SCHUBERT QUIVER GRASSMANNIANS

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ABSTRACT. Quiver Grassmannians are projective varieties parametrizing subrepresentations of given dimension in a quiver representation. We define a class of quiver Grassmannians generalizing those which realize degenerate flag varieties. We show that each irreducible component of the quiver Grassmannians in question is isomorphic to a Schubert variety. We give an explicit description of the set of irreducible components, identify all the Schubert varieties arising, and compute the Poincaré polynomials of these quiver Grassmannians.

INTRODUCTION

Let $R_\bullet = R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq V$ and $Q_\bullet = Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n \subseteq V$ be two flags in a complex vector space $V \simeq \mathbf{C}^N$, such that $Q_i \subseteq R_i$ for all $i = 1, 2, \cdots, n$ and such that they are both fixed by a Borel subgroup B of GL_N. In this paper we study varieties consisting of flags $E_\bullet$ in a partial flag manifold $\mathrm{Fl}_{(f_1, f_2, \cdots, f_n)}(V)$ of V such that $Q_i \subseteq E_i \subseteq R_i$ for all $i = 1, 2, \cdots, n$. By assumption, these subvarieties of $\mathrm{Fl}_{(f_1, f_2, \cdots, f_n)}(V)$ are B–stable, hence they are unions of finitely many Schubert varieties. We show that such varieties arise as suitable quiver Grassmannians of equioriented type A_n–quivers and we call them *Schubert quiver Grassmannians* (SQG for short). This observation allows, on one side, to study the geometry of SQGs via representation theory of quivers, and on the other side to study the geometry of such quiver Grassmannians using the theory of Schubert varieties.

Recall that, if Γ is a finite quiver with set of vertices Γ_0, and M is a finite–dimensional complex Γ–representation, then for a dimension vector $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{\Gamma_0}$, the *quiver Grassmannian* $\mathrm{Gr}_\mathbf{e}(M)$ is the (complex) variety of $\mathbf{e}$–dimensional sub–representations of M. In [3], the case when Γ is Dynkin (i.e. an orientation of a simply–laced Dynkin diagram) and $M = P \oplus I$ – the sum of a projective and an injective representation of Γ – was studied. More precisely, it was shown that the varieties $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ enjoy many nice properties. In particular, they are irreducible varieties endowed with the action of a large abelian unipotent group, acting with an open dense orbit. An example of such varieties is provided by the type A degenerate flag varieties [5], [6]. In this case, Γ is the equioriented type A quiver, P is the direct sum of all indecomposable projective representations and I is the direct sum of all indecomposable injective representations. It was shown

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in [1], [2] that the above mentioned abelian unipotent group can be identified with a certain subgroup of a Borel subgroup of the group SL_N (note that N here is much larger than the number of vertices of the initial quiver). Moreover, the quiver Grassmannian itself can be identified with a Schubert variety for SL_N. The main reason why the Borel subgroup appears is that the indecomposable direct summands of $M = P \oplus I$ form a chain (that is, a totally ordered subset) with respect to the partial ordering induced by the Auslander-Reiten quiver of Γ.

This observation motivates the following more general definition: Let Γ be a Dynkin quiver and let $M_1, \ldots, M_r$ be a set of indecomposable representations forming a chain with respect to the partial ordering induced by the Auslander-Reiten quiver. We call a direct sum $M = \bigoplus_{i=1}^r M_i^{\oplus a_i}$ catenoid.

From now on let Γ be a quiver of equioriented type A_n. In this case, there is a natural way to embed every quiver Grassmannian $\mathrm{Gr_e}(M)$ into a flag manifold: consider a minimal projective resolution

$$0 \longrightarrow Q \overset{\iota}{\longrightarrow} R \longrightarrow M \longrightarrow 0$$

of a representation M. Then $\mathrm{Gr_e}(M)$ is isomorphic (as a scheme) to the variety $\mathrm{Gr_f}(Q \dashrightarrow R)$ consisting of sub–representations of R of dimension vector $\mathbf{f} := \mathbf{e} + \mathbf{dim}\, Q$ containing $\iota(Q)$ (see Proposition 2.1). Since all the maps defining Q and R are injective, they induce flags inside the vector space R_n, and hence the quiver Grassmannian $\mathrm{Gr_e}(M)$ embeds into a flag manifold of R_n. We refer to this construction as the *natural embedding of a quiver Grassmannian inside a flag manifold*. Notice that $R_n \simeq \mathbf{C}^N$ where $N = a_1 + \cdots + a_r$ is the number of indecomposable direct summands of M. Our first result is the following.

Theorem 0.1. *The natural embedding of $\mathrm{Gr_e}(M)$ inside a flag manifold is stable under the action of a Borel subgroup $B_N \subset GL_N$ if and only if M is catenoid. In this case, $\mathrm{Gr_e}(M)$ is a SQG and every SQG arises in this way. The irreducible components of SQGs are Schubert varieties and can be explicitly described. The Schubert cells – the B_N-orbits – provide a cellular decomposition of $\mathrm{Gr_e}(M)$ and they are stable under the action of $\mathrm{Aut}(M)$; in particular the points of a given cell are all isomorphic as representations of Γ.*

It is a natural question to ask when a SQG is a Schubert variety, or, in other words, when it is irreducible. In general, a SQG can be factored into a product of several SQGs associated with representations of smaller quivers of equioriented type A. If this is not the case we say that the SQG is *simple* (see Definition 3.3). It is hence enough to answer this question for a simple SQG.

Theorem 0.2. *Let M be catenoid. A simple SQG $\mathrm{Gr_e}(M)$ is a Schubert variety if and only if there exists a projective representation P and*

an injective representation I such that M fits into an exact sequence $0 \rightarrowtail P \rightarrowtail M \twoheadrightarrow I \rightarrow 0$ and $\mathbf{e} = \dim P$. In this case, every point of the open B_N-orbit corresponds to a subrepresentation of M isomorphic to P.

Given a catenoid representation M and a dimension vector $\mathbf{e}$, we give a combinatorial description of the set of irreducible components of $\mathrm{Gr}_{\mathbf{e}}(M)$. For each irreducible component, we identify the partial flag variety containing the corresponding Schubert variety, and the corresponding Weyl group element. We show that the natural action of $\mathrm{Aut}(M)$ on $\mathrm{Gr}_{\mathbf{e}}(M)$ becomes, after the natural embedding into a flag manifold, the action of the parabolic subgroup of GL_N consisting of all automorphisms of R which fix $\iota(Q) \subseteq R$. We also compute the Poincaré polynomials of SQGs.

The paper is organized as follows. In Section 1 we fix the notation and introduce the objects of study. In Section 2 we provide a link to the theory of Schubert varieties. In Section 3 we describe the irreducible components of SQGs. In Section 4 we compute their Poincaré polynomials.

1. THE SETUP

1.1. **General definitions.** Let $\Gamma = \Gamma_n := 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$ be an equioriented quiver of type A_n, for some fixed integer $n \geq 1$. The indecomposable representations for this quiver are parametrized by (integer) subintervals $[i, j]$ of the interval $[1, n]$ $(1 \leq i \leq j \leq n)$. We denote the corresponding indecomposable representation by $M[i, j]$, and we say that the *support* of $M[i, j]$ is the interval $[i, j]$. Any representation M can be written in an essentially unique way as $M = \bigoplus_{1 \leq i \leq j \leq n} M[i, j]^{\oplus m_{ij}}$ for some non–negative integers m_{ij}. We denote the dimension vector $\mathbf{d} = (d_1, \cdots, d_n)$ of M by $\dim M$. We note that, for every vertex i, the projective cover of the simple representation $S_i = M[i, i]$ is $P_i := M[i, n]$, and its injective envelope is $I_i := M[1, i]$. The Euler form $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ of Γ is defined by

$$\langle \mathbf{e}, \mathbf{d} \rangle := \sum_{i=1}^{n} e_i d_i - \sum_{i=1}^{n-1} e_i d_{i+1}$$

for any $\mathbf{e}, \mathbf{d} \in \mathbb{Z}^n$.

The Auslander–Reiten quiver of Γ is described as follows: its vertices are parametrized by pairs (i, j) for $1 \leq i \leq j \leq n$, and the arrows are $(i, j) \to (i - 1, j)$ and $(i, j) \to (i, j - 1)$ (whenever the targets are well defined).

We are interested in the special class of Γ–representations defined as follows.

Definition 1.1. We say that a Γ–representation M is *catenoid* if all the distinct indecomposable direct summands of M belong to an oriented connected path of the Auslander–Reiten quiver of Γ.

Recall that a subset $\mathbf{p} \subset \{(i,j) : 1 \leq i \leq j \leq n\}$ is a *Dyck path* on $[1, n]$ if we can order the elements of $\mathbf{p}$ as $(p_1, \ldots, p_r)$ in such a way that $p_1 = (1, 1)$, $p_r = (n, n)$, and if $p_m = (i_m, j_m)$, then $p_{m+1} = (i_m, j_m + 1)$ or $p_{m+1} = (i_m + 1, j_m)$. We say that a representation M is supported on $\mathbf{p}$ if $(i, j) \in \mathbf{p}$ for each indecomposable direct summand $M[i, j]$ of M. The oriented connected paths of the Auslander-Reiten quiver of Γ are precisely the Dyck paths (on $[1, n]$). It follows that M is catenoid if and only if it is supported on a single Dyck path.

Another way to view this definition is the following. Consider the set $\mathcal{P}([1, n])$ of all connected sub-intervals of the interval $[1, n]$ (those are precisely the vertices of the Auslander-Reiten quiver of Γ). We endow $\mathcal{P}([1, n])$ with the following partial ordering:

$$(1.1) \qquad [i, j] \leq [k, \ell] \overset{\text{def}}{\Longleftrightarrow} i \leq k \text{ and } j \leq \ell$$

For example, for $n = 4$, we have:
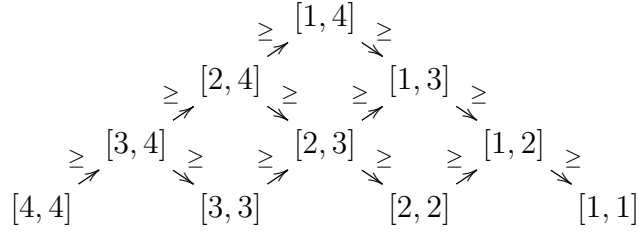
We have an induced partial order on the set of (isoclasses of) indecomposable Γ–representations:

$$(1.2) \qquad M[i, j] \leq M[k, \ell] \overset{\text{def}}{\Longleftrightarrow} [i, j] \leq [k, \ell].$$

Lemma 1.2. *$M[i, j] \leq M[k, \ell]$ if and only if there exists an oriented (connected) path from $M[k, \ell]$ to $M[i, j]$ in the Auslander–Reiten quiver of Γ. In particular, a representation M is catenoid if and only if the set of its distinct direct summands form a totally ordered set with respect to the partial order (1.2).*

Proof. Recall that the arrows of the Auslander–Reiten quiver are either of the form $(x, y) \to (x - 1, y)$ (pointing NE) or $(x, y) \to (x, y - 1)$ (pointing SE). Let (k, ℓ) and (i, j) be two vertices. It is hence clear that there is an oriented connected path from (k, ℓ) to (i, j) if and only if $i = k - s$ and $j = \ell - t$ for some $0 \leq s \leq k$ and $0 \leq t \leq \ell$. In other words, there is such a path from (k, ℓ) to (i, j) if and only if $[i, j] \leq [k, \ell]$ as desired. The rest follows. $\square$

Example 1.3. Examples of catenoid representations are $P \oplus I$, where P is projective and I is injective. In [3, 4], the quiver Grassmannians $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ were studied (see Example 3.9). Another class of examples is provided by those representations which are direct sums of one and two-dimensional indecomposables. The composition of the

linear maps forming such representations is zero and hence they are complexes (see Example 3.11).

2. Quiver Grassmannians associated with catenoid representations

We retain notation of the previous section. So, Γ is an equioriented quiver of type A_n. Given a Γ–representation M and a dimension vector $\mathbf{e} \in \mathbb{Z}_{\geq 0}^n$ we denote by $\mathrm{Gr}_{\mathbf{e}}(M)$ the projective varieties consisting of all subrepresentations of M of dimension vector $\mathbf{e}$ (see [3, Section 2] for details).

There is a natural way to realize $\mathrm{Gr}_{\mathbf{e}}(M)$ as a closed subvariety of a partial flag manifold, generalizing the idea of [1]. Let us describe this realization. A minimal projective resolution of M is given by

$$(2.1) \qquad 0 \to Q \overset{\iota}{\to} R \overset{\pi}{\to} M \to 0$$

where $R = \bigoplus_{1 \leq i \leq n} P_i^{[M,S_i]}$ and $Q = \bigoplus_{1 \leq i \leq n} P_i^{[M,S_i]^1}$, and we use the standard notation

$$[M,N] := \dim \mathrm{Hom}_\Gamma(M,N) \quad \text{and} \quad [M,N]^1 := \dim \mathrm{Ext}_\Gamma^1(M,N).$$

We note that the arrows of Γ act as injective linear maps on every projective $\mathbf{C}\Gamma$–module (recall that the category of Γ–representations is equivalent to the category of modules over the path algebra of Γ, which is the $\mathbf{C}$–vector space generated by paths of Γ with obvious multiplicative structure), and hence R is a partial flag $R = R_\bullet := R_1 \subseteq R_2 \subseteq \cdots \subseteq R_{n-1} \subseteq R_n$. Here the R_i's are finite dimensional vector spaces of dimension $\dim R_i =: r_i = \sum_{k \leq i}[M,S_k]$ (for all $i = 1, \cdots, n$). In particular the dimension N of the vector space R_n is precisely the number of indecomposable direct summands of M. Similarly, the representation Q is a partial flag $Q = Q_\bullet := Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_{n-1} \subseteq Q_n$, where Q_i is a vector subspace of R_i of dimension $\dim Q_i =: q_i = \sum_{k \leq i}[M,S_k]^1$ (for all $i = 1, \cdots, n$). Note that the vector space Q_1 is zero: indeed, $[M,S_1]^1 = 0$ since S_1 is injective.

For a vector space V of dimension m, and an increasing sequence $\mathbf{f} = (f_1, f_2, \cdots, f_n)$ of non–negative integers $0 \leq f_1 \leq f_2 \leq \cdots \leq f_n$ we denote by $\mathrm{Fl}_{\mathbf{f}}(V)$ the variety of partial flags $\mathcal{U}_\bullet := (\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \cdots \subseteq \mathcal{U}_n \subseteq V)$ in V with $\dim \mathcal{U}_i = f_i$.

We consider the dimension vector $\mathbf{f} := \mathbf{e} + \mathbf{dim}\, Q$, and the variety $\mathrm{Gr}_{\mathbf{f}}(Q \dashrightarrow R) \subset \mathrm{Fl}_{\mathbf{f}}(R_n)$ of partial flags $\mathcal{U}_\bullet$ in R_n such that

$$(2.2) \qquad \iota_j(Q_j) \subseteq \mathcal{U}_j \subseteq R_j \qquad (j = 1, \cdots, n).$$

The variety $\mathrm{Gr}_{\mathbf{f}}(Q \dashrightarrow R)$ is a closed subvariety of $\mathrm{Fl}_{\mathbf{f}}(R_n)$, and the following result provides the required closed embedding $\mathrm{Gr}_{\mathbf{e}}(M) \subset \mathrm{Fl}_{\mathbf{f}}(R_n)$.

Proposition 2.1. *The map $\varphi : \mathrm{Gr}_{\mathbf{e}}(M) \to \mathrm{Fl}_{\mathbf{e}+\mathbf{dim}\,Q}(R_n)$ defined by*

$$\varphi(N) = (\pi^{-1}(N))_1 \subseteq (\pi^{-1}(N))_2 \subseteq \cdots \subseteq (\pi^{-1}(N))_n$$

is a closed embedding. The image is the variety $\mathrm{Gr}_{\mathbf{f}}(Q{\dashrightarrow}R)$.

Proof. Since π is surjective, the map $N \mapsto \pi^{-1}(N)$ is an isomorphism of $\mathrm{Gr}_{\mathbf{e}}(M)$ with the variety of all sub–representations of R which contain $\mathrm{Ker}\,\pi = \iota(Q)$, and whose dimension vector is $\mathbf{f} = \mathbf{e} + \mathbf{dim}\,\mathrm{Ker}\pi$. It is straightforward to verify that it is a closed embedding (see e.g. [8, Section 3.2]). □

In the special case when the subvariety $\mathrm{Gr}_{\mathbf{f}}(Q{\dashrightarrow}R)$ of $\mathrm{Fl}_{\mathbf{f}}(R_n)$ is closed under the action of a Borel subgroup of $GL(R_n)$, we call it a *Schubert quiver Grassmannian* (SQG for short). The following result provides a characterization of SQGs.

Theorem 2.2. *The subvariety $\mathrm{Gr}_{\mathbf{f}}(Q{\dashrightarrow}R) \subseteq \mathrm{Fl}_{\mathbf{f}}(R_n)$ is closed under the action of a Borel subgroup of $GL(R_n)$ if and only if M is catenoid.*

Proof. We show that M is catenoid if and only if there exists a complete flag $F_\bullet$ which is a refinement of both $R_\bullet$ and $Q_\bullet$. In other words, M is catenoid if and only if there exists an ordered basis $\{v_1, \cdots, v_N\}$ of the vector space R_n such that $R_i = \mathrm{span}\{v_1, \cdots, v_{r_i}\}$ and $Q_i = \mathrm{span}\{v_1, \cdots, v_{q_i}\}$ for every $i = 1, \cdots, n$. This is equivalent to the fact that both $R_\bullet$ and $Q_\bullet$ are fixed by the Borel subgroup of upper triangular matrices, and hence the claim holds true.

If there exists such a basis, then the $\mathbf{C}\Gamma$–module structure on R is given naturally as follows: the underlying vector space is $\oplus_{i=1}^n R_i$ generated by the induced basis

$$\mathcal{B} = \{v_1^{(1)}, \cdots, v_{r_1}^{(1)}\} \cup \{v_1^{(2)}, \cdots, v_{r_2}^{(2)}\} \cup \cdots \cup \{v_1^{(n)}, \cdots, v_N^{(n)}\}$$

and the action of the path $(i \to i+1 \to \cdots \to j) \in \mathbf{C}\Gamma$ is given by the canonical embedding sending $v_k^{(i)} \mapsto v_k^{(j)}$. For every $k = 1, \cdots, N$, let $R(k)$ be the submodule generated by all the basis vectors $v_k^{(i)}$ for $i = i(k), \cdots, n$, where $i(k)$ is the minimal vertex of Γ such that $k \le r_{i(k)}$ so that $v_k^{(i(k))}$ is defined. Clearly, $R(k)$ is a projective direct summand of R isomorphic to $P_{i(k)}$. Similarly, the module Q admits, by hypothesis, a sub-basis $\mathcal{B}' \subset \mathcal{B}$, and we define the submodule $Q(k)$ similarly to $R(k)$. Then $Q(k)$ is isomorphic to the projective $P_{j(k)}$, where $j(k)$ is the minimal vertex such that $k \le q_{j(k)}$ so that $v_k^{(j(k))}$ belongs to Q. By construction, $Q(k)$ embeds into $R(k)$, and the quotient is $M[i(k), j(k)-1]$. Hence all the indecomposable direct summands of M are of the form $M[i(k), j(k)-1]$, for $k = 1, \cdots, N$. By construction, we have

$$i(k) \le i(s) \Longleftrightarrow k \le s \Longleftrightarrow j(k) \le j(s).$$

It follows that, given two distinct direct summands $M[i(k), j(k)-1]$ and $M[i(s), j(s)-1]$ of M such that $i(k) < i(s)$, we have $k < s$, and

hence $j(k) < j(s)$. This proves that all the direct summands of M are comparable with respect to the ordering (1.2), and hence M is catenoid by definition.

Now suppose that, conversely, $M = \bigoplus_{l=1}^{r} M(l)^{\oplus a_l}$ is catenoid with indecomposable direct summands $M(1), M(2), \cdots, M(r)$ ordered so that $M(l) < M(l+1)$ in the partial order (1.2). Their tops are the simples $S(i_1), S(i_2), \cdots, S(i_r)$ for some indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$, respectively, and their socles are $S(j_1), S(j_2), \cdots, S(j_r)$ for some vertices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq n$, respectively. A minimal projective resolution of M is obtained as the direct sum of the minimal projective resolutions of the $M(l)$'s, namely

$$0 \longrightarrow Q(l) \overset{\iota_l}{\longrightarrow} R(l) \overset{\pi_l}{\longrightarrow} M(l) \longrightarrow 0$$

where $R(l) = P_{i(l)}$, $Q_l = P_{j(l)+1}$ and ι_l and π_l are the canonical homomorphisms. For every vertex i, the vector space R_i is the direct sum of the one–dimensional vector spaces $R(1)_i, R(2)_i, \cdots, R(r_i)_i$, and hence it admits a basis $\mathcal{B}_i$ given by $v_1^{(i)}, v_2^{(i)}, \cdots, v_{r_i}^{(i)}$, where $v_l^{(i)}$ is the generator of $R(l)_i$. By construction, the vector subspace of R_i generated by $v_1^{(i)}, v_2^{(i)}, \cdots, v_{q_i}^{(i)}$ is Q_i. $\qquad\square$

2.1. **Group action on SQGs.** Let M be a Γ–representation with minimal projective resolution (2.1). Let $\mathbf{e}$ be a dimension vector for Γ and consider the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$. The group $\mathrm{Aut}(M)$ (consisting of Γ–automorphisms of M) naturally acts on $\mathrm{Gr}_{\mathbf{e}}(M)$. On the other hand, the closed subvariety $\mathrm{Gr}_{\mathbf{f}}(Q{\dashrightarrow}R)$ of $\mathrm{Fl}_{\mathbf{f}}(R_n)$ (where $\mathbf{f} := \mathbf{e} + \mathbf{dim}\, Q$) is acted upon by the following group:

$$\mathcal{P}_M := \{ f \in \mathrm{Aut}(R) | \, f \circ \iota = \iota \}$$

which consists of those automorphisms of R which fix $\iota(Q) \subseteq R$. Since $\mathrm{Gr}_{\mathbf{e}}(M)$ and $\mathrm{Gr}_{\mathbf{f}}(Q{\dashrightarrow}R)$ are isomorphic under the morphism φ of Proposition 2.1, it is natural to ask how those two actions are related. The next result says that they coincide.

Proposition 2.3. *There is an exact sequence of groups*

$$1 \longrightarrow K \longrightarrow \mathcal{P}_M \overset{\Psi}{\longrightarrow} \mathrm{Aut}(M) \longrightarrow 1$$

where $K = 1_R + \mathrm{Hom}_\Gamma(R, Q)$ and it acts trivially on $\mathrm{Gr}_{\mathbf{e}+\mathbf{dim}\, Q}(Q{\dashrightarrow}R)$. In particular the action of $\mathcal{P}_M$ on $\mathrm{Gr}_{\mathbf{e}+\mathbf{dim}\, Q}(Q{\dashrightarrow}R)$ coincides with the action of $\mathrm{Aut}(M)$ on $\mathrm{Gr}_{\mathbf{e}}(M)$ under the isomorphism φ.

Proof. We define the map Ψ as follows: let $f \in \mathcal{P}_M$. For $m \in M$ we define $\Psi(f)(m) := \pi \circ f(\pi^{-1}(m))$. First of all, we notice that Ψ is well–defined: indeed if $r, r' \in \pi^{-1}(m)$ we have $r - r' \in \iota(Q)$ and hence $\pi(f(r - r')) = \pi \circ f \circ \iota(r - r') = \pi \circ \iota(r - r') = 0$. Moreover Ψ is easily seen to be a homomorphism of groups. Let us show that Ψ is onto: let $g \in \mathrm{Aut}(M)$, then, since $\pi : R \to M$ is a minimal projective resolution

of M, there exists $f \in Aut(R)$ such that $g \circ \pi = \pi \circ f$ and thus $g = \Psi(f)$. Now, suppose that $\Psi(f) = 1_M$; this means that $\pi \circ f - \pi = 0$ or in other words: $\pi(f - 1_R) = 0$ which means that $f - 1_R \in Hom_\Gamma(R, Q)$.

Let $T \in \mathrm{Gr}_{\mathbf{e}+\mathbf{dim}\,Q}(Q \overset{\iota}{\rightarrowtail} R)$ (so that $\pi(T) \in \mathrm{Gr}_{\mathbf{e}}(M)$), and let $f \in \mathcal{P}$. We need to check that

$$\pi \circ f(T) = \Psi(f)(\pi(T)).$$

This follows immediately from the definition of Ψ and concludes the proof. $\qquad\square$

We now turn back our attention to catenoids. Let $M = \bigoplus_{l=1}^{r} M(l)^{\oplus a_l}$ be catenoid. Assume that each $M(l)$ is indecomposable and $M(l) < M(l+1)$ in the partial order (1.2). Let us denote by $N = a_1 + \cdots + a_r$ the number of indecomposable direct summands of M and we number them as $M_1 \leq M_2 \leq \cdots \leq M_N$ (each M_i is isomorphic to some $M(l)$). We fix an element $\phi_{i,j} \in \mathrm{Hom}_\Gamma(M_j, M_i)$, $1 \leq i, j \leq N$ in each homomorphism space (which is at most one–dimensional); we assume that $\phi_{i,j} \neq 0$ if $\mathrm{Hom}_\Gamma(M_j, M_i)$ is non zero. It is worth noting that ϕ_{ij} can be zero even if $i < j$. The elements $\phi_{i,j}$, considered as elements of $\mathrm{Hom}_\Gamma(M, M)$, span the vector space $\mathrm{Hom}_\Gamma(M, M)$. In particular, every element $\sum_{i,j} a_{ij} \phi_{ij}$ of $\mathrm{Aut}(M)$ comes from an $N \times N$ matrix $\sum_{i,j} a_{ij} E_{ij}$ (where E_{ij} is the elementary matrix with 1 in place (i, j) and zero elsewhere); so the whole automorphism group of M can be seen as the image of the standard parabolic subgroup $P \subset GL_N$ with blocks of sizes $a_1, \ldots, a_r$. We thus obtain the homomorphism $P \to Aut(M)$. In Proposition 2.5 we identify P with $\mathcal{P}_M$ of Proposition 2.3. We will need the following lemma.

Lemma 2.4. *The group* $\mathrm{Aut}(M)$ *is generated by its Levi part* $L(M) = \prod_{l=1}^{r} \mathrm{Aut}(M(l)^{a_l})$ *and its unipotent part* $U(M)$, *generated by the one-parameter subgroups* $\exp(z\phi_{i,j}, z \in \mathbb{C})$ *such that* M_i *and* M_j *are not isomorphic.*

Proof. An automorphism of M is completely determined by restrictions to the indecomposable summands. This implies the lemma. $\qquad\square$

We now provide another characterization of catenoids. Let $M = \bigoplus_{l=1}^{r} M(l)^{\oplus a_l}$ be a Γ–representation (not necessarily catenoid) with minimal projective resolution (2.1) and indecomposable direct summands $M_1, M_2, \cdots, M_N$. The automorphism group of R can be naturally embedded into GL_N as follows: We denote the projective cover of each M_i by P^i so that $R = P^1 \oplus P^2 \oplus \cdots \oplus P^N$. Without loss of generality, we can assume that $P^1 \leq P^2 \leq \cdots \leq P^N$ in the partial order (1.2). We denote by ρ_{ij} a generator of the vector space $\mathrm{Hom}_\Gamma(P^j, P^i)$ and by $\Omega = \{(i, j) | \rho_{ij} \neq 0\}$. Then an element $f \in \mathrm{Aut}(R)$ has the form $\sum_{(i,j) \in \Omega} b_{ij} \rho_{ij}$ and we send this element to the matrix $\sum_{i,j} b_{ij} E_{ij} \in GL_N$. Notice that $\mathrm{Hom}_\Gamma(P^j, P^i)$ is one-dimensional

for every $1 \leq i \leq j \leq N$, and hence the set of all matrices $\sum_{(i,j) \in \Omega} b_{ij} E_{ij}$ form a parabolic subgroup of GL_N that we denote by $\mathcal{R}$ ($\mathcal{R}$ contains the Borel subgroup of upper triangular matrices). Under the isomorphism $\text{Aut}(R) \to \mathcal{R}$, we denote by P the image of the subgroup $\mathcal{P}_M$ defined above.

Proposition 2.5. *Let M be a Γ–representation. Then M is catenoid if and only if P is a parabolic subgroup of GL_N. In this case P is the standard parabolic subgroup of GL_N associated with the partition $N = a_1 + \cdots + a_r$ with Levi decomposition $P = LU$. The restriction of Ψ to L is an isomorphism to $L(M)$ and $\Psi(U) = U(M)$.*

Proof. We know from the proof of Theorem 2.2 that M is catenoid if and only if the two flags $R_\bullet$ and $Q_\bullet$ are both stable under the action of a same Borel subgroup $B_N \subseteq GL_N$. In particular, $B_N \subseteq \mathcal{R}$ stabilizes $Q_\bullet$ and hence $B_N \subseteq P$.

In the following, we use the notation $\beta_k := a_1 + \cdots + a_k$. Let P' be the standard parabolic subgroup of GL_N associated with the partition $N = a_1 + \cdots + a_r$. We define the map $\xi : P' \to \text{Aut}(R)$ which sends an element $g = \sum_{i,j=1}^N g_{i,j} E_{i,j} \in P'$ to the element $\xi(g) := \sum_{i,j=1}^N g_{i,j} \rho_{i,j} \in \text{Aut}(R)$. First of all, ξ is well–defined: indeed every pair (i,j) of the form $\beta_k \leq j < i < \beta_{k+1}$ (for some $k = 1, 2, \cdots, r-1$) belongs to Ω since $P^i = P^j$ is the projective cover of $M(k)$ and hence $\rho_{ij} \neq 0$. Clearly, ξ is an injective homomorphism of groups. We claim that its image is $\mathcal{P}_M$. Indeed, we notice that the Borel subgroup $B_N \subset \mathcal{R} \simeq \text{Aut}(R)$ is contained in such image. It remains to show the following:

$$1 + \rho_{ij} \notin \mathcal{P}_M \text{ whenever } (i,j) \in \Omega \text{ and } j \leq \beta_k < i \text{ for some } k,$$

and

$$1 + \rho_{ij} \in \mathcal{P}_M \text{ whenever } (i,j) \in \Omega \text{ and } \beta_k \leq j < i < \beta_{k+1} \text{ for some } k.$$

To show the first statement, we take a pair $(i,j) \in \Omega$ such that $\beta_{l-1} < j \leq \beta_l \leq \beta_{k-1} < i \leq \beta_k$ for some l and k. Then P^j is the projective cover of $M(l)$ and P^i is the projective cover of $M(k)$ and $P^i = P^j$. So we have $M(l) < M(k)$ and they have the same top. This means that $M(l) = M[r,s]$ and $M(k) = M[r,t]$ for some $r \leq s < t$. But then $Q(l) = M[s+1,n]$ and $Q(k) = M[t+1,n]$ and hence the morphism ρ_{ij} does not stabilize $\iota(Q)$ since $[Q(k), Q(l)] = 0$.

To show the second statement we take a pair $(i,j) \in \Omega$ such that $\beta_k \leq j < i < \beta_{k+1}$. In this case $P^i = P^j$ and $Q^i = Q^j$ and hence ρ_{ij} stabilizes $\iota(Q)$.

To conclude the proof we notice that the map Ψ of Proposition 2.3 sends $1 + \rho_{ij} \in \mathcal{P}_M$ to $1 + \phi_{ij} \in \text{Aut}(M)$. $\qquad\square$

Remark 2.6. From now on we will freely use Propositions 2.3 and 2.5 and we will not distinguish between the action of B_N on $\text{Gr}_\mathbf{f}(Q \dashrightarrow R)$

and the action of $\mathrm{Aut}(M)$ on a SQG $\mathrm{Gr}_{\mathbf{e}}(M)$. We will freely say that B_N acts on $\mathrm{Gr}_{\mathbf{e}}(M)$ and that $\mathrm{Aut}(M)$ acts on $\mathrm{Gr}_{\mathbf{f}}(Q {\dashrightarrow} R)$.

3. Irreducible components of SQGs

By definition, a SQG is, in paticular, a closed B–stable subvariety of a partial flag variety, and hence its irreducible components are Schubert varieties. In this section we describe these Schubert varieties.

Let V be an N-dimensional vector space with a basis $v_1, \ldots, v_N$. We consider the algebraic group SL_N acting on V, the Borel subgroup B_N consisting of upper-triangular matrices, and the maximal torus T_N of diagonal matrices. To the vector space V we attach the representation $\mathbf{V}$ of A_n such that $\mathbf{V}_i = V$, and such that all the maps are identity maps (in other words, $\mathbf{V} \simeq P_1^{\oplus N}$).

Recall the projective resolution (2.1) of M. We note that the projective representation R is a sub–representation of $\mathbf{V}$. Let $\mathbf{dim}\, R = (r_1, \ldots, r_n)$ and $\mathbf{dim}\, Q = (q_1, \ldots, q_n)$. Let $W = \mathcal{S}_N$ be the Weyl group of SL_N. Given a dimension vector $\mathbf{e}$, we set $\mathbf{f} = \mathbf{e} + \mathbf{dim}\, Q$, so that $f_a = e_a + q_a$ for all $a = 1, 2, \cdots, n$.

Definition 3.1. An element $w \in \mathcal{S}_N$ is called $(M, \mathbf{e})$–*compatible* if

$$\{1, \ldots, q_a\} \subset w(\{1, \ldots, f_a\}) \subset \{1, \ldots, r_a\} \text{ for all } 1 \le a \le n.$$

We denote by $W(M, \mathbf{e}) \subset \mathcal{S}_N$ the set of all $(M, \mathbf{e})$–compatible elements.

Recall that, to an element $w \in \mathcal{S}_N$, one attaches the T_N-fixed point $p(w) \in \mathrm{Fl}_{\mathbf{e}+\mathbf{dim}\, Q}(V)$ which is the flag whose i–th vector space is span $\{v_{w(1)}, \cdots, v_{w(i)}\}$. Two elements w_1 and w_2 induce the same T_N-fixed point (that is, $p(w_1) = p(w_2)$) if and only if

$$w_1(\{1, \ldots, f_a\}) = w_2(\{1, \ldots, f_a\}), \text{ for all } a = 1, \ldots, n.$$

In this case we say that w_1 is equivalent to w_2. Let us pick one element of minimal length in each equivalence class and denote this set by $W^0(M, \mathbf{e})$.

We define a partial order on $W^0(M, \mathbf{e})$. For $1 \le a \le n$, let

$$w(\{1, \ldots, f_a\}) = \{k_1^a(w), \ldots, k_{f_a}^a(w)\} \text{ and } k_1^a(w) < \cdots < k_{f_a}^a(w).$$

Then we say that $w_1 \le w_2$ if $k_i^a(w_1) \le k_i^a(w_2)$ for all $1 \le a \le n$, $1 \le i \le f_a$. Note that this order depends on the equivalence classes of w_1 and w_2, but not on the elements themselves.

Lemma 3.2. *We have $w_1 \le w_2$ if and only if $p(w_1) \in \overline{B_N\, p(w_2)}$.*

Proof. The partial order put on $W^0(M, \mathbf{e})$ is nothing but the Bruhat order and the the claim is hence a well–known fact in the theory of Schubert varieties (see e.g. [7, Section 10.5]). □

Let $S(M, \mathbf{e}) \subset W^0(M, \mathbf{e})$ be the set of maximal elements in this poset. Lemma 3.2 implies that the SQG $\mathrm{Gr}_\mathbf{e}(M)$ is irreducible if and only if the set $S(M, \mathbf{e})$ consists of a single element. This case can be described explicitly in the following way. First of all, we perform some reductions: If $e_a > \dim M_a$ for some vertex $1 \leq a \leq n$, then our quiver Grassmannian $\mathrm{Gr}_\mathbf{e}(M)$ is empty. If $e_a = \dim M_a$ or $r_a \leq q_{a+1}$ (this is equivalent to the fact that the map $M_a \to M_{a+1}$ is zero), the SQG factors into the product of two quiver Grassmannians (again SQGs) for smaller equioriented type A quivers. Similarly, this is the case when $e_a = 0$ for some vertex a.

Definition 3.3. A SQG $\mathrm{Gr}_\mathbf{e}(M)$ is called *simple* if $0 < e_a < \dim M_a$ and $r_a > q_{a+1}$ for all vertices a.

It is clear from the remark above that every irreducible SQG is a product of simple irreducible SQGs (associated with quivers with a smaller number of vertices).

Theorem 3.4. *A simple SQG $\mathrm{Gr}_\mathbf{e}(M)$ is irreducible if and only if $r_a - e_a \geq r_{a+1} - e_{a+1}$ for all vertices a.*

Proof. $\mathrm{Gr}_\mathbf{e}(M)$ is irreducible if and only if $S(M, \mathbf{e})$ consists of a single element. We note that this happens if and only if there exists an element $w \in W(M, \mathbf{e})$ such that

$$w(\{1, \ldots, f_a\}) = \{1, \ldots, q_a\} \cup \{r_a, r_a - 1, \ldots, r_a - e_a + 1\}$$

for every vertex a. This is equivalent to $r_a - e_a \geq r_{a+1} - e_{a+1}$ for all a. Indeed, since $r_{a+1} > f_{a+1}$, the inclusion $w(\{1, \ldots, f_a\}) \supset \{r_a, r_a - 1, \ldots, r_a - e_a + 1\}$ implies $r_a - e_a + 1 \leq r_{a+1} - e_{a+1} + 1$. $\qquad \square$

Corollary 3.5. *If a simple SQG $\mathrm{Gr}_\mathbf{e}(M)$ is irreducible, the dimension of $\mathrm{Gr}_\mathbf{e}(M)$ is equal to the Euler form $\langle \mathbf{e}, \dim M - \mathbf{e} \rangle$, and there exists a short exact sequence*

$$0 \longrightarrow P \longrightarrow M \longrightarrow I \longrightarrow 0$$

where P is projective, I is injective, and $\mathbf{e} = \dim P$.

Proof. Assume that $r_a - e_a \geq r_{a+1} - e_{a+1}$ for all $a = 1, 2, \cdots, n-1$, so that $S(M, \mathbf{e})$ consists of a single element w. We show that the point $p(w)$ of $\mathrm{Fl}_\mathbf{f}(R_n)$ corresponds to a projective sub–representation P of M (under the map Ψ of Proposition 2.1), such that the quotient M/P is injective. By hypothesis, $e_a \leq e_{a+1}$ for all a, and hence there exists a projective representation P of dimension vector $\mathbf{e}$. We have $r_a - e_a \geq r_{a+1} - e_{a+1} > f_{a+1} - e_{a+1} = q_{a+1}$, and hence $e_a + q_{a+1} < r_a$; this implies that P embeds into M. To show that the quotient is injective, we note that $r_a - e_a - q_a \geq r_{a+1} - e_{a+1} - q_{a+1}$.

Finally, the dimension of $\mathrm{Gr}_{\mathbf{e}}(M)$ is equal to the dimension of the corresponding Schubert variety. This dimension is equal to

$$\sum_{a=1}^{n}(e_a - e_{a-1})(r_a - f_a) = \sum_{a=1}^{n}(e_a - e_{a-1})(d_a - e_a) = \langle \mathbf{e}, \dim M - \mathbf{e}\rangle,$$

with the convention that $e_0 := 0$.   □

Corollary 3.6. *If a simple SQG $\mathrm{Gr}_{\mathbf{e}}(M)$ is irreducible, then $e_a \leq e_{a+1}$ and $d_a - e_a \geq d_{a+1} - e_{a+1}$.*

The following result provides a representation–theoretic interpretation of Theorem 3.4 and shows that the necessary condition for irreducibility of a quiver Grassmannian given in [9, Theorem 5.1] is also sufficient for simple SQGs.

Corollary 3.7. *A simple SQG $\mathrm{Gr}_{\mathbf{e}}(M)$ is irreducible if and only if $[M, U] \leq \langle \mathbf{e}, \dim U\rangle$ for every non–injective indecomposable representation U.*

Proof. In view of Theorem 3.4, $\mathrm{Gr}_{\mathbf{e}}(M)$ is irreducible if and only if

(3.1) $r_a - e_a \geq r_{a+1} - e_{a+1}$ for all $a = 1, 2, \cdots, n-1$,

where $(r_1, \cdots, r_n)$ is the dimension vector of the projective cover R of M. Since $\pi : R \to M$ is surjective, we have $[M, U] \leq [R, U]$ for all U and it is hence enough to prove that the conditions (3.1) are satisfied if and only if the conditions

(3.2) $[R, U] \leq \langle \mathbf{e}, \dim U\rangle$ for all indecomposable non–injective U

are satisfied. Let $U = M[i, j]$ for some $2 \leq i \leq j \leq n$. We have $[R, U] = r_j - r_{i-1}$ and $\langle \mathbf{e}, \dim U\rangle = e_j - e_{i-1}$. If (3.1) holds, then $r_{i-1} - e_{i-1} \geq r_i - e_i \geq \cdots \geq r_j - e_j$, and hence (3.2) holds. Conversely, if (3.2) holds, then, by choosing $U = S_{a+1}$ ($a = 1, 2, \cdots, n-1$), we get $r_{a+1} - r_a \leq e_{a+1} - e_a$, and hence (3.1) holds.   □

Assume that $\mathrm{Gr}_{\mathbf{e}}(M)$ is irreducible and hence isomorphic to a Schubert variety. We determine explicitly the (unique) Weyl group element in $S(M, \mathbf{e})$. Denote by $s_i = (i, i+1)$ for $i = 1, \ldots, N-1$ the simple reflections in the symmetric group $\mathcal{S}_N$. For $1 \leq a \leq n-1$ let $m_a := (d_a - e_a) - (d_{a+1} - e_{a+1})$. We define a permutation π_a as

$$(s_{q_a+m_a} \ldots s_{q_a+1}) \ldots (s_{q_a+e_a+m_a-2} \ldots s_{q_a+e_a-1})(s_{q_a+e_a+m_a-1} \ldots s_{q_a+e_a}).$$

For example, for $a = n$ we have

$$\pi_n = (s_{r_n-e_n} \ldots s_{q_n+1}) \ldots (s_{r_n-2} \ldots s_{q_n+e_n-1})(s_{r_n-1} \ldots s_{q_n+e_n})$$

We note that the number of factors of π_a is equal to $e_a m_a$ (and equal to $e_n(d_n - e_n)$ for π_n).

Proposition 3.8. *Whenever a simple SQG $\mathrm{Gr}_{\mathbf{e}}(M)$ is irreducible, the unique element of $S(\mathbf{e}, M)$ is equal to $w = \pi_n \pi_{n-1} \cdots \pi_2 \pi_1$.*

Proof. We need to verify that

$$w(\{1, \ldots, q_a + e_a\}) = \{1, \ldots, q_a\} \cup \{r_a - e_a + 1, \ldots, r_a\},$$

and that w is the element of minimal length with this property. The first statement is proved by a direct computation (starting from $a = n$ and descending to $a = 1$), and the second statement follows from the equality

$$\sum_{a=1}^{n} e_a m_a = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle.$$

$\square$

Example 3.9 (Degenerate flag varieties). Let us consider the catenoid M which is the direct sum of all the indecomposable projective and the indecomposable injective Γ_n–representations:

$$M := P_1 \oplus P_2 \oplus \cdots \oplus P_n \oplus I_1 \oplus I_2 \oplus \cdots \oplus I_n.$$

Given the dimension vector $\mathbf{e} = (1, 2, \cdots, n)$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is isomorphic to the $\mathfrak{sl}_{n+1}$–degenerate flag variety [6, 3] and it is isomorphic to a Schubert variety X_{σ_n} [1] for a permutation $\sigma_n \in \mathcal{S}_{2n}$. Let us show that σ_n coincides with the word $w = w_n$ of Proposition 3.8 and hence the main result of [1] becomes a particular case of Theorem 3.4 and Proposition 3.8. First of all, M is clearly catenoid, $R = P_1^{n+1} \oplus P_2 \oplus \cdots \oplus P_n$ and $Q = P_2 \oplus \cdots \oplus P_n$ so that $r_a = n + a$, $q_a = a - 1$ and $m_a = 1$ for all $a = 1, 2, \cdots, n$. In particular, $r_a - e_a = n$ for all a, and hence $\mathrm{Gr}_{\mathbf{e}}(M)$ is an irreducible SQS (in view of Theorem 3.4). We have $\pi_a = s_a s_{a+1} \cdots s_{2a-1} \in \mathcal{S}_{2n}$ which is the permutation given by

$$\pi_a(k) = \begin{cases} k & \text{if } k \notin \{a, \cdots, 2a\} \\ a & \text{if } k = 2a \\ k+1 & \text{if } k \in \{a, \cdots, 2a-1.\} \end{cases}$$

By definition,

$$w = w_n = (s_n s_{n+1} \cdots s_{2n-1})(s_{n-1} s_n \cdots s_{2n-1}) \cdots (s_3 s_4 s_5)(s_2 s_3) s_1.$$

We claim that

$$(3.3) \qquad w_n(r) = \begin{cases} k & \text{if } r = 2k \\ n+1+k & \text{if } r = 2k+1. \end{cases}$$

To prove (3.3) one can proceed by induction on $n \geq 1$, by noting that $w_n = \pi_n w_{n-1}$ (after identifying $\mathcal{S}_{2n-2}$ with the subgroup of $\mathcal{S}_{2n}$ generated by $s_1, s_2, \cdots, s_{2n-3}$). Formula (3.3) shows that w_n is the permutation σ_n found in [1].

Now we consider the general case of reducible SQGs.

Theorem 3.10. *The T_N–fixed points of a SQG $\mathrm{Gr}_\mathbf{e}(M)$ are labelled by the elements of $W^0(M, \mathbf{e})$; the quiver Grassmannian $\mathrm{Gr}_\mathbf{e}(M)$ admits a cellular decomposition, the cells being the B_N–orbits of the T_N–fixed points. The irreducible components of $\mathrm{Gr}_\mathbf{e}(M)$ are parametrized by the elements of $S(M, \mathbf{e})$.*

Proof. Since $\mathrm{Gr}_\mathbf{e}(M)$ is B_N–stable, it is equal to a union of several Schubert varieties. Therefore, a cellular decomposition is provided by the B_N–orbits of the T_N-fixed points. Now it suffices to use Lemma 3.2. □

Example 3.11. Let $M = M[1,1] \oplus \bigoplus_{i=1}^{n-1} M[i, i+1] \oplus M[n,n]$. In particular, $\mathbf{dim}\, M = (2, \ldots, 2)$. Let $\mathbf{e} = (1, \ldots, 1)$. Then $N = n+1$, $q_a = a - 1$, $r_a = a + 1$, and the number of irreducible components of $\mathrm{Gr}_\mathbf{e}(M)$ (that is, the number of elements in $S(\mathbf{e}, M)$) is the n-th Fibonacci number. Indeed, assume that $w \in S(\mathbf{e}, M)$. Then, for all a, there are two possibilities: either $w(\{1, \ldots, a\}) = \{1, \ldots, a\}$ with no additional restrictions on $w(\{1, \ldots, a+1\})$, or

$$w(\{1, \ldots, a\}) = \{1, \ldots, a-1\} \cup \{a+1\},\ w(1, \ldots, a+1) = \{1, \ldots, a+1\}.$$

To any such w we attach a length n sequence of units and zeroes, where 0 appears if $w(\{1, \ldots, a\}) = \{1, \ldots, a\}$, and 0 appears otherwise. Now the $w \in S(\mathbf{e}, M)$ are labelled by the length n sequences such that the pair $(1, 1)$ is forbidden. The number of such sequences is exactly the Fibonacci number. We note that the irreducible component attached to such a sequence is isomorphic to the product of several copies of $\mathbb{P}^1$, and the number of copies is the number of 1's in the sequence.

4. Poincaré polynomials of SQGs

Let $\mathbf{p}$ be a Dyck path on $[1, n]$, and let $M = \bigoplus_{l=1}^r M(p_l)^{m(p_l)}$ be a representation supported on $\mathbf{p}$. The aim of this section is to compute the Poincaré polynomial in singular homology of the quiver Grassmannian $\mathrm{Gr}_\mathbf{e}(M)$.

Lemma 4.1. *Assume that all the indecomposable direct summands of M are isomorphic, that is, there exists a unique $p = (a, b) \in \mathbf{p}$ such that $m(p) > 0$. Then $\mathrm{Gr}_\mathbf{e}(M)$ is empty unless $e_i = 0$ for $i < a$, $i > b$ and $e_a \leq \cdots \leq e_b$. If $\mathrm{Gr}_\mathbf{e}(M)$ is non-empty, then the Poincaré polynomial of $\mathrm{Gr}_\mathbf{e}(M)$ is given by the q-multinomial coefficient*

$$P_{\mathrm{Gr}_\mathbf{e}(M)}(q) = \binom{m(p)}{e_a, \ldots, e_b}_q$$

Proof. If $e_i = 0$ for $i < a$, $i > b$ and $e_a \leq \cdots \leq e_b$, then the quiver Grassmannian is isomorphic to the partial flag variety $\mathrm{Fl}_{e_a, \ldots, e_b}(\mathbb{C}^{m(p)})$. The cells for the standard cellular decomposition are labeled by collections of subsets $K_\bullet = (K_a \subset \cdots \subset K_b)$ of a set of cardinality $m(p)$ such

that $\#K_i = e_i$; each cell is the orbit of the Borel subgroup through the T-fixed point defined by the sets $K_\bullet$. $\square$

Now let us consider the general case. We want to stratify $\mathrm{Gr}_\mathbf{e}(M)$ in such a way that each stratum is a fibration over a product of partial flag varieties, with fibers being affine spaces. Namely, we decompose $\mathbf{e}$ as a sum $\mathbf{e} = \mathbf{e}(1) + \cdots + \mathbf{e}(r)$ in such a way that

(i) $\mathbf{e}(l)$ is supported on p_l, that is, if $p_l = (i_l, j_l)$, then $\mathbf{e}(l)_a = 0$ unless $i_l \leq a \leq j_l$,

(ii) $\mathbf{e}(l)_a \leq \cdots \leq \mathbf{e}(l)_b \leq m(p_l)$ for all vertices $a \leq b$.

Then we obtain an embedding

$$(4.1) \qquad \prod_{l=1}^r \mathrm{Gr}_{\mathbf{e}(l)}(M(p_l)^{\oplus m(p_l)}) \subset \mathrm{Gr}_\mathbf{e}(M).$$

We denote the image of this embedding by $\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$.

Now we consider the subgroups $U(M), L(M) \subset Aut(M)$ (see Lemma 2.4). We recall that the groups $U(M)$ and $L(M)$ generate $\mathrm{Aut}(M)$, and that $\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$ is $L(M)$-invariant.

Theorem 4.2. *We have*

$$\mathrm{Gr}_\mathbf{e}(M) = \bigsqcup U(M)\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r)),$$

where the disjoint union is taken over all possible sequences $\mathbf{e}(1), \ldots, \mathbf{e}(r)$ satisfying (i) *and* (ii)*. Moreover, the map*

$$U(M)\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r)) \to \Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r)), \ \ gx \mapsto x$$

is an affine space fibration.

Proof. We need to prove that the union $\bigsqcup U(M)\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$ covers the whole quiver Grassmannian, and that the map above is indeed a fibration.

To prove the first claim, consider the embedding $\varphi : \mathrm{Gr}_\mathbf{e}(M) \to \mathrm{Fl}_{\mathbf{e}+\mathbf{dim}\,Q}(R_n)$ of Proposition 2.1, its image being the B_N–stable subvariety $\mathrm{Gr}_{\mathbf{e}+\mathbf{dim}\,Q}(Q \rightarrowtail R)$. Any point of this image lies in the B_N-orbit of some T_N-fixed point γ of $\mathrm{Fl}_{\mathbf{e}+\mathbf{dim}\,Q}(R_n)$. We note that γ belongs to some $\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$. Since $\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$ is $L(M)$-invariant, and B_N is generated by $L(M)$ and $U(M)$, the first claim follows.

Now we consider the map

$$U(M)\,\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r)) \to \Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r)), \qquad gx \mapsto x.$$

We fix a T_N–fixed point $\gamma \in \Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$ and write $\gamma = \prod_{l=1}^r \gamma(l)$ according to the factors of (4.1). There exists an open cell $C(l)$ in each $\mathrm{Gr}_{\mathbf{e}(l)}(M(p_l)^{\oplus m(p_l)})$, containing $\gamma(l)$. Now it is easy to see that the orbit $U(M)C(l)$ is canonically isomorphic to the product

$$C(l) \times \mathrm{Hom}_\Gamma(\gamma(l), \bigoplus_{s<l} M(p_s)^{\oplus m(p_s)})),$$

where $\gamma(l)$ is considered as a representation of Γ. $\square$

For $1 \leq l \leq r$ we define $\beta(l)_i = \sum_{s<l} m(p_s) \dim M(p_s)_i$.

Lemma 4.3. *The dimension $D(\mathbf{e}(1), \ldots, \mathbf{e}(r))$ of the fiber of the map $U(M)\Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r)) \to \Pi(\mathbf{e}(1), \ldots, \mathbf{e}(r))$ is equal to*

$$\sum_{l=1}^{r} \sum_{i=1}^{n} (e(l)_i - e(l)_{i-1})\beta(l)_i.$$

Proof. We fix a vector u in the vector space $(M(p_l)^{\oplus m(p_l)})_i$. We note that $\beta(l)_i$ is the dimension of the orbit $U(M)u$. This implies the claim of the lemma. $\square$

Corollary 4.4. *We have the following formula for the Poincaré polynomial of the SQG $\mathrm{Gr}_{\mathbf{e}}(M)$:*

$$P_{\mathrm{Gr}_{\mathbf{e}}(M)}(q) = \sum_{\mathbf{e}(1),\ldots,\mathbf{e}(r)} D(\mathbf{e}(1), \ldots, \mathbf{e}(r)) \prod_{l=1}^{r} \binom{m(p_l)}{\mathbf{e}(1), \ldots, \mathbf{e}(l)}_q$$

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