(j,n)} \sum_{L_0 \subset J} \int d\Lambda \prod B^\varepsilon \, \chi^{rec}_{L_0} \, \bar{\chi}^{rec}_{J\setminus L_0, J} \, f^\varepsilon_{0,S(J)}. \tag{4.17}$$

The set $J \setminus L_0$ is the maximal set of trees *without* external recollision and we are in the position to apply (4.9) (i.e. Lemma 4.4). This requires the introduction of the "mixed flow" first.

4.3.1.b Mixed backwards flow From the discussion in Sect. 4.2 it should be clear that, to treat (4.17), we need to define a *mixed* backwards flow, in which the particles of the trees in L_0 are evolved by taking into account all the recollisions among themselves, while the particles belonging to the trees in $J \setminus L_0$ are evolved through the flow $\tilde{\boldsymbol{\zeta}}^\varepsilon$, i.e. by ignoring their external recollisions (see Fig. 3 above). We shall indicate such a flow

$$(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\zeta}}^{(J\setminus L_0)}), \tag{4.18}$$

where $\boldsymbol{\zeta}^{(L_0)}$ is the flow of particles of the trees in L_0 and $\tilde{\boldsymbol{\zeta}}^{(J\setminus L_0)}$ is the flow of particles of the trees in $J \setminus L_0$. Note that we are ignoring the dependence on ε (now clear from the context) to unburden the notation.

Let $i \in H \subset J \setminus L_0$ and $K \subset J$. We introduce the following characteristic functions:

$$\chi^{ov}_{i,K} = 1$$

if and only if the tree i *overlaps* with some tree in $K \subset J$ in the dynamics (4.18) (in the sense that some particle in $S(i)$ reaches a distance smaller than ε from some other particle in $S(K)$); moreover we set

$$\chi^{ov}_{H,K} = \prod_{i\in H} \chi^{ov}_{i,K},$$
$$\bar{\chi}^{ov}_{i,K} = 1 - \chi^{ov}_{i,K},$$
$$\bar{\chi}^{ov}_{H,K} = \prod_{i\in H} \bar{\chi}^{ov}_{i,K}.$$

That is, $\chi^{ov}_{H,K} = 1$ if and only if all the trees in H overlap with some tree in K while $\bar{\chi}^{ov}_{H,K} = 1$ if and only if all the trees in H do not overlap with any tree in K.

Finally, we write (with a slight abuse w.r.t. the notation (3.29))

$$\prod B^{\varepsilon}(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\xi}}^{(J\backslash L_0)}) = \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{J\backslash L_0} \prod_{\substack{i=1\\(k_i\in L_0)}}^{n} \omega_i \cdot (v_{j+i} - \eta^{\varepsilon}_{k_i}(t_i)) \\ \times \prod_{\substack{i=1\\(k_i\in J\backslash L_0)}}^{n} \omega_i \cdot (v_{j+i} - \tilde{\eta}^{\varepsilon}_{k_i}(t_i)), \qquad (4.19)$$

where:

- $\mathbb{1}_{L_0}$ is the characteristic function ensuring that the particles created in the trees $\boldsymbol{\Gamma}_{L_0}$ do not overlap with other particles of the trees $\boldsymbol{\Gamma}_{L_0}$ at the moments of creation;
- $\tilde{\mathbb{1}}_{K\backslash L_0}$ is the characteristic function ensuring that the particles created in the tree Γ_i, $i \in K\backslash L_0$ do not overlap "internally" (i.e. with particles of the same tree) at the moments of creation.

With these definitions, the following trivial identity holds:

$$\left(\prod B^{\varepsilon}\, \chi^{rec}_{L_0}\, \bar{\chi}^{rec}_{J\backslash L_0, J}\, f^{\varepsilon}_{0,S(J)}\right)(\boldsymbol{\zeta}^{\varepsilon}) \\ = \left(\prod B^{\varepsilon}\, \chi^{rec}_{L_0}\, \bar{\chi}^{ov}_{J\backslash L_0, J}\, f^{\varepsilon}_{0,S(J)}\right)(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\xi}}^{(J\backslash L_0)}), \qquad (4.20)$$

which inserted into (4.17) leads to

$$f^{\varepsilon}_J(\mathbf{z}_J, t) = \sum_{n\geq 0} \sum_{\Gamma(j,n)} \sum_{L_0\subset J} \int d\Lambda \prod B^{\varepsilon}\, \chi^{rec}_{L_0}\, \bar{\chi}^{ov}_{J\backslash L_0, J}\, f^{\varepsilon}_{0,S(J)} \qquad (4.21)$$

with the integrand function calculated via the flow (4.18).

4.3.1.c Application of Lemma 4.4

Up to now, we just changed notation in (4.17).

Next we apply Lemma 4.4 to the case: $\mathcal{I} = J$, $\bar{\chi} = \bar{\chi}^{ov}$ and $L = J\backslash L_0$. We obtain

$$\bar{\chi}^{ov}_{J\backslash L_0, J} = \bar{\chi}^{ov}_{L, L\cup L_0} = \sum_{Q\subset L} R^{ov}(Q, L_0), \qquad (4.22)$$

where

$$|R^{ov}(Q, L_0)| \leq C^q q!\chi^{ov}_{Q, Q\cup L_0} \qquad (4.23)$$

for some $C > 0$.

Inserting the expansion in (4.21), we find

$$f_J^\varepsilon(\mathbf{z}_J,t)=\sum_{L_0\subset J}\sum_{Q\subset J\setminus L_0}\sum_{\substack{n\geq 0\\ \Gamma(j,n)}}\int d\Lambda\,\prod B^\varepsilon\chi_{L_0}^{rec}R^{ov}(Q,L_0)\,f_{0,S(J)}^\varepsilon. \tag{4.24}$$

Each tree in Q must obey an overlap-condition in order that $R^{ov}\neq 0$, while the trees in L_0 must recollide among themselves. In contrast, the trees $J\setminus(L_0\cup Q)$ are *free*, in the sense that there is no condition over them, so that they are not dynamically correlated (see the figure in Sect. 4.2—Step 1).

Of course, the latter are still correlated through the initial data $f_{0,S(J)}^\varepsilon$. Actually if the initial data were factorizing, then the algebraic part of our proof would finish here by extracting the leading term (namely, the trees which are free) which would reconstruct exactly the factorized part in (2.20).

4.3.2 *Expanding the initial correlation: final expression for $E_K(t)$*

To eliminate the additional correlation due to the initial datum, we expand it according to Property 2—Eq. (2.31)[4] with respect to the following tree-dependent partition of $S(J)$:

$$\{S_i\}_{i\in J\setminus(L_0\cup Q)},\ S(L_0\cup Q)$$

where $S_i=S(i)$ is the set of indices of the particles in the tree Γ_i. For this particular partition, (2.31) yields

$$f_{0,S(J)}^\varepsilon=\sum_{H\subset J\setminus(Q\cup L_0)}\left(\prod_{i\in H}f_{0,S(i)}^\varepsilon\right)\times\left(\bar{E}^0_{\{S_i\}_{i\in J\setminus H},S(Q\cup L_0)}+\bar{E}^0_{\{S_i\}_{i\in J\setminus H}}f_{0,S(Q\cup L_0)}^\varepsilon\right), \tag{4.25}$$

which holds in the extended phase space $\mathbb{R}^{6|S(J)|}$. Notice that we are assuming now the convention (2.6) for the initial data $f_{0,S(J)}^\varepsilon$, therefore we omit the characteristic functions $\bar{\chi}^0_{S(i)}$, $\bar{\chi}^0_{S(Q\cup L_0)}$.

Inserting this equation into (4.24) we readily find the final result

$$f_J^\varepsilon(t)=\sum_{H\subset J}(f_1^\varepsilon(t))^{\otimes H}E_{J\setminus H}(t), \tag{4.26}$$

[4] It is now clear that we need an expansion in the *extended* phase space because the mixed backwards flow (4.18) allows overlapping particles.

where the correlation error has the expression:

$$E_K(t) = \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \sum_{n\geq 0} \sum_{\Gamma(k,n)} \int d\Lambda \prod B^\varepsilon \chi^{rec}_{L_0} R^{ov}(Q, L_0) \\ \times \left(\bar{E}^0_{\{S(i)\}_{i\in K\setminus(Q\cup L_0)}, S(Q\cup L_0)} + \bar{E}^0_{\{S(i)\}_{i\in K\setminus(Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)} \right), \tag{4.27}$$

with $\int d\Lambda = \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{k,n})$ and with the integrand calculated through the mixed flow $(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\zeta}}^{(K\setminus L_0)})$.

4.3.3 Step 1.1: Reduction to energy functionals

Let us provide a first, preliminary estimate of E_K, by using the available informations on R^{ov} and on the initial data. As announced in Sect. 4.2 (Step 1.1), our purpose here is to replace the integrand in (4.27) with a simplified expression depending on the energy of flows only. To do this we shall introduce some cutoff parameters.

We recall first the estimates at disposal:

- from Lemma 4.4, one has the combinatorial bound (4.23) for $R^{ov}(Q, L_0)$;
- to control the initial data, we use Hypothesis 2.1 and (2.33) to obtain, for $k + n < \varepsilon^{-\alpha_0}$,

$$\left| \bar{E}^0_{\{S(i)\}_{i\in K\setminus(Q\cup L_0)}} f^\varepsilon_{0,S(Q\cup L_0)} \right| \leq z^{k+n}\, e^{-(\beta/2)\sum_{i\in S(K)} v_i^2} \\ \times \sum_{B\subset K\setminus(Q\cup L_0)} C^b b!\, \chi^0_{B,K\setminus(Q\cup L_0)} \varepsilon^{\gamma_0(k-q-l_0-b)} \tag{4.28}$$

and similar estimate for the other term in (4.27). Again we use the conventions $b = |B|$, $q = |Q|$, $l_0 = |L_0|$. Here $\chi^0_{B,K\setminus(Q\cup L_0)} = 1$ if and only if all the trees in B have a particle overlapping, at time zero, with some particle in a different tree belonging to $K\setminus(Q \cup L_0)$.

Inserting this information into (4.27) we establish the following result. Set

$$\mathcal{H}_K := \sum_{i\in S(K)} v_i^2$$

(twice) the energy of the trees in K and

$$F_{\theta_3} = F_{\theta_3}(K) := e^{-(\beta/2)\mathcal{H}_K}\, \mathbb{1}_{\mathcal{H}_K \leq \varepsilon^{-\theta_3}}. \tag{4.29}$$

Lemma 4.5 *Let $\theta_1, \theta_2, \theta_3 > 0$. There exist α, γ such that, for $k < \varepsilon^{-\alpha}$, $t < \bar{t}$ and ε sufficiently small,*

$$\int d\boldsymbol{v}_K |E_K(t)| \leq \frac{3\,\varepsilon^{\gamma k}}{4} + (zC)^k\, k!\, \varepsilon^{-\theta_1 k} \sum_{\substack{L_0, Q \\ \subset K \\ \text{disjoint}}} \varepsilon^{\gamma_0(k-q-l_0)}$$
$$\times \sum_{n=0}^{\log \varepsilon^{-\theta_2 k}} z^n \sum_{\Gamma(k,n)} \int d\Lambda\, d\boldsymbol{v}_k\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}\, F_{\theta_3}(K). \tag{4.30}$$

Formula (4.30) summarizes the combinatorial part of our proof. It says that, up to a small error, we reduce to a finite sum and integrals over compact sets of a bounded function. Moreover, for each pair L_0, Q, we gained from the initial data a factor $\varepsilon^{\gamma_0(k-l_0-q)} < \varepsilon^{\gamma(k-l_0-q)}$ if $\gamma < \gamma_0$. The remaining $\varepsilon^{\gamma(l_0+q)}$ necessary to achieve the main theorem must be obtained from the constraints $\chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}$ (each tree in Q overlaps with some other tree in K and each tree in L_0 recollides externally), requiring to control the "many-recollision integral" on which we focus in Sect. 4.3.4.

In (4.30), $C > 0$ is a pure constant as in Lemma 4.4 (but larger) not depending on any parameter introduced. The factor $\varepsilon^{-\theta_1 k}$ is the truncation on $\prod |B^\varepsilon|$. The characteristic functions $\mathbb{1}_{L_0}$, $\tilde{\mathbb{1}}_{K\setminus L_0}$ (ensuring well posedness of the mixed flow $(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\zeta}}^{(K\setminus L_0)})$ on which $\chi^{rec}_{L_0}\chi^{ov}_{Q,K}$ is evaluated) have been defined after (4.19).

The meaning of the several cutoffs introduced is summarized in the list that follows.

The proof of Lemma 4.5 is rather straightforward and is postponed to Appendix C.

4.3.3.a List of parameters We collect here, for the reader's convenience, a list of positive parameters entering in the proof of the main theorem. We indicate the point where they are precisely introduced, and the conditions they will have to obey according to our estimates.

- z, β fix the norm of the initial data (Hypothesis 2.1).
- α_0, γ_0 describe the correlation error estimates satisfied by the initial state (Hypothesis 2.2).
- The truncation on the physical space (formula (2.18)) is given by $\delta = \varepsilon^\theta$ with

$$\theta < 1/4 \tag{4.31}$$

(see comment (4), Sect. 2.4.1 for explanations). The reason of this specific choice is technical and related to Step 3 of the proof (Sect. 4.5.5).

- t^* is the limiting time of absolute convergence of the expansions involved in the proofs (Theorem 2.4) and it will be determined by z, β. We shall not optimize the value of t^* (for more details, see Sect. 4.8.2).
- $\varepsilon^{-\theta_1}$ is the truncation of cross-section factors in one-particle trees (Lemma 4.5 and Lemma C.1).
- θ_2 is the cutoff parameter bounding the number of creations in a collection of trees (formula (4.30)).
- θ_3 is the cutoff parameter controlling high energies (formula (4.29)).

For the sake of concreteness we fix now the latter technical cutoffs as follows:

$$\begin{cases} \theta_1 = a(\gamma_0) \\ \theta_2 = 1/(2\log(\bar{t}C(z,\beta)e)^{-1}) \ , \\ \theta_3 = 1/5 \end{cases} \tag{4.32}$$

where

$$a(\gamma_0) := \min[\gamma_0/2, 1/4] \tag{4.33}$$

and the constant $C(z, \beta)$ and the time $\bar{t}$ appear in Lanford's estimate, see (4.8).

- Finally, α, γ describe the correlation error bounds satisfied by the state at time t (Theorem 2.4). According to our estimates (Appendix C and Sect. 4.3.5), they will satisfy:

$$\begin{cases} \alpha < \min\left[\, \alpha_0 \;\; (1/3)a(\gamma_0) \;\; 1/4 - (1/3)a(\gamma_0) - (2/3)\theta \,\right] \\ \gamma < \min\left[\, a(\gamma_0) \;\; 3/4 - a(\gamma_0) - 2\theta \,\right] - 3\alpha \end{cases} . \tag{4.34}$$

These conditions are associated to the choice (4.32) above.

4.3.4 Many-recollision estimate

As explained after (4.30), we focus on the following result.

Proposition 4.6 *There exist constants $C_1 = C_1(z, \beta) > 0$ and $\gamma_1 > 0$ such that, for all $n \leq \varepsilon^{-3/4} \log \varepsilon^{-\theta_2}$, $Q, L_0 \subset K$, $Q \cap L_0 = \emptyset$ and ε sufficiently small,*

$$z^n \sum_{\Gamma(k,n)} \int d\Lambda \, d\boldsymbol{v}_k \, \mathbb{1}_{L_0} \, \tilde{\mathbb{1}}_{K\setminus L_0} \, \chi^{rec}_{L_0} \, \chi^{ov}_{Q,K} \, F_{\theta_3}(K)$$
$$\leq C_1^k \, k^k \, (n+k)^k \, (C_1 t)^n \, \varepsilon^{\gamma_1 \frac{q+l_0}{2}} \tag{4.35}$$

for $\boldsymbol{x}_k \in \mathcal{M}^x_k(\delta)$.

Sections 4.4 and 4.5 are devoted to the proof of Proposition 4.6.

Remark on γ_1. The coefficient γ_1 comes from the geometrical estimates of recollisions (Lemmas 4.10 and 4.12 below) which imply an arbitrary choice in the interval $\gamma_1 \in (0, \min[1, 2-4\theta-(5/2)\theta_3])$ (see (4.101)).

4.3.5 Proof of (2.21)

We show here how to conclude the proof of (2.21).

Let us fix the truncation parameters as in (4.32) and assume (4.31), (4.34) (see also the Remark in Appendix C).

Notice that (4.34) implies $\alpha < 3/4$ and hence $n \leq \varepsilon^{-3/4} \log \varepsilon^{-\theta_2}$ in (4.30). By Lemma 4.5 and Proposition 4.6, choosing $t^* < (eC_1)^{-1}$ we deduce that, for $t < t^*$,

$$\begin{aligned}
&\int_{\mathbb{R}^{3k}} d\mathbf{v}_k |E_K(t)| \\
&\leq \frac{3\,\varepsilon^{\gamma k}}{4} + (zC)^k\, k^k\, \varepsilon^{-\theta_1 k}\, 3^k\, C_1^k\, k^k \sum_{n\geq 0} (n+k)^k\, (C_1 t^*)^n\, \varepsilon^{\min[\gamma_0,\gamma_1/2]k} \\
&\leq \frac{3\,\varepsilon^{\gamma k}}{4} + \left(\sum_{n\geq 0} \frac{(n+k)^k}{k!} (C_1 t^*)^n\right) (3zCC_1)^k\, k^{3k}\, \varepsilon^{(\min[\gamma_0,\gamma_1/2]-\theta_1)k} \\
&\leq \frac{3\,\varepsilon^{\gamma k}}{4} + \left(\sum_{n\geq 0} (eC_1 t^*)^n\right) (3zCC_1 e)^k\, k^{3k}\, \varepsilon^{(\min[\gamma_0,\gamma_1/2]-\theta_1)k} \\
&\leq \frac{3\,\varepsilon^{\gamma k}}{4} + \left(\sum_{n\geq 0} (eC_1 t^*)^n\right) (3zCC_1 e)^k\, \varepsilon^{(\min[\gamma_0,\gamma_1/2]-\theta_1-3\alpha)k}. \qquad (4.36)
\end{aligned}$$

In the third step we applied (4.7) while in the fourth step we used $k < \varepsilon^{-\alpha}$. We conclude that (2.21) holds for ε small enough if

$$\gamma < \min[\gamma_0, \gamma_1/2] - \theta_1 - 3\alpha. \qquad (4.37)$$

By the above Remark on γ_1 and (4.32), this is ensured by

$$\gamma < \min[2a(\gamma_0), 3/4 - 2\theta] - a(\gamma_0) - 3\alpha. \qquad (4.38)$$

□

4.4 Step 2: Ordering of multiple recollisions

This section is devoted to the proof of Proposition 4.6. We follow the strategy described in 'Sect. 4.2—Step 2'. After suitable ordering in time of the recollision / overlap constraints, we will show in Sect. 4.4.2 that the estimate of a single recollision event (Lemma 4.10, proved in Sect. 4.5) allows to conclude the proof.

Let us focus on the constraint $\chi^{rec}_{L_0}\,\chi^{ov}_{Q,K}(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\zeta}}^{(K\setminus L_0)})$, where the flow in the argument is the mixed flow naturally entering in the definition of correlation error, Sect. 4.3.1.b. To simplify the proof, we will treat simultaneously recollisions and overlaps with a unique method.

Definition 4.7 (*Table of recollisions*) Let $L_0, Q \subset K$, $L_0 \cap Q = \emptyset$. A "table of recollisions" associated to (K, L_0, Q) is a set of couples

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \{(\alpha_1, \beta_1), \ldots, (\alpha_\ell, \beta_\ell)\},$$

with $\alpha_i \in L_0 \cup Q$ and $\beta_i \in K$, such that:

– $(\cup_{i=1}^{\ell}\alpha_i) \cup (\cup_{i=1}^{\ell}\beta_i) \supset L_0 \cup Q$;
– $\alpha_i \neq \alpha_1, \ldots, \alpha_{i-1}, \beta_1, \ldots, \beta_{i-1}$ for all $i = 1, \ldots, \ell$.

We call "bullet" a particle of type α and "target" a particle of type β.

According to this definition, the bullets are always new with respect to the previous array.

We shall apply the above definition to the case when the bullets α and the targets β are indices of the particles generating the trees $\Gamma_\alpha, \Gamma_\beta$ (see Fig. 5 below). Remind that an external recollision/overlap between Γ_α and Γ_β indicates a recollision/overlap between a pair of particles of the two trees.

Remark 1. All the particles in $L_0 \cup Q$ are either bullets or targets (or both) and each particle can be the target for several bullets. Conversely, each particle can be the bullet for at most one target, namely the α_i are all distinct and

$$|(\boldsymbol{\alpha}, \boldsymbol{\beta})| = \ell \geq (q + l_0)/2. \tag{4.39}$$

2. A rough bound on the total number of tables of recollision associated to (K, L_0, Q) is the following:

$$\sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \leq (q + l_0)!\, k^{q+l_0} \leq k!\, k^k. \tag{4.40}$$

3. We can construct one particular table with the following explicit procedure. Fix $(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\zeta}}^{(K\setminus L_0)})$ such that $\chi^{rec}_{L_0}\,\chi^{ov}_{Q,K} = 1$. The first backwards external

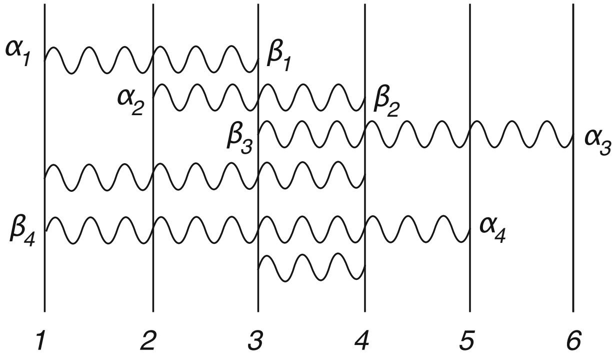


Fig. 5 A scheme for a table of recollisions associated to $K = \{1, 2, \ldots, 6\}$, $Q \cup L_0 = \{1, 2, 3, 5, 6\}$. Here $\ell = 4$. The *vertical lines* can be associated to particles (trees) in a backwards flow (time flowing *upwards*) and the *wavy lines* to their external recollisions/overlaps. In this case, the *fourth* and the *last wavy lines* represent recollisions (or overlaps) that do not appear in the table

recollision or overlap identifies the couple (α_1, β_1) (up to the exchange $\alpha_1 \leftrightarrow \beta_1$, if both particles belong to $L_0 \cup Q$). Going further backwards in time, we consider the first external recollision/overlap involving at least one tree in $L_0 \cup Q$ and different from α_1, β_1. This identifies the couple (α_2, β_2), with the following constraint. If one (and only one) of the two trees involved is α_1 or β_1, we set such tree $= \beta_2$, and its partner $= \alpha_2$. We iterate this procedure until all the particles in $L_0 \cup Q$ have received a name. See Fig. 5 for an example.

We shall say that a table of recollisions $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is "realized" if and only if the mixed dynamics $(\boldsymbol{\zeta}^{(L_0)}, \tilde{\boldsymbol{\zeta}}^{(K \setminus L_0)})$ is well defined (see the Remark on the existence of flows below) and, for all $i = 1, \ldots, \ell$:

(a) the *first* (backwards) recollision/overlap of the tree Γ_{α_i} occurs in $(0, t)$ with the tree Γ_{β_i};
(b) the *first* (backwards) recollision/overlap of the tree Γ_{α_i} occurs *in the past* with respect to the first (backwards) recollisions/overlaps of $\Gamma_{\alpha_{i'}}$, $i' < i$.

Lemma 4.8 *The following inequality holds true*:

$$\mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K \setminus L_0}\, \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K} \leq \sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \chi^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \tag{4.41}$$

where $\chi^{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ denotes the indicator function of the event for which the table of recollisions $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is realized.

Proof of Lemma 4.8 It follows immediately (by subadditivity) from Definition 4.7 and the Remark 3 above. ☐

The next lemma allows to estimate the integrations in the left hand side of (4.35) iteratively.

Lemma 4.9 *The following inequality holds true*:

$$\chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\left(\boldsymbol{\zeta}^{(L_0)},\tilde{\boldsymbol{\xi}}^{(K\setminus L_0)}\right) \le \prod_{i=1}^{\ell}\chi^{(\alpha_i,\beta_i)}\left(\boldsymbol{\zeta}^{(L_0\setminus\{\alpha_\ell,\alpha_{\ell-1},\dots,\alpha_i\})},\tilde{\boldsymbol{\xi}}^{(K\setminus(L_0\cup\{\alpha_\ell,\alpha_{\ell-1},\dots,\alpha_i\}))},\tilde{\boldsymbol{\xi}}^{(\alpha_i)}\right), \tag{4.42}$$

where $\chi^{(\alpha_i,\beta_i)} = 1$ *if and only if the first backwards overlap of the tree* Γ_{α_i} *occurs in* $(0,t)$ *with the tree* Γ_{β_i}.

Notice that the mixed flow in the argument of $\chi^{(\alpha_i,\beta_i)}$ is *not* the same as in the left hand side, namely the value of $\chi^{(\alpha_i,\beta_i)}$ is computed as if the trees $\{\alpha_\ell,\alpha_{\ell-1},\dots,\alpha_{i+1}\}$ were absent. The right hand side corresponds to an overcounting since the collection of recollision/overlap constraints realizing the table $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is not ordered in time (i.e. at this stage we drop, for simplicity, condition (b) above).

Remark (Existence of flows) The requirement of well posedness of the flows involved in the above expressions (forbidden overlaps at creations) is implicitly absorbed in the definition of $\chi^{(\alpha,\beta)}$, $\chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Note that $\chi^{(\alpha,\beta)}$ is a function of the indicated flow *only* through its history in the time interval (s,t) where s is the overlap time of α with β (if any, and zero otherwise). Therefore, existence of the flows in $(0,s)$ is not required. The same is true for $\chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, being in this case s the overlap time of α_ℓ with β_ℓ.

Proof of Lemma 4.9 We observe that

$$\chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\left(\boldsymbol{\zeta}^{(L_0)},\tilde{\boldsymbol{\xi}}^{(K\setminus L_0)}\right) = \chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\left(\boldsymbol{\zeta}^{(L_0\setminus\alpha_\ell)},\tilde{\boldsymbol{\xi}}^{(K\setminus(L_0\cup\alpha_\ell))},\tilde{\boldsymbol{\xi}}^{(\alpha_\ell)}\right), \tag{4.43}$$

that is we can always use the uncorrelated dynamics for the flow of the bullet. It follows that

$$\chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{\zeta}^{(L_0)},\tilde{\boldsymbol{\xi}}^{(K\setminus L_0)}) \le \chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})_{\ell-1}}\left(\boldsymbol{\zeta}^{(L_0\setminus\alpha_\ell)},\tilde{\boldsymbol{\xi}}^{(K\setminus(L_0\cup\alpha_\ell))}\right)\chi^{(\alpha_\ell,\beta_\ell)} \times\left(\boldsymbol{\zeta}^{(L_0\setminus\alpha_\ell)},\tilde{\boldsymbol{\xi}}^{(K\setminus(L_0\cup\alpha_\ell))},\tilde{\boldsymbol{\xi}}^{(\alpha_\ell)}\right) \tag{4.44}$$

where $(\boldsymbol{\alpha},\boldsymbol{\beta})_{\ell-1} = \{(\alpha_1,\beta_1),\dots,(\alpha_{\ell-1},\beta_{\ell-1})\}$ (and of course $(\boldsymbol{\alpha},\boldsymbol{\beta}) = (\boldsymbol{\alpha},\boldsymbol{\beta})_\ell$). Note that in the r.h.s. the overlap of Γ_{α_ℓ} with Γ_{β_ℓ} can occur at any time in $(0,t)$ (we forget item (b) above). The constraint $\chi^{(\boldsymbol{\alpha},\boldsymbol{\beta})_{\ell-1}}$ is now computed ignoring the history of the bullet tree Γ_{α_ℓ}.

Recursive application of (4.44) leads to the claim. □

4.4.1 Reordering of the integrations in (4.35)

Applying Lemmas 4.8 and 4.9 to the left hand side of (4.35), one finds

$$
\begin{aligned}
& z^n \sum_{\Gamma(k,n)} \int d\Lambda \, d\mathbf{v}_k \, \mathbb{1}_{L_0} \, \tilde{\mathbb{1}}_{K\setminus L_0} \, \chi^{rec}_{L_0} \, \chi^{ov}_{Q,K} \, F_{\theta_3}(K) \\
& \leq z^n \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\Gamma(k,n)} \int d\Lambda \, d\mathbf{v}_k \prod_{i=1}^{\ell} \chi^{(\alpha_i,\beta_i)} \, F_{\theta_3}(K) \\
& = z^n \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\substack{n_1,\dots,n_k, \\ n=\sum_i n_i}} \sum_{\Gamma_1,\dots,\Gamma_k} \int d\Lambda_1 \cdots d\Lambda_k \, d\mathbf{v}_k \prod_{i=1}^{\ell} \chi^{(\alpha_i,\beta_i)} \, F_{\theta_3}(K) \qquad (4.45)
\end{aligned}
$$

where the $\chi^{(\alpha_i,\beta_i)}$ is evaluated via the flow

$$
(\boldsymbol{\zeta}^{(L_0\setminus\{\alpha_\ell,\alpha_{\ell-1},\dots,\alpha_i\})}, \, \tilde{\boldsymbol{\zeta}}^{(K\setminus(L_0\cup\{\alpha_\ell,\alpha_{\ell-1},\dots,\alpha_i\}))}, \, \tilde{\boldsymbol{\zeta}}^{(\alpha_i)}).
$$

In the last equality, which follows from the factorization of trees (3.35), we introduced $d\Lambda_i = \int d\Lambda(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i})$.

The function F_{θ_3}, defined by (4.29), satisfies $F(K_1 \cup K_2) \leq F(K_1)F(K_2)$. Setting $A = K\setminus\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $\boldsymbol{\Gamma}_A = \{\Gamma_i\}_{i\in A}$ and $d\Lambda_A = \prod_{i\in A} \int d\Lambda(\mathbf{t}^i_{n_i}, \boldsymbol{\omega}^i_{n_i}, \mathbf{v}^i_{1,n_i})$, it follows that

$$
\begin{aligned}
& z^n \sum_{\Gamma(k,n)} \int d\Lambda \, d\mathbf{v}_k \, \mathbb{1}_{L_0} \, \tilde{\mathbb{1}}_{K\setminus L_0} \, \chi^{rec}_{L_0} \, \chi^{ov}_{Q,K} \, F_{\theta_3}(K) \\
& \leq z^n \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\substack{n_1,\dots,n_k, \\ n=\sum_i n_i}} \sum_{\boldsymbol{\Gamma}_A} \int d\Lambda_A \, d\mathbf{v}_A \, F_{\theta_3}(A) \\
& \quad \times \sum_{\Gamma_{\alpha_1}} \int d\Lambda_{\alpha_1} \, dv_{\alpha_1} \, \chi^{(\alpha_1,\beta_1)} \, F_{\theta_3}(\alpha_1) \sum_{\Gamma_{\alpha_2}} \int d\Lambda_{\alpha_2} \, dv_{\alpha_2} \cdots \\
& \quad \times \sum_{\Gamma_{\alpha_\ell}} \int d\Lambda_{\alpha_\ell} \, dv_{\alpha_\ell} \, \chi^{(\alpha_\ell,\beta_\ell)} \, F_{\theta_3}(\alpha_\ell). \qquad (4.46)
\end{aligned}
$$

4.4.2 Proof of Proposition 4.6

4.4.2.a Single-recollision estimate Our purpose is to estimate iteratively the integrals in (4.46). The result we need for the single step is given in the next lemma. Intuitively, it is depicted in Sect. 4.2—Step 3. In the figure, the trajectory on the left is the "target trajectory" with position and velocity $(\xi(s), \eta(s))$, $s \in (0, t)$. This trajectory is fixed and has bounded energy. It may actually include microscopic jumps in position (not shown in the figure), their total number being however limited by n. The "bullet trajectory" is the entire IBF, usual notation $\boldsymbol{\zeta}^{\varepsilon}(s)$, associated to a given tree (in the picture the bullet α). Its energy and number of creations are also bounded. At time t we assume a separation $\delta \gg \varepsilon$ between bullet and target. Then the goal is to measure the size of events where bullet trajectories reach a distance smaller than ε from the target.

Lemma 4.10 (Estimate of one external recollision) *Let $s \to \eta(s)$ be a piecewise constant function from $(0, t)$ to $\mathbb{R}^3$ such that $|\eta| \leq \varepsilon^{-\theta_3/2}$, with discontinuity points in the finite set $T = \{\tau_1, \tau_2, \ldots\}$. Let $s \to \xi(s)$ be a piecewise free trajectory in $\mathbb{R}^3$ with velocity*

$$\frac{d\xi}{ds} = \eta \tag{4.47}$$

except on T. In a subset of T, jumps of entity ε may occur ($|\xi(\tau_i^+) - \xi(\tau_i^-)| = \varepsilon$). Let $n \leq \varepsilon^{-3/4} \log \varepsilon^{-\theta_2}$ be the maximum number of such jumps. Fix $x_1 \in \mathbb{R}^3$ and assume $|x_1 - \xi(t)| > \delta = \varepsilon^{\theta}$. Then, there exist $D > 0$ and $\gamma_1 > 0$ such that, for all $n_1 \leq n$ and ε small enough,

$$\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\varepsilon})\, F_{\theta_3}(1) \leq (Dt)^{n_1}\, \varepsilon^{\gamma_1}, \tag{4.48}$$

where $\boldsymbol{\zeta}^{\varepsilon} = (\zeta_1^{\varepsilon}, \ldots, \zeta_{1+n_1}^{\varepsilon})$ is the IBF associated to the 1-particle, n_1-collision tree $\Gamma(1, n_1)$ with $\zeta_1^{\varepsilon}(t) = x_1$, and χ^{ov}_{ξ} is the indicator function of the event

$$\{\exists\, s \in (0, t) \mid \mathcal{D}(t - s) < \varepsilon\}$$

where, for $s \in (t_i, t_{i-1}]$, $\mathcal{D}(s) = \mathcal{D}[\boldsymbol{\zeta}^{\varepsilon}](s) := \min_{k=1,\ldots,i} |\xi(s) - \xi_k^{\varepsilon}(s)|$.

Note that it is implicit in the definition of χ^{ov}_{ξ} that the IBF is well defined (no internal overlaps at the creation times) up to the first s verifying the condition. The constant D depends only on the parameter β appearing in F_{θ_3}.

The Lemma is proved in Sect. 4.5.

4.4.2.b Virtual trajectories Following [30], we introduce a global notion of trajectory which will be convenient in the next subsection to apply iteratively the previous lemma. This will be also used in the geometrical estimate of Step 3.

Loosely speaking, a virtual trajectory is a trajectory of a given particle in the IBF (or other flow) *extended* up to time t. We shall use for it an upper-index notation. For instance, $\zeta^{\varepsilon,i}(s)$ coincides with $\zeta_i^{\varepsilon}(s)$ for $s > 0$ and up to the time of creation of i; thereafter is extended by the trajectory of its progenitor up to its creation time, and so on.

Definition 4.11 (*Virtual trajectory*) Consider particle i in the graph of a tree $\Gamma(k,n) = (k_1, \ldots, k_n)$. Let $\mathbf{t}_n = t_1, \ldots, t_n$ be the sequence of times associated to the nodes of the tree.

(i) A polygonal path p_i is uniquely defined if we walk on the tree by going forward in time, starting from the time-zero endpoint of line i and going up to the root-point at time t (e.g. Fig. 6).

(ii) Let $t_{i_1}, \ldots, t_{i_{n^i}}$ be the decreasing subsequence of $t_1, \ldots, t_n$, made of the times corresponding to the nodes met by following the path p_i (n^i being the number of such nodes, with the convention $i_0 = 0$, $t_{i_0} = t$). Then, for any backwards flow $\bar{\boldsymbol{\zeta}}$ which can be constructed from $\Gamma(k,n)$, $\mathbf{t}_n$,[5] we call **virtual trajectory associated to particle** i **in the flow**, and indicate it with upper indices $\bar{\zeta}^i(s) = (\bar{\xi}^i(s), \bar{\eta}^i(s)) \in \mathbb{R}^6$, $s \in [0,t]$, the trajectory given by:

$$\bar{\zeta}^i(s) = \begin{cases} \bar{\zeta}_i(s) & \text{for } s \in [0, t_{i_{n^i}}) \\ \bar{\zeta}_{k_{i_r}}(s) & \text{for } s \in [t_{i_r}, t_{i_{r-1}}), \quad 0 < r \leq n^i \end{cases} . \tag{4.49}$$

Note that the virtual trajectory is piecewise-free, and built up with pieces of trajectories of (different) particles of $\bar{\boldsymbol{\zeta}}$. Instantaneous jumps of entity ε occur at creation times, when the name of the particle in the flow $\bar{\boldsymbol{\zeta}}$ changes (e.g. t_{i_1}, t_{i_2} and t_{i_4} in Fig. 6). Only during the time of existence of particle i in the flow, $\bar{\zeta}^i(s) = \bar{\zeta}_i(s)$ holds.

*4.4.2.c Iterative estimate of multiple recollisions*We come back to (4.46) and focus on $\chi^{(\alpha_\ell,\beta_\ell)}$ (defined in Lemma 4.9). Remind the notation (3.39). If $\chi^{(\alpha_\ell,\beta_\ell)} = 1$, there exists $i \in S(\beta_\ell)$ such that $\tilde{\boldsymbol{\zeta}}^{(\alpha_\ell)}$ overlaps with the trajectory of particle i in the mixed flow $(\boldsymbol{\zeta}^{(L_0\backslash\alpha_\ell)}, \tilde{\boldsymbol{\zeta}}^{(K\backslash(L_0\cup\alpha_\ell))})$. In particular, α_ℓ overlaps with the virtual trajectory of i in the flow (Definition 4.11 applied to the mixed flow).

[5] In this definition, $\bar{\boldsymbol{\zeta}}$ can be either the IBF $\boldsymbol{\zeta}^{\varepsilon}$, the uncorrelated flow $\tilde{\boldsymbol{\zeta}}^{\varepsilon}$, the EBF $\boldsymbol{\zeta}^{\mathscr{E}}$ or a mixed flow. We shall use it in different contexts.

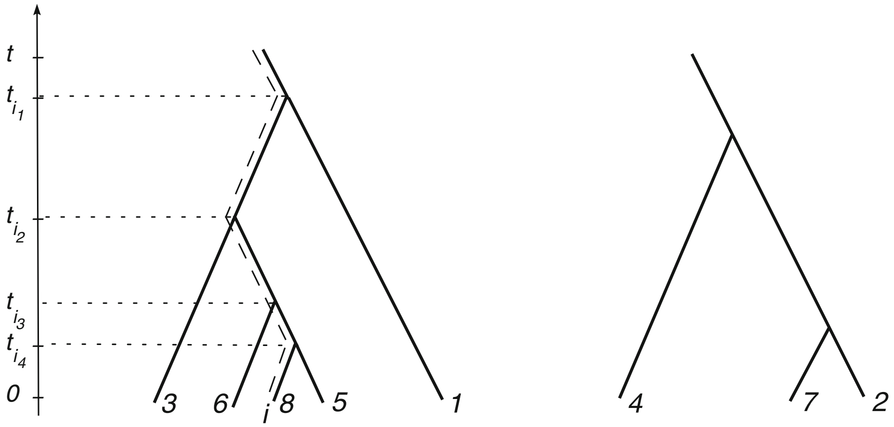


Fig. 6 The line closest to the *dashed line* is the path p_i in the tree $\Gamma(2, 6)$, with $i = 8$. The states of the particle associated to it via the flow $\bar{\zeta}$ form the"virtual trajectory of i"

We denote such virtual trajectory by $\hat{\zeta}^i = (\hat{\xi}^i, \hat{\eta}^i)$. Then,

$$\sum_{\Gamma_{\alpha_\ell}} \int d\Lambda_{\alpha_\ell}\, dv_{\alpha_\ell}\, \chi^{(\alpha_\ell,\beta_\ell)}\, F_{\theta_3}(\alpha_\ell)$$
$$\leq \sum_{i \in S(\beta_\ell)} \sum_{\Gamma_{\alpha_\ell}} \int d\Lambda_{\alpha_\ell}\, dv_{\alpha_\ell}\, \chi^{ov}_{\hat{\xi}^i}(\boldsymbol{\zeta}^\varepsilon)\, F_{\theta_3}(\alpha_\ell), \tag{4.50}$$

where the function χ^{ov} is defined in Lemma 4.10 and $\boldsymbol{\zeta}^\varepsilon$ is now the IBF associated to the 1-particle, n_{α_ℓ}-collision tree Γ_{α_ℓ}.

The presence of the functions F_{θ_3} in (4.46) ensures that $|\hat{\eta}^i| \leq \varepsilon^{-\theta_3/2}$. Furthermore, $\mathbf{x}_k \in \mathcal{M}^x_k(\delta)$ ensures $|x_{\alpha_\ell} - \hat{\xi}^i(t)| > \delta = \varepsilon^\theta$. Therefore, we are in the position to apply Lemma 4.10 and we deduce

$$\sum_{\Gamma_{\alpha_\ell}} \int d\Lambda_{\alpha_\ell}\, dv_{\alpha_\ell}\, \chi^{(\alpha_\ell,\beta_\ell)}\, F_{\theta_3}(\alpha_\ell) \leq (n_{\beta_\ell} + 1)\,(Dt)^{n_{\alpha_\ell}} \varepsilon^{\gamma_1}. \tag{4.51}$$

Inserting into (4.46) we obtain

$$z^n \sum_{\Gamma(k,n)} \int d\Lambda\, d\mathbf{v}_k\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K \setminus L_0}\, \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}\, F_{\theta_3}(K)$$
$$\leq \varepsilon^{\gamma_1} z^n \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\substack{n_1,\ldots,n_k,\\ n=\sum_i n_i}} (n_{\beta_\ell}+1)\,(Dt)^{n_{\alpha_\ell}} \sum_{\boldsymbol{\Gamma}_A} \int d\Lambda_A\, d\mathbf{v}_A\, F_{\theta_3}(A)$$

$$\times \sum_{\Gamma_{\alpha_1}} \int d\Lambda_{\alpha_1}\, dv_{\alpha_1}\, \chi^{(\alpha_1,\beta_1)}\, F_{\theta_3}(\alpha_1)$$
$$\cdots \sum_{\Gamma_{\alpha_{\ell-1}}} \int d\Lambda_{\alpha_{\ell-1}}\, dv_{\alpha_{\ell-1}}\, \chi^{(\alpha_{\ell-1},\beta_{\ell-1})}\, F_{\theta_3}(\alpha_{\ell-1}). \tag{4.52}$$

We repeat the above discussion *ad litteram* for $\chi^{(\alpha_{\ell-1},\beta_{\ell-1})}$, and so on up to $\chi^{(\alpha_1,\beta_1)}$. Since $\ell \geq (q+l_0)/2$, the result is:

$$z^n \sum_{\Gamma(k,n)} \int d\Lambda\, d\mathbf{v}_k\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}\, F_{\theta_3}(K)$$
$$\leq \varepsilon^{\gamma_1 \frac{q+l_0}{2}}\, z^n \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\substack{n_1,\ldots,n_k,\\ n=\sum_i n_i}} \prod_{i=1}^{\ell} (n_{\beta_i}+1)\, (Dt)^{n_{\alpha_i}} \sum_{\boldsymbol{\Gamma}_A} \int d\Lambda_A\, d\mathbf{v}_A\, F_{\theta_3}(A). \tag{4.53}$$

Using (4.7), the last sum over trees is bounded by $(const.)^{|A|}\,(e\,4\pi\,(2\pi/\beta)^{3/2}\,t)^{\sum_{i\in A} n_i}$. Since $A = K\setminus\{\alpha_1,\alpha_2,\ldots,\alpha_\ell\}$ and $\sum_{i=1}^{\ell} n_{\alpha_i} + \sum_{i\in A} n_i = n$, there holds

$$z^n \sum_{\Gamma(k,n)} \int d\Lambda\, d\mathbf{v}_k\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}\, F_{\theta_3}(K)$$
$$\leq \varepsilon^{\gamma_1 \frac{q+l_0}{2}}\, (C_1')^k\, (C_1' t)^n \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \sum_{\substack{n_1,\ldots,n_k,\\ n=\sum_i n_i}} \prod_{i=1}^{\ell} (n_{\beta_i}+1) \tag{4.54}$$

for suitable $C_1' = C_1'(z,\beta)$. We conclude that

$$z^n \sum_{\Gamma(k,n)} \int d\Lambda\, d\mathbf{v}_k\, \mathbb{1}_{L_0}\, \tilde{\mathbb{1}}_{K\setminus L_0}\, \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}\, F_{\theta_3}(K)$$
$$\leq \varepsilon^{\gamma_1 \frac{q+l_0}{2}}\, (C_1')^k\, (C_1' t)^n \sum_{\substack{\alpha_1,\ldots,\alpha_\ell\\ \alpha_i\neq\alpha_j}} \sum_{\substack{n_1,\ldots,n_k,\\ n=\sum_i n_i}} \prod_{i=1}^{\ell} \sum_{\beta_i\in K} (n_{\beta_i}+1)$$
$$= \varepsilon^{\gamma_1 \frac{q+l_0}{2}}\, (C_1')^k\, (C_1' t)^n \sum_{\substack{\alpha_1,\ldots,\alpha_\ell\\ \alpha_i\neq\alpha_j}} \sum_{\substack{n_1,\ldots,n_k,\\ n=\sum_i n_i}} (n+k)^\ell$$
$$\leq \varepsilon^{\gamma_1 \frac{q+l_0}{2}}\, (C_1)^{k+n}\, t^n\, k!\, (n+k)^k, \tag{4.55}$$

for suitable $C_1 > C_1'$. □

4.5 Step 3: Estimate of an external recollision

In this section, we prove Lemma 4.10. The proof is organized in seven steps which will be discussed in separate subsections. The main ideas are summarized in Sect. 4.2 (Step 3).

4.5.1 Substitution of the IBF with the EBF

The flow $\boldsymbol{\zeta}^{\varepsilon}$ in the left hand side of (4.48) involves internal recollisions, which is convenient to eliminate first. Let

$$\chi^{int} = \chi^{int}(\mathbf{t}_{n_1}, \boldsymbol{\omega}_{n_1}, \mathbf{v}_{1+n_1}) = 1 \tag{4.56}$$

if and only if:

- either an overlap at a creation time occurs (ill-defined $\boldsymbol{\zeta}^{\varepsilon}$), or
- the IBF $\boldsymbol{\zeta}^{\varepsilon}$ delivers an internal recollision.

Lemma 4.12 (Estimate of the internal recollision) *There exists a constant* $D > 0$ *such that, for any* $\gamma_1 < 1$ *and* ε *small enough,*

$$\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi^{int}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \le (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2}. \tag{4.57}$$

The control of the internal recollisions is well known (see [16,30]) and we postpone the proof to Appendix D. We note, incidentally, that the present estimate is optimal ($\gamma_1 \lesssim 1$).

Lemma 4.12 allows to bound the l.h.s. in (4.48) with a much simpler expression, i.e.

$$\begin{aligned}
&\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi_{\xi}^{ov}(\boldsymbol{\zeta}^{\varepsilon})\, F_{\theta_3}(1) \\
&\quad \le (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + \sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi_{\xi}^{ov}(\boldsymbol{\zeta}^{\varepsilon})\, (1 - \chi^{int})\, F_{\theta_3}(1) \\
&\quad \equiv (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + \sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi_{\xi}^{ov}(\boldsymbol{\zeta}^{\mathscr{E}})\, (1 - \chi^{int})\, F_{\theta_3}(1) \\
&\quad \le (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + \sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi_{\xi}^{ov}(\boldsymbol{\zeta}^{\mathscr{E}})\, F_{\theta_3}(1)
\end{aligned} \tag{4.58}$$

where $\boldsymbol{\zeta}^{\mathscr{E}} = (\zeta_1^{\mathscr{E}}, \ldots, \zeta_{1+n_1}^{\mathscr{E}})$ is the EBF associated to the 1-particle, n_1-collision tree $\Gamma(1, n_1)$, introduced in Sect. 3.5.2 (Fig. 3, (iii)).

4.5.2 Integration over virtual trajectories

We shall reduce the problem to the estimate of an integral spanning a single virtual trajectory (Sect. 4.4.2.b).

Condition $\chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\mathscr{E}}) = 1$ indicates the event "$\mathcal{D}[\boldsymbol{\zeta}^{\mathscr{E}}](t-s) < \varepsilon$ for some $s \in (0,t)$", which in turn implies

$$\text{“}|\xi(s) - \xi_i^{\mathscr{E}}(s)| < \varepsilon \;\text{ for some } i \in \{1,\ldots,n_1+1\} \text{ and some } s \in (0, t_{i-1})''. \tag{4.59}$$

Note now that such event depends actually not on the full EBF, but *just* on the virtual trajectory $\zeta^{\mathscr{E},i}$. Consequently, we may integrate out all the variables which are *not* entering in the construction of $\zeta^{\mathscr{E},i}(s)$.

According to Definition 4.11, for any given $\Gamma(1,n_1)$, i, calling n^i the number of nodes encountered by $\zeta^{\mathscr{E},i}$ and $i_1, i_2, \ldots, i_{n^i}$ their names (ordered as increasing sequence), the integration variables describing completely the virtual trajectory are:

$$v_1, t_{i_1}, \ldots, t_{i_{n^i}}, \omega_{i_1}, \ldots, \omega_{i_{n^i}}, v_{i_1}, \ldots, v_{i_{n^i}} \longrightarrow \zeta^{\mathscr{E},i}.$$

Since we are using upper indices for virtual trajectories, we rename the variables for convenience as

$$v_1, t^1, \ldots, t^{n^i}, \omega^1, \ldots, \omega^{n^i}, v^1, \ldots, v^{n^i} \longrightarrow \zeta^{\mathscr{E},i}.$$

With this notation and

$$\mathcal{H}_1^i := v_1^2 + \sum_{k=1}^{n^i} (v^k)^2, \tag{4.60}$$

we get

$$\begin{aligned} &\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\mathscr{E}})\, F_{\theta_3}(1) \\ &\quad \le \sum_{\Gamma(1,n_1)} \sum_{i=1}^{n_1+1} \frac{(D't)^{n_1-n^i}}{(n_1-n^i)!} \int_0^t dt^1 \int_0^{t^1} dt^2 \cdots \int_0^{t^{n^i-1}} dt^{n^i} \int d\omega^1 \cdots d\omega^{n^i} \\ &\qquad \times \int dv_1 dv^1 \cdots dv^{n^i}\, \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\, e^{-(\beta/2)\mathcal{H}_1^i}\, \mathbb{1}_{\mathcal{H}_1^i \le \varepsilon^{-\theta_3}}, \end{aligned} \tag{4.61}$$

where $D' = 4\pi\,(2\pi/\beta)^{3/2}$.

4.5.3 A change of variables: relative velocities

Let us focus on the last line in (4.61). To integrate over the characteristic function, it is convenient to use the variable v_1 together with the relative velocities at the creation times, which we introduce in what follows.

The virtual trajectory $\zeta^{\mathscr{E},i}$ has piecewise constant velocity, with n^i jumps at the creation times. We call

$$\eta^1, \eta^2, \ldots, \eta^{n^i+1}$$

the values assumed by the velocity, namely

$$\begin{aligned}&\eta^1 = v_1,\\&\eta^k \equiv \eta^{\mathscr{E},i}((t^{k-1})^-) \equiv \eta^{\mathscr{E},i}(s) \quad \text{for} \quad s \in (t^k, t^{k-1}).\end{aligned}$$

The relative velocities at creations are then:

$$\begin{aligned}&V_1 = v^1 - \eta^1,\\&V_2 = v^2 - \eta^2,\\&\ldots\\&V_{n^i} = v^{n^i} - \eta^{n^i}.\end{aligned}$$

Note that v^k are velocities of *added* particles at the moment of their creation. In particular, η^k is *independent* of v^k, so that the previous relations can be regarded as simple translations and

$$\begin{aligned}&\int dv_1 dv^1 \cdots dv^{n^i}\, \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\, e^{-(\beta/2)\mathcal{H}_1^i}\, \mathbb{1}_{\mathcal{H}_1^i \le \varepsilon^{-\theta_3}}\\&\quad = \int dv_1\, dV_1 \cdots dV_{n^i}\, \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\, e^{-(\beta/2)\mathcal{H}_1^i}\, \mathbb{1}_{\mathcal{H}_1^i \le \varepsilon^{-\theta_3}},\end{aligned} \tag{4.62}$$

where now $\zeta^{\mathscr{E},i}(s)$ and $\mathcal{H}_1^i$ have to be computed by using $V_1, \ldots, V_{n^i}$.

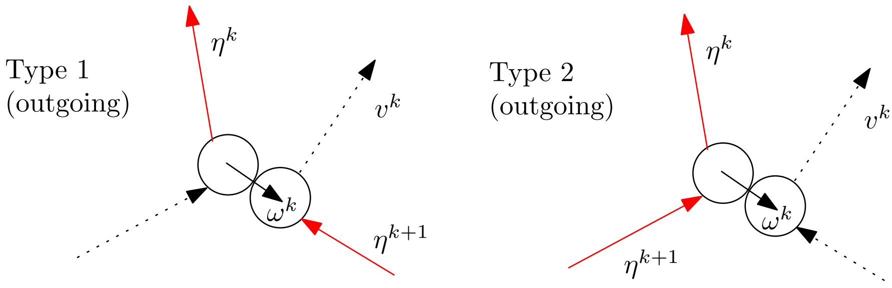
The energy function reads:

$$\mathcal{H}_1^i = v_1^2 + \sum_{k=1}^{n^i} (V_k + \eta^k)^2, \tag{4.63}$$

which we want to express completely in terms of the new integration variables. To do this, observe that each jump of velocity in the virtual trajectory $\zeta^{\mathscr{E},i}$, i.e.

$$\eta^k = \eta^{\mathscr{E},i}((t^k)^+) \to \eta^{k+1} = \eta^{\mathscr{E},i}((t^k)^-),$$

can be of two types, determined uniquely by the structure of the tree $\Gamma(1, n_1)$ (and corresponding for instance to nodes i_2 (type 1) and i_3 (type 2) in Fig. 6). That is:



- **Type** 1. The position jumps according to $\xi^{\mathscr{E},i}((t^k)^+) \to \xi^{\mathscr{E},i}((t^k)^-) = \xi^{\mathscr{E},i}((t^k)^+) + \varepsilon\omega^k$; the velocity jumps according to

$$\eta^{k+1} - \eta^k = \begin{cases} P^{\perp}_{\omega^k} V_k = V_k - \omega^k(\omega^k \cdot V_k) & (\omega^k \cdot V_k) \geq 0 \quad \text{(outgoing collision)} \\ V_k & (\omega^k \cdot V_k) < 0 \quad \text{(incoming collision)} \end{cases};$$

- **Type** 2. The position does not jump: $\xi^{\mathscr{E},i}((t^k)^+) = \xi^{\mathscr{E},i}((t^k)^-)$; the velocity jumps according to

$$\eta^{k+1} - \eta^k = \begin{cases} P^{\parallel}_{\omega^k} V_k = \omega^k(\omega^k \cdot V_k) & (\omega^k \cdot V_k) \geq 0 \quad \text{(outgoing collision)} \\ 0 & (\omega^k \cdot V_k) < 0 \quad \text{(incoming collision)} \end{cases}.$$

To have a compact notation, we write the above transformation as

$$\eta^{k+1} - \eta^k = P^k V_k \tag{4.64}$$

where P^k depends only on the tree $\Gamma(1, n_1)$ and on the variable ω^k. Hence the expression of the energy function (4.63) in terms of the new integration variables is

$$\mathcal{H}_1^i = \sum_{k=0}^{n^i} \left(V_k + \sum_{h=1}^{k-1} P^h V_h + v_1 \right)^2, \tag{4.65}$$

with the convention $V_0 = 0$.

The plan is now to bound $e^{-(\beta/2)\mathcal{H}_1^i}$, uniformly in v_1, with an integrable function of $V_1, \ldots, V_{n^i}$ and treat the recollision condition in (4.62) as a simpler condition *on the variable* v_1. The bound on $\mathcal{H}_1^i$ is included in the next section (and the boundedness of the corresponding Gaussian integral will be proved in Sect. 4.5.7), while the expression of the constraint as a v_1-tube is given in Sect. 4.5.5. Finally, the volume of the tube is estimated in Sect. 4.5.6.

4.5.4 Energy bounds

We collect here some energy estimates that will be used later on.

First, the lower bound on the energy function. Observe that

$$a_k := V_k + \sum_{h=1}^{k-1} P^h V_h \tag{4.66}$$

is a v_1-independent quantity and therefore

$$\begin{aligned}\inf_{v_1} \mathcal{H}_1^i &= \inf_{v_1} \sum_{k=0}^{n^i} (a_k + v_1)^2 \\ &= \inf_{v_1} \left(\sum_{k=0}^{n^i} a_k^2 + (n^i + 1) v_1^2 + 2v_1 \cdot \sum_{k=0}^{n^i} a_k \right) \\ &\geq \sum_{k=0}^{n^i} a_k^2 - \frac{\left(\sum_{k=0}^{n^i} a_k\right)^2}{n^i + 1}.\end{aligned} \tag{4.67}$$

Secondly, we derive some upper bounds on velocities and displacements in the virtual trajectory of the bullet.

The conservation of energy at collisions implies $|\eta^k|^2 + |v^k|^2 \geq |\eta^{k+1}|^2$. In particular, by (4.60), for any $k = 1, \ldots, n^i + 1$ one has

$$\mathcal{H}_1^i \geq v_1^2 + \sum_{q=1}^{k-1} (v^q)^2 = |\eta^1|^2 + \sum_{q=1}^{k-1} (v^q)^2 \geq |\eta^2|^2 + \sum_{q=2}^{k-1} (v^q)^2 \geq \cdots \geq |\eta^k|^2, \tag{4.68}$$

so that $\mathcal{H}_1^i \leq \varepsilon^{-\theta_3}$ leads to

$$|\eta^k| \leq \varepsilon^{-\theta_3/2} \tag{4.69}$$

for all k and

$$\left| \sum_{k=1}^{r} P^k V_k \right| \leq 2\varepsilon^{-\theta_3/2} \tag{4.70}$$

for all $r \in \{0, 1, \ldots, n^i\}$.

Moreover, for $s \in (t^{r+1}, t^r)$, the quantity

$$\sum_{k=1}^{r} P^k V_k(t^k - s) = (\eta^2 - \eta^1)(t^1 - s) + \cdots + (\eta^{r+1} - \eta^r)(t^r - s)$$
$$= -\eta^1(t - s) + \eta^1(t - t^1) + \eta^2(t^1 - t^2) + \cdots + \eta^{r+1}(t^r - s) \quad (4.71)$$

is bounded uniformly in s, r by

$$\sup_{s\in(0,t)} \left| \sum_{k=1}^{r} P^k V_k(t^k - s) \right| \le 2 \max_k |\eta^k| t \le 2t\varepsilon^{-\theta_3/2}. \quad (4.72)$$

Let

$$\mathcal{A} := \{(4.71) \text{ and } (4.73) \text{ are satisfied}\} \quad (4.73)$$

and notice that this is v_1-independent. Using (4.61), (4.62) and (4.67), we arrive to

$$\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi_\xi^{ov}(\boldsymbol{\zeta}^{\mathscr{E}})\, F_{\theta_3}(1)$$
$$\le \sum_{\Gamma(1,n_1)} \sum_{i=1}^{n_1+1} \frac{(D't)^{n_1-n^i}}{(n_1-n^i)!} \int_0^t dt^1 \int_0^{t^1} dt^2 \cdots \int_0^{t^{n^i-1}} dt^{n^i} \int d\omega^1 \cdots d\omega^{n^i}$$
$$\times \int dV_1 \cdots dV_{n^i}\, e^{-(\beta/2)\sum_k a_k^2 + \frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}\, \mathbb{1}_{\mathcal{A}}$$
$$\times \int dv_1 \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\, \mathbb{1}_{\mathcal{H}_1^i \le \varepsilon^{-\theta_3}}. \quad (4.74)$$

We shall study the latter integral in the next subsection.

4.5.5 The overlap constraint as an integral over "tubes"

We denote by $|\cdot|$ the volume of the set $\cdot$ in $\mathbb{R}^3$. We also introduce a small ε-dependent quantity

$$R^\varepsilon = \varepsilon^{1-\theta-\theta_3/2}\, 4t \quad (4.75)$$

and require $\theta + \theta_3/2 < 1$. Next we prove the following estimate.

Lemma 4.13 *In the assumptions of Lemma 4.10, one has*

$$\int dv_1 \mathbb{1}_{\{\inf_{s\in(0,t)} |\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\, \mathbb{1}_{\mathcal{H}_1^i \le \varepsilon^{-\theta_3}} \le \frac{|\mathcal{T}_\xi^\varepsilon|}{t^3}\, \mathbb{1}_{|\xi(t)-x_1| \le 3t\varepsilon^{-\theta_3/2}}, \quad (4.76)$$

where $\mathcal{T}_\xi^\varepsilon$ *is the region spanned by a ball of radius* R^ε *with center moving on the curve of parametric equation* $(\Delta(s))_{s\in(0,t-\varepsilon t/R^\varepsilon)}$, *defined by*

$$\Delta(s) := \frac{t}{t-s}\left[(\xi(s)-\xi(t)) + (\xi(t)-x_1) + \sum_{k=1}^{r} P^k V_k(t^k - s) - \sum_{k\le r}^{*} \varepsilon\,\omega^k\right] \tag{4.77}$$

for $s \in (t^{r+1}, t^r)$, *with the sum* $\sum^*$ *running over all the nodes of type* 1.

The above result holds for any choice of the variables determining $\zeta^{\mathscr{E},i}(s)$.

Proof of Lemma 4.13 Given $r \in \{0, 1, \ldots, n^i\}$, at time $s \in (t^{r+1}, t^r)$ (remind $t^0 \equiv t$) the virtual trajectory reads:

$$\begin{aligned}
\xi^{\mathscr{E},i}(s) &= x_1 - v_1(t-t^1) - \eta^2(t^1-t^2) - \cdots - \eta^{r+1}(t^r - s) + \sum_{k\le r}^{*} \varepsilon\,\omega^k \\
&= x_1 - v_1(t-s) - (\eta^2-\eta^1)(t^1-s) - \cdots - (\eta^{r+1}-\eta^r)(t^r-s) + \sum_{k\le r}^{*} \varepsilon\,\omega^k \\
&= x_1 - v_1(t-s) - \sum_{k=1}^{r} P^k V_k(t^k-s) + \sum_{k\le r}^{*} \varepsilon\,\omega^k.
\end{aligned} \tag{4.78}$$

By assumption, the velocity of the target is bounded as $|\eta| \le \varepsilon^{-\theta_3/2}$ and the same is true for the bullet (see (4.69)). Furthermore, both the trajectories of bullet and target may have at most n jumps of entity ε in position, where $n \le \varepsilon^{-3/4} \log \varepsilon^{-\theta_2}$ as fixed by Proposition 4.6. Namely, the displacements are bounded by

$$\begin{aligned}
\max\left(|\xi(s)-\xi(t)|, |\xi^{\mathscr{E},i}(s)-x_1|\right) &\le |\eta|(t-s) + n\,\varepsilon \\
&\le \varepsilon^{-\theta_3/2}(t-s) + \varepsilon^{1/4}\log\varepsilon^{-\theta_2}.
\end{aligned} \tag{4.79}$$

From this, we deduce two remarks on the overlap condition $\{\inf_{s\in(0,t)} |\xi(s) - \xi^{\mathscr{E},i}(s)| < \varepsilon\}$.

1. Time s realizing the condition can *not* be too close to t. In fact, since by hypothesis $|\xi(t) - x_1| \ge \varepsilon^\theta$,

$$\begin{aligned}
|\xi(s) - \xi^{\mathscr{E},i}(s)| &\ge |\xi(t)-x_1| - |\xi(s)-\xi(t)| - |\xi^{\mathscr{E},i}(s) - x_1| \\
&\ge \varepsilon^\theta - 2\varepsilon^{-\theta_3/2}(t-s) - 2\varepsilon^{1/4}\log\varepsilon^{-\theta_2},
\end{aligned} \tag{4.80}$$

which implies, through simple algebra,

$$(t-s) > \varepsilon^{\theta+\theta_3/2}/2 - \varepsilon^{1/4+\theta_3/2}\log\varepsilon^{-\theta_2} - \varepsilon^{1+\theta_3/2}/2, \tag{4.81}$$

and hence

$$(t-s) > \varepsilon^{\theta+\theta_3/2}/4 \tag{4.82}$$

if $\theta < 1/4$ and ε is small enough.
2. The condition implies that bullet and target are initially (i.e. at time t) not too far from each other, i.e.

$$\begin{aligned} |\xi(t)-x_1| &\le |\xi(s)-\xi^{\mathscr{E},i}(s)| + |\xi(s)-\xi(t)| + |\xi^{\mathscr{E},i}(s)-x_1| \\ &< \varepsilon + 2\varepsilon^{-\theta_3/2}(t-s) + 2\varepsilon^{1/4}\log\varepsilon^{-\theta_2} \\ &\le 3\,t\,\varepsilon^{-\theta_3/2}. \end{aligned} \tag{4.83}$$

Inserting now (4.78) and using (4.75), (4.82), the overlap condition assumes the form

$$\inf_{s\in(0,t-\varepsilon t/R^{\varepsilon})} \left|(\xi(s)-\xi(t)) + (\xi(t)-x_1) + v_1(t-s) + \sum_{k=1}^{r} P^k V_k(t^k-s) - \sum_{k\le r}^{*} \varepsilon\,\omega^k\right| < \varepsilon. \tag{4.84}$$

Alternatively, using the position variable

$$X := -v_1 t \tag{4.85}$$

and definition (4.77), one has:

$$\inf_{s\in(0,t-\varepsilon t/R^{\varepsilon})} \frac{t-s}{t}|X-\Delta(s)| < \varepsilon. \tag{4.86}$$

Thus, taking into account the above Remarks (1) and (2), conditions

$$\inf_{s\in(0,t-\varepsilon t/R^{\varepsilon})} |X-\Delta(s)| < R^{\varepsilon} \tag{4.87}$$

and (4.83) have to be both satisfied.

We conclude that

$$\begin{aligned} &\int dv_1 \mathbb{1}_{\{\inf_{s\in(0,t)}|\xi(s)-\xi^{\mathscr{E},i}(s)|<\varepsilon\}}\mathbb{1}_{\mathcal{H}_1^i} \\ &\le \frac{1}{t^3}\left(\int dX\,\mathbb{1}_{\{\inf_{s\in(0,t-\varepsilon t/R^{\varepsilon})}|X-\Delta(s)|<R^{\varepsilon}\}}\right)\mathbb{1}_{|\xi(t)-x_1|\le 3t\varepsilon^{-\theta_3/2}}. \end{aligned} \tag{4.88}$$

But Δ does *not* depend on X. Therefore, the integral in dX is nothing but the volume of the region $\mathcal{T}_\xi^\varepsilon$. □

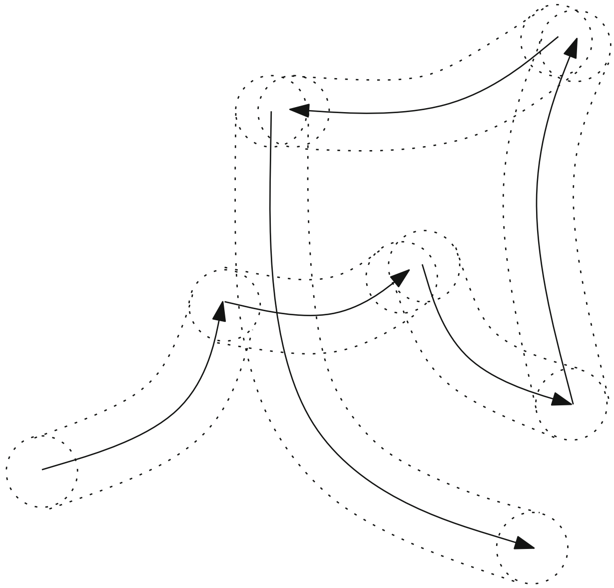


Fig. 7 Curve Δ. The boundary of the region $\mathcal{T}^{\varepsilon}_{\xi}$ is in *dotted lines*

4.5.6 Volume of $\mathcal{T}^{\varepsilon}_{\xi}$

The parametric curve Δ inherits its features from the trajectories of the bullet $\xi^{\mathscr{E},i}$ and of the target ξ, namely:

- $\Delta(s)$ is piecewise smooth, with singularity points in the set $T \cup \{t^1, t^2, \ldots, t^{n^i}\}$;
- at most n singular points $\tau^*_1, \tau^*_2, \ldots$ are jumps of entity ε;
- all the singular points are finite jumps in the velocity $\Delta'(s)$.

See e.g. Fig. 7.

Let L^{ε} be the length of the curve. If there were no jumps in position, $|\mathcal{T}^{\varepsilon}_{\xi}|$ would be bounded by $(4\pi/3)(R^{\varepsilon})^3 + \pi(R^{\varepsilon})^2 L^{\varepsilon}$. In fact, the volume of a tube with a cuspid in s is bounded by the volume of the smooth tube where we put $\Delta'(s^+)/|\Delta'(s^+)| = \Delta'(s^-)/|\Delta'(s^-)|$. Moreover, observe that n jumps in position produce an error in $|\mathcal{T}^{\varepsilon}_{\xi}|$ which is at most $(4\pi/3)(R^{\varepsilon})^3 n$. Therefore,

$$|\mathcal{T}^{\varepsilon}_{\xi}| \leq \pi(R^{\varepsilon})^2 L^{\varepsilon} + (4\pi/3)(R^{\varepsilon})^3(n+1). \tag{4.89}$$

Let us give a bound on L^{ε}. Denoting by $\bar{\Delta}$ the continuous parametric curve obtained from Δ by disregarding the positional jumps of entity ε, we can write

$$L^{\varepsilon} = \int_0^{t-\varepsilon t/R^{\varepsilon}} ds\, |\bar{\Delta}'(s)|. \tag{4.90}$$

If $s \in (t^{r+1}, t^r)$ and outside singularity points, one has

$$\bar{\Delta}'(s) = \frac{\bar{\Delta}(s)}{t-s} + \frac{t}{t-s}\left(\eta(s) - \sum_{k=1}^{r} P^k V_k\right). \tag{4.91}$$

We have now all the ingredients to provide a uniform bound on this, i.e.:

$$(t-s) > \varepsilon^{\theta+\theta_3/2}/4;$$
$$|\eta| \le \varepsilon^{-\theta_3/2};$$
$$|\xi(t) - x_1| \le 3t\varepsilon^{-\theta_3}(by(4.76));$$
$$\sup_{s\in(0,t)} \left|\sum_{k=1}^{r} P^k V_k(t^k - s)\right| \le 2t\varepsilon^{-\theta_3/2} \quad \text{and}$$
$$\left|\sum_{k=1}^{r} P^k V_k\right| \le 2\varepsilon^{-\theta_3/2} \text{ (we integrate over the set } \mathcal{A} \text{defined by (4.73)).}$$

We infer that

$$|\bar{\Delta}(s)| \le 4t\varepsilon^{-\theta-\theta_3/2}\left[\varepsilon^{-\theta_3/2}t + 3t\varepsilon^{-\theta_3/2} + 2t\varepsilon^{-\theta_3/2}\right] = 24\,t^2\varepsilon^{-\theta-\theta_3}, \tag{4.92}$$

$$|\bar{\Delta}'(s)| \le 4\varepsilon^{-\theta-\theta_3/2}\,24\,t^2\varepsilon^{-\theta-\theta_3} + 4t\varepsilon^{-\theta-\theta_3/2}\left(\varepsilon^{-\theta_3/2} + 2\varepsilon^{-\theta_3/2}\right)$$
$$= 96\,t^2\varepsilon^{-2\theta-3\theta_3/2} + 12\,t\varepsilon^{-\theta-\theta_3} \tag{4.93}$$

and hence

$$L^{\varepsilon} \le C\,t^3\,\varepsilon^{-2\theta-3\theta_3/2}, \tag{4.94}$$

where $C > 0$ is a pure constant.

Collecting (4.89), (4.94), (4.75) and $n \le \varepsilon^{-3/4}\log\varepsilon^{-\theta_2}$, we finally obtain

$$|\mathcal{T}^{\varepsilon}_{\xi}| \le C\,t^5\,\varepsilon^{2-4\theta-(5/2)\theta_3} \tag{4.95}$$

for some pure constant $C > 0$.

4.5.7 Conclusion

Equations (4.58), (4.74), (4.76) and (4.95) lead to

$$\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\varepsilon})\, F_{\theta_3}(1) \leq (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + C\, t^2\, \varepsilon^{2-4\theta-(5/2)\theta_3}$$
$$\times \sum_{\Gamma(1,n_1)} \sum_{i=1}^{n_1+1} \frac{(D't)^{n_1-n^i}}{(n_1-n^i)!} \int_0^t dt^1 \int_0^{t^1} dt^2 \cdots \int_0^{t^{n^i-1}} dt^{n^i} \int d\omega^1 \cdots d\omega^{n^i}$$
$$\times \int dV_1 \cdots dV_{n^i}\, e^{-(\beta/2)\sum_k a_k^2 + \frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}. \tag{4.96}$$

It remains to compute the Gaussian integral. This is conveniently done in the variables a_k. Note that $V_k \to a_k$, defined by (4.66), is a further translation. Then,

$$\int dV_1 \cdots dV_{n^i}\, e^{-(\beta/2)\sum_k a_k^2 + \frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}$$
$$= \int da_1 \cdots da_{n^i}\, e^{-(\beta/2)\sum_k a_k^2 + \frac{(\beta/2)}{n^i+1}(\sum_k a_k)^2}$$
$$\equiv I_{n^i}. \tag{4.97}$$

For $n \geq 1$, denoting $S_n = \sum_{k=1}^n a_k$, $Q_n = \sum_{k=1}^n a_k^2$,

$$I_n = \int da_1 \cdots da_n\, e^{-(\beta/2)Q_n + \frac{(\beta/2)}{n+1}S_n^2}$$
$$= \int da_1 \cdots da_{n-1}\, e^{-(\beta/2)Q_{n-1} + \frac{(\beta/2)}{n+1}S_{n-1}^2} \int da\, e^{-(\beta/2)a^2 + \frac{(\beta/2)}{n+1}a^2 + \frac{(\beta/2)}{n+1}2a\cdot S_{n-1}}$$
$$= \int da_1 \cdots da_{n-1}\, e^{-(\beta/2)Q_{n-1} + \frac{(\beta/2)}{n+1}S_{n-1}^2} \int da\, e^{-\frac{(\beta/2)n}{n+1}\left(a - \frac{S_{n-1}}{n}\right)^2} e^{+\frac{(\beta/2)}{n(n+1)}S_{n-1}^2}$$
$$= \int da_1 \cdots da_{n-1}\, e^{-(\beta/2)Q_{n-1} + \frac{(\beta/2)}{n+1}S_{n-1}^2} \left(\frac{2\pi}{\beta}\right)^{3/2} \left(\frac{n+1}{n}\right)^{3/2} e^{+\frac{(\beta/2)}{n(n+1)}S_{n-1}^2}$$
$$= I_{n-1} \left(\frac{2\pi}{\beta}\right)^{3/2} \left(\frac{n+1}{n}\right)^{3/2}. \tag{4.98}$$

Iterating n times up to $I_0 \equiv 1$, one gets the result

$$I_n = \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}n} (n+1)^{3/2}. \tag{4.99}$$

Replace this into (4.96). Then

$$\sum_{\Gamma(1,n_1)} \int d\Lambda\, dv_1\, \chi^{ov}_{\xi}(\boldsymbol{\zeta}^{\varepsilon})\, F_{\theta_3}(1) \le (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + C\, t^2\, \varepsilon^{2-4\theta-(5/2)\theta_3}$$
$$\times \sum_{\Gamma(1,n_1)} \sum_{i=1}^{n_1+1} \frac{(D't)^{n_1-n^i}}{(n_1-n^i)!} \frac{(D't)^{n^i}}{n^i!} (n^i+1)^{3/2}$$
$$\le (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + C\, t^2\, \varepsilon^{2-4\theta-(5/2)\theta_3} (2D't)^{n_1} \sum_{\Gamma(1,n_1)} \frac{(n_1+1)^{5/2}}{n_1!}$$
$$= (Dt)^{n_1} \frac{\varepsilon^{\gamma_1}}{2} + C\, t^2\, \varepsilon^{2-4\theta-(5/2)\theta_3} (2D't)^{n_1} (n_1+1)^{5/2}. \tag{4.100}$$

In the first term, $\gamma_1 < 1$ arbitrary (from Lemma 4.12). Restrict now

$$\gamma_1 < \min[1, 2-4\theta-(5/2)\theta_3]. \tag{4.101}$$

Lemma 4.10 is proved. □

4.6 Completion of the proof of Theorem 2.4

4.6.1 Proof of (2.23)

We insert $g^{\varepsilon}(t)$ into (2.20) by writing

$$f^{\varepsilon}_J(t) = \sum_{H\subset J} (f^{\varepsilon}_1(t))^{\otimes H} E_{J\setminus H}(t)$$
$$= \sum_{H\subset J} (f^{\varepsilon}_1(t) - g^{\varepsilon}(t) + g^{\varepsilon}(t))^{\otimes H} E_{J\setminus H}(t)$$
$$= \sum_{H\subset J} (g^{\varepsilon}(t))^{\otimes H} E^{\mathscr{E}}_{J\setminus H}(t), \tag{4.102}$$

where the Enskog error term of order k is

$$E^{\mathscr{E}}_K(t) := \sum_{Q\subset K} (f^{\varepsilon}_1(t) - g^{\varepsilon}(t))^{\otimes Q} E_{K\setminus Q}(t)$$
$$= \sum_{Q\subset K} (E^{\mathscr{E}}_1(t))^{\otimes Q} E_{K\setminus Q}(t). \tag{4.103}$$

Resorting to the respective BBGKY and Enskog tree expansions, Eqs. (3.36) and (3.43)–(3.44), the one-point Enskog error reads

$$\begin{aligned}
E_1^{\mathscr{E}}(t) &= \sum_{n=0}^{\infty} \sum_{\Gamma(1,n)} \int d\Lambda \left(\prod B^{\varepsilon} f_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) - \prod B^{\mathscr{E}} g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\mathscr{E}}(0)) \right) \\
&= \sum_{n=0}^{\infty} \sum_{\Gamma(1,n)} \int d\Lambda \prod B^{\varepsilon} \left(f_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) - g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) \right) \\
&\quad + \sum_{n=0}^{\infty} \sum_{\Gamma(1,n)} \int d\Lambda \left(\prod B^{\varepsilon} g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) - \prod B^{\mathscr{E}} g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\mathscr{E}}(0)) \right),
\end{aligned} \tag{4.104}$$

where $g_{0,1+n}^{\varepsilon} = f_0^{\otimes(1+n)}$.

Applying Hypothesis 2.2 first and then Hypotheses 2.3 and 2.1, the rate of convergence of the initial data is

$$\begin{aligned}
|f_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) - g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0))| &\leq |(f_{0,1}^{\varepsilon})^{\otimes(1+n)}(\boldsymbol{\zeta}^{\varepsilon}(0)) - g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0))| \\
&\quad + \varepsilon^{\gamma_0}(2z)^{1+n} e^{-(\beta/2)\sum_{i\in S(1)}(\eta_1^{\varepsilon}(0))^2} \\
&\leq 2\,\varepsilon^{\gamma_0}(2z)^{1+n} e^{-(\beta/2)\sum_{i\in S(1)}(\eta_1^{\varepsilon}(0))^2}
\end{aligned} \tag{4.105}$$

for ε small and $1+n < \varepsilon^{-\alpha_0}$ (same estimate with no ε^{γ_0} for larger n). Using this and the estimates of Lemma 4.2 and its proof:

$$\begin{aligned}
&\left| \sum_{n=0}^{\infty} \sum_{\Gamma(1,n)} \int d\Lambda \prod B^{\varepsilon} \left(f_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) - g_{0,1+n}^{\varepsilon}(\boldsymbol{\zeta}^{\varepsilon}(0)) \right) \right| \\
&\leq 2\varepsilon^{\gamma_0} \sum_{n=0}^{\infty} \sum_{\Gamma(1,n)} \int d\Lambda \prod |B^{\varepsilon}| (2z)^{1+n}\, e^{-(\beta/2)\sum_{i\in S(1)}(\eta_1^{\varepsilon}(0))^2} \\
&\quad + 2 \sum_{n\geq\varepsilon^{-\alpha_0}} \sum_{\Gamma(1,n)} \int d\Lambda \prod |B^{\varepsilon}| (2z)^{1+n}\, e^{-(\beta/2)\sum_{i\in S(1)}(\eta_1^{\varepsilon}(0))^2} \\
&\leq 3\,\varepsilon^{\gamma_0}\,\bar{C}\, e^{-(\beta/4)v_1^2}
\end{aligned} \tag{4.106}$$

where $\bar{C} = \bar{C}(2z, \beta) > 0$ and $t < \bar{t}$.

The last term in (4.104) is due to the differences among the IBF and the EBF. Since, in absence of internal recollisions of the IBF and of internal overlaps of the EBF, the two flows coincide, it holds that

$$\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\left(\prod B^{\varepsilon}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\varepsilon}(0))-\prod B^{\mathscr{E}}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right)$$
$$=\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\left(\prod B^{\varepsilon}\chi^{int}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\varepsilon}(0))-\prod B^{\mathscr{E}}\chi^{i.o.}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right) \tag{4.107}$$

where $\chi^{int} = \chi^{int}(\boldsymbol{\zeta}^{\varepsilon})$ and $\chi^{i.o.} = \chi^{i.o.}(\boldsymbol{\zeta}^{\mathscr{E}})$ are defined by (4.56) and (D.2) respectively. We use, in order, $|g^{\varepsilon}_{0,1+n}| \leq 2\,(2z)^{1+n}e^{-(\beta/2)\sum_{i\in S(1)}(\eta^{\varepsilon}_1(0))^2}$, Lemma C.1, Lemma 4.12 and (D.3) to deduce that

$$\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int dv_1 d\Lambda\left(\prod|B^{\varepsilon}|\chi^{int}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\varepsilon}(0))+\prod|B^{\mathscr{E}}|\chi^{i.o.}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))\right)$$
$$\leq \bar{C}'\,\varepsilon^{\theta_1}+\varepsilon^{\gamma_1-\theta_1}\,2\,z\sum_{n\geq 0}(2zDt)^n \tag{4.108}$$

for any $\gamma_1 \in (0, 1)$, arbitrary θ_1 and ε small enough. Up to constants depending on z, β, this is smaller than $\varepsilon^{\gamma_1/2}$ when $t < t^*$.

We conclude that $\int dv|E^{\mathscr{E}}_1(t)| \leq \varepsilon^{\gamma k}$ for any $\gamma < \min[\gamma_0, \gamma_1/2]$ and in particular for the γ appearing in (2.21) (remind (4.37)).

The final result follows readily from (4.103) and (2.21). □

4.6.2 Proof of (2.25)

From (4.102) one gets

$$f^{\varepsilon}_J(t) = \sum_{H\subset J}(f(t))^{\otimes H}E^{\mathcal{B}}_{J\setminus H}(t),$$
$$E^{\mathcal{B}}_K(t) := \sum_{Q\subset K}(g^{\varepsilon}(t)-f(t))^{\otimes Q}E^{\mathscr{E}}_{K\setminus Q}(t). \tag{4.109}$$

We resort once again to the tree expansions, Eqs. (3.43)–(3.44) and (3.47)–(3.48). By $g^{\varepsilon}_{0,1+n} = f_0^{\otimes(1+n)} = f_{0,1+n}$ and (3.51), we obtain

$$g^{\varepsilon}(t)-f(t)=\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\left(\prod B^{\mathscr{E}}g^{\varepsilon}_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))-\prod Bf_{0,1+n}(\boldsymbol{\zeta}(0))\right)$$
$$=\sum_{n=0}^{\infty}\sum_{\Gamma(1,n)}\int d\Lambda\prod B\left[f_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0))-f_{0,1+n}(\boldsymbol{\zeta}(0))\right]. \tag{4.110}$$

Using the Lipschitz-regularity assumption on f_0, (3.50) and

$$|\xi_i^{\mathscr{E}}(0) - \xi_i(0)| \leq n\,\varepsilon \tag{4.111}$$

(as follows e.g. from Fig. 3, (iii)–(iv)), one finds

$$\begin{aligned}&\left| f_{0,1+n}(\boldsymbol{\zeta}^{\mathscr{E}}(0)) - f_{0,1+n}(\boldsymbol{\zeta}(0)) \right| \\ &\quad \leq 2^{1+n}\,(\max(L,2z))^{1+n}\, e^{-(\beta/2)\sum_{i\in S(1)}(\eta_1^{\varepsilon}(0))^2}\,(n\varepsilon)\end{aligned} \tag{4.112}$$

for ε small and $n < \varepsilon^{-\alpha}$ with arbitrary $\alpha < 1$ (same estimate with no $(n\varepsilon)$ for larger n). Inserting into (4.110) and by further application of the estimates of Lemma 4.2, one proves

$$\int dv\,|g^{\varepsilon}(t) - f(t)| \leq \bar{C}''\varepsilon^{1-\alpha} \tag{4.113}$$

for suitable $\bar{C}'' = \bar{C}''(z,\beta,L)$ and $t < t^*$. Since α is here arbitrary, we conclude that this is smaller than $\varepsilon^{\gamma k}$ with γ as in (2.21).

Equation (2.25) follows from (4.109) and (2.23). □

4.7 Convergence of high order fluctuations

In this section we prove Theorem 2.5. Note preliminarily that, if the test functions $\varphi_1,\ldots,\varphi_j$ have disjoint supports, then the result follows immediately from Theorem 2.4. Indeed a simple algebra (see the remark after (4.122) below) leads to the identity

$$\mathbb{E}^{\varepsilon}\left[\prod_{i=1}^{j}\big(F_i(t) - \mathbb{E}^{\mathcal{B}}[\varphi_i(t)]\big)\right] = \int_{\mathbb{R}^{6j}} d\mathbf{z}_j\,\varphi(z_1)\cdots\varphi(z_j)\,E_j^{\mathcal{B}}(\mathbf{z}_j,t) \tag{4.114}$$

and hence to the result by observing that no δ-overlap occurs in the integrand of the r.h.s. for ε small (so (2.25) can be applied). Whenever the φ_i's have supports which are not disjoint, the estimate of the l.h.s. of (4.114) complicates considerably.

In the present section, we will work with the extended version of the correlation error E_k over $\mathbb{R}^{6k}$, defined by

$$f_J^{\varepsilon} = \sum_{K\subset J}\big(f_1^{\varepsilon}\big)^{\otimes K}\,\bar{E}_{J\setminus K}, \tag{4.115}$$

where $\bar{E}_K : \mathbb{R}^{6k} \to \mathbb{R}$ and (2.6) is used. Since no confusion arises, we shall denote $\bar{E}_k = E_k$.

4.7.1 Proof of Theorem 2.5

Let us replace, for the moment, $\mathbb{E}^{\mathcal{B}}[\varphi_i]$ by $\mathbb{E}^{\varepsilon}[F_i]$. Then we compute the *fluctuation* of order j, namely (1.9), i.e.

$$\mathbb{E}^{\varepsilon}\left[\prod_{i=1}^{j}\big(F_i(t)-\mathbb{E}^{\varepsilon}[F_i(t)]\big)\right]=\sum_{L\subset J}\left((-1)^l\prod_{i\in L}\mathbb{E}^{\varepsilon}[F_i(t)]\right)\mathbb{E}^{\varepsilon}\left[\prod_{i\in J\setminus L}F_i(t)\right]. \tag{4.116}$$

We use, again, $l=|L|$, $j=|J|$ and so on. From (1.7) and by symmetry of the state,

$$\begin{aligned}\mathbb{E}^{\varepsilon}[F_i(t)]&=\varepsilon^2\sum_{n\geq 0}\frac{1}{n!}\int_{\mathcal{M}_n}dz_1\cdots dz_n W_n^{\varepsilon}(\mathbf{z}_n,t)\sum_{j=1}^{n}\varphi_i(z_j)\\&=\varepsilon^2\int dz\,\varphi_i(z)\,\rho_1^{\varepsilon}(z,t)\\&=\int dz\,\varphi_i(z)\,f_1^{\varepsilon}(z,t),\end{aligned} \tag{4.117}$$

where we introduced the correlation function (2.7) and its rescaled version (2.11). Similarly, for $K=\{1,2,\ldots,k\}$,

$$\begin{aligned}\mathbb{E}^{\varepsilon}\left[\prod_{i\in K}F_i(t)\right]&=\varepsilon^{2k}\,\mathbb{E}^{\varepsilon}\left[\sum_{j_1,\ldots,j_k}\varphi_1(z_{j_1})\cdots\varphi_k(z_{j_k})\right]\\&=\varepsilon^{2k}\,\mathbb{E}^{\varepsilon}\left[\sum_{m=1}^{k}\sum_{\substack{P_1,\ldots,P_m\\ \cup_q P_q=K\\ |P_q|\geq 1\\ P_q\cap P_h=\emptyset,q\neq h}}\sum_{\substack{j_1,\ldots,j_m\\ j_q\neq j_h,q\neq h}}\prod_{q=1}^{m}\prod_{i\in P_q}\varphi_i(z_{j_q})\right]\\&=\sum_{m=1}^{k}\varepsilon^{2k-2m}\sum_{\substack{P_1,\ldots,P_m\\ \cup_q P_q=K\\ |P_q|\geq 1\\ P_q\cap P_h=\emptyset,q\neq h}}\int dz_1\cdots dz_m\prod_{q=1}^{m}\prod_{i\in P_q}\varphi_i(z_q)\,f_m^{\varepsilon}(z_1,\ldots,z_m,t).\end{aligned} \tag{4.118}$$

Observe that, in the evaluation of this integral, the observables $\{\varphi_i\}_{i\in P_q}$ are *contracted* in the variable z_q. We insert the two previous expressions into (4.116). Setting

$$\Phi_{P_q} = \prod_{i \in P_q} \varphi_i(z_q),$$

we find

$$\begin{aligned}
&\mathbb{E}^\varepsilon\left[\prod_{i=1}^{j}\big(F_i(t) - \mathbb{E}^\varepsilon[F_i(t)]\big)\right] \\
&\quad = \sum_{L\subset J}\left((-1)^l \prod_{i\in L}\int \varphi_i f_1^\varepsilon\right)\sum_{m=1}^{j-l}\varepsilon^{2j-2l-2m} \\
&\quad\quad \times \sum_{\substack{P_1,\dots,P_m \\ \cup_q P_q = J\setminus L \\ |P_q|\geq 1 \\ P_q\cap P_h=\emptyset, q\neq h}} \int d\mathbf{z}_M \prod_{q=1}^{m}\Phi_{P_q}\, f_m^\varepsilon(\mathbf{z}_M, t),
\end{aligned} \tag{4.119}$$

where $M = \{1, \dots, m\}$. Denoting $S = L \cup \{P_i;\ |P_i| = 1\}$,

$$\begin{aligned}
&\mathbb{E}^\varepsilon\left[\prod_{i=1}^{j}\big(F_i(t) - \mathbb{E}^\varepsilon[F_i(t)]\big)\right] = \sum_{S\subset J}\sum_{m=1}^{j-s}\varepsilon^{2j-2m-2s} \sum_{\substack{P_1,\dots,P_m \\ \cup_q P_q = J\setminus S \\ |P_q|\geq 2 \\ P_q\cap P_h=\emptyset, q\neq h}} \\
&\quad \times \int d\mathbf{z}_M\, d\mathbf{z}'_S \left(\prod_{i\in S}\varphi_i(z'_i)\right)\left(\prod_{q=1}^{m}\Phi_{P_q}\right) \\
&\quad \times \sum_{L\subset S}(-1)^l (f_1^\varepsilon)^{\otimes l}(\mathbf{z}'_L)\, f^\varepsilon_{m+s-l}(\mathbf{z}_M, \mathbf{z}'_{S\setminus L}).
\end{aligned} \tag{4.120}$$

Let us write this expression in terms of correlation errors. For any $S \subset K$, the following algebraic identities hold:

$$\begin{aligned}
&\sum_{L\subset S}(-1)^l (f_1^\varepsilon)^{\otimes L} f^\varepsilon_{K\setminus L} \\
&\quad = \sum_{L\subset S}(-1)^l (f_1^\varepsilon)^{\otimes L} \sum_{L'\subset K\setminus S}\delta_{L',\emptyset}(f_1^\varepsilon)^{\otimes L'} f^\varepsilon_{K\setminus(L\cup L')} \\
&\quad = \sum_{L\subset S}(-1)^l (f_1^\varepsilon)^{\otimes L} \sum_{L'\subset K\setminus S}\left(\sum_{L''\subset L'}(-1)^{l''}\right)(f_1^\varepsilon)^{\otimes L'} f^\varepsilon_{K\setminus(L\cup L')}
\end{aligned}$$

$$= \sum_{L \subset K \setminus S} (f_1^\varepsilon)^{\otimes L} \sum_{L' \subset K \setminus L} (-1)^{l'} (f_1^\varepsilon)^{\otimes L'} f^\varepsilon_{K \setminus (L \cup L')}$$
$$= \sum_{L \subset K \setminus S} (f_1^\varepsilon)^{\otimes L} E_{K \setminus L} \tag{4.121}$$

where in the last step we used the definition of correlation error, (1.12), extended in the whole space. Notice that in the fourth line we just renamed $L \cup L'' \to L'$, $L' \setminus L'' \to L$. Inserting (4.121) into (4.120), we obtain

$$\mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^\varepsilon [F_i(t)] \right) \right] = \sum_{S \subset J} \sum_{m=1}^{j-s} \varepsilon^{2j-2m-2s} \sum_{\substack{P_1, \dots, P_m \\ \cup_q P_q = J \setminus S \\ |P_q| \geq 2 \\ P_q \cap P_h = \emptyset, q \neq h}}$$
$$\times \int d\mathbf{z}_M \, d\mathbf{z}'_S \left(\prod_{i \in S} \varphi_i(z'_i) \right) \left(\prod_{q=1}^{m} \Phi_{P_q} \right)$$
$$\times \sum_{L \subset M} (f_1^\varepsilon)^{\otimes l} (\mathbf{z}'_L) E_{m+s-l}(\mathbf{z}_M, \mathbf{z}'_{S \setminus L}). \tag{4.122}$$

Remark (Observables with disjoint support) Assume that supp $\varphi_i \cap$ supp $\varphi_j = \emptyset$ for $i \neq j$. Then the above algebra becomes trivial, because no contractions are possible (it must be $z_{j_r} \neq z_{j_s}$ for $r \neq s$) and the only surviving term in (4.118) is $m = k$. Eqs. (4.119) and (1.12) lead immediately to

$$\mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^\varepsilon [F_i(t)] \right) \right] = \int_{\mathbb{R}^{6j}} d\mathbf{z}_j \, \varphi(z_1) \cdots \varphi(z_j) \, E_j(\mathbf{z}_j, t) \tag{4.123}$$

and the same computation with f_1^ε replaced by f leads to (4.114).

Observe that, actually, $2m \leq j - s$. Hence, by Proposition 2.6 (to be proved below)

$$\left| \mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^\varepsilon [F_i(t)] \right) \right] \right| \leq G^j \sum_{S \subset J} \sum_{m=1}^{j-s} \varepsilon^{j-s} \sum_{\substack{P_1, \dots, P_m \\ \cup_q P_q = J \setminus S \\ |P_q| \geq 2 \\ P_q \cap P_h = \emptyset, q \neq h}} \sum_{L \subset M} 2^l \varepsilon^{\gamma(m+s-l)}$$
$$\leq \varepsilon^{\gamma j} (8GC)^j j!$$
$$\leq \varepsilon^{\gamma' j} \tag{4.124}$$

for any $j < \varepsilon^{-\alpha'}$, $t < t^*$, $\gamma' < \gamma - \alpha'$ and ε small enough (having bounded the sum over partitions as in (B.6)).

On the other hand, $|\mathbb{E}^\varepsilon[F_i(t)] - \mathbb{E}^{\mathcal{B}}[\varphi_i(t)]| = |\int \varphi_i E_1^{\mathcal{B}}| \leq G\varepsilon^\gamma$ by Theorem 2.4. Therefore, slightly decreasing α', we conclude that

$$\begin{aligned}
&\sup_{j<\varepsilon^{-\alpha'}} \left| \mathbb{E}^\varepsilon \left[\prod_{i=1}^{j} \left(F_i(t) - \mathbb{E}^{\mathcal{B}}[\varphi_i(t)] \right) \right] \right| \\
&\quad \leq \sup_{j<\varepsilon^{-\alpha'}} \sum_{L\subset J} \left| \mathbb{E}^\varepsilon \left[\prod_{i\in L} \left(F_i(t) - \mathbb{E}^\varepsilon[F_i(t)] \right) \right] \right| (G\varepsilon^\gamma)^{j-l} \\
&\quad \leq \sup_{j<\varepsilon^{-\alpha'}} \sum_{L\subset J} \varepsilon^{\gamma' l} (G\varepsilon^\gamma)^{j-l},
\end{aligned} \tag{4.125}$$

which goes to zero as a power of ε. Theorem 2.5 is proved. □

In the proof of Theorem 2.5 we had to estimate the quantity

$$\int_{\mathbb{R}^{6k}} d\mathbf{z}_k \, \varphi(z_1) \cdots \varphi(z_k) \, E_k(\mathbf{z}_k, t). \tag{4.126}$$

Note that, by Theorem, 2.4 we control E_k only in the region

$$\mathcal{M}_k^x(\delta) = \{\mathbf{x}_k \in \mathbb{R}^{3k}, \;\; |x_i - x_j| > \delta, \; i \neq j\}.$$

Suppose now that $\mathbf{z}_k = (\mathbf{z}_{Q'}, \mathbf{z}_{K\setminus Q'})$, where

$$\mathbf{x}_{Q'} \in \mathcal{M}_{q'}^x(\delta) \cap \left\{ \mathbf{x}_{Q'} \in \mathbb{R}^{3q'}, \quad \min_{\substack{i\in Q' \\ j\in K\setminus Q'}} |x_i - x_j| > \delta \right\} \tag{4.127}$$

and $\mathbf{x}_{K\setminus Q'} \in \mathbb{R}^{3(k-q')}$ is some configuration lying in a small measure set with overlaps at distance δ. Then we cannot estimate brutally $|E_k|$ by $(const.)^k$, but we need to recover a small error $\varepsilon^{\gamma q'}$, relative to the non-overlapping configurations. That is, we need a natural improvement of Theorem 2.4 including the case in which δ-overlaps are admitted for a subset of $\mathbf{x}_k$. This is expressed by the following corollary, whose proof is deferred to Appendix E.

Corollary 4.14 *Under the assumptions of Theorem 2.4, let $Q' \subset K$. Then there exists a positive constant $C_2 = C_2(z, \beta)$ such that, for any $t < t^*$ and ε small enough,*

$$\int_{\mathbb{R}^{3q'}} d\mathbf{v}_{Q'} |E_K(t)| \leq C_2^k \left(\varepsilon^{\gamma k} + \varepsilon^{\gamma_1' q'} \varepsilon^{-\alpha_1 k} \right) \;\; \forall k < \varepsilon^{-\alpha}, \tag{4.128}$$

with $\gamma_1' = \min[\gamma_0, \gamma_1/2]$, $\alpha_1 = \theta_1 + 3\alpha$, $\boldsymbol{x}_{K\setminus Q'} \in \mathbb{R}^{3(k-q')}$ *and* $\boldsymbol{x}_{Q'}$ *as in* (4.127).

The parameters $\gamma, \gamma_0, \theta_1, \alpha, \delta$ and γ_1 are listed in Sects. 4.3.3 and 4.3.4.

By using the above corollary we achieve next the proof of Proposition 2.6.

4.7.2 *Proof of Proposition 2.6*

Let us fix $\delta = \varepsilon^\theta$ with θ in the interval

$$\theta \in (3/14, 1/4). \tag{4.129}$$

Furthermore, let

$$1 = \sum_{Q\subset K} \chi_Q^\delta \; \bar{\chi}_{K\setminus Q,K}^\delta, \tag{4.130}$$

where $\chi_Q^\delta = 1$ if and only if any particle with index in Q "δ-overlaps" with a different particle in Q, and $\bar{\chi}^0_{K\setminus Q,K} = 1$ if and only if all the particles in $K\setminus Q$ lie at distance strictly larger than δ from any other particle in K.

Inserting the partition of unity, (4.126) becomes

$$\sum_{Q\subset K} \int d\mathbf{z}_Q\, \chi_Q^\delta \left(\prod_{i\in Q} \varphi_i(z_i)\right) \int d\mathbf{z}_{K\setminus Q} \left(\prod_{i\in K\setminus Q} \varphi_i(z_i)\right) \bar{\chi}_{K\setminus Q,K}^\delta\, E_K(\mathbf{z}_k, t).$$

For $Q = \emptyset$ we can apply the main theorem, while, for $|Q| \geq 2$, we resort to Corollary 4.14. Taking the supremum over velocities of the test functions, one obtains

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{6k}} d\mathbf{z}_k\, \varphi(z_1)\cdots\varphi(z_k)\, E_K(\mathbf{z}_k, t)\right| \\
&\quad \leq G^k \varepsilon^{\gamma k} + \sum_{\substack{Q\subset K\\ |Q|>1}} G^{k-q} C_2^k \left(\varepsilon^{\gamma k} + \varepsilon^{\min[\gamma_0,\gamma_1/2](k-q)}\, \varepsilon^{-\theta_1 k - 3\alpha k}\right) \\
&\qquad \times \int d\mathbf{z}_Q\, \chi_Q^\delta \left(\prod_{i\in Q} |\varphi_i(z_i)|\right) \\
&\quad \leq G^k \varepsilon^{\gamma k} + G^{k-1} C_2^k \sum_{\substack{Q\subset K\\ |Q|>1}} \left(\varepsilon^{\gamma k} + \varepsilon^{\min[\gamma_0,\gamma_1/2](k-q)}\, \varepsilon^{-\theta_1 k - 3\alpha k}\right) \\
&\qquad \times \int dz_{i_*}\, \varphi_{i_*} \int d\mathbf{z}_{Q\setminus\{i_*\}} \chi_Q^\delta
\end{aligned}$$

$$
\begin{aligned}
&\le G^k \varepsilon^{\gamma k} + (GC_2)^k \sum_{\substack{Q \subset K \\ |Q|>1}} \left(\varepsilon^{\gamma k} + \varepsilon^{\min[\gamma_0, \gamma_1/2](k-q)} \, \varepsilon^{-\theta_1 k - 3\alpha k} \right) \\
&\quad \times (q-1)!(4\pi \delta^3/3)^{q-1} \\
&\le G^k \varepsilon^{\gamma k} + (GC_2 4\pi/3)^k \sum_{\substack{Q \subset K \\ |Q|>1}} \left(\varepsilon^{\gamma k} + \varepsilon^{\min[\gamma_0, \gamma_1/2](k-q)} \, \varepsilon^{-\theta_1 k - 3\alpha k} \right) \\
&\quad \times \varepsilon^{(3\theta - \alpha')(q-1)},
\end{aligned}
\tag{4.131}
$$

where i_* is an arbitrary element in Q. In the last step, we used $k < \varepsilon^{-\alpha'}$.

We choose

$$
\alpha' < 7\theta - 3/2. \tag{4.132}
$$

Then, by (4.101), we find $3\theta - \alpha' - \min[\gamma_0, \gamma_1/2] > 3\theta - 7\theta + 3/2 - 1/2 = -4\theta + 1 > 0$. In particular, using again (4.101) and reminding that $\theta_3 = 1/5$, $q \ge 2$,

$$
\begin{aligned}
&\left(3\theta - \alpha' - \min[\gamma_0, \gamma_1/2]\right) q - 3\theta + \alpha' \ge 6\theta - 2\alpha' - 2 + 4\theta \\
&\quad + (1/2) - 3\theta + \alpha' = 7\theta - \alpha' - 3/2 > 0.
\end{aligned}
$$

Therefore, the q-dependent factors in (4.131) are very small. The final result follows then from (4.37) for ε small enough. □

4.8 Concluding remarks

4.8.1 Truncated functions

In this paper we studied the kinetic theory of expansion (1.14) for a dilute gas of hard spheres. Similar expansions within the framework of kinetic theory have been considered in [5,17,26,27]. Moreover, they are very familiar in statistical mechanics. The standard example is given by the Ursell functions in the classical analysis of the equilibrium state in a gas at low (finite) density, chapter 4.4 of [33]. Typically, one expands in truncated functions

$$
f_J^\varepsilon = \sum_{1 \le m \le j} \sum_{\substack{P_1, \dots, P_m \\ \cup_q P_q = J \\ P_q \cap P_h = \emptyset, q \ne h \\ |P_q| \ge 1}} \prod_i f_{P_i}^{\varepsilon, T} \tag{4.133}
$$

and focuses on the decay properties of $f_j^{\varepsilon,T}$ and on related physical quantities. In connection to this, combinatorial methods have been intensively studied under the name of "cluster expansion", see e.g. [24].

Note that (4.133) defines implicitly the truncated functions, as we did for the correlation errors in Eqs. (1.11)–(1.14). The difference is that in (4.133) the sum runs over *all* partitions of j elements. A direct comparison with (1.14) yields:

$$E_J = \sum_{1 \le m \le j/2} \sum_{\substack{P_1,\dots,P_m \\ \cup_q P_q = J \\ P_q \cap P_h = \emptyset, q \neq h \\ |P_q| \ge 2}} \prod_i f_{P_i}^{\varepsilon,T}. \tag{4.134}$$

In other words, E_J is a "partially truncated correlation function" with respect to clusters of size at least 2.

The interpretation of (4.134) should be now transparent. The $f_j^{\varepsilon,T}$ measures events of j *maximally correlated* particles with at least $j - 1$ recollisions connecting all the particles. The E_j measures events of j *minimally correlated* particles with at least $\lceil j/2 \rceil$ recollisions, i.e. just one per particle. If ε^{γ_1} is the size of one single recollision, one expects $f_j^{\varepsilon,T} \sim \varepsilon^{\gamma_1(j-1)}$ and $E_j \sim \varepsilon^{\gamma_1 \lceil j/2 \rceil}$.

During the revision of a first version of the present paper,[6] a preprint appeared by Bodineau, Gallagher and Saint-Raymond including a derivation of the linearized Boltzmann equation for the two-dimensional hard disk gas at equilibrium, global in time [5]. Here a similar notion of "cumulant expansion" is introduced and the control of truncated functions is a crucial tool to reach arbitrary times. The combinatorial problem and the estimates of multiple recollisions are however different in this context.

4.8.2 *Time of validity*

Note that we did not optimize the time interval $(0, t^*)$ for which the main theorem holds. In fact, in Sect. 4.3.5 we used $t^* < (eC_1)^{-1}$ which can be strictly smaller than the value $\bar{t}$, appearing in Proposition 4.1 and ensuring Lanford's validity result. It is easy to extend our result up to $\bar{t}$ by paying the price of worst values of γ, α. It is enough to notice that in (4.36) we disregarded the truncation on n, i.e. $\sum_{n=0}^{\log \varepsilon^{-\theta_2 k}}$ (see Lemma 4.5). Substituting t^* by $\bar{t}$ in (4.36) and using

$$\sum_{n=0}^{\log \varepsilon^{-\theta_2 k}} (eC_1 \bar{t})^n \le \varepsilon^{-\theta_2 \log(\bar{C}_1)\, k}$$

for $\bar{C}_1 > \max(1, eC_1\bar{t})$, one obtains that condition (4.37) is replaced by

$$\gamma < \min[\gamma_0, \gamma_1/2] - \theta_1 - 3\alpha - \theta_2 \log(\bar{C}_1). \tag{4.135}$$

The final result follows for a different choice of the cutoff parameters.

[6] arXiv:1405.4676, 2014.

4.8.3 Canonical ensemble

The definition of "state" used in this paper includes the canonical $W^{\varepsilon}_{0,n} = 0$ for all $n \neq N$, $N \sim \varepsilon^{-2}$. Let us focus now on the BBGKY solution, (3.36), in the case of a state of this form. Even if we ignore the dynamical correlations (see the Remark in Sect. 3.4.2) and assume $W^{\varepsilon}_{0,N} = (w^{\varepsilon})^{\otimes N}$, the formula does not exhibit complete factorization. The reason is twofold:

1. we work with r.c.f. $f^{\varepsilon}_j = \varepsilon^{2j} N(N-1)\cdots(N-j+1)\,(w^{\varepsilon})^{\otimes j}$;
2. an additional correlation is present, given by the constraint $\sum n_i \leq N - j$.

The main advantage of using a grand canonical formalism is to get rid of these extra correlations.

Observe that the above effects have nothing to do with the dynamics and are uniquely determined by the special structure of the initial data. Actually our main result does cover a canonical state obeying the assumptions. However we have not verified, in a canonical case, Hypothesis 2.2, for which a more elaborate expansion than (A.7) seems to be necessary.

4.8.4 Spatial domain

Our results have been established in the whole space $\mathbb{R}^3$. A natural question is how to extend the analysis to the case of a region $\Lambda \subset \mathbb{R}^3$ with prescribed boundary conditions. We discuss the major points in what follows.

Assumptions on the boundary conditions ensuring existence and uniqueness of the n-particle flow have been studied in previous literature, e.g. [1,13,28]. Once the flow is well defined, the setting and the hierarchical formulas of Sects. 2 and 3 can be easily adapted, see for instance [3,35,36]. Note that, in the case of a bounded domain, all the sums over n (number of particles) become finite, both in the definition of correlation functions and in the related tree expansions (see also the states considered in Appendix A). Indeed, due to the hard sphere exclusion, $W^{\varepsilon}_n = 0$ for $n > N_{cp} =$ close-packing number.

However this does not produce any change in the combinatorics of Step 1. The graph expansion (Eq. (4.22)) is applied, as written, to (4.21) even when $n > N_{cp}$ (i.e., to zero terms). This produces non-zero error terms with overlapping trees and total number of created particles larger than N_{cp}. Such terms are the "close-packing correlation" which is therefore automatically taken into account by our method. Since $N_{cp} \sim \varepsilon^{-3}$ and α is certainly much smaller than 3, this correlation is just a part of the first error term in Lemma 4.5, related to the cutoff θ_2 (truncation on the number of creations in a collection of trees).

An extra difficulty comes from the geometrical estimates of recollisions, Step 3 of the proof. The case of a vessel of arbitrary geometry with reflecting and/or diffusive boundary conditions eludes our techniques.[7] On the other hand, the analysis of this paper can be easily adapted to some simple situation as the case of a gas contained in a parallelepiped with periodic or reflecting boundary conditions. Let us discuss this point in some more detail.[8]

Consider the gas confined in $\Lambda_0 = (0, L_1) \times (0, L_2) \times (0, L_3)$, $L_i > 0$ and assume periodic boundary conditions for the free flow. After a moment of thought, one realizes that the overlap condition appearing in Lemma 4.10, i.e. $\inf_s |\xi(s) - \xi_k^\varepsilon(s)| < \varepsilon$, can be represented as follows:

$$\mathbb{1}(\inf_s |\xi(s) - \xi_k^\varepsilon(s)| < \varepsilon) \leq \sum_{\boldsymbol{m}\in\mathbb{Z}^3} \mathbb{1}(\inf_s |\xi(\boldsymbol{m}; s) - \xi_k^\varepsilon(\mathbb{R}^3; s)| < \varepsilon) \quad (4.136)$$

where $\boldsymbol{m} = (m_1, m_2, m_3)$,

$$\xi(\boldsymbol{m}; s) = \xi(\mathbb{R}^3; s) + (m_1 L_1, m_2 L_2, m_3 L_3) \quad (4.137)$$

and $\xi(\mathbb{R}^3; s)$, $\xi_k^\varepsilon(\mathbb{R}^3; s)$ are the trajectories in $\mathbb{R}^3$ computed with *no* boundary conditions. In other words, if the bullet k hits the target ξ in the torus Λ_0, then the bullet k moving in the whole space hits some periodic copy of ξ, also moving in the whole space.

Since the time is finite and the velocities are bounded, there is no serious complication of the estimates of this paper. Reflecting boundary conditions are treated in the same way, but the periodic translations in (4.137) are replaced by reflections.

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Appendices

Appendix A: Chaotic states of hard spheres

We consider here the most natural construction of hard sphere states which factorize in the Boltzmann–Grad limit, and show that they satisfy the hypotheses stated in Sect. 2.3.

Let $\mathbf{W}_0^\varepsilon$ be the grand canonical state over $\mathcal{M}$ with system of densities

$$\frac{1}{n!} W_{0,n}^\varepsilon(\mathbf{z}_n) = \frac{1}{\mathcal{Z}_\varepsilon} \frac{e^{-\mu_\varepsilon} \mu_\varepsilon^n}{n!} f_0^{\otimes n}(\mathbf{z}_n), \quad (A.1)$$

7 Even an extension of Lanford's original proof to the more general cases has not been provided.

8 A discussion similar to the one that follows appears in [4, 15].

where $\varepsilon^2 \mu_\varepsilon = 1$,

$$\mathcal{Z}_\varepsilon = \sum_{n \geq 0} \frac{e^{-\mu_\varepsilon} \mu_\varepsilon^n}{n!} \mathcal{Z}_n^{can}, \tag{A.2}$$

and where the "canonical" normalization constant is

$$\mathcal{Z}_n^{can} = \int_{\mathcal{M}_n} d\mathbf{z}_n \, f_0^{\otimes n}(\mathbf{z}_n) = \int_{\mathbb{R}^{6n}} d\mathbf{z}_n \, f_0^{\otimes n}(\mathbf{z}_n) \prod_{1 \leq i < k \leq n} \bar{\chi}_{i,k}^0, \tag{A.3}$$

($\mathcal{Z}_0^{can} = 1$) with $\bar{\chi}_{i,k}^0$ the indicator function of the set $\{|x_i - x_k| > \varepsilon\}$. The function f_0 can be any probability density over $\mathbb{R}^3 \times \mathbb{R}^3$ satisfying $f_0(x, v) \leq (h(x)/2) e^{-(\beta/2)v^2}$, for some $h \in L^1(\mathbb{R}^3; \mathbb{R}^+)$ with $\operatorname{ess\,sup}_x h(x) = z$, and $z, \beta > 0$.

Remark – The state introduced is a "maximally factorized state" in the sense that the only correlations are due to the hard sphere exclusion. A Gibbs state in equilibrium statistical mechanics is of this form.
- The probability of finding n particles is $p_n = \mathcal{Z}_n^{can} \mathcal{Z}_\varepsilon^{-1} (1/n!) e^{-\mu_\varepsilon} \mu_\varepsilon^n$ and the distribution of the n particles $(\mathcal{Z}_n^{can})^{-1} f_0^{\otimes n}$.
- The asymptotic behaviour of the normalization constants can be proved to be $\mathcal{Z}_n^{can} \sim e^{-Cn^2\varepsilon^3}$ ($n \gg \varepsilon^{-2}$, $C > 0$) and $\mathcal{Z}_\varepsilon \sim e^{-C\varepsilon^{-1}}$ (see e.g. [30]).

Proposition A.1 *The state of the system defined by* (A.1) *admits r.c.f. satisfying Hypotheses* 2.1, 2.2 *and* 2.3.

Proof For this particular state, it is convenient to check (2.16)–(2.17) first. Since the only correlations are due to the exclusion, no combinatorial tools are required and a simple expansion of the non-overlap constraint suffices to reconstruct (2.16).

By definitions (2.7) and (2.11), the rescaled correlation functions are

$$f_{0,j}^\varepsilon(\mathbf{z}_j) = \frac{F^\varepsilon(\mathbf{z}_j)}{\mathcal{Z}_\varepsilon} f_0^{\otimes j}(\mathbf{z}_j), \tag{A.4}$$

where

$$F^\varepsilon(\mathbf{z}_j) = \sum_{n \geq 0} \frac{e^{-\mu_\varepsilon} \mu_\varepsilon^n}{n!} F_{can}^{j+n}(\mathbf{z}_j) \tag{A.5}$$

and

$$F_{can}^{j+n}(\mathbf{z}_j) = \int_{\mathbb{R}^{6n}} d\mathbf{z}_{j,n} f_0^{\otimes n}(\mathbf{z}_{j,n}) \left(\prod_{i=1}^{j} \prod_{k=j+1}^{j+n} \bar{\chi}_{i,k}^0 \right) \left(\prod_{j+1 \leq i < k \leq j+n} \bar{\chi}_{i,k}^0 \right) \tag{A.6}$$

($F_{can}^j(\mathbf{z}_j) = 1$).

For any $j, n \geq 1$, we rewrite $F_{can}^{j+n}(\mathbf{z}_j)$ by using

$$\prod_{i=1}^{j} \prod_{k=j+1}^{j+n} \bar{\chi}_{i,k}^{0} = \prod_{i=1}^{j} (1 - \chi_{i,J^c}^{0}) \tag{A.7}$$

where $J^c = \{j+1, \ldots, j+n\}$ and

$$\chi_{i,J^c}^{0} = \left(1 - \prod_{k=j+1}^{j+n} \bar{\chi}_{i,k}^{0}\right)$$
$$= \mathbb{1}_{\{\mathbf{z}_{j+n} \mid \exists k \in J^c \quad \text{such that } |x_i - x_k| \leq \varepsilon\}}.$$

Expanding the product in (A.7), we find

$$\prod_{i=1}^{j} \prod_{k=j+1}^{j+n} \bar{\chi}_{i,k}^{0} = \sum_{K \subset J} (-1)^k \chi_{K,J^c}^{0}, \tag{A.8}$$

with

$$\chi_{K,J^c}^{0} = \prod_{i \in K} \chi_{i,J^c}^{0}.$$

Inserting (A.8) into (A.6), we arrive to

$$f_{0,j}^{\varepsilon}(\mathbf{z}_j) = \sum_{L \subset J} f_0^{\otimes L}(\mathbf{z}_L) E_{J \setminus L}^{\mathcal{B},0}(\mathbf{z}_{J \setminus L}), \tag{A.9}$$

where $E_{\emptyset}^{\mathcal{B},0} = 1$ and, for $k \geq 1$,

$$E_K^{\mathcal{B},0}(\mathbf{z}_k) = (-f_0)^{\otimes k}(\mathbf{z}_k) \frac{1}{\mathcal{Z}_\varepsilon} \sum_{n \geq 1} \frac{e^{-\mu_\varepsilon} \mu_\varepsilon^n}{n!} \int d\mathbf{z}_{k,n}\, f_0^{\otimes n}(\mathbf{z}_{k,n})\, \chi_{K,K^c}^{0}$$
$$\prod_{k+1 \leq i < h \leq k+n} \bar{\chi}_{i,h}^{0}. \tag{A.10}$$

Let a be the maximum number of three-dimensional hard spheres that can be simultaneously overlapped by a single one, and $q = q(\mathbf{x}_k)$ the minimum number of different spheres in K^c necessary to satisfy the condition $\chi_{K,K^c}^{0} = 1$ (*any* sphere in K is overlapped by at least one sphere in K^c). Then $k/a \leq q \leq k$

and

$$\chi^0_{K,K^c} \le \sum_{\substack{Q\subset K^c\\|Q|=q}} \chi^0_{Q,K}. \tag{A.11}$$

It follows that

$$|E_K^{\mathcal{B},0}(\mathbf{z}_k)| \le (z/2)^k e^{-(\beta/2)\sum_{i\in K} v_i^2} \frac{1}{\mathcal{Z}_\varepsilon} \sum_{n\ge q} \frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!} \times \sum_{\substack{Q\subset K^c\\|Q|=q}} \int d\mathbf{z}_{k,n}\, f_0^{\otimes n}(\mathbf{z}_{k,n})\, \chi^0_{Q,K} \prod_{k+1\le i<h\le k+n} \bar{\chi}^0_{i,h}. \tag{A.12}$$

Note now that $\chi^0_{Q,K} = \prod_{i\in Q}\chi^0_{i,K}$ and, for all $i \in Q$,

$$\int dz_i\, f_0(z_i)\, \chi^0_{i,K} \le (z/2)(2\pi/\beta)^{3/2}\, k\, B\, \varepsilon^3 \tag{A.13}$$

where B is the volume of the unit ball. The remaining $n-q$ integration variables reconstruct $\mathcal{Z}^{can}_{n-q}$, so that we get

$$|E_K^{\mathcal{B},0}(\mathbf{z}_k)| \le (z/2)^k e^{-(\beta/2)\sum_{i\in K} v_i^2} \frac{1}{\mathcal{Z}_\varepsilon} \sum_{n\ge q} \frac{e^{-\mu_\varepsilon}\mu_\varepsilon^n}{n!} \binom{n}{q} k^q\, (C\varepsilon^3)^q\, \mathcal{Z}^{can}_{n-q}. \tag{A.14}$$

Here and below we indicate by C a positive constant, possibly changing from line to line and depending on z, β, a, B.

Using $\frac{1}{n!}\binom{n}{q}k^q \le \frac{(ke)^q}{q^q(n-q)!} \le C^q/(n-q)!$ and reminding (A.2), we deduce

$$\begin{aligned}|E_K^{\mathcal{B},0}(\mathbf{z}_k)| &\le (z/2)^k e^{-(\beta/2)\sum_{i\in K} v_i^2}(C\varepsilon)^q \\ &\le z^k e^{-(\beta/2)\sum_{i\in K} v_i^2}\, C^k\, \varepsilon^{k/a}.\end{aligned} \tag{A.15}$$

This implies the estimate (2.17) by choosing $\gamma_0' < 1/a$ and ε small enough.[9]

Hypotheses 2.1 and 2.3 follow immediately.

Finally, observe that Hypothesis 2.2 and (2.16)–(2.17) are equivalent. Indeed, starting from (2.16), setting $f_0^{\otimes H} = (f^\varepsilon_{0,1} - E_1^{\mathcal{B},0})^{\otimes H}$ and expanding, one finds formula (2.13) with

[9] The bad value of γ_0' is due to the uniform estimate in $\mathcal{M}_k(\varepsilon)$, which includes situations similar to close-packing. If the mutual distance between the particles in K is order 1, then $q = k$ and the above computation gives $\gamma_0' < 1$.

$$E_K^0 = \sum_{Q \subset K} (-1)^q (E_1^{\mathcal{B},0})^{\otimes Q} E_{K \setminus Q}^{\mathcal{B},0}, \tag{A.16}$$

hence (2.17) implies $|E_K^0| \leq 2^k \varepsilon^{\gamma_0' k} z^k \, e^{-(\beta/2)\sum_{i \in K} v_i^2} < \varepsilon^{\gamma_0 k} z^k \, e^{-(\beta/2)\sum_{i \in K} v_i^2}$ for any $\gamma_0 < \gamma_0'$ (and ε small). The proof of the inverse statement is similar (one finds $\gamma_0' < \gamma_0$). □

We conclude this appendix with the proof of the properties presented in Sect. 2.4.2.

Proof of Property 1 Hypothesis 2.2 is obtained from Property 1 in the case $S = \mathcal{J} = J$.

Let us show that Hypothesis 2.2 implies (2.29) for a generic partition of the set S. In this case, (2.28) is a rougher truncation and $E_{\mathcal{K}}^0$ takes into account only correlations among particles of *different* clusters.

Inverting (2.28) we find

$$E_{\mathcal{K}}^0 = \sum_{\mathcal{Q} \subset \mathcal{K}} (-1)^{|\mathcal{Q}|} \left(\prod_{S \in \mathcal{Q}} f_{0,S}^\varepsilon \right) f_{0,K \setminus Q}^\varepsilon . \tag{A.17}$$

We use the notation $K = \cup_{i \in \mathcal{K}} S_i$, $Q = \cup_{i \in \mathcal{Q}} S_i$ etc. By using (2.13), it follows that

$$E_{\mathcal{K}}^0 = \sum_{\mathcal{Q} \subset \mathcal{K}} (-1)^{|\mathcal{Q}|} \sum_{\substack{L_1, \ldots, L_{|\mathcal{Q}|} \\ L_r \subset S_{i_r}}} \prod_{r=1}^{|\mathcal{Q}|} E_{L_r}^0 \sum_{L_0 \subset K \setminus Q} E_{L_0}^0 \left(f_{0,1}^\varepsilon \right)^{\otimes L^c}, \tag{A.18}$$

where $i_1, \ldots, i_{|\mathcal{Q}|}$ are the indices of the clusters in $\mathcal{Q}$, and $L^c = K \setminus \cup_{r=0}^{|\mathcal{Q}|} L_r$. Note that the first sum is over subsets of clusters, while the other sums run over subsets of indices of particles. Setting $L = \cup_{r=0}^{|\mathcal{Q}|} L_r$ we notice that, for given $\mathcal{Q}$ and L, one has $L_r = L \cap S_{i_r}$ and $L_0 = L \cap (K \setminus Q)$. Therefore we rewrite (A.18) as

$$E_{\mathcal{K}}^0 = \sum_{L \subset K} \left(f_{0,1}^\varepsilon \right)^{\otimes K \setminus L} \sum_{\mathcal{Q} \subset \mathcal{K}} (-1)^{|\mathcal{Q}|} E_{L \cap (K \setminus Q)}^0 \prod_{r=1}^{|\mathcal{Q}|} E_{L \cap S_{i_r}}^0 . \tag{A.19}$$

Observe that, in the above sum, L must be such that $|L \cap S_i| > 0$ for all $i \in \mathcal{K}$. Otherwise if $L \cap S_i = \emptyset$ for some i, setting $S^* = S_i$, since $E_{L \cap S_i}^0 = 1$,

$$
\begin{aligned}
&\sum_{\substack{\mathcal{Q}\subset\mathcal{K}\\ S^*\in\mathcal{Q}}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} + \sum_{\substack{\mathcal{Q}\subset\mathcal{K}\\ S^*\notin\mathcal{Q}}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} \\
&\quad = - \sum_{\mathcal{Q}\subset\mathcal{K}\setminus\{S^*\}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} + \sum_{\mathcal{Q}\subset\mathcal{K}\setminus\{S^*\}} (-1)^{|\mathcal{Q}|} E^0_{L\cap(K\setminus Q)} \prod_{r=1}^{|\mathcal{Q}|} E^0_{L\cap S_{i_r}} \\
&\quad = 0. \qquad \text{(A.20)}
\end{aligned}
$$

As a consequence, using (2.12) and (2.14) in (A.19), we deduce

$$
\begin{aligned}
|E^0_{\mathcal{K}}| &\le 2^{|\mathcal{K}|} z^k e^{-(\beta/2)\sum_{i\in K} v_i^2} \sum_{\substack{L\subset K\\ L\cap S_i\neq\emptyset\ \forall i}} \varepsilon^{\gamma_0|L|} \\
&\le 4^k \varepsilon^{\gamma_0|\mathcal{K}|} z^k e^{-(\beta/2)\sum_{i\in K} v_i^2}, \qquad \text{(A.21)}
\end{aligned}
$$

so that (2.29) follows by reducing the values of γ_0, α_0. □

Proof of Property 2 We rewrite (2.30) as

$$
f^{\varepsilon}_{0,S} = \sum_{\mathcal{H}\subset\mathcal{J}} \bar{\chi}^0_{\mathcal{H},\mathcal{J}} \left(\prod_{i\in\mathcal{H}} \bar{\chi}^0_{S_i} f^{\varepsilon}_{0,S_i} \right) \bar{\chi}^0_{J\setminus H} E^0_{\mathcal{J}\setminus\mathcal{H}} \qquad \text{(A.22)}
$$

where $\bar{\chi}^0_{\mathcal{H},\mathcal{J}} = 1$ if and only if all the particles in S_i do not overlap with any other particle in S_k for any choice of $i \in \mathcal{H}, k \in \mathcal{J}, k \neq i$.

We expand now the exclusion constraint (using the ideas explained in Sect. 4.2—Step 1 in the context of dynamical correlations). By virtue of Lemma 4.4,

$$
\bar{\chi}^0_{\mathcal{H},\mathcal{J}} = \sum_{\mathcal{Q}\subset\mathcal{H}} R(\mathcal{Q}, \mathcal{J}\setminus\mathcal{H}) \qquad \text{(A.23)}
$$

and then we get

$$
|R(\mathcal{Q}, \mathcal{J}\setminus\mathcal{H})| \le C^{|\mathcal{Q}|} |\mathcal{Q}|!\, \chi^0_{\mathcal{Q},\mathcal{Q}\cup(\mathcal{J}\setminus\mathcal{H})}. \qquad \text{(A.24)}
$$

Inserting (A.23) in (A.22) we obtain (2.31) with

$$
\bar{E}^0_{\mathcal{K}} = \sum_{\substack{\mathcal{H}_1,\mathcal{H}_2\\ \mathcal{H}_1\cup\mathcal{H}_2=\mathcal{K}\\ \mathcal{H}_1\cap\mathcal{H}_2=\emptyset}} R(\mathcal{H}_1, \mathcal{H}_2) \left(\prod_{i\in\mathcal{H}_1} \bar{\chi}^0_{S_i} f^{\varepsilon}_{0,S_i} \right) \left(\bar{\chi}^0_{H_2} E^0_{\mathcal{H}_2} \right). \qquad \text{(A.25)}
$$

The bound (2.32) follows from (A.24). □

Appendix B: Graph expansion

We prove in this section the graph expansion Lemma. The strategy is explained informally in Sect. 4.2—Step 1, where $\bar{\chi}_{L,L\cup L_0}$ is the non-recollision condition of the trees in the set L (and χ is the overlap constraint).

Proof of Lemma 4.4 By addition/subtraction we find

$$\begin{aligned}\bar{\chi}_{L,L\cup L_0} &= 1 - \sum_{\substack{L_1,L_2\\ L_1\cup L_2=L\\ L_1\cap L_2=\emptyset\\ l_1\geq 1}} \chi_{L_1,L\cup L_0}\bar{\chi}_{L_2,L\cup L_0}\\ &= 1 - \sum_{\substack{L_1,L_2\\ L_1\cup L_2=L\\ L_1\cap L_2=\emptyset\\ l_1\geq 1}} \chi_{L_1,L_1\cup L_0}\bar{\chi}_{L_2,L_0\cup L_1\cup L_2}.\end{aligned} \tag{B.1}$$

Note that $l_1 = |L_1| > 0$ and $\chi_{L_1,L\cup L_0} = \chi_{L_1,L_1\cup L_0}$, because any vertex in L_2 is not connected. Iterating once,

$$\begin{aligned}\bar{\chi}_{L,L\cup L_0} = 1 - \sum_{\substack{L_1\subset L\\ l_1\geq 1}} \chi_{L_1,L_1\cup L_0}\\ + \sum_{\substack{L_1,L_2,L_3\\ L_1\cup L_2\cup L_3=L\\ L_i\cap L_j=\emptyset, i\neq j\\ l_1\geq 1, l_2\geq 1}} \chi_{L_1,L_0\cup L_1}\chi_{L_2,L_0\cup L_1\cup L_2}\bar{\chi}_{L_3,L_0\cup L_1\cup L_2\cup L_3}.\end{aligned} \tag{B.2}$$

Then, successive iterations yield the following expansion:

$$\bar{\chi}_{L,L\cup L_0} = \sum_{r=0}^{|L|}(-1)^r \sum_{\substack{L_1,\dots,L_r\\ \cup_i L_i\subset L\\ l_i\geq 1\\ L_i\cap L_j=\emptyset, i\neq j}} \chi_{L_1,L_0\cup L_1}\cdots\chi_{L_r,L_0\cup L_1\cdots\cup L_r}, \tag{B.3}$$

where the $r = 0$ term has to be interpreted as 1. I.e.

$$\bar{\chi}_{L,L\cup L_0} = \sum_{Q\subset L} R(Q, L_0), \tag{B.4}$$

with

$$R(Q, L_0) := \sum_{r=1}^{q} (-1)^r \sum_{\substack{L_1,\ldots,L_r \\ \cup_i L_i = Q \\ l_i \geq 1 \\ L_i \cap L_j = \emptyset, i \neq j}} \chi_{L_1, L_0 \cup L_1} \cdots \chi_{L_r, L_0 \cup L_1 \cdots \cup L_r}, \tag{B.5}$$

and $R(\emptyset, L_0) = 1$.

From this expression it follows that

$$\begin{aligned} |R(Q, L_0)| &\leq \sum_{r=1}^{q} \sum_{\substack{L_1,\ldots,L_r \\ \cup_i L_i = Q \\ l_i \geq 1 \\ L_i \cap L_j = \emptyset, i \neq j}} \chi_{Q, Q \cup L_0} \\ &\leq \chi_{Q, Q \cup L_0} \sum_{r=1}^{q} \sum_{\substack{l_1,\ldots,l_r \\ l_i \geq 1}} \frac{q!}{l_1! \cdots l_r!} \\ &\leq \chi_{Q, Q \cup L_0}\, q!\, C^q. \end{aligned} \tag{B.6}$$

□

Appendix C: Reduction to energy functionals

In this appendix we prove the technical result stated in Sect. 4.3.3. We divide the proof in three parts where we truncate respectively number of particles, energy, and cross-sections. The truncation errors are controlled by slight variants of Lanford's short time estimate.

Proof of Lemma 4.5 (a) We first use the bound (4.23) and the assumptions on the initial state (see (4.28)) to estimate E_K as given by (4.27). Notice that (4.28) can be applied for $k + n < \varepsilon^{-\alpha_0}$, which is ensured by $k < \varepsilon^{-\alpha}$, $n \leq \log \varepsilon^{-\theta_2 k}$ for arbitrary positive θ_2 and $\alpha < \alpha_0$, as soon as ε is small enough. We deduce:

$$\begin{aligned} \int d\mathbf{v}_K |E_K(t)| &\leq z^k C^k \sum_{\substack{L_0, Q, B \\ \subset K \\ \text{disjoint}}} q!\, b! \sum_{n=0}^{\log \varepsilon^{-\theta_2 k}} z^n \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda \\ &\times \prod |B^\varepsilon| \chi^{rec}_{L_0}\, \chi^{ov}_{Q,K}\, \chi^0_{B,K}\, \varepsilon^{\gamma_0 (k-q-l_0-b)} e^{-(\beta/2)\mathcal{H}_K} \\ &+ z^k C^k \sum_{\substack{L_0, Q \\ \subset K \\ \text{disjoint}}} q!\,(k-q-l_0)! \sum_{n > \log \varepsilon^{-\theta_2 k}} z^n \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon|\, e^{-(\beta/2)\mathcal{H}_K}. \end{aligned} \tag{C.1}$$

The symbol C is always used for pure positive constants. Note that, in the error produced by the truncation on n, the last line of (4.27) has been estimated simply by $z^{k+n} C^k e^{-(\beta/2)\mathcal{H}_K}(k-q-l_0)!$, as follows from (2.32), (A.17) and Hypothesis 2.1.

Proceeding exactly as in the proof of Lemma 4.2 (case $a=1$), the last term in (C.1) is bounded, for $t<\bar{t}$ (see (4.8)), by

$$
\begin{aligned}
& C^k\, k!\, (4\pi/\beta)^{\frac{3}{2}k}\, (C(z,\beta)e)^k \sum_{n>\log\varepsilon^{-\theta_2 k}} (\bar{t}\, C(z,\beta)e)^n \\
& \quad \le (C')^k\, k^k\, \varepsilon^{\theta_2 \log(\bar{t}C(z,\beta)e)^{-1} k} \\
& \quad \le (C')^k\, \varepsilon^{\theta_2 \log(\bar{t}C(z,\beta)e)^{-1} k - \alpha k} \le \varepsilon^{\gamma k}/4,
\end{aligned}
\tag{C.2}
$$

for a suitable $C' = C'(z,\beta) > 0$. In the last line we used $k < \varepsilon^{-\alpha}$,

$$
\gamma < \theta_2 \log(\bar{t}C(z,\beta)e)^{-1} - \alpha \tag{C.3}
$$

and ε small enough.

Since $\chi^{ov}_{Q,K}\, \chi^0_{B,K} \le \chi^{ov}_{Q\cup B,K}$ (overlap at time zero implies overlap in $[0,t]$), renaming $Q\cup B \to Q$, (C.1) yields

$$
\begin{aligned}
& \int d\mathbf{v}_K |E_K(t)| \\
& \quad \le z^k C^k k! \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \sum_{n=0}^{\log\varepsilon^{-\theta_2 k}} z^n \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda \\
& \qquad \times \prod |B^\varepsilon| \chi^{rec}_{L_0} \chi^{ov}_{Q,K} \varepsilon^{\gamma_0(k-q-l_0)} e^{-(\beta/2)\mathcal{H}_K} + \frac{\varepsilon^{\gamma k}}{4}.
\end{aligned}
\tag{C.4}
$$

(b) Next we truncate the integration domain to the sphere of energy smaller than $2\varepsilon^{-\theta_3}$, for arbitrary $\theta_3 > 0$. The corresponding error is bounded by

$$
\begin{aligned}
& C^k\, k! \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \sum_{n=0}^{\log\varepsilon^{-\theta_2 k}} z^{k+n} \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon|\, e^{-(\beta/2)\mathcal{H}_K}\, \mathbb{1}_{\mathcal{H}_K > \varepsilon^{-\theta_3}} \\
& \quad \le e^{-(\beta/4)\varepsilon^{-\theta_3}}\, C^k\, k! \sum_{\substack{L_0,Q \\ \subset K \\ \text{disjoint}}} \sum_{n=0}^{\log\varepsilon^{-\theta_2 k}} z^{k+n} \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda \prod |B^\varepsilon|\, e^{-(\beta/4)\mathcal{H}_K}
\end{aligned}
$$

$$\leq e^{-(\beta/4)\varepsilon^{-\theta_3}} C^k\, k!\, 4^k\, (8\pi/\beta)^{3k/2}\, (eC(z,\beta/2))^k \sum_{n=0}^{\log\varepsilon^{-\theta_2 k}} (eC(z,\beta/2)\bar{t})^n$$
$$\leq (C'')^k\, e^{-(\beta/4)\varepsilon^{-\theta_3}}\, k^k\, (C'')^{\log\varepsilon^{-\theta_2 k}} \tag{C.5}$$

for a suitable $C'' = C''(z,\beta) > 1$. From second to third line we repeated the proof of Lemma 4.2 with $a = 1$ and $\beta \to \beta/2$. Note that (C.5) is in turn bounded, for $k < \varepsilon^{-\alpha}$, by $(C'')^k\, e^{-(\beta/4)\varepsilon^{-\theta_3}+\varepsilon^{-\alpha}\log\varepsilon^{-\alpha}+\varepsilon^{-\alpha}\log\varepsilon^{-\theta_2}\log C''}$, which is smaller than $\varepsilon^{\gamma k}/4$ if $\theta_3 > \alpha$ and ε is small enough.

Remembering (4.29), it follows that

$$\int d\mathbf{v}_K |E_K(t)|$$
$$\leq z^k C^k k! \sum_{\substack{L_0, Q \\ \subset K \\ \text{disjoint}}} \sum_{n=0}^{\log\varepsilon^{-\theta_2 k}} z^n \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda$$
$$\times \prod |B^\varepsilon| \chi^{rec}_{L_0} \chi^{ov}_{Q,K} \varepsilon^{\gamma_0(k-q-l_0)} F_{\theta_3}(K) + \frac{2\varepsilon^{\gamma k}}{4}. \tag{C.6}$$

(c) Finally, we introduce a truncation of the cross-section factors $\prod |B^\varepsilon|$. We want actually to eliminate these factors from (C.6). Such a simplification of formulas will be very useful for the recollision estimates. (This procedure was already applied in [30].)

To this purpose we apply the following corollary of Lanford's short time estimate, Lemma 4.2, of which we adopt here the notation.

Lemma C.1 *Let $F \leq 1$ be any positive measurable function of the variables $\mathbf{z}_j, \mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}$. Let $N > 0$ and $\theta_1 > 0$. There exists $\bar{C}' > 0$ such that, for any $t < \bar{t}$,*

$$\int d\mathbf{v}_j \sum_{n=0}^{N} z^{j+n} \sum_{\Gamma(j,n)} \int d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n}) \left(\prod |B^\varepsilon|\right) e^{-(\beta/2)\sum_{i\in S(J)}(\eta_i^\varepsilon(0))^2} F$$
$$\leq (\bar{C}')^j \varepsilon^{\theta_1 j} + \varepsilon^{-\theta_1 j} \sum_{n=0}^{N} z^{j+n} \sum_{\Gamma(j,n)} \int d\mathbf{v}_j d\Lambda(\mathbf{t}_n, \boldsymbol{\omega}_n, \mathbf{v}_{j,n})$$
$$\times e^{-(\beta/2)\sum_{i\in S(J)}(\eta_i^\varepsilon(0))^2} \prod_{i=1}^{n} \mathbb{1}_{\{|\xi^\varepsilon_{j+i}(t_i)-\xi^\varepsilon_k(t_i)|>\varepsilon\ \forall k\neq k_i\}} F. \tag{C.7}$$

The result holds also when $B^\varepsilon, \boldsymbol{\zeta}^\varepsilon$ are replaced by $B^{\mathscr{E}}, \boldsymbol{\zeta}^{\mathscr{E}}$ (Enskog flow) or $B, \boldsymbol{\zeta}$ (Boltzmann flow).

Note that the last integral does not contain any more cross-section factors, the only residual being the characteristic function that prohibits overlaps at creation times.

To deduce the corollary, it is enough to observe that the integral on the l.h.s., when restricted to the set such that $\prod |B^\varepsilon| > \varepsilon^{-\theta_1 j}$, is bounded by $\varepsilon^{\theta_1 j}$ times the integral with respect to $d\mathbf{v}_j$ of the left hand side in (4.1) with $a = 2$. Applying Lemma 4.2, we obtain the result by taking $\bar{C}' = \bar{C}(4\pi/\beta)^{3/2}$.

Computing the l.h.s. in Lemma C.1 via the mixed flow (4.18) instead of the IBF causes, of course, no modification, except for the expression of the characteristic function in (C.7). Therefore we may apply the result to (C.6), which produces an error $C^k k!(\bar{C}')^k \varepsilon^{\theta_1 k} \le (C''')^k k^k \varepsilon^{\theta_1 k}$ for a suitable $C''' = C'''(z, \beta) > 0$ and arbitrary $\theta_1 > 0$. This is, in turn, smaller than $\varepsilon^{\gamma k}/4$ for $k < \varepsilon^{-\alpha}$, $\gamma < \theta_1 - \alpha$ and ε small enough.

We conclude that, for any $t < \bar{t}$,

$$\int d\mathbf{v}_K |E_K(t)| \le \frac{3\varepsilon^{\gamma k}}{4} + z^k C^k k!\, \varepsilon^{-\theta_1 k} \sum_{\substack{L_0, Q \\ \subset K \\ \text{disjoint}}} \sum_{n=0}^{\log \varepsilon^{-\theta_2 k}} z^n \sum_{\Gamma(k,n)} \int d\mathbf{v}_k d\Lambda\, \mathbb{1}_{L_0} \times \tilde{\mathbb{1}}_{K \setminus L_0}\, \chi^{rec}_{L_0} \chi^{ov}_{Q,K} \varepsilon^{\gamma_0 (k-q-l_0)} F_{\theta_3}(K), \tag{C.8}$$

where the characteristic functions $\mathbb{1}$ are those defined after (4.19). □

Remark (Choice of parameters) If we choose the parameters as in (4.32), then (4.34) ensures that all the conditions in the proof above are satisfied. In fact, in part (a) of the proof we just need to check (C.3) which reads $\gamma < (1/2) - \alpha$ and follows from $\gamma < a(\gamma_0) - 3\alpha < 1/4 - 3\alpha$. In part (b), the condition $\alpha < \theta_3 = 1/5$ follows from $\alpha < (1/3)a(\gamma_0) < 1/12$. Finally in part (c) the condition $\gamma < \theta_1 - \alpha = a(\gamma_0) - \alpha$ is guaranteed by $\gamma < a(\gamma_0) - 3\alpha$.

Appendix D: Estimate of internal recollisions

Proof of Lemma 4.12 It is convenient to use the Enskog backwards flow $\boldsymbol{\zeta}^{\mathscr{E}}$ introduced in Sect. 3.5.2. For any given value of the variables $(x_1, \Gamma(1, n_1), \mathbf{v}_{n_1+1}, \boldsymbol{\omega}_{n_1}, \mathbf{t}_{n_1})$, if the IBF $\boldsymbol{\zeta}^{\varepsilon}$ delivers an internal recollision, then the EBF $\boldsymbol{\zeta}^{\mathscr{E}}$ delivers an internal overlap (two particles of the flow having a distance smaller than ε). That is,

$$\chi^{int} \le \chi^{i.o.}(\boldsymbol{\zeta}^{\mathscr{E}}), \tag{D.1}$$

where

$$\chi^{i.o.} = \chi^{i.o.}(\boldsymbol{\zeta}^{\mathscr{E}}) = 1 \tag{D.2}$$

if and only if the EBF associated to the 1-particle tree exhibits at least one overlap between two particles. Therefore in what follows we shall focus on the proof of the estimate

$$\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda \, \chi^{i.o.} \, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \leq \frac{\varepsilon^{\gamma_1}}{2}(Dt)^{n_1}. \tag{D.3}$$

Remind that $d\Lambda = d\Lambda(\mathbf{t}_{n_1}, \boldsymbol{\omega}_{n_1}, \mathbf{v}_{1,1+n_1})$ and $\Gamma(1,n_1) = (k_1,\dots,k_{n_1})$.

We start with

$$\chi^{i.o.} \leq \sum_{s=2}^{n_1} \sum_{h=k_s,s+1} \sum_{\substack{i=1,\dots,s \\ i\neq k_s,s+1}} \chi^{i.o.}_{(i,h),s}, \tag{D.4}$$

where $\chi^{i.o.}_{(i,h),s} = \chi^{i.o.}_{(i,h),s}(\boldsymbol{\zeta}^{\mathscr{E}}) = 1$ if and only if:

(i) going backwards in time, the first overlap between particles i and h takes place at a time $\tau \in (0, t_s]$;

(ii) particles i and h move freely in (τ, t_s);

(iii) at time t_s

$$\eta_h^{\mathscr{E}}(t_s^-) \neq \eta_{k_s}^{\mathscr{E}}(t_s^+). \tag{D.5}$$

Notice that particle h is involved in the creation process at time t_s. See Fig. 8 below for a scheme of the possible situations and observe that, by virtue of (iii), we are excluding case 2 for incoming collision configurations at the creation time t_s.

From (D.3) to (D.4) one gets

$$\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda \, \chi^{i.o.} \, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}$$
$$\leq \sum_{\Gamma(1,n_1)} \sum_{s=2}^{n_1} \sum_{h=k_s,s+1} \sum_{\substack{i=1,\dots,s \\ i\neq k_s}} \int dv_1 d\Lambda \, \chi^{i.o.}_{(i,h),s} \, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2}. \tag{D.6}$$

Note that $\chi^{i.o.}_{(i,h),s}$ depends actually only on $\boldsymbol{\zeta}^{\mathscr{E}}_{1+s}$, hence we can immediately integrate out the node variables

$$t_{s+1},\dots,t_{n_1},\omega_{s+1}\dots,\omega_{n_1},v_{s+2},\dots,v_{1+n_1}$$

and sum over the tree variables $k_{s+1},\dots,k_{n_1}$. Applying (4.7),

$$\sum_{k_{s+1},\dots,k_{n_1}} \int d\mathbf{t}_{s,n_1-s} = (s+1)(s+2)\cdots(n_1)t^{n_1-s}/(n_1-s)! \leq e^{n_1}t^{n_1-s},$$

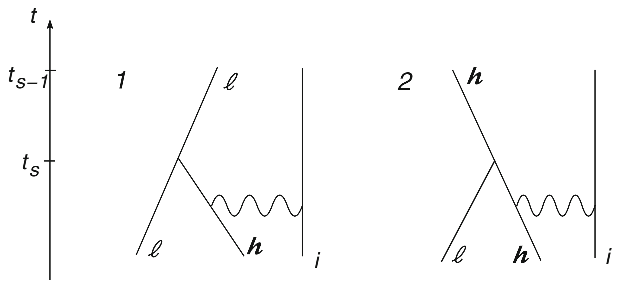

Fig. 8 Case 1: $h = s+1$, $k_s = \ell$. Case 2: $h = k_s$, $\ell = s+1$

thus we infer that

$$\sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi^{i.o.}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} \leq e^{n_1} \sum_{s=2}^{n_1} (D't)^{n_1-s} \sum_{\Gamma(1,s)} \sum_{h=k_s,s+1}$$
$$\times \sum_{\substack{i=1,\dots,s \\ i\neq k_s}} \cdot \int dv_1 d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\, \chi^{i.o.}_{(i,h),s}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2}, \tag{D.7}$$

where $D' = 4\pi\,(2\pi/\beta)^{3/2}$ and, in the last line, we are left with integrals associated to 1-particle, s-collision trees.

If $\chi^{i.o.}_{(i,h),s} = 1$, then there are two possibilities: either $h = s+1$ (h is created at t_s) or $k_s = h$ (h is the progenitor of $s+1$), see Fig. 8. Let us resort to the notation of virtual trajectories, to deal with both cases simultaneously (Definition 4.11, applied to $\bar{\boldsymbol{\zeta}} = \boldsymbol{\zeta}^{\mathscr{E}}$). We set

$$W = \eta_h^{\mathscr{E}}(t_s^-) - \eta_i^{\mathscr{E}}(t_s), \quad W_0 = \eta^{\mathscr{E},h}(t_s^+) - \eta_i^{\mathscr{E}}(t_s)$$

and

$$Y = \xi_h^{\mathscr{E}}(t_s^-) - \xi_i^{\mathscr{E}}(t_s), \quad Y_0 = \xi^{\mathscr{E},h}(t_{s-1}^-) - \xi_i^{\mathscr{E}}(t_{s-1}).$$

We remind that t^+, t^- denote the limit from the future (post-collision) or from the past (pre-collision) respectively. Note that (D.5) is, in this notation,

$$\eta^{\mathscr{E},h}(t_s^-) \neq \eta^{\mathscr{E},h}(t_s^+), \tag{D.8}$$

namely the virtual trajectory of particle h changes velocity at time t_s.

The overlap-condition implies

$$\inf_{\tau\in(0,t_s)} |Y - W\tau| \leq \varepsilon. \tag{D.9}$$

Put $\hat{W} = \frac{W}{|W|}$ if $W \neq 0$ and $W = (1, 0, 0)$ otherwise. Eq. (D.9) implies in turn

$$|Y \wedge \hat{W}| \leq \varepsilon,$$

i.e.

$$|(Y_0 - W_0 t_{s-1}) \wedge \hat{W} + (W_0 \wedge \hat{W}) t_s| \leq 2\,\varepsilon, \tag{D.10}$$

where the factor 2 takes into account the jump in position in the virtual trajectory of particle h at time t_s, case 1. Therefore, we may bound the last line in (D.7) by replacing $\chi^{i.o.}_{(i,h),s}$ with the indicator function of the events (D.10) and $W \neq W_0$ (which takes into account (D.8)).

By definition of the Enskog flow, Y_0 and W_0 do not depend on t_s (since they concern the previous history). Moreover, the velocities in $(0, t_s)$, which we denote

$$(\eta_1^-, \ldots, \eta_{s+1}^-) = (\eta_1^{\mathscr{E}}(t_s^-), \ldots, \eta_{s+1}^{\mathscr{E}}(t_s^-)), \tag{D.11}$$

are also independent of the times $t_1, \ldots, t_s$: they depend only on previous velocities and impact vectors. In particular, W does not depend on t_s, so that in (D.10) a *linear* relation in t_s appears. On the other hand, the integral in t_s over the condition (D.10) is bounded by $\min(t, 4\varepsilon|W_0 \wedge \hat{W}|^{-1})$. Hence, for an arbitrary $\gamma_1 \in (0, 1)$,

$$\begin{aligned}
&\sum_{\Gamma(1,s)} \sum_{h=k_s,s+1} \sum_{\substack{i=1,\ldots,s \\ i \neq k_s}} \int dv_1 d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\ \chi^{i.o.}_{(i,h),s}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2} \\
&\quad \leq (4/t)^{\gamma_1} t\, \varepsilon^{\gamma_1} \sum_{\Gamma(1,s)} \sum_{h=k_s,s+1} \sum_{\substack{i=1,\ldots,s \\ i \neq k_s}} \\
&\qquad \times \int dv_1 d\Lambda'(\mathbf{t}_{s-1}, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})\, \frac{1}{|W_0 \wedge \hat{W}|^{\gamma_1}}\, e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2},
\end{aligned} \tag{D.12}$$

where $d\Lambda'(\mathbf{t}_{s-1}, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})$ is the measure $d\Lambda(\mathbf{t}_s, \boldsymbol{\omega}_s, \mathbf{v}_{1,1+s})$ deprived of dt_s and multiplied, in case 2 of Fig. 8, by the characteristic function of $\omega_s \cdot (v_{1+s} - \eta^{\mathscr{E},h}(t_s^+)) > 0$ (coming from the condition $W \neq W_0$).

It remains to prove that the integral of the singular function $|W_0 \wedge \hat{W}|^{-\gamma_1}$ converges. To do so, let us first express W_0 in terms of the pre-collisional variables (D.11). Applying the elastic collision rule (2.3), one finds

$$\begin{aligned}
W_0 &= \left(\eta^{\mathscr{E},h}(t_s^+) - \eta^{\mathscr{E},h}(t_s^-)\right) + W \\
&= P_s W_\ell + W,
\end{aligned}$$

where

$$W_\ell = \eta_\ell^- - \eta_h^-$$

and

$$P_s X := \begin{cases} P^\perp_{\omega_s} X := X - \omega_s(\omega_s \cdot X) & \text{case1, outgoingcollision} \\ X & \text{case1, incomingcollision} \\ P^\parallel_{\omega_s} X := \omega_s(\omega_s \cdot X) & \text{case2, outgoingcollision} \\ 0 & \text{case2, incomingcollision} \end{cases}. \tag{D.13}$$

Cases 1, 2 are those in Fig. 8, while we remind that the incoming/outgoing collisions are depicted in Fig. 2 (here corresponding respectively to the negative / positive sign of the scalar product $\omega_s \cdot (v_{1+s} - \eta^{\mathscr{E},h}(t_s^+))$). Moreover, the "case" depends only on the structure of the chosen tree $\Gamma(1, s)$. It follows that

$$\frac{1}{|W_0 \wedge \hat{W}|} = \frac{1}{|P_s W_\ell \wedge \hat{W}|} \tag{D.14}$$

which we may insert into (D.12).

Next, we change variables according to $v_1, v_2, \ldots, v_{s+1} \to \eta_1^-, \ldots, \eta_{s+1}^-$. This is an invertible and measure-preserving transformation, for any fixed value of $\omega_1, \ldots, \omega_s$, (since the single hard-sphere collision (2.3) is so). Moreover, by the conservation of energy at collisions, $e^{-(\beta/2)\sum_{i=1}^{1+s} v_i^2} = e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}$. From (D.7), (D.12) and (D.14), we thus obtain

$$\begin{aligned} \sum_{\Gamma(1,n_1)} \int dv_1 d\Lambda\, \chi^{i.o.}\, e^{-(\beta/2)\sum_{i\in S(1)} v_i^2} &\le e^{n_1} \sum_{s=2}^{n_1} (D't)^{n_1-s}\, (4/t)^{\gamma_1} t\, \varepsilon^{\gamma_1}\, s!\, 2s \\ &\times \frac{t^{s-1}}{(s-1)!} \int d\boldsymbol{\omega}_s \int d\boldsymbol{\eta}_{s+1}^- \\ &\times \left(\frac{e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}}{|P^\perp_{\omega_s} W_1 \wedge \hat{W}|^{\gamma_1}} + \frac{e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}}{|W_1 \wedge \hat{W}|^{\gamma_1}} + \frac{e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}}{|P^\parallel_{\omega_s} W_1 \wedge \hat{W}|^{\gamma_1}} \right), \end{aligned} \tag{D.15}$$

where we renamed 1, 2, 3 particles ℓ, h, i respectively (hence $W_1 = \eta_1^- - \eta_2^-$, $W = \eta_2^- - \eta_3^-$).

Let us now give a bound of the explicit integral $\int d\boldsymbol{\eta}_{s+1}^- \frac{e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}}{|\tilde{P}_s W_1 \wedge \hat{W}|^{\gamma_1}}$, where $\tilde{P}_s W_1 = P^\perp_{\omega_s} W_1$, W_1 or $P^\parallel_{\omega_s} W_1$. Since $W^2 + W_1^2 \le 2(\eta_1^-)^2 + 4(\eta_2^-)^2 + 2(\eta_3^-)^2$, applying the translations $(\eta_1^-, \eta_2^-) \to (W_1 = \eta_1^- - \eta_2^-, W = \eta_2^- - \eta_3^-)$, we find

$$
\begin{aligned}
\int d\boldsymbol{\eta}_{s+1}^{-} \frac{e^{-(\beta/2)\sum_{i=1}^{1+s}(\eta_i^-)^2}}{|\tilde{P}_s W_1 \wedge \hat{W}|^{\gamma_1}} &\leq \int d\boldsymbol{\eta}_{s+1}^{-} e^{-(\beta/2)\sum_{i>3}(\eta_i^-)^2} \\
&\quad \times e^{-(\beta/4)(\eta_3^-)^2} \frac{e^{-(\beta/8)(W_1^2+W^2)}}{|\tilde{P}_s W_1 \wedge \hat{W}|^{\gamma_1}} \\
&= \int d\boldsymbol{\eta}_{2,s-1}^{-} e^{-(\beta/2)\sum_{i>3}(\eta_i^-)^2} e^{-(\beta/4)(\eta_3^-)^2} \int dW_1 dW \frac{e^{-(\beta/8)(W_1^2+W^2)}}{|\tilde{P}_s W_1 \wedge \hat{W}|^{\gamma_1}} \\
&\leq C_\beta^s \int dW_1 \frac{e^{-(\beta/8)W_1^2}}{|\tilde{P}_s W_1|^{\gamma_1}} \\
&\leq C_\beta^s C_{\beta,\gamma_1},
\end{aligned}
\tag{D.16}
$$

for suitable constants $C_\beta, C_{\beta,\gamma_1} > 0$ and for any $\gamma_1 < 1$ (with C_{β,γ_1} diverging in the case $\tilde{P}_s W_1 = P_{\omega_s}^{\parallel} W_1$ as $\gamma_1 \to 1$).

Inserting (D.16) into (D.15) and performing the sums, we obtain the final result. □

Appendix E: Proof of Corollary 4.14

The result follows from minor modifications in the proof of Theorem 2.4.

First of all, by Property 2, case $S = \mathcal{J} = J$, applied to the state with r.c.f. f_j^ε and correlation errors $\bar{E}_k \equiv E_k$,

$$
\begin{aligned}
|E_K| &\leq \sum_{H\subset K} \left(C^h\, h!\, \chi_{H,K}^0 \, \left(f_1^\varepsilon\right)^{\otimes H}\right) \left(\bar{\chi}_{K\setminus H}^0 |E_{K\setminus H}|\right) \\
&= \sum_{H\subset K\setminus Q'} \left(C^h\, h!\, \chi_{H,K}^0 \, \left(f_1^\varepsilon\right)^{\otimes H}\right) \left(\bar{\chi}_{K\setminus(Q'\cup H)}^0 |E_{K\setminus H}|\right).
\end{aligned}
\tag{E.1}
$$

Remind that $\chi_{H,K}^0 = 1$ if and only if any particle with index in H overlaps with a different particle in K, which implies $H \subset K\setminus Q'$. Moreover, $\bar{\chi}_{K\setminus H}^0 = 1$ if and only if all particles in $K\setminus H$ do not overlap among themselves, which implies $\bar{\chi}_{K\setminus H}^0 = \bar{\chi}_{K\setminus(Q'\cup H)}^0$. In particular,

$$
\begin{aligned}
\int_{\mathbb{R}^{3q'}} d\mathbf{v}_{Q'} |E_K(t)| &\leq \sum_{H\subset K\setminus Q'} \left(C^h\, h!\, \chi_{H,K}^0 \, \left(f_1^\varepsilon(t)\right)^{\otimes H}\right) \bar{\chi}_{K\setminus(Q'\cup H)}^0 \\
&\quad \times \int_{\mathbb{R}^{3q'}} d\mathbf{v}_{Q'} |E_{K\setminus H}(t)|
\end{aligned}
\tag{E.2}
$$

and we are allowed to insert expression (4.27) into (E.2). Note that this preparation is necessary, since (4.27) does apply only when the particles in $K \setminus H$ are sufficiently far from each other.

The estimate of the integral on the r.h.s. differs from that of the main theorem from the fact that we integrate only with respect to a subset of velocities $Q' \subset K$. Furthermore, we know that particles in Q' are at distance larger than δ from any other particle in K, but we have no information on the relative distance of particles in $K \setminus Q'$. We shall not repeat here the proof of Sects. 4.3–4.5, which applies unchanged, except for the following modifications.

1. In Lemma 4.5, one integrates only over $d\mathbf{v}_{Q'}$. However the integral over velocities is never used in the reduction to energy functionals (see Appendix C). Therefore one gets the same result apart from an overall $(const.)^k$. This produces the first term in (4.128).
2. In Proposition 4.6, one integrates only over $d\mathbf{v}_{Q'}$ and $\varepsilon^{\gamma_1 \frac{q+l_0}{2}} = \varepsilon^{\gamma_1 \frac{|Q \cup L_0|}{2}}$ has to be replaced by $\varepsilon^{\gamma_1 \frac{|(Q \cup L_0) \cap Q'|}{2}}$. Indeed in the proof of the proposition, Sect. 4.4.2.c, when the bullet α_i is outside Q', Lemma 4.10 cannot be applied. Instead of estimate (4.51), one uses then the simple estimate
$$\sum_{\Gamma_{\alpha_i}} \int d\Lambda_{\alpha_i}\, \chi^{(\alpha_i,\beta_i)}\, F_{\theta_3}(\alpha_i) \le (D't)^{n_{\alpha_i}}. \tag{E.3}$$
3. In (4.36), one integrates only over $d\mathbf{v}_{Q'}$ and, by virtue of the previous two points, one gets $\varepsilon^{\min[\gamma_0,\gamma_1/2]q'}$ instead of $\varepsilon^{\min[\gamma_0,\gamma_1/2]k}$. This produces the second term in (4.128). □

References

1. Alexander, R.K.: The infinite hard sphere system. Ph.D.Thesis, Dep. of Mathematics, University of California at Berkeley (1975)
2. Aoki, K., Pulvirenti, M., Simonella, S., Tsuji, T.: Backward clusters, hierarchy and wild sums for a hard sphere system in a low-density regime. Math. Models Methods Appl. Sci. **25**(5), 995–1010 (2015)
3. van Beijeren, H., Lanford III, O.E., Lebowitz, J.L., Spohn, H.: Equilibrium time correlation functions in the low-density limit. J. Stat. Phys. **22**, 2 (1980)
4. Bodineau, T., Gallagher, I., Saint-Raymond, L.: The Brownian motion as the limit of a deterministic system of hard-spheres. Invent. Math. **203**(2), 493–553 (2016)
5. Bodineau, T., Gallagher, I., Saint-Raymond, L.: From hard sphere dynamics to the Stokes–Fourier equations: an L^2 analysis of the Boltzmann–Grad limit (2015) arXiv:1511.03057v2
6. Bogolyubov, N.N.: Studies in Statistical Mechanics. In: de Boer, J., Uhlenbeck, G.E. (eds.) Studies in Statistical Mechanics. Interscience, New York (1962)
7. Bogolyubov, N.N.: Microscopic solutions of the Boltzmann–Enskog equation in kinetic theory for elastic balls. Theor. Math. Phys. **24**, 804–807 (1975)

8. Boltzmann, L.: Lectures on gas theory. English edition annotated by S. Brush, University of California Press, Berkeley (1964) (reprint)
9. Caprino, S., De Masi, A., Presutti, E., Pulvirenti, M.: A derivation of the Broadwell equation. Commun. Math. Phys. **135**(3), 443–465 (1991)
10. Caprino, S., Pulvirenti, M.: The Boltzmann–Grad limit for a one-dimensional Boltzmann equation in a stationary state. Commun. Math. Phys. **177**(1), 63–81 (1996)
11. Caprino, S., Pulvirenti, M., Wagner, W.: Stationary particle systems approximating stationary solutions to the Boltzmann equation. SIAM J. Math. Anal. **29**(4), 913–934 (1998)
12. Cercignani, C.: On the Boltzmann equation for rigid spheres. Transp Theory Stat. Phys. **2**, 211–225 (1972)
13. Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases. Applied Mathematical Sciences, vol. 106. Springer, New York (1994)
14. Di Perna, R., Lions, P.-L.: On the Cauchy problem for the Boltzmann equation: global existence and weak stability. Ann. Math. **2**(130), 312–366 (1989)
15. Esposito, R., Pulvirenti, M.: Rigorous validity of the Boltzmann equation for a thin layer of a rarefied gas. Kinet. Relat. Mod. **3**(2), 281–297 (2010)
16. Gallagher, I., Saint Raymond, L., Texier, B.: From Newton to Boltzmann: Hard Spheres and Short-Range Potentials. Zurich Lectures in Advanced Mathematics, vol. 18. EMS (2014)
17. Gerasimenko, V.I., Gapyak, I.V.: Hard sphere dynamics and the Enskog equation. Kinet. Relat. Models **5**(3), 459–484 (2012)
18. Grad, H.: On the kinetic theory of rarefied gases. Commun. Pure Appl. Math. **2**(4), 331–407 (1949)
19. Grad, H.: Principles of the kinetic theory of gases. In: Flügge, S. (ed.) Handbuch der Physik vol. 12, pp. 205–294. Springer (1958)
20. Illner, R., Pulvirenti, M.: Global validity of the Boltzmann equation for a two-dimensional rare gas in the vacuum. Commun. Math. Phys. **105**, 189–203 (1986)
21. Illner, R., Pulvirenti, M.: Global validity of the Boltzmann equation for a two- and three-dimensional rare gas in vacuum: erratum and improved result. Commun. Math. Phys. **121**, 143–146 (1989)
22. Illner, R., Pulvirenti, M.: A derivation of the BBGKY-hierarchy for hard sphere particle systems. Transp Theory Stat. Phys. **16**, 997–1012 (1987)
23. King, F.: BBGKY hierarchy for positive potentials. Ph.D. Thesis, Department of Mathematics, Univ. California, Berkeley (1975)
24. Kotecký, R.: Cluster Expansions. Encyclopedia of Mathematical Physics, vol. 1. Elsevier, Oxford (2006)
25. Lanford, O.E.: Time evolution of large classical systems. In: Moser, J. (ed.) Dynamical Systems, Theory and Applications. Lecture Notes in Physics, vol. 38, pp. 1–111. Springer, Berlin (1975)
26. Lukkarinen, J., Marcozzi, M.: Wick polynomials and time-evolution of cumulants. J. Math. Phys. **57**, 083301 (2016). doi:10.1063/1.4960556
27. Lukkarinen, J., Marcozzi, M., Nota, A.: Summability of joint cumulants of nonindependent lattice fields (2016) arXiv:1601.08163
28. Marchioro, C., Pellegrinotti, A., Presutti, E., Pulvirenti, M.: On the dynamics of particles in a bounded region: a measure theoretical approach. J. Math. Phys. **17**, 647 (1976)
29. Mouhot, C., Villani, C.: Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. Arch. Ration. Mech. Anal. **173**(2), 35–43 (2004)
30. Pulvirenti, M., Saffirio, C., Simonella, S.: On the validity of the Boltzmann equation for short-range potentials. Rev. Math. Phys. **26**, 2 (2014)
31. Pulvirenti, M., Simonella, S.: On the evolution of the empirical measure for the hard-sphere dynamics. Bull. Inst. Math. Acad. Sin. **10**(2), 171–204 (2015)
32. Ruelle, D.: States of classical statistical mechanics. J. Math. Phys. **8**, 1657–1668 (1967)
33. Ruelle, D.: Statistical Mechanics. Rigorous Results. W. A. Benjamin Inc., New York (1969)

34. Simonella, S.: Evolution of correlation functions in the hard sphere dynamics. J. Stat. Phys. **155**(6), 1191–1221 (2014)
35. Spohn, H.: Fluctuations around the Boltzmann equation. J. Stat. Phys. **26**, 2 (1981)
36. Spohn, H.: Fluctuation theory for the Boltzmann equation. In: Lebowitz, J.L., Montroll, E.W. (eds.) Nonequilibrium Phenomena I: The Boltzmann Equation. North–Holland Pub. Co., Amsterdam (1983)
37. Spohn, H.: Boltzmann equation and Boltzmann hierarchy. In: Cercignani, C. (ed.) Kinetic Theories and the Boltzmann Equation. Lecture Notes in Mathematics, vol. 1048, pp. 207–220. Springer, Berlin (1984)
38. Spohn, H.: On the integrated form of the BBGKY hierarchy for hard spheres (1985). arXiv:math-ph/0605068v1
39. Spohn, H.: Large Scale Dynamics of Interacting Particles. Texts and Monographs in Physics. Springer, Heidelberg (1991)
40. Uchiyama, K.: Derivation of the Boltzmann equation from particle dynamics. Hiroshima Math. J. **18**, 245–297 (1988)
41. Ukai, S.: On the existence of global solutions of mixed problem for non-linear Boltzmann equation. Proc. Jpn. Acad. **50**, 179–184 (1974)
42. Ukai, S.: Les solutions globales de l'équation de Boltzmann dans lespace tout entier et dans le demi-espace. C.R. Acad. Sci. Paris **282**(6), 317–320 (1976)
43. Ukai, S.: The Boltzmann–Grad limit and Cauchy–Kovalevskaya theorem. Jpn. J. Ind. Appl. Math. **18**, 383–392 (2001)
44. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: Friedlander, S., Serre, D. (eds.) Handbook of Mathematical Fluid Dynamics, vol. I, pp. 71–305. North-Holland, Amsterdam (2002)