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BALANCED VISCOSITY SOLUTIONS TO A RATE-INDEPENDENT COUPLED ELASTO-PLASTIC DAMAGE SYSTEM*

VITO CRISMALE† AND RICCARDA ROSSI‡

Abstract. A rate-independent model coupling small-strain associative elasto-plasticity and damage is studied via a *vanishing-viscosity* analysis with respect to all the variables describing the system. This extends the analysis performed for the same system in [V. Crismale and G. Lazzaroni, *Calc. Var. Partial Differential Equations*, 55 (2016), 17], where a vanishing-viscosity regularization involving only the damage variable was set forth. In the present work, an additional approximation featuring vanishing plastic hardening is introduced in order to deal with the vanishing viscosity in the plastic variable. Different regimes are considered, leading to different notions of Balanced Viscosity solutions for the perfectly plastic damage system, and for its version with hardening.

Key words. rate-independent systems, variational models, vanishing viscosity, BV solutions, damage, elasto-plasticity

AMS subject classifications. 35A15, 34A60, 35Q74, 74C05

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1. Introduction.

In this paper we address the analysis of a rate-independent system coupling small-strain associative elasto-plasticity and damage. We construct weak solutions for the related initial-boundary value problem via a *vanishing-viscosity* regularization that affects all the variables describing the system. Before entering into the details of this procedure, let us briefly illustrate the rate-independent model we are interested in.

In a time interval $[0,T]$, for a bounded open Lipschitz domain $\Omega\subset\mathbb{R}^n$, $n\in\{2,3\}$, and time-dependent volume and surface forces f and g, we consider a PDE system coupling the evolution of the *displacement* $u:(0,T)\times\Omega\to\mathbb{R}^n$, of the *elastic* and *plastic strains* $e:(0,T)\times\Omega\to\mathbb{M}^{n\times n}_{\mathrm{sym}}$ and $p:(0,T)\times\Omega\to\mathbb{M}^{n\times n}_{\mathrm{D}}$, and of a *damage variable* $z:(0,T)\times\Omega\to[0,1]$ that assesses the soundness of the material: for $z(t,x)=1$ ($z(t,x)=0$, respectively) the material is in the undamaged (fully damaged, respectively) state, at the time $t\in(0,T)$ and "locally" around the point $x\in\Omega$. In fact, the PDE system consists of

- the momentum balance

$$-\operatorname{div}\sigma=f\quad\text{in }\Omega\times(0,T)\,,\qquad \sigma\mathrm{n}=g\text{ on }\Gamma_{\mathrm{Neu}}\times(0,T)\tag{1.1a}$$

(with Γ_{Neu} the Neumann part of the boundary $\partial\Omega$), where the *stress* tensor is given by

$$\sigma=\mathbb{C}(z)e\quad\text{in }\Omega\times(0,T),\tag{1.1b}$$

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†Dipartimento di Matematica, Università Roma-I "La Sapienza," 00185 Rome, Italy (vito.crismale@uniroma1.it).

‡DIMI, Università degli studi di Brescia, via Branze 38, 25133 Brescia, Italy (riccarda.rossi@unibs.it).

and the kinematic admissibility condition for the *strain* $\mathrm{E}(u) = \frac{\nabla u + \nabla u^T}{2}$ reads

$$\mathrm{E}(u) = e + p \quad \text{in } \Omega \times (0,T)\,; \tag{1.1c}$$

- the flow rule for the damage variable z

$$\partial \mathrm{R}(\dot z) + A_{\mathrm{m}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e \quad \text{in } \Omega \times (0,T), \tag{1.1d}$$

  where, above and in (1.1e), the symbol ∂ denotes the convex analysis subdifferential of the density of dissipation potential

$$\mathrm{R} : \mathbb{R} \to [0,+\infty] \;\; \text{defined by} \;\; \mathrm{R}(\eta) := \begin{cases} |\eta| & \text{if } \eta \leq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

  encompassing the unidirectionality in the evolution of damage, A_{m} is the m-Laplacian operator, with $\mathrm{m} > \frac{n}{2}$, and W is a suitable nonlinear, possibly nonsmooth, function;
- the flow rule for the plastic tensor

$$\partial_{\dot p} \mathrm{H}(z, \dot p) \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega \times (0,T), \tag{1.1e}$$

  with σ_{D} the deviatoric part of the stress tensor σ and $\mathrm{H}(z,\cdot)$ the density of plastic dissipation potential; $\mathrm{H}(z,\cdot)$ is the support function of the *constraint set* $K(z)$. The PDE system is supplemented with initial conditions and the boundary conditions

$$u = w \text{ on } \Gamma_{\mathrm{Dir}} \times (0,T), \qquad \partial_n z = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.1f}$$

  with Γ_{Dir} the Dirichlet part of the boundary $\partial\Omega$.

Let us highlight that the damage variable z influences both the Hooke tensor $\mathbb{C}$, which determines the elastic stiffness of the material, and the constraint set K for the deviatoric part of the stress, which is such that the material undergoes plastic deformations only if σ_{D} reaches the boundary ∂K. By our choice of the dissipation potential R, the variable z is forced to decrease in time: it is then usual to assume that $[0,1] \ni z \mapsto \mathbb{C}(z)$ is nonincreasing and that $[0,1] \ni z \mapsto K(z)$ is nondecreasing, with respect to the natural ordering for positive definite tensors and to the inclusion of sets (cf. section 2 for the precise assumptions).

The elasto-plastic damage model (1.1), which reduces to the Prandtl–Reuss model for perfect plasticity (cf., e.g., [DMDM06, Sol09, FG12, Sol14]) if no dependence on damage is assumed, was first proposed and studied in [AMV14, AMV15]. Subsequently, in [Cri16] (with refinements in [CO18]; see also [CO19]), the existence of Energetic solutions à la Mielke and Theil (cf. [MT99, MT04]) was proved. We recall that this weak solvability concept for rate-independent processes, also known as *quasistatic evolution* (cf., e.g., [DMT02]), consists of (i) a *global stability* condition, which prescribes that at each process time the current configuration minimizes the sum of the total internal energy and the dissipation potential; (ii) an energy-dissipation balance featuring the variation of the internal energy between the current and the initial times, the total dissipated energy, and the work of the external loadings. Thus, the energetic formulation is derivative-free and hence very flexible and suitable for limit passage procedures. In the framework of energetic-type solution concepts, the study of models coupling damage and plasticity indeed seems to have attracted some attention in recent years: in this respect, we may, e.g., quote [Cri17] for a damage model coupled with strain-gradient plasticity, as well as [BMR12, BRRT16, RT17, RV17] for plasticity with hardening, [RV16] accounting also for damage healing, [MSZ] for finite-strain plasticity with damage, and [DRS19] for perfect plasticity and damage in viscoelastic solids in a dynamical setting.

System (1.1), however, has been analyzed also from a perspective different from that of Energetic solutions. Indeed, despite their manifold advantages, Energetic solutions have a catch: when the energy functional driving the system is nonconvex, Energetic solutions as functions of time may have "too early" and "too long" jumps between energy wells (cf., e.g., [KMZ08, Example 6.3], [MRS09, Example 6.1]) and the full characterization of Energetic solutions to one-dimensional rate-independent systems from [RS13]. Essentially, this is due to the rigidity of the global stability condition that involves the global, rather than the local, energy landscape. These considerations have motivated the quest of alternative weak solvability notions for rate-independent systems. In this paper, we focus on notions obtained by a *vanishing-viscosity* approximation of the original rate-independent process.

The vanishing-viscosity approach stems from the idea that rate-independent processes originate in the limit of systems governed by two time scales: the inner scale of the system and the time scale of the external loadings. The latter scale is considerably slower than the former, but it is dominant, and from its viewpoint viscous dissipation is negligible. But viscosity is expected to reenter into the picture in the description of the system behavior at jumps, which should indeed be considered as *viscous transitions* between metastable states; cf. [EM06]. Thus, one selects those solutions to the original rate-independent system that arise as limits of solutions to the viscously regularized system. What is more, following an idea from [EM06], in order to capture the viscous transition path between two jump points one reparameterizes the viscous trajectories and performs the vanishing-viscosity analysis for curves in an extended phase space that also comprises the rescaling function. For this, it is crucial to control the length (or a "generalized length") of the viscous curves, uniformly w.r.t. the viscosity parameter. This limit procedure then leads to reparameterized solutions (functions of an "artificial" time variable $s \in [0, S]$) of the original rate-independent system, such that the reparameterized state variable(s) is (are) coupled with a rescaling function $\mathsf{t} : [0, S] \to [0, T]$ that takes values in the original time interval. In this way, equations for the paths connecting the left and right limits (stable states, themselves) of the system at a jump point may be derived; the (possibly viscous) path followed by the (reparameterized) limit solution at a jump point is also accounted for in a suitable energy-dissipation balance. Furthermore, the solution concept obtained by vanishing viscosity is supplemented by a first-order, *local* stability condition, which holds in the "artificial" time intervals corresponding to those in which the system does not jump in the original (fast) timescale.

Moving from the pioneering [EM06], in [MRS09, MRS12a, MRS16a] (cf. also [Neg14]) this idea has been formalized in an abstract setting, codifying the properties of these "vanishing-viscosity solutions" in the notion of *Balanced Viscosity* (hereafter often shortened as BV) solution to a rate-independent system. In parallel, the vanishing-viscosity technique has been developed and refined in various concrete applications, ranging from plasticity (cf., e.g., [DDS11, BFM12, FS13]), to damage, fracture, and fatigue (see, for instance, [KMZ08, LT11, KRZ13, Alm17, CL17, ACO19, ALL19]).

For the present elasto-plastic damage system (1.1), the vanishing-viscosity approach was first addressed in [CL16]. There, BV solutions to system (1.1) were constructed by passing to the limit in the viscously regularized system featuring viscosity *only in the flow rule for the damage variable* z. Namely, the momentum balance (1.1a) (with (1.1b) and (1.1c)) and the plastic flow rule were coupled with the *rate-dependent* subdifferential inclusion

$$\partial\mathrm{R}(\dot z)+\varepsilon\dot z+A_{\mathrm{m}}(z)+W'(z)\ni-\tfrac{1}{2}\mathbb{C}'(z)e:e\qquad\text{in }\Omega\times(0,T)$$

(with $0<\varepsilon\ll 1$), in place of (1.1d). Accordingly, the dissipation potential governing (1.1) was augmented by a viscosity contribution featuring the L^2-norm for the damage rate $\dot z$. Actually, in [CL16] the authors succeeded in deriving estimates (uniform w.r.t. the viscosity parameter ε) for the length of the viscous solutions $(z_\varepsilon)_\varepsilon$ in the H^{m}-norm, even ($H^{\mathrm{m}}(\Omega)$ being the reference space for the damage variable). Relying on these bounds and on the reparameterization procedure described above, they obtained a notion of BV solution such that only viscosity in z (possibly) enters in the description of the transition path followed by the system at jumps. Accordingly, this is reflected in the energy-dissipation balance satisfied by BV solutions.

Nonetheless, jumps in the other variables are not excluded during jumps for z, and the "reduced" vanishing-viscosity approach carried out in [CL16] does not provide information on the (possibly) viscous trajectories followed by those variables at jumps.

This has motivated us to develop a *"full" vanishing-viscosity approach* to system (1.1). Namely, we have approximated (1.1) by a viscously regularized system featuring a viscosity contribution for the plastic and the displacement variables, besides the damage variable and, correspondingly, obtained a notion of Balanced Viscosity solution for (1.1).

The "full" vanishing-viscosity approach. Upon viscously regularizing all variables u, z, and p, the scenario turns out to be more complicated than the one in [CL16] from an analytical point of view. The first challenge is related to the derivation of (uniform, w.r.t. the viscosity parameter) estimates for the length (in a suitable sense) of the viscous solutions. Quasistatic evolutions for perfect plasticity *without damage*, which are known to be Lipschitz in time, can be approximated by viscoplastic evolutions à la Perzyna [Per71] (where an L^2-viscous regularization for the plastic variable p is added), as detailed in [Sol14]. However, in the present case with damage, a Perzyna-type viscous regularization for p does not lead to any a priori length estimate for p with respect to the norm of its reference space, i.e., the space $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ of bounded Radon measures with values in $\mathbb{M}^{n\times n}_{\mathrm{D}}$.

On the one hand, this could be due to the fact that the usual techniques for proving a priori estimates in the vanishing-viscosity framework, based on testing the viscously regularized equations with the time derivatives of the corresponding variables, seem suitable to get good length estimates only in Hilbert spaces. Now, estimates for p in Hilbert spaces contained in $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, such as $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, would be unnatural and incompatible with the concentration effects (in space) that one would see in the limiting, perfectly plastic, evolution.

On the other hand, adding *directly*, to the plastic flow rule, a viscous regularization that features the L^2-norm, stronger than the one in the reference space $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, does not seem to be the right procedure from a heuristic point of view when the evolution may display jumps. Indeed, the idea associated with the vanishing-viscosity approach is to let the system explore the energy landscape around the starting configuration and choose an arrival configuration that is preferable from an energetic viewpoint, but close enough in terms of the viscosity norm (this becomes more evident on the level of the time discretization of the viscous system; cf. (4.1)). When viscosity vanishes, the evolution still keeps track of this procedure during jumps. Therefore, in this respect it is reasonable to take, for the viscous regularization, a norm that is not stronger than the reference norm. In this way, the system is free to detect the updated configuration in all the reference space. This is the case, for instance, of the L^2-viscous regularization for z, whose reference space is $H^{\mathrm{m}}(\Omega)$, used for the damage flow rule here and in [KRZ13, CL16, Neg19].

In order to mimic this approach also for the variable p (and, consequently, for u),

- we have introduced a further hardening regularizing term to the plastic flow rule, tuned by a parameter $\mu > 0$, and in this way, the reference space for the plastic strain p becomes $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$;
- we have addressed a viscous regularization for p such that the viscosity parameter ε is modulated by an additional parameter ν with $\nu \le \mu$.

All in all, we consider the following *rate-dependent* system for damage coupled with viscoplasticity, featuring the three parameters ε, ν, $\mu > 0$:

- the viscous (albeit *quasistatic*, as inertial forces are neglected), momentum balance

(1.2a)
$$-\mathrm{div}(\varepsilon\nu\mathbb{D}\mathrm{E}(\dot u)+\sigma) = f \text{ in } \Omega\times(0,T),\ \ (\varepsilon\nu\mathbb{D}\mathrm{E}(\dot u)+\sigma)\mathrm{n} = g \ \text{ in } \Gamma_{\mathrm{Neu}}\times(0,T)$$

(with $\mathbb{D}$ a fixed positive-definite fourth-order tensor), coupled with the expression for σ from (1.1b) and the kinematic admissibility condition (1.1c);

- the rate-dependent damage flow rule for z

$$\partial\mathrm{R}(\dot z) + \varepsilon\dot z + A_{\mathrm{m}}(z) + W'(z) \ni -\tfrac{1}{2}\mathbb{C}'(z)e : e \quad \text{in } \Omega\times(0,T); \tag{1.2b}$$

- the viscous flow rule for the plastic tensor

$$\partial_{\dot p}\mathrm{H}(z,\dot p) + \varepsilon\nu\dot p + \mu p \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega\times(0,T). \tag{1.2c}$$

The system is supplemented with the boundary conditions (1.1f). We highlight that viscosity for the u variable has been encompassed in the stress tensor (in accord with *Kelvin–Voigt rheology*) through the term $\mathrm{E}(\dot u)$. In fact, the other possible choice, $\dot e$, would not have preserved the *gradient structure* of the system, which is crucial for our analysis.

Let us emphasize that, for the rate-dependent system with hardening (i.e., with fixed $\varepsilon,\nu,\mu > 0$) both the reference space and the viscosity space for p are $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$. Furthermore, the choice $\nu \le \mu$ (one could take $\nu \le C\mu$ as well) guarantees that we do not lose the desired "order" between viscosity and reference norm for p as ν, μ vanish. This has enabled us to derive a priori estimates for the viscous solutions that are uniform not only w.r.t. ε but also w.r.t. μ (and ν). By the way, we observe that the technique of [CL16] to derive length estimates does not work with $\nu > 0$.

We will refer to ν as a *rate* parameter. Indeed, for fixed $\nu > 0$ and $\varepsilon \downarrow 0$, the displacement and the plastic strain converge to equilibrium and rate-independent evolution, respectively, at the same rate at which the damage parameter converges to rate-independent evolution. When $\varepsilon \downarrow 0$ and $\nu \downarrow 0$ *simultaneously*, relaxation to equilibrium and rate-independent behavior occurs at a faster rate for u and p than for z. The vanishing-viscosity analysis then acquires a *multirate* character. Balanced Viscosity to *multirate* systems have been explored in an abstract, albeit finite-dimensional setting, in [MRS16b] (cf. the forthcoming [MR21] for the extension to the infinite-dimensional setup).

Our results. In what follows, we will address *three different problems*.

First of all, we will carry out the vanishing-viscosity analysis of (1.2) as $\varepsilon \downarrow 0$ with $\mu > 0$ fixed. This will lead to the existence of (two different types of) Balanced Viscosity solutions to a rate-independent system for damage and plasticity *with hardening*, consisting of (1.1a), (1.1b), (1.1d), (1.1c), (1.1f) coupled with

$$\partial_{\dot{p}} \mathrm{H}(z, \dot{p}) + \mu p \ni \sigma_{\mathrm{D}} \quad \text{in } \Omega \times (0, T). \tag{1.3}$$

In fact, we will consider two cases in the vanishing-viscosity analysis as $\varepsilon \downarrow 0$ with $\mu > 0$ fixed:

1. First, we will keep the rate parameter $\nu > 0$ fixed, so that (u, z, p) relax to equilibrium (for u) and rate-independent evolution (for z and p) with the same rate. In this way, we will prove the existence of BV *solutions to the rate-independent system with hardening* (1.1a), (1.1b), (1.1d), (1.1c), (1.1f) (1.3); see Definition 6.2 and Theorem 6.8.
2. Second, we will let $\nu \downarrow 0$ together with $\varepsilon \downarrow 0$, so that u and p relax to equilibrium and rate-independent evolution faster than z, relaxing to rate-independent evolution. In this way, we will obtain BV *solutions to the multi-rate system with hardening* (1.1a), (1.1b), (1.1d), (1.1c), (1.1f) (1.3); see Definition 6.10 and Theorem 6.13.

Balanced Viscosity solutions to the rate-independent system with hardening arising from the "full" vanishing-viscosity approach are parameterized curves $(\mathsf{t}, \mathsf{u}, \mathsf{z}, \mathsf{p})$ defined on an "artificial" time interval (with t the rescaling function) that satisfy a suitable (scalar) energy-dissipation balance encoding all information on the evolution of the system. This is in accord with the notion that has been codified, in an abstract (finite-dimensional) setup, in [MRS16b]. More in general, this solution concept stems from a variational approach to gradient flows and general gradient systems; indeed, it is in the same spirit as the notion of *curve of maximal slope* [AGS08]. The energy-dissipation balance characterizing (parameterized) BV solutions features a *vanishing-viscosity contact potential*, namely a functional $\mathcal{M} = \mathcal{M}(\mathsf{t}, \mathsf{q}, \mathsf{t}', \mathsf{q}')$ (hereafter, we will often use q as a place-holder for the triple $(\mathsf{u}, \mathsf{z}, \mathsf{p})$), whose expression (and notation) depends on the different regimes considered.

In all cases, $\mathcal{M}$ encodes the possible onset of viscous behavior of the system at jumps. Indeed, in the "artificial" time, jumps occur at instants at which the rescaled slow time variable t is frozen, i.e., $\mathsf{t}' = 0$. Now, (only) at the jump instants the system may not satisfy (a weak version of the) first-order stability conditions in the variables u, p, z, and for this it dissipates energy in a way that is described by the specific expression of $\mathcal{M}$ for $\mathsf{t}' = 0$. In particular, we have as follows:

(i) For the BV solutions obtained via vanishing viscosity with $\nu > 0$ fixed, the contact potential $\mathcal{M}(\mathsf{t}, \mathsf{q}, 0, \mathsf{q}')$ features a term with the (viscous) $H^1 \times L^2 \times L^2$-norm of *the full triple* $(\mathsf{u}', \mathsf{p}', \mathsf{z}')$. While referring to section 6.1 for more comments, here we highlight that the expression of $\mathcal{M}$ reflects the fact that, at a jump, the system may be switched to a regime where viscous dissipation in the three variables intervenes "in the same way." This mirrors the fact that the variables u, z, p relax to static equilibrium and rate-independent evolution with the same rate.

(ii) For the BV solutions obtained in the limit as ε, $\nu \downarrow 0$ jointly, in the expression of $\mathcal{M}(\mathsf{t}, \mathsf{q}, 0, \mathsf{q}')$ two distinct terms account for the roles of the rates $(\mathsf{u}', \mathsf{p}')$ and of z'. A careful analysis, carried out in section 6.2, in particular shows that, at a jump, z is frozen until u, p have reached the elastic equilibrium/attained the local stability condition, respectively. This reflects the fact that u, p relax to equilibrium/rate-independent behavior faster than z, hence the multirate character of the evolution.

The above considerations can be easily inferred from the PDE characterization of (parameterized) BV solutions that we provide in Propositions 6.4 and 6.11; we also refer to Remarks 6.7 and 6.12 for further comments and for a comparison between the two notions of solutions for the system with hardening.

After the discussion of plasticity with fixed hardening,

(3) we will consider the case when also μ vanishes and thus address the asymptotic analysis of system (1.2) as the parameters ε, ν, $\mu \downarrow 0$ *simultaneously*. With our main result, Theorem 7.9, we will prove that, after a suitable reparameterization, viscous solutions converge to Balanced Viscosity solutions for the perfectly plastic system (1.1) that differ from the ones obtained in [CL16] in this respect: the description of the trajectories during jumps may possibly involve viscosity in *all* the variables u, p, z. Since ε and ν vanish jointly, the system has again a multirate character.

However, in the perfectly plastic case the situation is more complex than for the case with hardening. Indeed, for perfect plasticity the reference function space for (the rescaled plastic strain) p is $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ instead of $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, while the viscous dissipation that (possibly) intervenes at jumps features the L^2-norm of $\dot{\mathsf{p}}$. In particular, at jumps the expression of the contact potential $\mathcal{M}$ guarantees that p is in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ and u is in $H^1(\Omega;\mathbb{R}^n)$, which is reminiscent of the approximation through plastic hardening. The change in the functional framework occurring at the jump regime has important consequences for the analysis. On the one hand, we have to exploit density arguments and equivalent characterizations of the stability conditions to pass from the $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$-framework to the $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$-setting. On the other hand, a suitable reparameterization and abstract tools are needed to reveal more spatial regularity for u and p along jumps, in the spirit of [MRS16a, subsection 7.1] (cf. section 7 for more details). Another interesting point is that the present approximation through plasticity with hardening completely alleviates the need for a classical Kohn–Temam duality between stress and plastic strain, so we can use only the duality in [FG12] and therefore we do not have to impose more regularity on Ω or more regularity on the external loading (cf. Remark 7.1).

Plan of the paper. In section 2 we fix all the standing assumptions on the constitutive functions and on the problem data and prove some preliminary results. Section 3 focuses on the gradient structure that underlies the rate-dependent system (1.2) and that is at the core of its vanishing-viscosity analysis. Based on this structure, we set out to prove the existence of solutions to (1.2) by passing to the limit in a carefully devised time-discretization scheme. A series of a priori estimates on the time-discrete solutions are proved in section 4. Such bounds serve as a basis both for the existence proof for the viscous problem and for its vanishing-viscosity analysis. Indeed, in Proposition 4.4 we obtain estimates for the total variation of the discrete solutions that are uniform w.r.t. the viscosity parameter ε and w.r.t. ν and, in some cases, μ as well. For this, the condition $\nu \leq \mu$ plays a crucial role. Such bounds will lead to the estimates on the lengths of the curves needed for the arclength repameterizations and the vanishing-viscosity limit passages. We then derive the existence of solutions for the viscous system (1.2) in section 5. This is the common ground for the subsequent analysis as either some or all parameters vanish. The limit passages in (1.2) with $\mu > 0$ fixed are carried out in section 6: in particular, section 6.1 focuses on the analysis as $\varepsilon \downarrow 0$ with fixed $\nu > 0$, while the limit as $\varepsilon,\ \nu \downarrow 0$ is discussed in section 6.2. The limit passage as $\varepsilon, \nu, \mu \downarrow 0$ is performed in section 7. Therein, we do not reparameterize the viscous solutions by their "classical" arclength but by an *energy-dissipation* arclength that somehow encompasses the onset, for the limiting BV solutions, of rate-dependent behavior and additional spatial regularity during jumps.

2. Setup for the rate-dependent and rate-independent systems. In this section we establish the setup and the assumptions on the constitutive functions and on the problem data, both for the rate-dependent system (1.2) and for its rate-independent limits. Namely, we will propose a framework of conditions fitting

- *both* the rate-independent process with hardening, i.e., that obtained by taking the vanishing-viscosity limit of (1.2) as $\varepsilon \downarrow 0$ (and, possibly, $\nu \downarrow 0$ in the *multi-rate* case), with $\mu > 0$ fixed, *and*
- the rate-independent process for perfect plasticity and damage (i.e., that obtained in the further limit passage as $\mu \downarrow 0$).

Further definitions and auxiliary results for the perfectly plastic damage system will be expounded in section 7.

First of all, let us fix some notation that will be used throughout the paper.

Notation 2.1 (general notation and preliminaries). Given a Banach space X, we will denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X^* and X (and, for simplicity, also between $(X^n)^*$ and X^n). We will just write $\langle \cdot, \cdot \rangle$ for the inner Euclidean product in $\mathbb{R}^n$. Analogously, we will indicate by $\| \cdot \|_X$ the norm in X and often use the same symbol for the norm in X^n, as well, and just write $| \cdot |$ for the Euclidean norm in $\mathbb{R}^m$, $m \geq 1$. We will denote by $B_r(0)$ the open ball of radius r, centered at 0, in the Euclidean space $X = \mathbb{R}^m$.

We will denote by $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ the space of the symmetric $(n\times n)$-matrices and by $\mathbb{M}^{n\times n}_{\mathrm{D}}$ the subspace of the deviatoric matrices with null trace. In fact, $\mathbb{M}^{n\times n}_{\mathrm{sym}} = \mathbb{M}^{n\times n}_{\mathrm{D}} \oplus \mathbb{R} I$ (I denoting the identity matrix), since every $\eta \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be written as $\eta = \eta_{\mathrm{D}} + \frac{\mathrm{tr}(\eta)}{n} I$ with η_{D} the orthogonal projection of η into $\mathbb{M}^{n\times n}_{\mathrm{D}}$. We will refer to η_{D} as the deviatoric part of η. We write for $\mathrm{Sym}(\mathbb{M}^{n\times n}_{\mathrm{D}}; \mathbb{M}^{n\times n}_{\mathrm{D}})$ the set of symmetric endomorphisms on $\mathbb{M}^{n\times n}_{\mathrm{D}}$.

We will often use the shorthand notation $\| \cdot \|_{L^p}$, $1 \leq p < +\infty$, for the L^p-norm on the space $L^p(O; \mathbb{R}^m)$, with O a measurable subset of $\mathbb{R}^n$, and analogously we will write $\| \cdot \|_{H^1}$. We will denote by $\mathrm{M}_{\mathrm{b}}(O; \mathbb{R}^m)$ the space of bounded Radon measures on O with values in $\mathbb{R}^m$.

As already mentioned in the introduction, as in [KRZ13, CL16] the mechanical energy will encompass a gradient regularizing contribution for the damage variable, featuring the bilinear form

$$a_{\mathrm{m}} : H^{\mathrm{m}}(\Omega) \times H^{\mathrm{m}}(\Omega) \to \mathbb{R}, \tag{2.1}$$

$$a_{\mathrm{m}}(z_1, z_2) := \int_\Omega \int_\Omega \frac{\big(\nabla z_1(x) - \nabla z_1(y)\big) \cdot \big(\nabla z_2(x) - \nabla z_2(y)\big)}{|x-y|^{n+2(\mathrm{m}-1)}} \,\mathrm{d}x \,\mathrm{d}y \quad \text{with } \mathrm{m} \in \left(\frac{n}{2}, 2\right). \tag{2.2}$$

We will denote by $A_{\mathrm{m}} : H^{\mathrm{m}}(\Omega) \to H^{\mathrm{m}}(\Omega)^*$ the associated operator, viz.

$$\langle A_{\mathrm{m}}(z), w \rangle_{H^{\mathrm{m}}(\Omega)} := a_{\mathrm{m}}(z, w) \quad \text{for every } z,\, w \in H^{\mathrm{m}}(\Omega).$$

We recall that $H^{\mathrm{m}}(\Omega)$ is a Hilbert space with the inner product $\langle z_1, z_2 \rangle_{H^{\mathrm{m}}(\Omega)} := \int_\Omega z_1 z_2 \,\mathrm{d}x + a_{\mathrm{m}}(z_1, z_2)$. Since we assume that $m > \frac{n}{2}$, we have the compact embedding $H^{\mathrm{m}}(\Omega) \Subset \mathrm{C}(\overline{\Omega})$.

Whenever working with a real function v defined on a space-time cylinder $\Omega \times (0,T)$ and differentiable w.r.t. time a.e. on $\Omega \times (0,T)$, we will denote by $\dot{v} : \Omega \times (0,T) \to \mathbb{R}$ its (almost everywhere defined) partial time derivative. However, as soon as we consider v as a (Bochner) function from $(0,T)$ with values in a suitable Lebesgue/Sobolev space X (with the Radon–Nikodým property) and v is in the space $\mathrm{AC}([0,T];X)$, we will denote by $v' : (0,T) \to X$ its (almost everywhere defined) time derivative.

Finally, we will use the symbols c, c', C, C', etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities.

Let us recall some basic facts about the space $\mathrm{BD}(\Omega)$ of *functions of bounded deformations*, defined by

$$\mathrm{BD}(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^n) \,:\, \mathrm{E}(u) \in \mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\}, \tag{2.3}$$

where $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ is the space of bounded Radon measures on Ω with values in $\mathbb{M}^{n\times n}_{\mathrm{sym}}$, with norm $\|\lambda\|_{\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} := |\lambda|(\Omega)$ and $|\lambda|$ the variation of the measure. Recall that, by the Riesz representation theorem, $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ can be identified with the dual of the space $\mathrm{C}_0(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. The space $\mathrm{BD}(\Omega)$ is endowed with the graph norm

$$\|u\|_{\mathrm{BD}(\Omega)} := \|u\|_{L^1(\Omega;\mathbb{R}^n)} + \|\mathrm{E}(u)\|_{\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})},$$

which makes it a Banach space. It turns out that $\mathrm{BD}(\Omega)$ is the dual of a normed space; cf. [TS80].

In addition to the strong convergence induced by $\|\cdot\|_{\mathrm{BD}(\Omega)}$, the duality from [TS80] defines a notion of weak* convergence on $\mathrm{BD}(\Omega)$: a sequence $(u_k)_k$ converges weakly* to u in $\mathrm{BD}(\Omega)$ if $u_k \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^n)$ and $\mathrm{E}(u_k) \overset{*}{\rightharpoonup} \mathrm{E}(u)$ in $\mathrm{M_b}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. The space $\mathrm{BD}(\Omega)$ is contained in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$; every bounded sequence in $\mathrm{BD}(\Omega)$ has a weakly* converging subsequence and, furthermore, a subsequence converging weakly in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$ and strongly in $L^p(\Omega;\mathbb{R}^n)$ for every $1 \le p < \frac{n}{n-1}$.

Finally, we recall that for every $u \in \mathrm{BD}(\Omega)$ the trace $u|_{\partial\Omega}$ is well defined as an element in $L^1(\partial\Omega;\mathbb{R}^n)$ and that (cf. [Tem83, Proposition 2.4, Remark 2.5]) a Poincaré-type inequality holds:

$$\exists\, C > 0 \;\; \forall\, u \in \mathrm{BD}(\Omega) : \;\; \|u\|_{L^1(\Omega;\mathbb{R}^n)} \le C \left(\|u\|_{L^1(\Gamma_{\mathrm{Dir}};\mathbb{R}^n)} + \|\mathrm{E}(u)\|_{\mathrm{M}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} \right). \tag{2.4}$$

2.1. Assumptions and preliminary results.

The reference configuration. Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded Lipschitz domain. The minimal assumption for our analysis is that Ω is a *geometrically admissible multiphase domain* in the sense of [FG12, subsection 1.2] with only *one phase*, that is, $i = 1$ therein, where $(\Omega_i)_i$ is a partition corresponding to the phases. Referring still to [FG12], this corresponds to assuming that the Dirichlet boundary Γ_{Dir} is a nonempty open set in the relative topology of $\partial\Omega$, with (relative) boundary $\partial|_{\partial\Omega}\Gamma_{\mathrm{Dir}}$ *admissible* in the sense of [FG12, (6.20)]. As observed in [FG12, Theorem 6.5], a sufficient condition for this is the so-called Kohn–Temam condition, which we recall below and assume throughout the paper:

(2.Ω) $\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \Sigma$ with Γ_{Dir}, Γ_{Neu}, Σ pairwise disjoint,

(2.5) Γ_{Dir} and Γ_{Neu} relatively open in $\partial\Omega$, with $\partial\Gamma_{\mathrm{Dir}} = \partial\Gamma_{\mathrm{Neu}} = \Sigma$ their relative boundary in $\partial\Omega$,

(2.6) such that Σ is of class C^2 with $\mathcal{H}^{n-1}(\Sigma) = 0$, and $\partial\Omega$ is Lipschitz and of class C^2 in a neighborhood of Σ.

We will work with the spaces

$$H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) := \left\{u \in H^1(\Omega;\mathbb{R}^n) \colon u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\right\}, \tag{2.7}$$

$$\widetilde{\Sigma}(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \colon \mathrm{div}(\sigma) \in L^2(\Omega;\mathbb{R}^n)\}\,. \tag{2.8}$$

For $\sigma \in \widetilde{\Sigma}(\Omega)$ one may define the distribution $[\sigma \mathrm{n}]$ on $\partial\Omega$ by

$$\langle[\sigma\mathrm{n}],\psi\rangle_{\partial\Omega} := \langle \mathrm{div}(\sigma),\psi\rangle_{L^2} + \langle\sigma,\mathrm{E}(\psi)\rangle_{L^2} \tag{2.9}$$

for $\psi \in H^1(\Omega;\mathbb{R}^n)$. It is known (see, e.g., [KT83, Theorem 1.2] or [DMDM06, (2.24)]) that $[\sigma\mathrm{n}] \in H^{-1/2}(\partial\Omega;\mathbb{R}^n)$ and that if $\sigma \in \mathrm{C}^0(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ the distribution $[\sigma\mathrm{n}]$ coincides with $\sigma\mathrm{n}$, that is, the pointwise product matrix-normal vector in $\partial\Omega$. With each $\sigma \in \widetilde{\Sigma}(\Omega)$ we associate an elliptic operator in $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*$ denoted by $-\mathrm{Div}(\sigma)$ and defined by

$$\langle -\mathrm{Div}(\sigma), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)} := \langle -\mathrm{div}(\sigma), v\rangle_{L^2(\Omega;\mathbb{R}^n)} + \langle[\sigma\mathrm{n}],v\rangle_{H^{1/2}(\partial\Omega;\mathbb{R}^n)} = \int_\Omega \sigma : \mathrm{E}(v)\,\mathrm{d}x \tag{2.10}$$

for all $v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$, where the equality above is an integration by parts formula based on the divergence theorem.

The elasticity and viscosity tensors. We assume that the elastic tensor $\mathbb{C} : [0,+\infty) \to \mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}};\mathbb{M}^{n\times n}_{\mathrm{sym}})$ fulfills the following conditions:

$$\mathbb{C} \in \mathrm{C}^{1,1}([0,+\infty);\mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}};\mathbb{M}^{n\times n}_{\mathrm{sym}}))\,, \tag{2.11}$$

$$z \mapsto \mathbb{C}(z)\xi : \xi \text{ is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}\,, \tag{2.12}$$

$$\exists\,\gamma_1,\,\gamma_2 > 0 \ \ \forall\, z \in [0,+\infty)\ \forall\,\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}} \ : \quad \gamma_1|\xi|^2 \leq \mathbb{C}(z)\xi : \xi \leq \gamma_2|\xi|^2\,, \tag{2.13}$$

$$\mathbb{C}(z)\xi := \mathbb{C}_{\mathrm{D}}(z)\xi_{\mathrm{D}} + \kappa(z)(\mathrm{tr}\,\xi)I \quad \text{with } \begin{cases} \mathbb{C}_{\mathrm{D}} \in L^\infty(0,1;\mathrm{Sym}(\mathbb{M}^{n\times n}_{\mathrm{D}};\mathbb{M}^{n\times n}_{\mathrm{D}}))\,,\\ \kappa \in L^\infty(0,1)\,. \end{cases} \tag{2.14}$$

Again, observe that (2.14) is relevant for the perfectly plastic damage system, only. Even in that context, (2.14) is not needed for the analysis, but it is just assumed for mechanical reasons, since purely volumetric deformations do not affect plastic behavior.

We introduce the stored elastic energy $\mathcal{Q} : L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \times \mathrm{C}^0(\overline{\Omega}) \to \mathbb{R}$

$$\mathcal{Q}(z,e) := \frac{1}{2}\int_\Omega \mathbb{C}(z)e : e\,\mathrm{d}x\,. \tag{2.15}$$

As for the viscosity tensor $\mathbb{D}$, we require that

$$\mathbb{D} \in \mathrm{C}^0(\overline{\Omega}; \mathrm{Sym}(\mathbb{M}^{n\times n}_{\mathrm{D}}; \mathbb{M}^{n\times n}_{\mathrm{D}})) \text{ and} \tag{2.16}$$

$$\exists\, \delta_1,\, \delta_2 > 0\ \forall x \in \Omega\ \ \forall A \in \mathbb{M}^{d\times d}_{\mathrm{sym}} : \qquad \delta_1 |A|^2 \leq \mathbb{D}(x)A : A \leq \delta_2 |A|^2. \tag{2.17}$$

For later use, we introduce the dissipation potential

$$\mathcal{V}_{2,\nu}(v) := \frac{\nu}{2} \int_\Omega \mathbb{D}\mathrm{E}(v) : \mathrm{E}(v)\,\mathrm{d}x\,. \tag{2.18}$$

Throughout the paper, we will use that $\mathbb{D}$ induces an equivalent (by a Korn–Poincaré-type inequality) Hilbert norm on $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$, namely

$$\|u\|_{H^1,\mathbb{D}} := \left(\int_\Omega \mathbb{D}\mathrm{E}(u) : \mathrm{E}(u)\,\mathrm{d}x \right)^{1/2} \quad \text{with } \|u\|_{H^1,\mathbb{D}} \leq K_{\mathbb{D}} \|\mathrm{E}(u)\|_{L^2} \text{ for } u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n), \tag{2.19}$$

and the "dual norm"

$$\|\eta\|_{(H^1,\mathbb{D})^*} := \left(\int_\Omega \mathbb{D}^{-1}\xi : \xi \right)^{1/2} \forall\, \eta \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^* \text{ with } \eta = \mathrm{Div}(\xi) \text{ for some } \xi \in \widetilde{\Sigma}(\Omega)\,. \tag{2.20}$$

The overall mechanical energy. Besides the elastic energy $\mathcal{Q}$ from (2.15) and the regularizing, nonlocal gradient contribution featuring the bilinear form a_{m}, the mechanical energy functional will feature a further term acting on the damage variable z, with density W satisfying

$$W \in \mathrm{C}^2((0,+\infty);\mathbb{R}^+) \cap \mathrm{C}^0([0,+\infty);\mathbb{R}^+ \cup \{+\infty\})\,, \tag{2.21}$$

$$s^{2n} W(s) \to +\infty \text{ as } s \to 0^+\,, \tag{2.22}$$

where $W \in \mathrm{C}([0,+\infty);\mathbb{R}^+ \cup \{+\infty\})$ means that $W(0) = \infty$ and $W(z) \uparrow +\infty$ if $z \downarrow 0$ as prescribed by (2.22). Clearly, these requirements on W force z to be strictly positive (cf. also the upcoming Remark 3.2); consequently, the material never reaches the most damaged state at any point. We also have the contribution of a time-dependent loading $F : [0,T] \to H^1(\Omega;\mathbb{R}^n)^*$, specified in (2.39b) below, which subsumes the volume and the surface forces f and g. All in all, the energy functional driving the rate-dependent and rate-independent systems *with hardening* is $\mathcal{E}_\mu \colon [0,T] \times H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) \times H^{\mathrm{m}}(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \to \mathbb{R} \cup \{+\infty\}$, defined for $\mu > 0$ by

$$\begin{aligned} \mathcal{E}_\mu(t,u,z,p) := {}& \mathcal{Q}(z, \mathrm{E}(u{+}w(t)){-}p) + \int_\Omega \left(W(z){+}\frac{\mu}{2}|p|^2 \right) \mathrm{d}x \\ & + \frac{1}{2} a_{\mathrm{m}}(z,z) - \langle F(t), u + w(t) \rangle_{H^1(\Omega;\mathbb{R}^n)} \end{aligned} \tag{2.23}$$

with w the time-dependent Dirichlet loading specified in (2.41) ahead.

The plastic dissipation potential and the overall plastic dissipation functional. The plastic dissipation potential reflects the constraint that the admissible stresses belong to given constraint sets. In turn, such sets depend on the damage variable z: this, and the z-dependence of the matrix $\mathbb{C}(z)$ of elastic coefficients, provides a strong coupling between the plastic and the damage flow rules.

More precisely, in a softening framework, following the footsteps of [CL16] we require that the constraint sets $(K(z))_{z\in[0,+\infty)}$ fulfill

$$K(z) \subset \mathbb{M}^{n\times n}_{\mathrm{D}} \text{ is closed and convex for all } z\in[0,+\infty), \tag{2.24}$$

$$\exists\, 0<\bar r<\bar R \quad \forall\, 0\le z_1\le z_2\ : \qquad B_{\bar r}(0)\subset K(z_1)\subset K(z_2)\subset B_{\bar R}(0), \tag{2.25}$$

$$\exists\, C_K>0 \quad \forall\, z_1,\, z_2\in[0,+\infty)\ : \qquad d_{\mathscr H}(K(z_1),K(z_2))\le C_K|z_1-z_2|, \tag{2.26}$$

with $d_{\mathscr H}$ the Hausdorff distance between two subsets of $\mathbb{M}^{n\times n}_{\mathrm{D}}$, defined by

$$d_{\mathscr H}(K_1,K_2):=\max\left(\sup_{x\in K_1}\mathrm{dist}(x,K_2),\ \sup_{x\in K_2}\mathrm{dist}(x,K_1)\right),$$

and $\mathrm{dist}(x,K_i):=\min_{y\in K_i}|x-y|$, $i=1,2$. We now introduce the support function $H:[0,+\infty)\times\mathbb{M}^{n\times n}_{\mathrm{D}}\to[0,+\infty)$ defined by

$$H(z,\pi):=\sup_{\sigma\in K(z)}\sigma:\pi \qquad \forall (z,\pi)\in[0,+\infty)\times\mathbb{M}^{n\times n}_{\mathrm{D}}\,. \tag{2.27}$$

It was shown in [CL16, Lemma 2.1] that, thanks to (2.24)–(2.26), H enjoys the following properties:

$$H \text{ is continuous,} \tag{2.28a}$$

$$0\le H(z_2,\pi)-H(z_1,\pi) \qquad \text{for all } 0\le z_1\le z_2 \text{ and all } \pi\in\mathbb{M}^{n\times n}_{\mathrm{D}} \text{ with } |\pi|=1\,, \tag{2.28b}$$

$$\exists\, C_K>0\ \forall\, z_1,\, z_2\in[0,+\infty)\ \forall\,\pi\in\mathbb{M}^{n\times n}_{\mathrm{D}} \qquad |H(z_2,\pi)-H(z_1,\pi)|\le C_K|\pi||z_2-z_1|\,, \tag{2.28c}$$

$$\pi\mapsto H(z,\pi) \text{ is convex and 1-positively homogeneous for all } z\in[0,1]\,, \tag{2.28d}$$

$$\bar r|\pi|\le H(z,\pi)\le\bar R|\pi|\,. \tag{2.28e}$$

As observed in [CL16], properties (2.24)–(2.26) are satisfied by constraint sets in the "multiplicative form" $K(z)=V(z)K(1)$, with $V\in\mathrm{C}^{1,1}([0,+\infty))$ nondecreasing and such that $\overline{m}\le V(z)\le\overline{M}$ for all $z\in[0,+\infty)$ and some $\overline{m}$, $\overline{M}>0$.

The *plastic dissipation potential* $\mathcal{H}\colon \mathrm{C}^0(\overline\Omega;[0,+\infty))\times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\to\mathbb{R}$ is defined by

$$\mathcal{H}(z,\pi):=\int_\Omega H(z(x),\pi(x))\,\mathrm{d}x\,. \tag{2.29}$$

Clearly, it follows from (2.28a)–(2.28e) that

$$\pi\mapsto\mathcal{H}(z,\pi) \text{ is convex and positively one-homogeneous for every } z\in\mathrm{C}^0(\overline\Omega;[0,+\infty)), \tag{2.30a}$$

$$\bar r\|\pi\|_1\le\mathcal{H}(z,\pi)\le\bar R\|\pi\|_1 \quad \forall z\in\mathrm{C}^0(\overline\Omega;[0,+\infty)) \text{ and } \pi\in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}), \tag{2.30b}$$

$$0\le\mathcal{H}(z_2,\pi)-\mathcal{H}(z_1,\pi) \quad \forall z_1\le z_2\in\mathrm{C}^0(\overline\Omega;[0,+\infty)) \quad \forall\pi\in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,, \tag{2.30c}$$

$$|\mathcal{H}(z_2,\pi)-\mathcal{H}(z_1,\pi)|\le C_K\|z_1-z_2\|_{L^\infty(\Omega)}\|\pi\|_1 \qquad \forall z_1,z_2\in\mathrm{C}^0(\overline\Omega;[0,+\infty)),\ \pi\in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,. \tag{2.30d}$$

Let us introduce the set

$$\widetilde{\mathcal{K}}_z(\Omega) := \{\sigma \in \widetilde{\Sigma}(\Omega)\colon \sigma_{\mathrm{D}}(x) \in K(z(x)) \text{ for a.e. } x \in \Omega\}\,. \tag{2.31}$$

By standardly approximating (in the L^1-norm) π by piecewise constant functions, we show that if $z \mapsto K(z)$ is constant, namely $K(z) \equiv \overline{K} \subset \mathbb{M}^{n\times n}_{\mathrm{D}}$, then

$$\mathcal{H}(z,\pi) = \sup_{\sigma \in \widetilde{K}_z(\Omega)} \langle \sigma_D, \pi\rangle_{L^1}\,. \tag{2.32}$$

For a general map $z \mapsto K(z)$ the argument in [Sol09, Theorem 3.6, Corollary 3.8] shows that (2.32) still holds.

The convex analysis subdifferential $\partial_\pi\mathcal{H}\colon \mathrm{C}^0(\overline{\Omega};[0,+\infty))\times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \rightrightarrows L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, given by

$$\omega \in \partial_\pi\mathcal{H}(z,\pi) \quad \text{if and only if } \ \mathcal{H}(z,\varrho) - \mathcal{H}(z,\pi) \geq \int_\Omega \omega(\varrho-\pi)\,\mathrm{d}x \ \forall \varrho \in L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$$

fulfills

$$\omega \in \partial_\pi\mathcal{H}(z,\pi) \quad \text{if and only if} \quad \omega(x) \in \partial H(z(x),\pi(x)) \quad \text{for a.a. } x\in\Omega\,, \tag{2.33}$$

where, with slight abuse, $\partial_\pi\mathcal{H}$ denotes the subdifferential of $\mathcal{H}$ w.r.t. the second variable.

The rate-dependent system (1.2) with the viscously regularized plastic flow rule (1.2c) features the dissipation potential $\mathcal{H}^{\mathrm{tot}}_\nu : \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \to [0,+\infty)$ defined by

$$\mathcal{H}^{\mathrm{tot}}_\nu(z,\pi) := \mathcal{H}(z,\pi) + \mathcal{H}_{2,\nu}(\pi) \qquad \text{with } \mathcal{H}_{2,\nu}(\pi) := \frac{\nu}{2}\|\pi\|^2_{L^2(\Omega)}\,. \tag{2.34}$$

By the sum rule for convex analysis subdifferentials (cf., e.g., [AE84, Corollary IV.6]), the subdifferential $\partial_\pi\mathcal{H}\colon \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \rightrightarrows L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ is given by

$$\partial_\pi\mathcal{H}^{\mathrm{tot}}_\nu(z,\pi) = \partial_\pi\mathcal{H}(z,\pi) + \{\nu\pi\} \qquad \forall \pi \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}),\ z \in \mathrm{C}^0(\overline{\Omega};[0,+\infty))\,. \tag{2.35}$$

The damage dissipation potential. We consider the damage dissipation density $\mathrm{R} : \mathbb{R} \to [0,+\infty]$ defined by

$$\begin{aligned} &\mathrm{R}(\zeta) := P(\zeta) + I_{(-\infty,0]}(\zeta) \quad \text{with} \\ &P(\zeta) := -\kappa\zeta \text{ and } I_{(-\infty,0]} \text{ the indicator function of } (-\infty,0] \end{aligned}$$

so that

$$\mathrm{R}(\zeta) := \begin{cases} -\zeta & \text{if } \zeta \leq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

With R we associate the dissipation potential $\mathcal{R} : L^1(\Omega) \to [0,+\infty]$ defined by $\mathcal{R}(\zeta) := \int_\Omega \mathrm{R}(\zeta(x))\,\mathrm{d}x$. In fact, since the flow rule for the damage variable will be posed in $H^{\mathrm{m}}(\Omega)^*$ (cf. (3.3b) ahead), it will be convenient to consider the restriction of $\mathcal{R}$ to the space $H^{\mathrm{m}}(\Omega)$ which, with a slight abuse of notation, we will denote by the same symbol, namely

$$\begin{aligned} &\mathcal{R} : H^{\mathrm{m}}(\Omega) \to [0,+\infty], \\ &\mathcal{R}(\zeta) = \int_\Omega \mathrm{R}(\zeta(x))\,\mathrm{d}x = \mathcal{P}(\zeta) + \mathcal{I}(\zeta) \text{ with } \begin{cases} \mathcal{P}(\zeta) := \int_\Omega P(\zeta(x))\,\mathrm{d}x, \\ \mathcal{I}(\zeta) := \int_\Omega I_{(-\infty,0]}(\zeta(x))\,\mathrm{d}x. \end{cases} \end{aligned} \tag{2.36}$$

The viscously regularized damage flow rule (1.2b) in fact features the dissipation potential
(2.37)
$$\mathcal{R}^{\mathrm{tot}}: H^{\mathrm{m}}(\Omega)\to[0,+\infty],\qquad \mathcal{R}^{\mathrm{tot}}(\zeta)=\mathcal{R}(\zeta)+\mathcal{R}_2(\zeta)\qquad \text{with } \mathcal{R}_2(\zeta):=\frac12\|\zeta\|^2_{L^2(\Omega)}\,.$$

We will denote by $\partial\mathcal{R}: H^{\mathrm{m}}(\Omega)\rightrightarrows H^{\mathrm{m}}(\Omega)^*$ and $\partial\mathcal{R}^{\mathrm{tot}}: H^{\mathrm{m}}(\Omega)\rightrightarrows H^{\mathrm{m}}(\Omega)^*$ the subdifferentials of $\mathcal{R}$ and $\mathcal{R}^{\mathrm{tot}}$ in the sense of convex analysis. Observe that $\mathrm{dom}(\partial\mathcal{R})=\mathrm{dom}(\partial\mathcal{R}^{\mathrm{tot}})=H^{\mathrm{m}}_{-}(\Omega)$. We will provide explicit formulae for both subdifferentials in Lemma 2.4 at the end of this section.

The initial data, the body forces, and the Dirichlet loading. We will consider initial data

$$\begin{aligned}&u_0\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n),\qquad z_0\in H^{\mathrm{m}}(\Omega)\text{ with } W(z_0)\in L^1(\Omega)\text{ and } z_0\le 1\text{ in }\overline{\Omega},\\ &p_0\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}).\end{aligned}\tag{2.38}$$

The assumptions that we require on the volume and surface forces depend on the type of plasticity considered. In the analysis of systems with hardening we will require the following conditions on the volume force f and the assigned traction g:

$$f\in H^1(0,T;L^2(\Omega;\mathbb{R}^n)),\qquad g\in H^1(0,T;H^{1/2}(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)^*).\tag{2.39a}$$

To shorten notation, we will often incorporate the forces f and g into the induced total load, namely the function $F\colon[0,T]\to H^1(\Omega;\mathbb{R}^n)^*$ defined at $t\in(0,T)$ by

$$\langle F(t),v\rangle_{H^1(\Omega;\mathbb{R}^n)}:=\langle f(t),v\rangle_{L^2(\Omega;\mathbb{R}^n)}+\langle g(t),v\rangle_{H^{1/2}(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)}\tag{2.39b}$$

for all $v\in H^1(\Omega;\mathbb{R}^n)$.

In turn, for handling the perfectly plastic damage system of section 7 we will require that

$$f\in H^1(0,T;L^n(\Omega;\mathbb{R}^n))\,,\quad g\in H^1(0,T;L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^n))\,,\tag{2.39c}$$

so that F turns out to take values in $\mathrm{BD}(\Omega)^*$, defining

$$\langle F(t),v\rangle_{\mathrm{BD}(\Omega)}:=\langle f(t),v\rangle_{L^{n/(n-1)}(\Omega;\mathbb{R}^n)}+\langle g(t),v\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)}$$

for all $v\in\mathrm{BD}(\Omega)$. Both for the analysis of the system with hardening and of the perfectly plastic one, we will assume a *uniform safe load condition*, namely that there exists

$$\rho\in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))\quad\text{with}\quad \rho_{\mathrm{D}}\in H^1(0,T;L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))\tag{2.39d}$$

and there exists $\alpha>0$ such that for every $t\in[0,T]$ (recall (2.9))

$$-\operatorname{div}(\varrho(t))=f(t)\text{ a.e. on }\Omega\,,\qquad [\varrho(t)\mathrm{n}]=g(t)\text{ on }\Gamma_{\mathrm{Neu}}\,,\tag{2.39e}$$

$$\rho_{\mathrm{D}}(t,x)+\xi\in K\qquad\text{for a.a. } x\in\Omega\text{ and for every }\xi\in\mathbb{M}^{n\times n}_{\mathrm{sym}}\text{ s.t. }|\xi|\le\alpha\,.\tag{2.39f}$$

Assumption (2.39c) will be crucial in the derivation of a priori estimates uniform with respect to the parameter μ in Proposition 4.3, while with (2.39a) the estimates would depend on $\mu>0$; cf. also Remark 4.6 ahead. Combining (2.39a) with (2.39d)–(2.39f) gives $-\mathrm{Div}(\varrho(t))=F(t)$ in $H^1_{\mathrm{Dir}}(\Omega,\mathbb{R}^n)^*$, while if (2.39c) holds, then (2.39d)–(2.39f) yield $-\widehat{\mathrm{Div}}(\varrho(t)) = F(t)$ for all $t \in [0,T]$ (where the operator $-\widehat{\mathrm{Div}}$ will be introduced in (7.5)). For later use, we notice that, thanks to (2.32), it is easy to deduce that for all $t \in [0,T]$

$$\mathcal{H}(z,p) - \int_\Omega \rho_{\mathrm{D}}(t) p \,\mathrm{d}x \geq \alpha \|p\|_{L^1(\Omega)} \,. \tag{2.40}$$

As for the time-dependent Dirichlet loading w, we will require that

$$w \in H^1(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n)). \tag{2.41}$$

Remark 2.2. In fact, the analysis of the *rate-independent* system for damage and plasticity, with or without hardening, would just require $w \in \mathrm{AC}([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))$ so that, upon taking the vanishing-viscosity limit as $\varepsilon \downarrow 0$ of system (1.2), we could approximate a loading $w \in \mathrm{AC}([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))$ with a sequence $(w_\varepsilon)_\varepsilon \subset H^1(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n))$. The same applies to the time regularity of the forces. However, to avoid overburdening the exposition we have preferred not to pursue this path.

Generally, in what follows we will tacitly assume the validity of all the above conditions and omit explicitly invoking them in the various results, with a few exceptions.

Remark 2.3 (rewriting the driving energy functional). By the safe load condition (2.39e) and the integration by parts formula in (2.10) applied to $u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$, $\mathcal{E}_\mu$ can be rewritten as

$$\begin{aligned}\mathcal{E}_\mu(t,u,z,p) = \mathcal{Q}(z,e(t)) &+ \int_\Omega \Big(W(z) + \frac{\mu}{2}|p|^2\Big)\,\mathrm{d}x \\ &+ \frac{1}{2} a_{\mathrm{m}}(z,z) - \int_\Omega \rho(t)\mathrm{E}(u)\,\mathrm{d}x - \langle F(t), w(t)\rangle_{H^1(\Omega;\mathbb{R}^n)},\end{aligned}$$

where we have highlighted the *elastic part* of the strain tensor $\mathrm{E}(u{+}w(t))$,

$$e(t) := \mathrm{E}(u{+}w(t)) - p \,. \tag{2.42}$$

We now introduce the functional

$$\begin{aligned}\mathcal{F}_\mu(t,u,z,p) := \mathcal{Q}(z,e(t)) &+ \int_\Omega \Big(W(z) + \frac{\mu}{2}|p|^2\Big)\,\mathrm{d}x \\ &+ \frac{1}{2} a_{\mathrm{m}}(z,z) - \int_\Omega \rho(t)(e(t) - \mathrm{E}(w(t))\,\mathrm{d}x - \langle F(t), w(t)\rangle_{H^1(\Omega;\mathbb{R}^n)} \,.\end{aligned} \tag{2.43}$$

Then, taking into account that $\int_\Omega (\rho{-}\rho_{\mathrm{D}}) p \,\mathrm{d}x = 0$, we have

$$\mathcal{E}_\mu(t,u,z,p) = \mathcal{F}_\mu(t,u,z,p) - \int_\Omega \rho_{\mathrm{D}}(t) p \,\mathrm{d}x \,. \tag{2.44}$$

In the following result we clarify the expression of the subdifferentials $\partial\mathcal{R}$ and $\partial\mathcal{R}^{\mathrm{tot}}$; these basic facts will be useful, for instance, in the proof of Lemma 3.6.

LEMMA 2.4. *We have the following representation formula for the subdifferential* $\partial\mathcal{I}\colon H^{\mathrm{m}}(\Omega) \rightrightarrows H^{\mathrm{m}}(\Omega)^*$ *of the functional* $\mathcal{I}$ *from* (2.36)*: for all* $\zeta \in H^{\mathrm{m}}_-(\Omega) := \{v \in H^{\mathrm{m}}(\Omega) : \ v \leq 0 \text{ in } \Omega\}$

$$\chi \in \partial\mathcal{I}(\zeta) \qquad \text{if and only if} \qquad \langle \chi, w - \zeta\rangle_{H^{\mathrm{m}}(\Omega)} \leq 0 \qquad \forall\, w \in H^{\mathrm{m}}_-(\Omega) \,. \tag{2.45}$$

Moreover, for all $\zeta \in H^{\mathrm{m}}_{-}(\Omega)$ there holds

$$\partial\mathcal{R}(\zeta) = \partial\mathcal{P}(\zeta) + \partial\mathcal{I}(\zeta) = -\kappa + \partial\mathcal{I}(\zeta), \tag{2.46}$$

$$\partial\mathcal{R}^{\mathrm{tot}}(\zeta) = \partial\mathcal{R}(\zeta) + \{\zeta\}, \tag{2.47}$$

where $-\kappa$ stands for the functional $H^{\mathrm{m}}(\Omega) \ni \zeta \mapsto \int_\Omega(-\kappa)\zeta(x)\,\mathrm{d}x$, and we simply write ζ, in place of $J(\zeta)$ ($J : L^1(\Omega) \to H^{\mathrm{m}}(\Omega)^*$ denoting the Riesz mapping).

Proof. Formula (2.45) is in fact the definition of $\partial\mathcal{I}(\zeta)$, whereas (2.46) and (2.47) follow the sum rule for convex subdifferentials; cf., e.g., [AE84, Corollary IV.6]. □

3. The gradient structure of the viscous system. In this section we are going to establish the functional setup in which the (Cauchy problem for the) *rate-dependent* system with hardening (i.e., with $\mu > 0$) (1.2) is formulated and, accordingly, specify the notion of solution we are interested in. This will enable us to unveil the gradient structure underlying system (1.2), which will have a twofold outcome:

1. Exploiting this structure we will show that (1.2) can be equivalently reformulated in terms of an energy-dissipation inequality, which is in turn equivalent to an energy-dissipation *balance*. This observation will simplify the proof of existence of viscous solutions, carried out in section 5.
2. The energy-dissipation *balance* will be at the core of the vanishing-viscosity analysis (with $\nu > 0$ fixed) performed in section 6, as well as of the vanishing-hardening analysis carried out in section 7.

System (1.2) involves the rescaled dissipation potentials $\mathcal{V}_{\varepsilon,\nu} : H^1(\Omega;\mathbb{R}^n) \to [0,+\infty)$, $\mathcal{H}_{\varepsilon,\nu} : \mathrm{C}^0(\overline{\Omega};[0,+\infty)) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \to [0,+\infty)$, and $\mathcal{R}_\varepsilon : H^{\mathrm{m}}(\Omega) \to [0,+\infty]$ defined by

$$\begin{gathered}\mathcal{V}_{\varepsilon,\nu}(v) := \frac{1}{\varepsilon}\mathcal{V}_{2,\nu}(\varepsilon v), \quad \mathcal{H}_{\varepsilon,\nu}(z,\pi) := \frac{1}{\varepsilon}\mathcal{H}^{\mathrm{tot}}_{\nu}(z,\varepsilon\pi) = \mathcal{H}(z,\pi) + \frac{1}{\varepsilon}\mathcal{H}_{2,\nu}(z,\varepsilon\pi),\\ \mathcal{R}_\varepsilon(\zeta) := \frac{1}{\varepsilon}\mathcal{R}^{\mathrm{tot}}(\varepsilon\zeta) = \mathcal{R}(\zeta) + \frac{1}{\varepsilon}\mathcal{R}_2(\varepsilon\zeta)\end{gathered} \tag{3.1}$$

with $\mathcal{V}_{2,\nu}$, $\mathcal{H}_{2,\nu}$, and $\mathcal{R}_2$, from (2.18), (2.34), and (2.37), respectively. With $H_{\varepsilon,\nu}$ we will denote the density of the integral functional $\mathcal{H}_{\varepsilon,\nu}$. We are now in a position to provide the variational formulation of (the Cauchy problem for) system (1.2).

Problem 3.1. Find a triple (u,z,p) with

$$\begin{aligned}&u \in H^1(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)), \quad z \in H^1(0,T;H^{\mathrm{m}}(\Omega)) \text{ with } W(z) \in L^\infty(0,T;L^1(\Omega)),\\ &p \in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})),\end{aligned} \tag{3.2}$$

such that, with $e(t) := \mathrm{E}(u(t)+w(t)) - p(t)$ and $\sigma(t) := \mathbb{C}(z(t))e(t)$, there holds

$$-\operatorname{Div}\big(\varepsilon\nu\mathbb{D}\mathrm{E}(u'(t)) + \sigma(t)\big) = F(t) \qquad \text{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*, \tag{3.3a}$$

$$\partial\mathcal{R}_\varepsilon(z'(t)) + A_{\mathrm{m}}(z(t)) + W'(z(t)) \ni -\frac{1}{2}\mathbb{C}'(z)e(t) : e(t) \qquad \text{in } H^{\mathrm{m}}(\Omega)^*, \tag{3.3b}$$

$$\partial_\pi H_{\varepsilon,\nu}(z(t),p'(t)) + \mu p(t) \ni \big(\sigma(t)\big)_{\mathrm{D}} \qquad \text{a.e. in } \Omega \tag{3.3c}$$

for almost all $t \in (0,T)$, joint with the initial conditions

$$u(0) = u_0 \text{ in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n), \qquad z(0) = z_0 \text{ in } H^{\mathrm{m}}(\Omega), \qquad p(0) = p_0 \text{ in } L^2(\Omega). \tag{3.4}$$

Remark 3.2. A few observations on formulation (3.3) are in order:

1. As shown in [CL16, Lemma 3.3], from the requirement $W(z) \in L^\infty(0,T; L^1(\Omega))$ we deduce the strict positivity property

$$\exists\, m_0 > 0 \quad \forall\,(x,t) \in \overline{\Omega} \times [0,T] : \qquad z(x,t) \geq m_0. \tag{3.5}$$

2. In view of (3.5) and of (2.21), we have that $W'(z) \in \mathrm{C}^0(\overline{\Omega} \times [0,T])$. The term featuring in (3.3b) has to be understood as the image of $W'(z(t)) \in \mathrm{C}^0(\overline{\Omega})$ under the Riesz mapping with values in $H^{\mathrm{m}}(\Omega)^*$.
3. By the monotonicity of $t \mapsto z(x,t)$ and the requirement that $z_0 \leq 1$ in $\overline{\Omega}$, we immediately infer that $z(x,t) \leq 1$ for all $(x,t) \in \overline{\Omega} \times [0,T]$ which, combined with (3.5), is consistent with the physical meaning of the damage variable.

The requirement $W(z) \in L^\infty(0,T;L^1(\Omega))$ which, as shown by Remark 3.2, has an important impact on the properties of the solution component z is in turn consistent with the gradient structure of system (3.3) with respect to the driving energy $\mathcal{E}_\mu$ from (2.23). To reveal this structure, it will be convenient to introduce the following notation for the triple (u,z,p) of state variables and the associated state space:

$$q := (u,z,p) \in \mathbf{Q} := H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) \times H^{\mathrm{m}}(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,. \tag{3.6}$$

With slight abuse of notation, we will write both $\mathcal{E}_\mu(t,u,z,p)$ and $\mathcal{E}_\mu(t,q)$.

Lemma 3.3. *For every $\mu > 0$ the proper domain of $\mathcal{E}_\mu : [0,T] \times \mathbf{Q} \to \mathbb{R} \cup \{+\infty\}$ is*

$$\mathbf{D}_T := [0,T] \times \mathbf{D} \qquad \text{with } \mathbf{D} = \{(u,z,p) \in \mathbf{Q} :\ z > 0 \text{ in } \overline{\Omega}\}.$$

For all $t \in [0,T]$, the functional $q \mapsto \mathcal{E}_\mu(t,q)$ is Fréchet differentiable on $\mathbf{D}$, with Fréchet differential

$$\begin{aligned} &\mathrm{D}_q\mathcal{E}_\mu(t,q) \\ &\quad = (\mathrm{D}_u\mathcal{E}_\mu(t,u,z,p), \mathrm{D}_z\mathcal{E}_\mu(t,u,z,p), \mathrm{D}_p\mathcal{E}_\mu(t,u,z,p)) \\ &\quad = \big(-\mathrm{Div}(\sigma(t)) - F(t), A_{\mathrm{m}}(z) + W'(z) + \tfrac{1}{2}\mathbb{C}'(z)e(t) : e(t), \mu p - \sigma_{\mathrm{D}}(t))\big) \in \mathbf{Q}^*. \end{aligned} \tag{3.7}$$

Furthermore, for all $q \in \mathbf{Q}$ the function $t \mapsto \mathcal{E}_\mu(t,q)$ is in $H^1(0,T)$, with

$$\partial_t\mathcal{E}_\mu(t,q) = \int_\Omega \sigma(t) : \mathrm{E}(w'(t))\,\mathrm{d}x - \langle F'(t), u+w(t)\rangle_{H^1(\Omega;\mathbb{R}^n)} - \langle F(t), w'(t)\rangle_{H^1(\Omega)} \tag{3.8}$$

for a.a. $t \in (0,T)$. Finally, the following chain-rule property holds: for all $q \in H^1(0,T;\mathbf{Q})$ with $\sup_{t\in[0,T]} |\mathcal{E}_\mu(t,q(t))| < +\infty$,

$$\begin{aligned} &\text{the mapping } t \mapsto \mathcal{E}_\mu(t,q(t)) \text{ is in } \mathrm{AC}([0,T]), \text{ and} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\mu(t,q(t)) = \langle \mathrm{D}_q\mathcal{E}_\mu(t,q(t)), q'(t)\rangle_{\mathbf{Q}} + \partial_t\mathcal{E}_\mu(t,q(t)) \qquad \text{for a.a. } t \in (0,T). \end{aligned} \tag{3.9}$$

Proof. First of all, (3.7) gives the Gâteaux differential of $\mathcal{E}_\mu(t,\cdot)$: we will just check the formula for $\mathrm{D}_u\mathcal{E}_\mu(t,u,z,p)$ by observing that, since $\mathcal{E}_\mu(t,\cdot,z,p)$ is convex, we have that $\eta = \mathrm{D}_u\mathcal{E}_\mu(t,u,z,p)$ if and only if it holds that $\mathcal{E}_\mu(t,v,z,p) - \mathcal{E}_\mu(t,u,z,p) \geq \langle \eta, v-u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)}$ or, equivalently, that

$$\begin{aligned} &\frac{1}{2}\int_\Omega \mathbb{C}(z)(\mathrm{E}(v{+}w(t)){-}p) : (\mathrm{E}(v{+}w(t)){-}p)\,\mathrm{d}x - \frac{1}{2}\int_\Omega \sigma(t){:}\, e(t)\,\mathrm{d}x - \langle F(t), v{-}u\rangle_{H^1_{\mathrm{Dir}}} \\ &\quad \geq \langle \eta, v{-}u\rangle_{H^1_{\mathrm{Dir}}} \end{aligned} \tag{3.10}$$

for all $v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$ (using the shorthand notation $\langle\cdot,\cdot\rangle_{H^1_{\mathrm{Dir}}}$). Ultimately, (3.10) holds true if and only if

$$\langle \eta, \tilde v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)} = \int_\Omega \sigma(t) : \mathrm{E}(\tilde v)\,\mathrm{d}x - \langle F(t), \tilde v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)}$$

for all $\tilde v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$. In order to check the Fréchet differentiability, it is enough to prove the continuity property

$$(q_n = (u_n, z_n, p_n) \to q = (u,z,p) \text{ in } \mathbf{Q}) \implies (\mathrm{D}_q\mathcal{E}_\mu(t,q_n) \to \mathrm{D}_q\mathcal{E}_\mu(t,q) \text{ in } \mathbf{Q}^*)\,. \tag{3.11}$$

For this, we observe that $z_n \to z$ in $H^{\mathrm{m}}(\Omega)$ implies $z_n \to z$ in $\mathrm{C}^0(\overline{\Omega})$ and, thus, $\mathbb{C}(z_n) \to \mathbb{C}(z)$ and $\mathbb{C}'(z_n) \to \mathbb{C}'(z)$ in $L^\infty(\Omega;\mathrm{Lin}(\mathbb{M}^{n\times n}_{\mathrm{sym}};\mathbb{M}^{n\times n}_{\mathrm{sym}}))$. Therefore, we have $\mathbb{C}(z_n)e_n(t) \to \mathbb{C}(z)e(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, which gives $\mathrm{D}_u\mathcal{E}_\mu(t,u_n,z,p) \to \mathrm{D}_u\mathcal{E}_\mu(t,u,z,p)$. We also find that $\mathbb{C}'(z_n)e_n(t){:}e_n(t) \to \mathbb{C}'(z)e(t){:}e(t)$ in $L^1(\Omega)$, hence we have the convergence $\mathrm{D}_z\mathcal{E}_\mu(t,u_n,z_n,p_n) \to \mathrm{D}_z\mathcal{E}_\mu(t,u,z,p)$ in $H^{\mathrm{m}}(\Omega)^*$. We easily have $\mathrm{D}_p\mathcal{E}_\mu(t,u_n,z_n,p_n) \to \mathrm{D}_p\mathcal{E}_\mu(t,u,z,p)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, which concludes the proof of (3.11).

By standard arguments we conclude (3.8) and (3.9). This finishes the proof. □

Let us now introduce the overall dissipation potential $\Psi_\nu : \mathbf{Q}\times\mathbf{Q} \to [0,+\infty]$

$$\Psi_\nu(q,q') := \mathcal{V}_{2,\nu}(u') + \mathcal{R}^{\mathrm{tot}}(z') + \mathcal{H}^{\mathrm{tot}}_\nu(z,p') \tag{3.12}$$

and its rescaled version $\Psi_{\varepsilon,\nu}(q,q') := \dfrac{1}{\varepsilon}\Psi_\nu(q,\varepsilon q') = \mathcal{V}_{\varepsilon,\nu}(u') + \mathcal{R}_\varepsilon(z') + \mathcal{H}_{\varepsilon,\nu}(z,p')\,.$

Taking into account (3.7), it is then a standard matter to reformulate Problem 3.1 in these terms: *find* $q \in H^1(0,T;\mathbf{Q})$ *with* $\sup_{t\in(0,T)}|\mathcal{E}_\mu(t,q(t))| < +\infty$ *solving the generalized gradient system*

$$\partial_{q'}\Psi_{\varepsilon,\nu}(q(t),q'(t)) + \mathrm{D}_q\mathcal{E}_\mu(t,q(t)) \ni 0 \qquad \text{in } \mathbf{Q}^* \quad \text{for a.a. } t\in(0,T). \tag{3.13}$$

This reformulation allows us to easily obtain the energy-dissipation balance underlying system (3.3), which is in fact *equivalent* to (3.13). Indeed, arguing as in [MRS13] (this observation is, however, at the core of the variational approach to gradient flows; cf. [AGS08]), we observe that (3.13), namely $-\mathrm{D}_q\mathcal{E}_\mu(t,q) \in \partial_{q'}\Psi_{\varepsilon,\nu}(q,q')$, is equivalent, by standard convex analysis results, to the identity

$$\Psi_{\varepsilon,\nu}(q(t),q'(t)) + \Psi^*_{\varepsilon,\nu}(q(t),-\mathrm{D}_q\mathcal{E}_\mu(t,q(t))) = \langle -\mathrm{D}_q\mathcal{E}_\mu(t,q(t)), q'(t)\rangle_{\mathbf{Q}} \tag{3.14}$$

for a.a. $t\in(0,T)$, with $\Psi^*_{\varepsilon,\nu} : \mathbf{Q}\times\mathbf{Q}^* \to [0,+\infty]$, $\Psi^*_{\varepsilon,\nu}(q,\xi) := \sup_{v\in\mathbf{Q}}(\langle\xi,v\rangle_{\mathbf{Q}} - \Psi_{\varepsilon,\nu}(q,v))$ the Fenchel–Moreau conjugate of $\Psi_{\varepsilon,\nu}(q,\cdot)$. By the definition of $\Psi^*_{\varepsilon,\nu}$, the $\geq$ estimate in (3.14) is automatically verified. Therefore, (3.14) is in fact equivalent to the $\leq$ estimate

$$\begin{aligned}\Psi_{\varepsilon,\nu}(q(t),q'(t)) + \Psi^*_{\varepsilon,\nu}(q(t),-\mathrm{D}_q\mathcal{E}_\mu(t,q(t))) &\leq \langle -\mathrm{D}_q\mathcal{E}_\mu(t,q(t)), q'(t)\rangle_{\mathbf{Q}} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\mu(t,q(t)) + \partial_t\mathcal{E}_\mu(t,q(t))\end{aligned} \tag{3.15}$$

for a.a. $t\in(0,T)$, where the latter identity follows from the chain rule (3.9). In fact, it is immediate to check that (3.13) is equivalent to the integrated versions of (3.14) and of (3.15). The latter reads

$$\begin{aligned}(3.16)\qquad &\int_0^t \left(\Psi_{\varepsilon,\nu}(q(r),q'(r)) + \Psi^*_{\varepsilon,\nu}(q(r),-\mathrm{D}_q\mathcal{E}_\mu(r,q(r)))\right)\mathrm{d}r + \mathcal{E}_\mu(t,q(t))\\ &\leq \mathcal{E}_\mu(0,q(0)) + \int_0^t \partial_t\mathcal{E}_\mu(r,q(r))\,\mathrm{d}r\end{aligned}$$

for all $t \in [0,T]$. Observe that for $\xi = (\eta,\chi,\omega) \in \mathbf{Q}^*$ we have

(3.17)

$$\Psi^*_{\varepsilon,\nu}(q,\xi) = \mathcal{V}^*_{\varepsilon,\nu}(\eta) + \mathcal{R}^*_\varepsilon(\chi) + \mathcal{H}^*_{\varepsilon,\nu}(z,\omega)$$

$$\text{with}\quad \begin{cases}\mathcal{V}^*_{\varepsilon,\nu}(\eta) = \dfrac{1}{2\varepsilon\nu}\displaystyle\int_\Omega \mathbb{D}^{-1}\tau\colon\tau\,\mathrm{d}x \quad \text{for } \eta = -\mathrm{Div}(\tau) \quad \text{and } \tau \in \widetilde{\Sigma}(\Omega),\\ \mathcal{R}^*_\varepsilon(\chi) = \dfrac{1}{2\varepsilon}\widetilde{d}^2_{L^2(\Omega)}(\chi,\partial\mathcal{R}(0)) := \dfrac{1}{2\varepsilon}\displaystyle\min_{\gamma\in\partial\mathcal{R}(0)} \mathfrak{f}_2(\chi-\gamma)\,,\\ \mathcal{H}^*_{\varepsilon,\nu}(z,\omega) = \dfrac{1}{2\varepsilon\nu}d^2_{L^2}(\omega,\partial_\pi\mathcal{H}(z,0)) := \dfrac{1}{2\varepsilon\nu}\displaystyle\min_{\rho\in\partial_\pi\mathcal{H}(z,0)} \|\omega-\rho\|^2_{L^2(\Omega)}\,,\end{cases}$$

where $\widetilde{\Sigma}(\Omega)$ is from (2.8),

$$\mathfrak{f}_2 : H^{\mathrm{m}}(\Omega)^* \to [0,+\infty] \text{ is defined by}\quad \mathfrak{f}_2(\beta) := \begin{cases}\|\beta\|^2_{L^2(\Omega)} & \text{if } \beta \in L^2(\Omega)\,,\\ +\infty & \text{if } \beta \in H^{\mathrm{m}}(\Omega)^*\setminus L^2(\Omega)\,,\end{cases}$$

and observe that the min in the definition of $\widetilde{d}_{L^2(\Omega)}(\chi,\partial\mathcal{R}(0))$ is attained as soon as $\widetilde{d}_{L^2(\Omega)}$ is finite. Indeed, we have calculated

$$\begin{aligned}\mathcal{V}^*_{\varepsilon,\nu}(-\mathrm{Div}(\tau)) &= \sup_{v\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)} \left(\langle -\mathrm{Div}(\tau),v\rangle_{H^1(\Omega;\mathbb{R}^n)} - \mathcal{V}_{\varepsilon,\nu}(v)\right)\\ &= \sup_{v\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)} \left(\int_\Omega \tau\colon \mathrm{E}(v)\,\mathrm{d}x - \frac{\varepsilon\nu}{2}\int_\Omega \mathbb{D}\mathrm{E}(v)\colon \mathrm{E}(v)\,\mathrm{d}x\right)\\ &= \frac{1}{2\varepsilon\nu}\int_\Omega \mathbb{D}^{-1}\tau\colon\tau\,\mathrm{d}x,\end{aligned}$$

whereas the formulae for $\mathcal{R}^*_\varepsilon$ and $\mathcal{H}^*_{\varepsilon,\nu}$ follow from the inf-sup convolution formula; cf., e.g., [IT79, Theorem 3.3.4.1]. Therefore, we may calculate explictly the second contribution to the left-hand side of (3.16). Indeed, recalling that, by (2.39e), we have $F(t) = -\mathrm{Div}(\rho(t))$, we find that

$$\begin{aligned}\mathcal{V}^*_{\varepsilon,\nu}(-\mathrm{D}_u\mathcal{E}_\mu(r,u(r),z(r),p(r))) = \mathcal{V}^*_{\varepsilon,\nu}(\mathrm{Div}(\sigma(r))+F(r)) &= \mathcal{V}^*_{\varepsilon,\nu}(\mathrm{Div}(\sigma(r)-\rho(r)))\\ &= \frac{1}{2\varepsilon\nu}\int_\Omega \mathbb{D}^{-1}(\sigma(r)-\rho(r))\colon(\sigma(r)-\rho(r))\,\mathrm{d}x\end{aligned}$$

(where $\sigma(r) = \mathbb{C}(z(r))e(r)$). All in all, we arrive at the following result, which will play a key role for the analysis of the rate-dependent system (1.2), since it provides a characterization of solutions to the viscous Problem 3.1.

Proposition 3.4. *The following properties are equivalent for a triple* $q = (u,z,p) \in H^1(0,T;\mathbf{Q})$ *fulfilling the initial conditions* (3.4)*:*

1. *q is a solution of Problem* 3.1*;*
2. *q fulfills the* energy-dissipation upper estimate

$$
\begin{aligned}
&\mathcal{E}_\mu(t,q(t)) + \int_0^t \left(\mathcal{V}_{\varepsilon,\nu}(u'(r)) + \mathcal{R}_\varepsilon(z'(r)) + \mathcal{H}_{\varepsilon,\nu}(z(r),p'(r))\right) \mathrm{d}r \\
&\quad + \int_0^t \Big[\mathcal{V}^*_{\varepsilon,\nu}\Big(\mathrm{Div}(\sigma(r)) + F(r)\Big) \\
&\qquad + \mathcal{R}^*_\varepsilon\Big(-A_{\mathrm{m}}(z(r)) - W'(z(r)) - \tfrac{1}{2}\mathbb{C}'(z(r))e(r):e(r)\Big) \\
&\qquad + \mathcal{H}^*_{\varepsilon,\nu}\Big(z(r), -\mu p(r) + \sigma_{\mathrm{D}}(r)\Big)\Big] \mathrm{d}r \\
&\leq \mathcal{E}_\mu(0,q_0) + \int_0^t \partial_t \mathcal{E}_\mu(r,q(r))\,\mathrm{d}r;
\end{aligned}
\tag{3.18}
$$

3. *q fulfills* (3.18) *as an* energy-dissipation balance*, integrated on any interval $[s,t] \subset [0,T]$.*

With the upcoming result we exhibit a further characterization of solutions to the viscous system that will be useful for the vanishing-viscosity analyses carried out in sections 6 and 7, borrowing an idea from [CL16]. Proposition 3.5 indeed shows that the *energy-dissipation balance* (3.18) can be rewritten in terms of the functionals

$$
\begin{aligned}
&\mathcal{N}^\mu_{\varepsilon,\nu}(t,q,q') := \mathcal{R}(z') + \mathcal{H}(z,p') + \mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon,\nu}(t,q,q'), \qquad \text{where} \\
&\mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon,\nu}(t,q,q') := \mathcal{D}_\nu(q')\,\mathcal{D}^{*,\mu}_\nu(t,q) \qquad \text{with} \\
&\mathcal{D}_\nu(q') := \sqrt{\nu\|u'(t)\|^2_{H^1,\mathbb{D}} + \|z'(t)\|^2_{L^2} + \nu\|p'(t)\|^2_{L^2}} \\
&\mathcal{D}^{*,\mu}_\nu(t,q) \\
&\quad := \sqrt{\frac{1}{\nu}\,\|-\mathrm{D}_u\mathcal{E}_\mu(t,q)\|^2_{(H^1,\mathbb{D})^*} + \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0))^2 + \frac{1}{\nu}\,d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q),\partial_\pi\mathcal{H}(z,0))^2}.
\end{aligned}
\tag{3.19}
$$

Proposition 3.5. *Along a solution $q \in H^1(0,T;\mathbf{Q})$ there holds for a.a. $r \in (0,T)$,*

$$
\begin{aligned}
&\mathcal{V}_{\varepsilon,\nu}(u'(r)) + \mathcal{R}_\varepsilon(z'(r)) + \mathcal{H}_{\varepsilon,\nu}(z(r),p'(r)) + \mathcal{V}^*_{\varepsilon,\nu}\Big(\mathrm{Div}(\sigma(r)) + F(r)\Big) \\
&\quad + \mathcal{R}^*_\varepsilon\Big(-A_{\mathrm{m}}(z(r)) - W'(z(r)) - \tfrac{1}{2}\mathbb{C}'(z(r))e(r):e(r)\Big) + \mathcal{H}^*_{\varepsilon,\nu}\Big(z(r), -\mu p(r) + \sigma_{\mathrm{D}}(r)\Big) \\
&= \mathcal{N}^\mu_{\varepsilon,\nu}(r,q(r),q'(r)) \\
&= \mathcal{R}(z'(r)) + \mathcal{H}(z(r),p'(r)) + \varepsilon\Big(\nu\|u'(r)\|^2_{H^1,\mathbb{D}} + \|z'(r)\|^2_{L^2} + \nu\|p'(r)\|^2_{L^2}\Big)\,.
\end{aligned}
\tag{3.20}
$$

In particular, a curve $q \in H^1(0,T;\mathbf{Q})$ is a solution to the Cauchy problem, Problem 3.1*, if and only if it satisfies for every $t \in [0,T]$ the energy-dissipation balance*

$$
\mathcal{E}_\mu(t,q(t)) + \int_0^t \mathcal{N}^\mu_{\varepsilon,\nu}(r,q(r),q'(r))\,\mathrm{d}r = \mathcal{E}_\mu(0,q_0) + \int_0^t \partial_t\mathcal{E}_\mu(r,q(r))\,\mathrm{d}r\,. \tag{3.21}
$$

Proof. First, we have that for a.e. $r \in (0,T)$

$$
\begin{aligned}
&\mathcal{N}^\mu_{\varepsilon,\nu}(r,q(r),q'(r)) = \mathcal{R}(z'(r)) + \mathcal{H}(z(r),p'(r)) + \mathcal{D}_\nu(q'(r))\,\mathcal{D}^{*,\mu}_\nu(r,q(r)) \\
&\leq \mathcal{R}(z'(r)) + \mathcal{H}(z(r),p'(r)) + \frac{\varepsilon}{2}\mathcal{D}^2_\nu(q'(r)) + \frac{1}{2\varepsilon}(\mathcal{D}^{*,\mu}_\nu(r,q(r)))^2 \\
&\leq \langle -\mathrm{D}_q\mathcal{E}_\mu(r,q(r)), q'(r)\rangle_{\mathbf{Q}}\,,
\end{aligned}
\tag{3.22}
$$

by the Cauchy inequality and (3.14), which holds along the solutions.

Let us now prove the converse inequality. Consider a measurable selection $r \mapsto \gamma(r) \in \partial\mathcal{R}(0)$ fulfilling

$$\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(r,q(r)),\partial\mathcal{R}(0)) = \|-\mathrm{D}_z\mathcal{E}_\mu(r,q(r)) - \gamma(r)\|_{L^2(\Omega)}$$

(observe that the existence of γ is guaranteed by the fact that $\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(r,q(r)), \partial\mathcal{R}(0)) < +\infty$ for almost all $r \in (0,T)$ and that $r \mapsto \mathrm{D}_z\mathcal{E}_\mu(r,q(r))$ is measurable). Analogously, let $r \mapsto \rho(r) \in \partial_\pi\mathcal{H}(z(r),0)$ fulfill

$$d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(r,q(r)),\partial_\pi\mathcal{H}(z(r),0)) = \|-\mathrm{D}_z\mathcal{E}_\mu(r,q(r)) - \rho(r)\|_{L^2(\Omega)} .$$

Then, we have (using shorter notation for the duality pairings between $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$ and $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*$, $H^{\mathrm{m}}(\Omega)$, and $H^{\mathrm{m}}(\Omega)^*$, and for the scalar product in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$)

(3.23)

$$\begin{aligned}
&\langle -\mathrm{D}_q\mathcal{E}_\mu(r,q(r)),q'(r)\rangle_{\mathbf{Q}} \\
&= \langle -\mathrm{D}_u\mathcal{E}_\mu(r,q(r)),u'(r)\rangle_{H^1} + \langle -\mathrm{D}_z\mathcal{E}_\mu(r,q(r)),z'(r)\rangle_{H^{\mathrm{m}}} + \langle -\mathrm{D}_p\mathcal{E}_\mu(r,q(r)),p'(r)\rangle_{L^2} \\
&\le \|u'(r)\|_{H^1,\mathbb{D}}\|-\mathrm{D}_u\mathcal{E}_\mu(r,q(r))\|_{(H^1,\mathbb{D})^*} + \langle -\mathrm{D}_z\mathcal{E}_\mu(r,q(r)) - \gamma(r),z'(r)\rangle_{H^{\mathrm{m}}} \\
&\quad + \langle \gamma(r),z'(r)\rangle_{H^{\mathrm{m}}} + \langle -\mathrm{D}_p\mathcal{E}_\mu(r,q(r)) - \rho(r),p'(r)\rangle_{L^2} + \langle \rho(r),p'(r)\rangle_{L^2} \\
&\overset{(1)}{\le} \|u'(r)\|_{H^1,\mathbb{D}}\|-\mathrm{D}_u\mathcal{E}_\mu(r,q(r))\|_{(H^1,\mathbb{D})^*} + \|z'(r)\|_{L^2}\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(r,q(r)),\partial\mathcal{R}(0)) \\
&\quad + \mathcal{R}(z'(r)) + \|p'(r)\|_{L^2}d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(r,q(r)),\partial_\pi\mathcal{H}(z(r),0)) + \mathcal{H}(z(r),p'(r)) \\
&\overset{(2)}{\le} \mathcal{R}(z'(r)) + \mathcal{H}(z(r),p'(r)) + \mathcal{D}_\nu(q'(r))\,\mathcal{D}^{*,\mu}_\nu(r,q(r)) = \mathcal{N}^\mu_{\varepsilon,\nu}(r,q(r),q'(r)),
\end{aligned}$$

where (1) follows from the very definition of $\partial\mathcal{R}(0)$ and $\partial\mathcal{H}(z(r),0)$ combined with the fact that $\mathcal{R}(0) = \mathcal{H}(z(r),0) = 0$, and (2) follows from the Cauchy–Schwarz inequality.

Then, all inequalities in (3.22) are equalities whence, in particular, we conclude that for a.a. $r \in (0,t)$

(3.24)

$$\mathcal{D}^{*,\mu}_\nu(r,q(r)) = \varepsilon\mathcal{D}_\nu(q'(r)) \quad \text{and} \quad \mathcal{N}^\mu_{\varepsilon,\nu}(r,q(r),q'(r)) = \langle -\mathrm{D}_q\mathcal{E}_\mu(r,q(r)),q'(r)\rangle_{\mathbf{Q}} .$$

This shows (3.20) and concludes the proof. □

We now study the semicontinuity properties of the distance-type functionals introduced in (3.17) that also enter the definition of $\mathcal{D}^{*,\mu}_\nu$. We will make use of the norms $\|\cdot\|_{(H^1,\mathbb{D})}$, $\|\cdot\|_{(H^1,\mathbb{D})^*}$ from (2.19), (2.20), and refer to the space $H^{\mathrm{m}}_-(\Omega)$ and the functional $\kappa : H^{\mathrm{m}}(\Omega) \to \mathbb{R}$ introduced in Lemma 2.4.

Lemma 3.6. *Let $\mu > 0$ be fixed. For any $(t,q) = (t,(u,z,p)) \in [0,T] \times \mathbf{Q}$ there holds*

(3.25a)
$$\|\mathrm{D}_u\mathcal{E}_\mu(t,q)\|_{(H^1,\mathbb{D})^*} = \sup_{\substack{\eta_u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\\ \|\eta_u\|_{(H^1,\mathbb{D})}\le 1}} \langle -\mathrm{Div}(\sigma(t)) - F(t),\eta_u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)} ,$$

(3.25b)
$$\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0))^2 = \sup_{\substack{\eta_z \in H^{\mathrm{m}}_-(\Omega)\\ \|\eta_z\|_{L^2}\le 1}} \langle A_{\mathrm{m}}(z) + W'(z) + \tfrac{1}{2}\mathbb{C}'(z)e(t) : e(t) + \kappa, -\eta_z\rangle_{H^{\mathrm{m}}(\Omega)} ,$$

(3.25c)
$$d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q),\partial_\pi\mathcal{H}(z,0)) = \sup_{\substack{\eta_p \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\\ \|\eta_p\|_{L^2}\le 1}} \Big(\langle \sigma_{\mathrm{D}}(t) - \mu p,\eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} - \mathcal{H}(z,\eta_p)\Big) .$$

Hence, for all $(t_k,q_k)_k,(t,q)\in[0,T]\times\mathbf{Q}$ with $t_k\to t$ and $q_k\rightharpoonup q$ in $\mathbf{Q}$ we have that

$$\|\mathrm{D}_u\mathcal{E}_\mu(t,q)\|_{(H^1,\mathbb{D})^*}\le\liminf_{k\to0}\|\mathrm{D}_u\mathcal{E}_\mu(t_k,q_k)\|_{(H^1,\mathbb{D})^*}\,,\tag{3.26a}$$

$$\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0))\le\liminf_{k\to0}\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(t_k,q_k),\partial\mathcal{R}(0))\,,\tag{3.26b}$$

$$d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q),\partial_\pi\mathcal{H}(z,0))\le\liminf_{k\to0}d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t_k,q_k),\partial_\pi\mathcal{H}(z_k,0))\,.\tag{3.26c}$$

Proof. (3.25): The well-known fact that $\|-\phi_u\|_{(H^1,\mathbb{D})^*}=\sup\{\langle-\phi_u,\eta_u\rangle_{(H^1,\mathbb{D})}\colon\|\eta_u\|_{(H^1,\mathbb{D})}\le1\}$ yields (3.25a). As for (3.25b), one has

$$\widetilde{d}_{L^2(\Omega)}(-\phi_z,\partial\mathcal{R}(0))=\sup\left\{\langle-\phi_z+\kappa,\eta_z\rangle_{H^{\mathrm{m}}(\Omega)}\colon\eta_z\in H^{\mathrm{m}}_-(\Omega)\,,\ \|\eta_z\|_{L^2(\Omega)}\le1\right\}.$$

This follows from [CL16, Remark 4.3, Lemma 4.4]; in fact, the set G from [CL16] equals the set $-\partial\mathcal{J}(\zeta)=-\partial\mathcal{R}(\zeta)-\kappa$ in the notation of the present paper (see Lemma 2.4). Finally, we have

$$\begin{aligned}
&\sup\left\{\langle-\phi_p,\eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}-\mathcal{H}(\mathsf{z},\eta_p)\colon\|\eta_p\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\le1\right\}\\
&=\sup_{\eta_p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\left\{\langle-\phi_p,\eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}-\mathcal{H}(\mathsf{z},\eta_p)-I_{B_{L^2}}(\eta_p)\right\}\\
&=\left(\mathcal{H}(\mathsf{z},\cdot)+I_{B_{L^2}}\right)^*(-\phi_p)=\min_{\eta_p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\left\{\mathcal{H}(\mathsf{z},\cdot)^*(\eta_p)+\|-\phi_p-\eta_p\|_{L^2}\right\}\\
&=\min_{\eta_p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\left\{\|-\phi_p-\eta_p\|_{L^2}+I_{\partial_\pi\mathcal{H}(\mathsf{z},0)}(\eta_p)\right\}=d_{L^2}(-\phi_p,\partial_\pi\mathcal{H}(\mathsf{z},0))\,,
\end{aligned}$$

where $I_{B_{L^2}}$ is the indicator function of the closed unit ball in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ (namely, $I_{B_{L^2}}(\eta_p)=0$ if $\|\eta_p\|_{L^2}\le1$ and $I_{B_{L^2}}(\eta_p)=+\infty$ otherwise). Hence (3.26c) follows, recalling (3.7).

(3.26): In order to show the lower semicontinuity properties (3.26), we notice that, for fixed $\eta_u\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$ and $\eta_p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, the functions $(t,q)\mapsto\langle-\mathrm{Div}(\sigma(t))-F(t),\eta_u\rangle$ and $(t,q)\mapsto\langle\sigma_{\mathrm{D}}(t)-\mu p,\eta_p\rangle-\mathcal{H}(z,\eta_p)$ in (3.25a) and (3.25c) (here we abbreviate the notation for the duality products) are continuous with respect to the convergence of t and the weak convergence in $\mathbf{Q}$. For this, we rely on assumptions (2.$\mathbb{C}$), on (2.28c), and also on the fact that if $q_k\rightharpoonup q$ in Q, then $z_k\to z$ in $\mathrm{C}^0(\overline{\Omega})$.

Moreover, for fixed $\eta_z\in H^{\mathrm{m}}_-(\Omega)$ the function $(t,q)\mapsto\langle A_{\mathrm{m}}(z)+W'(z)+\frac12\mathbb{C}'(z)e:e+\kappa,-\eta_z\rangle$ is semicontinuous with respect to the convergence of t and the weak convergence in $\mathbf{Q}$: the contribution $(t,q)\mapsto\langle A_{\mathrm{m}}(z)+W'(z)+\kappa,-\eta_z\rangle$ is continuous, recalling (2.1), (2.$\mathbb{C}$), (2.W), while $(t,q)\mapsto\langle\mathbb{C}'(z)e:e,-\eta_z\rangle$ is lower semicontinuous, since $-\eta_z\ge0$ (cf. also [CL16, (4.48) and (4.52)]).

Therefore we get (3.26) since, by (3.25), we are taking supremums of lower semicontinuous functions. ∎

4. Time discretization. In this section we discretize the rate-dependent system (1.2) and, again exploiting its underlying gradient structure, we derive a series of estimates on the discrete solutions that are uniform w.r.t. the discretization parameter τ, as well as the parameters ε, ν, and μ. Therefore,

- we will use these estimates to pass to the limit in the discretization scheme, for ε, ν, and μ fixed, and construct a solution to Problem 3.1 in section 5;
- since the viscous solutions to system (1.2) thus obtained will enjoy estimates uniform w.r.t. ε and ν, we will resort to them in the vanishing-viscosity analyses as $\varepsilon\downarrow0$ and $\varepsilon,\nu\downarrow0$, for $\mu>0$ fixed, carried out in section 6;

- the estimates that are also uniform w.r.t. $\mu > 0$ will be inherited by the viscous solutions. Therefore, we will exploit them to perform the *joint* vanishing-viscosity and vanishing-hardening analysis in section 7, as well.

We construct time-discrete solutions to the Cauchy problem for the rate-dependent system for damage and plasticity (1.2) by solving the following time-incremental minimization problems: for fixed ε, ν, $\mu > 0$, we consider a uniform partition $\{0 = t_\tau^0 < \cdots < t_\tau^N = T\}$ of the time interval $[0,T]$ with fineness $\tau = t_\tau^{k+1} - t_\tau^k = T/N$. We will use the notation $\eta_\tau^k := \eta(t_\tau^k)$ for $\eta \in \{w, F\}$. The elements $(q_\tau^k)_{0\le k\le N} = (u_\tau^k, z_\tau^k, p_\tau^k)_{0\le k\le N}$ are determined by

$$u_\tau^0 := u_0, \qquad z_\tau^0 := z_0, \qquad p_\tau^0 := p_0$$

and, for $k \in \{1, \dots, N\}$, by solving the time-incremental problems

$$\begin{aligned}
q_\tau^k \in \operatorname{Argmin}&\Big\{\tau\Psi_{\varepsilon,\nu}\left(q, \frac{q-q_\tau^{k-1}}{\tau}\right) + \mathcal{E}_\mu(t_\tau^k, q) \ : \ q \in \mathbf{Q}\Big\} \\
= \operatorname{Argmin}&\Big\{\frac{\varepsilon}{2\tau}\left(\int_\Omega \nu\mathbb{D}(\mathrm{E}(u)-\mathrm{E}(u_\tau^{k-1})) : (\mathrm{E}(u)-\mathrm{E}(u_\tau^{k-1}))\,\mathrm{d}x\right. \\
&\qquad \left. + \|z-z_\tau^{k-1}\|_{L^2}^2 + \nu\|p-p_\tau^{k-1}\|_{L^2}^2\right) \\
&\quad + \mathcal{R}(z-z_\tau^{k-1}) + \mathcal{H}(z, p-p_\tau^{k-1}) \\
&\quad + \mathcal{E}_\mu(t_\tau^k, u, p, z) \colon u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\,,\ z \in H^{\mathrm{m}}(\Omega)\,,\ p \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\Big\}\,.
\end{aligned} \tag{4.1}$$

Notice that, to shorten notation, we omit writing the dependence of the minimizers $(q_\tau^k)_{k=1}^N$ on the positive parameters ε, ν, and μ.

Remark 4.1. Taking into account that $\mathcal{R}(z-z_\tau^{k-1}) = \mathcal{P}(z-z_\tau^{k-1}) + \mathcal{I}(z-z_\tau^{k-1})$ with $\mathcal{P}$ and $\mathcal{I}$ from (2.36), it is immediate to check that the minimum problem (4.1) reformulates as

$$\begin{aligned}
q_\tau^k \in \operatorname{Argmin}\Bigg\{&\frac{\varepsilon}{2\tau}\Bigg(\int_\Omega \nu\mathbb{D}(\mathrm{E}(u)-\mathrm{E}(u_\tau^{k-1})) : (\mathrm{E}(u)-\mathrm{E}(u_\tau^{k-1}))\,\mathrm{d}x \\
&\qquad + \|z-z_\tau^{k-1}\|_{L^2}^2 + \nu\|p-p_\tau^{k-1}\|_{L^2}^2\Bigg) \\
&- \int_\Omega \kappa z\,\mathrm{d}x + \mathcal{H}(z, p-p_\tau^{k-1}) \\
&+ \mathcal{E}_\mu(t_\tau^k, u, p, z) \colon (u,z,p) \in \mathbf{Q},\ z \le z_\tau^{k-1} \text{ in } \Omega\Bigg\}\,.
\end{aligned}$$

Observe that, upon setting $\nu = \mu = 0$, the above problem does coincide with the time-incremental minimization scheme used to construct solutions to the viscous system in [CL16].

The existence of a minimizing triple for (4.1) relies on the coercivity properties of the functional $\mathcal{E}_\mu$, specified in Lemma 4.2 below. Let us highlight that the coercivity estimates below are *uniform* w.r.t. the hardening parameter $\mu \in [0,1]$, and in particular they are valid also for $\mu = 0$. This will have a key role in the derivation of a priori estimates on the viscous solutions uniform w.r.t. μ as well.

LEMMA 4.2. *There exist constants c_E, $C_E > 0$ such that for all $\mu \in [0,1]$ and $(t,u,z,p) \in [0,T] \times \mathbf{Q}$*

$$\begin{aligned} &\mathcal{E}_\mu(t,u,z,p) + \mathcal{H}(z,p) + \|z\|_{L^2(\Omega)} \\ &\geq c_E \Big(\|e(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \|z\|_{H^{\mathrm{m}}(\Omega)} + \mu^{1/2}\|p\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} \\ &\qquad + \mu^{1/2}\|u\|_{H^1(\Omega;\mathbb{R}^n)} + \|p\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} \Big) - C_E\,. \end{aligned} \tag{4.2}$$

Proof. In the following lines, we will use that $\mathcal{E}_\mu$ rewrites as $\mathcal{E}_\mu(t,u,z,p) = \mathcal{F}_\mu(t,u,z,p) - \int_\Omega \rho_{\mathrm{D}}(t) p \,\mathrm{d}x$; cf. (2.44). Now, taking into account (2.13) and the positivity of W, we easily have that

$$\begin{aligned} \mathcal{F}_\mu(t,u,z,p) &\geq \frac{\gamma_1}{2}\|e(t)\|_{L^2}^2 + \frac{\mu}{2}\|p\|_{L^2}^2 + \frac{1}{2}a_{\mathrm{m}}(z,z) - \frac{1}{2\gamma_1}\|\rho(t)\|_{L^2}^2 \\ &\geq \frac{\gamma_1}{2}\|e(t)\|_{L^2}^2 + \frac{\mu}{2}\|p\|_{L^2}^2 + \frac{1}{2}a_{\mathrm{m}}(z,z) - C_\rho. \end{aligned} \tag{4.3}$$

By (2.40), we deduce that

$$\mathcal{E}_\mu(t,u,z,p) + \mathcal{H}(z,p) \geq c \left(\|e(t)\|_{L^2}^2 + \mu\|p\|_{L^2}^2 + a_{\mathrm{m}}(z,z) + \|p\|_{L^1} \right) - C,$$

and (4.2) easily follows by a Korn–Poincar inequality for $u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$. □

By virtue of Lemma 4.2 and the direct method of calculus of variations, problem (4.1) does admit a solution $(q_\tau^k)_{0\leq k\leq N} = (u_\tau^k, z_\tau^k, p_\tau^k)_{0\leq k \leq N}$. Moreover, we set

$$e_\tau^k := \mathrm{E}(u_\tau^k + w_\tau^k) - p_\tau^k \qquad \text{and} \qquad \sigma_\tau^k := \mathbb{C}(z_\tau^k) e_\tau^k. \tag{4.4}$$

For $\eta \in \{q,u,e,z,p,\sigma,w,F\}$, we will use the shorthand notation

$$\dot{\eta}_\tau^k := \frac{\eta_\tau^k - \eta_\tau^{k-1}}{\tau} \qquad \text{for } k \in \{0,\ldots,N\}\,. \tag{4.5}$$

In addition, the following piecewise constant and piecewise linear interpolation functions will be used;

$$\begin{aligned} &\overline{\eta}_\tau(t) := \eta_\tau^k \text{ for } t \in (t_\tau^{k-1}, t_\tau^k], \ \ \underline{\eta}_\tau(t) := \eta_\tau^{k-1} \text{ for } t \in [t_\tau^{k-1}, t_\tau^k), \\ &\eta_\tau(t) := \eta_\tau^{k-1} + \frac{t - t_\tau^{k-1}}{\tau}(\eta_\tau^k - \eta_\tau^{k-1}) \text{ for } t \in [t_\tau^{k-1}, t_\tau^k] \end{aligned}$$

with $\overline{\eta}_\tau(0) := \eta_0$, $\eta_\tau(T) := \eta_\tau^k$. Furthermore, we will use the notation

$$\begin{aligned} \overline{t}_\tau(r) &= t_\tau^k && \text{for } r \in (t_\tau^{k-1}, t_\tau^k], \\ \underline{t}_\tau(r) &= t_\tau^{k-1} && \text{for } r \in [t_\tau^{k-1}, t_\tau^k). \end{aligned}$$

Relying on the sum rule from [Mor06, Proposition 1.107], we see that the minimizers $(q_\tau^k)_{k=1}^N$ for (4.1) satisfy the Euler–Lagrange equation

$$\partial_{q'}\Psi_{\varepsilon,\nu}\left(q_\tau^k, \frac{q_\tau^k - q_\tau^{k-1}}{\tau}\right) + \tau\,\partial_q \Psi_{\varepsilon,\nu}\left(q_\tau^k, \frac{q_\tau^k - q_\tau^{k-1}}{\tau}\right) \ni -\mathrm{D}_q \mathcal{E}_\mu(t_\tau^k, q_\tau^k) \qquad \text{in } \mathbf{Q}^* \tag{4.6}$$

for $k = 1,\ldots,N$, where, with a slight abuse of notation, we have denoted by $\partial_q \Psi_{\varepsilon,\nu}$ the *Fréchet subdifferential* of $q \mapsto \Psi_{\varepsilon,\nu}(q,q')$, i.e., the multivalued operator $\partial_q \Psi_{\varepsilon,\nu} : \mathbf{Q} \times \mathbf{Q} \rightrightarrows \mathbf{Q}^*$ defined by

$$\xi \in \partial_q \Psi_{\varepsilon,\nu}(q,q') \quad \text{if and only if} \quad \lim_{w\to q} \frac{\Psi_{\varepsilon,\nu}(w,q') - \Psi_{\varepsilon,\nu}(q,q') - \langle \xi, w-q\rangle_{\mathbf{Q}}}{\|w-q\|_{\mathbf{Q}}} \geq 0\,.$$

Now, $\partial_q \Psi_{\varepsilon,\nu}$ in fact reduces to the Fréchet subdifferential $\partial_z \mathcal{H} : \mathrm{C}^0(\overline{\Omega}) \times L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \rightrightarrows \mathrm{M}(\Omega)$. Hence, the term $\tau\partial_q\Psi_{\varepsilon,\nu}(q_\tau^k,\dot{q}_\tau^k)$ in (4.6) leads to the contribution $\tau\partial_z\mathcal{H}(z_\tau^k, \dot{\pi}_\tau^k) \in \mathrm{M}(\Omega) \subset H^{\mathrm{m}}(\Omega)^*$ that features in the discrete flow rule for the damage variable; cf. (4.7b) below. Taking into account Lemma 3.3, (4.6) in fact translates into the system, for all $k \in \{1,\ldots,N\}$,

(4.7a)
$$-\operatorname{Div}\big(\varepsilon\nu\mathbb{D}\mathrm{E}(\dot{u}_\tau^k)+\sigma_\tau^k\big) = F_\tau^k \qquad \text{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*,$$
(4.7b)
$$\partial\mathcal{R}_\varepsilon(\dot{z}_\tau^k) + A_{\mathrm{m}}(z_\tau^k) + W'(z_\tau^k) + \tau\,\partial_z H_{\varepsilon,\nu}(z_\tau^k,\dot{p}_\tau^k) \ni -\frac{1}{2}\mathbb{C}'(z_\tau^k)e_\tau^k : e_\tau^k \quad \text{in } H^{\mathrm{m}}(\Omega)^*,$$
(4.7c)
$$\partial_\pi H_{\varepsilon,\nu}(z_\tau^k,\dot{p}_\tau^k) + \mu p_\tau^k \ni \big(\sigma_\tau^k\big)_{\mathrm{D}} \qquad \text{a.e. in } \Omega.$$

For later use, let us rewrite system (4.7) in terms of the piecewise constant and linear interpolants of the discrete solutions, also taking into account the structure formulae (2.35) and (2.47): we have

$$\text{(4.8a)}\qquad -\operatorname{Div}\big(\varepsilon\nu\mathbb{D}\mathrm{E}(u_\tau')+\overline{\sigma}_\tau\big) = \overline{F}_\tau \qquad \text{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*,$$
$$\text{(4.8b)}\qquad \begin{aligned} &\overline{\chi}_\tau+\varepsilon z_\tau'+A_{\mathrm{m}}(\overline{z}_\tau)+W'(\overline{z}_\tau)+\tau\overline{\lambda}_\tau = -\frac{1}{2}\mathbb{C}'(\overline{z}_\tau)\overline{e}_\tau : \overline{e}_\tau \qquad \text{in } H^{\mathrm{m}}(\Omega)^* \\ &\qquad \text{with } \overline{\chi}_\tau \in \partial\mathcal{R}(z_\tau'),\ \overline{\lambda}_\tau \in \partial_z H_{\varepsilon,\nu}(\overline{z}_\tau,p_\tau'), \end{aligned}$$
$$\text{(4.8c)}\qquad \begin{aligned} &\overline{\omega}_\tau + \varepsilon\nu p_\tau' + \mu\overline{p}_\tau = (\overline{\sigma}_\tau)_{\mathrm{D}} \qquad \text{a.e. in } \Omega \\ &\qquad \text{with } \overline{\omega}_\tau \in \partial_\pi H(\overline{z}_\tau,p_\tau') \end{aligned}$$

almost everywhere in $(0,T)$.

Proposition 4.3 below collects the first set of a priori estimates for the discrete solutions. Essentially, these estimates are obtained from the basic energy estimate following from choosing the competitor $q = q_\tau^{k-1}$ in the minimum problem (4.1), which leads to

$$\text{(4.9)}\quad \mathcal{E}_\mu(t_\tau^k,q_\tau^k) + \tau\Psi_{\varepsilon,\nu}\left(q_\tau^k, \frac{q_\tau^k-q_\tau^{k-1}}{\tau}\right) \le \mathcal{E}_\mu(t_\tau^{k-1},q_\tau^{k-1}) + \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t\mathcal{E}_\mu(s,q_\tau^{k-1})\,\mathrm{d}s\,.$$

Let us mention in advance that, in Proposition 4.8 ahead, we will derive a finer discrete energy-dissipation inequality, which will be the starting point for the limit passage as $\tau \downarrow 0$.

Proposition 4.3 (basic energy estimates). *There exists a constant $C_1 > 0$, independent of ε, μ, ν, $\tau > 0$, such that the following estimates hold:*

(4.10a)
$$\sup_{t\in[0,T]}\Bigg(\|\overline{e}_\tau(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \|\overline{p}_\tau(t)\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} + \|\overline{u}_\tau(t)\|_{\mathrm{BD}(\Omega)} + \|\overline{z}_\tau(t)\|_{H^{\mathrm{m}}(\Omega)} \\ + \int_\Omega W(\overline{z}_\tau(t))\,\mathrm{d}x + \sqrt{\mu}\|\overline{p}_\tau(t)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} + \sqrt{\mu}\|\overline{u}_\tau(t)\|_{H^1(\Omega;\mathbb{R}^n)}\Bigg) \le C_1\,,$$
(4.10b)
$$\int_0^T \Big(\|p_\tau'(s)\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} + \|z_\tau'(s)\|_{L^1(\Omega)}\Big)\,\mathrm{d}s \le C_1\,,$$

(4.10c)
$$\varepsilon \int_0^T \left(\nu \|u_\tau'(s)\|^2_{H^1(\Omega;\mathbb{R}^n)} + \nu \|p_\tau'(s)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} + \|z_\tau'(s)\|^2_{L^2(\Omega)} \right) \mathrm{d}s \leq C_1 \,.$$

Therefore, there exists $m_0 > 0$*, independent of* ε*,* ν*,* μ*,* $\tau > 0$*, such that*

$$\overline{z}_\tau(x,t) \geq m_0 \,, \qquad z_\tau(x,t) \geq m_0 \quad \forall (x,t) \in [0,T] \times \overline{\Omega} \,. \tag{4.11}$$

Proof. It is immediate to check that the time-incremental minimization problem (4.1) is equivalent to

$$\begin{aligned} q_\tau^k \in \mathrm{Argmin} \Bigg\{ & \frac{\varepsilon}{2\tau} \left(\int_\Omega \nu \, \mathbb{D}(\mathrm{E}(u) - \mathrm{E}(u_\tau^{k-1})) : (\mathrm{E}(u) - \mathrm{E}(u_\tau^{k-1})) \, \mathrm{d}x \right. \\ & \left. + \|z - z_\tau^{k-1}\|^2_{L^2} + \nu \|p - p_\tau^{k-1}\|^2_{L^2} \right) \\ & + \mathcal{R}(z - z_\tau^{k-1}) + \mathcal{H}(z_\tau^k, p - p_\tau^{k-1}) \\ & - \int_\Omega (\rho_\tau^k(t))_{\mathrm{D}} (p - p_\tau^{k-1}) \, \mathrm{d}x + \mathcal{F}_\mu(t_\tau^k, u, p, z) \colon (u,z,p) \in \mathbf{Q} \Bigg\} \end{aligned}$$

with $\mathcal{F}_\mu$ from (2.43). Then, considering the analogue of estimate (4.9) and summing it up with respect to the index $k = 1, \dots, j$, with j arbitrary in $\{1, \dots, N\}$, we find

$$\begin{aligned} & \mathcal{F}_\mu(t_\tau^j, q_\tau^j) + \sum_{k=1}^j \left[\tau \Psi_{\varepsilon,\nu} \left(q_\tau^k, \frac{q_\tau^k - q_\tau^{k-1}}{\tau} \right) - \int_\Omega (\rho_\tau^k(t))_{\mathrm{D}} (p_\tau^k - p_\tau^{k-1}) \, \mathrm{d}x \right] \\ & \leq \mathcal{F}_\mu(0, q_\tau^0) + \sum_{k=1}^j \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_t \mathcal{F}_\mu(s, q_\tau^{k-1}) \, \mathrm{d}s. \end{aligned} \tag{4.12}$$

On the one hand, again thanks to (2.40) we have that

$$\tau \Psi_{\varepsilon,\nu} \left(q_\tau^k, \frac{q_\tau^k - q_\tau^{k-1}}{\tau} \right) - \int_\Omega (\rho_\tau^k(t))_{\mathrm{D}} (p_\tau^k - p_\tau^{k-1}) \, \mathrm{d}x \geq \tau \widetilde{\Psi}_{\varepsilon,\nu} \left(\frac{q_\tau^k - q_\tau^{k-1}}{\tau} \right)$$

with $\widetilde{\Psi}_{\varepsilon,\nu}(q') := \mathcal{V}_{\varepsilon,\nu}(u') + \mathcal{R}_\varepsilon(z') + \alpha \|p'\|_{L^1}$.

On the other hand, since $\partial_t \mathcal{F}_\mu(t,q) = \int_\Omega \sigma(t) : \mathrm{E}(w'(t)) \, \mathrm{d}x - \int_\Omega \rho'(t)(e(t) - \mathrm{E}(w(t)) \, \mathrm{d}x - \partial_t(\langle F(t), w(t) \rangle_{H^1})$, we easily find also in view of (2.13), of (2.39d)–(2.39e), and of (2.41) that

$$\begin{aligned} & |\partial_t \mathcal{F}_\mu(t,q)| \leq \mathcal{L}(t) \|e\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \tilde{\mathcal{L}}(t) \qquad \text{with} \\ & \begin{cases} \mathcal{L}(t) := C \left(\|w'(t)\|_{H^1} + \|\varrho'(t)\|_{L^2} \right) \in L^1(0,T) \,, \\ \tilde{\mathcal{L}}(t) := C' \|F'(t)\|_{(H^1)^*} \in L^1(0,T). \end{cases} \end{aligned}$$

From (4.12) we then gather that

(4.13)
$$
\begin{aligned}
&\mathcal{F}_\mu(t_\tau^j, q_\tau^j) + \sum_{k=1}^{j} \tau \widetilde{\Psi}_{\varepsilon,\nu}(q_\tau^k, \dot{q}_\tau^k) \\
&\leq \mathcal{F}_\mu(0, q_\tau^0) + \sum_{k=1}^{j} \|e_\tau^{k-1}\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} \int_{t_\tau^{k-1}}^{t_\tau^k} \mathcal{L}(s)\,\mathrm{d}s + \int_0^T \tilde{\mathcal{L}}(t)\,\mathrm{d}t \\
&\overset{(1)}{\leq} \mathcal{F}_1(0, q_\tau^0) + \|e_\tau^0\| \|\mathcal{L}\|_{L^1(0,T)} + \sum_{k=1}^{j} \left(\mathcal{F}_\mu(t_\tau^{k-1}, q_\tau^{k-1}) + C_\rho + \frac{1}{2\gamma_1} \right) \int_{t_\tau^{k-1}}^{t_\tau^k} \mathcal{L}(s)\,\mathrm{d}s \\
&\qquad + \int_0^T \tilde{\mathcal{L}}(t)\,\mathrm{d}t \\
&\overset{(2)}{\leq} C + \frac{2}{\gamma_1} \sum_{k=0}^{j-1} \left(\mathcal{F}_\mu(t_\tau^k, q_\tau^k) + C_\rho \right) \int_{t_\tau^k}^{t_\tau^{k+1}} \mathcal{L}(s)\,\mathrm{d}s + \int_0^T \tilde{\mathcal{L}}(t)\,\mathrm{d}t,
\end{aligned}
$$

where (1) and (2) follow from the fact that, by (2.38) and $\mu \in [0,1]$, it holds that $\mathcal{F}_\mu(0, q_\tau^0) \leq \mathcal{F}_1(0, q_\tau^0) \leq C$ uniformly in μ and $\tau > 0$, as well as from estimate (4.3). We are now in a position to apply a version of the discrete Gronwall lemma (cf., e.g., Lemma A.1 ahead) to conclude that

$$
\mathcal{F}_\mu(t_\tau^j, q_\tau^j) + C_\rho \leq C' \exp\left(\frac{2}{\gamma_1} \sum_{k=0}^{j-1} \int_{t_\tau^k}^{t_\tau^{k+1}} \mathcal{L}(s)\,\mathrm{d}s \right) \leq C,
$$

where the latter estimate follows from (2.39d)–(2.39e) and (2.41). All in all, from (4.13) we conclude that

$$
\exists\, C > 0 \quad \forall\, \varepsilon,\, \nu,\, \mu,\, \tau > 0 \quad \forall\, j \in \{1, \ldots, N\}, \qquad |\mathcal{F}_\mu(t_\tau^j, q_\tau^j)| + \sum_{k=1}^{j} \tau \widetilde{\Psi}_{\varepsilon,\nu}(q_\tau^k, \dot{q}_\tau^k) \leq C.
$$

In particular, we find that $\|p_\tau^j\|_{L^1(\Omega)} \leq C$. Then, recalling that $\mathcal{E}_\mu(t,q) = \mathcal{F}_\nu(t,q) - \int_\Omega \rho_{\mathrm{D}}(t) p \,\mathrm{d}x$ and that $\rho_{\mathrm{D}} \in L^\infty(0,T; L^\infty(\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}))$, and using (2.30b), it is immediate to check that

$$
\exists\, C > 0 \quad \forall\, \varepsilon,\, \nu,\, \mu,\, \tau > 0 : \qquad \sup_{t\in[0,T]} \left| \mathcal{E}_\mu(\bar{t}_\tau(t), \overline{q}_\tau(t)) \right| + \int_0^T \Psi_{\varepsilon,\nu}(\overline{q}_\tau(s), q_\tau'(s))\,\mathrm{d}s \leq C\,.
$$

Then, estimates (4.10b) and (4.10c) immediately follow, while (4.10a) ensues due to the coercivity property (4.2). Let us additionally mention that the estimates for $\overline{e}_\tau$ and $\overline{p}_\tau$ entail a bound for $\mathrm{E}(\overline{u}_\tau)$ in $L^\infty(0,T; L^1(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}))$, which yields the bound for $\overline{u}_\tau$ in $L^\infty(0,T; \mathrm{BD}(\Omega))$ via the Poincaré-type inequality (2.4). Property (4.11) can be deduced from $\sup_{t\in[0,T]} \int_\Omega W(\overline{z}_\tau(t))\,\mathrm{d}x \leq C_1$ (cf. (4.10a)) arguing as in [CL16, Lemma 3.3]; cf. also Remark 3.2. □

The following step is the derivation of *enhanced* a priori estimates for the discrete solutions $(q_\tau)_\tau = (u_\tau, z_\tau, p_\tau)_\tau$. With Proposition 4.3 we have obtained for $(q_\tau)_\tau$ an a priori estimate in $H^1(0,T; \mathbf{Q})$ that blows up as $\varepsilon,\, \nu \downarrow 0$; it will be used to conclude the existence of viscous solutions to system (1.2) for ε, ν, and $\mu > 0$ fixed. Now, with Proposition 4.4 below,

1. we prove a set of *enhanced* a priori estimates, *uniform* in τ, ν, μ, and blowing up as $\varepsilon \downarrow 0$, that will ensure the existence of solutions to the viscous system with higher temporal regularity than that guaranteed by Proposition 4.3;
2. we obtain a set of a priori estimates, also *uniform in* ε, that will be at the basis of the vanishing-viscosity analyses carried out in section 6, as well as of the vanishing-hardening limit passage in section 7.

All of these estimates will hold under the further condition that $\nu \le \mu$, which is consistent

- both with the situation in which the hardening parameter μ is kept fixed, the viscosity parameter ε vanishes, and either ν is kept fixed (cf. section 6.1), or ν vanishes along with ε (cf. section 6.2);
- and with the case where we perform joint vanishing-viscosity and vanishing-hardening analysis for the viscous solutions (cf. section 7).

We prove Proposition 4.4 under the following additional conditions on the initial data $q_0 = (u_0, z_0, p_0)$:

$$
\begin{aligned}
&\mathrm{D}_q\mathcal{E}_\mu(0,q_0) = (\mathrm{D}_u\mathcal{E}_\mu(0,u_0,z_0,p_0), \mathrm{D}_z\mathcal{E}_\mu(0,u_0,z_0,p_0), \mathrm{D}_p\mathcal{E}_\mu(0,u_0,z_0,p_0)) \\
&\quad = \big(-\mathrm{Div}(\sigma_0) - F(0), A_{\mathrm{m}}(z_0) + W'(z_0) + \tfrac{1}{2}\mathbb{C}'(z_0)e_0 : e_0, \mu p_0 - (\sigma_0)_{\mathrm{D}}\big) \\
&\quad \in L^2(\Omega;\mathbb{R}^n \times \mathbb{R} \times \mathbb{M}^{n\times n}_{\mathrm{D}}).
\end{aligned} \tag{4.14}
$$

Proposition 4.4 (enhanced a priori estimates). *Under the assumptions of section* 2, *suppose in addition that the initial data* (u_0, z_0, p_0) *fulfill conditions* (4.14). *Then, for* $\frac{\tau}{\varepsilon}$ *small enough, we have that*

1. *there exists a constant* $C_2^\varepsilon > 0$, *independent of* τ, ν, $\mu > 0$, *with* $C_2^\varepsilon \uparrow +\infty$ *as* $\varepsilon \downarrow 0$, *such that for all* τ, ν, $\mu > 0$ *with* $\underline{\nu \le \mu}$ *there holds*

$$
\begin{aligned}
&\sqrt{\mu}\|\dot u_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} + \|\dot z_\tau\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \sqrt{\mu}\|\dot p_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))} \le C_2^\varepsilon, \\
&\|\dot e_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))} + \|\dot z_\tau\|_{L^2(0,T;H^{\mathrm{m}}(\Omega))} \le C_2^\varepsilon;
\end{aligned} \tag{4.15a}
$$

2. *there exists a constant* $C_2 > 0$, *independent of* ε, τ, ν, $\mu > 0$, *such that for all* τ, ε, ν, $\mu > 0$ *with* $\underline{\nu \le \mu}$ *there holds*

$$
\begin{aligned}
&\|\dot e_\tau\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))} + \|\dot z_\tau\|_{L^1(0,T;H^{\mathrm{m}}(\Omega))} + \sqrt{\mu}\|\dot p_\tau\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))} \\
&\quad + \sqrt{\mu}\|\dot u_\tau\|_{L^1(0,T;H^1(\Omega;\mathbb{R}^n))} \le C_2\,.
\end{aligned} \tag{4.15b}
$$

As we will see in Remark 4.6 later on, assuming only (2.39a) in place of (2.39c), estimates (4.15) hold for two constants $C_2^{\varepsilon,\mu}$ and C_2^μ depending also on $\mu > 0$.

Outline of the proof. Our argument will be split into the following steps:

1. The first step basically corresponds to "differentiating w.r.t. time" the discrete Euler–Lagrange equations/subdifferential inclusions satisfied by the discrete solutions and testing them by $\dot u_\tau^k$, $\dot z_\tau^k$, $\dot p_\tau^k$, respectively. In practice, we will do so with the discrete equations for u_τ^k and p_τ^k (i.e., (4.7a) and (4.7c)), while, instead of working with the discrete flow rule (4.7b) for z (and dealing with the Fréchet subdifferential term therein), we will resort to (4.16) and (4.17) below, which are a key consequence of the minimum problem (4.1). We will add up the resulting relations and perform suitable calculations.
2. Next, we perform a suitable estimate of $\|\dot p_\tau^k\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}$. The key role of this calculation is commented upon in Remark 4.7 ahead.

3. We will rearrange the estimate obtained in steps 1–2.
4. The tasks in steps 1–3 are addressed by working with the discrete Euler–Lagrange system (4.7) for $k \in \{2, \dots, N_\tau\}$. In this step, we will separately treat the case $k = 1$.
5. We will apply the Gronwall lemma, Lemma A.2, to get estimates (4.15a), blowing up as $\varepsilon \downarrow 0$.
6. We will apply the Gronwall-type Lemma A.3 to get estimates (4.15b), uniform w.r.t. ε, ν, $\mu > 0$.

We will also use the following result.

LEMMA 4.5 (see [CL16, Lemma 3.4]). *The minimizers* $(q_\tau^k)_{k=1}^{N_\tau}$ *of* (4.1) *satisfy for all* $k \in \{1, \dots, N_\tau\}$ *and* $\zeta \in H^{\mathrm{m}}(\Omega)$

$$\mathcal{R}(\zeta) + \varepsilon \int_\Omega \dot{z}_\tau^k \zeta \, \mathrm{d}x + a_{\mathrm{m}}(z_\tau^k, \zeta) + \int_\Omega \left(W'(z_\tau^k) + \frac{1}{2}\mathbb{C}'(z_\tau^k) e_\tau^k e_\tau^k \right) \zeta \geq 0, \tag{4.16}$$

$$\mathcal{R}(\dot{z}_\tau^k) + \varepsilon \|\dot{z}_\tau^k\|_{L^2}^2 + a_{\mathrm{m}}(z_\tau^k, \dot{z}_\tau^k) + \int_\Omega \left(W'(z_\tau^k) + \frac{1}{2}\mathbb{C}'(z_\tau^k) e_\tau^k e_\tau^k \right) \dot{z}_\tau^k \leq C_K \tau \|\dot{z}_\tau^k\|_{L^\infty} \|\dot{p}_\tau^k\|_{L^1} \tag{4.17}$$

with C_K *from* (2.28c).

Proof of Proposition 4.4.

Step 1*:* For $k \in \{2, \dots, N_\tau\}$, let us subtract (4.7a) at step $k-1$ from (4.7a) at step k. Testing the resulting relation by $\dot{u}_\tau^k$, we obtain

$$\underbrace{\int_\Omega \varepsilon\nu \mathbb{D}\mathrm{E}(\dot{u}_\tau^k - \dot{u}_\tau^{k-1}) : \mathrm{E}(\dot{u}_\tau^k) \, \mathrm{d}x}_{\doteq I_1} + \underbrace{\int_\Omega (\sigma_\tau^k - \sigma_\tau^{k-1}) : \mathrm{E}(\dot{u}_\tau^k) \, \mathrm{d}x}_{\doteq I_2} = \underbrace{\langle F_\tau^k - F_\tau^{k-1}, \dot{u}_\tau^k \rangle_{H^1(\Omega;\mathbb{R}^n)}}_{\doteq I_3} . \tag{4.18}$$

Since $\int_\Omega \mathbb{D}\mathrm{E}(u_1) : (\mathrm{E}(u_1) - \mathrm{E}(u_2)) \, \mathrm{d}x \geq \|u_1\|_{H^1,\mathbb{D}} (\|u_1\|_{H^1,\mathbb{D}} - \|u_2\|_{H^1,\mathbb{D}}) \geq \frac{1}{2}\|u_1\|_{H^1,\mathbb{D}}^2 - \frac{1}{2}\|u_2\|_{H^1,\mathbb{D}}^2$, we have

$$I_1 \geq \varepsilon\nu \|\dot{u}_\tau^k\|_{H^1,\mathbb{D}} (\|\dot{u}_\tau^k\|_{H^1,\mathbb{D}} - \|\dot{u}_\tau^{k-1}\|_{H^1,\mathbb{D}}).$$

As for I_2, we use that $\mathrm{E}(\dot{u}_\tau^k) = \dot{e}_\tau^k + \dot{p}_\tau^k - \mathrm{E}(\dot{w}_\tau^k)$ and that $\sigma_\tau^k = \mathbb{C}(z_\tau^k) e_\tau^k$ (cf. (4.4)), so that

$$\sigma_\tau^k - \sigma_\tau^{k-1} = \mathbb{C}(z_\tau^k)(e_\tau^k - e_\tau^{k-1}) + \left(\mathbb{C}(z_\tau^k) - \mathbb{C}(z_\tau^{k-1})\right) e_\tau^{k-1}. \tag{4.19}$$

Therefore,

$$I_2 = \underbrace{\int_\Omega \mathbb{C}(z_\tau^k)(e_\tau^k - e_\tau^{k-1}) : \dot{e}_\tau^k \, \mathrm{d}x}_{\doteq I_{2,1}} + \underbrace{\int_\Omega \left(\mathbb{C}(z_\tau^k) - \mathbb{C}(z_\tau^{k-1})\right) e_\tau^{k-1} : \dot{e}_\tau^k \, \mathrm{d}x}_{\doteq I_{2,2}} + \underbrace{\int_\Omega (\sigma_\tau^k - \sigma_\tau^{k-1}) \dot{p}_\tau^k \, \mathrm{d}x}_{\doteq I_{2,3}} - \underbrace{\int_\Omega (\sigma_\tau^k - \sigma_\tau^{k-1}) \mathrm{E}(\dot{w}_\tau^k) \, \mathrm{d}x}_{\doteq I_{2,4}} .$$

Now, we have that

$$I_{2,1} = \tau \int_\Omega \mathbb{C}(z_\tau^k)\dot{e}_\tau^k : \dot{e}_\tau^k \,\mathrm{d}x \geq \gamma_1 \tau \|\dot{e}_\tau^k\|_{L^2}^2$$

by (2.13), whereas, since the mapping $z \mapsto \mathbb{C}(z)$ is Lipschitz continuous,

$$|I_{2,2}| \leq C\tau \|\dot{z}_\tau^k\|_{L^\infty} \|e_\tau^{k-1}\|_{L^2} \|\dot{e}_\tau^k\|_{L^2} \leq C\tau \|\dot{z}_\tau^k\|_{L^\infty} \|\dot{e}_\tau^k\|_{L^2},$$

where the last estimate follows from the previously obtained (4.10a). While the term $I_{2,3}$ will be canceled in the next lines, again relying on (4.19) and the Lipschitz continuity of $\mathbb{C}$, we estimate

$$\begin{aligned} |I_{2,4}| &\leq \tau \|\mathbb{C}(z_\tau^k)\|_{L^\infty} \|\dot{e}_\tau^k\|_{L^2} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + C\tau \|\dot{z}_\tau^k\|_{L^\infty} \|e_\tau^{k-1}\|_{L^2} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} \\ &\leq C\tau \left(\|\dot{e}_\tau^k\|_{L^2} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{z}_\tau^k\|_{L^\infty} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} \right), \end{aligned}$$

where the latter estimate again follows from (4.10a). Finally, recalling that $F \in H^1(0,T;\mathrm{BD}(\Omega)^*)$, we may estimate

$$\begin{aligned} |I_3| &\leq \tau \|\dot{F}_\tau^k\|_{\mathrm{BD}(\Omega)^*} \|\dot{u}_\tau^k\|_{\mathrm{BD}(\Omega)} \leq C\tau \|\dot{F}_\tau^k\|_{\mathrm{BD}(\Omega)^*} \|\mathrm{E}(\dot{u}_\tau^k)\|_{L^1} \\ &\leq C\tau \|\dot{F}_\tau^k\|_{\mathrm{BD}(\Omega)^*} \left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^1} + \|\dot{e}_\tau^k\|_{L^1} + \|\dot{p}_\tau^k\|_{L^1} \right), \end{aligned} \tag{4.20}$$

where the second estimate follows from Poincaré's inequality for $\mathrm{BD}(\Omega)$ (cf. (2.4)), and the very last one follows from the fact that $\mathrm{E}(\dot{u}_\tau^k) = \dot{e}_\tau^k + \dot{p}_\tau^k - \mathrm{E}(\dot{w}_\tau^k)$. All in all, combining the above calculations with (4.18), we conclude that

(4.21)

$$\begin{aligned} &\varepsilon\nu \|\dot{u}_\tau^k\|_{H^1,\mathbb{D}} (\|\dot{u}_\tau^k\|_{H^1,\mathbb{D}} - \|\dot{u}_\tau^{k-1}\|_{H^1,\mathbb{D}}) + \gamma_1 \tau \|\dot{e}_\tau^k\|_{L^2}^2 + \int_\Omega (\sigma_\tau^k - \sigma_\tau^{k-1})_{\mathrm{D}} \dot{p}_\tau^k \,\mathrm{d}x \\ &\leq C\tau \Big(\|\dot{z}_\tau^k\|_{L^\infty} \|\dot{e}_\tau^k\|_{L^2} + \|\dot{e}_\tau^k\|_{L^2} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{z}_\tau^k\|_{L^\infty} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{F}_\tau^k\|_{\mathrm{BD}^*} \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^1} \\ &\qquad + \|\dot{F}_\tau^k\|_{\mathrm{BD}^*} \|\dot{e}_\tau^k\|_{L^1} + \|\dot{F}_\tau^k\|_{\mathrm{BD}^*} \|\dot{p}_\tau^k\|_{L^1} \Big). \end{aligned}$$

Let us now consider estimate (4.17) at step k and subtract from it (4.16) at step $k-1$ (recall that $k \in \{2, \ldots, N_\tau\}$), with the test function $\beta := \dot{z}_\tau^k$. We thus obtain

(4.22)

$$\begin{aligned} &\underbrace{\mathcal{R}(\dot{z}_\tau^k) - \mathcal{R}(\dot{z}_\tau^k)}_{=0} + \varepsilon \int_\Omega (\dot{z}_\tau^k - \dot{z}_\tau^{k-1}) \dot{z}_\tau^k \,\mathrm{d}x + a_{\mathrm{m}}(z_\tau^k - z_\tau^{k-1}, \dot{z}_\tau^k) \\ \leq\ &\underbrace{\int_\Omega \left[W'(z_\tau^{k-1}) - W'(z_\tau^k) \right] \dot{z}_\tau^k \,\mathrm{d}x}_{\doteq I_4} + \underbrace{\frac{1}{2} \int_\Omega \left[\mathbb{C}'(z_\tau^{k-1}) - \mathbb{C}'(z_\tau^k) \right] e_\tau^k : e_\tau^k \, \dot{z}_\tau^k \,\mathrm{d}x}_{\doteq I_5} \\ &- \underbrace{\frac{1}{2} \int_\Omega \left(\mathbb{C}'(z_\tau^{k-1}) e_\tau^k : e_\tau^k - \mathbb{C}'(z_\tau^{k-1}) e_\tau^{k-1} : e_\tau^{k-1} \right) \dot{z}_\tau^k \,\mathrm{d}x}_{\doteq I_6} + C_K \tau \|\dot{z}_\tau^k\|_{L^\infty} \|\dot{p}_\tau^k\|_{L^1}. \end{aligned}$$

Now, recall that, by (4.11), $0 < m_0 \leq \dot{z}_\tau^k \leq 1$ for all $k \in \{0, \ldots, N_\tau\}$. Since the restriction of W' to $[m_0, 1]$ is Lipschitz continuous, we conclude that

$$|I_4| \leq C \int_\Omega |z_\tau^k - z_\tau^{k-1}| |\dot{z}_\tau^k| \,\mathrm{d}x \leq C\tau \|\dot{z}_\tau^k\|_{L^2}^2;$$

by the Lipschitz continuity of $\mathbb{C}'$ we have that

$$|I_5| \le C \int_\Omega |z_\tau^k - z_\tau^{k-1}||e_\tau^k|^2 |\dot z_\tau^k| \,\mathrm{d}x \le C\tau \|\dot z_\tau^k\|_{L^\infty}^2 \|e_\tau^k\|_{L^2}^2 \le C\tau \|\dot z_\tau^k\|_{L^\infty}^2 \,,$$

the latter estimate due to (4.10a); finally,

$$|I_6| \le C \int_\Omega |e_\tau^k + e_\tau^{k-1}||e_\tau^k - e_\tau^{k-1}||\dot z_\tau^k| \,\mathrm{d}x \le C\tau \|\dot e_\tau^k\|_{L^2} \|\dot z_\tau^k\|_{L^\infty} \,,$$

where we have used that $\|\mathbb{C}(z_\tau^k)\|_{L^\infty} \le C$, and again the previously proved (4.10a). Inserting the above estimates into (4.22) leads to

(4.23)

$$\varepsilon \|\dot z_\tau^k\|_{L^2} \left(\|\dot z_\tau^k\|_{L^2} - \|\dot z_\tau^{k-1}\|_{L^2} \right) + \tau a_{\mathrm{m}}(\dot z_\tau^k, \dot z_\tau^k) \le C\tau \|\dot z_\tau^k\|_{L^\infty} \left(\|\dot z_\tau^k\|_{L^\infty} + \|\dot e_\tau^k\|_{L^2} + \|\dot p_\tau^k\|_{L^1} \right).$$

Prior to working with (4.7c), let us specify that it reformulates as

$$\omega_\tau^k + \varepsilon\nu \dot p_\tau^k + \mu p_\tau^k = (\sigma_\tau^k)_{\mathrm{D}} \qquad \text{for some } \omega_\tau^k \in \partial_\pi H(z_\tau^k, \dot p_\tau^k) \qquad \text{a.e. in } \Omega \tag{4.24}$$

(cf. (4.8c)). We subtract (4.24), written at step $k-1$, from (4.24) at step k, and test the resulting relation by $\dot p_\tau^k$. This leads to

$$\begin{aligned} &\underbrace{\int_\Omega (\omega_\tau^k - \omega_\tau^{k-1}) \dot p_\tau^k \,\mathrm{d}x}_{\doteq I_7} + \varepsilon\nu \int_\Omega (\dot p_\tau^k - \dot p_\tau^{k-1}) \dot p_\tau^k \,\mathrm{d}x + \mu \int_\Omega (p_\tau^k - p_\tau^{k-1}) \dot p_\tau^k \,\mathrm{d}x \\ &\quad = \int_\Omega (\sigma_\tau^k - \sigma_\tau^{k-1})_{\mathrm{D}} \dot p_\tau^k \,\mathrm{d}x \,. \end{aligned} \tag{4.25}$$

From the 1-homogeneity of H and the fact that $\omega_\tau^k \in \partial_\pi H(z_\tau^k, \dot p_\tau^k)$ and $\omega_\tau^{k-1} \in \partial_\pi H(z_\tau^{k-1}, \dot p_\tau^{k-1})$ a.e. in Ω, it follows that

$$\int_\Omega \omega_\tau^k \dot p_\tau^k \,\mathrm{d}x = \mathcal{H}(z_\tau^k, \dot p_\tau^k), \qquad \int_\Omega \omega_\tau^{k-1} \dot p_\tau^k \,\mathrm{d}x \le \mathcal{H}(z_\tau^{k-1}, \dot p_\tau^k)\,.$$

Therefore, by (2.30c) we conclude that

$$|I_7| \le \left| \mathcal{H}(z_\tau^k, \dot p_\tau^k) - \mathcal{H}(z_\tau^{k-1}, \dot p_\tau^k) \right| \le C_K' \tau \|\dot z_\tau^k\|_{L^\infty} \|\dot p_\tau^{k-1}\|_{L^1} \,.$$

All in all, from (4.25) we infer that

$$\begin{aligned} &\varepsilon\nu \|\dot p_\tau^k\|_{L^2} \left(\|\dot p_\tau^k\|_{L^2} - \|\dot p_\tau^{k-1}\|_{L^2} \right) + \mu\tau \|\dot p_\tau^k\|_{L^2}^2 \\ &\quad \le \int_\Omega (\sigma_\tau^k - \sigma_\tau^{k-1})_{\mathrm{D}} \dot p_\tau^k \,\mathrm{d}x + C\tau \|\dot z_\tau^k\|_{L^\infty} \|\dot p_\tau^{k-1}\|_{L^1} \,. \end{aligned} \tag{4.26}$$

Summing up (4.21), (4.23), and (4.26), adding $\tau \|\dot z_\tau^k\|_{L^2}^2$ to both sides of the inequality, and observing the cancellation of one term, we conclude that

(4.27)

$$\begin{aligned} &\varepsilon\nu \|\dot u_\tau^k\|_{H^1,\mathbb{D}} \left(\|\dot u_\tau^k\|_{H^1,\mathbb{D}} - \|\dot u_\tau^{k-1}\|_{H^1,\mathbb{D}} \right) + \varepsilon \|\dot z_\tau^k\|_{L^2} \left(\|\dot z_\tau^k\|_{L^2} - \|\dot z_\tau^{k-1}\|_{L^2} \right) \\ &\quad + \varepsilon\nu \|\dot p_\tau^k\|_{L^2} \left(\|\dot p_\tau^k\|_{L^2} - \|\dot p_\tau^{k-1}\|_{L^2} \right) + \bar\zeta\tau \left(\|\dot e_\tau^k\|_{L^2}^2 + \|\dot z_\tau^k\|_{H^{\mathrm{m}}}^2 + \mu \|\dot p_\tau^k\|_{L^2}^2 \right) \\ &\le C\tau \Big(\|\dot z_\tau^k\|_{L^\infty} \|\dot e_\tau^k\|_{L^2} + \|\dot e_\tau^k\|_{L^2} \|\mathrm{E}(\dot w_\tau^k)\|_{L^2} + \|\dot z_\tau^k\|_{L^\infty} \|\mathrm{E}(\dot w_\tau^k)\|_{L^2} + \|\dot F_\tau^k\|_{\mathrm{BD}^*} \|\mathrm{E}(\dot w_\tau^k)\|_{L^1} \\ &\qquad + \|\dot F_\tau^k\|_{\mathrm{BD}^*} \|\dot e_\tau^k\|_{L^1} + \|\dot F_\tau^k\|_{\mathrm{BD}^*} \|\dot p_\tau^k\|_{L^1} + \|\dot z_\tau^k\|_{L^\infty}^2 + \|\dot z_\tau^k\|_{L^\infty} \|\dot p_\tau^k\|_{L^1} \Big) \end{aligned}$$

with $\bar\zeta = \min\{\gamma_1, 1\}$.

Step 2*:* Let us now estimate $\|\dot{p}_\tau^k\|_{L^1}$ for $k \in \{2,\ldots,N_\tau\}$. We observe that

(4.28)
$$
\begin{aligned}
\alpha\|\dot{p}_\tau^k\|_{L^1} &\overset{(1)}{\leq} \mathcal{H}(z_\tau^k,\dot{p}_\tau^k) - \int_\Omega (\rho_\tau^k)_{\mathrm{D}}\dot{p}_\tau^k \,\mathrm{d}x \\
&\overset{(2)}{=} \mathcal{H}(z_\tau^k,\dot{p}_\tau^k) + \int_\Omega \rho_\tau^k : (\dot{e}_\tau^k - \mathrm{E}(\dot{w}_\tau^k))\,\mathrm{d}x - \int_\Omega \rho_\tau^k : \mathrm{E}(\dot{u}_\tau^k)\,\mathrm{d}x \\
&\overset{(3)}{=} \mathcal{H}(z_\tau^k,\dot{p}_\tau^k) + \int_\Omega \rho_\tau^k : (\dot{e}_\tau^k - \mathrm{E}(\dot{w}_\tau^k))\,\mathrm{d}x - \langle F_\tau^k, \dot{u}_\tau^k\rangle_{\mathrm{BD}(\Omega)} \\
&\overset{(4)}{=} -\varepsilon\nu\|\dot{p}_\tau^k\|_{L^2}^2 - \mu\int_\Omega p_\tau^k\dot{p}_\tau^k\,\mathrm{d}x + \int_\Omega (\sigma_\tau^k)_{\mathrm{D}}\dot{p}_\tau^k\,\mathrm{d}x + \int_\Omega \rho_\tau^k : (\dot{e}_\tau^k - \mathrm{E}(\dot{w}_\tau^k))\,\mathrm{d}x \\
&\qquad - \int_\Omega \sigma_\tau^k : \mathrm{E}(\dot{u}_\tau^k)\,\mathrm{d}x - \varepsilon\nu\int_\Omega \mathbb{D}\mathrm{E}(\dot{u}_\tau^k) : \mathrm{E}(\dot{u}_\tau^k)\,\mathrm{d}x \\
&\overset{(5)}{\leq} -\mu\int_\Omega p_\tau^k\dot{p}_\tau^k\,\mathrm{d}x + \int_\Omega (\rho_\tau^k - \sigma_\tau^k) : (\dot{e}_\tau^k - \mathrm{E}(\dot{w}_\tau^k))\,\mathrm{d}x \\
&\leq \sqrt{\mu}\|p_\tau^k\|_{L^2}\sqrt{\mu}\|\dot{p}_\tau^k\|_{L^2} + \|\rho_\tau^k - \sigma_\tau^k\|_{L^2}\|\dot{e}_\tau^k - \mathrm{E}(\dot{w}_\tau^k)\|_{L^2} \\
&\overset{(6)}{\leq} C\left(\|\dot{e}_\tau^k\|_{L^2} + \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \sqrt{\mu}\|\dot{p}_\tau^k\|_{L^2}\right),
\end{aligned}
$$

where (1) follows from (2.40), (2) is due to the fact that $\dot{p}_\tau^k = \mathrm{E}(\dot{u}_\tau^k + \dot{w}_\tau^k) - \dot{e}_\tau^k$, (3) follows from the integration by parts formula (2.10) observing that $\dot{u}_\tau^k \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)$ and that $F_\tau^k = -\mathrm{Div}(\varrho_\tau^k)$ by (2.39e), (4) ensues from testing (4.7a) by $\dot{u}_\tau^k$ and (4.7c) by $\dot{p}_\tau^k$, (5) from the fact that $-\varepsilon\nu\|\dot{p}_\tau^k\|_{L^2}^2 \leq 0$ and $-\varepsilon\nu\int_\Omega \mathbb{D}\mathrm{E}(\dot{u}_\tau^k) : \mathrm{E}(\dot{u}_\tau^k)\,\mathrm{d}x \leq 0$, and again from $\mathrm{E}(\dot{u}_\tau^k) = \dot{e}_\tau^k + \dot{p}_\tau^k - \mathrm{E}(\dot{w}_\tau^k)$, and (6) is due to the fact that $\rho \in L^\infty(0,T;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and to the previously obtained estimates for $\overline{\sigma}_\tau$ and $\sqrt{\mu}\overline{p}_\tau$ in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$; cf. (4.10a).

In view of (4.28), estimate (4.27) is rewriten as

$$
\begin{aligned}
&\varepsilon\nu\|\dot{u}_\tau^k\|_{H^1,\mathbb{D}}(\|\dot{u}_\tau^k\|_{H^1,\mathbb{D}} - \|\dot{u}_\tau^{k-1}\|_{H^1,\mathbb{D}}) + \varepsilon\|\dot{z}_\tau^k\|_{L^2}\left(\|\dot{z}_\tau^k\|_{L^2} - \|\dot{z}_\tau^{k-1}\|_{L^2}\right) \\
&\quad + \varepsilon\nu\|\dot{p}_\tau^k\|_{L^2}\left(\|\dot{p}_\tau^k\|_{L^2} - \|\dot{p}_\tau^{k-1}\|_{L^2}\right) + \bar{\zeta}\tau\left(\|\dot{e}_\tau^k\|_{L^2}^2 + \|\dot{z}_\tau^k\|_{H^{\mathrm{m}}}^2 + \mu\|\dot{p}_\tau^k\|_{L^2}^2\right) \\
&\quad \leq C\tau\|\dot{e}_\tau^k\|_{L^2}\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + C\tau\|\dot{z}_\tau^k\|_{L^\infty}(\|\dot{e}_\tau^k\|_{L^2} + \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{z}_\tau^k\|_{L^\infty}) \\
&\qquad + C\tau(\|\dot{F}_\tau^k\|_{\mathrm{BD}^*} + \|\dot{z}_\tau^k\|_{L^\infty})\left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{e}_\tau^k\|_{L^2} + \sqrt{\mu}\|\dot{p}_\tau^k\|_{L^2}\right).
\end{aligned}
\tag{4.29}
$$

Step 3*:* Let us introduce the vector

$$
v_k := (\sqrt{\nu}\|\dot{u}_\tau^k\|_{H^1,\mathbb{D}}, \|\dot{z}_\tau^k\|_{L^2}, \sqrt{\nu}\|\dot{p}_\tau^k\|_{L^2}).
$$

Then, observe that the first three terms on the left-hand side of (4.29) rewrite as $\varepsilon\langle v_k, v_k - v_{k-1}\rangle$. For the fourth term we have the estimate

$$
\begin{aligned}
&\bar{\zeta}\tau\left(\|\dot{e}_\tau^k\|_{L^2}^2 + \|\dot{z}_\tau^k\|_{H^{\mathrm{m}}}^2 + \mu\|\dot{p}_\tau^k\|_{L^2}^2\right) \\
&\overset{(1)}{\geq} c\tau\left(\|\dot{e}_\tau^k\|_{L^2}^2 + \|\dot{z}_\tau^k\|_{H^{\mathrm{m}}}^2 + \mu\|\dot{p}_\tau^k\|_{L^2}^2 + \mu\|\mathrm{E}(\dot{u}_\tau^k)\|_{L^2}^2\right) - C\tau\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2}^2 \\
&\overset{(2)}{\geq} \tilde{\zeta}\tau\left(\|\dot{e}_\tau^k\|_{L^2}^2 + \|\dot{z}_\tau^k\|_{H^{\mathrm{m}}}^2 + \mu\|\dot{p}_\tau^k\|_{L^2}^2 + \nu\|\dot{u}_\tau^k\|_{H^1,\mathbb{D}}^2\right) - C\tau\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2}^2,
\end{aligned}
$$

where for (1) we have used that $\mu\|\mathrm{E}(\dot{u}_\tau^k)\|_{L^2}^2 \leq 3\mu\|\dot{e}_\tau^k\|_{L^2}^2 + 3\mu\|\dot{p}_\tau^k\|_{L^2}^2 + 3\mu\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2}^2$, while (2) ensues from (2.19) and from the fact that $\mu\|\mathrm{E}(\dot{u}_\tau^k)\|_{L^2}^2 \geq \nu\|\mathrm{E}(\dot{u}_\tau^k)\|_{L^2}^2$ (since, by assumption, $\nu \leq \mu$), with the constant $\tilde{\zeta}$ fulfilling $\tilde{\zeta}(3K_{\mathbb{D}}^2 + 1) \leq \bar{\zeta}$ with $K_{\mathbb{D}}$ from (2.19).

As for the right-hand side of (4.29), we will crucially use the compact embedding of $H^{\mathrm{m}}(\Omega)$ into $L^\infty(\Omega)$, which ensures that

$$\forall\,\delta>0\;\exists\,C_\delta>0\;\forall\,\zeta\in H^{\mathrm{m}}(\Omega):\qquad \|\zeta\|_{L^\infty}^2\le\delta\|\zeta\|_{H^{\mathrm{m}}}^2+C_\delta\|\zeta\|_{L^1}^2\,. \tag{4.30}$$

Therefore, also by Young's inequality we have the estimate

$$\begin{aligned}&\|\dot z_\tau^k\|_{L^\infty}\left(\|\dot e_\tau^k\|_{L^2}+\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}+\|\dot z_\tau^k\|_{L^\infty}\right)\\&\quad\le\delta(\|\dot z_\tau^k\|_{H^{\mathrm{m}}}^2+\|\dot e_\tau^k\|_{L^2}^2)+C_\delta(\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}^2+\|\dot z_\tau^k\|_{L^1}^2)\end{aligned}$$

for some suitable constant $\delta>0$ to be specified later on.

All in all, from (4.29) we deduce

$$\begin{aligned}&\varepsilon A_k(A_k-A_{k-1})+\tilde\zeta\tau B_k^2\le C\tau\left(1+C_k^2\right)+C\tau\|\dot z_\tau^k\|_{L^1}^2\\&\qquad+C\tau\|\dot z_\tau^k\|_{L^\infty}\left(\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}+\|\dot e_\tau^k\|_{L^2}+\sqrt{\mu}\|\dot p_\tau^k\|_{L^2}\right)+C\delta\tau B_k^2,\end{aligned} \tag{4.31}$$

where we have used the place-holders A_k, B_k, and C_k defined by

$$\begin{aligned}&A_k^2:=|v_k|^2=\nu\|\dot u_\tau^k\|_{H^1,\mathbb{D}}^2+\|\dot z_\tau^k\|_{L^2}^2+\nu\|\dot p_\tau^k\|_{L^2}^2,\\&B_k^2:=\|\dot e_\tau^k\|_{L^2}^2+\|\dot z_\tau^k\|_{H^{\mathrm{m}}}^2+\mu\|\dot p_\tau^k\|_{L^2}^2+\mu\|\dot u_\tau^k\|_{H^1,\mathbb{D}}^2\,,\quad C_k^2:=\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}^2+\|\dot F_\tau^k\|_{\mathrm{BD}^*}^2\end{aligned}$$

and estimated

$$\begin{aligned}&C\tau\|\dot e_\tau^k\|_{L^2}\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}\le\delta\tau B_k^2+C\tau\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}^2,\\&C\tau\|\dot F_\tau^k\|_{\mathrm{BD}^*}\left(\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}+\|\dot e_\tau^k\|_{L^2}+\sqrt{\mu}\|\dot p_\tau^k\|_{L^2}\right)\le\delta\tau B_k^2+C\tau\|\dot F_\tau^k\|_{\mathrm{BD}^*}^2+C\tau\|\mathrm{E}(\dot w_\tau^k)\|_{L^2}^2\end{aligned} \tag{4.32}$$

via Young's inequality. Therefore, choosing $\delta>0$ in (4.31) small enough in such a way as to absorb the term $C\delta\tau B_k^2$ on the left-hand side, we arrive at

$$\begin{aligned}&\varepsilon A_k(A_k-A_{k-1})+\frac{\tilde\zeta}{2}\tau B_k^2\\&\le C\tau(1+C_k^2)+C\tau\|\dot z_\tau^k\|_{L^1}^2+C\tau\|\dot z_\tau^k\|_{L^\infty}\left(\|\dot e_\tau^k\|_{L^2}+\sqrt{\mu}\|\dot p_\tau^k\|_{L^2}\right)\,.\end{aligned} \tag{4.33}$$

Again relying on (4.30) we estimate the last term on the right-hand side of (4.33) by

$$\begin{aligned}&C\tau\|\dot z_\tau^k\|_{L^\infty}\left(\|\dot e_\tau^k\|_{L^2}+\sqrt{\mu}\|\dot p_\tau^k\|_{L^2}\right)\\&\quad\le C\tau\delta\|\dot z_\tau^k\|_{H^{\mathrm{m}}}B_k+C_\delta\tau\|\dot z_\tau^k\|_{L^1}B_k\le C\tau\delta B_k^2+C\tau\delta' B_k^2+C\tau\|\dot z_\tau^k\|_{L^1}^2.\end{aligned}$$

Choosing the constants δ and δ' such that $C\tau(\delta+\delta')\le\frac{\tilde\mu}{4}\tau$ and using that $\|\dot z_\tau^k\|_{L^1}\le A_k$, we obtain that, for every $k\in\{2,\ldots,N_\tau\}$,

$$\varepsilon A_k(A_k-A_{k-1})+\frac{\tilde\zeta}{4}\tau B_k^2\le C\tau(1+C_k^2)+C\tau\|\dot z_\tau^k\|_{L^1}^2\le C\tau(1+C_k^2)+C\tau A_k\|\dot z_\tau^k\|_{L^1}\,. \tag{4.34}$$

Step 4: Let us now address the case $k=1$. To start with, let us set $u_\tau^{-1}:=u_0$, $z_\tau^{-1}:=z_0$, and $p_\tau^{-1}:=p_0$, so that

$$\dot u_\tau^0=\tfrac{u_\tau^0-u_\tau^{-1}}{\tau}=0\text{ and, analogously, }\dot z_\tau^0=0\text{ and }\dot p_\tau^0=0. \tag{4.35}$$

We test (4.7a), with $k=1$, by $\dot u_\tau^1$. With an easy algrebraic manipulation we obtain

$$
\begin{aligned}
&\int_\Omega \varepsilon\nu \mathbb{D}\mathrm{E}(\dot u_\tau^1)\mathrm{E}(\dot u_\tau^1)\,\mathrm{d}x + \int_\Omega (\sigma_\tau^1 - \sigma_\tau^0) : \mathrm{E}(\dot u_\tau^1)\,\mathrm{d}x \\
&= \langle F_\tau^1 - F_\tau^0, \dot u_\tau^1\rangle_{H^1} - \int_\Omega \sigma_\tau^0 : \mathrm{E}(\dot u_\tau^1)\,\mathrm{d}x + \langle F_\tau^0, \dot u_\tau^1\rangle_{H^1}\,.
\end{aligned}
\tag{4.36}
$$

Repeating the very same calculations as throughout (4.19) and the subsequent formulae, we arrive at (cf. (4.21))

$$
\begin{aligned}
&\varepsilon\nu\|\dot u_\tau^1\|_{H^1,\mathbb{D}}(\|\dot u_\tau^1\|_{H^1,\mathbb{D}} - \|\dot u_\tau^0\|_{H^1,\mathbb{D}}) + \gamma_1\tau\|\dot e_\tau^1\|_{L^2}^2 + \int_\Omega (\sigma_\tau^1-\sigma_\tau^0)_{\mathrm{D}}\dot p_\tau^1\,\mathrm{d}x \\
&\le C\tau\Big(\|\dot z_\tau^1\|_{L^\infty}\|\dot e_\tau^1\|_{L^2} + \|\dot e_\tau^1\|_{L^2}\|\mathrm{E}(\dot w_\tau^1)\|_{L^2} + \|\dot z_\tau^1\|_{L^\infty}\|\mathrm{E}(\dot w_\tau^1)\|_{L^2} + \|\dot F_\tau^1\|_{\mathrm{BD}^*}\|\mathrm{E}(\dot w_\tau^1)\|_{L^1} \\
&\qquad + \|\dot F_\tau^1\|_{\mathrm{BD}^*}\|\dot e_\tau^1\|_{L^1} + \|\dot F_\tau^1\|_{\mathrm{BD}^*}\|\dot p_\tau^1\|_{L^1}\Big) + \left|\int_\Omega (F(0) + \mathrm{Div}(\sigma_0))\dot u_\tau^1\,\mathrm{d}x\right|,
\end{aligned}
\tag{4.37}
$$

where we have used that, by definition, $\dot u_\tau^0$, and exploited the fact that $F(0) + \mathrm{Div}(\sigma_0) \in L^2(\Omega;\mathbb{R}^n)$ by (4.14) to rewrite the last two terms on the right-hand side of (4.36).

We now write (4.17) for $k=1$. With algebraic manipulations, also taking into account that $\dot z_\tau^0 = 0$ and using (4.14), we arrive at

$$
\begin{aligned}
&\mathcal{R}(\dot z_\tau^1) + \varepsilon\int_\Omega (\dot z_\tau^1 - \dot z_\tau^0)\dot z_\tau^1\,\mathrm{d}x + a_{\mathrm{m}}(z_\tau^1 - z_\tau^0, \dot z_\tau^1) \\
&\le \int_\Omega \left[W'(z_\tau^1) - W'(z_\tau^0)\right]\dot z_\tau^1\,\mathrm{d}x + \frac12\int_\Omega \left[\mathbb{C}'(z_\tau^0) - \mathbb{C}'(z_\tau^1)\right] e_\tau^1 : e_\tau^1\,\dot z_\tau^1\,\mathrm{d}x \\
&\quad - \frac12\int_\Omega \left(\mathbb{C}'(z_\tau^0)e_\tau^1 : e_\tau^1 - \mathbb{C}'(z_\tau^0)e_\tau^0 : e_\tau^0\right)\dot z_\tau^1\,\mathrm{d}x \\
&\quad + C_K\tau\|\dot z_\tau^1\|_{L^\infty}\|\dot p_\tau^1\|_{L^1} + \left|\int_\Omega \left(A_{\mathrm{m}} z_\tau^0 + W'(z_\tau^0) + \tfrac12\mathbb{C}'(z_\tau^0)e_\tau^0 : e_\tau^0\right)\dot z_\tau^1\,\mathrm{d}x\right|.
\end{aligned}
\tag{4.38}
$$

With the same calculations as throughout (4.22)–(4.23) we conclude that

$$
\begin{aligned}
&\varepsilon\|\dot z_\tau^1\|_{L^2}\left(\|\dot z_\tau^1\|_{L^2} - \|\dot z_\tau^0\|_{L^2}\right) + \tau a_{\mathrm{m}}(\dot z_\tau^1, \dot z_\tau^1) \\
&\le C\tau\|\dot z_\tau^1\|_{L^\infty}\left(\|\dot z_\tau^1\|_{L^\infty} + \|\dot e_\tau^1\|_{L^2} + \|\dot p_\tau^1\|_{L^1}\right) \\
&\quad + \left|\int_\Omega \left(A_{\mathrm{m}} z_\tau^0 + W'(z_\tau^0) + \tfrac12\mathbb{C}'(z_\tau^0)e_\tau^0 : e_\tau^0\right)\dot z_\tau^1\,\mathrm{d}x\right|.
\end{aligned}
\tag{4.39}
$$

Finally, we test (4.24), written for $k=1$, with $\dot p_\tau^1$. Taking into account that, by construction, $\dot p_\tau^0 = 0$, this leads to

$$
\begin{aligned}
&\int_\Omega \omega_\tau^1 \dot p_\tau^1\,\mathrm{d}x + \varepsilon\nu\int_\Omega (\dot p_\tau^1 - \dot p_\tau^0)\dot p_\tau^1\,\mathrm{d}x + \mu\int_\Omega (p_\tau^1 - p_\tau^0)\dot p_\tau^1\,\mathrm{d}x \\
&\quad = \int_\Omega (\sigma_\tau^1 - \sigma_\tau^0)_{\mathrm{D}}\dot p_\tau^1\,\mathrm{d}x + \int_\Omega (\sigma_\tau^0 - \mu p_\tau^0)\dot p_\tau^1\,\mathrm{d}x\,.
\end{aligned}
$$

With the same computations as for (4.26), we obtain

$$
\begin{aligned}
&\mathcal{H}(\dot z_\tau^1, \dot p_\tau^1) + \varepsilon\nu\|\dot p_\tau^1\|_{L^2}\left(\|\dot p_\tau^1\|_{L^2} - \|\dot p_\tau^0\|_{L^2}\right) + \mu\tau\|\dot p_\tau^1\|_{L^2}^2 \\
&\le \int_\Omega (\sigma_\tau^1 - \sigma_\tau^0)_{\mathrm{D}}\dot p_\tau^1\,\mathrm{d}x + \left|\int_\Omega (\sigma_\tau^0 - \mu p_\tau^0)\dot p_\tau^1\,\mathrm{d}x\right|.
\end{aligned}
\tag{4.40}
$$

We add up (4.37), (4.39), and (4.40). The very same calculations as throughout Steps 2 and 3 lead to

$$\varepsilon A_1(A_1-A_0) + \frac{\tilde{\zeta}}{4}\tau B_1^2 \leq C\tau(1+C_1^2) + C\tau A_1\|\dot{z}_\tau^1\|_{L^1} + F_1\,.$$

Here, the term F_1 subsumes the very last contributions on the right-hand sides of (4.37), (4.39), and (4.40): in fact, for later use we introduce the place-holder

$$\begin{aligned} F_k := &\left|\int_\Omega \mathrm{D}_u\mathcal{E}_\mu(0,u_0,z_0,p_0)\dot{u}_\tau^1\,\mathrm{d}x\right| + \left|\int_\Omega \mathrm{D}_z\mathcal{E}_\mu(0,u_0,z_0,p_0)\dot{z}_\tau^1\,\mathrm{d}x\right| \\ &+ \left|\int_\Omega \mathrm{D}_p\mathcal{E}_\nu(0,u_0,z_0,p_0)\dot{p}_\tau^1\,\mathrm{d}x\right|\,. \end{aligned}$$

Then, (4.34) extends to the index $k=1$, and we ultimately get the relation, for all $k \in \{1,\ldots,N_\tau\}$,

(4.41)

$$\varepsilon A_k(A_k-A_{k-1}) + \frac{\tilde{\zeta}}{4}\tau B_k^2 \leq C\tau + C\tau\|\dot{z}_\tau^k\|_{L^1}^2 \leq C\tau(1+C_k^2) + C\tau A_k\|\dot{z}_\tau^k\|_{L^1} + \delta_{1,k}F_1\,.$$

Step 5*:* From (4.41) we infer

$$\frac{1}{2}A_k^2 - \frac{1}{2}A_{k-1}^2 + \frac{\tilde{\zeta}}{4\varepsilon}\tau B_k^2 \leq \frac{C}{\varepsilon}\tau\left(1+A_k^2+C_k^2\right) + \frac{\delta_{1,k}}{\varepsilon}F_1 \qquad \forall\, k \in \{1,\ldots,N_\tau\}\,.$$

so that adding up the above relations we obtain (recall $A_0 = 0$ by (4.35))

$$A_k^2 + \sum_{j=1}^k \frac{\tilde{\zeta}}{\varepsilon}\tau B_j^2 \leq \frac{1}{\varepsilon}F_1 + \frac{2CT}{\varepsilon} + \frac{2C}{\varepsilon}\sum_{j=1}^k \tau C_j^2 + \frac{2C}{\varepsilon}\sum_{j=1}^k \tau A_j^2\,. \tag{4.42}$$

We are now in a position to apply Lemma A.2 in Appendix A, with the choices $a_k := A_k^2$, $\Lambda := \frac{2CT}{\varepsilon} + \frac{2C}{\varepsilon}\sum_{j=1}^k \tau C_j^2 + \frac{1}{\varepsilon}F_1$, and $b = \frac{2C\tau}{\varepsilon}$: hence we need to assume, e.g., $\tau/\varepsilon < 1/(4C)$, so that $b<1$. Then, (A.2) gives (notice that $\sum_{j=1}^k \tau C_j^2 \leq C'$ in view of (2.39c) and (2.41))

$$\begin{aligned} \sup_{k=1,\ldots,N_\tau} A_k^2 &\leq \frac{1}{1-\tau}\left(A_0^2 + \frac{2CT}{\varepsilon} + \frac{F_1}{\varepsilon} + \frac{2C}{\varepsilon}\sum_{j=1}^k \tau C_j^2\right)\exp\left(\frac{b}{1-b}k\right) \\ &\overset{(*)}{\leq} 2\left(A_0^2 + \frac{2CT}{\varepsilon} + \frac{2C}{\varepsilon}\sum_{j=1}^k \tau C_j^2 + \frac{1}{\varepsilon}F_1\right)\exp\left(\tfrac{4CT}{\varepsilon}\right) \doteq S_\varepsilon^1, \end{aligned} \tag{4.43}$$

so that $S_\varepsilon^1 \uparrow +\infty$ as $\varepsilon \downarrow 0$, where estimate $(*)$ is true for, say, $\tau \in [0,1/2]$. Plugging the above estimate into (4.42) we obtain

$$\sum_{k=1}^{N_\tau} \tau B_k^2 \leq S_\varepsilon^2 \qquad \text{with } S_\varepsilon^2 \uparrow +\infty \text{ as } \varepsilon \downarrow 0. \tag{4.44}$$

Clearly, (4.43) and (4.44) give estimates (4.15a).

Step 6*:* Using that $B_k \geq A_k$, from (4.41) we deduce

$$A_k(A_k - A_{k-1}) + \frac{\tilde{\zeta}\tau}{4\varepsilon}A_k^2 + \frac{\tilde{\zeta}\tau}{4\varepsilon}B_k^2 \leq C\frac{\tau}{\varepsilon}(1+C_k^2) + \frac{\delta_{1,k}}{\varepsilon}F_1 + C\frac{\tau}{\varepsilon}A_k\|\dot{z}_\tau^k\|_{L^1} \tag{4.45}$$

for all $k \in \{1, \dots, N_\tau\}$. Hence, we are in position to apply the forthcoming Lemma A.3 with the choices $a_k := A_k$, $M_k := B_k$, $\gamma := \frac{\tilde{\zeta}\tau}{4\varepsilon}$, $c_k := C_k$, and $R_k := \frac{4C}{\tilde{\zeta}}\|\dot{z}_\tau^k\|_{L^1}$ and suitable choices for the constants c and ρ (notice that $A_0 = 0$ by construction, $R_k \leq \overline{c}A_k$, and we now have to take $\tau/\varepsilon < 1/(2\overline{c})$). From (A.4), along with $\sum_{k=1}^{N_\tau}\tau C_k^2 \leq C'$ (by (2.39c) and (2.41)) and (4.10b), we infer

$$\exists\, S_2 > 0 \ \ \forall\, \tau, \varepsilon, \nu > 0\ : \qquad \sum_{k=1}^{N_\tau}\tau B_k \leq S_2\,. \tag{4.46}$$

Then, estimate (4.15b) ensues. This concludes the proof. □

Remark 4.6. In the case in which only (2.39a) holds in place of (2.39c), we have only $F \in H^1(0,T;(H^1(\Omega;\mathbb{R}^n)^*)$, so that in place of (4.20) we may infer only (with shorter notation for the norms)

$$|I_3| \leq C\tau\|\dot{F}_\tau^k\|_{(H^1)^*}\left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{e}_\tau^k\|_{L^2} + \|\dot{p}_\tau^k\|_{L^2}\right)\,.$$

This affects the second inequality in (4.32), which is now replaced by

$$\begin{aligned} &C\tau\|\dot{F}_\tau^k\|_{(H^1)^*}\left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{e}_\tau^k\|_{L^2} + \|\dot{p}_\tau^k\|_{L^2}\right) \\ &\quad \leq \delta\tau\left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2}^2 + \|\dot{e}_\tau^k\|_{L^2}^2 + \|\dot{p}_\tau^k\|_{L^2}^2\right) + C\tau\|\dot{F}_\tau^k\|_{(H^1)^*}^2 \end{aligned}$$

Now, since $\left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2}^2 + \|\dot{e}_\tau^k\|_{L^2}^2 + \|\dot{p}_\tau^k\|_{L^2}^2\right)$ equals $\sqrt{\mu}^{-1}B_k^2$, we may only control

$$C\tau\|\dot{F}_\tau^k\|_{(H^1)^*}\left(\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \|\dot{e}_\tau^k\|_{L^2} + \|\dot{p}_\tau^k\|_{L^2}\right) \leq \delta\tau B_k^2 + C_\mu\tau\|\dot{F}_\tau^k\|_{(H^1)^*}^2\,,$$

where C_μ depends on μ, too, and blows up as $\mu \downarrow 0$. We could argue in the very same way for the rest of the proof, but the constant affect also the other estimates such as (4.33), that now have to contain constants depending also on μ on the right-hand side. Thus, we end up proving (4.15) with a constant depending also on μ.

Remark 4.7. Estimate (4.28), giving

$$\|\dot{p}_\tau^k\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} \leq C\left(\|\dot{e}_\tau^k\|_{L^2} + \|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2} + \sqrt{\mu}\|\dot{p}_\tau^k\|_{L^2}\right)\,, \tag{4.47}$$

is fundamental since it allows us to estimate $\|\dot{p}_\tau^k\|_{L^1}$ by means of the term B_k and of $\|\mathrm{E}(\dot{w}_\tau^k)\|_{L^2}$ In this way, the terms containing $\|\dot{p}_\tau^k\|_{L^1}$ can be partly absorbed into the left-hand side. If we did not resort to estimate (4.28), we would have to deal with the term $C\tau\|\dot{p}_\tau^k\|_{L^1}^2$ on the right-hand side of (4.41), which would be controlled only by considering constants depending on μ, as explained in Remark 4.6 above.

The last result of this section provides a *discrete* version of the energy-dissipation upper estimate (3.18).

PROPOSITION 4.8 (discrete energy-dissipation upper estimate). *The piecewise constant and linear interpolants of the discrete solutions* $(u_\tau^k, z_\tau^k, p_\tau^k)_{k=1}^N$ *fulfill*

(4.48)

$$
\begin{aligned}
&\mathcal{E}_\mu(t, u_\tau(t), z_\tau(t), p_\tau(t)) + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} (\mathcal{V}_{\varepsilon,\nu}(u_\tau'(r)) + \mathcal{R}_\varepsilon(z_\tau'(r)) + \mathcal{H}_{\varepsilon,\nu}(\overline{z}_\tau(r), p_\tau'(r)))\,\mathrm{d}r \\
&\quad + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \Big(\mathcal{V}^*_{\varepsilon,\nu}(\mathrm{Div}(\overline{\sigma}_\tau(r)) + \overline{F}_\tau(r)) + \mathcal{R}^*_\varepsilon(-A_{\mathrm{m}}(\overline{z}_\tau(r)) \\
&\quad - W'(\overline{z}_\tau(r)) - \tfrac{1}{2}\mathbb{C}'(\overline{z}_\tau(r))\overline{e}_\tau(r) : \overline{e}_\tau(r) - \tau\overline{\lambda}_\tau(r)\big) \\
&\quad + \mathcal{H}^*_{\varepsilon,\nu}(\underline{z}_\tau(r), -\mu\overline{p}_\tau(r) + (\overline{\sigma}_\tau(r))_{\mathrm{D}})\Big)\,\mathrm{d}r \\
&\quad \le \mathcal{E}_\mu(s, u_\tau(s), z_\tau(s), p_\tau(s)) \\
&\quad + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega \mathbb{C}(z_\tau(r))(\mathrm{E}(u_\tau(r) + w(r)) - p_\tau(r)) : \mathrm{E}(w'(r))\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle F'(r), u_\tau(r) + w(r)\rangle_{H^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}r - \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle F(r), w'(r)\rangle_{H^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}r + \mathfrak{R}_\tau(s,t),
\end{aligned}
$$

where $\overline{\lambda}_\tau$ *is a selection in* $\partial_z\mathcal{H}_{\varepsilon,\nu}(\overline{z}_\tau, p_\tau')$ *fulfiling the Euler–Lagrange equation* (4.8b), *the remainder term is*

(4.49)

$$
\begin{aligned}
\mathfrak{R}_\tau(s,t) := C_3 \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} &\big(\|\overline{u}_\tau - u_\tau\|_{H^1(\Omega)} + \|\overline{z}_\tau - z_\tau\|_{H^{\mathrm{m}}(\Omega)} + \|\overline{p}_\tau - p_\tau\|_{L^2(\Omega)} + \|\overline{w}_\tau - w\|_{H^1(\Omega)}\big) \\
&\times \big(\|u_\tau'\|_{H^1(\Omega)} + \|z_\tau'\|_{H^{\mathrm{m}}(\Omega)} + \|p_\tau'\|_{L^2(\Omega)}\big)\,\mathrm{d}r
\end{aligned}
$$

and the constant C_3, *uniform w.r.t.* ε, ν, μ, τ, *only depends on the constant* C_1 *from* (4.10).

Proof. With the very same calculations as in the proof of [KRZ15, Lemma 6.1], also based on the convex analysis arguments leading to (3.14), from the Euler–Lagrange equation (4.6) we deduce that the interpolants $\overline{q}_\tau$, $\underline{q}_\tau$, and q_τ fulfill

$$
\begin{aligned}
&\mathcal{E}_\mu(t, q_\tau(t)) + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(\Psi_{\varepsilon,\nu}(\overline{q}_\tau(r), q_\tau'(r)) + \Psi^*_{\varepsilon,\nu}(\overline{q}_\tau(r), -\mathrm{D}_q\mathcal{E}_\mu(\overline{t}_\tau(r), \overline{q}_\tau(r))) \\
&\qquad - \tau\partial_q\Psi_{\varepsilon,\nu}(\overline{q}_\tau(r), q_\tau'(r))\big)\,\mathrm{d}r \\
&= \mathcal{E}_\mu(s, q_\tau(s)) + \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \partial_t\mathcal{E}_\mu(r, q_\tau(r))\,\mathrm{d}r \\
&\quad - \underbrace{\int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle \mathrm{D}_q\mathcal{E}_\mu(\overline{t}_\tau(r), \overline{q}_\tau(r)) - \mathrm{D}_q\mathcal{E}_\mu(r, q_\tau(r)), q_\tau'(r)\rangle_{\mathbf{Q}}\,\mathrm{d}r}_{\doteq R_\tau(s,t)}\,.
\end{aligned}
\tag{4.50}
$$

Then, taking into account (4.8), it is immediate to check that the left-hand side of (4.50) translates into the left-hand side of (4.48). Analogously, taking into account the explicit calculation (3.8) of $\partial_t\mathcal{E}_\mu$, we see that the first two terms on the right-hand side of (4.50) correspond to the first four terms on the right-hand side of (4.48). We now estimate the remainder term $R_\tau(s,t)$ as follows. First of all, we observe that

$$|R_\tau(s,t)| \le |R^1_\tau(s,t)| + |R^2_\tau(s,t)| + |R^3_\tau(s,t)|.$$

Then,
(4.51)
$$
\begin{aligned}
|R^1_\tau(s,t)| &= \left| \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \langle \mathrm{Div}(\mathbb{C}(\overline{z}_\tau)\overline{e}_\tau) - \mathrm{Div}(\mathbb{C}(z_\tau)(\mathrm{E}(u_\tau+w)-p_\tau), u_\tau' \rangle_{H^1(\Omega)} \right| \mathrm{d}r \\
&\leq \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \left| \int_\Omega (\mathbb{C}(\overline{z}_\tau) - \mathbb{C}(z_\tau))\overline{e}_\tau : \mathrm{E}(u_\tau')\,\mathrm{d}x \right| \mathrm{d}r \\
&\quad + \int_{\overline{t}_\tau(s)}^{\overline{t}_\tau(t)} \left| \int_\Omega \mathbb{C}(z_\tau)(\overline{e}_\tau - (\mathrm{E}(u_\tau+w)-p_\tau)) : \mathrm{E}(u_\tau')\,\mathrm{d}x \right| \mathrm{d}r \\
&\overset{(1)}{\leq} C \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(\|\overline{z}_\tau - z_\tau\|_\infty \|\overline{e}_\tau\|_{L^2} \|\mathrm{E}(u_\tau')\|_{L^2} + \|z_\tau\|_{L^\infty} \|\overline{e}_\tau - (\mathrm{E}(u_\tau+w)-p_\tau)\|_{L^2} \|\mathrm{E}(u_\tau')\|_{L^2} \big)\,\mathrm{d}r \\
&\overset{(2)}{\leq} C \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(\|\overline{u}_\tau - u_\tau\|_{H^1} + \|\overline{z}_\tau - z_\tau\|_{H^\mathrm{m}}) + \|\overline{p}_\tau - p_\tau\|_{L^2} + \|\overline{w}_\tau - w\|_{H^1} \big) \|\mathrm{E}(u_\tau')\|_{L^2}\,\mathrm{d}r\,,
\end{aligned}
$$

where (1) ensues from (2.11), estimate (4.10a) and the continuous embedding $H^\mathrm{m}(\Omega) \subset \mathrm{C}(\overline{\Omega})$. For (2) we have again used the latter embedding along with the identity $\overline{e}_\tau = \mathrm{E}(\overline{u}_\tau + \overline{w}_\tau) - \overline{p}_\tau$. Second,
(4.52)
$$
\begin{aligned}
&|R^2_\tau(s,t)| \\
&\quad = \Big| \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \Big(a_\mathrm{m}(\overline{z}_\tau - z_\tau, z_\tau') + \int_\Omega (W'(\overline{z}_\tau) - W'(z_\tau)) z_\tau'\,\mathrm{d}x \\
&\qquad + \frac{1}{2} \int_\Omega \Big(\mathbb{C}'(\overline{z}_\tau)\overline{e}_\tau : \overline{e}_\tau - \mathbb{C}'(z_\tau)(\mathrm{E}(u_\tau+w)-p_\tau) : (\mathrm{E}(u_\tau+w)-p_\tau) \Big) z_\tau'\,\mathrm{d}x \Big)\,\mathrm{d}r \Big| \\
&\quad \overset{(3)}{\leq} C \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \Big(\|\overline{z}_\tau - z_\tau\|_{H^\mathrm{m}} \|z_\tau'\|_{H^\mathrm{m}} + \|\overline{z}_\tau - z_\tau\|_2 \|z_\tau'\|_{L^2} \\
&\qquad\qquad + \|\overline{u}_\tau - u_\tau\|_{H^1} \|z_\tau'\|_{H^\mathrm{m}} + \|\overline{p}_\tau - p_\tau\|_{L^2} \|z_\tau'\|_{H^\mathrm{m}} + \|\overline{w}_\tau - w\|_{H^1} \|z_\tau'\|_{H^\mathrm{m}} \Big)\,\mathrm{d}r\,,
\end{aligned}
$$

where for (3) we have used that, since $\overline{z}_\tau,\ z_\tau \in [m_0,1]$ by property (3.5) and W is of class C^2 on $[m_0,1]$, it is possible to estimate $\|W'(\overline{z}_\tau) - W'(z_\tau)\|_{L^2} \leq C\|\overline{z}_\tau - z_\tau\|_2$. We have also estimated

$$
\begin{aligned}
&\|(\mathbb{C}'(\overline{z}_\tau)\overline{e}_\tau : \overline{e}_\tau - \mathbb{C}'(z_\tau)(\mathrm{E}(u_\tau+w)-p_\tau) : (\mathrm{E}(u_\tau+w)-p_\tau))\, z_\tau'\|_{L^1} \\
&\quad \leq C\|\overline{z}_\tau - z_\tau\|_{L^\infty} \|\overline{e}_\tau\|^2_{L^2} \|z_\tau'\|_{L^\infty} \\
&\qquad + \|z_\tau\|_\infty \|\overline{e}_\tau + (\mathrm{E}(u_\tau+w)-p_\tau)\|_{L^2} \|\overline{e}_\tau - (\mathrm{E}(u_\tau+w)-p_\tau)\|_{L^2} \|z_\tau'\|_{L^\infty} \\
&\quad \leq C\|\overline{z}_\tau - z_\tau\|_\infty \|z_\tau'\|_{L^\infty} + \|\overline{e}_\tau - (\mathrm{E}(u_\tau+w)-p_\tau)\ \|_{L^2} \|z_\tau'\|_{L^\infty}
\end{aligned}
$$

thanks to (2.11) and estimate (4.10a); subsequently, we have estimated $\|\overline{e}_\tau - (\mathrm{E}(u_\tau + w)-p_\tau)\|_{L^2}$ as we did for (4.51). All in all, this leads to (4.52). Third, we see that
(4.53)
$$
\begin{aligned}
&|R^3_\tau(s,t)| \\
&\quad = \left| \int_{\underline{t}_\tau(s)}^{\overline{t}_\tau(t)} \int_\Omega (\mu\overline{p}_\tau - \mu p_\tau + (\mathbb{C}(z_\tau)(\mathrm{E}(u_\tau+w)-p_\tau))_\mathrm{D} - (\overline{\sigma}_\tau)_\mathrm{D})\, p_\tau'\,\mathrm{d}x\,\mathrm{d}r \right| \\
&\quad \overset{(4)}{\leq} C \int_{\overline{t}_\tau(s)}^{\overline{t}_\tau(t)} \big(\|\overline{u}_\tau - u_\tau\|_{H^1(\Omega)} + \|\overline{z}_\tau - z_\tau\|_{H^\mathrm{m}(\Omega)} + \|\overline{p}_\tau - p_\tau\|_{L^2} + \|\overline{w}_\tau - w\|_{H^1} \big) \|p_\tau'\|_{L^2}\,\mathrm{d}r\,.
\end{aligned}
$$

Here, (4) is due to (2.11) and the previously obtained estimates (4.10a), which also enter into the estimate

$$
\begin{aligned}
&\|(\mathbb{C}(z_\tau)(\mathrm{E}(u_\tau+w)-p_\tau)-\mathbb{C}(\overline{z}_\tau)\overline{e}_\tau)_{\mathrm{D}}p'_\tau\|_1 \\
&\le \|\mathbb{C}(z_\tau)-\mathbb{C}(\overline{z}_\tau)\|_\infty\|\overline{e}_\tau\|_{L^2}\|p'_\tau\|_{L^2}+\|\mathbb{C}(\overline{z}_\tau)\|_\infty\|\overline{e}_\tau-(\mathrm{E}(u_\tau+w)-p_\tau)\|_{L^2}\|p'_\tau\|_{L^2} \\
&\le C\left(\|\overline{z}_\tau-z_\tau\|_\infty+\|\overline{u}_\tau-u_\tau\|_{H^1(\Omega)}+\|\overline{p}_\tau-p_\tau\|_{L^2}+\|\overline{w}_\tau-w\|_{H^1}\right)\|p'_\tau\|_{L^2}\,.
\end{aligned}
$$

Combining (4.51)–(4.53) with (4.50), we conclude the proof. ∎

5. Existence of solutions to the viscous problem. This section focuses on the existence of solutions to Problem 3.1 for fixed $\varepsilon>0$, $\nu>0$, and $\mu>0$. Besides the standing assumptions from section 2.1, our existence result, Theorem 5.1 below, will require conditions (4.14) on the initial data (u_0,z_0,p_0). In fact, to prove the *sole* existence of solutions for Problem 3.1 (and for the vanishing-viscosity analysis in section 6), it would be enough to assume (2.39a) in place of (2.39c) since, for $\mu>0$ fixed, estimates (4.15), with constants depending on μ (cf. also Remark 4.6) would be sufficient. However, condition (2.39c) ensures that the solutions we exhibit in Theorem 5.1 enjoy the upcoming estimates (5.3) uniformly w.r.t. ε, ν, *and* μ. This will be at the basis of the vanishing-hardening analysis in section 7.

THEOREM 5.1. *Under the assumptions in section* 2, *and* (4.14) *as well, Problem* 3.1 *admits a solution triple* (u,z,p) *enjoying the additional regularity and summability properties*

$$
\begin{aligned}
&u\in W^{1,\infty}(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)), \qquad z\in H^1(0,T;H^{\mathrm{m}}(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega)),\\
&p\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})).
\end{aligned}
\tag{5.1}
$$

Moreover, the triple (u,z,p) *fulfills*

$$
\int_0^T \mathcal{N}^\mu_{\varepsilon,\nu}(r,q(r),q'(r))\,\mathrm{d}r\le C_4 \tag{5.2}
$$

for a constant $C_4>0$ *independent of* ε, μ, $\nu>0$. *Additionally, we have the following bounds* uniformly *w.r.t. all parameters* ε, ν, *and* μ *provided that* $\underline{\nu\le\mu}$ *(recall that* $e:=\mathrm{E}(u+w)-p$*):*

$$
\begin{aligned}
&\|e\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))}+\|z\|_{W^{1,1}(0,T;H^{\mathrm{m}}(\Omega))}+\sqrt{\mu}\|p\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))}\\
&+\sqrt{\mu}\|u\|_{W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^n))}+\|p\|_{W^{1,1}(0,T;L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))}\le C_5\,.
\end{aligned}
\tag{5.3}
$$

Proof. By virtue of Proposition 3.4, it is sufficient to show that the piecewise constant and linear interpolants of the discrete solutions constructed in section 4 converge to a triple (u,z,p) fulfiling the initial conditions (3.4) and the energy-dissipation upper estimate (3.18). For this, we will take the limit of the discrete energy-dissipation inequality (4.48), using that, thanks to (2.39c) and (2.41),

$$
\overline{F}_{\tau_k}\to F\quad\text{in } H^1(0,T;\mathrm{BD}(\Omega)^*)\,,\qquad \overline{w}_{\tau_k}\to w\quad\text{in } H^1(0,T;H^1(\Omega;\mathbb{R}^n))\,. \tag{5.4}
$$

Estimates (5.3) will be inherited by (u,z,p) and e from the analogous bounds for the approximate solutions via lower semicontinuity arguments. Accordingly, the proof is split into three steps.

Step 1*: Compactness.* Let us consider a null sequence $\tau_k\downarrow 0$ and, accordingly, the discrete solutions $(\overline{u}_{\tau_k},u_{\tau_k},\overline{z}_{\tau_k},z_{\tau_k},\overline{p}_{\tau_k},p_{\tau_k})_k$, along with $(\overline{e}_{\tau_k},e_{\tau_k})_k$. It follows from estimates (4.10) and (4.15a), combined with standard weak compactness arguments and Aubin–Lions-type compactness results (cf., e.g., [Sim87]), that there exists a triple (u, z, p) fulfiling (5.1), such that the following convergences hold:

$$
\begin{aligned}
&\text{(5.5a)} \quad u_{\tau_k} \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n)), \qquad u_{\tau_k} \to u \quad \text{in } \mathrm{C}^0([0,T];Y),\\
&\text{(5.5b)} \quad z_{\tau_k} \overset{*}{\rightharpoonup} z \quad \text{in } H^1(0,T;H^{\mathrm{m}}(\Omega)), \qquad z_{\tau_k} \to z \quad \text{in } \mathrm{C}^0([0,T];Z),\\
&\text{(5.5c)} \quad p_{\tau_k} \overset{*}{\rightharpoonup} p \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})), \qquad p_{\tau_k} \to p \quad \text{in } \mathrm{C}^0([0,T];W)
\end{aligned}
$$

for any Banach spaces Y, Z, and W such that $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n) \Subset Y$, $H^{\mathrm{m}}(\Omega) \Subset Z$ (in particular, for $Z = \mathrm{C}^0(\overline{\Omega})$), and $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \Subset W$. Hence, we have that

$$
\text{(5.6)} \quad u_{\tau_k}(t) \rightharpoonup u(t) \text{ in } H^1(\Omega;\mathbb{R}^n), \quad z_{\tau_k}(t) \rightharpoonup z(t) \text{ in } H^{\mathrm{m}}(\Omega), \quad p_{\tau_k}(t) \rightharpoonup p(t) \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})
$$

for all $t \in [0,T]$. Furthermore, it follows from estimates (4.15a) that

$$
\text{(5.7a)} \quad
\begin{aligned}
&\|\overline{u}_{\tau_k} - u_{\tau_k}\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} \leq \tau_k \|u'_{\tau_k}\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} \to 0 \qquad \text{as } k \to +\infty,\\
&\|\overline{z}_{\tau_k} - z_{\tau_k}\|_{L^\infty(0,T;H^{\mathrm{m}}(\Omega))} \leq \tau_k \|z'_{\tau_k}\|_{L^2(0,T;H^{\mathrm{m}}(\Omega))} \to 0 \qquad \text{as } k \to +\infty,\\
&\|\overline{p}_{\tau_k} - p_{\tau_k}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))} \leq \tau_k \|p'_{\tau_k}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))} \to 0 \qquad \text{as } k \to +\infty,
\end{aligned}
$$

and we have the very same estimates for $\underline{u}_{\tau_k}$, $\underline{z}_{\tau_k}$, and $\underline{p}_{\tau_k}$. Therefore, the pointwise convergences (5.6) hold for the sequences $\overline{u}_{\tau_k}$, $\underline{u}_{\tau_k}$, $\overline{z}_{\tau_k}$, $\underline{z}_{\tau_k}$ $\overline{p}_{\tau_k}$, and $\underline{p}_{\tau_k}$, as well. Since $w \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n))$, it is not difficult to check that, likewise,

$$
\text{(5.7b)} \quad \|\overline{w}_{\tau_k} - w_{\tau_k}\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} \leq \tau_k \|w'\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))} \to 0 \qquad \text{as } k \to +\infty\,.
$$

As a consequence of (5.4), (5.5), and (5.7a), we also have that

$$
\text{(5.8)} \quad
\begin{aligned}
&\overline{e}_{\tau_k} = \mathrm{E}(\overline{u}_{\tau_k} + \overline{w}_{\tau_k}) - \overline{p}_{\tau_k} \overset{*}{\rightharpoonup} e := \mathrm{E}(u+w) - p \qquad && \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))\,,\\
&\overline{e}_{\tau_k}(t) \rightharpoonup e(t) && \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \quad \forall t \in [0,T]\,.
\end{aligned}
$$

Then, it turns out that $\overline{\sigma}_{\tau_k} \overset{*}{\rightharpoonup} \sigma$ in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$, since

$$
\text{(5.9)} \quad (\overline{\sigma}_{\tau_k} - \sigma) = (\mathbb{C}(\overline{z}_{\tau_k}) - \mathbb{C}(z))\overline{e}_{\tau_k} + \mathbb{C}(z)(\overline{e}_{\tau_k} - e) \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})),
$$

since $\|\mathbb{C}(\overline{z}_{\tau_k}) - \mathbb{C}(z)\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))} \to 0$ by the Lipschitz continuity of $\mathbb{C}$, combined with convergences (5.5b) and (5.7a). Finally, let us observe that, by the Lipschitz continuity of the functional $z \mapsto \mathcal{H}(z,\pi)$ (cf. (2.30d)), and [Mor06, Proposition 1.85], the function $\overline{\lambda}_\tau$ featuring in the argument of $\mathcal{R}^*_\varepsilon$ on the left-hand side of (4.48) fulfills

$$
\|\overline{\lambda}_\tau\|_{L^\infty(0,T;\mathrm{M}(\Omega))} \leq C_K \|p'_\tau\|_{L^\infty(0,T;L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))},
$$

so that, by virtue of estimate (4.15a),

$$
\text{(5.10)} \qquad \tau_k \overline{\lambda}_{\tau_k} \to 0 \quad \text{in } L^\infty(0,T;H^{\mathrm{m}}(\Omega)^*) \quad \text{as } k \to +\infty.
$$

Since $\eta_{\tau_k}(0) = \eta_0$ for $\eta \in \{u, z, p\}$, it follows from convergences (5.6) that the triple (u, z, p) complies with the initial conditions (3.4).

Step 2: Limit passage in (4.48). Since we aim at (3.18), it is sufficient to take the limit of (4.48) written for $s = 0$ and $t = T$. We start by discussing the limit passage on the left-hand side of (4.48). Relying on the convergences from Step 1, easily check that

$$\liminf_{k\to+\infty} \mathcal{E}_\mu(\overline{t}_{\tau_k}(T), \overline{u}_{\tau_k}(T), \overline{z}_{\tau_k}(T), \overline{p}_{\tau_k}(T)) \geq \mathcal{E}_\mu(T, u(T), z(T), p(T)) .$$

In view of convergences (5.5), we immediately have

$$\liminf_{k\to+\infty} \int_0^T \left(\mathcal{V}_{\varepsilon,\nu}(u'_{\tau_k}(r)) + \mathcal{R}_\varepsilon(z'_{\tau_k}(r))\right) \mathrm{d}r \geq \int_0^T \left(\mathcal{V}_{\varepsilon,\nu}(u'(r)) + \mathcal{R}_\varepsilon(z'(r))\right) \mathrm{d}r.$$

It follows from (2.30d) that

$$\begin{aligned}\int_0^T &\left|\mathcal{H}(\underline{z}_{\tau_k}(t), p'_{\tau_k}(t)) - \mathcal{H}(z(t), p'_{\tau_k}(t))\right| \mathrm{d}t \\ &\leq \|\underline{z}_{\tau_k} - z\|_{L^\infty(0,T;L^\infty(\Omega))} \|p'_{\tau_k}\|_{L^1(0,T;L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))} \to 0\end{aligned}$$

as $k \to +\infty$, since $\underline{z}_{\tau_k} \to z$ in $L^\infty(0,T;L^\infty(\Omega))$ by (5.5b) and (5.7). On the other hand, by (5.5c) we have

$$\liminf_{k\to+\infty} \int_0^T \mathcal{H}(z(t), p'_{\tau_k}(t))\,\mathrm{d}t \geq \int_0^T \mathcal{H}(z(t), p'(t))\,\mathrm{d}t .$$

Therefore,

$$\begin{aligned}&\liminf_{k\to+\infty} \int_0^T \mathcal{H}_{\varepsilon,\nu}(\underline{z}_{\tau_k}(t), p'_{\tau_k}(t))\,\mathrm{d}t \\ &\quad\geq \liminf_{k\to+\infty} \int_0^T \mathcal{H}(\underline{z}_{\tau_k}(t), p'_{\tau_k}(t))\,\mathrm{d}t + \liminf_{k\to+\infty} \frac{\varepsilon\nu}{2} \int_0^T \|p'_{\tau_k}(t)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\,\mathrm{d}t \\ &\quad\geq \int_0^T \mathcal{H}(z(t), p'(t))\,\mathrm{d}t + \frac{\varepsilon\nu}{2} \int_0^T \|p'(t)\|^2_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\,\mathrm{d}t = \int_0^T \mathcal{H}_{\varepsilon,\nu}(z(t), p'(t))\,\mathrm{d}t .\end{aligned}$$

By (5.9), $\mathrm{Div}(\mathbb{C}(\overline{z}_{\tau_k})\overline{e}_{\tau_k}) = \mathrm{Div}(\overline{\sigma}_{\tau_k}) \overset{*}{\rightharpoonup} \mathrm{Div}(\sigma) = \mathrm{Div}(\mathbb{C}(z)e)$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)^*)$. Therefore, also in view of (5.4) and by the convexity of $\mathcal{V}^*_{\varepsilon,\nu}$, we find that

$$\begin{aligned}\liminf_{k\to+\infty} &\int_0^T \mathcal{V}^*_{\varepsilon,\nu}(\mathrm{Div}(\mathbb{C}(\overline{z}_{\tau_k}(r))\overline{e}_{\tau_k}(r)) + \overline{F}_{\tau_k}(r))\,\mathrm{d}r \\ &\geq \int_0^T \mathcal{V}^*_{\varepsilon,\nu}(\mathrm{Div}(\mathbb{C}(z(r))e(r)) + F(r))\,\mathrm{d}r .\end{aligned}$$

We now observe that

$$\begin{aligned}&\liminf_{k\to+\infty} \int_0^T \mathcal{R}^*_\varepsilon(-A_{\mathrm{m}}(\overline{z}_{\tau_k}(r)) - W'(\overline{z}_{\tau_k}(r)) - \tfrac{1}{2}\mathbb{C}'(\overline{z}_{\tau_k}(r))\overline{e}_{\tau_k}(r) : \overline{e}_{\tau_k}(r) - \tau_k\overline{\lambda}_{\tau_k}(r))\,\mathrm{d}r \\ &\overset{(1)}{\geq} \int_0^T \liminf_{k\to+\infty} \mathcal{R}^*_\varepsilon(-A_{\mathrm{m}}(\overline{z}_{\tau_k}(r)) - W'(\overline{z}_{\tau_k}(r)) - \tfrac{1}{2}\mathbb{C}'(\overline{z}_{\tau_k}(r))\overline{e}_{\tau_k}(r) : \overline{e}_{\tau_k}(r) - \tau_k\overline{\lambda}_{\tau_k}(r))\,\mathrm{d}r \\ &\overset{(2)}{\geq} \int_0^T \mathcal{R}^*_\varepsilon(-A_{\mathrm{m}}(z(r)) - W'(z(r)) - \tfrac{1}{2}\mathbb{C}'(z(r))e(r) : e(r))\,\mathrm{d}r,\end{aligned}$$

where (1) follows from the Fatou lemma and (2) from the fact that

$$\begin{aligned}
&\liminf_{k\to+\infty} \mathcal{R}_\varepsilon^*(-A_{\mathrm{m}}(\overline{z}_{\tau_k}(r)) - W'(\overline{z}_{\tau_k}(r)) - \tfrac{1}{2}\mathbb{C}'(\overline{z}_{\tau_k}(r))\overline{e}_{\tau_k}(r) : \overline{e}_{\tau_k}(r) - \tau_k \overline{\lambda}_{\tau_k}(r)) \\
&\geq \sup_{\zeta\in H^{\mathrm{m}}_-(\Omega)} \Big(\liminf_{k\to+\infty} \langle A_{\mathrm{m}}(\overline{z}_{\tau_k}(r)) + W'(\overline{z}_{\tau_k}(r)) + \tfrac{1}{2}\mathbb{C}'(\overline{z}_{\tau_k}(r))\overline{e}_{\tau_k}(r) : \overline{e}_{\tau_k}(r) + \tau_k \overline{\lambda}_{\tau_k}(r), -\zeta\rangle_{H^{\mathrm{m}}(\Omega)} \\
&\qquad - \mathcal{R}_\varepsilon(\zeta)\Big) \\
&\geq \sup_{\zeta\in H^{\mathrm{m}}_-(\Omega)} \Big(\langle A_{\mathrm{m}}(z(r)) + W'(z(r)) + \tfrac{1}{2}\mathbb{C}'(z(r))e(r) : e(r), -\zeta\rangle_{H^{\mathrm{m}}(\Omega)} - \mathcal{R}_\varepsilon(\zeta)\Big)
\end{aligned}$$

for all $r \in [0,T]$ by (5.6), (5.8), and (5.10). In the end, we have that

$$\liminf_{k\to+\infty} \int_0^T \mathcal{H}^*_{\varepsilon,\nu}(\underline{z}_{\tau_k}(r), -\mu\overline{p}_{\tau_k}(r) + (\overline{\sigma}_{\tau_k}(r))_{\mathrm{D}})\,\mathrm{d}r \geq \int_0^T \mathcal{H}^*_{\varepsilon,\nu}(z(r), -\mu p(r) + (\sigma(r))_{\mathrm{D}})\,\mathrm{d}r$$

by convergences (5.5) and (5.9), and a version of the Ioffe theorem; cf., e.g. [Val90, Theorem 21]. The latter result applies since

1. the mapping $H^{\mathrm{m}}(\Omega) \ni z \mapsto \mathcal{H}^*(z,\omega)$ is lower semicontinuous for all $\omega \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ (as $\mathcal{H}^*(z,\omega) = \sup_{\pi\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}(\int_\Omega \omega\pi\,\mathrm{d}x - \mathcal{H}(z,\pi))$ and the maps $z \mapsto -\mathcal{H}(z,\pi)$ are continuous by (2.30d)), and thus $z \in H^{\mathrm{m}}(\Omega) \mapsto \mathcal{H}^*_{\varepsilon,\nu}(z,\omega)$ is also lower semicontinuous;
2. the mapping $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}) \ni \pi \mapsto \mathcal{H}^*_{\varepsilon,\nu}(z,\pi)$ is convex.

As for the right-hand side of (4.48), clearly we have $\mathcal{E}_\mu(0, \overline{u}_{\tau_k}(0), \overline{z}_{\tau_k}(0), \overline{p}_{\tau_k}(0)) = \mathcal{E}_\mu(0, u_0, z_0, p_0)$ for all $k \in \mathbb{N}$. The power terms converge, too, as we have

$$\begin{aligned}
&\int_0^T \int_\Omega \mathbb{C}(z_{\tau_k}(r))(\mathrm{E}(u_{\tau_k}(r) + w(r)) - p_{\tau_k}(r)) : \mathrm{E}(w'(r))\,\mathrm{d}x\,\mathrm{d}r \\
&\overset{(1)}{\to} \int_0^T \int_\Omega \mathbb{C}(z(r))e(r) : \mathrm{E}(w'(r))\,\mathrm{d}x\,\mathrm{d}r, \\
&-\int_0^T \langle F'(r), u_{\tau_k}(r) + w(r)\rangle_{H^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}r \overset{(2)}{\to} -\int_0^T \langle F'(r), u(r) + w(r)\rangle_{H^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}r
\end{aligned}$$

with (1) due to convergences (5.5) and to the fact that, by the Lipschitz continuity of $\mathbb{C}$, $\mathbb{C}(z_{\tau_k}) \to \mathbb{C}(z)$ in $L^\infty(0,T;L^\infty(\Omega))$, and (2) again due to (5.5). In a completely analogous way the last-but-one term on the right-hand side of (4.48) passes to the limit. Finally, we estimate the remainder term $\mathfrak{R}_\tau(0,T)$ from (4.49) via

$$\begin{aligned}
&\mathfrak{R}_\tau(0,T) \\
&\leq C_3 \left(\|\overline{u}_{\tau_k} - u_\tau\|_{L^\infty(0,T;H^1(\Omega))} + \|\overline{z}_\tau - z_\tau\|_{L^\infty(0,T;H^{\mathrm{m}}(\Omega))} \right. \\
&\quad \left. + \|\overline{p}_\tau - p_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{w}_\tau - w\|_{L^\infty(0,T;H^1(\Omega))} \right) \\
&\quad \times \int_0^T \left(\|u'_{\tau_k}\|_{H^1(\Omega)} + \|z'_{\tau_k}\|_{H^{\mathrm{m}}(\Omega)} + \|p'_{\tau_k}\|_{L^2(\Omega)} \right)\mathrm{d}r \\
&\overset{(3)}{\leq} C\tau_k \to 0 \qquad \text{as } k \to +\infty
\end{aligned} \tag{5.11}$$

with (3) due to (5.7) and the fact that $w \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^n))$, and estimates (4.15). This concludes the proof of the energy-dissipation upper estimate (3.18).

Step 3*: Proof of* (5.3)*.* Estimates (5.3) follow from the analogous bounds (4.10) and (4.15b) for the discrete solutions via convergences (5.5) and (5.8) and lower semicontinuity arguments. We conclude observing that (5.2) follows from the reformulation (3.21) of the energy-dissipation balance, and the fact that $\int_0^t \partial_t \mathcal{E}_\mu$ is uniformly bounded w.r.t. ε, ν, μ, thanks to the assumptions on F and w and the previously proven (5.3). □

6. The vanishing-viscosity limit with fixed hardening parameter. This section focuses on the limit passage in the viscous system (1.2) as $\varepsilon \downarrow 0$ and, possibly, $\nu \downarrow 0$, while the hardening parameter $\mu > 0$ *is kept fixed*. In fact we will distinguish the two cases:

1. $\varepsilon \downarrow 0$ and $\nu > 0$ is kept fixed, addressed in section 6.1, in which the vanishing-viscosity analysis will lead to the existence of Balanced Viscosity solutions to the *rate-independent* system for damage and plasticity *with hardening* (cf. Theorem 6.8 ahead);
2. ε, $\nu \downarrow 0$, addressed in section 6.2, in which we will obtain Balanced Viscosity solutions to the *multirate* system for damage and plasticity *with hardening* (cf. Theorem 6.13 later on).

Notation 6.1. We will denote by $(q_{\varepsilon,\nu})_{\varepsilon,\nu} = (u_{\varepsilon,\nu}, z_{\varepsilon,\nu}, p_{\varepsilon,\nu})_{\varepsilon,\nu}$ a family of solutions to Problem 3.1 for $\mu > 0$ fixed, with initial and external data independent of ε and ν and satisfying the conditions listed in section 2.

Prior to distinguishing the case in which $\nu > 0$ is fixed from that in which $\nu \downarrow 0$, let us establish the common ground for the vanishing-viscosity analysis. Following the well-established reparameterization technique pioneered in [EM06], we will suitably reparameterize the viscous solutions $(q_{\varepsilon,\nu})_{\varepsilon,\nu}$, observe that the rescaled functions $(\mathsf{q}_{\varepsilon,\nu})_{\varepsilon,\nu}$ comply with a reparameterized version of the energy-dissipation balance corresponding to (3.18), and pass to the limit in it as $\varepsilon \downarrow 0$ and $\nu > 0$ is fixed (see section 6.1), and as ε, $\nu \downarrow 0$ (see section 6.2).

Rescaling. We introduce the arclength function $s_{\varepsilon,\nu} : [0, T] \to [0, S_{\varepsilon,\nu}]$ (with $S_\varepsilon := s_{\varepsilon,\nu}(T)$) defined by

(6.1)
$$
\begin{aligned}
s_{\varepsilon,\nu}(t) :&= \int_0^t \left(1 + \|q'_{\varepsilon,\nu}(\tau)\|_{\mathbf{Q}}\right) \mathrm{d}\tau \\
&= \int_0^t \Big(1 + \|u'_{\varepsilon,\nu}(\tau)\|_{H^1(\Omega;\mathbb{R}^n)} + \|z'_{\varepsilon,\nu}(\tau)\|_{H^{\mathrm{m}}(\Omega)} + \|p'_{\varepsilon,\nu}(\tau)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\Big) \, \mathrm{d}\tau \,.
\end{aligned}
$$

It follows from estimate (5.3) that $\sup_{\varepsilon,\nu} S_{\varepsilon,\nu} < +\infty$. We now define

$$
\begin{aligned}
&\mathsf{t}_{\varepsilon,\nu} : [0, S_{\varepsilon,\nu}] \to [0, T], && \mathsf{t}_{\varepsilon,\nu} := s_{\varepsilon,\nu}^{-1}, \\
&\mathsf{u}_{\varepsilon,\nu} : [0, S_{\varepsilon,\nu}] \to H^1(\Omega; \mathbb{R}^n), && \mathsf{u}_{\varepsilon,\nu} := u_{\varepsilon,\nu} \circ \mathsf{t}_{\varepsilon,\nu} \\
&\mathsf{z}_{\varepsilon,\nu} : [0, S_{\varepsilon,\nu}] \to H^{\mathrm{m}}(\Omega), && \mathsf{z}_{\varepsilon,\nu} := z_{\varepsilon,\nu} \circ \mathsf{t}_{\varepsilon,\nu}, \\
&\mathsf{p}_{\varepsilon,\nu} : [0, S_{\varepsilon,\nu}] \to L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{D}}), && \mathsf{p}_{\varepsilon,\nu} := p_{\varepsilon,\nu} \circ \mathsf{t}_{\varepsilon,\nu},
\end{aligned}
\tag{6.2a}
$$

and set $\mathsf{q}_{\varepsilon,\nu} := (\mathsf{u}_{\varepsilon,\nu}, \mathsf{z}_{\varepsilon,\nu}, \mathsf{p}_{\varepsilon,\nu})$. In what follows, with slight abuse of notation we will often write $\mathcal{E}(t, q)$ in place of $\mathcal{E}(t, u, z, p)$. We also introduce

$$
\begin{aligned}
&\mathsf{e}_{\varepsilon,\nu} : [0, S_{\varepsilon,\nu}] \to L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}), && \mathsf{e}_{\varepsilon,\nu} := e_{\varepsilon,\nu} \circ \mathsf{t}_{\varepsilon,\nu} = \mathrm{E}(\mathsf{u}_{\varepsilon,\nu} + (w \circ \mathsf{t}_{\varepsilon,\nu})) - \mathsf{p}_{\varepsilon,\nu}, \\
&\sigma_{\varepsilon,\nu} : [0, S_{\varepsilon,\nu}] \to L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}), && \sigma_{\varepsilon,\nu} := \sigma_{\varepsilon,\nu} \circ \mathsf{t}_{\varepsilon,\nu} = \mathbb{C}(\mathsf{z}_{\varepsilon,\nu}) \mathsf{e}_{\varepsilon,\nu}.
\end{aligned}
\tag{6.2b}
$$

The parameterized energy-dissipation balance. A straightforward calculation on the energy-dissipation balance corresponding to (3.18) yields that the reparameterized viscous solutions $(\mathsf{u}_{\varepsilon,\nu}, \mathsf{z}_{\varepsilon,\nu}, \mathsf{p}_{\varepsilon,\nu})$, along with the rescaling functions $\mathsf{t}_{\varepsilon,\nu}$, fulfill

(6.3)
$$
\begin{aligned}
&\mathcal{E}(\mathsf{t}_{\varepsilon,\nu}(s_2), \mathsf{q}_{\varepsilon,\nu}(s_2)) + \int_{s_1}^{s_2} \mathsf{t}'_{\varepsilon,\nu} \left[\mathcal{V}_{\varepsilon,\nu}\left(\frac{\mathsf{u}'_{\varepsilon,\nu}}{\mathsf{t}'_{\varepsilon,\nu}}\right) + \mathcal{R}_{\varepsilon,\nu}\left(\frac{\mathsf{z}'_{\varepsilon,\nu}}{\mathsf{t}'_{\varepsilon,\nu}}\right) + \mathcal{H}_{\varepsilon,\nu}\left(\mathsf{z}_{\varepsilon,\nu}, \frac{\mathsf{p}'_{\varepsilon,\nu}}{\mathsf{t}'_{\varepsilon,\nu}}\right)\right] \mathrm{d}\tau \\
&\quad + \int_{s_1}^{s_2} \mathsf{t}'_{\varepsilon,\nu} \Big[\mathcal{V}^*_{\varepsilon,\nu}\Big(\mathrm{Div}\big(\sigma_{\varepsilon,\nu} + \rho(\mathsf{t}_{\varepsilon,\nu})\big)\Big) + \mathcal{R}^*_{\varepsilon}\Big(-A_{\mathrm{m}}(\mathsf{z}_{\varepsilon,\nu}) - W'(\mathsf{z}_{\varepsilon,\nu}) - \tfrac{1}{2}\mathbb{C}'(\mathsf{z}_{\varepsilon,\nu})\mathsf{e}_{\varepsilon,\nu} : \mathsf{e}_{\varepsilon,\nu}\Big) \\
&\qquad\qquad + \mathcal{H}^*_{\varepsilon,\nu}\Big(\mathsf{z}_{\varepsilon,\nu}, -\nu \mathsf{p}_{\varepsilon,\nu} + (\sigma_{\varepsilon,\nu})_{\mathrm{D}}\Big)\Big] \mathrm{d}\tau \\
&= \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}(s_1), \mathsf{q}_{\varepsilon,\nu}(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}, \mathsf{q}_{\varepsilon,\nu})\, \mathsf{t}'_{\varepsilon,\nu} \,\mathrm{d}\tau
\end{aligned}
$$

for all $0 \le s_1 \le s_2 \le S_{\varepsilon,\nu}$, where we have used that $F \circ \mathsf{t}_{\varepsilon,\nu} = -\mathrm{Div}(\rho \circ \mathsf{t}_{\varepsilon,\nu})$ by condition (2.39e).

Let us now introduce a functional $\mathcal{M}^\mu_{\varepsilon,\nu} = \mathcal{M}^\mu_{\varepsilon,\nu}(t, q, t', q')$ subsuming the terms featuring in the integrals on the left-hand side of (6.3). In order to motivate our definition of $\mathcal{M}^\mu_{\varepsilon,\nu}$ (cf. (6.5) below), we recall the definitions (3.1) of the functionals $\mathcal{V}_{\varepsilon,\nu}$, $\mathcal{R}_\varepsilon$, and $\mathcal{H}_{\varepsilon,\nu}$, so that

$$
\begin{aligned}
\mathcal{V}_{\varepsilon,\nu}\left(\frac{\mathsf{u}'_{\varepsilon,\nu}}{\mathsf{t}'_{\varepsilon,\nu}}\right) &= \frac{\varepsilon\nu}{2(\mathsf{t}'_{\varepsilon,\nu})^2} \|\mathsf{u}'_{\varepsilon,\nu}\|^2_{H^1,\mathbb{D}}\,, \\
\mathcal{R}_\varepsilon\left(\frac{\mathsf{z}'_{\varepsilon,\nu}}{\mathsf{t}'_{\varepsilon,\nu}}\right) &= \frac{1}{\mathsf{t}'_{\varepsilon,\nu}} \mathcal{R}(\mathsf{z}'_{\varepsilon,\nu}) + \frac{\varepsilon}{2(\mathsf{t}'_{\varepsilon,\nu})^2} \|\mathsf{z}'_{\varepsilon,\nu}\|^2_{L^2}\,, \\
\mathcal{H}_{\varepsilon,\nu}\left(\mathsf{z}_{\varepsilon,\nu}, \frac{\mathsf{p}'_{\varepsilon,\nu}}{\mathsf{t}'_{\varepsilon,\nu}}\right) &= \frac{1}{\mathsf{t}'_{\varepsilon,\nu}} \mathcal{H}(\mathsf{z}_{\varepsilon,\nu}, \mathsf{p}'_{\varepsilon,\nu}) + \frac{\varepsilon\nu}{2(\mathsf{t}'_{\varepsilon,\nu})^2} \|\mathsf{p}'_{\varepsilon,\nu}\|^2_{L^2}\,.
\end{aligned}
$$

Moreover, we take into account the expressions (3.17) of the conjugates and the fact that the arguments of $\mathcal{V}^*_{\varepsilon,\nu}$, $\mathcal{R}^*_\varepsilon$, and $\mathcal{H}^*_{\varepsilon,\nu}$ in (6.3) involve the derivatives $-\mathrm{D}_x \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}, \mathsf{u}_{\varepsilon,\nu}, \mathsf{z}_{\varepsilon,\nu}, \mathsf{p}_{\varepsilon,\nu})$ for $x = u$, $x = z$, and $x = p$, respectively. In particular, in view of (2.20) we have

$$
\mathcal{V}^*_{\varepsilon,\nu}(\mathrm{Div}(\sigma_{\varepsilon,\nu} + \rho(\mathsf{t}_{\varepsilon,\nu}))) = \frac{1}{2\varepsilon\nu} \| -\mathrm{D}_u \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}, \mathsf{u}_{\varepsilon,\nu}, \mathsf{z}_{\varepsilon,\nu}, \mathsf{p}_{\varepsilon,\nu})\|^2_{(H^1,\mathbb{D})^*}\,. \tag{6.4}
$$

All in all, the functional $\mathcal{M}^\mu_{\varepsilon,\nu} : [0,T] \times \mathbf{Q} \times (0,+\infty) \times \mathbf{Q} \to [0,+\infty]$ encompassing the integrands on the left-hand side of (6.3) reads

(6.5)
$$
\begin{aligned}
&\mathcal{M}^\mu_{\varepsilon,\nu}(t, q, t', q') := \mathcal{R}(z') + \mathcal{H}(z, p') + \mathcal{M}^{\mu,\mathrm{red}}_{\varepsilon,\nu}(t, q, t', q') \qquad \text{with } \mathcal{M}^{\mu,\mathrm{red}}_{\varepsilon,\nu} \text{ defined by} \\
&\mathcal{M}^{\mu,\mathrm{red}}_{\varepsilon,\nu}(t, q, t', q') := \frac{\varepsilon}{2t'} \mathcal{D}_\nu(q')^2 + \frac{t'}{2\varepsilon} (\mathcal{D}^{*,\mu}_\nu(t,q))^2\,,
\end{aligned}
$$

with the functionals $\mathcal{D}_\nu$ and $\mathcal{D}^{*,\mu}_\nu$ from (3.19), namely

(6.6)
$$
\begin{aligned}
&\mathcal{D}_\nu(q') := \sqrt{\nu\|u'(t)\|^2_{H^1,\mathbb{D}} + \|z'(t)\|^2_{L^2} + \nu\|p'(t)\|^2_{L^2}}\,.. \\
&\mathcal{D}^{*,\mu}_\nu(t,q) \\
&\quad := \sqrt{\frac{1}{\nu}\|-\mathrm{D}_u \mathcal{E}_\mu(t,q)\|^2_{(H^1,\mathbb{D})^*} + \widetilde{d}_{L^2}(-\mathrm{D}_z \mathcal{E}_\mu(t,q), \partial\mathcal{R}(0))^2 + \frac{1}{\nu} d_{L^2}(-\mathrm{D}_p \mathcal{E}_\mu(t,q), \partial_\pi \mathcal{H}(z,0))^2}\,.
\end{aligned}
$$

Therefore, the rescaled solutions $(\mathsf{t}_{\varepsilon,\nu}, \mathsf{q}_{\varepsilon,\nu})_{\varepsilon,\nu}$ satisfy

- the *parameterized* energy-dissipation balance for every $s_1, s_2 \in [0, S_{\varepsilon,\nu}]$,

$$
\begin{aligned}
&\mathcal{E}(\mathsf{t}_{\varepsilon,\nu}(s_2), \mathsf{q}_{\varepsilon,\nu}(s_2)) + \int_{s_1}^{s_2} \mathcal{M}^{\mu}_{\varepsilon,\nu}(\mathsf{t}_{\varepsilon,\nu}(\tau), \mathsf{q}_{\varepsilon,\nu}(\tau), \mathsf{t}'_{\varepsilon,\nu}(\tau), \mathsf{q}'_{\varepsilon,\nu}(\tau))\, \mathrm{d}\tau \\
&\quad = \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}(s_1), \mathsf{q}_{\varepsilon,\nu}(s_1)) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}(\tau), \mathsf{q}_{\varepsilon,\nu}(\tau))\, \mathsf{t}'_{\varepsilon,\nu}(\tau)\, \mathrm{d}\tau
\end{aligned} \tag{6.7}
$$

(which rephrases (6.3));
- the *normalization* condition, for a.a. $s \in (0, S_{\varepsilon,\nu})$.
(6.8)
$\mathsf{t}'_{\varepsilon,\nu}(s) + \|\mathsf{q}'_{\varepsilon,\nu}(\tau)\|_{\mathbf{Q}} = \mathsf{t}'_{\varepsilon,\nu}(s) + \|\mathsf{u}'_{\varepsilon,\nu}(s)\|_{H^1} + \|\mathsf{z}'_{\varepsilon,\nu}(s)\|_{H^{\mathrm{m}}} + \|\mathsf{p}'_{\varepsilon,\nu}(s)\|_{L^2} \equiv 1\,.$

Finally, it follows from (3.21) that the reparameterized viscous solutions $(\mathsf{u}_{\varepsilon,\nu}, \mathsf{z}_{\varepsilon,\nu}, \mathsf{p}_{\varepsilon,\nu})$ fulfill for all $0 \le s_1 \le s_2 \le S_{\varepsilon,\nu}$

$$
\begin{aligned}
&\mathcal{E}_\mu(\mathsf{t}_{\varepsilon,\nu}(s_2), \mathsf{q}_{\varepsilon,\nu}(s_2)) + \int_{s_1}^{s_2} \mathcal{N}^{\mu}_{\varepsilon,\nu}(\mathsf{t}_{\varepsilon,\nu}(\tau), \mathsf{q}_{\varepsilon,\nu}(\tau), \mathsf{q}'_{\varepsilon,\nu}(\tau))\, \mathrm{d}\tau \\
&\quad = \mathcal{E}_\mu(\mathsf{t}_{\varepsilon,\nu}(s_1), \mathsf{q}_{\varepsilon,\nu}(s_1) + \int_{s_1}^{s_2} \partial_t \mathcal{E}(\mathsf{t}_{\varepsilon,\nu}, \mathsf{q}_{\varepsilon,\nu})\, \mathsf{t}'_{\varepsilon,\nu}\, \mathrm{d}\tau\,.
\end{aligned} \tag{6.9}
$$

Indeed, it will be in (6.9) that we will perform the vanishing-viscosity limit passages. With the very same arguments as in the proof of Theorem 5.1 (cf. (5.2)), it is immediate to deduce from (6.9) that

$$
\exists\, C > 0\ \forall\, \varepsilon, \nu > 0 : \quad \int_0^S \mathcal{N}^{\mu}_{\varepsilon,\nu}(\mathsf{t}_{\varepsilon,\nu}(\tau), \mathsf{q}_{\varepsilon,\nu}(\tau), \mathsf{q}'_{\varepsilon,\nu}(\tau))\, \mathrm{d}\tau \le C. \tag{6.10}
$$

6.1. The vanishing-viscosity analysis as $\varepsilon \downarrow 0$ and $\nu > 0$ is fixed. Throughout this section we will keep the rate parameter $\nu > 0$ *fixed*. In order to signify this and simplify notation, we will drop the dependence on ν of the viscous solutions and simply write

$$
(\mathsf{t}_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon, \mathsf{p}_\varepsilon) \quad \text{in place of} \quad (\mathsf{t}_{\varepsilon,\nu}, \mathsf{u}_{\varepsilon,\nu}, \mathsf{z}_{\varepsilon,\nu}, \mathsf{p}_{\varepsilon,\nu}).
$$

Since the variables u and p relax to equilibrium and rate-independent evolution with the same rate with which z relaxes to rate-independent behavior, a Γ-convergence argument and the comparison with the general results from [MRS16b, MR21] lead us to expect that any parameterized curve $(\mathsf{t}, \mathsf{q}) = (\mathsf{t}, \mathsf{u}, \mathsf{z}, \mathsf{p})$ arising as a limit point of the family $(\mathsf{t}_\varepsilon, \mathsf{q}_\varepsilon)_\varepsilon$ as $\varepsilon \downarrow 0$ will satisfy the energy-dissipation (upper) estimate

$$
\begin{aligned}
&\mathcal{E}_\mu(\mathsf{t}(S), \mathsf{q}(S)) + \int_0^S \mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(\tau), \mathsf{q}(\tau), \mathsf{t}'(\tau), \mathsf{q}'(\tau))\, \mathrm{d}\tau \\
&\quad \le \mathcal{E}_\mu(\mathsf{t}(0), \mathsf{q}(0)) + \int_0^S \partial_t \mathcal{E}_\nu(\mathsf{t}(\tau), \mathsf{q}(\tau))\, \mathsf{t}'(\tau)\, \mathrm{d}\tau
\end{aligned} \tag{6.11}
$$

with $S = \lim_{\varepsilon \downarrow 0} S_\varepsilon$ and the functional $\mathcal{M}^{\mu}_{0,\nu} : [0,T] \times \mathbf{Q} \times [0,+\infty) \times \mathbf{Q} \to [0,+\infty]$ defined by (as usual, here $q = (u, z, p)$)

$$\mathcal{M}^{\mu}_{0,\nu}(t,q,t',q') := \mathcal{R}(z') + \mathcal{H}(z,p') + \mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(t,q,t',q') \,, \quad \text{where}$$

(6.12a)
$$\text{if } t'>0\,, \quad \mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(t,q,t',q') := \begin{cases} 0 & \text{if } \begin{cases} -\mathrm{D}_u\mathcal{E}_\mu(t,q) = 0\,, \\ -\mathrm{D}_z\mathcal{E}_\mu(t,q) \in \partial\mathcal{R}(0)\,, \text{ and} \\ -\mathrm{D}_p\mathcal{E}_\mu(t,q) \in \partial_\pi\mathcal{H}(z,0)\,, \end{cases} \\ +\infty & \text{otherwise,} \end{cases}$$

(6.12b)
$$\text{if } t'=0\,, \quad \mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(t,q,0,q') := \mathcal{D}_\nu(q')\,\mathcal{D}^{*,\mu}_\nu(t,q)$$

(recall (6.6) for the definition of the functionals $\mathcal{D}_\nu$ and $\mathcal{D}^{*,\mu}_\nu$). Observe that the product $\mathcal{D}_\nu(q')\,\mathcal{D}^{*,\mu}_\nu(t,q)$ contains the term $\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}(t,q),\partial\mathcal{R}(0))$ which, in principle, need not be finite at all $(t,q) \in [0,T]\times\mathbf{Q}$ since, in general, we only have $\mathrm{D}_z\mathcal{E}(t,q)$, $\partial\mathcal{R}(0) \subset H^{\mathrm{m}}(\Omega)^*$. Let us then clarify that

(6.13)
$$\text{if } \mathcal{D}^{*,\mu}_\nu(t,q) = +\infty \text{ and } \mathcal{D}_\nu(q') = 0, \text{ in (6.12b) we mean } \mathcal{D}_\nu(q')\,\mathcal{D}^{*,\mu}_\nu(t,q) := +\infty.$$

Following [MRS16b, MR21], we will refer to the functional $\mathcal{M}^{\mu}_{0,\nu}$ from (6.12) as *vanishing-viscosity contact potential*. Observe that we keep on highlighting the dependence of $\mathcal{M}^{\mu}_{0,\nu}$ on the (fixed) parameters ν and μ for later use in section 7.

Our definition of *Balanced Viscosity* solution to the rate-independent system with hardening (1.1a), (1.1b), (1.1d), (1.1f), (1.3) features (6.11) as a *balance*, satisfied along *any* subinterval of a given interval $[0,S]$. Along the lines of [MRS16b] we give the following.

DEFINITION 6.2. *We say that a parameterized curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathrm{AC}([0,S];[0,T]\times\mathbf{Q})$ *is a* (parameterized) BV *solution to the rate-independent system with hardening* (1.1a), (1.1b), (1.1d), (1.1f), (1.3) *if* $\mathsf{t}\colon [0,S]\to[0,T]$ *is nondecreasing and* (t,q) *fulfills the energy-dissipation balance*

$$\begin{aligned} \mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau \\ = \mathcal{E}_\mu(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau \end{aligned} \tag{6.14}$$

for all $0\le s\le S$. *We call a* BV *solution* (t,q) nondegenerate *if, in addition, there holds for almost all* $s\in(0,S)$

(6.15)
$$\mathsf{t}'(s) + \|\mathsf{q}'(s)\|_{\mathbf{Q}} = \mathsf{t}'(s) + \|\mathsf{u}'(s)\|_{H^1(\Omega;\mathbb{R}^n)} + \|\mathsf{z}'(s)\|_{H^{\mathrm{m}}(\Omega)} + \|\mathsf{p}'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} > 0\,.$$

Remark 6.3. We have defined BV solutions following the general setting where they are only required to be absolutely continuous in the reparametrized variable s. Nonetheless, up to a further reparametrization by arclength, one obtains curves that are Lipschitz in s. In fact, notice that in Theorem 6.8 below we obtain the existence of a BV solution $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in W^{1,\infty}([0,S];[0,T]\times\mathbf{Q})$. The Lipschitz regularity here is a consequence of the normalization condition (6.8) used in the approximation.

We postpone a discussion of the nondegeneracy condition (6.15) to the upcoming Remark 6.9.

As we will see, the energy-dissipation balance (6.14) encodes all the information on the evolution of the parameterized curve (t,q) and, in particular, on the onset of rate-dependent behavior in the jump regime (i.e., when $\mathsf{t}' = 0$). While postponing further comments to Remark 6.7, let us only record here the fact that the expression of $\mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(t,q,0,q')$ shows that, at a jump, viscous behavior for the variables u, z, and p emerges "in the same way," since the viscous terms related to each variable equally contribute to $\mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(t,q,0,q')$. This aspect will be further explored in Remark 6.7.

The *main result of this subsection*, Theorem 6.8 ahead, shows that the parameterized solutions $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)_\varepsilon$ of the viscous system (1.2) converge to a BV solution to the rate-independent system with hardening (1.1a), (1.1b), (1.1d), (1.3). Our proof will crucially rely on the following characterization of BV solutions that is in the same spirit as [MRS12a, Proposition 5.3], [MRS16a, Corollary 4.5].

Proposition 6.4. *For a parameterized curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathrm{AC}([0,S];[0,T]\times\mathbf{Q})$ *with* $\mathsf{t}\colon[0,S]\to[0,T]$ *nondecreasing the following properties are equivalent:*

1. (t,q) *is a* BV *solution to the rate-independent system with hardening;*
2. (t,q) *fulfills the energy-dissipation upper estimate* (6.11)*;*
3. (t,q) *fulfills the* contact condition *for a.a.* $s\in(0,S)$

$$\mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) = \langle -\mathrm{D}_q\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{q}'(s)\rangle_{\mathbf{Q}}\,. \tag{6.16}$$

The proof is based on the following key chain-rule estimate

Lemma 6.5. *Along any parameterized curve*

$$(\mathsf{t},\mathsf{q})\in\mathrm{AC}([0,S];[0,T]\times\mathbf{Q}) \ \textit{s.t.}\ \mathcal{M}^{\mu}_{0,\nu}(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}')<+\infty \ \textit{a.e. in}\ (0,S),\ \textit{there holds}$$
$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) + \partial_t\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) &= \langle -\mathrm{D}_q\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{q}'(s)\rangle_{\mathbf{Q}} \\ &\le \mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \quad \textit{for a.a.}\ s\in(0,S)\,. \end{aligned} \tag{6.17}$$

Proof. The first equality in (6.17) directly follows from the chain rule (3.9). To deduce the second estimate, we start by observing that, from $\mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))<+\infty$, we gather that

$$\text{if } \mathsf{t}'(s)=0 \text{ then } \quad \begin{cases} -\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) = 0\,, \\ -\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) \in \partial\mathcal{R}(0)\,, \\ -\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) \in \partial_\pi\mathcal{H}(z,0)\,, \end{cases}$$

while if $\mathsf{t}'(s)>0$, then

$$\|\mathsf{z}'(s)\|_{L^2}\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0)) \le \mathcal{D}_\nu(\mathsf{q}'(s))\,\mathcal{D}^{*,\mu}_\nu(\mathsf{t}(s),\mathsf{q}(s))\,.$$

In each case, we have

$$\|\mathsf{z}'(s)\|_{L^2}\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0)) < +\infty$$

whence $\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0))<+\infty$ if $\mathsf{z}'(s)\neq0$. If $\mathsf{z}'(s)=0$, taking into account our convention (6.13) and the fact that $\mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))<+\infty$ we again get $\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0))<+\infty$.

After this preliminary discussion, it is sufficient to observe that

$$\begin{aligned} &\langle -\mathrm{D}_q\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{q}'(s)\rangle_{\mathbf{Q}} \\ &\le \|\mathsf{u}'(s)\|_{H^1,\mathbb{D}}\|{-\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))}\|_{(H^1,\mathbb{D})^*} + \|\mathsf{z}'(s)\|_{L^2}\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0)) \\ &\quad + \mathcal{R}(\mathsf{z}'(s)) + \|\mathsf{p}'(s)\|_{L^2}d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\partial_\pi\mathcal{H}(\mathsf{z}(s),0)) + \mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) \end{aligned}$$

(cf. (3.23)) in order to conclude (6.17). ∎

We are now in a position to carry out the proof of Proposition 6.4.

Proof of Proposition 6.4. Let us suppose that (t,q) complies with (6.11). Integrating (6.17) in time gives the converse inequality and thus the desired balance (6.14).

Clearly, combining the *contact condition* (6.16) with the chain rule (3.9) leads to (6.14). The converse implication is also true thanks to inequality (6.17). This concludes the proof. ☐

Adapting the arguments for [MRS16b, Theorem 5.3] to the present context, we may now obtain a characterization of BV solutions in terms of a system of subdifferential inclusions that has the same structure as the viscous system (1.2), but where the viscous terms in the single equations can be "activated" only at jumps.

Proposition 6.6. *If* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathrm{AC}([0,S];[0,T]\times\mathbf{Q})$ *is a* BV *solution to the rate-independent system with hardening* (1.1a), (1.1b), (1.1d), (1.1f), (1.3)*, then there exists a measurable function* $\lambda : [0,S] \to [0,1]$ *such that*

$$\mathsf{t}'(s)\lambda(s) = 0 \quad \textit{for a.a. } s \in (0,S) \tag{6.18}$$

and (t,q) *satisfies for a.a.* $s \in (0,S)$ *the system of subdifferential inclusions*

(6.19a)

$$\lambda(s)\mathrm{D}\mathcal{V}_{2,\nu}(\mathsf{u}'(s)) + (1-\lambda(s))\,\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) = 0 \quad \textit{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*\,,$$

(6.19b)

$$(1{-}\lambda(s))\,\partial\mathcal{R}(\mathsf{z}'(s)) + \lambda(s)\,\mathrm{D}\mathcal{R}_2(\mathsf{z}'(s)) + (1{-}\lambda(s))\,\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) \ni 0 \qquad \textit{in } H^{\mathrm{m}}(\Omega)^*,$$

(6.19c)

$$\begin{aligned}(1{-}\lambda(s))\,\partial_\pi\mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s)) + \lambda(s)\,\mathrm{D}\mathcal{H}_{2,\nu}(\mathsf{p}'(s))&\\ + (1{-}\lambda(s))\,\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)) \ni 0 \quad &\textit{in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&-\lambda(s)\,\mathrm{Div}\big(\nu\mathbb{D}\mathrm{E}(\mathsf{u}'(s))\big) - (1{-}\lambda(s))\,\big(\mathrm{Div}(\sigma(s)) + F(\mathsf{t}(s))\big) = 0 \qquad \textit{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*,\\ &(1{-}\lambda(s))\,\partial\mathcal{R}(\mathsf{z}'(s)) + \lambda(s)\,\mathsf{z}'(s)\\ &+ (1{-}\lambda(s))\left(A_{\mathrm{m}}(\mathsf{z}(s)) + W'(\mathsf{z}(s)) + \frac{1}{2}\mathbb{C}'(\mathsf{z}(s))\mathsf{e}(s):\mathsf{e}(s)\right) \ni 0 \qquad \textit{in } H^{\mathrm{m}}(\Omega)^*,\\ &(1{-}\lambda(s))\,\partial_\pi H(\mathsf{z}(s),\mathsf{p}'(s)) + \lambda(s)\,\nu\mathsf{p}'(s) + (1{-}\lambda(s))\,\big(\nu\mathsf{p}(s) - \big(\sigma(s)\big)_{\mathrm{D}}\big) \ni 0 \quad \textit{a.e. in } \Omega\,.\end{aligned}$$

Conversely, if (t,q) *satisfies* (6.19)*, with* λ *as in* (6.18)*, and the map* $s \mapsto \mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))$ *is absolutely continuous on* $[0,S]$*, then* (t,q) *is a* BV *solution.*

Remark 6.7. A few comments on the mechanical interpretation of system (6.19) are in order. Due to the switching condition (6.18), the coefficient λ can be nonnull only if $\mathsf{t}'(s) = 0$, namely the system is jumping in the (slow) external time scale. When the system does not jump, the evolution of the variables z and p is rate-independent, and u "follows them" staying at elastic equilibrium. At a jump, the system may switch to a viscous regime where viscous dissipation intervenes in the evolution of the three variables u, z, p modulated by the *same* coefficient λ. This reflects the fact that, in the vanishing-viscosity approximation u, z, p relax to their limiting evolution with the same rate. We notice that if $\lambda = 1$ in an interval of $(0,S)$, then the system does not evolve in that interval. This cannot be the case for nondegenerate BV solutions (cf. (6.15)); such a nondegeneracy condition can be guaranteed by a further time reparameterization (see Remark 6.9).

Proof. The proof is a straightforward adaptation of the argument for [MRS16b, Theorem 5.1]. Thus, we will only recapitulate it here, referring to [MRS16b] for all details. The key point is to use that, by Proposition 6.4, a parameterized curve (t,q) is a BV solution if and only if it fulfills (6.16), namely for almost all $s \in (0,S)$

$$(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)) \in \Sigma := \{(t,q,t',q') \in [0,T]\times \mathbf{Q} \times [0,+\infty)\times \mathbf{Q} \,:\, \mathcal{M}^{\mu}_{0,\nu}(t,q,t',q') = \langle -\mathrm{D}_q \mathcal{E}_\mu(t,q), q'\rangle_{\mathbf{Q}}\}. \tag{6.20}$$

Then, [MRS16b, Proposition 5.1] provides a characterization of the contact set Σ. Such a characterization holds in our infinite-dimensional context as well, and it allows us to describe Σ as the union of two sets that encompass elastic equilibrium for u and rate-independent evolution for (z,p) on the one hand and viscous evolution for all three variables on the other hand. Namely,

$$\Sigma = \mathsf{E_u R_{z,p}} \cup \mathsf{V_{u,z,p}} \tag{6.21}$$

with the sets

$$\begin{aligned}
\mathsf{E_u R_{z,p}} &:= \big\{(t,q,t',q') \,:\, t'>0,\ \mathrm{D}_u\mathcal{E}_\mu(t,q)=0, \\
&\qquad\qquad \partial\mathcal{R}(z') + \mathrm{D}_z\mathcal{E}_\mu(t,q) \ni 0,\ \partial\mathcal{H}(z,p') + \mathrm{D}_p\mathcal{E}_\mu(t,q)\ni 0\big\},\\
\mathsf{V_{u,z,p}} &:= \{(t,q,t',q') \,:\, t'=0 \text{ and} \\
&\qquad \exists\,\lambda\in[0,1] \,:\, \left\{\begin{array}{l} \lambda \mathrm{D}\mathcal{V}_{2,\nu}(u') + (1{-}\lambda)\mathrm{D}_u\mathcal{E}_\mu(t,q) = 0,\\ (1{-}\lambda)\partial\mathcal{R}(z') + \lambda\mathrm{D}\mathcal{R}_2(z') + (1{-}\lambda)\mathrm{D}_z\mathcal{E}_\mu(t,q) \ni 0,\\ (1{-}\lambda)\partial\mathcal{H}(z,p') + \lambda\mathrm{D}\mathcal{H}_{2,\nu}(p') + (1{-}\lambda)\mathrm{D}_p\mathcal{E}_\mu(t,q)\ni 0 \end{array}\right\}.
\end{aligned}$$

Combining (6.20) and (6.21) leads to (6.19). ◻

We conclude this section with our existence result for BV solutions, obtained as limits of a family $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)_\varepsilon = (\mathsf{t}_\varepsilon,\mathsf{u}_\varepsilon,\mathsf{z}_\varepsilon,\mathsf{p}_\varepsilon)_\varepsilon$ of (reparameterized) viscous solutions to Problem 3.1.

In order to properly state our convergence result, we recall that, for $s_\varepsilon : [0,T]\to[0,S_\varepsilon]$ as in (6.1), the sequence $(S_\varepsilon)_\varepsilon$ is bounded thanks to (4.15b). Moreover, $S_\varepsilon \geq T$ for every $\varepsilon>0$. Hence,

$$\text{there is a sequence } \varepsilon_k \downarrow 0 \text{ and } S>0 \text{ such that } S_{\varepsilon_k}\to S. \tag{6.22}$$

If $S_\varepsilon < S$, we extend $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)$ to $[0,S]$ by setting $(\mathsf{t}_\varepsilon(s),\mathsf{q}_\varepsilon(s)) = (\mathsf{t}_\varepsilon(S_\varepsilon){+}s{-}S_\varepsilon, \mathsf{q}_\varepsilon(S_\varepsilon))$ for $s\in(S_\varepsilon,S]$.

We are now in a position to show the existence of a BV solution to the rate-independent system with hardening (1.1a), (1.1b), (1.1d), (1.1f), (1.3). The proof is based on approximation by means of solutions to Problem 3.1. The general scheme follows the steps of [MRS16b, MR21]. Some technical points, arising when dealing with the coupled plastic-damage system, are treated as in [CL16, Theorem 5.4], which features the viscosity *only* in the damage variable and not in the plastic variable.

Theorem 6.8. *Under the assumptions of section* 2 *and* (4.14)*, let* $(\varepsilon_k)_k$ *be as in* (6.22)*. Then, there exist a (not relabeled) subsequence* $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})_k = (\mathsf{t}_{\varepsilon_k},\mathsf{u}_{\varepsilon_k},\mathsf{z}_{\varepsilon_k})$ *and a Lipschitz curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in W^{1,\infty}([0,S];[0,T]{\times}\mathbf{Q})$ *such that*

1. *for all* $s\in[0,S]$ *the following convergences hold as* $k\to+\infty$*:*

$$\begin{aligned}
&\mathsf{t}_{\varepsilon_k}(s)\to\mathsf{t}(s),\ \ \mathsf{u}_{\varepsilon_k}(s)\rightharpoonup\mathsf{u}(s) \ \text{in } H^1(\Omega;\mathbb{R}^n),\\
&\mathsf{z}_{\varepsilon_k}(s)\rightharpoonup\mathsf{z}(s) \ \text{in } H^{\mathrm{m}}(\Omega),\ \ \mathsf{p}_{\varepsilon_k}(s)\rightharpoonup\mathsf{p}(s) \ \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}});
\end{aligned} \tag{6.23}$$

2. (t,q) *is a* BV *solution to the rate-independent system with hardening according to Definition* 6.2*.*

Arguments that are, by now, standard (detailed in, e.g., [MRS12a, MRS16a, MRS16b]) would also allow us to prove, a posteriori, the convergence of the energy terms and of the energy-dissipation integrals in (6.7) to their analogues in (6.14); the same is true for the other forthcoming convergence results, i.e., Theorems 6.13 and 7.9. However, to avoid overburdening the exposition we have preferred to overlook this point.

Proof. The proof is divided into three steps. First, we find a limiting parameterized curve by compactness arguments, then we deduce the finiteness of the vanishing-viscosity contact potential when $\mathsf{t}'>0$, namely when there are no jumps in the fast time scale, and eventually we prove the energy-dissipation upper estimate (6.11).

Step 1*: Compactness.* Let $(q_\varepsilon)_\varepsilon=(u_\varepsilon,z_\varepsilon,p_\varepsilon)_\varepsilon$ be a family of solutions to Problem 3.1. Let $s_\varepsilon:[0,T]\to[0,S_\varepsilon]$ be as in (6.1) and $(\mathsf{t}_\varepsilon,\mathsf{q}_\varepsilon)=(\mathsf{t}_\varepsilon,\mathsf{u}_\varepsilon,\mathsf{z}_\varepsilon,\mathsf{p}_\varepsilon)$ be as in (6.2). By (4.15b), S_ε is uniformly bounded in ε; moreover, $S_\varepsilon\geq T$ for every ε. Therefore, there is a sequence $\varepsilon_k\to0^+$ and $S>0$ such that $S_{\varepsilon_k}\to S$. Henceforth, we will write $(\mathsf{t}_k,\mathsf{q}_k)$ in place of $(\mathsf{t}_{\varepsilon_k},\mathsf{q}_{\varepsilon_k})$, and we will not relabel subsequences.

By (6.8), the sequence $(\mathsf{t}_k,\mathsf{q}_k)_k$ is equibounded in $W^{1,\infty}(0,S;[0,T]\times\mathbf{Q})$. Therefore, arguing as in Step 1 of the proof of Theorem 5.1 above (and in particular resorting to the compactness results from [Sim87]), we obtain a limit curve (t,q) such that (up to a subsequence, not relabeled) the following convergences hold as $k\to+\infty$:

$$\mathsf{t}_k\overset{*}{\rightharpoonup}\mathsf{t}\quad\text{in }W^{1,\infty}(0,S;[0,T])\,,\qquad\qquad \mathsf{u}_k\overset{*}{\rightharpoonup}\mathsf{u}\quad\text{in }W^{1,\infty}(0,S;H^1(\Omega;\mathbb{R}^n))\,,\tag{6.24a}$$

$$\mathsf{z}_k\overset{*}{\rightharpoonup}\mathsf{z}\quad\text{in }W^{1,\infty}(0,S;H^{\mathrm{m}}(\Omega))\,,\qquad\qquad \mathsf{z}_k\to\mathsf{z}\quad\text{in }\mathrm{C}^0([0,S];\mathrm{C}^0(\overline{\Omega}))\,,\tag{6.24b}$$

$$\mathsf{p}_k\overset{*}{\rightharpoonup}\mathsf{p}\quad\text{in }W^{1,\infty}(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}));\tag{6.24c}$$

the pointwise convergences in (6.23) hold as well.

We now define

$$s_-(t):=\sup\{s\in[0,S]:\ \mathsf{t}(s)<t\}\quad\text{for }t\in(0,T]\,,\tag{6.25a}$$

$$s_+(t):=\inf\{s\in[0,S]:\ \mathsf{t}(s)>t\}\quad\text{for }t\in[0,T)\,,\tag{6.25b}$$

and $s_-(0):=0$, $s_+(T):=S$. Then we have

$$\begin{aligned} s_-(t)\leq\liminf_{k\to+\infty}s_k(t)&\leq\limsup_{k\to+\infty}s_k(t)\\ &\leq s_+(t)\quad\text{and}\quad\mathsf{t}(s_-(t))=t=\mathsf{t}(s_+(t))\quad\text{for every }t\in[0,T]\,,\end{aligned}$$

$$s_-(\mathsf{t}(s))\leq s\leq s_+(\mathsf{t}(s))\quad\text{for every }s\in[0,S]\,.$$

Moreover the set

$$\mathcal{S}:=\{t\in[0,T]:\ s_-(t)<s_+(t)\}\tag{6.25c}$$

is at most countable. Set

$$\mathcal{U}:=\{s\in[0,S]:\ \mathsf{t}\text{ is constant in a neighborhood of }s\}\,,\tag{6.25d}$$

then

$$\mathcal{U} = \bigcup_{t \in \mathcal{S}} (s_-(t), s_+(t)) \,.$$

For future convenience (see Step 3 below) we remark that the original functions (u_k, z_k, p_k) satisfy, for every $t \in [0,T] \setminus \mathcal{S}$,

$$u_k(t) \rightharpoonup \mathsf{u}(s_-(t)) = \mathsf{u}(s_+(t)) \text{ in } H^1(\Omega;\mathbb{R}^n)\,, \tag{6.26a}$$

$$z_k(t) \rightharpoonup \mathsf{z}(s_-(t)) = \mathsf{z}(s_+(t)) \text{ in } H^{\mathrm{m}}(\Omega)\,, \tag{6.26b}$$

$$p_k(t) \rightharpoonup \mathsf{p}(s_-(t)) = \mathsf{p}(s_+(t)) \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,. \tag{6.26c}$$

Step 2: Finiteness of $\mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))$ *when* $\mathsf{t}'(\tau)>0$. We prove that

$$\begin{cases} -\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)) = 0\,, \\ -\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \in \partial\mathcal{R}(0) \\ -\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \in \partial_\pi\mathcal{H}(z,0), \end{cases} \qquad \text{for a.a. } \tau \in A := \{\tau\in(0,S)\,:\, \mathsf{t}'(\tau)>0\}\,, \tag{6.27}$$

i.e., the configuration is stable where t grows. This is equivalent to showing that $\mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))$ is finite for a.a. $\tau \in A$.

Preliminarily, we observe that

$$\limsup_{k\to+\infty} \mathsf{t}_k'(\tau) > 0 \qquad \text{for a.a. } \tau \in A. \tag{6.28}$$

This can be shown with the very same arguments as for the proof of [CL16, (5.18)].

By (3.26) and convergences (6.23) we have that

$$0 \le \mathcal{D}^{*,\mu}_\nu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \le \liminf_{k\to+\infty} \mathcal{D}^{*,\mu}_\nu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) \qquad \forall \tau \in [0,S]. \tag{6.29}$$

Moreover, by (3.24), written for $t = \mathsf{t}_k(\tau)$ we obtain

$$\begin{aligned} \mathcal{D}^{*,\mu}_\nu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) &= \mathcal{D}^{*,\mu}_\nu(\mathsf{t}_k(\tau),q_k(\mathsf{t}_k(\tau))) \\ &= \varepsilon_k \mathcal{D}_\nu(q_k'(\mathsf{t}_k(\tau))) \\ &= \varepsilon_k \sqrt{\nu\|u_k'(\mathsf{t}_k(\tau))\|^2_{H^1,\mathbb{D}} + \|z_k'(\mathsf{t}_k(\tau))\|^2_{L^2} + \nu\|p_k'(\mathsf{t}_k(\tau))\|^2_{L^2}} \\ &= \frac{\varepsilon_k}{\mathsf{t}_k'(\tau)} \sqrt{\nu\|\mathsf{u}_k'(\tau)\|^2_{H^1,\mathbb{D}} + \|\mathsf{z}_k'(\tau)\|^2_{L^2} + \nu\|\mathsf{p}_k'(\tau)\|^2_{L^2}} \\ &\le \frac{\varepsilon_k}{\mathsf{t}_k'(\tau)} \qquad \text{for a.a. } \tau \in (0,S), \end{aligned} \tag{6.30}$$

where the last estimate follows from the normalization condition (6.8) and since $\nu \le 1$. Combining (6.28), (6.29), and (6.30), we ultimately find

$$0 \le \mathcal{D}^{*,\mu}_\nu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \le \liminf_{k\to+\infty} \mathcal{D}^{*,\mu}_\nu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) = 0 \qquad \text{for a.a. } \tau \in A,$$

which implies (6.27).

In particular, we obtain that $\mathcal{M}^{\mu}_{0,\nu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))$ is finite, and equals $\mathcal{R}(\mathsf{z}'(\tau)) + \mathcal{H}(\mathsf{z}(\tau),\mathsf{p}'(\tau))$, for a.a. $\tau \in A$. Let us remark also that

$$[0,S] \ni \tau \mapsto \mathcal{D}^{*,\mu}_\nu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \text{ is lower semicontinuous} \tag{6.31}$$

by (3.26) and the fact that $(\mathsf{t},\mathsf{q}) \in W^{1,\infty}(0,S;[0,T]\times\mathbf{Q})$. In particular, the set

$$A^\circ := \{\tau \in [0,S] \colon \mathcal{D}^{*,\mu}_\nu(\mathsf{t}(\tau),\mathsf{q}(\tau)) > 0\} \text{ is open and included in } [0,S]\setminus A\,. \tag{6.32}$$

Step 3: The energy-dissipation upper estimate. By (6.9) we have

$$\begin{aligned} &\mathcal{E}_\mu(\mathsf{t}_k(S),\mathsf{q}_k(S)) + \int_0^S \mathcal{N}^\mu_{\varepsilon,\nu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau))\,\mathrm{d}\tau \\ &\quad = \mathcal{E}_\mu(\mathsf{t}_k(0),\mathsf{q}_k(0)) + \int_0^S \partial_t\mathcal{E}_\mu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,\mathsf{t}_k'(\tau)\,\mathrm{d}\tau\,. \end{aligned} \tag{6.33}$$

In order to obtain (6.11), we will pass to the lim inf in (6.33), using the lower semicontinuity of $\mathcal{E}_\mu$ and the previously proved (6.29).

We first prove the lower semicontinuity estimate

$$\int_{A^\circ} \mathcal{M}^{\mu,\mathrm{red}}_{0,\nu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau \leq \liminf_{k\to+\infty}\int_{A^\circ} \mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon_k,\nu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,, \tag{6.34}$$

where the set A° has been introduced in (6.32). By (3.19)

$$\mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon_k,\nu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau)) = \mathcal{D}_\nu(\mathsf{q}_k'(s))\,\mathcal{D}^{*,\mu}_\nu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,. \tag{6.35}$$

Then, estimate (6.34) follows from Lemma B.1 in Appendix B. Indeed, we apply it combining (6.29) with convergences (6.24), which imply that

$$(\sqrt{\nu}\mathsf{u}_\varepsilon,\mathsf{z}_\varepsilon,\sqrt{\nu}\mathsf{p}_\varepsilon) \rightharpoonup (\sqrt{\nu}\mathsf{u},\mathsf{z},\sqrt{\nu}\mathsf{p}) \quad \text{in } W^{1,\infty}(0,S;\mathrm{Q})$$

(recall that $\nu > 0$ is fixed). Hence, on the one hand we have that

$$m_k(\tau) := \mathcal{D}_\nu(\mathsf{q}_k'(\tau)) = \sqrt{\nu\|\mathsf{u}_k'(\tau)\|^2_{H^1,\mathbb{D}}+\|\mathsf{z}_k'(\tau)\|^2_{L^2}+\nu\|\mathsf{p}_k'(\tau)\|^2_{L^2}} \overset{*}{\rightharpoonup} m(\tau) \text{ in } L^\infty(0,S)$$

and

$$m(\tau) \geq \sqrt{\nu\|\mathsf{u}'(\tau)\|^2_{H^1,\mathrm{D}} + \|\mathsf{z}'(\tau)\|^2_{L^2} + \nu\|\mathsf{p}'(\tau)\|^2_{L^2}} = \mathcal{D}_\nu(\mathsf{q}'(\tau)) \qquad \text{for a.a. } \tau \in (0,S).$$

On the other hand, the sequence $h_k(\tau) := \mathcal{D}^{*,\mu}_\nu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))$ satisfies the first condition in (B.1). Therefore, by Lemma B.1 we have the desired estimate (6.34).

Moreover, by (6.24) and Ioffe theorem (cf. [Val90, Theorem 21]) it is not difficult to see that

$$\int_0^S \mathcal{R}(\mathsf{z}'(\tau)) + \mathcal{H}(\mathsf{z}(\tau),\mathsf{p}'(\tau))\,\mathrm{d}\tau \leq \liminf_{k\to+\infty}\int_0^S \mathcal{R}(\mathsf{z}_\varepsilon'(\tau)) + \mathcal{H}(\mathsf{z}_\varepsilon(\tau),\mathsf{p}_k'(\tau))\,\mathrm{d}\tau\,. \tag{6.36}$$

As for the right-hand side, notice that

$$\int_0^S \partial_t\mathcal{E}_\mu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,\mathsf{t}_k'(\tau)\,\mathrm{d}\tau = \int_0^T \partial_t\mathcal{E}_\mu(\tau,q_k(\tau))\,\mathrm{d}\tau\,. \tag{6.37}$$

By (6.26),

$$\begin{aligned} &\partial_t\mathcal{E}_\mu(\tau,q_k(\tau)) \\ &= \int_\Omega \mathbb{C}(z)\,e_k(\tau) : \mathrm{E}(w'(\tau))\,\mathrm{d}x - \langle F'(\tau),u_k(\tau)+w(\tau)\rangle_{H^1(\Omega;\mathbb{R}^n)} - \langle F(\tau),w'(\tau)\rangle_{H^1(\Omega;\mathbb{R}^n)} \\ &\longrightarrow \partial_t\mathcal{E}_\mu(\tau,\mathsf{q}(s_-(\tau))) \quad \text{for every } \tau \in [0,T]\setminus\mathcal{S}\,. \end{aligned}$$

Hence by dominated convergence

(6.38)
$$\int_0^T \partial_t \mathcal{E}_\mu(\tau, q_k(\tau))\,\mathrm{d}\tau \to \int_0^T \partial_t \mathcal{E}_\mu(\tau, \mathsf{q}(s_-(\tau)))\,\mathrm{d}\tau = \int_0^S \partial_t \mathcal{E}_\mu(\mathsf{t}(\tau), \mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau\,,$$

where we have used the fact that $\mathsf{t}'(s) = 0$ for a.e. $s \in \mathcal{U}$ and $s_-(\mathsf{t}(s)) = s$ for a.e. $s \in [0,S] \setminus \mathcal{U}$ (see Step 1 above).

We now collect (6.27) and (6.32)–(6.38) to conclude the energy-dissipation upper estimate (6.11). By the characterization provided by Proposition 6.4, we deduce that the curve (t, q) is a BV solution. This concludes the proof. □

Remark 6.9. It is an open problem to prove that the reparameterized viscous solutions converge to a *nondegenerate* (in the sense of (6.15)) BV solution. Nonetheless, following [MRS09, Remark 2] any degenerate BV solution (t, q) can be reparameterized to a nondegenerate one $(\tilde{\mathsf{t}}, \tilde{\mathsf{q}}) = (\tilde{\mathsf{t}}, \tilde{\mathsf{u}}, \tilde{\mathsf{z}}, \tilde{\mathsf{p}})$ by setting

(6.39)
$$\begin{aligned}
&m : [0,S] \to [0,+\infty), && m(s) := \int_0^s (\mathsf{t}'(\tau) + \|\mathsf{q}'(\tau)\|_{\mathbf{Q}})\,\mathrm{d}\tau, \quad \tilde{S} := m(S),\\
&r : [0,\tilde{S}] \to [0,S], && r(\mu) := \inf\{s \geq 0 \,:\, m(s) = \mu\},\\
&\tilde{\mathsf{t}} : [0,\tilde{S}] \to [0,T], && \tilde{\mathsf{t}}(\mu) := \mathsf{t}(r(\mu)), \qquad \tilde{\mathsf{q}} : [0,\tilde{S}] \to \mathbf{Q} \quad \tilde{\mathsf{q}}(\mu) := \mathsf{q}(r(\mu))\,.
\end{aligned}$$

With the very same calculations as in [MRS09, Remark 2] (cf. also [KRZ13, Remark 5.2]), one sees that

$$(\tilde{\mathsf{t}}, \tilde{\mathsf{q}}) \in W^{1,\infty}(0,\tilde{S}; [0,T] \times \mathbf{Q}) \quad \text{with} \ \ \tilde{\mathsf{t}}'(\mu) + \|\tilde{\mathsf{q}}'(\mu)\|_{\mathbf{Q}} \equiv 1 \text{ a.e. in } (0,\tilde{S})$$

and that $(\tilde{\mathsf{t}}, \tilde{\mathsf{q}})$ is still a BV solution in the sense of Definition 6.2.

6.2. The vanishing-viscosity analysis as $\boldsymbol{\varepsilon}$, $\boldsymbol{\nu} \downarrow 0$. We now address the asymptotic analysis of Problem 3.1 as *both* the viscosity parameter ε *and* the rate parameter ν tend to zero. Accordingly, throughout this section we will revert to the notation $(\mathsf{t}_{\varepsilon,\nu}, \mathsf{q}_{\varepsilon,\nu})_{\varepsilon,\nu}$ for a family of reparameterized viscous solutions.

Again, it is to be expected that any limit curve (t, q) of the family $(\mathsf{t}_{\varepsilon,\nu}, \mathsf{q}_{\varepsilon,\nu})_{\varepsilon,\nu}$ as ε, $\nu \downarrow 0$ will satisfy the analogue of the energy-dissipation inequality (6.11), however featuring, in the present context, a *different vanishing-viscosity contact potential* that reflects the multirate character of the problem, and in particular the fact that u and p relax to equilibrium and rate-independent evolution, respectively, at a faster rate than z relaxing to rate-independent evolution. For consistency of notation, we will denote this new contact potential $\mathcal{M}_{0,0}^\mu$. It will turn out (in analogy with the results from [MRS16b, MR21]) that $\mathcal{M}_{0,0}^\mu : [0,T] \times \mathbf{Q} \times [0,+\infty) \times \mathbf{Q} \to [0,+\infty]$ is given by

$$\begin{aligned}
\mathcal{M}_{0,0}^\mu(t,q,t',q') = \mathcal{M}_{0,0}^\mu(t,u,z,p,t',u',z',p') &:= \mathcal{R}(z') + \mathcal{H}(z,p')\\
&\quad + \mathcal{M}_{0,0}^{\mu,\mathrm{red}}(t,u,z,p,t',u',z',p')
\end{aligned}$$

with $\mathcal{M}_{0,0}^{\mu,\mathrm{red}}$ given by (6.12a) if $t' > 0$. Instead, in place of (6.12b) we have, if $t' = 0$,

(6.40a)
$$\mathcal{M}_{0,\nu}^{\mu,\mathrm{red}}(t,q,0,q') := \begin{cases} \mathcal{D}(u',p')\,\mathcal{D}^{*,\mu}(t,q) & \text{if } z' = 0,\\ \|z'\|_{L^2}\,\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(t,q), \partial\mathcal{R}(0)) & \text{if } \mathcal{D}^{*,\mu}(t,q) = 0,\\ +\infty & \text{if } \|z'\|_{L^2}\mathcal{D}^{*,\mu}(t,q) > 0, \end{cases}$$

where we have used the notation

(6.40b)
$$\begin{aligned}
\mathcal{D}(u',p') &:= \sqrt{\|u'\|_{H^1,\mathbb{D}}^2 + \|p'\|_{L^2(\Omega;\mathbb{M}_{\mathrm{D}}^{n\times n})}^2},\\
\mathcal{D}^{*,\mu}(t,q) &:= \sqrt{\|-\mathrm{D}_u\mathcal{E}_\mu(t,q)\|_{(H^1,\mathbb{D})^*}^2 + d_{L^2}(-\mathrm{D}_p\mathcal{E}_\mu(t,q), \partial_\pi\mathcal{H}(z,0))^2}\,.
\end{aligned}$$

Again, in the case in which $z'=0$ and $\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0))=+\infty$, in (6.40a) we set

$$\|z'\|_{L^2}\,\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(t,q),\partial\mathcal{R}(0)):=+\infty\,.$$

The multirate character of the vanishing-viscosity approximation addressed in this case is already apparent in the expression for $\mathcal{M}_{0,0}^{\mu,\mathrm{red}}(t,q,t',q')$ at $t'=0$. Indeed, $\mathcal{M}_{0,0}^{\mu,\mathrm{red}}(t,q,0,q')$ is finite only either if $z'=0$ (i.e., z is frozen) or if $\mathcal{D}^{*,\mu}(t,q)=0$, which entails that u is at equilibrium and p fulfills the local stability condition $-\mathrm{D}_p\mathcal{E}_\mu(t,q)\in\partial_\pi\mathcal{H}(z,0)$; cf. Remark 6.12 later on for further comments.

Accordingly, we give the following.

Definition 6.10. *We call a parameterized curve* $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in\mathrm{AC}([0,S];[0,T]\times\mathbf{Q})$ *a* (parameterized) Balanced Viscosity *solution to the* multirate *system with hardening* (1.1a), (1.1b), (1.1d), (1.1f), (1.3) *if* $\mathsf{t}\colon[0,S]\to[0,T]$ *is nondecreasing and* (t,q) *fulfills for all* $0\le s\le S$ *the energy-dissipation balance*

$$\begin{aligned}&\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))+\int_0^s\mathcal{M}_{0,0}^{\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau\\&=\mathcal{E}_\mu(\mathsf{t}(0),\mathsf{q}(0))+\int_0^s\partial_t\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau\,.\end{aligned}\tag{6.41}$$

We say that (t,q) *is* nondegenerate *if it fulfills* (6.15).

The very analogue of Proposition 6.4 holds for BV solutions to the multirate system as well, based on the chain-rule estimate

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))+\partial_t\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))\\&=\langle-\mathrm{D}_q\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{q}'(s)\rangle_{\mathbf{Q}}\le\mathcal{M}_{0,0}^{\mu}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\quad\text{for a.a. } s\in(0,S),\end{aligned}$$

which can be shown along any parameterized curve $(\mathsf{t},\mathsf{q})\in\mathrm{AC}([0,S];[0,T]\times\mathbf{Q})$ such that $\mathcal{M}_{0,0}^{\mu}(\mathsf{t}(s,\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))<\infty$ for almost all $s\in(0,S)$ by adapting the arguments for Lemma 6.5; see also Proposition 3.4.

Likewise, we have a differential characterization for BV solutions in the sense of Definition 6.10 analogous to the characterization from Proposition 6.6. In system (6.43) below, we have used the shorthand $\mathcal{H}_2$ for the plastic dissipation potential $\mathcal{H}_{2,1}$ (cf. (2.34)).

Proposition 6.11. *A parameterized curve* $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in\mathrm{AC}([0,S];[0,T]\times\mathbf{Q})$ *is a* BV *solution to the* multirate *system with hardening if and only if there exist two functions* $\lambda_{\mathsf{u},\mathsf{p}}\,\lambda_{\mathsf{z}}:[0;S]\to[0;1]$ *such that*

$$\mathsf{t}'(s)\lambda_{\mathsf{u},\mathsf{p}}(s)=\mathsf{t}'(s)\lambda_{\mathsf{z}}(s)=0\qquad\text{for a.a. } s\in(0,S),\tag{6.42a}$$

$$\lambda_{\mathsf{u},\mathsf{p}}(s)(1-\lambda_{\mathsf{z}}(s))=0\qquad\text{for a.a. } s\in(0,S),\tag{6.42b}$$

and (t,q) *satisfies for a.a.* $s\in(0,S)$ *the system of subdifferential inclusions*

$$\lambda_{\mathsf{u},\mathsf{p}}(s)\mathrm{D}\mathcal{V}_2(\mathsf{u}'(s))+(1-\lambda_{\mathsf{u},\mathsf{p}}(s))\,\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))=0\quad\text{in } H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*\,,\tag{6.43a}$$

$$(1-\lambda_{\mathsf{z}}(s))\,\partial\mathcal{R}(\mathsf{z}'(s))+\lambda_{\mathsf{z}}(s)\,\mathrm{D}\mathcal{R}_2(\mathsf{z}'(s))+(1-\lambda_{\mathsf{z}}(s))\,\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))\ni0\ \text{in } H^{\mathrm{m}}(\Omega)^*\,,\tag{6.43b}$$

$$\begin{aligned}&(1-\lambda_{\mathsf{u},\mathsf{p}}(s))\,\partial_\pi\mathcal{H}(\mathsf{z}(s),\mathsf{p}'(s))+\lambda_{\mathsf{u},\mathsf{p}}(s)\,\mathrm{D}\mathcal{H}_2(\mathsf{p}'(s))\\&\qquad+(1-\lambda_{\mathsf{u},\mathsf{p}}(s))\,\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t}(s),\mathsf{q}(s))\ni0\quad\text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,.\end{aligned}\tag{6.43c}$$

Like the argument for Proposition 6.6, the *proof* is again based on the analysis of the structure of the contact set associated with $\mathcal{M}^{\mu}_{0,0}$ (cf. (6.20)), which in turn can be characterized by adapting the arguments from the proof of [MRS16b, Proposition 5.1].

Remark 6.12. Along the lines of [MRS16b, Remark 5.4], we observe that system (6.43) reflects the fact that u and p relax to equilibrium and rate-independent evolution, respectively, faster than z. Indeed, at a jump (corresponding to $\mathsf{t}'=0$, hence the coefficients $\lambda_{\mathsf{u},\mathsf{p}}$ and λ_{z} can be nonzero), due to (6.42b) either $\lambda_{\mathsf{z}}=1$, and then $\mathsf{z}'=0$, or $\lambda_{\mathsf{u},\mathsf{p}}=0$, which gives that u is at equilibrium and p satisfies the local stability condition $-\mathrm{D}_p\mathcal{E}_\mu(t,q)\in\partial_\pi\mathcal{H}(z,0)$. Namely, at a jump z cannot change until u has reached the equilibrium and p attained the stable set $\partial_\pi\mathcal{H}(z,0)$. After that, z may evolve either rate-independently (if $\lambda_{\mathsf{z}}=0$) or governed by viscosity (if $\lambda_{\mathsf{z}}\in(0,1)$).

With our final result we prove the convergence of the reparameterized viscous solutions $(\mathsf{t}_{\varepsilon,\nu},\mathsf{q}_{\varepsilon,\nu})_{\varepsilon,\nu}$ to a BV solution of the multirate system as *both* ε and ν tend to zero. As observed right before the statement of Theorem 6.8, we may suppose that the curves $(\mathsf{t}_{\varepsilon,\nu},\mathsf{q}_{\varepsilon,\nu})$ are defined in a fixed interval $(0,S)$.

THEOREM 6.13. *Under the assumptions of section* 2 *and* (4.14)*, for all vanishing sequences* $(\varepsilon_k)_k$ *and* $(\nu_k)_k$ *such that* $S_{\varepsilon_k,\nu_k}\to S$ *there exist a (not relabeled) subsequence* $(\mathsf{t}_{\varepsilon_k,\nu_k},\mathsf{q}_{\varepsilon_k,\nu_k})_k$ *and a Lipschitz curve* $(\mathsf{t},\mathsf{q})\in W^{1,\infty}([0,S];[0,T]\times\mathbf{Q})$ *such that convergences* (6.23) *hold as* $k\to+\infty$ *and* (t,q) *is a* BV *solution to the* multirate *system with hardening according to Definition* 6.10.

Proof. In the proof of this result, we will mainly highlight the differences with respect to the argument for Theorem 6.8, without repeating the analogous passages. Hereafter, we will consider sequences ε_k, $\nu_k\to 0$ and write $(\mathsf{t}_k,\mathsf{q}_k)_k$ in place of $(\mathsf{t}_{\varepsilon_k,\nu_k},\mathsf{q}_{\varepsilon_k,\nu_k})_{\varepsilon_k,\nu_k}$, and we will not relabel subsequences.

Step 1: *Compactness.* As in the proof of Theorem 6.8, we conclude that there exist a subsequence and $(\mathsf{t},\mathsf{q})\in W^{1,\infty}([0,S];[0,T]\times\mathbf{Q})$ such that the analogues of (6.24)–(6.26) hold.

Step 2: *Finiteness of* $\mathcal{M}^{\mu,\mathrm{red}}_{0,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))$ *when* $\mathsf{t}'(\tau)>0$. As in Step 2 of Theorem 6.8, we introduce the set $A:=\{\tau\in[0,S]\colon \mathsf{t}'(\tau)>0\}$ and show that

$$\mathcal{M}^{\mu,\mathrm{red}}_{0,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))<+\infty \qquad \text{for a.a.}\ t\in A,$$

which yields the (local) stability condition (6.27) for a.a. $t\in A$. To do so, as in (6.28) we observe that $\limsup_{k\to+\infty}\mathsf{t}_k'(\tau)>0$. We now use equality (3.24) at $r=\mathsf{t}_k(\tau)$ and $q(r)=\mathsf{q}_k(\tau)$, and thus we get

$$\begin{aligned}\mathcal{D}^{*;\mu}_{\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) &= \mathcal{D}^{*;\mu}_{\nu_k}(\mathsf{t}_k(\tau),q_k(\mathsf{t}_k(\tau)))\\ &=\varepsilon_k\sqrt{\nu_k\|u_k'(\mathsf{t}_k(\tau))\|^2_{H^1,\mathbb{D}}+\|z_k'(\mathsf{t}_k(\tau))\|^2_{L^2}+\nu_k\|p_k'(\mathsf{t}_k(\tau))\|^2_{L^2}}\\ &=\frac{\varepsilon_k}{\mathsf{t}_k'(\tau)}\sqrt{\nu_k\|\mathsf{u}_k'(\tau)\|^2_{H^1,\mathbb{D}}+\|\mathsf{z}_k'(\tau)\|^2_{L^2}+\nu_k\|\mathsf{p}_k'(\tau)\|^2_{L^2}}\\ &\le\frac{\varepsilon_k}{\mathsf{t}_k'(\tau)}\qquad\text{for a.a.}\ \tau\in(0,S),\end{aligned}\tag{6.44}$$

where, again, the last estimate follows from the normalization conditions (6.8) and since $\nu\le 1$. We observe that the right-hand side of (6.44) goes to 0 as $k\to+\infty$.

We now deduce the local stability (6.27) at almost all $s\in(0,S)$. Indeed, if, say, there holds $\|\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau))\|_{(H^1,\mathbb{D})^*}>0$, then we get by the semicontinuity inequality (3.26a) that $\liminf_{k\to+\infty} \|\mathrm{D}_u \mathcal{E}_\mu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\|_{(H^1,\mathbb{D})^*} > 0$. Recalling the definition of $\mathcal{D}^{*,\mu}_{\nu_k}$ from (3.19), and since $\nu_k \to 0$, this would give that $\liminf_{k\to+\infty} \mathcal{D}^{*,\mu}_{\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) = +\infty$, which contradicts (6.44). Thus $\mathrm{D}_u\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)) = 0$. In the same way we get $-\mathrm{D}_p\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \in \partial_\pi\mathcal{H}(z,0)$, while, if $-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)) \notin \partial\mathcal{R}(0)$, we would get $\liminf_{k\to+\infty} \mathcal{D}^{*,\mu}_{\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) > 0$, which still would contradict (6.44).

Moreover, in view of (3.26) and of the regularity of (t,q), we have that the sets

$$B^\circ_\mu := \{\tau\in[0,S] \colon \mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau)) > 0\} \text{ and}$$
$$C^\circ_\mu := \{\tau\in[0,S] \colon \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)),\partial\mathcal{R}(0)) > 0\} \tag{6.45}$$

are open and included in $[0,S]\setminus A$.

Step 3: *The energy-dissipation upper estimate (*6.41*).* By the analogue of Proposition 6.4, in order to conclude that (t,q) is a BV solution to the multirate system with hardening it is sufficient to obtain (6.41) as an upper estimate $\le$. With this aim, as in Step 3 of Theorem 6.8, we start from the analogues of (6.33) and (6.35). First of all, it holds that for a.e. $\tau\in(0,S)$

$$\mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau)) \ge \frac{1}{\sqrt{\nu_k}}\|\mathsf{z}_k'(\tau)\|_{L^2}\,\mathcal{D}^{*,\mu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,, \tag{6.46}$$

recalling (3.19) and (6.35). Now, we may apply Lemma B.1 with the choices $I := B^\circ$, $m_k = \|\mathsf{z}_k'\|_{L^2}$ such that $m_k \overset{*}{\rightharpoonup} m$ in $L^\infty(0,S)$ and $m \ge \|\mathsf{z}'\|_{L^2}$ a.e. in $(0,S)$, and with $h_k := \mathcal{D}^{*,\mu}(\mathsf{t}_k,\mathsf{q}_k)$, $h := \mathcal{D}^{*,\mu}(\mathsf{t},\mathsf{q})$. Indeed, observe that

$$\liminf_{k\to+\infty} \mathcal{D}^{*,\mu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) \ge \mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau)) \qquad \forall\tau\in[0,S], \tag{6.47}$$

thanks to (6.24b) and the lower semicontinuity properties (3.26). We thus obtain that

$$\int_{B^\circ_\mu} \|\mathsf{z}'(\tau)\|_{L^2}\,\mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathrm{d}\tau \le \liminf_{k\to\infty}\int_{B^\circ_\mu} \|\mathsf{z}_k'(\tau)\|_{L^2}\,\mathcal{D}^{*,\mu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,\mathrm{d}\tau\,. \tag{6.48}$$

Since $\nu_k\to0$, from (6.46) and (6.48) we deduce that $\mathsf{z}'(\tau)=0$ for a.e. $\tau\in B^\circ$, that is,

$$\mathsf{z}'(\tau)\,\mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau)) = 0 \quad \text{ for a.a. } \tau\in(0,S)\,. \tag{6.49}$$

In view of the definition (6.40a) of $\mathcal{M}^{\mu,\mathrm{red}}_{0,0}$, (6.49) yields that

$$\mathcal{M}^{\mu,\mathrm{red}}_{0,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau)) = \mathcal{D}(\mathsf{u}'(\tau),\mathsf{p}'(\tau))\,\mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau)) \qquad \text{a.e. in } B^\circ\,. \tag{6.50}$$

By (3.19), (6.35), and an easy algebraic calculation we obtain that

$$\mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau)) \ge \mathcal{D}(\mathsf{u}_k'(\tau),\mathsf{p}_k'(\tau))\,\mathcal{D}^{*,\mu}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,. \tag{6.51}$$

Then, again by Lemma B.1, applied thanks to (6.24) and (3.26), we deduce

$$\begin{aligned}\int_{B^\circ} \mathcal{M}^{\mu,\mathrm{red}}_{0,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau &= \int_{B^\circ} \mathcal{D}(\mathsf{u}'(\tau),\mathsf{p}'(\tau))\,\mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathrm{d}\tau \\ &\le \liminf_{k\to+\infty}\int_{B^\circ} \mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,.\end{aligned} \tag{6.52}$$

Let us now consider the set $C_\mu^\circ \setminus B_\mu^\circ$, where $\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)),\partial\mathcal{R}(0)) > 0$ with $\mathcal{D}^{*,\mu}(\mathsf{t}(\tau),\mathsf{q}(\tau)) = 0$; cf. (6.32). Starting from (3.19) and (6.35), we estimate

$$\mathcal{N}_{\varepsilon_k,\nu_k}^{\mu,\mathrm{red}}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau)) \geq \|\mathsf{z}_k'(\tau)\|_{L^2}\,\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)),\partial\mathcal{R}(0))\,. \tag{6.53}$$

We then employ Lemma B.1 with $I := C_\mu^\circ \setminus B_\mu^\circ$, $m_k := \|\mathsf{z}_k'\|_{L^2}$ such that $m_k \overset{*}{\rightharpoonup} m \geq \|\mathsf{z}'\|_{L^2}$ in $L^\infty(0,S)$, $h_k := \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)),\partial\mathcal{R}(0))$, $h := \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)),\partial\mathcal{R}(0))$. Again, we obtain that $\liminf_{k\to+\infty} h_k(\tau) \geq h(\tau)$ for all $\tau \in [0,S]$ by (6.24) and (3.26b). Thus, with Lemma B.1 we get

$$\begin{aligned}
&\int_{(0,S)\setminus B_\mu^\circ} \mathcal{M}_{0,0}^{\mu,\mathrm{red}}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau \\
&\quad = \int_{C_\mu^\circ\setminus B_\mu^\circ} \|\mathsf{z}'(\tau)\|_{L^2}\,\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}(\tau),\mathsf{q}(\tau)),\partial\mathcal{R}(0))\,\mathrm{d}\tau \\
&\quad \leq \liminf_{k\to+\infty} \int_{C_\mu^\circ\setminus B_\mu^\circ} \|\mathsf{z}_k'(\tau)\|_{L^2}\,\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_\mu(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)),\partial\mathcal{R}(0))\,\mathrm{d}\tau \\
&\quad \leq \liminf_{k\to+\infty} \int_{C_\mu^\circ\setminus B_\mu^\circ} \mathcal{N}_{\varepsilon_k,\nu_k}^{\mu,\mathrm{red}}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,.
\end{aligned} \tag{6.54}$$

All in all, collecting (6.52) and (6.54) we conclude

$$\int_0^S \mathcal{M}_{0,0}^{\mu,\mathrm{red}}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau \leq \liminf_{k\to+\infty}\int_0^S \mathcal{N}_{\varepsilon_k,\nu_k}^{\mu,\mathrm{red}}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,.$$

The remainder of the proof (namely the semicontinuity of the other terms in $\mathcal{M}_{0,0}^\mu$ and of the driving energy $\mathcal{E}_\mu$, and the continuity of the power term) follows as in Step 3 of Theorem 6.8. In this way, we conclude the limit passage in the energy-dissipation balance (6.33), obtaining the desired (6.41) as an upper estimate $\leq$. The proof is then completed. ∎

7. The vanishing-hardening limit. We now address the limit passage in the viscous system (1.2) as the three parameters ε, ν, μ vanish simultaneously. For this, we will combine the techniques from section 6, with the functional-analytic tools related to the passage from plasticity with hardening to perfect plasticity. In what follows,

- we will establish the setup for the perfectly plastic model, recalling results from [DMDM06, FG12];
- we will introduce a suitable "energy-dissipation" arclength reparameterization of viscous solutions; in combination with the energy-dissipation balance (6.7), the thus reparameterized solutions will satisfy a normalization condition, whence the key estimates will stem, as well as the specific temporal and spatial regularity properties of the limiting *admissible parameterized curves* (cf. Definition 7.2);
- we will properly define the vanishing-viscosity contact potential relevant for BV solutions (cf. Definition 7.3) to the perfectly plastic system, taking care of the technicalities related to the new functional setup;
- we will address the properties of BV solutions and in particular characterize them in terms of an energy-dissipation *upper estimate* in Proposition 7.7. Such characterization will play a key role in the proof of our existence result, Theorem 7.9 ahead.

Let us now first fix the setup for the perfectly plastic system. We mention in advance that the space for the displacements will be $\mathrm{BD}(\Omega)$ and the space for the plastic strains will be $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$, i.e., the space of bounded Radon measures on $\Omega\cup\Gamma_{\mathrm{Dir}}$ with values in $\mathbb{M}^{n\times n}_{\mathrm{D}}$; this reflects the fact that, now, the plastic strain p is a measure that can concentrate on Lebesgue-negligible sets.

Setup adapted for perfect plasticity: The state space. The state space for the perfectly plastic system with damage is

$$
\begin{aligned}
\mathbf{Q}_{\mathrm{PP}} := \{ & q=(u,z,p)\in \mathrm{BD}(\Omega)\times H^{\mathrm{m}}(\Omega)\times \mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})\ : \\
& e:=\mathrm{E}(u)-p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}),\ u\odot \mathrm{n}\,\mathscr{H}^{n-1}+p=0 \text{ on } \Gamma_{\mathrm{Dir}}\},
\end{aligned}
\tag{7.1}
$$

where n is the normal vector to $\partial\Omega$ and $\odot$ the symmetrized tensorial product. Observe that the condition $u\odot \mathrm{n}\,\mathscr{H}^{n-1}+p=0$ relaxes the homogeneous Dirichlet condition $u=0$ on Γ_{Dir}.

Setup adapted for perfect plasticity: The plastic dissipation potential. We extend the plastic dissipation potential $\mathcal{H}(z,\cdot)$ to the reference space $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$. We define $\mathcal{H}_{\mathrm{PP}}\colon \mathrm{C}^0(\overline{\Omega};[0,1])\times \mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})\to\mathbb{R}$ by

$$
\mathcal{H}_{\mathrm{PP}}(z,\pi) := \int_{\Omega\cup\Gamma_{\mathrm{Dir}}} H\left(z(x),\frac{\mathrm{d}\pi}{\mathrm{d}\mu}(x)\right)\mathrm{d}\mu(x)\,,
\tag{7.2}
$$

where H is defined in (2.27), $\mu\in \mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ is a positive measure such that $\pi\ll\mu$, and $\frac{\mathrm{d}\pi}{\mathrm{d}\mu}$ is the Radon–Nikodým derivative of π with respect to μ; since $H(z(x),\cdot)$ is one-homogeneous, the definition is actually independent of μ. We refer to [GS64] for the theory of convex functions of measures. By [AFP05, Proposition 2.37], the functional $p\mapsto\mathcal{H}_{\mathrm{PP}}(z,p)$ is convex and positively one-homogeneous for every $z\in\mathrm{C}^0(\overline{\Omega};[0,1])$. In particular, $\mathcal{H}_{\mathrm{PP}}(z,p_1+p_2)\le\mathcal{H}_{\mathrm{PP}}(z,p_1)+\mathcal{H}_{\mathrm{PP}}(z,p_2)$ for every $z\in\mathrm{C}^0(\overline{\Omega};[0,1])$ and $p_1,p_2\in\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$. Since $|\frac{\mathrm{d}p}{\mathrm{d}|p|}(x)|=1$ for $|p|$-a.e. $x\in\Omega\cup\Gamma_{\mathrm{Dir}}$, by (2.28) we have

$$
r\|p\|_1\le\mathcal{H}_{\mathrm{PP}}(z,p)\le R\|p\|_1\,,
$$

where we denote by $\|\cdot\|_1$ the total variation of a measure (in the case of p on $\Omega\cup\Gamma_{\mathrm{Dir}}$), and

$$
0\le\mathcal{H}_{\mathrm{PP}}(z_2,p)-\mathcal{H}_{\mathrm{PP}}(z_1,p)\le C'_K\|z_1-z_2\|_{L^\infty}\|p\|_1\quad\text{for } 0\le z_1\le z_2\le 1\,.
$$

Therefore, by Reshetnyak's lower semicontinuity theorem, if $(z_k)_k$ and $(p_k)_k$ are sequences in $\mathrm{C}^0(\overline{\Omega};[0,1])$ and $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$ such that $z_k\to z$ in $\mathrm{C}^0(\overline{\Omega})$ and $p_k\rightharpoonup p$ weakly* in $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$, then

$$
\mathcal{H}_{\mathrm{PP}}(z,p)\le\liminf_{k\to+\infty}\mathcal{H}_{\mathrm{PP}}(z_k,p_k)\,.
$$

Stress-strain duality. Let us recall the notion of stress-strain duality, relying on [KT83], [DMDM06], and the more recent extension to Lipschitz boundaries [FG12], to which we refer for the properties mentioned below. First of all, we recall the definition (in the sense of [DMDM06]) of *admissible displacement and strains* $A(w)$ associated with a function $w\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, i.e.,

$$
\begin{aligned}
A(w):=\{&(u,e,p)\in\mathrm{BD}(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\times\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})\colon\\
&\mathrm{E}(u)=e+p \text{ in } \Omega,\ p=(w-u)\odot\mathrm{n}\,\mathscr{H}^{n-1}\text{ on }\Gamma_{\mathrm{Dir}}\}\,.
\end{aligned}
$$

We also recall the *space of admissible plastic strains*

$$\Pi(\Omega) := \{p \in \mathrm{M_b}(\Omega \cup \Gamma_{\mathrm{Dir}}; \mathbb{M}^{n\times n}_{\mathrm{D}}) \colon \exists\, (u,w,e) \in \mathrm{BD}(\Omega) \times H^1(\mathbb{R}^n;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \text{ s.t. } (u,e,p) \in A(w)\}\,.$$

We then define

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \colon \mathrm{div}(\sigma) \in L^n(\Omega;\mathbb{R}^n)\,,\ \sigma_{\mathrm{D}} \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\}$$

and, for $\sigma \in \Sigma(\Omega)$ and $p \in \Pi(\Omega)$, we set

$$\langle[\sigma_{\mathrm{D}} : p],\varphi\rangle := -\int_\Omega \varphi\sigma\cdot(e-\mathrm{E}(w))\,\mathrm{d}x - \int_\Omega \sigma\cdot[(u-w)\odot\nabla\varphi]\,\mathrm{d}x - \int_\Omega \varphi\,(\mathrm{div}(\sigma))\cdot(u-w)\,\mathrm{d}x \tag{7.3}$$

for every $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, where u and e are such that $(u,e,p) \in A(w)$; the definition is indeed independent of u and e. If $\sigma \in \Sigma(\Omega)$ and $p \in \Pi(\Omega)$, then $\sigma \in L^r(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ for every $r < \infty$, and $[\sigma_{\mathrm{D}} : p]$ is a bounded Radon measure such that $\|[\sigma_{\mathrm{D}} : p]\|_1 \le \|\sigma_{\mathrm{D}}\|_{L^\infty}\|p\|_1$ in $\mathbb{R}^n$. Considering the restriction of this measure to $\Omega \cup \Gamma_{\mathrm{Dir}}$, we also define

$$\langle\sigma_{\mathrm{D}} \,|\, p\rangle := [\sigma_{\mathrm{D}} : p](\Omega \cup \Gamma_{\mathrm{Dir}})\,.$$

By (7.3) and taking into account that $u \in \mathrm{BD}(\Omega) \subset L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, if $[\sigma\mathrm{n}] \in L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)$ (recall (2.9)) and (2.Ω) holds, then we have the integration by parts formula

$$\langle\sigma_{\mathrm{D}} \,|\, p\rangle = -\langle\sigma, e-\mathrm{E}(w)\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \langle-\mathrm{div}\,\sigma, u-w\rangle_{L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} + \langle[\sigma\mathrm{n}], u-w\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)} \tag{7.4}$$

for every $\sigma \in \Sigma(\Omega)$ and $(u,e,p) \in A(w)$. Thus, defining for $\sigma \in \Sigma(\Omega)$ the functional $-\widehat{\mathrm{Div}}(\sigma) \in \mathrm{BD}(\Omega)^*$ via

$$\langle-\widehat{\mathrm{Div}}(\sigma), v\rangle_{\mathrm{BD}(\Omega)} := \langle-\mathrm{div}(\sigma), v\rangle_{L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} + \langle[\sigma\mathrm{n}], v\rangle_{L^1(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)} \tag{7.5}$$

for all $v \in \mathrm{BD}(\Omega)$, we have that (7.4) reads as

$$\langle\sigma_{\mathrm{D}} \,|\, p\rangle = -\langle\sigma, e-\mathrm{E}(w)\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \langle-\widehat{\mathrm{Div}}(\sigma), u-w\rangle_{\mathrm{BD}(\Omega)}. \tag{7.6}$$

For $z \in \mathrm{C}^0(\overline{\Omega})$ let

$$\mathcal{K}_z(\Omega) := \{\sigma \in \Sigma(\Omega) \colon \sigma_{\mathrm{D}}(x) \in K(z(x)) \text{ for a.e. } x \in \Omega\}\,. \tag{7.7}$$

Since the multifunction $z \in [0,1] \mapsto K(z)$ is continuous, from [FG12, Proposition 3.9] (which holds also if $\mathrm{div}(\sigma)$ is not identically 0) it follows that for every $\sigma \in \mathcal{K}_z(\Omega)$

$$H\left(z, \frac{\mathrm{d}p}{\mathrm{d}|p|}\right)|p| \ge [\sigma_{\mathrm{D}} : p] \quad \text{as measures on } \Omega \cup \Gamma_{\mathrm{Dir}}\,. \tag{7.8}$$

In particular, we have

$$\mathcal{H}_{\mathrm{PP}}(z,p) \ge \sup_{\sigma\in\mathcal{K}_z(\Omega)} \langle\sigma_{\mathrm{D}} \,|\, p\rangle \qquad \text{for every } p \in \Pi(\Omega). \tag{7.9}$$

Remark 7.1. In [FG12, Remark 2.9] the authors explain that in the presence of external forces one has to resort to the classic (deviatoric) stress-(plastic) strain duality, provided by [KT83] and employed in several papers, e.g., [DMDM06], to put in duality $\varrho_D(t)$ and $p \in \Pi(\Omega)$. Such a duality requires one of the two following conditions, alternatively: either (1) $\varrho \in \mathrm{AC}(0,T;\mathrm{C}^0(\overline{\Omega};\mathbb{M}^{n\times n}_{\mathrm{D}}))$ or (2) Ω globally of class C^2. The use of the Kohn–Temam duality seems to be needed to infer that (7.9) holds as an equality, which in turn implies the analogue of (2.40) for $\mathcal{H}_{\mathrm{PP}}$, $\mathcal{K}_z(\Omega)$, $p \in \mathrm{M_b}$ in place of $\mathcal{H}$, $\widetilde{\mathcal{K}}_z(\Omega)$, $p \in L^1$. However, by our approximation procedure via plasticity with hardening, we just need to use the coercivity condition (2.40) in the a priori estimates for the solutions of the system with plastic hardening (cf. (4.28)), together with (7.9) to pass to the limit. For this reason we do not assume any further regularity on either ϱ or on Ω.

The energy functional. The energy functional $\mathcal{E}_0$ driving the perfectly plastic system has an expression analogous to the functional $\mathcal{E}_\mu$ (2.23) for the system with hardening where μ is formally set equal to 0. Indeed, it consists of the contributions of the elastic energy, of the potential energy for the damage variable, and of the time-dependent volume and surface forces. Then $\mathcal{E}_0 : [0,T] \times \mathbf{Q}_{\mathrm{PP}} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{E}_0(t,u,z,p) := \mathcal{Q}(z,e(t)) + \int_\Omega W(z)\,\mathrm{d}x + \frac{1}{2}a_{\mathrm{m}}(z,z) - \langle F(t), u + w(t)\rangle_{\mathrm{BD}(\Omega)}\,,$$

where we have highlighted the elastic part $e(t) = \mathrm{E}(u{+}w(t)) - p$ of the strain tensor. Since $\varrho(t)$ from (2.39d) is in $\Sigma(\Omega)$ and $F(t) = -\widehat{\mathrm{Div}}(\varrho(t))$ for all $t \in [0,T]$ by (2.39e), we may employ (7.6) to rewrite $\mathcal{E}_0$ as (cf. (2.44))

(7.10)
$$\begin{aligned}
&\mathcal{E}_0(t,u,z,p) = \mathcal{F}_0(t,z,e(t)) - \langle \rho_{\mathrm{D}}(t)\,|\,p\rangle \qquad \text{with}\\
&\mathcal{F}_0(t,z,e) := \mathcal{Q}(z,e) + \int_\Omega W(z)\,\mathrm{d}x\\
&\qquad\qquad + \frac{1}{2}a_{\mathrm{m}}(z,z) - \int_\Omega \rho(t)(e - \mathrm{E}(w(t)))\,\mathrm{d}x - \langle F(t), w(t)\rangle_{\mathrm{BD}(\Omega)}\,.
\end{aligned}$$

Energy-dissipation arclength reparameterization. As already mentioned, we will obtain Balanced Viscosity solutions to the perfectly plastic system by taking the joint vanishing-viscosity and hardening limit of (reparameterized) viscous solutions to Problem 3.1. Thus, let $(q^\mu_{\varepsilon,\nu})_{\varepsilon,\nu,\mu} = (u^\mu_{\varepsilon,\nu}, z^\mu_{\varepsilon,\nu}, p^\mu_{\varepsilon,\nu})_{\varepsilon,\nu,\mu}$ be a family of solutions to Problem 3.1. We are going to reparameterize them by the *energy-dissipation arclength* $\widetilde{s}^\mu_{\varepsilon,\nu} : [0,T] \to [0, \widetilde{S}^\mu_{\varepsilon,\nu}]$ (with $\widetilde{S}^\mu_{\varepsilon,\nu} := \widetilde{s}^\mu_{\varepsilon,\nu}(T)$) defined by

(7.11)
$$\begin{aligned}
\widetilde{s}^\mu_{\varepsilon,\nu}(t) := \int_0^t \Big(&1{+}\sqrt{\mu}\|{u^\mu_{\varepsilon,\nu}}'(\tau)\|_{H^1(\Omega;\mathbb{R}^n)}\\
&+\|{z^\mu_{\varepsilon,\nu}}'(\tau)\|_{H^{\mathrm{m}}(\Omega)}{+}\|{p^\mu_{\varepsilon,\nu}}'(\tau)\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}{+}\sqrt{\mu}\|{p^\mu_{\varepsilon,\nu}}'(\tau)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\\
&+\|{e^\mu_{\varepsilon,\nu}}'(\tau)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}{+}\mathcal{D}_\nu({u^\mu_{\varepsilon,\nu}}'(\tau),{p^\mu_{\varepsilon,\nu}}'(\tau))\,\mathcal{D}^{*,\mu}_\nu(\tau,q^\mu_{\varepsilon,\nu}(\tau))\Big)\,\mathrm{d}\tau
\end{aligned}$$

with $\mathcal{D}_\nu$ and $\mathcal{D}^{*,\mu}_\nu$ from (6.6). We will comment on the choice of the arclength function $\widetilde{s}^\mu_{\varepsilon,\nu}$ below. By estimates (5.3) and (6.10) we have that $\sup_{\varepsilon,\nu,\mu}\widetilde{S}^\mu_{\varepsilon,\nu} \le C$. As in (6.2), we set

$$\begin{aligned}
&\mathsf{t}^\mu_{\varepsilon,\nu} := (\widetilde{s}^\mu_{\varepsilon,\nu})^{-1}, \quad \mathsf{q}^\mu_{\varepsilon,\nu} := q^\mu_{\varepsilon,\nu} \circ \mathsf{t}^\mu_{\varepsilon,\nu} = (\mathsf{u}^\mu_{\varepsilon,\nu}, \mathsf{z}^\mu_{\varepsilon,\nu}, \mathsf{p}^\mu_{\varepsilon,\nu}), \quad \mathsf{e}^\mu_{\varepsilon,\nu} := e^\mu_{\varepsilon,\nu} \circ \mathsf{t}^\mu_{\varepsilon,\nu},\\
&\mathsf{\sigma}^\mu_{\varepsilon,\nu} := \sigma^\mu_{\varepsilon,\nu} \circ \mathsf{t}^\mu_{\varepsilon,\nu}
\end{aligned}$$

that we may assume defined on a fixed interval $[0,S]$, with $S := \lim_{\varepsilon,\nu,\mu\downarrow 0} \widetilde{S}^\mu_{\varepsilon,\nu}$ (the limit is intended along a suitable subsequence).

The very same calculations as in section 6 show that the rescaled solutions $(\mathsf{t}^\mu_{\varepsilon,\nu}, \mathsf{q}^\mu_{\varepsilon,\nu})_{\varepsilon,\nu,\mu}$ and the curves $(\mathsf{e}^\mu_{\varepsilon,\nu})_{\varepsilon,\nu,\mu}$ satisfy the parameterized energy-dissipation balance (6.7) as well as the normalization condition for almost all $s \in (0, S^\mu_{\varepsilon,\nu})$,

$$
\begin{aligned}
&\mathsf{t}^{\mu}_{\varepsilon,\nu}{}'(s)+\sqrt{\mu}\|\mathsf{u}^{\mu}_{\varepsilon,\nu}{}'(s)\|_{H^1(\Omega;\mathbb{R}^n)}\\
&\quad+\|\mathsf{z}^{\mu}_{\varepsilon,\nu}{}'(s)\|_{H^{\mathrm{m}}(\Omega)}+\sqrt{\mu}\|\mathsf{p}^{\mu}_{\varepsilon,\nu}{}'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}+\|\mathsf{p}^{\mu}_{\varepsilon,\nu}{}'(s)\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\\
&\quad+\|\mathsf{e}^{\mu}_{\varepsilon,\nu}{}'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}+\mathcal{D}_\nu(u^{\mu}_{\varepsilon,\nu}{}'(s),\mathsf{p}^{\mu}_{\varepsilon,\nu}{}'(s))\,\mathcal{D}^{*,\mu}_\nu(\mathsf{t}^{\mu}_{\varepsilon,\nu}(s),\mathsf{q}^{\mu}_{\varepsilon,\nu}(s))=1\,.
\end{aligned}
\tag{7.12}
$$

The choice of $\widetilde{s}^\mu_{\varepsilon,\nu}$ is precisely motivated by the need to ensure the validity of (7.12); in the lines below we are going to hint at the role of the term $\mathcal{D}_\nu\,\mathcal{D}^{*,\mu}_\nu$, while that of the contributions modulated by $\sqrt{\mu}$ will be evident in the proof of the upcoming Theorem 7.9. Let us also mention in advance that, in analogy with section 6, we will pass to the limit as ε, ν, $\mu \downarrow 0$ in the energy balance

$$
\begin{aligned}
&\mathcal{E}_\mu(\mathsf{t}^{\mu}_{\varepsilon,\nu}(S),\mathsf{q}^{\mu}_{\varepsilon,\nu}(S))+\int_0^S \mathcal{N}^{\mu}_{\varepsilon,\nu}(\mathsf{t}^{\mu}_{\varepsilon,\nu}(\tau),\mathsf{q}^{\mu}_{\varepsilon,\nu}(\tau),\mathsf{q}^{\mu}_{\varepsilon,\nu}{}'(\tau))\,\mathrm{d}\tau\\
&\quad=\mathcal{E}_\mu(0,\mathsf{q}_0)+\int_0^S \partial_t\mathcal{E}_\mu(\mathsf{t}^{\mu}_{\varepsilon,\nu}(\tau),\mathsf{q}^{\mu}_{\varepsilon,\nu}(\tau))\,\mathsf{t}^{\mu}_{\varepsilon,\nu}{}'(\tau)\,\mathrm{d}\tau\,.
\end{aligned}
\tag{7.13}
$$

The vanishing-viscosity contact potential for the perfectly plastic system. Clearly, upon taking the limit of the viscous system as the parameters ε, ν, $\mu \downarrow 0$, we are in particular addressing the case in which the viscosity in the momentum equation and in the plastic flow rule tends to zero with a higher rate than the viscosity in the damage flow rule. Therefore, the analysis carried out in section 6.2 would lead us to expect, for the limiting system, a notion of BV solution featuring a vanishing-viscosity potential (that will be denoted by $\mathcal{M}^0_{0,0}$ for consistency of notation), with the same structure as that from (6.40), but associated with the driving energy $\mathcal{E}_0$ for the perfectly plastic system. Specifically, one would envisage to deal with the quantity $\mathcal{D}^*(t,q) := \mathcal{D}^{*,0}(t,q)$ encompassing the $(H^1)^*$-norm of $\mathrm{D}_u\mathcal{E}_0$, and the L^2-distance of $\mathrm{D}_p\mathcal{E}_0$ from the stable set; cf. (6.40b). However, such quantities are no longer well defined for the functional $\mathcal{E}_0$, defined on $[0,T]\times\mathbf{Q}_{\mathrm{PP}}$ (while the L^2-distance of $\mathrm{D}_z\mathcal{E}_0$ from the stable set still makes sense). Therefore, in order to introduce the vanishing-viscosity potential $\mathcal{M}^0_{0,0}$ for the perfectly plastic system, we first introduce suitable "surrogates" of the $(H^1)^*$-norm of $\mathrm{D}_u\mathcal{E}_0$, and the L^2-distance from the stable set of $\mathrm{D}_p\mathcal{E}_0$. In accord with the representation formulae from Lemma 3.6, we set, for $(t,q)\in[0,T]\times\mathbf{D}$ and $\sigma(t)=\mathbb{C}(z)e(t)=\mathbb{C}(z)(\mathrm{E}(u+w(t))-p)$,

$$
\mathcal{S}_u\mathcal{E}_0(t,q):=\sup_{\substack{\eta_u\in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)\\ \|\eta_u\|_{(H^1,\mathbb{D})}\le 1}}\langle -\mathrm{Div}(\sigma(t))-F(t),\eta_u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)}\,,
\tag{7.14a}
$$

$$
\mathcal{W}_p\mathcal{E}_0(t,q):=\sup_{\substack{\eta_p\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\\ \|\eta_p\|_{L^2}\le 1}}\Big(\langle\sigma_{\mathrm{D}}(t),\eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}-\mathcal{H}(z,\eta_p)\Big)\,.
\tag{7.14b}
$$

We then set

$$
\mathcal{D}^*(t,q):=\sqrt{(\mathcal{S}_u\mathcal{E}_0(t,q))^2+(\mathcal{W}_p\mathcal{E}_0(t,q))^2}\,.
\tag{7.14c}
$$

Notice that the above expressions are well-defined since, by the definition of $\mathbf{Q}_{\rm PP}$, $e(t)$ and, a fortiori, $\sigma(t)$ are elements in $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$.

Thus, we are in a position to define the vanishing-viscosity contact potential $\mathcal{M}^0_{0,0}: [0,T]\times\mathbf{Q}_{\rm PP}\times[0,+\infty)\times\mathbf{Q}_{\rm PP}\to[0,+\infty]$ via

$$\mathcal{M}^0_{0,0}(t,q,t',q') := \mathcal{R}(z') + \mathcal{H}_{\rm PP}(z,p') + \mathcal{M}^{0,\rm red}_{0,0}(t,q,t',q'), \tag{7.15a}$$

where for $q=(u,z,p)$ and $q'=(u',z',p')$ we have

(7.15b)
$$\text{if } t'>0\,,\quad \mathcal{M}^{0,\rm red}_{0,0}(t,q,t',q') := \begin{cases} 0 & \text{if } \begin{cases} \mathcal{S}_u\mathcal{E}_0(t,q)=0\,,\\ \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0))=0\,, \text{ and}\\ \mathcal{W}_p\mathcal{E}_0(t,q)=0\,,\end{cases}\\ +\infty & \text{otherwise,}\end{cases}$$

(7.15c)
$$\mathcal{M}^{0,\rm red}_{0,0}(t,q,0,q') := \begin{cases} \mathcal{D}(u',p')\,\mathcal{D}^*(t,q) & \text{if } z'=0,\\ \|z'\|_{L^2}\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0)) & \text{if } \mathcal{D}^*(t,q)=0,\\ +\infty & \text{if } \|z'\|_{L^2}\mathcal{D}^*(t,q)>0\,.\end{cases}$$

In particular, observe that, once again, the expression of $\mathcal{M}^{0,\rm red}_{0,0}(t,q,t',q')$ for $t'>0$ enforces a "relaxed" form of equilibrium for u with the condition $\mathcal{S}_u\mathcal{E}_0(t,q)=0$, the local stability condition $\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0))=0$ for z, and a "relaxed" form of local stability for p via $\mathcal{W}_p\mathcal{E}_0(t,q)=0$; cf. Lemma 7.4 and Remark 7.5 ahead. Recalling that $\mathcal{D}(u',p') := \sqrt{\|u'\|^2_{H^1}+\|p'\|^2_{L^2}}$ (cf. (6.40b)), the product $\mathcal{D}(u',p')\,\mathcal{D}^*(t,q)$ is well defined as soon as $u'\in H^1(\Omega;\mathbb{R}^n)$ and $p'\in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm D})$; otherwise, we intend $\mathcal{D}(u',p')\,\mathcal{D}^*(t,q):=+\infty$.

Admissible parameterized curves. We are now in a position to introduce the class of parameterized curves enjoying the temporal and spatial integrability/ "regularity" properties of the curves that are limits of reparameterized viscous solutions as ε, ν, $\mu\downarrow 0$. Basically, such properties are motivated by the a priori estimates that the rescaled viscous solutions inherit from the normalization condition (7.12). In particular, let us highlight that (7.12) provides a (uniform-in-time) bound for the quantity $\mathcal{D}_\nu({u^\mu_{\varepsilon,\nu}}',{\mathsf{p}^\mu_{\varepsilon,\nu}}')\,\mathcal{D}^{*,\mu}_\nu(\mathsf{t}^\mu_{\varepsilon,\nu},\mathsf{q}^\mu_{\varepsilon,\nu})$. Recall that $\mathcal{D}_\nu({u^\mu_{\varepsilon,\nu}}',{\mathsf{p}^\mu_{\varepsilon,\nu}}')$ controls the H^1-norm of ${u^\mu_{\varepsilon,\nu}}'$ and the L^2-norm of ${p^\mu_{\varepsilon,\nu}}'$. Therefore, for the limiting parameterized curves $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})$, from such a bound one expects to infer, "away" from the set where $\{\mathcal{D}^{*,\mu}_\nu(\mathsf{t},\mathsf{q})=0\}$, additional *spatial regularity* for u' and p' in addition to that provided by the estimate for $\|{\mathsf{u}^\mu_{\varepsilon,\nu}}'\|_{\rm BD}+\|{\mathsf{p}^\mu_{\varepsilon,\nu}}'\|_{L^1}$. All of this is reflected in the following definition, where we introduce the notion of *admissibile parameterized curve* for the perfectly plastic system, in the spirit of [MRS16a, Definition 4.1].

DEFINITION 7.2. *A curve* $(\mathsf{t},\mathsf{q})=(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\colon[0,S]\to[0,T]\times\mathbf{Q}_{\rm PP}$ *is an* admissibile parameterized curve for the perfectly plastic system *if* $\mathsf{t}\colon[0,S]\to[0,T]$ *is nondecreasing and*

$$(\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p})\in \mathrm{AC}\big([0,S];[0,T]\times\mathrm{BD}(\Omega)\times H^{\rm m}(\Omega)\times\mathrm{M_b}(\Omega\cup\Gamma_{\rm Dir};\mathbb{M}^{n\times n}_{\rm D})\big)\,, \tag{7.16a}$$

$$\mathsf{e}=\mathrm{E}(\mathsf{u}+w(\mathsf{t}))-\mathsf{p}\in\mathrm{AC}([0,S];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))\,, \tag{7.16b}$$

$$(\mathsf{u},\mathsf{p})\in\mathrm{AC_{loc}}(B^\circ;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm D}))\,, \tag{7.16c}$$

where $B^\circ := \{s \in (0,S) \colon \mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) > 0\}$,

(7.16d) $\quad \mathsf{t}$ *is constant in every connected component of* B° .

We will write $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{PP}})$.

Let us point out that along an admissible curve $s \mapsto (\mathsf{t}(s),\mathsf{q}(s))$ we always have

$$\mathcal{D}(\mathsf{u}'(s),\mathsf{p}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) < \infty \qquad \text{for a.a. } s \in B^\circ .$$

Balanced Viscosity solutions arising in the joint vanishing-viscosity and hardening limit and their properties. We are now in a position to give the following definition.

Definition 7.3. *A curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \colon [0,S] \to [0,T]\times\mathbf{Q}_{\mathrm{PP}}$ *is a* (parameterized) Balanced Viscosity (BV, for short) solution to the multirate system for perfect plasticity and damage (1.1) *if*

- (t,q) *is an admissible parameterized curve in the sense of Definition* 7.2*;*
- (t,q) *fulfills the energy-dissipation balance*

$$\begin{aligned}&\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) + \int_0^s \mathcal{M}_{0,0}^0(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))\,\mathrm{d}\tau\\ &= \mathcal{E}_0(\mathsf{t}(0),\mathsf{q}(0)) + \int_0^s \partial_t\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathsf{t}'(\tau)\,\mathrm{d}\tau\end{aligned} \tag{7.17}$$

for all $0 \le s \le S$.

We say that (t,q) *is nondegenerate if it fulfills*

$$\mathsf{t}' + \|\mathsf{z}'\|_{H^{\mathrm{m}}(\Omega)} + \|\mathsf{p}'\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} + \|\mathsf{e}'\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} > 0 \qquad \textit{a.e. in } (0,S)\,.$$

As for BV solutions to the system with hardening, we have a characterization of BV solutions in terms of the upper energy-dissipation estimate $\le$ in (7.17); cf. Proposition 7.7 ahead. Such characterization will rely on the chain-rule estimate in the forthcoming Lemma 7.6 that, in turn, hinges on the following technical result that mimics [DMDM06, Proposition 3.5].

Lemma 7.4. *Suppose that* $\mathcal{S}_u\mathcal{E}_0(t,q) = \mathcal{W}_p\mathcal{E}_0(t,q) = 0$ *at some* $(t,q) \in [0,T] \times \mathbf{Q}_{\mathrm{PP}}$*. Then, for* $\sigma(t) = \mathbb{C}(z)e(t)$*, we have that*

$$\sigma(t) \in \mathcal{K}_z(\Omega)\,, \quad -\mathrm{div}\,\sigma(t) = f(t) \ \textit{a.e. in } \Omega\,, \quad [\sigma(t)\mathrm{n}] = g(t) \ \mathscr{H}^{n-1}\textit{-a.e. on } \Gamma_{\mathrm{Neu}}\,. \tag{7.18}$$

Proof. Since $\mathcal{S}_u\mathcal{E}_0(t,q) = 0$, we have that $-\mathrm{Div}(\sigma(t)) = F(t)$ in $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^n)^*$. Recalling the form (2.39b) of F, we get that $-\mathrm{div}(\sigma(t)) = f \in L^n(\Omega;\mathbb{R}^n)$ a.e. in Ω and $[\sigma(t)\mathrm{n}] = g(t) \in L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^n)$.

Moreover, since $\mathcal{W}_p\mathcal{E}_0(t,q) = 0$ and $H(z,\cdot)$ is positively 1-homogeneous, we get that for every $\eta_p \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$

$$-\mathcal{H}(z,-\eta_p) \le \langle\sigma_{\mathrm{D}}(t),\eta_p\rangle_{L^2} \le \mathcal{H}(z,\eta_p) \tag{7.19}$$

(where $\langle\cdot,\cdot\rangle_{L^2}$ is shorthand for the duality in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$). Then we may argue as in the proof of [DMDM06, Proposition 3.5]: in (7.19) we choose the test function $\eta(x) := 1_B(x)\xi$, with $B \subset \Omega$ an arbitrary Borel set and an arbitrary $\xi \in \mathbb{M}^{n\times n}_{\mathrm{D}}$. In this way, we get

$$-H(z(x),-\xi) \le \sigma_{\mathrm{D}}(t,x)\cdot\xi \le H(z(x),\xi) \quad \text{for a.a. } x \text{ in } \Omega\,.$$

Then $\sigma_{\mathrm{D}}(t,x) \in \partial_p H(z(x),0) = K(z(x))$ for a.a. $x \in \Omega$, so that $\sigma(t) \in \mathcal{K}_z(\Omega)$ and the proof is concluded. ∎

Remark 7.5. Conditions (7.18), expressed along BV solutions, correspond to the stability conditions in u and p ($ev1$) and ($ev2$) in the definition of the so-called rescaled quasistatic viscosity evolutions in [CL16, Definition 5.1]. Moreover, the identity $\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0)) = 0$ corresponds to the *Kuhn–Tucker inequality* ($ev3$) therein. Notice that these three conditions hold in the set $\{s\in(0,S)\colon \mathsf{t}'(s)>0\}$; cf. (7.15b).

We are now in a position to prove the chain-rule estimate involving $\mathcal{M}^{0,\mathrm{red}}_{0,0}$.

Lemma 7.6. *Along any* admissible *parameterized curve*

(7.20)
$$(\mathsf{t},\mathsf{q})\in\mathscr{A}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{PP}}) \text{ such that } \mathcal{M}^0_{0,0}(\mathsf{t},\mathsf{q},\mathsf{t}',\mathsf{q}'))<+\infty \text{ a.e. in } (0,S),$$
$$\text{we have that } s\mapsto\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) \text{ is absolutely continuous on } [0,S]$$
$$\text{and there holds for a.a.} s\in(0,S)$$
$$-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))+\partial_t\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\mathsf{t}'(s)\le\mathcal{M}^0_{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s))\,.$$

Proof. By the regularity of admissible parameterized curves we easily deduce that the function $s\mapsto\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))$ is absolutely continuous on $[0,S]$. Its derivative is given by (cf. Lemma 3.3)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) = \partial_t\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\mathsf{t}'(s)&+\langle\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^m}\\ &-\langle\sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle-\langle\widehat{\mathrm{Div}}(\sigma(s))+F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{\mathrm{BD}}\end{aligned}$$

for $\partial_t\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)) = \langle\sigma(s),\mathrm{E}(w'(\mathsf{t}(s)))\rangle_{L^2}-\langle F'(\mathsf{t}(s)),\mathsf{u}(s)+w(\mathsf{t}(s))\rangle_{\mathrm{BD}}-\langle F(\mathsf{t}(s)), w'(\mathsf{t}(s))\rangle_{\mathrm{BD}}$. Therefore, (7.20) follows if we prove that

(7.21)
$$\begin{aligned}&-\langle\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^m}+\langle\sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle+\langle\widehat{\mathrm{Div}}(\sigma(s))+F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{\mathrm{BD}}\\ &\le\mathcal{M}^0_{0,0}(\mathsf{t}(s),\mathsf{q}(s),\mathsf{t}'(s),\mathsf{q}'(s)).\end{aligned}$$

Let us then show (7.21). For a.e. $s\in(0,S)$ it holds that

(7.22)
$$-\langle\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\mathsf{z}'(s)\rangle_{H^m}\le\mathcal{R}(\mathsf{z}'(s))+\|\mathsf{z}'(s)\|_{L^2}\,\widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s)),\partial\mathcal{R}(0))$$

(cf. the calculations in the proof of Lemma 6.5). Let us estimate the two remaining terms, distinguishing the two cases of $\mathsf{t}'(s)=0$ and $\mathsf{t}'(s)>0$.

If $\mathsf{t}'(s)=0$ and $s\in B^\circ$, since $\mathsf{u}'(s)\in H^1(\Omega;\mathbb{R}^n)$ and $\mathsf{p}'(s)\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ a.e. in B° we have that

$$\begin{aligned}-\langle\widehat{\mathrm{Div}}(\sigma(s))+F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{\mathrm{BD}}&=-\langle\mathrm{Div}(\sigma(s))+F(\mathsf{t}(s)),\mathsf{u}'(s)\rangle_{H^1}\\ &\le\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\|\mathsf{u}'(s)\|_{H^1}\,,\end{aligned}\tag{7.23}$$

$$\langle\sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle-\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s))\le\mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))\,\|\mathsf{p}'(s)\|_{L^2}\,.\tag{7.24}$$

If $\mathsf{t}'(s)=0$ and $s\notin B^\circ$, then $\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))=\mathcal{W}_p\mathcal{E}_0(\mathsf{t}(s),\mathsf{q}(s))=0$, so that Lemma 7.4 together with (7.9) imply that

$$-\widehat{\mathrm{Div}}(\sigma(s))=F(\mathsf{t}(s))\text{ in }\mathrm{BD}(\Omega)^*\,,\qquad\langle\sigma_{\mathrm{D}}(s)\,|\,\mathsf{p}'(s)\rangle\le\mathcal{H}_{\mathrm{PP}}(\mathsf{z}(s),\mathsf{p}'(s))\,.\tag{7.25}$$

If $\mathsf{t}'(s)>0$, again we have (7.25).

Collecting (7.22), (7.23), (7.24), (7.25), recalling the definition of $\mathcal{M}^0_{0,0}$ (7.15), and employing the Cauchy–Schwarz inequality, we deduce (7.21) and then conclude the proof. ∎

As a straightforward corollary of Lemma 6.17, we have the desired characterization of BV solutions.

PROPOSITION 7.7. *For an admissible parameterized curve* $(\mathsf{t},\mathsf{q}) \in \mathscr{A}([0,S];[0,T] \times \mathbf{Q}_{\mathrm{PP}})$ *(in the sense of Definition* 7.2*) the following properties are equivalent:*

1. (t,q) *is a* BV *solution to the multirate system for perfect plasticity;*
2. (t,q) *fulfills the upper estimate* $\leq$ *in* (7.17).

We now give a lower semicontinuity result that will be useful in the proof of Theorem 7.9.

LEMMA 7.8. *Let* $t_k \to t$ *in* $[0,T]$, $\mu_k \to 0$, $(q_k)_k = (u_k, z_k, p_k)_k \subset \mathbf{Q}_{\mathrm{PP}}$ *such that the following convergences hold as* $k \to +\infty$*:* $q_k \rightharpoonup q = (u,z,p)$ *in* $\mathbf{Q}_{\mathrm{PP}}$, $e(t_k) = \mathrm{E}(u_k + w(t_k)) - p_k \to e(t) = \mathrm{E}(u + w(t)) - p$ *in* $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$*, and* $\mu_k\, p_k \to 0$ *in* $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$*. Then*

$$\mathcal{S}_u\mathcal{E}_0(t,q) \leq \liminf_{k\to+\infty} \|\mathrm{D}_u\mathcal{E}_{\mu_k}(t_k,q_k)\|_{(H^1,\mathbb{D})^*}\,, \tag{7.26a}$$

$$\widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_0(t,q),\partial\mathcal{R}(0)) \leq \liminf_{k\to+\infty} \widetilde{d}_{L^2}(-\mathrm{D}_z\mathcal{E}_{\mu_k}(t_k,q_k),\partial\mathcal{R}(0))\,, \tag{7.26b}$$

$$\mathcal{W}_p\mathcal{E}_0(t,q) \leq \liminf_{k\to+\infty} d_{L^2}(-\mathrm{D}_p\mathcal{E}_{\mu_k}(t_k,q_k),\partial_\pi\mathcal{H}(z_k,0))\,. \tag{7.26c}$$

Proof. It is immediate to see that, under the assumed convergences, setting $\sigma(t_k) = \mathbb{C}(z_k)e(t_k)$ and $\sigma(t) = \mathbb{C}(z)e(t)$, for fixed $\eta_u \in H^1(\Omega;\mathbb{R}^n)$ we have that

$$\langle -\mathrm{Div}(\sigma(t_k)) - F(t_k), \eta_u\rangle_{H^1(\Omega;\mathbb{R}^n)} \to \langle -\mathrm{Div}(\sigma(t)) - F(t), \eta_u\rangle_{H^1(\Omega;\mathbb{R}^n)}\,.$$

Furthermore, for fixed $\eta_p \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ there holds

$$\langle \sigma_{\mathrm{D}}(t_k) - \mu_k\, p_k, \eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} - \mathcal{H}(z_k,\eta_p) \to \langle \sigma_{\mathrm{D}}(t), \eta_p\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})} - \mathcal{H}(z,\eta_p)\,.$$

Passing to the supremums, we obtain (7.26a) and (7.26c). As for (7.26b), this follows as in (3.26b) since one only employs the convergence $z_k \rightharpoonup z$ in $H^m(\Omega)$, encompassed in the hypothesis $q_k \rightharpoonup q$ in $\mathbf{Q}_{\mathrm{PP}}$. ☐

Existence of BV solutions to the multirate system for perfect plasticity. We are now ready to state and prove our existence result for BV solutions in the sense of Definition 7.3. In order to simplify notation, we fix vanishing sequences $(\varepsilon_k)_k$, $(\nu_k)_k$, $(\mu_k)_k$ with $\nu_k \leq \mu_k$ and denote by $(\mathsf{t}_k)_k$, $(\mathsf{q}_k)_k$, $(\mathsf{e}_k)_k$, $(\upsigma_k)_k$ the sequences $(\mathsf{t}^{\mu_k}_{\varepsilon_k,\nu_k})_k$, $(\mathsf{q}^{\mu_k}_{\varepsilon_k,\nu_k})_k$, $(\mathsf{e}^{\mu_k}_{\varepsilon_k,\nu_k})_k$, $(\upsigma^{\mu_k}_{\varepsilon_k,\nu_k})_k$, respectively.

THEOREM 7.9. *Under the assumptions of section* 2 *and* (4.14) *for all vanishing sequences* $(\varepsilon_k)_k$, $(\nu_k)_k$, $(\mu_k)_k$ *with* $\nu_k \leq \mu_k$ *for every* $k \in \mathbb{N}$ *there exist a (not relabeled) subsequence* $(\mathsf{t}_k,\mathsf{q}_k)$ *and a curve* $(\mathsf{t},\mathsf{q}) = (\mathsf{t},\mathsf{u},\mathsf{z},\mathsf{p}) \in \mathscr{A}([0,S];[0,T]\times\mathbf{Q}_{\mathrm{PP}})$ *such that*

1. *for all* $s \in [0,S]$ *the following convergences hold as* $k \to +\infty$*:*

$$\begin{aligned} &\mathsf{t}_k(s) \to \mathsf{t}(s)\,, \quad \mathsf{u}_k(s) \overset{*}{\rightharpoonup} \mathsf{u}(s) \text{ in } \mathrm{BD}(\Omega)\,, \quad \mathsf{z}_k(s) \rightharpoonup \mathsf{z}(s) \text{ in } H^{\mathrm{m}}(\Omega)\,,\\ &\mathsf{e}_k(s) \rightharpoonup \mathsf{e}(s) \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\,, \quad \mathsf{p}_k(s) \rightharpoonup \mathsf{p}(s) \text{ in } \mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})\,; \end{aligned} \tag{7.27}$$

2. *there exists* $\overline{C} > 0$ *such that for a.e.* $s \in (0,S)$ *there holds*

$$\begin{aligned} &\mathsf{t}'(s) + \|\mathsf{u}'(s)\|_{\mathrm{BD}(\Omega)} + \|\mathsf{z}'(s)\|_{H^{\mathrm{m}}(\Omega)} + \|\mathsf{p}'(s)\|_{\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})}\\ &\qquad + \|\mathsf{e}'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})} + \mathcal{D}(\mathsf{u}'(s),\mathsf{p}'(s))\,\mathcal{D}^*(\mathsf{t}(s),\mathsf{q}(s)) \leq \overline{C}\,; \end{aligned} \tag{7.28}$$

3. (t,q) *is a Balanced Viscosity solution to the multirate system for perfect plasticity* (1.1) *in the sense of Definition* 7.3.

Proof. As done for Theorems 6.8 and 6.13, we divide the proof into three steps.

Step 1: Compactness. Let $(\mathsf{t}_k,\mathsf{q}_k)_k$ be a sequence as in the statement. It follows from the normalization condition (7.12) that there exists $C>0$ such that for a.a. $s\in(0,S)$

$$\begin{aligned}\mathsf{t}_k'(s)+\|\mathsf{u}_k'(s)\|_{W^{1,1}(\Omega;\mathbb{R}^n)}+&\|\mathsf{z}_k'(s)\|_{H^{\mathrm{m}}(\Omega)}+\|\mathsf{p}_k'(s)\|_{L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}\\ &+\sqrt{\mu_k}\,\|\mathsf{p}_k'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})}+\|\mathsf{e}_k'(s)\|_{L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})}\leq C\,,\end{aligned}\tag{7.29}$$

where the estimate for $\|\mathsf{u}_k'\|_{W^{1,1}}$ (recall that $\mathsf{u}_k'\in H^1(\Omega;\mathbb{R}^n)$) ensues from the fact that $\mathrm{E}(\mathsf{u}_k')=\mathsf{e}_k'+\mathsf{p}_k'$ is bounded in $L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ combined with Korn's inequality.

Clearly, in the estimates for u_k and p_k we may pass from $W^{1,1}(\Omega;\mathbb{R}^n)$ and $L^1(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$ to the (duals of separable spaces) $\mathrm{BD}(\Omega)$ and $\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})$. Therefore, we are in a position to apply the compactness results from [Sim87] to get that there exists $(\mathsf{t},\mathsf{q})\in W^{1,\infty}([0,S];[0,T]\times\mathrm{BD}(\Omega)\times H^{\mathrm{m}}(\Omega)\times\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}}))$, and $\mathsf{e}\in W^{1,\infty}(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$, such that, along a not relabeled subsequence,

$$\mathsf{t}_k\overset{*}{\rightharpoonup}\mathsf{t}\quad\text{in }W^{1,\infty}(0,S;[0,T])\,,\qquad \mathsf{u}_k\overset{*}{\rightharpoonup}\mathsf{u}\quad\text{in }W^{1,\infty}(0,S;\mathrm{BD}(\Omega))\,,\tag{7.30a}$$

$$\mathsf{z}_k\overset{*}{\rightharpoonup}\mathsf{z}\quad\text{in }W^{1,\infty}(0,S;H^{\mathrm{m}}(\Omega))\,,\qquad \mathsf{z}_k\to\mathsf{z}\quad\text{in }\mathrm{C}^0([0,S];\mathrm{C}^0(\overline{\Omega}))\,,\tag{7.30b}$$

$$\mathsf{e}_k\overset{*}{\rightharpoonup}\mathsf{e}\quad\text{in }W^{1,\infty}(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))\,,\tag{7.30c}$$

$$\mathsf{p}_k\overset{*}{\rightharpoonup}\mathsf{p}\quad\text{in }W^{1,\infty}(0,S;\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}}))\,.\tag{7.30d}$$

It can be checked that $\mathsf{e}=\mathrm{E}(\mathsf{u}+w(\mathsf{t}))-\mathsf{p}$. In particular, the pointwise convergences (7.27) hold. Notice also that

$$\sqrt{\mu_k}\,\mathsf{p}_k\overset{*}{\rightharpoonup}0\quad\text{in }W^{1,\infty}(0,S;L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))\,,\tag{7.30e}$$

so that for every $s\in[0,S]$

$$\sqrt{\mu_k}\,\mathsf{p}_k(t)\rightharpoonup0\quad\text{in }L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})\,.\tag{7.30f}$$

Next, we introduce the functions s^- and s^+ and the sets $\mathcal{S}$, $\mathcal{U}$ in the same way as in Step 1 in Theorem 6.8 (cf. (6.25)); we readily deduce the following convergences for all $t\in[0,T]$:

$$u^{\mu_k}_{\varepsilon_k,\nu_k}(t)\overset{*}{\rightharpoonup}\mathsf{u}(s_-(t))=\mathsf{u}(s_+(t))\text{ in }\mathrm{BD}(\Omega)\,,\tag{7.31a}$$

$$z^{\mu_k}_{\varepsilon_k,\nu_k}(t)\rightharpoonup\mathsf{z}(s_-(t))=\mathsf{z}(s_+(t))\text{ in }H^{\mathrm{m}}(\Omega)\,,\tag{7.31b}$$

$$p^{\mu_k}_{\varepsilon_k,\nu_k}(t)\overset{*}{\rightharpoonup}\mathsf{p}(s_-(t))=\mathsf{p}(s_+(t))\text{ in }\mathrm{M_b}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{n\times n}_{\mathrm{D}})\,.\tag{7.31c}$$

Step 2: Finiteness of $\mathcal{M}^{0,\mathrm{red}}_{0,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))$ when $\mathsf{t}'(\tau)>0$. In view of the definition (7.15) of $\mathcal{M}^{0,\mathrm{red}}_{0,0}$, the function $\tau\mapsto\mathcal{M}^{0,\mathrm{red}}_{0,0}(\mathsf{t}(\tau),\mathsf{q}(\tau),\mathsf{t}'(\tau),\mathsf{q}'(\tau))$ is finite in the set $A:=\{s\in[0,S]\,:\,\mathsf{t}'(s)>0\}$ if and only if

$$\mathcal{S}_u\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau))=0\,,\ -\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau))\in\partial\mathcal{R}(0)\,,\ \mathcal{W}_p\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau))=0\text{ for a.a. }\tau\in A\,.\tag{7.32}$$

By (3.24) we obtain

(7.33)
$$\begin{aligned}\mathcal{D}^{*,\mu_k}_{\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau)) &= \mathcal{D}^{*,\mu_k}_{\nu_k}(\mathsf{t}_k(\tau),q_k(\mathsf{t}_k(\tau)))\\ &= \varepsilon_k\sqrt{\nu_k\|u_k'(\mathsf{t}_k(\tau))\|^2_{H^1,\mathbb{D}}+\|z_k'(\mathsf{t}_k(\tau))\|^2_{L^2}+\nu_k\|p_k'(\mathsf{t}_k(\tau))\|^2_{L^2}}\\ &= \frac{\varepsilon_k}{\mathsf{t}_k'(\tau)}\sqrt{\nu_k\|\mathsf{u}_k'(\tau)\|^2_{H^1,\mathbb{D}}+\|\mathsf{z}_k'(\tau)\|^2_{L^2}+\nu_k\|\mathsf{p}_k'(\tau)\|^2_{L^2}}\\ &\le \frac{\varepsilon_k}{\mathsf{t}_k'(\tau)} \qquad \text{for a.a. } \tau\in(0,S),\end{aligned}$$

where in the last estimate we exploited the normalization condition (7.12) and the fact that $\nu_k \le \mu_k$. Moreover, one sees as in (6.28) that $\limsup_{k\to+\infty}\mathsf{t}_k'(\tau)>0$ for a.e. $\tau\in A$. Since ε_k, $\nu_k\downarrow 0$, by Lemma 7.8 (notice that its assumptions are satisfied by the convergences (7.27) and (7.30f), also recalling that $\mu_k\to 0$) and an argument analogous to that in Step 2 of Theorem 6.13 we deduce (7.32).

Step 3: The energy-dissipation upper estimate (7.17). In view of the characterization provided by Proposition 7.7, to conclude that (t,q) is a BV solution in the sense of Definition 7.3 it is sufficient to show that (t,q) is an admissible parameterized curve as in Definition 7.2 and that it fulfills (7.17) as an upper estimate. First of all, we show that

(7.34)
$$\mathsf{z}'(\tau)\,\mathcal{D}^*(\mathsf{t}(\tau),\mathsf{q}(\tau))=0 \quad \text{for a.a. } \tau\in(0,S)\,.$$

This follows arguing similarly to what was done in Step 2 of Theorem 6.13. We start from (6.46) and then observe that (cf. (6.47))

(7.35)
$$\liminf_{k\to+\infty}\mathcal{D}^{*,\mu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\ge\mathcal{D}^*(\mathsf{t}(\tau),\mathsf{q}(\tau)) \qquad \forall\tau\in[0,S],$$

due to (7.27), (7.30f), (7.26a), and (7.26c). Then, applying Lemma B.1 with the analogous choices and arguments as in the proofs of Theorems 6.8 and 6.13, also relying on Lemma 7.8 we conclude that

(7.36)
$$\begin{aligned}\int_{B^\circ}\|\mathsf{z}'(\tau)\|_{L^2}\,\mathcal{D}^*(\mathsf{t}(\tau),\mathsf{q}(\tau))\,\mathrm{d}\tau &\le \liminf_{k\to\infty}\int_{B^\circ}\|\mathsf{z}_k'(\tau)\|_{L^2}\,\mathcal{D}^{*,\mu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\,\mathrm{d}\tau\\ &\le\sqrt{\nu_k}\liminf_{k\to\infty}\mathcal{N}^{\mu,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau),\mathsf{q}_k'(\tau))\,\mathrm{d}\tau=0\end{aligned}$$

with B° from (7.16c). Then, (7.34) ensues.

In analogy with (6.45), we also introduce the set

(7.37)
$$C^\circ:=\{\tau\in[0,S]\colon \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(\tau),\mathsf{q}(\tau)),\partial\mathcal{R}(0))>0\}\,.$$

By Lemma B.2 applied with the choices $X:=[0,1]\times[0,T]\times\mathbf{Q}_{\mathrm{PP}}$, $I:=[0,S]$, $B:=B^\circ$, $v_k(\tau):=(\mu_k,\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))$, $v(\tau):=(\mu,\mathsf{t}(\tau),\mathsf{q}(\tau))$, and with the function $f:X\to[0,+\infty]$ defined by

$$f(\mu,t,q):=\begin{cases}\mathcal{D}^*_\mu(t,q) & \text{if } \mu>0,\\ \mathcal{D}^*(t,q) & \text{if } \mu=0;\end{cases}$$

indeed, thanks to Lemma 7.8 the function f is weakly* lower semicontinuous on $X:=[0,1]\times[0,T]\times\mathbf{Q}_{\mathrm{PP}}$. Thus, we obtain that for any compact subset K° of B° there exist $c>0$ and $\overline{k}\in\mathbb{N}$ such that

$$\mathcal{D}^{*,\mu_k}(\mathsf{t}_k(\tau),\mathsf{q}_k(\tau))\ge c \qquad \text{for every } k\ge\overline{k}\,,\tau\in K^\circ\,.$$

By the normalization condition (7.12) (recall the notation for t_k, q_k) we obtain that for $k \geq \overline{k}$

$$\mathcal{D}(\mathsf{u}_k'(\tau), \mathsf{p}_k'(\tau)) \leq \frac{1}{c} \qquad \text{for a.a.}\, \tau \in K^\circ,$$

so that u_k and p_k are equi-Lipschitz in K° with values in $H^1(\Omega;\mathbb{R}^n)$ and $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}})$, respectively. Therefore, we deduce that $\mathsf{u} \in W^{1,\infty}(K^\circ; H^1(\Omega;\mathbb{R}^n))$ and that $\mathsf{p} \in W^{1,\infty}(K^\circ; L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))$. By the arbitrariness of K° we conclude that $(\mathsf{u},\mathsf{p}) \in W^{1,\infty}_{\mathrm{loc}}(B^\circ; H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{D}}))$, and then (t,q) is an admissible parameterized curve in the sense of Definition 7.2. Moreover, again for every $K^\circ \Subset B^\circ$ we have that the sequence $(\mathcal{D}(\mathsf{u}_k', \mathsf{p}_k'))_k$ converges weakly to some d in $L^\infty(K^\circ)$, with $d \geq \mathcal{D}(\mathsf{u}', \mathsf{p}')$ a.e. in K°. Then, we are again in a position to apply Lemma B.1, deducing (in view of the arbitrariness of $K^\circ \subset B^\circ$)

$$\begin{aligned}
&\int_{B^\circ} \mathcal{D}(\mathsf{u}'(\tau), \mathsf{p}'(\tau))\, \mathcal{D}^*(\mathsf{t}(\tau), \mathsf{q}(\tau))\,\mathrm{d}\tau \\
&\quad\leq \liminf_{k\to+\infty} \int_{B^\circ} \mathcal{D}(\mathsf{u}_k'(\tau), \mathsf{p}_k'(\tau))\, \mathcal{D}^{*,\mu_k}(\mathsf{t}_k(\tau), \mathsf{q}_k(\tau))\,\mathrm{d}\tau \\
&\quad\leq \liminf_{k\to+\infty} \int_{B^\circ} \mathcal{N}^{\mu_k,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau), \mathsf{q}_k(\tau), \mathsf{t}_k'(\tau), \mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,,
\end{aligned} \tag{7.38}$$

the last estimate due to (6.51). Then, estimate (7.28) follows by lower semicontinuity arguments.

Finally, we repeat the arguments for (6.54), obtaining that

$$\begin{aligned}
&\int_{(0,S)\setminus B^\circ} \mathcal{M}^{0,\mathrm{red}}_{0,0}(\mathsf{t}(\tau), \mathsf{q}(\tau), \mathsf{t}'(\tau), \mathsf{q}'(\tau))\,\mathrm{d}\tau \\
&\quad= \int_{C^\circ\setminus B^\circ} \|\mathsf{z}'(\tau)\|_{L^2}\, \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_0(\mathsf{t}(\tau), \mathsf{q}(\tau)), \partial\mathcal{R}(0))\,\mathrm{d}\tau \\
&\quad\overset{(1)}{\leq} \liminf_{k\to\infty} \int_{C^\circ\setminus B^\circ} \|\mathsf{z}_k'(\tau)\|_{L^2}\, \widetilde{d}_{L^2(\Omega)}(-\mathrm{D}_z\mathcal{E}_{\mu_k}(\mathsf{t}_k(\tau), \mathsf{q}_k(\tau)), \partial\mathcal{R}(0))\,\mathrm{d}\tau \\
&\quad\leq \liminf_{k\to+\infty} \int_{C^\circ\setminus B^\circ} \mathcal{N}^{\mu_k,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau), \mathsf{q}_k(\tau), \mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,.
\end{aligned} \tag{7.39}$$

Observe that (1) follows from the very same argument as in the proof of Theorem 6.13, now employing (7.26b) in place of (3.26b).

It follows from (7.38) and (7.39), also recalling the definition of B°, that

$$\int_0^S \mathcal{M}^{0,\mathrm{red}}_{0,0}(\mathsf{t}(\tau), \mathsf{q}(\tau), \mathsf{t}'(\tau), \mathsf{q}'(\tau))\,\mathrm{d}\tau \leq \liminf_{k\to+\infty} \int_0^S \mathcal{N}^{\mu_k,\mathrm{red}}_{\varepsilon_k,\nu_k}(\mathsf{t}_k(\tau), \mathsf{q}_k(\tau), \mathsf{q}_k'(\tau))\,\mathrm{d}\tau\,.$$

Combining the above lower semicontinuity estimate with the limit passage in the terms with driving energy and in the power term (which is standard and goes as in section 6), we succeed in taking the limit in the energy-dissipation inequality (6.9) to conclude the desired validity of the energy-dissipation upper estimate $\leq$ in (7.17). This finishes the proof of Theorem 7.9. ∎

Appendix A. Discrete Gronwall-type lemmas. Here we collect, for the reader's convenience, the discrete Gronwall-type results that have been exploited in the proof of the a priori estimates from Proposition 4.3.

LEMMA A.1. *Let* B, $\tau > 0$, $N_\tau \in \mathbb{N}$, $(a_k)_{k=1}^{N_\tau}$, $(b_k)_{k=1}^{N_\tau} \subset [0, +\infty)$ *fulfill*

$$a_k \leq B + \sum_{j=0}^{k-1} a_j b_j \quad \forall k \in \{1, \dots, N_\tau\}.$$

Then, there holds

$$a_k \leq B \exp\left(\sum_{j=0}^{k-1} b_j\right) \quad \forall k \in \{1, \dots, N_\tau\}. \tag{A.1}$$

LEMMA A.2 (see [RS06, Lemma 4.5]). *Let* $N_\tau \in \mathbb{N}$ *and* b, λ, $\Lambda \in (0, +\infty)$ *fulfill* $1 - b \geq \frac{1}{\lambda} > 0$*; let* $(a_k)_{k=1}^{N_\tau} \subset [0, +\infty)$ *satisfy*

$$a_k \leq \Lambda + b \sum_{j=1}^{k} a_j \qquad \forall k \in \{1, \dots, N_\tau\}.$$

Then, there holds

$$a_k \leq \lambda \Lambda \exp(\lambda b k) \quad \forall k \in \{1, \dots, N_\tau\}. \tag{A.2}$$

The following lemma generalizes [KRZ13, Lemma 4.1], and its proof is based on the calculations developed in for [CL16, Proposition 3.8] (see also [ACO19, Proposition 3.5]); that it is why we will only partially carry out the argument, and we will refer to [CL16] for more details.

LEMMA A.3. *Let* $\{a_k\}_{k=0}^{N_\tau}$, $\{M_k\}_{k=1}^{N_\tau}$, $\{r_k\}_{k=1}^{N_\tau}$, $\{c_k\}_{k=0}^{N_\tau}$ ρ *and* η *be nonnegative numbers,* $\varepsilon, \tau > 0$ *with* $\gamma := \kappa_1 \tau / \varepsilon \leq 1$ *for some* $\kappa_1 > 0$ *and* $N_\tau \in \mathbb{N}$, $N_\tau \tau = T$*. Assume that* $a_0 = 0$, $r_k \leq \kappa_2 a_k$ *for some* $\kappa_2 > 1$ *and that for* $1 \leq k \leq N_\tau$ *it holds that*

$$a_k(a_k - a_{k-1}) + \gamma a_k^2 + \gamma M_k^2 \leq \eta^2 \gamma \left(1 + c_k^2 + \tfrac{\delta_{1,k}}{\tau\varepsilon} \rho^2\right) + \gamma a_k r_k. \tag{A.3}$$

Then, if $\gamma = \kappa_1 \tau / \varepsilon \leq 1/(2\kappa_2)$*, there exists a constant* $C = C(\eta, T) > 0$ *not depending on any of the other above quantities such that*

$$\sum_{k=1}^{N_\tau} \tau M_k \leq C \left(T + \rho + \sum_{k=1}^{N_\tau} \tau c_k^2 + \sum_{k=1}^{N_\tau} \tau r_k\right). \tag{A.4}$$

Proof. For $2 \leq k \leq N_\tau$, we can recast (A.3) in the same form as [CL16, inequality between (3.35) and (3.36)], namely

$$2a_k'(a_k' - a_{k-1'}) + 2\zeta (a_k')^2 + (b_k')^2 \leq (c_k')^2 + 2a_k' d_k' \,.$$

For this, it is sufficient to replace a_k, γ, γM_k^2, $\eta^2\gamma(1 + c_k^2)$, and r_k in (A.3) (observe that $\delta_{1,k} = 0$ for $k \in \{2, \dots, N_\tau\}$), by, respectively, $a_k'/\sqrt{2}$, $\zeta = \overline{C}\tau/\varepsilon$, $(b_k')^2$, $(c_k')^2$, and $d_k'/\sqrt{2}$, with a universal constant $\overline{C}$. Following exactly the argument in [CL16, Proposition 3.8] and then rewriting [CL16, (3.41)] in the present setup, we get that

$$\sum_{k=2}^{N_\tau} \tau M_k \leq C \left(T + \varepsilon\, a_1 + \sum_{k=2}^{N_\tau} \tau c_k^2 + \sum_{k=2}^{N_\tau} \tau r_k\right). \tag{A.5}$$

Let us now estimate τM_1 and εa_1 by (A.3) for $k=1$. Notice that, since $a_0=0$ and using the cauchy inequality $2a_1 r_1 \le a_1^2 + r_1^2$, we derive that $M_1^2 \le \eta^2(1+c_1^2+\frac{\rho^2}{\tau\varepsilon}) + r_1^2$. Multiplying by τ^2, recalling $\tau<\varepsilon$, and taking the square root we obtain, for a suitable C, that

$$\tau M_1 \le C\tau(1+c_1+r_1)+\varrho \le C\tau(2+c_1^2+r_1)+\varrho\,. \tag{A.6}$$

We are then left to estimate $\varepsilon\, a_1$. We again start from (A.3) for $k=1$: recalling that $a_0=0$, we then have

$$a_1^2+\gamma a_1^2+\gamma M_1^2 \le \eta^2\gamma\left(1+c_1^2+\frac{1}{\tau\varepsilon}\rho^2\right)+\gamma a_1 r_1\,. \tag{A.7}$$

Then, we use the conditions $r_k \le \kappa_2 a_k$ and $k_1\tau/\varepsilon < 1/(2k_2)$ to get $\gamma a_1\, r_1 \le \frac{a_1^2}{2}$, which can be absorbed into the left-hand side of (A.7). Multiplying by ε^2 we get $\varepsilon^2 a_1^2 \le c^2\tau\varepsilon(1+c_1^2)+c^2\varrho^2$ for some $c>0$, so that

$$\varepsilon a_1 \le C\left(1+\sqrt{\varepsilon\,\tau c_1^2}+\varrho\right) \le C(2+\varepsilon\,\tau c_1^2+\varrho)\,. \tag{A.8}$$

Collecting (A.5), (A.6), and (A.8), up to modifying C we conclude (A.4). ◻

Appendix B. Two abstract results. We first recall an abstract lemma from [MRS16a] and [MRS12b] (to which we refer for the proof).

Lemma B.1. *Let I be a measurable subset of $\mathbb{R}$ and let h_n, h, m_n, $m\colon I\to[0,+\infty]$ be measurable functions for $n\in\mathbb{N}$ that satisfy*

$$h(x) \le \liminf_{n\to+\infty} h_n(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } x\in I, \qquad m_n \rightharpoonup m \quad \text{in } L^1(I)\,. \tag{B.1}$$

Then

$$\int_I h(x)m(x)\,\mathrm{d}x \le \liminf_{n\to+\infty}\int_I h_n(x)m_n(x)\,\mathrm{d}x\,.$$

Let us now consider a result that is applied in the proof of Theorem 7.9.

Lemma B.2. *Let $X=Y^*$, for Y a separable Banach space, $I=[a,b]\subset\mathbb{R}$, $f\colon X\to[0,+\infty]$ be weakly* lower semicontinuous, and let $(v_k)_k$ be a sequence of functions $v_k\colon I\to X$ satisfying*

$$\begin{aligned} &\exists\, C>0\ \forall\, t,s\in I:\quad \|v_k(t)-v_k(s)\|_X \le C|t-s|\,,\\ &v_k(t) \overset{*}{\rightharpoonup} v(t) \qquad \text{in } X \quad \forall t\in I\,. \end{aligned} \tag{B.2}$$

Then, for every compact subset $K\subset B:=\{t\in I\colon f(v(t))>0\}$ there exist $c>0$ and $\bar{k}\in\mathbb{N}$ such that

$$f(v_k(t)) \ge c \qquad \text{for every } k\ge\bar{k},\ t\in K\,. \tag{B.3}$$

Proof. By (B.2) and the pointwise weak* convergence to v, we deduce that

$$\|v(t)-v(s)\|_X \le C|t-s| \qquad \text{for every } t,s\in I, \tag{B.4}$$

that the set $V:=\bigcup_k v_k(I)\cup v(I)$ is bounded in X and that

$$\lim_{k\to+\infty}\sup_{t\in I} d_{w^*}(v_k(t),v(t))=0 \tag{B.5}$$

with d_{w^*} the metric inducing the weak* topology on the bounded set V (here we use the separability of Y and refer to the compactness arguments by [Sim87]).

Now let K be as in the statement. By (B.4) we get that $v(K)$ is compact in X, and since $K \subset B$ we have that $v(K) \subset \{f > 0\} \doteq \{f > 0\}$ the set $\{x \in X : f(x) > 0\}$. Then we can find an open set A such that $v(K) \subset A \subset \overline{A} \subset \{f > 0\}$. We deduce, employing the lower semicontinuity of f, that

$$f(A) \subset [c, +\infty] \tag{B.6}$$

for a suitable constant $c > 0$. Thanks to (B.5) and the fact that $d_{w*}(v_1, v_2) \leq C^*\|v_1 - v_2\|_X$, for a suitable C^* and every v_1 and $v_2 \in V$, choosing ε small enough we get that $v_k(K) \subset A$ for $k \geq \overline{k}$. Therefore, by (B.6) we conclude (B.3). ☐

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REFERENCES

[ACO19] R. Alessi, V. Crismale, and G. Orlando, *Fatigue effects in elastic materials with variational damage models: A vanishing viscosity approach*, J. Nonlinear Sci., 29 (2019), pp. 1041–1094.

[AE84] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Pure Appl. Math. (New York), John Wiley & Sons, New York, 1984.

[AFP05] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York, 2005.

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[ALL19] S. Almi, L. Lazzaroni, and I. Lucardesi, *Crack growth by vanishing viscosity in planar elasticity*, Math. Eng., 2 (2020), pp. 141–173.

[Alm17] S. Almi, *Energy release rate and quasi-static evolution via vanishing viscosity in a fracture model depending on the crack opening*, ESAIM Control Optim. Calc. Var., 23 (2017), pp. 791–826.

[AMV14] R. Alessi, J.J. Marigo, and S. Vidoli, *Gradient damage models coupled with plasticity and nucleation of cohesive cracks*, Arch. Ration. Mech. Anal., 214 (2014), pp. 575–615.

[AMV15] R. Alessi, J.J. Marigo, and S. Vidoli, *Gradient damage models coupled with plasticity: variational formulation and main properties*, Mech. Mater. B, 80 (2015), pp. 351–367.

[BFM12] J.-F. Babadjian, G. Francfort, and M.G. Mora, *Quasistatic evolution in non-associative plasticity—the cap model*, SIAM J. Math. Anal., 44 (2012), pp. 245–292.

[BMR12] S. Bartels, A. Mielke, and T. Roubíček, *Quasi-static small-strain plasticity in the limit of vanishing hardening and its numerical approximation*, SIAM J. Numer. Anal., 50 (2012), pp. 951–976.

[BRRT16] E. Bonetti, E. Rocca, R. Rossi, and M. Thomas, *A rate-independent gradient system in damage coupled with plasticity via structured strains*, in Gradient Flows: From Theory to Application, ESAIM Proc. Surveys 54, EDP, Les Ulis, 2016, pp. 54–69.

[CL16] V. Crismale and G. Lazzaroni, *Viscous approximation of quasistatic evolutions for a coupled elastoplastic-damage model*, Calc. Var. Partial Differential Equations, 55 (2016), 17.

[CL17] V. Crismale and G. Lazzaroni, *Quasistatic crack growth based on viscous approximation: a model with branching and kinking*, NoDEA Nonlinear Differential Equations Appl., 24 (2017), 7.

[CO18] V. Crismale and G. Orlando, *A Reshetnyak-type lower semicontinuity result for linearised elasto-plasticity coupled with damage in* $W^{1,n}$, NoDEA Nonlinear Differential Equations Appl., 25 (2018), 16.

[CO19] V. Crismale and G. Orlando, *A lower semicontinuity result for linearised elasto-plasticity coupled with damage in* $W^{1,\gamma}$, $\gamma > 1$, Math. Eng., 2 (2020), pp. 101–118.

[Cri16] V. Crismale, *Globally stable quasistatic evolution for a coupled elastoplastic-damage model*, ESAIM Control Optim. Calc. Var., 22 (2016), pp. 883–912.

[Cri17] V. Crismale, *Globally stable quasistatic evolution for strain gradient plasticity coupled with damage*, Ann. Mat. Pura Appl. (4), 196 (2017), pp. 641–685.

[DDS11] G. Dal Maso, A. DeSimone, and F. Solombrino, *Quasistatic evolution for cam-clay plasticity: a weak formulation via viscoplastic regularization and time rescaling*, Calc. Var. Partial Differential Equations, 40 (2011), pp. 125–181.

[DMDM06] G. Dal Maso, A. DeSimone, and M.G. Mora, *Quasistatic evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal., 180 (2006), pp. 237–291.

[DMOT16] G. Dal Maso, G. Orlando, and R. Toader, *Fracture models for elasto-plastic materials as limits of gradient damage models coupled with plasticity: the antiplane case*, Calc. Var. Partial Differential Equations, 55 (2016), Art. 45, 39.

[DMT02] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures: Existence and approximation results*, Arch. Ration. Mech. Anal., 162 (2002), pp. 101–135.

[DRS19] E. Davoli, T. Roubíček, and U. Stefanelli, *Dynamic perfect plasticity and damage in viscoelastic solids*, ZAMM Z. Angew. Math. Mech., 99 (2019), e201800161.

[EM06] M. Efendiev and A. Mielke, *On the rate-independent limit of systems with dry friction and small viscosity*, J. Convex Anal., 13 (2006), pp. 151–167.

[FG12] G.A. Francfort and A. Giacomini, *Small-strain heterogeneous elastoplasticity revisited*, Comm. Pure Appl. Math., 65 (2012), pp. 1185–1241.

[FS13] G. Francfort and U. Stefanelli, *Quasistatic evolution for the Armstrong-Frederick hardening-plasticity model*, Appl. Maths. Res. Express, 2 (2013), pp. 297–344.

[GS64] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J., 31 (1964), pp. 159–178.

[IT79] A.D. Ioffe and V.M. Tihomirov, *Theory of Extremal Problems*, Stud. Math. Appl. 6, North-Holland, Amsterdam, 1979.

[KMZ08] D. Knees, A. Mielke, and C. Zanini, *On the inviscid limit of a model for crack propagation*, Math. Models Methods Appl. Sci., 18 (2008), pp. 1529–1569.

[KRZ13] D. Knees, R. Rossi, and C. Zanini, *A vanishing viscosity approach to a rate-independent damage model*, Math. Models Methods Appl. Sci., 23 (2013), pp. 565–616.

[KRZ15] D. Knees, R. Rossi, and C. Zanini, *A quasilinear differential inclusion for viscous and rate-independent damage systems in non-smooth domains*, Nonlinear Anal. Real World Appl., 24 (2015), pp. 126–162.

[KT83] R. Kohn and R. Temam, *Dual spaces of stresses and strains, with applications to Hencky plasticity*, Appl. Math. Optim., 10 (1983), pp. 1–35.

[LT11] G. Lazzaroni and R. Toader, *A model for crack propagation based on viscous approximation*, Math. Models Methods Appl. Sci., 21 (2011), pp. 2019–2047.

[Mor06] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation.* II, Grundlehren Math. Wiss. 331, Springer-Verlag, Berlin, 2006.

[MR21] A. Mielke and R. Rossi, *Balanced viscosity solutions to infinite-dimensional multi-rate systems*, in preparation.

[MRS09] A. Mielke, R. Rossi, and G. Savaré, *Modeling solutions with jumps for rate-independent systems on metric spaces*, Discrete Contin. Dyn. Syst., 25 (2009), pp. 585–615.

[MRS12a] A. Mielke, R. Rossi, and G. Savaré, *BV solutions and viscosity approximations of rate-independent systems*, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 36–80.

[MRS12b] A. Mielke, R. Rossi, and G. Savaré, *Variational convergence of gradient flows and rate-independent evolutions in metric spaces*, Milan J. Math., 80 (2012), pp. 381–410.

[MRS13] A. Mielke, R. Rossi, and G. Savaré, *Nonsmooth analysis of doubly nonlinear evolution equations*, Calc. Var. Partial Differential Equations, 46 (2013), pp. 253–310.

[MRS16a] A. Mielke, R. Rossi, and G. Savaré, *Balanced viscosity (BV) solutions to infinite-dimensional rate-independent systems*, J. Eur. Math. Soc. (JEMS), 18 (2016), pp. 2107–2165.

[MRS16b] A. Mielke, R. Rossi, and G. Savaré, *Balanced-viscosity solutions for multi-rate systems*, J. Phys. Conf. Ser., (2016), 010210.

[MSZ] D. Melching, R. Scala, and J. Zeman, *Damage model for plastic materials at finite strains*, ZAMM Z. Angew. Math. Mech., 99 (2019), e201800032.

[MT99] A. Mielke and F. Theil, *A mathematical model for rate-independent phase transformations with hysteresis*, in Proceedings of the Workshop on Models of Continuum Mechanics in Analysis and Engineering, H.-D. Alber, R.M. Balean, and R. Farwig, eds., Shaker-Verlag, Aachen, 1999, pp. 117–129.

[MT04] A. Mielke and F. Theil, *On rate-independent hysteresis models*, NoDEA Nonlinear Differential Equations Appl., 11 (2004), pp. 151–189.

[Neg14] M. Negri, *Quasi-static rate-independent evolutions: Characterization, existence, approximation and application to fracture mechanics*, ESAIM Control Optim. Calc. Var., 20 (2014), pp. 983–1008.

[Neg19] M. Negri, *A unilateral L^2-gradient flow and its quasi-static limit in phase-field fracture by an alternate minimizing movement*, Adv. Calc. Var., 12 (2019), pp. 1–29.

[Per71] P. Perzyna, *Thermodynamic theory of Viscoplasticity*, Adv. Appl. Mech., 11 (1971), pp. 313–354.

[RS06] R. Rossi and G. Savaré, *Gradient flows of non convex functionals in Hilbert spaces and applications*, ESAIM Control Optim. Calc. Var., 12 (2006), pp. 564–614.

[RS13] R. Rossi and G. Savaré, *A characterization of energetic and BV solutions to one-dimensional rate-independent systems*, Discrete Contin. Dyn. Syst. Ser. S, 6 (2013), pp. 167–191.

[RT17] R. Rossi and M. Thomas, *Coupling rate-independent and rate-dependent processes: existence results*, SIAM J. Math. Anal., 49 (2017), pp. 1419–1494.

[RV16] T. Roubíček and J. Valdman, *Perfect plasticity with damage and healing at small strains, its modeling, analysis, and computer implementation*, SIAM J. Appl. Math., 76 (2016), pp. 314–340.

[RV17] T. Roubíček and J. Valdman, *Stress-driven solution to rate-independent elasto-plasticity with damage at small strains and its computer implementation*, Math. Mech. Solids, 22 (2017), pp. 1267–1287.

[Sim87] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl., 146 (1987), pp. 65–96.

[Sol09] F. Solombrino, *Quasistatic evolution problems for nonhomogeneous elastic plastic materials*, J. Convex Anal., 16 (2009), pp. 89–119.

[Sol14] F. Solombrino, *Quasistatic evolution in perfect plasticity for general heterogeneous materials*, Arch. Ration. Mech. Anal., 212 (2014), pp. 283–330.

[Tem83] R. Temam, *Problèmes mathématiques en plasticité*, Méthodes Mathématiques de l'Informatique 12, Gauthier-Villars, Montrouge, 1983.

[TS80] R. Temam and G. Strang, *Duality and relaxation in the variational problems of plasticity*, J. Mécanique, 19 (1980), pp. 493–527.

[Val90] M. Valadier, *Young measures*, in Methods of Nonconvex Analysis (Varenna, 1989), Springer, Berlin, 1990, pp. 152–188.