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S.I. : RECENT DEVELOPMENTS IN FINANCIAL MODELING AND RISK MANAGEMENT

Fair prices under a unified lattice approach for interest rate derivatives

Giacomo Morelli[1]

Abstract
An open question in interest rates derivative pricing is whether the price of the contracts should be computed by means of a multi-curve approach (different yield curves for discounting and forwarding) or by using a single curve (just one yield curve both for discounting and forwarding). The answer is of primary importance for financial markets as it allows to define a class of *fair* contracts. This paper calculates and compares the price of a simple swap within both multi-curve and single curve approaches and proposes a generalization of the lattice approach, which is usually used to approximate short interest rate models in the multi-curve framework. As an example, I show how to use the Black et al. (Financ Anal J 46(1):33–39, 1990) interest rate model on binomial lattice in multi-curve framework and calculate the price of the 2–8 period swaption with a single (LIBOR) curve and two-curve (OIS+LIBOR) approaches. Such technique can be used for pricing any interest rate based contract.

Keywords Interest rates · Single curve · Multiple curve · Derivative pricing · Fair contracts

JEL Classification C02 · C60 · C63

1 Introduction

For interest rate derivative pricing, 2007 crisis was a turning point. Before the financial crisis in 2007, the pricing of interest rate derivatives was a clear case of a framework that researchers agreed on [see Brigo and Mercurio (2006)]: a single risk free yield curve was used, reflecting at present time the cost of future cash flows (known as discounting yield curve) as well as

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the level of forward rates (known as forward yield curve). The crisis affected interest rate derivatives valuation leading to the introduction of multiple curve for interest rate derivative pricing, i.e., the use of different yield curves for discounting and forwarding.

This paper aims to close the dilemma about the computation of the fair price for interest rate derivatives: single or multi-curve approach ? It derives a price for interest rate derivatives which is *fair* in that whatever the complexity of the contract, the parties involved obtain an equitative exchange of the cash flows. Moreover, it generalizes the multi-curve approach for lattice models. I consider the Black et al. (1990) interest rate model on binomial lattice and calculate the price of the 2–8 period swaption in a single (LIBOR) curve and two-curve (OIS+LIBOR) approaches using 5 years historical data taken from the Bank of England. Such generalization can be used for pricing any type of interest rates contract.

The effects of the financial crisis in the private and public sectors spread out in the world-wide economy through the financial markets. Inevitably, companies and banks have struggled to meet the liquidity and credit requirement that are highly related to their ability of trading financial products [see Cont and Minca (2016)] The interest rate derivatives market is the largest derivatives market in the world. The Bank for International Settlements estimates that the notional amount outstanding in June 2012[1] were US $ 494 trillion for OTC interest rate contracts and US $342 trillion for OTC interest rate swaps. According to the International Swaps and Derivatives Association, 80% of the world's top 500 companies as of April 2003 used interest rate derivatives[2] to control their cash flows. This compares with 75% for foreign exchange options, 25% for commodity options, and 10% for stock options.

In the single-curve framework, a single yield curve was used for both discounting and forwarding. That is, the same instruments were used to derive all the curves: the discount curve, the spot curve, the forward curve. For instance, either LIBOR or EURIBOR rates were used for both discounting and forwarding future cash flows. Since August 2007, primary interest rates of the interbank market such as LIBOR, EURIBOR, EONIA, and Federal Funds rates started displaying large basis spreads that raised up to 200 basis points. Similar divergences were also found among swap rates with different floating leg tenors[3]. Such trends are captured in Fig. 1. The financial community has thus been pushed to develop a new theoretical framework including a larger set of relevant risk factors into the models used for derivatives' pricing and risk analysis. A single yield curve was no longer adequate to derive discount factors and forward rates as it emerged a need to account for the increasing credit and liquidity risks as well as for richer dynamics than before for instruments with different maturities [see Mercurio (2008) and Bianchetti and Carlicchi (2011)]. One of the solutions put into practice was the use of multiple separated yield curves for discounting and forwarding. Within this a new framework, rates such as LIBOR or EURIBOR are used for forwarding but these rates are no longer considered as a proxy of the risk free rate.[4] A very recent discussion is then about eliminating LIBOR rates both in Europe and in the U.S. in 2021. Andrew Bailey, chief executive of the Financial Conduct Authority, made the announcement after admitting banks no longer want to participate in setting the rate. The discussion is still open and substitute rates are in the process of being proposed. However, at

[1] The statistic derives from the Semiannual OTC derivative statistics of the Bank for International Settlements.

[2] The most popular vanilla interest rate derivatives are interest rate swaps (fixed-for-floating), cross currency swaps, interest rate caps, swaptions (options on swaps), bond options, forward rate agreements, interest rate futures.

[3] This is called the swap coupon frequency.

[4] Indeed, the best current proxy of the risk free rate is the Overnight Indexed Swap (OIS) Rate—EONIA in the Euro-zone—which is a unique discounting curve for all tenor forward curves commonly used to discount collateralized derivative deals.

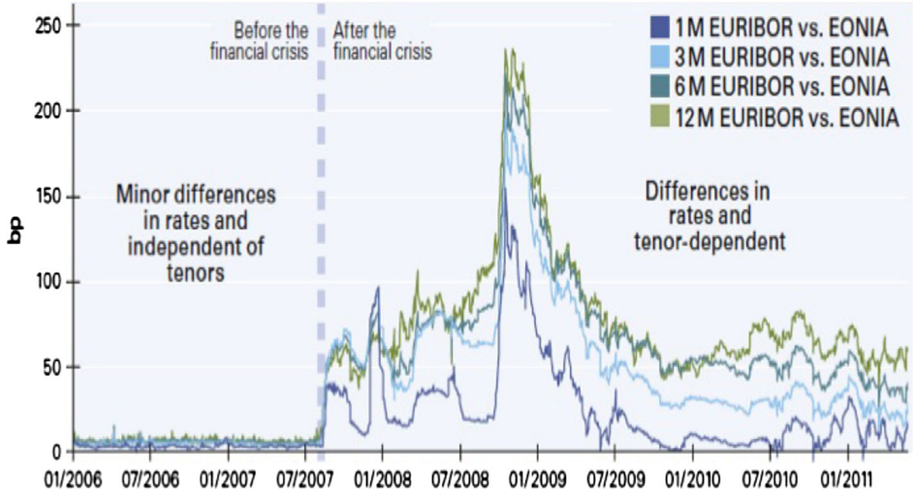

Fig. 1 Money market rates before and after the financial crisis

present time the London interbank offered rate underpins the financial system and is used to price more than $ 350tn of financial products around the world.

Multi-curve models for pricing and valuation represent a general term for the description of circumstances in which discount curves and forward curves differ. There are several reasons for this. First, price discovery in financial markets in the course of the financial crisis takes into account differences in tenors which result in different tenor specific interest rates. This difference is also known as “tenor basis spread”. Second, changes in market conventions and institutional changes within financial markets drive the implementation of so called “OIS-discounting” for collateralized trades. During the financial crisis the spreads between interest rates of different tenors widened and have since that time remained on a significant level. Consequently the dependence of interest rates on tenors for the valuation of derivatives became significant. Also, multiple curve approach tries to eliminate the problem of pricing and hedging plain vanilla single-currency interest rate derivatives using multiple distinct yield curves for market coherent estimation of discount factors and forward rates with different underlying rate tenors: longer tenors are riskier and forwards related to longer tenors should be priced higher. Indeed, a portfolio of interest rate derivatives with different underlying tenors requires thus separate forward curves. Multi-curve approaches for pricing derivatives, especially interest rate derivatives, are currently in the process of being implemented by banks in order to adapt to changes in market practice and this paper lies the foundations of merging the multi-curve framework with a lattice approach in a simple way.

This paper contributes to two strands of literature about the main developments in the theory and computational aspects of pricing interest rate swaps (IRS)[5].

The first literature is that of the methodology proposed in pricing IRS. Before the financial crisis, the pricing of interest rate swaps was solid and agreed phenomenon and the literature on this issue solely focused on using the single curve for the both legs (fixed and floating), and bootstrapping was used for the yield curve construction [see i.e., Ron (2000), Boenkost and Schmidt (1991), Brigo and Mercurio (2006) and Hull (2009)]. The main issue has came

[5] IRS is a contract in which two counterparties agree to exchange interest payments of different character based on an underlying notional principle amount that is not exchanged.

up with the crises. The increasing spread between different yield curves called for a new method to make the pricing of the interest rate derivatives more reliable and precise. Brigo (2008) introduce a new LIBOR market model based on modeling the joint evolution of FRA rates and forward rates belonging to the discount curve and obtain the dynamics of FRA rates under different measures in closed form formulas for caplets and swaptions derived in the log-normal and Heston (1993) cases. A first idea for the multi-curve framework is in the discussions on the effects of changing the discounting curve by Henrard (2010) who proposes a coherent valuation framework for derivatives based on different LIBOR tenors still using the traditional bootstrapping technique, though assuming the discounting curve as given. Ametrano and Bianchetti (2009) proposes a bootstrap market segmentation and swap rates within each tenor separately, making the model subject to arbitrage. An extended version of this model to avoid arbitrage opportunities is in Bianchetti (2008). Similar approaches have been used by Chibane and Sheldon (2009) and Kijima et al. (2009). Mercurio (2008) constructs a LIBOR market model using the the joint evolution of forward rates (FRA) and implied forward rates. Johannes and Sundaresan (2007) and Whittal (2010) extend the multi-curve framework by taking the effect of collateralization into account. The works of Fujii et al. (2011) and Fujii et al. (2010) provide a new framework to build a coherent term structure in the presence of differential base and provides a multi-currency environment in a framework where forward rates are restored to incorporate the effect of the differential base, instead of building a yield curve by bootstrapping from different instruments of the liquid market. Bianchetti (2010) and Bianchetti and Carlicchi (2011) focus on the fixed income market and analyzes the most relevant empirical evidences regarding the divergence between LIBOR and OIS rates, the explosion of Basis Swaps spreads, and the diffusion of collateral agreements and CSA-discounting in terms of credit and liquidity effects. Recently, Karouzakis et al. (2018) propose a convexity adjustment for constant maturity swaps in a multi-curve framework.

The second literature this paper belongs is that of the theoretical models used for describing the evolution of interest rates. and the behavior of the interest rate derivatives. One of the frameworks is the Heath et al. (1990) (HJM) that considers the evolution of the instantaneous forward rate. Term-structure models are essential for the valuation of interest rate dependent claims. Although term-structure experts have produced a variety of useful models, they involve complex mathematics, which limits their accessibility to investment practitioners who are not engaged in this area of specialization. For a general overview of the implementation of term structure lattice models we refer to Sochacki and Buetow (2001). The other major framework looks at the evolution of the short rate (instantaneous spot rate). Ho and Lee (1986) is the first no-aribtrage model. Hull and White (1990) extend the Ho and Lee (1986) model to allow for mean reversion, Kalotay et al. (1993) assume a log-normal distribution that and eliminates the problem of negative short rates, Black and Karasinski (1991) is an extension of the Kalotay et al. (1993) model that controls the growth in the short rate, and Black et al. (1990) permits independent and time-varying spot-rate volatilities. Short rates models can be divided into two groups. One group is that of the factor short rate models. In this group, a single stochastic factor—the short rate—determines the future evolution of all interest rates. The other group is constituted by the multi-factor models, where other stochastic factors are present along with the short rate. I consider one factor short rate models.

The paper is organized as follows. The models and the hypothesis are introduced in Sect. 2. Section 3 presents the contribution. The implementation and the results are discussed in Sect. 4. Section 5 concludes.

2 Theoretical models and hypothesis

I briefly recall notions of interest rates, such as spot rate, instantaneous spot rate, forward rate, instantaneous forward rate and also difference between their simply and continuously compounding which will be useful later.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(W_t)_{t\geq 0}$ a one dimensional Brownian motion and $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the filtration to which W_t is adapted. Denote $P(t, T)$ the price of zero coupon bond in t with maturity at T. The simply-compounded spot interest rate is defined by

$$L(t, T) = \frac{1 - P(t, T)}{\tau(t, T)P(t, T)}, \tag{1}$$

where $\tau(t, T)$ is the year fraction between time t and T. A simply compounded investment of one currency unit at time t will grow by the amount $(1 + \frac{L(t;T)}{m}^m)$ in m periods of time during a year. As $m \to \infty$ this quantity converges. This way one receives continuously compounded spot rate $L_c(t, T) = -\ln \frac{P(t,T)}{\tau(t,T)}$. The instantaneous sport rate (or short rate) is defined as

$$r_s(t) = \lim_{T\to t+} L_c(t, T). \tag{2}$$

This is a spot rate between two infinitesimally small time periods, which is why such a continuously compounded rate is used in the stochastic differential equations (SDE) of short interest rate models. For $\{t < T < S\}$, the simply compounded forward interest rate with maturity S and expiry T is given by

$$F(t, T, S) = \frac{1}{\tau(T, S)} \left(\frac{P(t, T)}{P(t, S)} - 1 \right). \tag{3}$$

Continuously compounded forward rate is given by $F_c(t, T, S) = -\frac{\ln P(t,S) - \ln P(t,T)}{\tau(T,S)}$. The instantaneous forward interest rate can be defined using similar considerations as for the spot rate,

$$r_f(t) = \lim_{S\to T+} F_c(t, T, S) = -\frac{\partial \log P(t, T)}{\partial T} \tag{4}$$

2.1 Swap

Among several types of swaps,[6] I consider coupon swaps only in which case the cash flows are made up of two legs. The fixed leg (fixed rate interest rate payments) is the stream of cash flows made by a swap buyer whereas the floating leg (floating rate interest rate payments) is the stream of cash flows made by a swap seller. Let N be a notional principle amount, K the a fixed interest rate and $F(T^F_{j-1}, T^F_j)$ a forward interest rate defined by (3). The swap fixed leg pays at each time $\{T^K_1, \ldots, T^K_m\}$ and the swap floating leg pays at each time $\{T^F_1, \ldots, T^F_n\}$. Both fixed and floating legs terminate simultaneously, so $T^K_m = T^F_n$. Without loss of generality, I consider no upfront payment to enter a swap agreement. This way, the fixed rate of the swap can be found using no-arbitrage consideration and a condition that the

[6] Coupon swaps exchange fixed rate for floating rate instruments in the same currency, Basis swaps exchange of floating rate for floating rate instruments in the same currency, Cross Currency interest rate swaps exchange of fixed rate instruments in one currency for floating rate in another.

present value of the fixed leg

$$PV_{fixed} = N \sum_{j=1}^{m} \tau(T_{j-1}^K, T_j^K) K P(0, T_j^K) \tag{5}$$

should equal the present value of the the floating leg

$$PV_{floating} = N \sum_{j=1}^{n} \tau(T_{j-1}^F, T_j^F) F(T_{j-1}^F, T_j^F) P(0, T_j^F). \tag{6}$$

The non-arbitrage expression is then given by

$$K = \frac{\sum_{j=1}^{m} \tau(T_{j-1}^F, T_j^F) F(T_{j-1}^F, T_j^F) P(0, T_j^F)}{\sum_{j=1}^{m} \tau(T_{j-1}^K, T_j^K) P(0, T_j^K)}. \tag{7}$$

Which forward curve and discount curve should be used? A multi-curve framework considers different yield curves for discounting and forwarding. For example, LIBOR/EURIBOR can be used for forwarding and EONIA for discounting. Following Mercurio (2008) and Bianchetti and Carlicchi (2011) both discounting and forwarding yield curves should be obtained from vanilla interest rate instruments with homogeneous corresponding underlying rate tenors. In this case, T^K for discounting curve and T^F for forward curve.

In a single-curve approach, when only one yield curve is used (i.e., LIBOR or EURIBOR) for both discounting and forwarding, (7) reduces to

$$K_{single} = \frac{1 - P(0, T_j^K)}{\sum_{j=1}^{m} \tau(T_{j-1}^K, T_j^K) P(0, T_j^K)}. \tag{8}$$

2.2 Swaption

The plain-vanilla option in the interest rate market is the European swaption. A swaption gives the right for a buyer to enter at time $T_a^F = T_c^K$ an IRS with payment times for the floating and fixed legs given, respectively, by $\{T_{a+1}^F, \ldots, T_n^F\}$ and $\{T_{c+1}^K, \ldots, T_m^K\}$, with $T_n^F = T_m^K$ and K fixed rate. In a multi-curve framework, the swaption payoff at the time $T_a^F = T_c^K$ is[7]

$$\text{Payoff} = (K_{a,n,c,m}(T_a^F) - K)^+ \sum_{j=c+1}^{m} \tau(T_c^K, T_j^K) P(T_c^K, T_j^K), \tag{9}$$

where $K_{a,n,c,m}(T_a^F)$ is the forward swap rate, derived in (7)[8]

$$K_{a,n,c,m}(t) = \frac{\sum_{j=a+1}^{n} \tau(T_{j-1}^F, T_j^F) F(t; T_{j-1}^F, T_j^F) P(t, T_j^F)}{\sum_{j=c+1}^{m} \tau(T_{j-1}^K, T_j^K) P(t, T_j^K)}. \tag{10}$$

The payoff (9) can be priced under the risk neutral measure $\mathbb{Q}$ whose associated numeraire is the annuity $\sum_j = c + 1^m \tau(T_{j-1}^K, T_j^K) P(t, T_j^K)$. I get

7 We follow the Mercurio (2008).

8 A different notation has been used to stress the fact that $K_{a,n,c,m}$ is the rate computed in (7) whereas K is the rate agreed in the European swaption contract.

$$\begin{aligned}&\mathrm{Payoff}(t; K; T^F_{a+1}, \ldots, T^F_n, T^K_{c+1}, \ldots, T^K_m)\\&= \sum_{j=c+1}^{m} \tau(T^K_{j-1}, T^K_j) P(t, T^K_j) \mathbb{E}^{\mathbb{Q}}\left[(K_{a,n,c,m}(T^F_a) - K)^+ | \mathcal{F}_t\right].\end{aligned} \tag{11}$$

In the multi-curve approach, the forward swap rate (10) depends on all yield curves and correspondingly has very complicated dynamics. Assuming that, under $\mathbb{Q}$, the volatility evolves according to a driftless geometric Brownian motion,

$$dK_{a,n,c,m}(t) = \sigma_{a,n,c,m} K_{a,n,c,m}(t) dW(t), \tag{12}$$

the payoff (11) can be explicitly calculated, as a generalized Black-Scholes formula

$$\begin{aligned}&PS(t; K; T^F_{a+1}, \ldots, T^F_n, T^K_{c+1}, \ldots, T^K_m)\\&= \sum_{j=c+1}^{m} \tau(T^K_{j-1}, T^K_j) P(t, T^K_j) Bl\left(K, K_{a,n,c,m}, \sigma_{a,n,c,m}\sqrt{T^F_a - t}, w\right),\end{aligned} \tag{13}$$

where the Black-Scholes function is defined as

$$\begin{aligned}Bl(K, S, \sigma, w) &= wS\Phi(wd_1) - Kw\Phi(wd_2),\\ d_1 &= \frac{0.5\sigma^2 - \ln(K/S)}{\sigma}, \qquad d_2 = \frac{-0.5\sigma^2 - \ln(K/S)}{\sigma}.\end{aligned}$$

The problem of (13) is in deriving and justifying the yield curve for the forward swap rate (10), which is not always feasible as it depends on both discount and forward yield curves.

3 Towards a unified approach

I propose to describe separately the discount and forward yield curves in the framework of one of the short rate models. Generally, one can choose different short rate models for the discount and forward interest rates. Then, an approximation of continuous time models on discrete lattice allows to calculate interest rate trees—binomial or trinomial—for both discount and forward rates, and calculate the cash flows tree using both forward and discount interest rate trees.

Interest rates changes through time according to a stochastic behaviour[9] and are usually modelled by stochastic differential equations (SDE). The most general form of SDE for one factor short rate model is the following

$$df(r_t) = (\theta_t + \rho_t g(r_t))dt + \sigma(r_t, t)dW_t, \tag{14}$$

where f and g are suitably chosen functions, $\theta \in \mathbb{R}$ is the drift of the short rate determined by the market $\rho \in \mathbb{R}$ is the 'tendency to an equilibrium short rate' (mean reversion rate) that can be chosen by the user of the model or inferred by the market and σ is the local volatility of the short rate. Specifically, I consider the Black et al. (1990) (BDT). The equation describing the interest rate dynamics in the BDT model has $f(r) = g(r) = \ln(r)$ and $\rho(t) = \frac{d\ln(\sigma(t))}{dt} = \frac{\sigma'_t}{\sigma_t}$. That is, the short rate in the BDT model follows the lognormal process

$$d\ln r_t = \left(\theta_t + \frac{\sigma'_t}{\sigma_t}\ln r_t\right)dt + \sigma_t dW_t. \tag{15}$$

[9] For example, see Tse (1995) and Ball and Torous (1999).

In order to use this model for empirical applications it can be approximated to the discrete space of binomial tree. Clewlow and Strickland (1998) show that for each step i in the binomial tree, (15) gives an expression for the short rate at each node in the tree. That is,

$$r_{i,j} = a_i e^{\sigma_{i,j}\sqrt{\Delta t}}, \tag{16}$$

where j numerate different possible states for every step i, $\ln(a_i)$ is a drift parameter and can be found from the calibration to the observed term-structure of corresponding market spot rates whereas $\sigma_{i,j}$ is a volatility of the forward rate and can be found using historical data (historical volatility) or can be estimated from the current security's prices (implied volatility).

Following Arrow and Debreu (1954), an elementary security is defined as a security that pays 1 at time i and state j and 0 at every other time and state. Denoting by $P_e(i, j)$ its price at time 0, then the elementary security satisfies the forward equations

$$\begin{aligned} P_e(k+1, s) &= \frac{P_e(k, s-1)}{2(1+r_{k,s-1}} + \frac{P_e(k, s)}{2(1+r_{k,s})}, \quad 0 < s < k+1, \\ P_e(k+1, 0)) &= \frac{P_e(k, 0)}{2(1+r_{k,0})}, \\ P_e(k+1, k+1)) &= \frac{P_e(k, k)}{2(1+r_{k,k})}. \end{aligned}$$

In order to calibrate the BDT model to the observed term-structure, consider an n-period lattice and let $(s_1, \ldots, s_n)$ be the term-structure observed in the market, assuming that spot rates are compounded per period. Then,

$$\frac{1}{(1+s_i)^i} = \sum_{j=0}^{i} P_e(i, j). \tag{17}$$

Equation 17 simply says that today (time 0) market price of a zero coupon bond, which pays 1 at period i, should be equal to its price calculated by using the binomial tree. Using the forward equations, the right-hand side of (17) is

$$\begin{aligned} \frac{1}{(1+s_i)^i} &= \frac{P_e(i-1, 0)}{2(1+a_{i-1})} + \sum_{j=0}^{i-1} \left(\frac{P_e(i-1, j)}{2(1+a_{i-1}e^{b_{i-1}j})} + \frac{P_e(i-1, j-1)}{2(1+a_{i-1}e^{b_{i-1}(j-1)})} \right) \\ &\quad + \frac{P_e(i-1, j-1)}{2(1+a_{i-1}e^{b_{i-1}(j-1)})}. \end{aligned} \tag{18}$$

Equation (18) is used to solve iteratively for the a_i's.

4 Results and discussion

4.1 Single versus multi-curve

I use the data available on the Bank of England website for continuously compounded sterling overnight index swap (OIS) spot rates, which is available out to 5 years horizon, and sterling interbank LIBOR spot rates, which is available out to 25 years horizon. I find the prices of

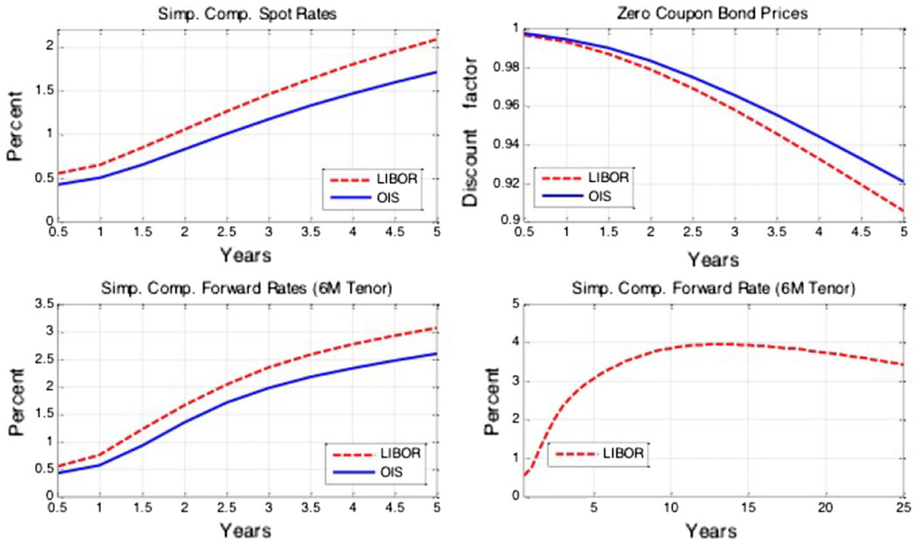

Fig. 2 Calibration of simply compounded spot rates, forward rates, and zero coupon bond prices computed with LIBOR up to 5 years horizon and with OIS up to 25 years horizon starting from June 1st 2014

zero coupon bonds (discounting curves), constructed simply compounded spot curves [using (1)], and simply compounded forward curves [using (3)] with 6 month tenor for both OIS rate and LIBOR. The OIS rate is currently considered as a proxy for a risk free rate and Fig. 2 shows that both spot and forward OIS yield curves lies below the corresponding LIBOR yield curves meaning that LIBOR rate is higher and correspondingly riskier. This fact is reflected in the price of corresponding zero coupon bonds: those that correspond to LIBOR have lower price, ensuring higher rate of return. It can also be seen that the spread between LIBOR and OIS rate increases for both spot and forward rates, meaning that the increase of the risk is when the maturity increases. For a simply compounded sport rate, such a spread increases from 0.12% (12 bp) for 0.5 year maturity to 0.37% (37 bp) for 5 year maturity whereas for a simply compounded forward rate with 6 month tenor, this spread increases from 0.12% (12 bp) for 0.5 year maturity to 0.48% (48 bp) for 5 year maturity. Another interesting feature is the shape of the forward rate on a 25 year horizon. The forward rate increases up to some point (12–13 year maturity) and then starts decreasing. A possible economic explanation is that the demand for long term maturity securities is high in the market, which leads an increase in their prices and correspondingly a decrease in the interest rate. This may be a signal that investors try to invest into long term securities possibly expecting some bad news in short and medium terms.

The following example shows clear evidence of the dilemma about pricing with single or multi-curve approach. I consider a swap with the maturity of 5 years and both fixed and floating rate (LIBOR) payments made semiannually (6M tenor). Within the single-curve approach[10] I find that the swap fixed rate is equal to 1.97% and within the multi-curve approach[11] it equals 1.98%. The two approaches give a difference of 0.01% (1 bp), which is significant.

[10] We use LIBOR for both discounting and forwarding.

[11] We use OIS rate for discounting and LIBOR for forwarding.

4.2 Towards a uniform approach

I propose a generalization of the lattice approach which is used to approximate the short interest rate models for multi-curve framework. The main idea of such generalization is to use different binomial/trinomial interest rate trees for forwarding and discounting. The procedure goes as follows. First, choose the short rate models which is use to describe the dynamics of the forward and discount interest rates. Then, using observed market term-structure, calibrate the model parameters and construct the binomial/trinomial trees separately for discounting and forwarding interest rates. Finally, using separate trees, calculate the cash flows to finish the task of the interest rate derivative valuation.

I show how the procedure works by calculating the price of the 2–8 payer swaption, an option that expires in 2 periods (1 year in the case considered) to enter an 8-period swap with both fixed and floating legs paying semiannually in period 3 through period 10. Floating payment are based on the prevailing LIBOR rate of the previous months. For this particular problem, consider the semi-annual fixed rate[12] which is set at 1% (corresponding to annual fixed rate of 2%), close to what was found in Sect. 4.1. I choose the BDT model as it allows for non-negative interest rates and short rate volatility through time.

The Macro Financial Analysis Division of the Bank of England estimates three kinds of yield curves for the UK on a daily basis. The first is based on yields on UK government bonds (gilts), the second is based on sterling interbank rates (LIBOR) and the last one is based on yields on instruments linked to LIBOR and based on sterling overnight index swap (OIS) rates which are instruments that settle on overnight unsecured interest rates (the SONIA rate in the UK).

The main steps to calibrate the BDT model are presented in order to price the interest rate derivatives.(18) is used to solve iteratively for the a_i's as follows:

- Set $i = 1$ in (18) and note that $P_e(0; 0) = 1$ to see that $a_0 = s_1$.
- Use the forward equations to find $P_e(1; 0)$ and $P_e(1; 1)$
- Set $i = 2$ in (18) and solve for Set a_i
- Continue to iterate forward until all a_i's have been found

Binomial trees for the LIBOR and OIS rates

In order to construct the binomial trees, I convert continuously compounded LIBOR and OIS rate, which are taken from the Bank of England, to semi-annual simply compounded interest rates and then calibrate the BDT model to match the market interest rates. I use past 5 years historical volatility from the data provided by the bank of England in the calibration of binomial tree. Specifically, considering LIBOR the procedure becomes:

- Set $a_0 = s_1 = 0.0028$
- Use the forward equations to find $P_e(1; 0) = P_e(1; 1) = 0.4986$
- Set $i = 2$ in (18) and solve for a_2
- Continue to iterate forward until all a_i's have been found

Using (16) I find the binomial trees for both LIBOR and OIS rate.

Cash flow trees in single curve and multi-curve approaches

In order to construct the cash-flow tree for the swaption in a multi-curve approach one needs both LIBOR and OIS interest rate trees constructed above. I use LIBOR interest rate tree to compare fixed rate with the floating rate for every period. Due to the difference in fixed

[12] I consider LIBOR rate.

LIBOR

									0.0161
								0.0154	0.0160
							0.0146	0.0154	0.0160
						0.0136	0.0146	0.0153	0.0159
					0.0124	0.0136	0.0145	0.0152	0.0158
				0.0108	0.0123	0.0136	0.0145	0.0152	0.0158
			0.0088	0.0107	0.0123	0.0135	0.0144	0.0151	0.0157
		0.0065	0.0088	0.0107	0.0123	0.0135	0.0144	0.0151	0.0156
	0.0040	0.0065	0.0087	0.0107	0.0122	0.0134	0.0143	0.0150	0.0156
0.0028	0.0040	0.0065	0.0087	0.0107	0.0122	0.0134	0.0143	0.0150	0.0155

Fig. 3 2–8 Swaption price under the LIBOR rate Binomial Tree

OIS

									0.0136
								0.0130	0.0135
							0.0124	0.0130	0.0135
						0.0115	0.0123	0.0129	0.0134
					0.0104	0.0115	0.0123	0.0129	0.0134
				0.0089	0.0104	0.0114	0.0122	0.0128	0.0133
			0.0071	0.0089	0.0103	0.0114	0.0122	0.0128	0.0133
		0.0059	0.0071	0.0089	0.0103	0.0114	0.0122	0.0128	0.0132
	0.0020	0.0059	0.0071	0.0089	0.0103	0.0113	0.0121	0.0127	0.0132
0.0022	0.0020	0.0059	0.0070	0.0089	0.0103	0.0113	0.0121	0.0127	0.0131

Fig. 4 2–8 Swaption price under the OIS rate Binomial Tree

and floating rates the present value of the cash-flows is discounted and for this purpose I use OIS interest rate tree. Also, recall that at every node interest rate can develop into one of two equally likely possible (binomial) states.

The graphs below represent the 2–8 swaption price within different settings.The final current price of the swaption at $t = 0$ is reported in the first columns of the following binomial lattices and is computed as the sum of all possible, properly discounted and weighted future cash-flows, within both single curve[13] and multi-curve frameworks for a notional swaption amount of a one unit. In particular, Figs. 3 and 4 compare the 2–8 swaption price computed by using LIBOR rate and OIS rate. A comparison between the single and multi curve approach is reported in Figs. 5 and 6 showing that the swaption price in a multiple curve method, which is 0.0027, is higher than the one in the single curve framework, which is 0.0025. The difference can be explained by the fact that OIS rate is lower comparing to LIBOR, in turn leading to a lower discounting.

5 Conclusion

This paper studied the influence of the modern multi-curve approach on the interest rate derivative pricing by calculating and comparing the price of a simple swap in both multi-curve and single curve approaches. I found that the pricing of IRS within the two approaches leads to a difference of 1 basis point. Closing the dilemma about which curve to use for

[13] I used LIBOR tree for both discounting and forwarding.

Single curve

									0.0060
								0.0112	0.0059
							0.0155	0.0111	0.0059
						0.0188	0.0153	0.0110	0.0058
					0.0208	0.0186	0.0152	0.0108	0.0057
				0.0212	0.0206	0.0184	0.0150	0.0107	0.0057
			0.0197	0.0210	0.0204	0.0182	0.0149	0.0106	0.0056
		0.0160	0.0194	0.0207	0.0201	0.0180	0.0147	0.0105	0.0056
	0.0098	0.0157	0.0192	0.0205	0.0199	0.0178	0.0145	0.0104	0.0055
0.0025	0.0095	0.0154	0.0189	0.0202	0.0197	0.0176	0.0144	0.0103	0.0054

Fig. 5 2–8 Swaption price in a single-curve framework

Multiple curve

									0.0060
								0.0112	0.0059
							0.0156	0.0111	0.0059
						0.0189	0.0154	0.0110	0.0058
					0.0210	0.0187	0.0153	0.0109	0.0057
				0.0214	0.0207	0.0185	0.0151	0.0108	0.0057
			0.0199	0.0212	0.0205	0.0183	0.0149	0.0106	0.0056
		0.0162	0.0197	0.0209	0.0203	0.0181	0.0148	0.0105	0.0056
	0.0101	0.0159	0.0194	0.0206	0.0200	0.0179	0.0146	0.0104	0.0055
0.0027	0.0098	0.0156	0.0191	0.0204	0.0198	0.0177	0.0144	0.0103	0.0054

Fig. 6 2–8 Swaption price in a multiple-curve framework

pricing interest rates derivatives, I proposed a generalization of the lattice approach, which is used to approximate the short interest rate models, in a multi-curve framework. This way fair contracts arise. I showed how to use the Black et al. (1990) interest rates model on binomial lattice in multi-curve framework and calculated the price of the 2-8 period swaption in a single (LIBOR) curve and two-curve (OIS+LIBOR) approaches. This technique can be used to give fair pricing of any interest rate instruments. The proposals of eliminating LIBOR rates both in Euorpe and in the U.S. raise the question to investigate how the example presented will look like when using one of the new risk-free rates that will replace the LIBOR rate. That will be addressed in a future project.

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