# General conditioned and aimed information on fuzzy setting

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*Abstract:* In this paper our investigation on aimed information, started in 2011, will be completed on fuzzy setting. Here will be given a form of information for fuzzy sets, when it is conditioned and aimed. This information is called *general,*because it is defined without using probability or fuzzy measure.

*Key–Words: Fuzzy sets, information, conditioning information, aimed information*

## 1 Introduction

By using the concept of general information (i.e. information without probability or fuzzy measure [7, 8, 5]), the definition of conditional information [10, 11] and aimed information [12] have been introduced for crisp sets.

It is possible to move to fuzzy setting. In fact the goal of this paper is to introduce a form of general information $J$ conditioned and aimed by two different sets, independent of each other with respect to $J$ ($J-$independence).

This measure can be useful when we want to measure information of a set of people with different levels of the same illness, treated with different dose of a medicament.

The paper is organized in the following way. Sect.2 contains some preliminaries. In Sect.3 in fuzzy setting will be introduced the definition of general conditional information with a given aim, by means of axioms. The properties of this information are traslated in a system of functional equations [1, 2]. In Sect.4 the problem is solved, finding a class of solutions and a particular solution in $J-$independent case. Sect.5 is devoted to the conclusion.

## 2 Preliminaires

Let $X$ be an abstact space and $\mathcal{F}$ the $\sigma-$algebra of all fuzzy sets of $X$, such that $(X, \mathcal{F})$ is measurable. Basic notions, notations and operation on fuzzy sets can be found in [14, 9]. Now, the definition of measure of general information for fuzzy sets is recalled [6].

**Definition 1** *Measure of general information $J(\cdot)$ is a mapping $J(\cdot) : \mathcal{F} \to [0, +\infty]$ such that $\forall F, F' \in \mathcal{F}$ :*

$(i) F \supset F' \to J(F) \leq J(F')$,

$(ii) J(\emptyset) = +\infty, \quad J(X) = 0.$

Given a measure of general information $J$ and $K, K' \in \mathcal{F}$ with $K \neq K'$, $K \cap K' \neq \emptyset$, $K$ and $K'$ are said $J-$ independent (i.e. independent of each other with respect to $J$) if

$(iii)$ $J(K \cap K') = J(K) + J(K')$.

## 3 Statement of the problem

In this paragraph will be introduced measure of general information when it is conditioned by a given event $H$ and it is aimed by a different event $S$ . From now on, the following assumption is considered:

$$\text{let } H, S \in \mathcal{F}, \mathrm{H} \neq \mathrm{S}, \tag{1}$$

$$J(H) \neq +\infty, J(S) \neq +\infty,$$

$H$ and $S$ are calling *conditioning* and *aiming* events, respectively. Now, given a conditioning and aiming sets as in (1), it is introduced the definition of general information of the set $F \in \mathcal{F}$ conditioned by $H$ with the aim $S$: this information will be denoted by $J_H(F \to S)$.

**Definition 2** *Given $H$ and $S$ as in (1), measure of general information conditioned by $H$ with the aim $S$ is a mapping*

$$J_H(\cdot \to S) : \mathcal{F} \to [0, +\infty]$$

*such that* $\forall F, F' \in \mathcal{F}$ :

$(l)\ F \supset F' \to J_H(F \to S) \le J_H(F' \to S)$,
$(ll)\ J_H(\emptyset \to S) = +\infty, \quad J_H(X \to S) = 0.$

Given a measure $J_H(\cdot \to S)$ as in Def.3, $K, K' \in \mathcal{F}$ with $K \neq K', K \cap K' \neq \emptyset$, $K$ and $K'$ are said $J-$ *conditional independent with the aim* $S$ (i.e. independent of each other with respect to $J$ conditioned by $H$ with the aim $S$) if

$$(lll)\ J_H((K \cap K') \to S) =$$

$$J_H(K \to S) + J_H(K' \to S).$$

### 3.1 The function $\Phi$

With the assumption (1), our study considers that measure $J_H(\cdot \to S)$ of $F \in \mathcal{F}$ depends on $J(F), J(H), J(S), J(F \cap H), J(F \cap S)$. So, one will find a function $\Phi$ such that:

$$J_H(F \to S) = \tag{2}$$

$$\Phi\left(J(F), J(H), J(S), J(F \cap H), J(F \cap S)\right),$$

with $\Phi : T \to [0, +\infty]$ and $T$ will be specified later. Putting: $x = J(F), y = J(H), z = J(S), u = J(F \cap H), v = J(F \cap S)$, with $x, u, v \in [0, +\infty], y, z \in [0, +\infty), x \le u, y \le u, x \le v, z \le v$, from (2) it is

$$J_H(F \to S) = \Phi\left(x, y, z, u, v\right) \tag{3}$$

and $T = \{(x, y, z, u, v)/x, u, v \in [0, +\infty], y, z \in [0, +\infty), x \le u, y \le u, x \le v, z \le v\}$.

Moreover, setting $x' = J(F'), u' = J(F' \cap H), v' = J(F' \cap S)$, with $x', u', v' \in [0, +\infty], x' \le u', x' \le v'$, the properties $[(l)-(ll)]$ of $J_H(\cdot \to S)$ are traslated in the following system of functional equations:

$$\begin{cases} (e_1)\ \ \Phi(x, y, z, u, v) \le \Phi(x', y, z, u', v') \\ \ \text{if} \quad x \le x', u \le u', v \le v'\ , \\ (e_2)\ \ \Phi(+\infty, y, z, +\infty, +\infty) = +\infty\ , \\ (e_3)\ \ \Phi(0, y, z, y, z) = 0\ . \end{cases}$$

# 4 Solution of the problem

## 4.1 General case

For the system $[(e_1) - (e_3)]$ it is

**Proposition 3** *A class of solution of the system* $[(e_1) - (e_3)]$ *is*

$$\Phi_h(x, y, z, u, v) = \tag{4}$$

$$h^{-1}\left(h(x) - h(y) - h(z) + h(u) + h(v)\right)$$

*where h is any continuous, strictly increasing function* $h : [0, +\infty] \to [0, +\infty]$ *with* $h(0) = 0, h(+\infty) = +\infty$.

**Proof:** The prof follows easily from the properties of the function $h$.

From (3) and (4), given $H$ and $S$ as in (1), measure of general information of any fuzzy set $F$ conditioned by $H$ with the aim $S$ is

$$J_H(F \to S) = h^{-1}\Big(h(J(F)) - h(J(H)) - \tag{5}$$

$$h(J(S)) + h(J(F \cap H) + h(J(F \cap S)\Big)$$

where $h$ is any continuous, strictly increasing function $h : [0, +\infty] \to [0, +\infty]$ with $h(0) = 0, h(+\infty) = +\infty$.

## 4.2 $J-$independence

In the case of $J-$independence the system $[(e_1) - (e_3)]$ must be completed with an extra equation deduced by the property $(lll)$ :

$$(e_4)\ \ \Phi\left(t + t', y, z, t + t' + y, t + t' + z\right) =$$

$$\Phi\left(t, y, z, t + y, t + z\right) + \Phi\left(t', y, z, t' + y, t' + z\right),$$

where $t = J(K), t' = J(K'), t, t' \in [0, +\infty]$.

Among all $h$ of the Proposition 3, only differentiable functions are considered. Here it is used the same procedure of [13].

The equation $[(e_4)]$ is

$$h^{-1}(h(t+t') - h(y) - h(z) + h(t+t'+y) + h(t+t'+z))$$

$$= h^{-1}(h(t) - h(y) - h(z) + h(t + y) + h(t + z)) +$$

$$h^{-1}(h(t') - h(y) - h(z) + h(t' + y) + h(t' + z)).$$

Now, the function $h$ will be characterized.

Putting $y = z$,

$$h^( h(t+t') - h(y) - h(y) + h(t+t'+y) + h(t+t'+y))$$

$$= h^{-1}(h(t) - h(y) - h(y) + h(t + y) + h(t + y)) +$$

$$h^{-1}(h(t') - h(y) - h(y) + h(t' + y) + h(t' + y)),$$

i.e. it is

$$h^{-1}\left(2\,h(t + t' + y) + h(t + t) - 2\,h(y)\right) = \tag{6}$$

$$h^{-1}\left(2\,h(t + y) + h(t) - 2\,h(y)\right) +$$

$$h^{-1}\left(2\,h(t' + y) + h(t') - 2\,h(y)\right).$$

Setting

$$\varphi(t, y) = h^{-1}\left(2\,h(t + y) + h(t) - 2\,h(y)\right) \tag{7}$$

the equation (6) becomes

$$\varphi(t+t',y)=\varphi(t,y)+\varphi(t',y). \tag{8}$$

Fixed $y=y^*$, the (8) is the classical Cauchy equation [1], whose solution is the continuous function $\varphi$ :

$$\varphi(t,y^*)=\lambda(y^*)t. \tag{9}$$

So, from (7),

$$\lambda(y^*)t=h^{-1}\Big(2\,h(t+y^*)+h(t)-2\,h(y^*)\Big)\quad i.e.$$

$$h\Big(\lambda(y^*)t\Big)\ =2\,h(t+y^*)+h(t)-2\,h(y^*). \tag{10}$$

If $y^*=0$, as $h(0)=0$, from (10), one has

$$h\Big(\lambda(0)t\Big)\ =2\,h(t)+h(t),\ \ i.e.$$

$$h\Big(\lambda(0)t\Big)\ =3\,h(t). \tag{11}$$

Taking inspiration by [1, 2, 3, 4] one will prove that

$$h\Big(\lambda(0)t\Big)\ =3\,h(t)\Longrightarrow\lambda(0)=3. \tag{12}$$

Set $\lambda(0)=c$, from (11), one will solve the equation

$$h(c\,t)=3\,h(t); \tag{13}$$

by differentiating $ch'(c\,t)=3\,h'(t)$ from which

$$\frac{c\,h'(c\,t)}{h(c\,t)}=\frac{h'(t)}{h(t)}. \tag{14}$$

Setting

$$v(t)=\frac{h'(t)}{h(t)}, \tag{15}$$

the (14) is

$$v(c\,t)=\frac{v(t)}{c},\quad\forall\,t. \tag{16}$$

The function $v(t)=\frac{1}{t}$ is the unique solution admitting a Laurent expansion about 0. By substituing in (15), one obtain the equation

$$\frac{h'(t)}{h(t)}=\frac{1}{t} \tag{17}$$

whose solution is

$$h(t)=k\,t,t\in[0,+\infty],k>0. \tag{18}$$

By substituing (18) in (13), it is $c=\lambda(0)=3$. So, the function $h$ satisfies the following condition:

$$h(3\,t)=3\,h(t). \tag{19}$$

From (10),

$$\varphi(x,t)=3\,t=h^{-1}\Big(2\,h(t+y)+h(t)-2\,h(y)\Big)$$

$$i.e.\ \ h(3\,t)=2\,h(t+y)+h(t)-2\,h(y),$$

taking into account (19), it is

$$3\,h(t)=2\,h(t+y)-2\,h(y)+h(t)$$

$$i.e.\ \ h(t)+h(y)=h(t+y),$$

which is the classical Cauchy equation [1], whose solution is

$$h(x)=c\,x,c>0. \tag{20}$$

Now, it is possible to give the following

**Proposition 4** *The solution of the system* $[(e_1)-(e_4)]$ *is*

$$\Phi(x,y,z,u,v)=x-y-z+u+v. \tag{21}$$

**Proof:** It is easy to check that (21) holds, by applying (20) in the (4).

In the independent case, given $H$ and $S$ as in (1), from (21), information of any set $A\in\mathcal{A}$ conditioned by $H$ with the aim $S$ is

$$J_H(A\to S)=J(A)-J(H)-J(S)+ \tag{22}$$

$$J(A\cap H)+J(A\cap S).$$

# 5 Conclusion

First, by axiomatic way, it has been defined general conditional information with an aim, on fuzzy setting. By using its properties, it has been possible to find a class of this measure (5).

Then, taking into account the $J-$independence property, it has been obtained a particular measure (22).

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