Sampled-data reduction of nonlinear input-delayed dynamics

Mattia Mattioni *Student Member, IEEE,* Salvatore Monaco *Fellow, IEEE* and Dorothée Normand-Cyrot *Fellow, IEEE*

***Abstract*—A reduction approach on the discrete-time equivalent model of a nonlinear input delayed system is proposed to design a sampled-data stabilizing feedback. Global asymptotic stability of the feedback system is so achieved by solving the problem over the reduction state. Stabilization of the reduced dynamics is obtained through Input-Lyapunov Matching. Connections with prediction-based methods are established. A simulated example illustrates the performances.**

***Index Terms*—Sampled-data control, Delay systems, Algebraic/geometric methods**

I. INTRODUCTION

When dealing with time delay systems, a huge number of challenges arise from both theoretical and practical problems (see, among others, [1], [2], [3], [4] and references therein). In particular, two main classes of delays have been identified: discrete delays when the model depends on retarded variables at time $t-\tau$ ($\tau > 0$ denotes the delay length); distributed delays, when the model explicitly depends on the story of the retarded variables over the interval $[t-\tau, t[$.

This work is concerned with systems affected by discrete delays over the input variables. Despite the wide literature, a lot of questions still remain unanswered, even for Linear Time Invariant (LTI) systems. This is mainly linked with the fact that the retarded system is intrinsically infinite dimensional. Different prediction and reduction-based design approaches have been proposed (e.g., [5], [6], [7], [8], [9]). In the first case, the stabilizing feedback is deduced by computing the delay-free feedback over the future trajectories of the system on the time window $[t, t+\tau[$. In the second case, the design of the reduction-based control is lead to a somehow *equivalent* reduced delay-free dynamics in a sense that depends on the control purpose.

More recently, an increasing focus has been devoted to sampled-data time-delay systems (e.g, [10], [11]) when assuming that the control is piecewise constant and measures are available at discrete-time instants. This interest is mainly motivated by the fact that the retarded infinite dimensional continuous-time system admits a finite dimensional equivalent sampled-data model whenever there exists an explicit relation among the delay and the sampling period. In this context, several approaches have been proposed based on reduction (e.g., [12], only for the LTI case), prediction or, more recently, Immersion and Invariance methods (e.g., [13], [14] for nonlinear systems). In the latter cases, the design is based on the assumption that the delay-length is an entire multiple of the sampling period (i.e., $\tau = N\delta$ for some $N \in \mathbb{N}$). This assumption has been recently relaxed in [15] by considering *non-entire* delays (namely, $\tau = N\delta + \sigma$ for some $N \in \mathbb{N}$ and $\sigma \in [0, \delta[$) and extending the prediction method to a non-entire time interval of length $N\delta + \sigma$. Moreover, to improve robustness, an Immersion and Invariance (I&I) approach, with the delay free dynamics corresponding to the &I target dynamics, has been proposed in in [16].

In spite of that, predictor-based strategies are hard to extend to much general classes of time-delay systems as, for example, LTI dynamics affected by multichannel delay ([17], [18]). Following the work by Artstein [6], and for the first time at the best of authors' knowledge, a *sampled-data reduction*-based method is proposed in this paper for stabilizing nonlinear systems affected by input delay. Differently from the work in [15], the present strategy qualifies for extension to a larger class of time-delay systems, such as the multichannel case. Finally, the sampled-data design over the reduced model simplifies the task and allows the computation of approximate solutions that are actually implemented in practice.

Our contribution is two-fold: first, we define a *discrete-time reduction variable* exhibiting a delay-free dynamics which identifies the discrete-time *reduced model*; secondly, we prove that any discrete-time feedback stabilizing the reduced model guarantees stabilization at the sampling instants of the original system. The design of the control law is pursued via a suitably defined Input-Lyapunov Matching (ILM) problem ([19], [20]) when assuming smooth stabilizability of the delay-free system. It is also shown that a suitable choice of the reduction-based control enables one to recover the prediction-based feedback proposed in [15].

In Section II the problem is set and the instrumental definitions provided while the main result is stated in Section III. In Section IV, the design of the control law is developed and the case of LTI system is detailed as a case study in Section V. Simulations on the van der Pol oscillator are discussed in Section VI. Final comments in Section VII conclude the paper.

Notations and definitions: All the functions and vector fields defining the dynamics are assumed smooth over the

Mattia Mattioni thanks the *Université Franco-Italienne/Università Italo-Francese* (UFI/UIF) for supporting his mobility from France to Italy within his PhD program.

M. Mattioni and S. Monaco are with Dipartimento di Ingegneria Informatica, Automatica e Gestionale *A. Ruberti* (Università di Roma *La Sapienza*); Via Ariosto 25, 00185 Rome, Italy {mattia.mattioni,salvatore.monaco}@uniroma1.it.

M. Mattioni and D. Normand-Cyrot are with Laboratoire des Signaux et Systèmes (L2S, CNRS); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France {mattioni,cyrot}@l2s.centralesupelec.fr

respective definition spaces. M_U (resp. M_U^I) denotes the space of measurable and locally bounded functions $u:\mathbb{R}\to U$ ($u:I\to U$, $I\subset\mathbb{R}$) with $U\subseteq\mathbb{R}$. $\mathscr{U}_\delta\subseteq M_U$ denotes the set of piecewise constant functions over time intervals of length $\delta\in]0,T^*[$, a finite time interval; i.e. $\mathscr{U}_\delta=\{u\in M_U \text{ s.t. } u(t)=u_k, \forall t\in[k\delta,(k+1)\delta[; k\geq 0\}$. Id and I denote the identity function and matrix respectively. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f=\sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. Given two vector fields f and g, $ad_f g=[f,g]$ and iteratively $ad_f^i g=[f,ad_f^{i-1}g]$. The operator $e^{\mathrm{L}_f}Id$ denotes the associated Lie series operator, $e^{\mathrm{L}_f}:=\mathrm{I}+\sum_{i\geq 1}\frac{\mathrm{L}_f^i}{i!}$. Given any smooth function $h:\mathbb{R}^n\to\mathbb{R}$ then $e^{\mathrm{L}_f}h(x)=h(e^{\mathrm{L}_f}Id\big|_x)$. The composition of functions is denoted by "$\circ$". A function $R(x,\delta)=O(\delta^p)$ is said of order $\delta^p; p\geq 1$ if whenever it is defined it can be written as $R(x,\delta)=\delta^{p-1}\tilde{R}(x,\delta)$ and there exist a function $\theta\in\mathscr{K}_\infty$ and $\delta^*>0$ s. t. $\forall\delta\leq\delta^*$, $|\tilde{R}(x,\delta)|\leq\theta(\delta)$.

II. Problem statement

Consider the nonlinear time-delay system

$$\dot{x}(t)=f(x(t))+g(x(t))u(t-\tau) \tag{1}$$

with $x\in\mathbb{R}^n$, $u\in\mathbb{R}$ and $f(0)=0$ and the *delay-free system*

$$\dot{x}(t)=f(x(t))+g(x(t))u(t). \tag{2}$$

The following standing assumptions are set: the delay-free system is forward complete so implying forward completeness of (1) [14]; denoting by δ the sampling period, measurements are available at sampling instants $t=k\delta$, $k\geq 0$ and $u\in\mathscr{U}_\delta$; the time delay τ is fixed, known and such that $\tau=N\delta+\sigma$ for some $N\in\mathbb{N}$ and $\sigma\in[0,\delta[$.

Denoting $x_k:=x(k\delta)$ and $u_k:=u(k\delta)$, one computes the sampled-data equivalent model of (1) as

$$\begin{aligned}x_{k+1}=&F^{\delta-\sigma}(\cdot,u_{k-N})\circ F^{\sigma}(x_k,u_{k-N-1})\\=&F^{\delta-\sigma}(F^{\sigma}(x_k,u_{k-N-1}),u_{k-N})\\:=&F^{\delta}(\sigma,x_k,u_{k-N-1},u_{k-N})\end{aligned} \tag{3}$$

with $F^\theta(x,u)=e^{\theta(\mathrm{L}_f+u\mathrm{L}_g)}Id\big|_x$ and $u\in\mathscr{U}_\delta$.

Remark 2.1: When $N=0$ and $\sigma=0$, (3) recovers the sampled-data equivalent model to the delay-free (2) [21]; i.e.

$$x_{k+1}=F^{\delta}(x_k,u_k)=e^{\delta(\mathrm{L}_f+u_k\mathrm{L}_g)}Id\big|_{x_k}. \tag{4}$$

Roughly speaking, from (3) one deduces that a discrete delay affecting (1) is transformed into a distributed delay on the equivalent discrete-time model (3).

The aim of this work is to characterize a *discrete-time reduction variable* (or simply *reduction*), say y, which exhibits a discrete-time delay-free dynamics (the *discrete-time reduced model*) with the property that any of its stabilizing controller achieves stabilization of (3) in turn (i.e. sampled-data stabilization of the original system (1)).

Definition 2.1 (S-GAS): The equilibrium of a continuous-time dynamics $\dot{x}=f(x)$ is sampled-data globally asymptotically stable at the sampling instants $t=k\delta$ $(k\geq 0)$, if the equilibrium of its discrete-time equivalent dynamics $x_{k+1}=e^{\delta\mathrm{L}_f}Id\big|_{x_k}$ is globally asymptotically stable (GAS).

III. Main result

A. The case $\tau=\sigma$, $(N=0)$

When $N=0$, the sampled model (3) reduces to

$$x_{k+1}=F^{\delta}(\sigma,x_k,u_{k-1},u_k)=F^{\delta-\sigma}(F^{\sigma}(x_k,u_{k-1}),u_k). \tag{5}$$

Accordingly, one can define the mapping

$$y_k=F_0^{-\sigma}(F^{\sigma}(x_k,u_{k-1})) \tag{6}$$

with $F_0^\theta(x)=e^{\theta\mathrm{L}_f}Id\big|_x$ as a candidate reduction for (5). Computing (6) one-step ahead, one gets

$$y_{k+1}=F_0^{-\sigma}(F^{\sigma}(x_{k+1},u_k)). \tag{7}$$

By rewriting (5) in terms of the reduction (6), one has

$$x_{k+1}=\bar{F}^{\delta}(\sigma,y_k,u_k) \tag{8}$$

with

$$\begin{aligned}\bar{F}^{\delta}(\sigma,y_k,u_k)=&F^{\delta-\sigma}(\cdot,u_k)\circ F_0^{\sigma}(y_k)\\=&e^{\sigma\mathrm{L}_f}e^{(\delta-\sigma)(\mathrm{L}_f+u_k\mathrm{L}_g)}Id\big|_{y_k}.\end{aligned}$$

By substituting the above mappings into (7), one concludes that the dynamics of (6) is delay-free so that (6) is actually a reduction for (5). More in detail, the reduced model takes the form

$$y_{k+1}=F_r^{\delta}(\sigma,y_k,u_k) \tag{9}$$

with

$$\begin{aligned}F_r^{\delta}(\sigma,y,u):=&F_0^{-\sigma}\circ F^{\delta}(\cdot,u)\circ F_0^{\sigma}(y)\\=&e^{\sigma\mathrm{L}_f}e^{\delta(\mathrm{L}_f+u\mathrm{L}_g)}e^{-\sigma\mathrm{L}_f}Id\big|_y.\end{aligned}$$

Proposition 3.1: Any feedback $u_k=\alpha(y_k)$ achieving GAS of the origin of (9) ensures GAS the origin of (5) and, thus, S-GAS of (1). Furthermore, suppose that $y_k=0$, $k\geq\bar{k}$, then x_k goes to 0 in exactly $\bar{k}+1$ steps.

Proof. Consider the original dynamics (5) equivalently rewritten in the form (8). First, we write the original dynamics (8) and the reduced model (9) as a strict-feedforward interconnection over $\mathbb{R}^n\times\mathbb{R}^n$ of the form

$$x_{k+1}=\bar{F}^{\delta}(\sigma,y_k,u_k) \tag{10a}$$

$$y_{k+1}=F_r^{\delta}(\sigma,y_k,u_k). \tag{10b}$$

Now, consider any feedback $u_k=\alpha(y_k)$ that makes the origin of the reduced model (10b) GAS and define the transformation $\zeta_k=x_k-\phi(y_k,y_{k-1})$ with

$$\phi(y_k,y_{k-1})=F^{-\sigma}(\cdot,\alpha(y_{k-1}))\circ F_0^{\sigma}(y_k).$$

Under $u_k=\alpha(y_k)$, one has that $\phi(y_{k+1},y_k)=\bar{F}^{\delta}(\sigma,y_k,\alpha(y_k))$. so implying that, in the (ζ,y) coordinates, the dynamics (10) in closed-loop rewrites as the composition of two decoupled dynamics

$$\begin{aligned}\zeta_{k+1}=&0\\y_{k+1}=&F_r^{\delta}(\sigma,y_k,\alpha(y_k))\end{aligned}$$

with GAS equilibrium at the origin. Consequently, GAS of the origin of the original system (5) (equivalently, (10a)) follows. By virtue of the feedforward structure, if $y_k=0$ for any $k\geq\bar{k}$, then $x_k=0$ for $k\geq\bar{k}+1$. $\triangleleft$

B. The case $\tau = N\delta + \sigma$, $(N > 0)$

The definition of the reduction is generalized to $N \geq 0$ as follows.

Proposition 3.2: Consider the continuous-time system (1) and let (3) be its sampled-data equivalent model. The map

$$y_k = F_0^{-\tau} \circ F^{\delta}(\cdot, u_{k-1}) \circ \cdots \circ F^{\delta}(\cdot, u_{k-N}) \circ F^{\sigma}(x_k, u_{k-N-1}) \quad (11)$$

defines a *reduction* for (3) evolving according to the reduced dynamics

$$y_{k+1} = F_r^{\delta}(\tau, y_k, u_k) \quad (12)$$

with

$$F_r^{\delta}(\tau, y, u) := F_0^{-\tau} \circ F^{\delta}(\cdot, u) \circ F_0^{\tau}(y) = e^{\tau \mathrm{L}_f} e^{\delta(\mathrm{L}_f + u\mathrm{L}_g)} e^{-\tau \mathrm{L}_f} Id\big|_y.$$

Proof. Computing (11) one-step ahead, we get

$$y_{k+1} = F_0^{-\tau} \circ F^{\delta}(\cdot, u_k) \circ \cdots \circ F^{\delta}(\cdot, u_{k-N+1}) \circ F^{\sigma}(x_{k+1}, u_{k-N}) \quad (13)$$

while (5) rewrites as

$$x_{k+1} = \bar{F}^{\delta}(\sigma, y_k, u_{k-1}, \ldots, u_{k-N}) \quad (14)$$

with

$$\bar{F}^{\delta}(\sigma, y_k, u_{k-1}, \ldots, u_{k-N}) := F^{-\sigma}(\cdot, u_{k-N}) \circ F^{-\delta}(\cdot, u_{k-N+1}) \circ \cdots \circ F^{-\delta}(\cdot, u_{k-1}) \circ F_0^{\tau}(y_k).$$

and, for $N = 1$, $\bar{F}^{\delta}(\sigma, y_k, u_{k-1}) := F^{-\sigma}(\cdot, u_{k-1}) \circ F_0^{\tau}(y_k)$. By substituting (14) into (13) one gets the result. ◁

Remark 3.1: Again, when $\tau = 0$, $y \equiv x$ and the reduction dynamics (12) recovers the sampled-data delay-free one (4).

Remark 3.2: By exploiting the Lie exponential, (11) rewrites as

$$y_k = x_k + \sum_{s_1 + \cdots + s_{N+2} > 0} \frac{(-1)^{s_2} \sigma^{s_1 + s_2} \delta^{s_3 + \cdots + s_{N+2}}}{s_1! \ldots s_{N+2}!} \times \mathrm{L}_{f + u_{k-N-1} g}^{s_{N+2}} \ldots \mathrm{L}_{f + u_{k-1} g}^{s_1} \mathrm{L}_f^{s_2} Id\big|_{x_k}.$$

Remark 3.3: By expanding (12), one gets

$$y_{k+1} = e^{\delta \mathrm{L}_f}(y_k) + \delta u_k e^{\tau \mathrm{ad}_f} g(y_k) + O(u^2)$$

so explicitly recovering the Lie controllability directions $\mathrm{ad}_f^j g$ and their Lie brackets describing the sampled-data reduced dynamics (12) which is delay-free but generally nonlinear in the control u_k.

Proposition 3.1 extends to this case as follows.

Theorem 3.1: Consider the continuous-time system (1) with sampled-data equivalent model (3). Define the reduction y in the form (11) evolving according to (12). Then, any feedback $u_k = \alpha(y_k)$ achieving GAS of the origin of (12), ensures GAS (resp., S-GAS) of the origin of (3) (resp., (1)). Furthermore, suppose that $y_k = 0$ for $k \geq \bar{k}$, then x_k converges to 0 in exactly $\bar{k} + N + 1$ steps.

Proof. The proof proceeds along the lines of the one of Proposition 3.1 by considering (14) and exploiting the cascade structure

$$x_{k+1} = \bar{F}^{\delta}(\sigma, y_k, u_{k-1}, \ldots, u_{k-N}), \quad y_{k+1} = F_r^{\delta}(\tau, y_k, u_k).$$

◁

Remark 3.4: The results in Proposition 3.2 and Theorem 3.1 hold in the case of entire delays (i.e., when $\sigma = 0$) so providing an alternative solution to the one presented in [14].

According to the previous result, stabilization of the reduced dynamics (12) ensures S-GAS in closed-loop of the original system (1). In the following, a possible choice of the feedback $u_k = \alpha(y_k)$ is proposed.

IV. On the design of the sampled-data feedback

The following assumption is introduced.

A. There exists a smooth continuous-time feedback $u(t) = \gamma(x(t))$ ensuring GAS of the equilibrium of the delay-free (2) with radially-unbounded strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that $\mathrm{L}_g V(x) \neq 0$ for any $x \neq 0$.

As proved in [19], Assumption **A** implies the existence of a smooth sampled-data feedback stabilizing the origin of the delay-free system (4). Such a feedback is inferred via the notion of Input-Lyapunov Matching (ILM, [19], [20]).

Theorem 4.1 ([19]): Let the delay-free dynamics (2) fulfil Assumption **A**. Then, there exists $\gamma^{\delta} : \mathbb{R}^n \to \mathbb{R}$ as the unique solution $u_k = \gamma^{\delta}(x_k)$, for any $x_k = x(k\delta)$, of the ILM equality

$$e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} V(x)\big|_{x_k} - V(x_k) = \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f + \gamma(\cdot) g} V(x(s)) \mathrm{d}s \quad (16)$$

with $x(s) = e^{s \mathrm{L}_{f + \gamma(\cdot) g}} Id\big|_{x_k}$. Moreover, $\gamma^{\delta}(x)$ admits the power expansion

$$\gamma^{\delta}(x) = \gamma(x) + \sum_{i \geq 0} \frac{\delta^i}{(i+1)!} \gamma_i(x). \quad (17)$$

As a consequence, $u_k = \gamma^{\delta}(x_k)$ ensures GAS (resp. S-GAS) of the closed-loop delay-free dynamics (4) (resp., (2)).

In the following, we will show that Assumption **A** is sufficient to ensure the existence of a sampled-data reduction-based feedback yielding S-GAS of the equilibrium of the retarded system (1).

A. Reduction-based stabilization via ILM

The idea is to construct a sampled-data feedback over the dynamics (9) to ensure matching (at any sampling instant) of the Lyapunov function $V(x(t))$ along the closed-loop delay-free dynamics (2) when $u(t) = \gamma(x(t))$. For, we recall that when $\tau = 0$, $x \equiv y$ so that the following result can be stated.

Theorem 4.2: Consider the time-delay system (1) under Assumption **A** and let (3) be its sampled-data equivalent model. Introduce the reduction y as in (11) with reduced dynamics (12). Then, there exists a smooth mapping $K^{\delta}(\tau,\cdot):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ of the form

$$K^{\delta}(\tau,y)=\gamma(y)+\sum_{i+j>0}\frac{\delta^i\tau^j}{(i+1)!j!}K_{ij}(y) \tag{18}$$

that is the unique solution $u_k=K^{\delta}(\tau,y_k)$ of the ILM equality

$$V(F_r^{\delta}(\tau,y_k,u_k))-V(y_k)=\int_{k\delta}^{(k+1)\delta}\mathrm{L}_{f+\gamma(\cdot)g}V(y(s))\mathrm{d}s \tag{19}$$

for any $k\geq 0$ and $y(s)=e^{(s-k\delta)\mathrm{L}_{f+\gamma(\cdot)g}}Id\big|_{y_k}$. Moreover, the feedback $u_k=K^{\delta}(\tau,y_k)$ makes the closed-loop equilibrium of (3) (resp. (1)) GAS (resp. S-GAS).

Proof. The existence of a unique solution to (19) in the form (18) is deduced from direct application of the Implicit Function Theorem provided that Assumption **A** holds. Concerning closed-loop stability, because of matching, one gets that

$$\int_{k\delta}^{(k+1)\delta}\mathrm{L}_{f+\gamma(\cdot)g}V(y(s))\mathrm{d}s=\int_{k\delta}^{(k+1)\delta}\mathrm{L}_{f+\gamma(\cdot)g}V(x(s))\mathrm{d}s<0$$

with $x(s)=e^{(s-k\delta)L_{f+\gamma(\cdot)g}}Id\big|_{x_k}$. Thus, by construction, one has that

$$V(F_r^{\delta}(\tau,y_k,K^{\delta}(\tau,y_k)))-V(y_k)<0.$$

Thus, when $u_k=K^{\delta}(\tau,y_k)$, (12) has a GAS equilibrium of the origin. From Theorem 3.1, one concludes that such a feedback ensures GAS of (3) (resp., S-GAS of (1)) in closed-loop. ◁

Remark 4.1: The final feedback $u_k=K^{\delta}(\tau,y)$ is smoothly parametrized by both δ and τ. When $\tau\to 0$, (19) coincides with (16) so implying that $K^{\delta}(0,x)=\gamma^{\delta}(x)$ in (17).

B. About approximate solutions

Theorem 4.2 proves that whenever one can compute a stabilizing smooth feedback for the continuous-time delay-free system (2), sampled-data stabilization in closed-loop of the time-delay dynamics (1) can be pursued by combining reduction-based and ILM arguments. Though, the final feedback comes in the form of a series expansions in powers of δ and τ. As a consequence, exact solutions cannot be computed in general and only approximation of (18) can be implemented in practice.

Definition 4.1: An *approximate solution of order* p $K_N^{\delta[p]}(\tau,\cdot)$ to (19) is defined as the truncation of the series (18) at any finite $p:=i+j$ in $\delta^i\tau^j$; i.e.,

$$K^{\delta[p]}(\tau,y)=\sum_{i=0,j=0}^{j+i=p}\frac{\delta^i\tau^j}{(i+1)!j!}K_{ij}(y).$$

Each term K_{ij} can be computed via an iterative procedure by developing both sides of (19) and equating the terms with the same power $\delta^i\tau^j$. Accordingly, at each step, one has to solve a linear equation in the unknown K_{ij} as a function of the previous terms. For the first terms one gets

$$K_{01}=\frac{\gamma(y)}{\mathrm{L}_gV}\mathrm{L}_{ad_fg}V,\quad K_{20}=\ddot{\gamma}(y)+\frac{\dot{\gamma}(y)}{2\mathrm{L}_gV}\mathrm{L}_{ad_fg}V \tag{20a}$$

$$\begin{aligned}K_{02}=&\frac{2K_{01}}{\mathrm{L}_gV}\mathrm{L}_{ad_fg}V-\frac{\gamma(x)}{\mathrm{L}_gV}(\mathrm{L}_g\mathrm{L}_f^2-2\mathrm{L}_f\mathrm{L}_g\mathrm{L}_f+\mathrm{L}_f^2\mathrm{L}_g)V\\ K_{10}=&\dot{\gamma}(y)=\mathrm{L}_{f+\gamma g}\gamma(y)\end{aligned} \tag{20b}$$

$$\begin{aligned}K_{11}=&-\frac{K_{01}}{\mathrm{L}_gV}(\mathrm{L}_g\mathrm{L}_f+\mathrm{L}_f\mathrm{L}_g)V-\frac{2K_{00}K_{01}}{\mathrm{L}_gV}\mathrm{L}_g^2V+\\ &\frac{K_{10}}{\mathrm{L}_gV}\mathrm{L}_{ad_fg}V-\frac{K_{00}}{\mathrm{L}_gV}(\mathrm{L}_g\mathrm{L}_f^2-\mathrm{L}_f^2\mathrm{L}_g)V\\ &-\frac{K_{00}^2}{\mathrm{L}_gV}(\mathrm{L}_g^2\mathrm{L}_f-\mathrm{L}_f\mathrm{L}_g^2)V\end{aligned} \tag{20c}$$

with $\ddot{\gamma}(y)=\mathrm{L}^2_{f+\gamma g}\gamma(y)$.

Although global results are in general lost under approximate solutions, those control still yield interesting properties in closed-loop, such as practical-GAS or Input-to-State Stability ([14], [20] and [22]).

C. Reduction and prediction-based stabilization

In the sequel a comparison with respect to the predictor-based approach proposed in [15] is developed. As a matter of fact, by suitably defining $u=\alpha(y)$ in Theorem 3.1, the predictor feedback is recovered. For this purpose, we note that the reduction variable y_k in (11) rewrites as $y_k=F_0^{-\tau}(x(k\delta+\tau))$ where

$$x(k\delta+\tau)=F^{\delta}(\cdot,u_{k-1})\circ\cdots\circ F^{\delta}(\cdot,u_{k-N})\circ F^{\sigma}(x_k,u_{k-N-1}).$$

defines the prediction of the state at $t=k\delta+\tau$ from x_k.

Based on the above relation, it turns out that reduction can be interpreted as prediction of the state at $t=k\delta+\tau$ that is projected backward via the free evolution $F_0^{-\tau}(\cdot)$; namely, $y_k=F_0^{-\tau}(x(k\delta+\tau))$.

The following statement settles the result in [15] in terms of reduction.

Theorem 4.3: Consider the time-delay system (1) under Assumption **A** and let (5) be its equivalent sampled-data model. Let the reduction state y in (11) evolve according to (12). Then, the feedback $u_k=\gamma^{\delta}(F_0^{\tau}(y_k))$, where $\gamma^{\delta}:\mathbb{R}^n\to\mathbb{R}$ is computed as the unique solution to (16), ensures GAS of (1) at the time instants $t=k\delta+\sigma$ with $k\geq 0$.

Proof. In order to prove the result, one has to prove that the feedback $u_k=\gamma^{\delta}(F_0^{\tau}(y_k))$ coincides with the predictor-based feedback proposed in [15]. For, introduce the coordinates change $z_k=F_0^{\tau}(y_k)$ so that $u_k=\gamma^{\delta}(z_k)$ while the dynamics (12) takes the form $z_{k+1}=F^{\delta}(z_k,u_k)$ with $F^{\delta}(z_k,u_k)=e^{\delta(\mathrm{L}_f+u_k\mathrm{L}_g)}Id\big|_{z_k}$. Thus, the predictor feedback is recovered. Since γ^{δ} is the solution of an ILM problem, $u=\gamma^{\delta}(z)$ stabilizes the predictor dynamics. Thus, such a feedback ensures S-GAS of (1) in closed-loop. ◁

By virtue of the above result, we note that, whenever the system (1) is driftless, the reduction and predictor-based solutions coincide.

Contrarily to prediction the reduction-based feedback only requires the knowledge of the state at the sampling instants. Indeed, the former control is based on the knowledge of the state at the inter sampling instant $t = k\delta + \sigma$ that is not available from measures. Thus, the feedback in [15] needs a further prediction over the inter sampling interval.

Moreover, the prediction-based controller [15] ensures sampled-data stabilization at the inter sampling instants $t = k\delta + \sigma$ ($k \geq 0$) while the proposed reduction feedback ensures stabilization at the sampling instants $t = k\delta$ and, thus, S-GAS. By virtue of this, the prediction-based control should be more sensible to the variation of σ and, thus, on τ.

V. LTI SYSTEMS AS A CASE STUDY

Consider the case in which (1) is a LTI system

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) \tag{21}$$

under the standing assumptions presented in Section II plus

$\mathbf{A_L}$. the couple (A, B) is controllable.

The sampled-data equivalent model of (21) is provided by

$$x_{k+1} = A^{\delta}x_k + A^{\delta-\sigma}B^{\sigma}u_{k-N-1} + B^{\delta-\sigma}u_{k-N}. \tag{22}$$

reducing to, for $\tau = 0$,

$$x_{k+1} = A^{\delta}x_k + B^{\delta}u_k \tag{23}$$

with $A^s = e^{sA}$, $B^s = \int_0^s e^{A\mu}\mathrm{d}\mu B$ and $A^{\delta-\sigma}B^{\sigma} + B^{\delta-\sigma} = B^{\delta}$.

Remark 5.1: Assumption $\mathbf{A_L}$ is necessary and sufficient to guarantee that the delay-free sampled-data couple (A^{δ}, B^{δ}) is controllable almost everywhere [23]. This can be relaxed by only requiring stabilizability of the couple (A, B) without affecting our result.

Accordingly, Theorems 3.2 specifies as follows.

Corollary 5.1: Consider the LTI system (21) under Assumption $\mathbf{A_L}$. Then,

$$y_k = x_k + A^{-\sigma}B^{\sigma}u_{k-N-1} + \sum_{j=k-N}^{k-1} A^{(k-N-j-1)\delta-\sigma}B^{\delta}u_j \tag{24}$$

is a reduction for (22) evolving according to the dynamics

$$y_{k+1} = A^{\delta}y_k + A^{-\tau}B^{\delta}u_k. \tag{25}$$

From Theorem 3.1, it turns out that, whenever (25) is controllable, one can compute a control $u_k = F^{\delta}y_k$ so that $A^{\delta} + A^{-\tau}B^{\delta}F^{\delta}$ is Schur and, as a consequence, (21) is S-GAS in closed-loop. As a consequence, the problem of stabilizing the retarded system is reconduced to assigning the eigenvalues of the reduced model.

In the following, it is shown that controllability of the delay-free continuous-time system ensures controllability (almost everywhere) of (25).

Proposition 5.1: Consider the LTI system (21) under Assumption $\mathbf{A_L}$ and introduce the reduction (24) with dynamics (25). Then, (25) is controllable almost everywhere and any feedback $u_k = F^{\delta}y_k$ such that $A^{\delta} + A^{-\tau}B^{\delta}F^{\delta}$ is Schur ensures that (22) (resp., (21)) is GAS (resp., S-GAS).

Proof. One has to show that (25) is controllable. By computing the controllability matrix $\mathscr{R}(A^{\delta}, A^{-\tau}B^{\delta})$, one can easily verify that $\mathscr{R}(A^{\delta}, A^{-\tau}B^{\delta}) = A^{-\tau}\mathscr{R}(A^{\delta}, B^{\delta})$ where $\mathscr{R}(A^{\delta}, B^{\delta})$ denotes the nonsingular controllability matrix of the delay-free system (23). Thus, one can compute a control $u_k = F^{\delta}y_k$ so that $A^{\delta} + A^{-\tau}B^{\delta}F^{\delta}$ is Schur. In order to guarantee asymptotic stability of (22), introduce the auxiliary states $v = \mathrm{col}(v^1, \ldots, v^{N+1})$ with $v^i_k = u_{k-N+i}$ for $i = 1, \ldots, N+1$ and consider the extended (x, v, y)-dynamics under $u_k = F^{\delta}y_k$

$$\begin{pmatrix} x_{k+1} \\ v_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & A_{12} & A^{\delta} \\ \mathbf{0} & \hat{A} & \hat{B}F^{\delta} \\ \mathbf{0} & \mathbf{0} & A^{\delta} + A^{-\tau}B^{\delta}F^{\delta} \end{pmatrix} \begin{pmatrix} x_k \\ v_k \\ y_k \end{pmatrix}$$

with

$$A_{12} = \begin{pmatrix} 0 & -A^{-\sigma}B^{\sigma} & -A^{-(\delta+\sigma)}B^{\delta} & \ldots & -A^{-(N-1)\delta-\sigma}B^{\delta} \end{pmatrix}$$
$$\hat{A} = \begin{pmatrix} \mathbf{0} & \mathrm{I}_{\mathrm{N}\times\mathrm{N}} \\ 0 & \mathbf{0} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

It is clear that the overall dynamical matrix is Schur so proving the result. ◁

VI. THE VAN DER POL OSCILLATOR

Consider the case of the van der Pol oscillator whose dynamics is provided by

$$\dot{x}_1 = x_2 - x_2^2 u(t-\tau), \quad \dot{x}_2 = u(t-\tau) \tag{26}$$

with $x = \mathrm{col}(x_1, x_2)$, $\tau = \delta + \sigma$ and sampled-data equivalent model described in [14]-[15]. Accordingly, the sampled-data reduction state $y = \mathrm{col}(y_1, y_2)$ gets the form

$$\begin{aligned} y_1 =& x_1 - \frac{\sigma^3}{3}u^3_{k-2} - \sigma(x_2 + \delta u_{k-1} + \sigma u_{k-2}) + \delta(x_2 + \sigma u_{k-2} \\ & - u_{k-1}(x_2 + \sigma u_{k-2})^2) - \sigma(u_{k-2}x_2^2 - x_2) - \frac{\delta^3}{3}u^3_{k-1} - \\ & \frac{\sigma^2}{2}u_{k-2}(2u_{k-2}x_2 - 1) - \frac{\delta^2}{2}u_{k-1}(2u_{k-1}(x_2 + \sigma u_{k-2}) - 1) \\ y_2 =& x_2 + \sigma u_{k-2} + (\delta - \sigma)u_{k-1} \end{aligned}$$

so evolving according to

$$\begin{aligned} y_{1k+1} =& y_1 + \delta(y_1 - y_1^2 u - (\sigma + \delta)u) + \frac{\delta^2}{2}(1 - 2y_2 u)u - \frac{\delta^3}{3}u^3 \\ y_{2k+1} =& y_2 + \delta u. \end{aligned}$$

For feedback design, it was shown in [15] that (26) verifies Assumption $\mathbf{A}$ with $\gamma(x) = -3x_1 - \frac{x_1^3}{3} - x_2$ and Lyapunov function $V(x) = x_1^2 + \frac{x_1^4}{3} + x_1x_2 + \frac{1}{2}x_2^2$. Accordingly, the result in Theorem 4.2 applies and one can compute the resulting feedback $u_k = K^{\delta}(\delta + \sigma, y_k)$.

Partial simulations are reported in Figure 1 providing an interesting comparison of the closed-loop performances yielded by the approximate reduction-based (RB) and prediction-based (PB, [15]) feedback laws. In particular, the approximate control law $u_k = K^{\delta[2]}(\sigma, y)$ of Theorem 4.2 has been applied. Although further simulations show that both strategies behave similarly for small δ, prediction-based control yields degrading performances (1) as δ and σ increase. Moreover, further simulations underline that the evolutions of the Lyapunov

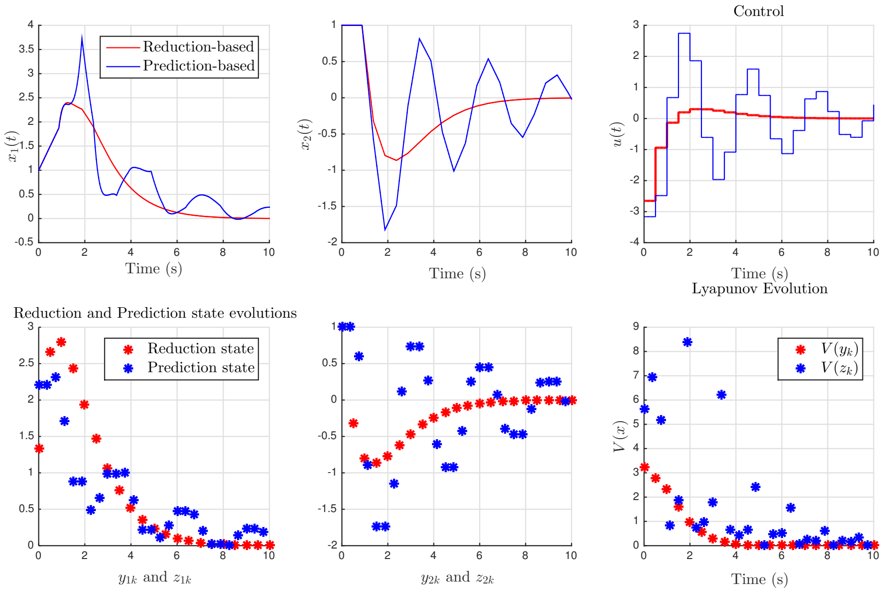

Fig. 1. $\delta = 0.5$ s, $N = 1$ and $\sigma = 0,4$ s.

function under reduction-based feedback are decreasing, at the sampling instants, even for higher values of the sampling period.

VII. Conclusions

This paper introduces a sampled-data reduction approach for stabilizing nonlinear dynamics affected by non-entire input delay as a generalization of the prediction-based methodologies presented in [14] and [15]. Further investigations will address robustness with respect to variations of the delay length and extensions to more general classes of time-delayed systems. Finally work is in progress toward nonlinear time-delay discrete-time dynamics.

References

[1] R. Sipahi, T. Vyhlídal, S.-I. Niculescu, and P. Pepe, Eds., *Time Delay Systems: Methods, Applications and New Trends*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag Berlin Heidelberg, 2012, vol. 423.

[2] W. Michiels and S. Niculescu, *Stability, control, and computation for time-delay systems. An eigenvalue based approach*, ser. Advances in Design and Control. Philadelphia: SIAM Society for Industrial and Applied Mathematics, 2014.

[3] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*, ser. Systems & Control: Foundations & Applications. E. Fridman, 2014.

[4] I. Karafyllis, M. Malisoff, F. Mazenc, and P. Pepe, Eds., *Recent Results on Nonlinear Delay Control Systems*, ser. Advances in Delays and Dynamics. Springer International Publishing, 2016, vol. 4.

[5] N. Bekiaris-Liberis and M. Krstic, "Lyapunov stability of linear predictor feedback for distributed input delays," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 655–660, March 2011.

[6] Z. Artstein, "Linear systems with delayed controls: A reduction," *IEEE Transactions on Automatic Control*, vol. 27, no. 4, pp. 869–879, Aug 1982.

[7] F. Mazenc and M. Malisoff, "Local stabilization of nonlinear systems through the reduction model approach," *IEEE Transactions on Automatic Control*, vol. 59, no. 11, pp. 3033–3039, Nov 2014.

[8] F. Mazenc, S. I. Niculescu, and M. Krstic, "Some remarks on Lyapunov-Krasovskii functionals and reduction approach for input delay compensation," in *2012 American Control Conference (ACC)*, June 2012, pp. 6229–6234.

[9] A. Pietrabissa, "A multi-model reference control approach for bandwidth-on-demand protocols in satellite networks," *Control Engineering Practice*, vol. 16, no. 7, pp. 847–860, 2008.

[10] P. Pepe, "Stabilization in the sample-and -hold sense of nonlinear retarded systems," *SIAM Journal on Control and Optimization*, vol. 52, no. 5, pp. 3053–3077, 2014.

[11] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Sampled-data stabilisation of a class of state-delayed nonlinear dynamics," in *2015 54th IEEE Conference on Decision and Control (CDC)*, 2015, pp. 5695–5700.

[12] F. Mazenc and D. Normand-Cyrot, "Reduction model approach for linear systems with sampled delayed inputs," *IEEE Transactions on Automatic Control*, vol. 58(5), pp. 1263 – 1268, 2013.

[13] I. Karafyllis and M. Krstic, "Numerical schemes for nonlinear predictor feedback," *Mathematics of Control, Signals, and Systems*, vol. 26, pp. 519–546, 2014.

[14] S. Monaco, D. Normand-Cyrot, and M. Mattioni, "Sampled-data stabilization of nonlinear dynamics with input delays through immersion and invariance," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2561–2567, May 2017.

[15] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Further results on sampled-data time-delay systems," in *20th IFAC World Congress*, 2017.

[16] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and adaptive control with applications*. Springer Publishing Company, 2008.

[17] D. Tsubakino, M. Krstic, and T. R. Oliveira, "Exact predictor feedbacks for multi-input {LTI} systems with distinct input delays," *Automatica*, vol. 71, pp. 143 – 150, 2016.

[18] T. R. Oliveira, M. Krstić, and D. Tsubakino, "Extremum seeking for static maps with delays," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1911–1926, April 2017.

[19] S. Monaco, D. Normand-Cyrot, and F. Tiefensee, "Sampled-data stabilization; a pbc approach," *IEEE Trans. on Automatic Control*, vol. 56, pp. 907–912, 2011.

[20] V. Tanasa, S. Monaco, and D. Normand-Cyrot, "Backstepping control under multi-rate sampling," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1208–1222, May 2016.

[21] S. Monaco and D. Normand-Cyrot, "Issues on nonlinear digital systems," *Semi-Plenary Conference, European Journal of Control*, vol. 7, no. 2-3, pp. 160–178, 2001.

[22] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Immersion and invariance stabilization of strict-feedback dynamics under sampling," *Automatica*, vol. 76, pp. 78 – 86, 2017.

[23] E. D. Sontag, *Mathematical control theory: deterministic finite dimensional systems*. Springer Science & Business Media, 2013, vol. 6.