RANDOM WALKS ON QUASI ONE DIMENSIONAL LATTICES: LARGE DEVIATIONS AND FLUCTUATION THEOREMS

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Abstract. Several stochastic processes modeling molecular motors on a linear track are given by random walks (not necessarily Markovian) on quasi 1d lattices and share a common regenerative structure. Analyzing this abstract common structure, we derive information on the large fluctuations of the stochastic process by proving large deviation principles for the first–passage times and for the position. We focus our attention on the Gallavotti–Cohen–type symmetry of the position rate function (fluctuation theorem), showing its equivalence with the independence of suitable random variables. In the special case of Markov random walks, we show that this symmetry is universal only inside a suitable class of quasi 1d lattices.

Keywords: Markov chain, Random time change, Large deviation principle, Gallavotti–Cohen–type symmetry, Fluctuation Theorem, Molecular motor.

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1. Introduction

Molecular motors are proteins working as nanomachines [23]: they usually convert chemical energy coming from ATP hydrolysis to produce mechanical work fundamental e.g. for cargo transport inside the cell, cell division, genetic transcription, muscle contraction. Molecular motors are object of intensive study in biology and biophysics. They are crucial in several fundamental biological processes and are also relevant from a theoretic viewpoint in statistical physics, since they are small systems operating inside an environment with large thermal fluctuations (differently from macroscopic motors) and in a out–of–equilibrium regime. We concentrate here on the large family of molecular motors working in a non–cooperative way and moving along the cytoskeletal filaments, which are given by polarized homogeneous polymers.

The theoretical study of molecular motors has been developed from two main modelizations. In the so called *Brownian ratchet* model [24, 38, 39] the dynamics of the molecular motor is given by a one–dimensional diffusion in a switching force field (i.e. the force field changes at random times). The other paradigm, on which we concentrate here, is given by continuous time random walks[1] on quasi linear graphs having a periodic structure [18, 19, 26, 27, 28, 29, 44]. We call these graphs *quasi 1d lattices*, since they are obtained by gluing together several copies of a fundamental cell in a linear fashion. The geometric complexity of the fundamental cell (given by a finite, oriented and connected graph) corresponds to the conformational transformations of the molecular motor in its mechanochemical cycle. The simplest example is given by a random walk on $\mathbb{Z}$ with periodic jump rates [12] (in this case the fundamental cell is given by an interval with

[1]By *random walks* we mean stochastic jump processes on a given graph. When we restrict to random walks given by Markov chains [37] (hence with exponential waiting times), we call them *Markov random walks*

N sites, N being the periodicity), while random walks on other classes of quasi 1d lattices (parallel–chain models and divided–chain models) have been studied motivated by experimental evidence of a richer structure [8, 9, 25].

In a companion paper [15] we have studied in full generality both the asymptotic velocity (law of large numbers) and the gaussian fluctuations (invariance principle) for random walks on quasi 1d lattices. We focus here on their large deviations. Large deviations and Gallavotti–Cohen–type symmetries (also called *fluctuation theorems*), which are given by special identities satisfied by the rate function, have received in the last decade much attention inside non–equilibrium statistical physics of small systems and in particular for molecular motors (cf. [1, 2, 17, 31, 42, 43] and references therein).

We treat random walks on quasi 1d lattices in full generality. All relevant information concerning the position of the random walk is encoded in an associated random walk on $\mathbb{Z}$ with nearest neighbor jumps and typically non–exponential holding times, that we call *skeleton process*. We derive for the latter the LDP for the first–passage times as well as for the position (cf. Theorem 1). We also obtain a detailed qualitative analysis of the rate functions of the above LDPs (cf. Theorem 2 and Proposition 5.3). The tools developed in this part are fundamental to investigate the Gallavotti–Cohen symmetry (shortly, GC symmetry) of the form $I(\vartheta) = I(-\vartheta) + c\vartheta$, where I is the LD rate function for the position of the skeleton process, $\vartheta \in \mathbb{R}$ and c is a suitable constant. The GC symmetry has been derived in [30, 31] for Markov random walks on $\mathbb{Z}$ with periodic rates of period 2. These random walks and their large deviations have been analyzed in [30] by matrix methods, allowing to study also an enriched process taking into account the ATP consumed by the molecular motor. We restrict here to the molecular motor position (i.e. the skeleton process) and show that the GC symmetry pointed out in [30] cannot hold for a generic Markov random walk on a quasi 1d lattice. Indeed, we show that there exists a class of quasi 1d lattices (called $(\underline{v}, \overline{v})$*–minimal*) such that the GC symmetry is verified for any choice of the rates, while outside that class the GC symmetry is violated for Lebesgue any choice of the rates. This result implies that a priori one cannot expect to observe this symmetry even if nanotechnology would allow the observations of large deviations. Moreover, it answers the basic question of how universal the GC symmetry discovered by [30] in a simple model is. The relevance of both these issues (possible experimental evidence and universality) has been stressed also in [31]. In [16] we will continue our analysis discussing more in detail the connection with the GC functional [32] and why the validity of the GC symmetry for the above class of quasi 1d lattices is indeed a consequence of a universal symmetry for algebraic currents [17]. In [16] we will also consider some examples.

We conclude this introduction with some comments on technical aspects. When considering Markov random walks the proof of the position LD principle is simpler, obtained by the Gärtner–Ellis theorem [22] and by generalizing the matrix approach introduced by [30] (cf. Theorem 3). On the other hand it gives no insight on the mechanism leading to the GC symmetry. The results, presented in Section 2, concerning the LD principles for first–passage times and for the position (Theorems 1 and 2) hold also for non–Markov random walks on quasi 1d lattices and indeed for stochastic processes on quasi 1d lattices with a suitable regenerative structure (Theorems 6 and 7). More precisely, they hold for stochastic processes $(Z_t)_{t\in\mathbb{R}_+}$ obtained as follows. Consider a sequence $(w_i, \tau_i)_{i\geq 1}$ of i.i.d. 2d vectors with values in $\mathbb{R} \times (0, +\infty)$. Defining $W_m := \sum_{i=1}^m w_i$ and $\mathcal{T}_m := \sum_{i=1}^m \tau_i$ for $m \geq 0$ integer, set $Z_t := W_{\max\{m\geq 0:\, \mathcal{T}_m \leq t\}}$. Sums of i.i.d. random variables have many nice properties and random time changes are not troublesome for what concerns the LLN and the invariance principles [15]. On the other hand, the derivation of the LDP for $(Z_t)_{t\in\mathbb{R}_+}$ from the large deviation properties of $(W_m)_{m\geq 0}$ and $(\mathcal{T}_m)_{m\geq 0}$ is much more delicate. In [13] a LDP is obtained under the condition that the τ_i's have finite logarithmic moment generating function. This condition is not satisfied when considering Markov random walks on quasi 1d lattices, hence in our case the results of [13], an the similar ones of [41], cannot be applied. In the context of LDPs for processes under random time changes we also mention the new progresses obtained in [33, 36]. Restricting to the case $w_i \in \{-1, 1\}$ (which covers the applications to molecular motors), the process $(Z_t)_{t\in\mathbb{R}_+}$ becomes a random walk on $\mathbb{Z}$ with generic holding times (not necessarily exponential). Following the main scheme presented in [10] we derive the LDP for the process $(Z_t)_{t\in\mathbb{R}_+}$. We point out some technical issues making our analysis different from [10]: we allow correlations between w_i and τ_i (absent in [10]), moreover the minimum in the support of the law of τ_i can be zero or positive (the first case is excluded in [10]). Hence, although we have no random environment (thus of course simplifying the analysis) in our case there is a richer scenario for the possible behavior of the rate functions of the process $(Z_t)_{t\in\mathbb{R}_+}$ and of the associated first–passage times, and this behavior has to be investigated and kept in consideration in order to prove LDPs (see Section 5.2).

The theorems concerning the GC symmetry are the most innovative ones from a mathematical viewpoint. Using the above LD analysis, in Theorems 4 and 8 we prove several characterizations of the GC symmetry for $(Z_t)_{t\in\mathbb{R}_+}$, including the fact that it holds if and only if w_i and τ_i are independent, thus clarifying the probabilistic mechanism leading to the GC symmetry. Using the above characterizations, we study the GC symmetry for Markov random walks (Theorem 5). The validity of the GC symmetry for Markov random walks on $(\underline{v}, \overline{v})$–minimal 1d lattices is derived by introducing a special path transformation and comparing the original paths with the transformed ones. On the other hand, the proof of the almost everywhere breaking of the GC symmetry outside the class of $(\underline{v}, \overline{v})$–minimal quasi 1d lattices is based on complex analysis methods.

2. Random walks on quasi 1d lattices

We start by defining quasi 1d lattices. Consider first a finite oriented graph $G = (V, E)$, V being the set of vertices and E being the set of oriented edges, $E \subset \{(v, w) : v \neq w \text{ in } V\}$. We fix in V two vertices $\underline{v}, \overline{v}$. We assume that the oriented graph G is connected, i.e. for any $v, w \in V$ there is an oriented path in G from v to w. Then the quasi 1d lattice $\mathcal{G}$ associated to the triple $\big(G, \underline{v}, \overline{v}\big)$ is the oriented graph obtained by gluing together countable copies of G such that the point $\overline{v}$ of one copy is identified with the point $\underline{v}$ of the next copy. To give a formal definition, we define $\mathcal{G}$ as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ as follows (see Figure 1):

$$\begin{aligned} &\mathcal{V} := \{v_n := (v, n) \in (V \setminus \{\overline{v}\}) \times \mathbb{Z}\} \\ &\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \,, \end{aligned}$$

where

$$\begin{aligned} &\mathcal{E}_1 := \{(v_n, w_n) \,:\, (v, w) \in E \,,\; n \in \mathbb{Z}\} \,, \\ &\mathcal{E}_2 := \cup_{n\in\mathbb{Z}} \big\{(v_n, \underline{v}_{n+1}) \,:\, (v, \overline{v}) \in E\big\} \,, \\ &\mathcal{E}_3 := \cup_{n\in\mathbb{Z}} \big\{(\underline{v}_{n+1}, v_n) \,:\, (\overline{v}, v) \in E\big\} \,. \end{aligned}$$

To simplify notation we set

$$n_* := \underline{v}_n \,, \qquad n \in \mathbb{Z} \,.$$

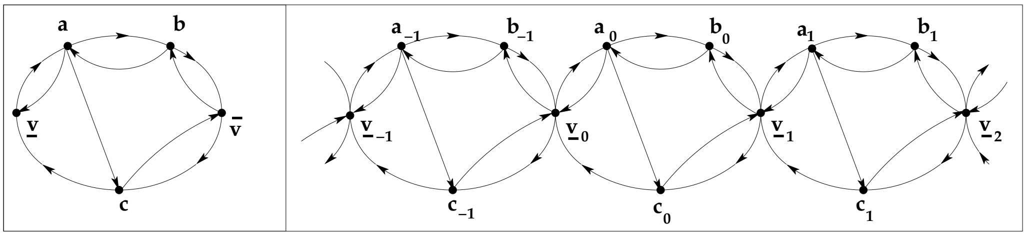

FIGURE 1. The graph $G = (V, E)$ with vertices $\underline{v}, \overline{v}$ (left) and the associated quasi 1d lattice $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (right)

On the graph $\mathcal{G}$ we define the shift $\mathcal{T} : \mathcal{V} \to \mathcal{V}$ as $\mathcal{T}(v_n) = v_{n+1}$. Note that the graph $\mathcal{G}$ is left invariant by the action of $\mathcal{T}$. We can now define the class of stochastic processes on quasi 1d lattices we are interested in:

Definition 2.1. *Given a quasi 1d lattice $\mathcal{G}$ associated to the triple $\big(G, \underline{v}, \overline{v}\big)$, we consider a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ with paths in the Skohorod space $D(\mathbb{R}_+;\mathcal{V})$ starting at any site n_* (we denote by $\mathbb{P}_{n_*}$ the associated law on $D(\mathbb{R}_+;\mathcal{V})$) and fulfilling the following properties:*

(i) *for each $n \in \mathbb{Z}$, $\mathbb{P}_{n_*}$–a.s., jumps are possible only along the edges in $\mathcal{E}$,*
(ii) *for each $n \in \mathbb{Z}$, when $(X_t)_{t\in\mathbb{R}_+}$ is sampled with law $\mathbb{P}_{n_*}$ then the law of $(\mathcal{T}(X_t))_{t\in\mathbb{R}_+}$ equals $\mathbb{P}_{(n+1)_*}$,*
(iii) *defining S as the random time*

$$S := \inf\{t \geq 0 \,:\, X_t \in \{-1_*, 1_*\}\} \,, \tag{1}$$

it holds $\mathbb{E}_{0_*}(S) < \infty$.
(iv) *under $\mathbb{P}_{0_*}(\cdot \,|\, X_S = \pm 1_*)$ the random path $(X_{S+t})_{t\in\mathbb{R}_+}$ is independent from $(X_t)_{t\in[0,S]}$ and has law $\mathbb{P}_{\pm 1_*}$.*

In the applications, typically $(X_t)_{t\in\mathbb{R}_+}$ is a Markov random walk:

Lemma 2.2. *Let $(X_t)_{t\in\mathbb{R}_+}$ be a Markov random walk with state space $\mathcal{V}$ and with positive jump rates $r(x, y)$, $(x, y) \in \mathcal{E}$, such that*

$$r(x, y) = r(\mathcal{T}x, \mathcal{T}y) \,. \tag{2}$$

Then the above random walk is well defined for all times t (no explosion takes place), fulfills the properties of Definition 2.1 and moreover $\mathbb{E}_{0_*}(e^{\lambda S}) < +\infty$ for $\lambda > 0$ small enough.

The proof of the above lemma is simple and therefore omitted. The finite exponential moments for λ small follow from the exponential decay of hitting probabilities for irreducible Markov chains with finite state space.

We point out that in the applications another relevant example is given by a random walk $(X_t)_{t\in\mathbb{R}_+}$ on the graph $\mathcal{G}$ with non exponential holding times (cf. [27]).

Note that the states n_*'s behave as gates which have to be crossed by the stochastic process X_t in order to move from one fundamental cell to the neighboring ones in the quasi 1d lattice $\mathcal{G}$. In the applications to molecular motors, each site n_* corresponds to a spot in the n–monomer of the polymeric filament where the molecular motor can bind. The other states v_n correspond to intermediate conformational states that the molecular motor achieves in its mechanochemical transformations, which are described by jumps along edges in $\mathcal{E}$. In particular, states v_n do not encode only a spatial position and jumps do not necessarily correspond to spatial jumps.

We now introduce the fundamental object of our investigation:

Definition 2.3. *Given the stochastic process X as in Definition 2.1, the* skeleton process *$X^* = (X^*_t)_{t\in\mathbb{R}_+}$ is defined as $X^*_t := \Phi(X_\iota)$ where $\Phi(n_*) = n$ and*

$$\iota := \sup\{s \in [0,t] : X_t = n_* \text{ for some } n \in \mathbb{Z}\} .$$

X^*_t has values in $\mathbb{Z}$ and records the last visited state of the form n_* up to time t.

In the applications to molecular motors, the process $(X^*_t)_{t\in\mathbb{R}_+}$ contains all the relevant information, indeed it allows to determine the position of the molecular motor up to an error of the same order of the monomer size.

3. Main results for random walks on quasi 1d lattices

Let S be the random time defined in (1). As proved in [15], one can easily obtained a strong law of large numbers for the skeleton process since $\mathbb{E}_{0_*}(S) < +\infty$ (cf. Theorems 1 and 2 in [15]):

$$\lim_{t\to\infty} \frac{X^*_t}{t} = \frac{\mathbb{P}_{0_*}(X_S = 1_*) - \mathbb{P}_{0_*}(X_S = -1_*)}{\mathbb{E}_{0_*}(S)} =: v\,, \qquad \mathbb{P}_{0_*}\text{-a.s.}\,. \tag{3}$$

In [15] we study also the gaussian fluctuations of the skeleton process, proving an invariance principle if $\mathbb{E}_{0_*}(S^2) < +\infty$. We concentrate here on large deviations.

3.1. **Large deviations.** From now on, in addition to the requirements in Definition 2.1, we assume that

$$\mathbb{P}_{0_*}(X_S = 1_*) > 0 \text{ and } \mathbb{P}_{0_*}(X_S = -1_*) > 0\,, \tag{4}$$

which holds for molecular motors.

Theorem 1. *Consider the process $(X_t)_{t\in\mathbb{R}_+}$ starting at 0_*. Call T_n the first time the skeleton process hits $n \in \mathbb{Z}$, i.e.*

$$T_n := \inf\{t \in \mathbb{R}_+ : X^*_t = n\} \in [0, +\infty]\,. \tag{5}$$

Then the following holds:

(i) *As $n \to \pm\infty$ the random variables $T_n / |n|$ satisfy a LDP with speed $|n|$ and convex rate function*

$$J_\pm(u) := \sup_{\lambda\in\mathbb{R}} \{\lambda u - \log \varphi_\pm(\lambda)\}\,, \qquad u \in \mathbb{R}\,, \tag{6}$$

where

$$\varphi_\pm(\lambda) := \mathbb{E}_{0_*}\left(e^{\lambda T_{\pm 1}} \mathbb{1}(T_{\pm 1} < \infty)\right) \in (0, +\infty]\,, \qquad \lambda \in \mathbb{R}\,. \tag{7}$$

The rate function $J_\pm$ is good[2] *if and only if $\mathbb{P}_{0_*}(T_1 < \infty) \neq \mathbb{P}_{0_*}(T_{-1} < \infty)$.*

(ii) *As $t \to +\infty$, the random variables X^*_t/t satisfy a LDP with speed t and good and convex rate function I given by*

$$I(\vartheta) = \begin{cases} \vartheta J_+(1/\vartheta) & \text{if} \quad \vartheta > 0\,, \\ |\vartheta| J_-(1/|\vartheta|) & \text{if} \quad \vartheta < 0\,, \end{cases} \tag{8}$$

and $I(0) = \lim_{\vartheta\to 0} I(\vartheta)$.

[2] We use the same terminology of [11]

Theorem 1 is an immediate consequence of Lemma 4.1 and Theorem 6 in Section 4. Since T_n can take value $+\infty$, the meaning of the LDP for $T_n/|n|$ as $n \to \pm\infty$ is the following: for each close subset $\mathcal{C} \subset \mathbb{R}$ and each open subset $\mathcal{O} \subset \mathbb{R}$ it holds

$$\limsup_{n\to\pm\infty} \frac{1}{|n|} \log \mathbb{P}_{0_*}\left(\frac{T_n}{|n|} \in \mathcal{C}\right) \leq -\inf_{\mathcal{C}} J_\pm \,,$$
$$\liminf_{n\to\pm\infty} \frac{1}{|n|} \log \mathbb{P}_{0_*}\left(\frac{T_n}{|n|} \in \mathcal{O}\right) \geq -\inf_{\mathcal{O}} J_\pm \,.$$

We now collect information on the qualitative behavior of the rate function I. The qualitative behavior of the rate functions J_-, J_+ is described in Proposition 5.3 in Section 5. Here we concentrate on the rate function I since the large deviations of X^*_t/t are more relevant in the applications.

Definition 3.1. *We define $\alpha_\pm$ as the minimum of the support of the law of $T_{\pm 1}$.*

We point out that $\alpha_\pm$ is the minimum of the support of the Borel measure $A \to \mathbb{P}_{0_*}(S \in A\,,\ X_S = \pm 1_*)$ (see Prop. 4.3 in Section 4). Below $1/\alpha_\pm$ is intended to be $+\infty$ if $\alpha_\pm = 0$. Note that $\alpha_\pm = 0$ in the case of Markov random walks.

Theorem 2. *The following holds:*

(i) *I is infinite outside $\left[-\frac{1}{\alpha_-}, \frac{1}{\alpha_+}\right]$, I is finite and C^1 on $\left(-\frac{1}{\alpha_-}, \frac{1}{\alpha_+}\right)$, moreover it is smooth on $(-1/\alpha_-, 1/\alpha_+) \setminus \{0\}$.*

(ii) *The following holds:*

$$\lim_{\vartheta \nearrow \frac{1}{\alpha_+}} I(\vartheta) = \begin{cases} +\infty & \text{if } \mathbb{P}_{0_*}(T_1 = \alpha_+) = 0 \\ I\left(\frac{1}{\alpha_+}\right) < \infty & \text{otherwise.} \end{cases} \tag{9}$$

$$\lim_{\vartheta \searrow -\frac{1}{\alpha_-}} I(\vartheta) = \begin{cases} +\infty & \text{if } \mathbb{P}_{0_*}(T_{-1} = \alpha_-) = 0 \\ I\left(-\frac{1}{\alpha_-}\right) < \infty & \text{otherwise.} \end{cases} \tag{10}$$

(iii) *The derivative of I satisfies $\lim_{\vartheta \nearrow \frac{1}{\alpha_+}} I'(\vartheta) = +\infty$ and $\lim_{\vartheta \searrow -\frac{1}{\alpha_-}} I'(\vartheta) = -\infty$.*

(iv) *I is lower semicontinuous and convex on $\mathbb{R}$, it is strictly convex on $\left(-\frac{1}{\alpha_-}, \frac{1}{\alpha_+}\right)$.*

(v) *I has a unique global minimum, which is given by 0 and is attained at $v \in (-1/\alpha_-, 1/\alpha_+)$, where v is the asymptotic velocity defined in (3). Moreover I is strictly decreasing on $(-1/\alpha_-, v)$ and is strictly increasing on $(v, 1/\alpha_+)$.*

Theorem 2 is an immediate consequence of Lemma 4.1 and Theorem 7 in Section 4.

When the process X is a Markov random walk with periodic rates (i.e. satisfying (2)), the derivation of the large deviation principle is simpler. In this case given $x \in \mathcal{V}$ we set $r(x) := \sum_{y:(x,y)\in\mathcal{E}} r(x,y)$ and, given $v \neq w$ in $V \setminus \{\overline{v}\}$, we set (using the convention that $r(y,z) = 0$ if $(y,z) \notin \mathcal{E}$)

$$r(v) := r(v_n)\,,\ \ r_-(w,v) := r(w_{n-1}, v_n)\,,\ \ r_0(w,v) := r(w,v)\,,\ \ r_+(w,v) := r(w_{n+1}, v_n)\,.$$

The above definition is well posed due to (2). Finally, given $\lambda \in \mathbb{R}$ we consider the finite matrix $\mathcal{A}(\lambda)$, with entries parameterized by $(V \setminus \{\overline{v}\}) \times (V \setminus \{\overline{v}\})$, defined as :

$$\mathcal{A}_{v,w}(\lambda) := \begin{cases} -r(v) & \text{if } v = w\,, \\ e^\lambda r_-(w,v) + r_0(w,v) + e^{-\lambda} r_+(w,v) & \text{if } v \neq w\,. \end{cases} \tag{11}$$

Applying Gäertner–Ellis Theorem we will derive the following result:

Theorem 3. *Suppose X is a Markov random walk on the quasi 1d lattice $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with transition rates satisfying* (2)*. Then, as $t \to +\infty$, the random variables X_t^*/t satisfy a large deviation principle with speed t and convex and good rate function $I(\vartheta)$ given by*

$$I(\vartheta) = \sup_{\lambda \in \mathbb{R}} \{\vartheta\lambda - \Lambda(\lambda)\}\,, \qquad \vartheta \in \mathbb{R}$$

where $\Lambda(\lambda)$ is the finite value

$$\Lambda(\lambda) := \max\{\Re(\gamma) \,:\, \gamma \text{ eigenvalue of } \mathcal{A}(\lambda)\}$$

and the matrix $\mathcal{A}(\lambda)$ is defined (11)*.*

3.2. **Fluctuation theorems (Gallavotti–Cohen type symmetry).**

Theorem 4. *The following facts are equivalent:*

(i) *For some $c \in \mathbb{R}$ the Gallavotti–Cohen type symmetry*[3] *$I(\vartheta) = I(-\vartheta) + c\vartheta$ holds for all $\vartheta \in \mathbb{R}$;*

(ii) *The random variables X_S and S are independent.*

Moreover, when (i),(ii) hold it must be $c = \log \frac{\mathbb{P}_{0_*}(X_S = -1_*)}{\mathbb{P}_{0_*}(X_S = 1_*)}$.

Theorem 4 is an immediate consequence of Lemma 4.1 and Theorem 8 in Section 4.

We continue our investigation of the Gallavotti–Cohen type symmetry (shortly, GC symmetry) as in Theorem 4 restricting now to Markov random walks $(X_t)_{t\in\mathbb{R}_+}$ on quasi 1d lattices. Recall that we write $r(x, y)$, $(x, y) \in \mathcal{E}$, for the positive jump rates of the Markov random walk and assume the periodicity (2) to hold. We restrict our discussion to the case of fundamental graphs $G = (V, E)$ such that

$$(x, y) \in E \text{ if and only if } (y, x) \in E\,, \tag{12}$$

which is the standard setting in the investigation of GC symmetry for Markov chains [32].

In what follows, given an edge $(u, v) \in E$ in the fundamental graph $G = (V, E)$, we define

$$r(u, v) = r(\pi(u), \pi(v))\,, \tag{13}$$

where π is the map $V \to \mathcal{V}$ defined as $\pi(u) = u_0$ if $u \neq \overline{v}$ and $\pi(\overline{v}) = \underline{v}_1$. Note that, fixed positive numbers $a(e)$, $e \in E$, it is univocally determined a Markov random walk on $\mathcal{G}$ whose rates satisfy (2) and such that $r(e) = a(e)$ for all $e \in E$. We call it the Markov random walk induced by $a(e)$, $e \in E$.

We introduce a special class of graphs G which includes in particular trees. Recall that G has connected support when disregarding the orientation of the edges.

Definition 3.2. *We say that the graph $G = (V, E)$ is $(\underline{v}, \overline{v})$*–minimal *if it satisfies* (12) *and moreover there is a unique path $\gamma_* = (z_0, z_1, \ldots, z_n)$ such that* (i) *$z_0 = \underline{v}$,* (ii) *$z_n = \overline{v}$,* (iii) *$(z_i, z_{i+1}) \in E$ and* (iv) *the points $z_0, \ldots, z_n$ are all distinct.*

Note that, given a generic fundamental graph $G = (V, E)$, there exists at least one path $\gamma = (z_0, z_1, \ldots, z_n)$ satisfying the above properties (i),...,(iv). Indeed, since G is connected, there exists a path from $\underline{v}$ to $\overline{v}$. Take such a path and prune iteratively the loops. Since each time a loop is pruned away the length of the path decreases, after a finite number of prunes one gets a minimal path satisfying the above properties (i),...,(iv).

Now suppose that $G = (V, E)$ is $(\underline{v}, \overline{v})$*–minimal* and take two points $z_i \neq z_j$ (the z_k's are as in the Def. 3.2). Then it cannot exist a path from z_i to z_j whose intermediate points are in $V \setminus \{z_0, z_1, \ldots, z_n\}$. In particular, the graph G must be as in Fig. 2 (due to property

[3] Sum is thought in $[0, +\infty]$

(12) we only draw the support of G, disregarding orientation). More precisely, the graph is given by the linear chain γ_* of Def. 3.2, to which one attaches some subgraphs, in such a way that each attached subgraph has exactly one point in common with γ_*.

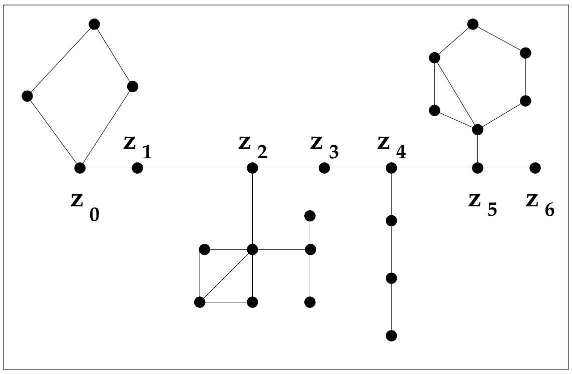

FIGURE 2. A $(\underline{v},\overline{v})$–*minimal* graph $G=(V,E)$

Theorem 5. *Suppose that X is a Markov random walk and that $G=(V,E)$ is a graph satisfying* (12)*. If G is* $(\underline{v},\overline{v})$–minimal*, then the random variables S and X^*_S are independent, and in particular the large deviation rate function I associated to the skeleton process X^* satisfies the Gallavotti–Cohen type symmetry*

$$I(\vartheta)=I(-\vartheta)-\Delta\,\vartheta\,,\qquad \forall \vartheta\in\mathbb{R}\,, \tag{14}$$

where

$$\Delta=\log\frac{r(z_0,z_1)r(z_1,z_2)\cdots r(z_{n-1},z_n)}{r(z_1,z_0)r(z_2,z_1)\cdots r(z_n,z_{n-1})} \tag{15}$$

and $\gamma_*=(z_0,z_1,z_2,\dots,z_{n-1},z_n)$ is the path in Definition 3.2.

Vice versa, if G is not $(\underline{v},\overline{v})$–minimal then the vectors $\big(r(e)\,:\,e\in E\big)\in(0,+\infty)^E$ for which the induced random walk on $\mathcal{G}$ satisfies (14) *for some constant Δ (depending on the numbers $r(e)$, $e\in E$) has zero Lebesgue measure in $(0,+\infty)^E$.*

The proof of the above theorem is given in Section 9.

4. Random time changes of cumulative processes

As already mentioned, the results presented in Section 3 hold in a more general context that we now describe. Consider a sequence $(w_i,\tau_i)_{i\geq 1}$ of i.i.d. 2d vectors with values in $\mathbb{R}\times(0,+\infty)$. For each integer $m\geq 1$ we define

$$W_m:=w_1+w_2+\cdots+w_m\,, \tag{16}$$

$$\mathcal{T}_m:=\tau_1+\tau_2+\cdots+\tau_m\,. \tag{17}$$

We set $W_0=\mathcal{T}_0=0$. Note that $\lim_{m\to\infty}\mathcal{T}_m=+\infty$ a.s. As a consequence, we can univocally define a.s. a random process $\{\nu(t)\}_{t\in\mathbb{R}_+}$ with values in $\{0,1,2,3,\dots\}$ such that

$$\mathcal{T}_{\nu(t)}\leq t<\mathcal{T}_{\nu(t)+1}\,,\qquad t\geq 0\,. \tag{18}$$

Note that $\nu(t)=\max\{m\in\mathbb{N}:\,\mathcal{T}_m\leq t\}$. Finally, we define the process $Z:[0,\infty)\to\mathbb{R}$ as

$$Z_t:=W_{\nu(t)}\,. \tag{19}$$

Note that $Z_0 = 0$. The resulting process $Z = (Z_t)_{t\in\mathbb{R}_+}$ is therefore obtained from the cumulative process $(W_m)_{m\geq 0}$ by a random time change, and generalizes the concept of (time–homogeneous) random walk on $\mathbb{R}$. For example, if w_i and τ_i are independent and τ_i is an exponential variable of parameter λ, then the process Z is a continuous time Markov random walk with exponential holding times of parameter λ and with jump probability given by the law of w_i. If $\tau_i \equiv 1$ for all i, then $Z_t = W_{\lfloor t\rfloor}$ ($\lfloor\cdot\rfloor$ denoting the integer part) and $(Z_n)_{n\in\mathbb{N}}$ is a discrete time Markov random walk on $\mathbb{R}$ with jump probability given by the law of w_i.

Due to Definitions 2.1 and 2.3 the skeleton process X^* is indeed a special case of process Z (recall the definition of the random time S given in (1)):

Lemma 4.1. *Consider a sequence $(w_i, \tau_i)_{i\geq 1}$ of i.i.d. vectors, with the same law of the random vector $\big(X^*_S, S\big) \in \{-1, 1\} \times (0, +\infty)$ when the random walk $(X_t)_{t\in\mathbb{R}_+}$ starts at 0_*. We define $(Z_t)_{t\in\mathbb{R}_+}$ as the stochastic process built from $(w_i, \tau_i)_{i\geq 1}$ according to* (19)*. Then $(Z_t)_{t\in\mathbb{R}_+}$ has the same law of $(X^*_t)_{t\in\mathbb{R}_+}$ with $X^*_0 = 0$.*

The proof of the above lemma is very simple and therefore omitted. We recall also the LLN discussed in [15][Appendix A]:

Proposition 4.2 (FS)**.** *If $\mathbb{E}(\tau_i) < \infty$, then almost surely $\lim_{t\to\infty} \frac{Z_t}{t} = v := \frac{\mathbb{E}(w_i)}{\mathbb{E}(\tau_i)}$.*

We now state our main results for $(Z_t)_{t\in\mathbb{R}_+}$:

Theorem 6. *[LDP] Suppose that*
- (A1) $w_i \in \{-1, 1\}$ *a.s.*
- (A2) $\mathbb{P}(w_1 = +1) > 0$ *and* $\mathbb{P}(w_1 = -1) > 0$.

Set $T_n := \inf\{t \in \mathbb{R}_+ : Z_t = n\} \in [0, +\infty]$. Define $J_\pm$ and $\varphi_\pm$ as in (6) *and* (7)*. Then the following holds:*

- (i) *As $n \to \pm\infty$ the random variables $T_n / |n|$ satisfy a LDP with speed $|n|$ and convex rate function $J_\pm$. The rate function $J_\pm$ is good if and only if $\mathbb{P}(T_1 < \infty) \neq \mathbb{P}(T_{-1} < \infty)$.*
- (ii) *As $t \to +\infty$, the random variables Z_t/t satisfy a LDP with speed t and good and convex rate function I given by*

$$I(\vartheta) = \begin{cases} \vartheta J_+(1/\vartheta) & \text{if} \quad \vartheta > 0\,, \\ |\vartheta| J_-(1/|\vartheta|) & \text{if} \quad \vartheta < 0\,, \end{cases} \tag{20}$$

and $I(0) = \lim_{\vartheta\to 0} I(\vartheta)$.

The proof of the above result is given in Sections 5 and 6.

We introduce the functions $f_\pm$ on $\mathbb{R}$ as

$$f_\pm(\lambda) := \mathbb{E}\big(e^{\lambda\tau_1}\mathbb{1}(w_1 = \pm 1)\big) \in (0, +\infty]\,. \tag{21}$$

Note that $f_\pm(\lambda) > 0$ under Assumption (A2). We call $\alpha_\pm$ the minimum value in the support of the law of $T_{\pm 1}$.

Proposition 4.3. *Under Assumptions (A1) and (A2) of the previous theorem the following holds:*

- (i) *The function $\varphi_\pm(\lambda)$ satisfies*

$$\varphi_\pm(\lambda) = \frac{1 - \sqrt{1 - 4f_-(\lambda)f_+(\lambda)}}{2f_\mp(\lambda)} \tag{22}$$

for $\lambda \leq \lambda_c$, where $\lambda_c \in [0,+\infty)$ is the unique value of λ such that $f_-(\lambda)f_+(\lambda) = 1/4$, while $\varphi_\pm(\lambda) = +\infty$ for $\lambda > \lambda_c$.

(ii) *Consider the measure $\mu_\pm$ on $[0,+\infty)$ such that $\mu_\pm(A) = \mathbb{P}(\tau_1 \in A, w_1 = \pm 1)$ for any Borel $A \subset \mathbb{R}$. Then $\alpha_\pm$ is the minimum value in the support of $\mu_\pm$. Moreover $\mathbb{P}(T_{\pm 1} = \alpha_\pm) = \mathbb{P}(\tau_1 = \alpha_\pm, w_1 = \pm 1)$.*

The proof is given at the beginning of Subsection 5.2.

The qualitative behavior of the rate function $I(\vartheta)$ is described by the following theorem (for the qualitative behavior of $J_\pm$ see Proposition 5.3):

Theorem 7. *Theorem 2 remains valid in the present more general context, with v defined as in Proposition 4.2.*

We conclude with a result on the presence of a Gallavotti–Cohen type symmetry in the rate function I:

Theorem 8. *The following facts are equivalent:*

(i) *There exists a constant $c \in \mathbb{R}$ such that the Gallavotti–Cohen type symmetry*

$$I(\vartheta) = I(-\vartheta) + c\vartheta \tag{23}$$

holds for all $\vartheta \in \mathbb{R}$;

(ii) *fixed i, the random variables w_i, τ_i are independent;*

(iii) *the functions $\varphi_+(\lambda)$ and $\varphi_-(\lambda)$ are proportional where finite, that is:*

$$\varphi_+(\lambda) = C\varphi_-(\lambda) \quad \text{for all } \lambda \leq \lambda_c\,.$$

Moreover, if we let $p := \mathbb{P}(w_i = 1)$ and $q := \mathbb{P}(w_i = -1)$ (with $p, q > 0$ by Assumption (A2)), then $C = p/q$ and $c = \log(q/p) = -\log C$.

The proof of this result is given in Section 7.

5. Proof of Theorem 6–(i) and Theorem 7

In this section we prove Item (i) of Theorem 6 and we study the behavior of the functions $I, J_\pm$ defined in Theorem 6. In particular, we prove Theorem 7 at the end of this section.

5.1. **Proof of Item (i) in Theorem 6.** For $n \geq 1$ the random variable T_n has the same law of $\sum_{k=1}^n \hat\tau_n$, where $\hat\tau_n$'s are i.i.d. random variables taking value in $(0,+\infty]$, distributed as T_1. The thesis can then be obtained from Cramér Theorem. We give the proof in the case $n \to \infty$. Call $\alpha := \mathbb{P}(T_1 < \infty)$ and note that $P(T_n < \infty) = \alpha^n$. Then for each subset $\mathcal{A} \subset \mathbb{R}$ we can write $\mathbb{P}(T_n/n \in \mathcal{A}) = \alpha^n \mathbb{P}(T_n/n \in \mathcal{A} \,|\, T_n < \infty)$. Now we observe that, conditioning on the event $T_n < \infty$, T_n can be represented as $\sum_{k=1}^n \hat\tau_n'$, where the real random variables τ_n' are i.i.d. and distributed as T_1 conditioned to be finite. In conclusion $\mathbb{P}(T_n/n \in \mathcal{A}|T_n < \infty) = P(\frac{1}{n}\sum_{k=1}^n \hat\tau_n' \in \mathcal{A})$. At this point one only need to apply Cramér Theorem for i.i.d. real random variables (cf. [11][Th. 2.2.3]) observing that the moment generating function of $\hat\tau_n'$ is φ_+/α. The fact that $J_\pm$ is good if and only if $\mathbb{P}(T_1 < \infty) \neq \mathbb{P}(T_{-1} < \infty)$ is proved in the next Subsection (see Remark 5.4 below).

5.2. **Qualitative study of the functions $J_\pm(\vartheta)$, $I(\vartheta)$.** In this subsection we first prove some properties of the function $I(\vartheta)$ defined in Theorem 6 by (20) and the identity $I(0) = \lim_{\vartheta\to 0} I(\vartheta)$. In the next subsection we will indeed prove that $I(\vartheta)$ is the rate function of the LDP for Z_t/t.

We start by proving Proposition 4.3.

Proof of Proposition 4.3. Let us prove Point (i). Recall the definition of the positive functions $f_\pm$ given in (21). Distinguishing on the value of w_1 we can write

$$T_1 = \mathbb{1}(w_1 = 1)\tau_1 + \mathbb{1}(w_1 = -1)(\tau_1 + T_1' + T_1'') \tag{24}$$

where T_1', T_1'' are independent random variables, independent from w_1, τ_1 and distributed as T_1 (roughly, T_1' is the time for Z to go from -1 to 0 and T_1'' is the time for Z to go from 0 to 1). The above identity implies that

$$\varphi_+(\lambda) = f_+(\lambda) + f_-(\lambda)\varphi_+(\lambda)^2 \,. \tag{25}$$

From this we deduce that $\varphi_+(\lambda) < +\infty$ if and only if $f_+(\lambda)f_-(\lambda) \leq 1/4$, and moreover in this case (22) holds. By symmetry, one obtains that the same condition implies $\varphi_-(\lambda) < +\infty$. Trivially $f_\pm$ is increasing, $\lim_{\lambda\to-\infty} f_\pm(\lambda) = 0$ and $\lim_{\lambda\to+\infty} f_\pm(\lambda) = +\infty$. Moreover, $f_\pm(\lambda)$ is smooth and strictly increasing on the open set $\{f_\pm < +\infty\}$. As a consequence there exists a unique value $\lambda_c \in \mathbb{R}$ such that $f_+(\lambda)f_-(\lambda) = 1/4$ and therefore $f_+(\lambda)f_-(\lambda) \leq 1/4$ for $\lambda \leq \lambda_c$. Since trivially $\varphi_+(\lambda) < +\infty$ for $\lambda \leq 0$ it must be $\lambda_c \geq 0$. This completes the proof of Point (i).

We now move to Point (ii). To see that $\mathbb{P}(\tau_1 < \alpha_+, w_1 = 1) = 0$ observe that by (24) $\{w_1 = 1\} \subseteq \{T_1 = \tau_1\}$ and therefore $\mathbb{P}(\tau_1 < \alpha_+, w_1 = 1) = \mathbb{P}(T_1 < \alpha_+, w_1 = 1) \leq \mathbb{P}(T_1 < \alpha_+) = 0$. To get the thesis it remains to show that $\forall \varepsilon > 0$ $\mathbb{P}(\tau_1 \in [\alpha_+, \alpha_+ + \varepsilon), w_1 = 1) > 0$. Assume the contrary. Then there exists $\hat\varepsilon > 0$ such that $\mathbb{P}(\tau_1 \in [\alpha_+, \alpha_+ + \hat\varepsilon), w_1 = 1) = 0$. By definition of α_+, on the other hand, we have $\mathbb{P}(T_1 \in [\alpha_+, \alpha_+ + \hat\varepsilon)) > 0$. Combining this with the decomposition in (24), we find

$$\begin{aligned}
0 < \mathbb{P}(T_1 < \alpha_+ + \hat\varepsilon) &= \mathbb{P}(\tau_1 < \alpha_+ + \hat\varepsilon, w_1 = 1) + \mathbb{P}(\tau_1 + T_1' + T_1'' < \alpha_+ + \hat\varepsilon, w_1 = -1) \\
&\leq \mathbb{P}(\tau_1 < \alpha_+ + \hat\varepsilon, w_1 = -1)\mathbb{P}(T_1 < \alpha_+ + \hat\varepsilon)^2 \\
&\leq \mathbb{P}(w_1 = -1)\mathbb{P}(T_1 < \alpha_+ + \hat\varepsilon)^2 \,.
\end{aligned}$$

Dividing both sides by the positive quantity $\mathbb{P}(T_1 < \alpha_+ + \hat\varepsilon)$ and recalling that by (A2) $\mathbb{P}(w_1 = -1) < 1$, we get the contradiction and this concludes the proof. □

We now focus on the behavior $\log\varphi_\pm$. Recall the definition of $\lambda_c, \alpha_\pm$ given in Proposition 4.3.

Lemma 5.1. *The following holds:*

(i) $\log\varphi_\pm$ *is strictly increasing and continuous on* $(-\infty, \lambda_c]$, *convex and smooth on* $(-\infty, \lambda_c)$ *and moreover* $\lim_{\lambda\to-\infty} \log\varphi_\pm(\lambda) = -\infty$*;*

(ii) *the second derivative* $(\log\varphi_\pm)''$ *is strictly positive on* $(-\infty, \lambda_c)$ *(in particular* $(\log\varphi_+(\lambda))'$ *is strictly increasing on* $(-\infty, \lambda_c)$ *) and*

$$\lim_{\lambda\to-\infty} (\log\varphi_\pm(\lambda))' = \alpha_\pm \,, \tag{26}$$

$$\lim_{\lambda\nearrow\lambda_c} (\log\varphi_\pm(\lambda))' = +\infty \,. \tag{27}$$

Proof. The proof of Point (i) is rather standard (see Lemma 2.2.5 in [11], the fact that $\log\varphi_\pm$ is strictly increasing on $(-\infty, \lambda_c]$ and convex follows also from Point (ii)).

We prove Point (ii) restricting to φ_+ without loss of generality. Note that, for $\lambda < \lambda_c$,

$$(\log\varphi_+(\lambda))'' = \frac{\mathbb{E}\left(T_1^2 e^{\lambda T_1}\mathbb{1}(T_1 < \infty)\right)}{\mathbb{E}\left(e^{\lambda T_1}\mathbb{1}(T_1 < \infty)\right)} - \frac{\mathbb{E}\left(T_1 e^{\lambda T_1}\mathbb{1}(T_1 < \infty)\right)^2}{\mathbb{E}\left(e^{\lambda T_1}\mathbb{1}(T_1 < \infty)\right)^2} = \mathrm{Var}_{\mathbb{Q}}(T_1) \,,$$

where $\mathbb{Q}$ it the probability defined as $\mathbb{Q}(A) = \mathbb{E}\left(\mathbb{1}(A)e^{\lambda T_1}\mathbb{1}(T_1<\infty)\right)/\mathbb{E}\left(e^{\lambda T_1}\mathbb{1}(T_1<\infty)\right)$. Since T_1 is non constant $\mathbb{Q}$–a.s. by Assumption (A1), we conclude that $(\log\varphi_+(\lambda))''>0$ for $\lambda<\lambda_c$, hence $(\log\varphi_+(\lambda))'$ on $(-\infty,\lambda_c)$ is strictly increasing.

We first derive (26) in the case $\alpha_+=0$. It is convenient to prove the thesis for a generic nonnegative random variable T_1, non necessarily defined as in Theorem 6. Suppose first that $\mathbb{P}(T_1=0)>0$. Since $\varphi_+(\lambda)\geq\mathbb{P}(T_1=0)$, while $\lim_{\lambda\to-\infty}\varphi'(\lambda)=\lim_{\lambda\to-\infty}\mathbb{E}(T_1e^{\lambda T_1}\mathbb{1}(T_1<\infty))=0$ by the monotone convergence theorem, we get (26).

We now consider the case $\mathbb{P}(T_1=0)=0$, thus implying $\mathbb{P}(T_1\in(0,\varepsilon))>0$ for all $\varepsilon>0$. We fix any $c>0$ and take $\lambda<-1/c$. By this choice it holds $\sup_{x\geq c}xe^{\lambda x}=ce^{\lambda c}$. Moreover we fix $c_1,c_2:0<c_1<c_2<c$ such that $\mathbb{P}(c_1\leq T_1\leq c_2)>0$. Define:

$$\begin{aligned}
e_1(\lambda)&:=\mathbb{E}(T_1e^{\lambda T_1}\mathbb{1}(c\leq T_1<\infty))\leq ce^{\lambda c}\,,\\
e_2(\lambda)&:=\mathbb{E}(T_1e^{\lambda T_1}\mathbb{1}(T_1<c))\geq c_1e^{\lambda c_2}\mathbb{P}(c_1\leq T_1\leq c_2)\,,\\
e_3(\lambda)&:=\mathbb{E}(e^{\lambda T_1}\mathbb{1}(c\leq T_1<\infty))\leq e^{\lambda c}\,,\\
e_4(\lambda)&:=\mathbb{E}(e^{\lambda T_1}\mathbb{1}(T_1<c))\geq e^{\lambda c/2}\mathbb{P}(T_1<c/2)>0\,.
\end{aligned}$$

By the previous bounds we have $\lim_{\lambda\to-\infty}e_1(\lambda)/e_2(\lambda)=0$ and $\lim_{\lambda\to-\infty}e_3(\lambda)/e_4(\lambda)=0$. In conclusion

$$\begin{aligned}
0\leq\lim_{\lambda\to-\infty}(\log\varphi_+(\lambda))'&=\lim_{\lambda\to-\infty}\frac{\varphi_+'(\lambda)}{\varphi_+(\lambda)}=\lim_{\lambda\to-\infty}\frac{e_2(\lambda)\big(1+e_1(\lambda)/e_2(\lambda)\big)}{e_4(\lambda)\big(1+e_3(\lambda)/e_4(\lambda)\big)}\\
&=\lim_{\lambda\to-\infty}\frac{e_2(\lambda)}{e_4(\lambda)}=\lim_{\lambda\to-\infty}\frac{\mathbb{E}(T_1e^{\lambda T_1}\mathbb{1}(T_1<c))}{\mathbb{E}(e^{\lambda T_1}\mathbb{1}(T_1<c))}\leq c\,.
\end{aligned}$$

Since $c>0$ is arbitrary we get (26).

To complete the proof of (26) it remains to discuss the case $\alpha_+>0$. To this aim note that 0 is the minimum in the support of the law of $\hat{T}_1:=T_1-\alpha_+$. Hence, by what just proven, it holds $\lim_{\lambda\to-\infty}(\log\hat{\varphi}_+(\lambda))'=0$, where $\hat{\varphi}_+(\lambda):=\mathbb{E}(e^{\lambda\hat{T}_1}\mathbb{1}(\hat{T}_1<\infty))$. Since $\varphi_+(\lambda)=e^{\lambda\alpha_+}\hat{\varphi}_+(\lambda)$, we get (26).

To conclude the proof of Point (ii) we need to justify (27). Since by Point (i) $\log\varphi_+$ is smooth and convex on $(-\infty,\lambda_c)$, the derivative $(\log\varphi_+(\lambda))'=\varphi_+'(\lambda)/\varphi_+(\lambda)$ is increasing and therefore the limit in (27) exists. Moreover, since $\lim_{\lambda\nearrow\lambda_c}\varphi_+(\lambda)=\varphi_+(\lambda_c)<\infty$, we only need to show that $\lim_{\lambda\nearrow\lambda_c}\varphi_+'(\lambda)=+\infty$. To this aim recall (25). Differentiating such identity for $\lambda<\lambda_c$ (note that everything is finite and smooth) we get

$$\big(1-2f_-(\lambda)\varphi_+(\lambda)\big)\varphi_+'(\lambda)=f_+'(\lambda)+f_-'(\lambda)\varphi_+(\lambda)^2\,. \tag{28}$$

By the monotone convergence theorem we get that $f_-(\lambda),\varphi_+(\lambda)$ and the derivative $f_\pm'(\lambda)=\mathbb{E}\big(\tau_1e^{\lambda\tau_1}\mathbb{1}(w_1=\pm1)\big)$ converge to $f_-(\lambda_c),\varphi_+(\lambda_c)$ and $f_\pm'(\lambda_c)$ respectively as $\lambda\nearrow\lambda_c$. Observing that

$$\varphi_+(\lambda_c)=\frac{1}{2f_-(\lambda_c)} \tag{29}$$

due to (22) and the identity $f_+(\lambda_c)f_-(\lambda_c)=1/4$, we get that $1-2f_-(\lambda)\varphi_+(\lambda)$ converges to zero as $\lambda\nearrow\lambda_c$. On the other hand, as $\lambda\nearrow\lambda_c$ the r.h.s. of (28) converges to its value at λ_c, which is nonzero. The limit (27) is therefore the only possibility as $\lambda\nearrow\lambda_c$ in (28). □

Recall the definition of the asymptotic velocity v given in (3).

Remark 5.2. *Taking the expectation in* (24) *and in the analogous expression for* T_{-1}, *one concludes that* $\mathbb{E}(T_{\pm 1}) < +\infty$ *implies that* $\mathbb{E}(T_{\pm 1})\,\mathbb{E}(w_1) = \pm\mathbb{E}(\tau_1)$. *Since* $\mathbb{E}(\tau_1) \neq 0$, *we conclude that if* $\mathbb{E}(T_{\pm 1}) < \infty$ *then* $\mathbb{E}(w_1) \neq 0$ *and* $\mathbb{E}(T_{\pm 1}) = \pm\mathbb{E}(\tau_1)/\mathbb{E}(w_1) = \pm 1/v$.

From Lemma 5.1 we deduce the qualitative behavior of the rate function $J_\pm(\vartheta) := \sup_{\lambda\in\mathbb{R}}\{\lambda\vartheta - \log\varphi_\pm(\lambda)\}$:

Proposition 5.3. *The following holds:*

(i) $J_\pm$ *is lower semicontinuous, convex and takes values in* $[0, +\infty]$.

(ii) $J_\pm$ *is finite on* $(\alpha_\pm, +\infty)$ *and infinite on* $(-\infty, \alpha_\pm)$.

(iii) $J_\pm$ *is smooth on* $(\alpha_\pm, +\infty)$ *and the derivative* $J'_\pm$ *satisfies* $\lim_{\vartheta\to+\infty} J'_\pm(\vartheta) = \lambda_c$. *In particular,* $\lim_{\vartheta\to+\infty} J_\pm(\vartheta) = +\infty$ *if* $\lambda_c > 0$.

(iv) *If* $\lambda_c = 0$ *then* $J_\pm$ *is strictly decreasing on* $(\alpha_\pm, +\infty)$. *If* $\lambda_c > 0$ *then there exist* $\vartheta_c^\pm \in (\alpha_\pm, +\infty)$ *such that* $J_\pm$ *is strictly decreasing on* $(\alpha_\pm, \vartheta_c^\pm)$, *strictly increasing on* $(\vartheta_c^\pm, +\infty)$. *Moreover:*
   - $v = 0$ *if and only if* $\lambda_c = 0$,
   - *if* $v > 0$ *then* $\vartheta_c^+ = 1/v$ *and* $J_+(\vartheta_c^+) = 0$,
   - *if* $v < 0$ *then* $\vartheta_c^- = -1/v$ *and* $J_-(\vartheta_c^-) = 0$.

(v) *The value* $J_\pm(\alpha_\pm)$ *admits the following characterization*

$$J_\pm(\alpha_\pm) = \lim_{\vartheta\searrow\alpha_\pm} J_\pm(\vartheta) = \begin{cases} +\infty & \text{if } \mathbb{P}(T_{\pm 1} = \alpha_\pm) = 0\,, \\ < \infty & \text{otherwise}\,. \end{cases} \tag{30}$$

Remark 5.4. *Due to the above result,* $J_\pm$ *is a good rate function (i.e.* $\{J_\pm \leq u\}$ *is compact for all* $u \in \mathbb{R}$*) if and only if* $\lambda_c > 0$.

Proof. Without loss of generality we prove the above statements only for J_+.

The proof of Point (i) is standard (cf. [11][Ch.2]) and we omit it. We now prove Point (ii). The fact that $J_+(\vartheta) = \infty$ for $\vartheta \leq 0$ follows from Lemma 5.1, Item (i). We now show that if $\alpha_+ > 0$ then $J_+(\vartheta) = \infty$ also for $\vartheta \in (0, \alpha_+)$. For such ϑ, by (26) in Lemma 5.1 it holds

$$\lim_{\lambda\to-\infty} \frac{\log\varphi_+(\lambda)}{\lambda\vartheta} = \lim_{\lambda\to-\infty} \frac{(\log\varphi_+)'(\lambda)}{\vartheta} = \frac{\alpha_+}{\vartheta} > 1$$

and therefore

$$J_+(\vartheta) \geq \lim_{\lambda\to-\infty} \lambda\vartheta\left(1 - \frac{\log\varphi_+(\lambda)}{\lambda\vartheta}\right) = +\infty\,.$$

Take now $\vartheta > \alpha_+$. Since by Lemma 5.1 $(\log\varphi_+)'(\lambda)$ is a strictly increasing function which takes values in $(\alpha_+, +\infty)$, there exists a unique $\tilde\lambda_+(\vartheta)$ such that

$$\vartheta = (\log\varphi_+)'(\tilde\lambda_+(\vartheta))\,. \tag{31}$$

Then $J_+(\vartheta) = \vartheta\tilde\lambda_+(\vartheta) - \log\varphi_+(\tilde\lambda_+(\vartheta))$ which is finite. This concludes the proof of (ii).

We now move to Point (iii). Observe that, by (31) and Lemma 5.1, $\tilde\lambda_+$ is the inverse function of $(\log\varphi_+)'$. By Lemma 5.1 $(\log\varphi_\pm)'$ is smooth on $(-\infty, \lambda_c)$ and $(\log\varphi_\pm)'' > 0$ on $(-\infty, \lambda_c)$. Hence, by the implicit function theorem, the function $\tilde\lambda_+$ is smooth on $(\alpha_+, +\infty)$ and tending to λ_c as $\vartheta \to +\infty$ (see (27)). Hence J_+ is smooth on $(\alpha_+, +\infty)$ and

$$J'_+(\vartheta) = \vartheta\tilde\lambda'_+(\vartheta) + \tilde\lambda_+(\vartheta) - (\log\varphi_+)'(\tilde\lambda_+(\vartheta))\tilde\lambda'_+(\vartheta) = \tilde\lambda_+(\vartheta)\,, \tag{32}$$

thus implying that $\lim_{\vartheta\to+\infty} J'_+(\vartheta) = \lambda_c$. This concludes the proof of Item (iii).

We now prove Point (iv). By Lemma 5.1 $\tilde\lambda_+$ is strictly increasing on $(\alpha_+,+\infty)$, $\lim_{\vartheta\searrow\alpha_+}\tilde\lambda_+(\vartheta)=-\infty$ and $\lim_{\vartheta\to\infty}\tilde\lambda_+(\vartheta)=\lambda_c$. If $\lambda_c=0$, then $\tilde\lambda_+$ must be negative and from (32) we conclude that J_+ is strictly decreasing on $(\alpha_+,+\infty)$. If $\lambda_c>0$, then there exists a unique ϑ_c^+ such that $\tilde\lambda_+(\vartheta_c^+)=0$, $\tilde\lambda_+$ is negative on the left of ϑ_c^+ and is positive on the right of ϑ_c^+. Hence, by (32) we see that J_+ has a unique minimum at $\vartheta=\vartheta_c^+$. In this case, from (31) we have

$$\vartheta_c^+=(\log\varphi_+)'(0)=\frac{\varphi_+'(0)}{\varphi_+(0)}=\frac{\mathbb{E}(T_1\mathbb{1}(T_1<\infty))}{\mathbb{P}(T_1<\infty)}\,.$$

If $v>0$, then by the LLN in Proposition 4.2 we get that $T_1<\infty$ a.s. and $\vartheta_c^+=\mathbb{E}(T_1)=1/v$ (cf. Remark 5.2). Hence, recalling that $\tilde\lambda_+(\vartheta_c^+)=0$,

$$\inf_{\vartheta\in\mathbb{R}}J_+(\vartheta)=J_+(\vartheta_c^+)=\tilde\lambda_+(\vartheta_c^+)\vartheta_c^+-\log\varphi_+(\tilde\lambda_+(\vartheta_c^+))=-\log\mathbb{P}(T_1<\infty)=0\,.$$

The case $v<0$ can be treated similarly. We conclude the proof by showing that $v=0\Leftrightarrow\lambda_c=0$. Trivially, $v=0\Leftrightarrow\mathbb{P}(w_1=\pm1)=\frac12\Leftrightarrow\mathbb{P}(w_1=1)\mathbb{P}(w_1=-1)=\frac14$. The last identity can be rewritten as $f_+(0)f_-(0)=\frac14$, where the function f_+,f_- are defined as in (21). Due to the characterization of λ_c given after (29), we conclude that the last identity is equivalent to $\lambda_c=0$.

To derive Point (v) we note that by Point (iv) the limit in (30) exists. We first assume $\mathbb{P}(T_1=\alpha_+)=0$. Taking $\delta>0$ and $\lambda<0$ we can bound

$$\varphi_+(\lambda)\le e^{\lambda\alpha_+}\mathbb{P}(T_1\le\alpha_++\delta)+e^{\lambda(\alpha_++\delta)}\mathbb{P}(T_1>\alpha_++\delta)\,,$$

thus implying

$$J_+(\alpha_+)\ge\lambda\alpha_+-\log\varphi_+(\lambda)\ge-\log\big[\mathbb{P}(T_1\le\alpha_++\delta)+e^{\lambda\delta}\mathbb{P}(T_1>\alpha_++\delta)\big]\,.$$

To get that $J_+(\alpha_+)=+\infty$ it is enough to take first the limit $\lambda\to-\infty$ and afterwards the limit $\delta\to0$. Since J_+ is also l.s.c. one has $\lim_{\vartheta\searrow\alpha_+}J_+(\vartheta)\ge J_+(\alpha_+)$ and therefore one gets (30).

Assume, on the other hand, that $\mathbb{P}(T_1=\alpha_+)>0$. The fact that $J_+(\alpha_+)<\infty$ follows by the LDP for T_n (cf. Subsection 5.1) and the characterization of α_+ given in Proposition 4.3–(ii). Indeed we can bound

$$\begin{aligned}-J_+(\alpha_+)&\ge\limsup_{n\to\infty}\frac1n\log\mathbb{P}\left(\frac{T_n}{n}=\alpha_+\right)=\limsup_{n\to\infty}\frac1n\log\mathbb{P}(\tau_1=\alpha_+,w_1=1)^n\\&=\log\mathbb{P}(\tau_1=\alpha_+,w_1=1)=\log\mathbb{P}(T_1=\alpha_+)>-\infty\,.\end{aligned}$$

To see that J_+ is right–continuous at α_+ observe that by lower semicontinuity $J_+(\alpha_+)\le\lim_{\vartheta\searrow\alpha_+}J_+(\vartheta)$. We claim that $J_+(\alpha_+)\ge\lim_{\vartheta\searrow\alpha_+}J_+(\vartheta)$. Indeed, fixed $\alpha_0>\alpha_+$, by convexity it holds

$$J_+(\alpha_+)\ge\frac{1}{1-\lambda}J_+\big((1-\lambda)\alpha_++\lambda\alpha_0\big)-\frac{\lambda}{1-\lambda}J_+(\alpha_0)\,.$$

The claim then follows from the monotonicity of J_+ on the right of α_+. Combining the last observations we get $\lim_{\vartheta\searrow\alpha_+}J_+(\vartheta)=J_+(\alpha_+)<\infty$ and this concludes the proof of Point (v). □

We now move to the study of the function $I(\vartheta)$ defined on $\mathbb{R}\setminus\{0\}$ as

$$I(\vartheta)=\begin{cases} I_+(\vartheta):=\sup_{\lambda\in\mathbb{R}}\{\lambda-\vartheta\log\varphi_+(\lambda)\},\ \vartheta>0\,,\\ I_-(\vartheta):=\sup_{\lambda\in\mathbb{R}}\{\lambda+\vartheta\log\varphi_-(\lambda)\},\ \vartheta<0\,.\end{cases} \tag{33}$$

Lemma 5.5. *It holds*

$$\lim_{\vartheta\nearrow 0} I_-(\vartheta)=\lim_{\vartheta\searrow 0} I_+(\vartheta)=\lambda_c\,, \tag{34}$$

$$\lim_{\vartheta\nearrow 0} I'_-(\vartheta)=\lim_{\vartheta\searrow 0} I'_+(\vartheta)\,. \tag{35}$$

In particular, the definition of $I(\vartheta)$ in Theorem 6 is well posed and $I(0)=\lambda_c$. Moreover, I is finite and C^1 on $(-1/\alpha_-,1/\alpha_+)$, and it is smooth on $(-1/\alpha_-,1/\alpha_+)\setminus\{0\}$.

Proof. For any $\vartheta>0$ we have $I(\vartheta)=\sup_{\lambda\leq\lambda_c}\{\lambda-\vartheta\log\varphi_+(\lambda)\}$ since $\varphi_+(\lambda)=+\infty$ if $\lambda>\lambda_c$. Moreover, always by Lemma 5.1, for $0<\vartheta<1/\alpha_+$ the above supremum is attained at the unique value $\lambda_+(\vartheta)<\lambda_c$ such that

$$(\log\varphi_+)'\big(\lambda_+(\vartheta)\big)=1/\vartheta\,, \tag{36}$$

thus implying that $\lambda_+(\vartheta)$ is a strictly decreasing function and $\lim_{\vartheta\searrow 0}\lambda_+(\vartheta)=\lambda_c$ (due to Lemma 5.1). In particular, $I(\vartheta)=\lambda_+(\vartheta)-\vartheta\log\varphi_+\big(\lambda_+(\vartheta)\big)$ is finite on $(0,1/\alpha_+)$ and moreover

$$\lim_{\vartheta\searrow 0} I(\vartheta)=\lim_{\vartheta\searrow 0}\{\lambda_+(\vartheta)-\vartheta\log\varphi_+(\lambda_+(\vartheta))\}=\lambda_c$$

since $\lim_{\lambda\nearrow\lambda_c}\log\varphi_+(\lambda)=\log\varphi_+(\lambda_c)$ which is finite due to (29). This concludes the proof of (34) for I_+. By similar arguments one gets that, given $\vartheta\in(-1/\alpha_-,0)$ there is a unique value $\lambda_-(\vartheta)$ solving the equation

$$(\log\varphi_-)'\big(\lambda_-(\vartheta)\big)=-1/\vartheta\,. \tag{37}$$

The function λ_- is strictly increasing on $(-1/\alpha_-,0)$ where it holds $I(\vartheta)=\lambda_-(\vartheta)+\vartheta\log\varphi_-(\lambda_-(\vartheta))$. As above one gets that $\lim_{\vartheta\nearrow 0}I_-(\vartheta)=\lambda_c$, hence (34). Note that (34) implies that I is well defined in Theorem 6 and that $I(0)=\lambda_c$. By the previous results we conclude also that I is finite on $(-1/\alpha_-,1/\alpha_+)$.

Let us now prove (35) and that I is C^1 on $(-1/\alpha_-,1/\alpha_+)\setminus\{0\}$. By the implicit function theorem and Lemma 5.1, the function $(0,1/\alpha_+)\ni\vartheta\to\lambda_+(\vartheta)\in(-\infty,\lambda_c)$ is smooth. In particular, using (36), I_+ is smooth on $(0,1/\alpha_+)$ where it holds

$$\begin{aligned} I'_+(\vartheta)&=\frac{\mathrm{d}}{\mathrm{d}\vartheta}\big(\lambda_+(\vartheta)-\vartheta\log\varphi_+(\lambda_+(\vartheta))\big)\\ &=\lambda'_+(\vartheta)-\log\varphi_+(\lambda_+(\vartheta))-\vartheta\cdot(\log\varphi_+)'(\lambda_+(\vartheta))\cdot\lambda'_+(\vartheta)\\ &=-\log\varphi_+(\lambda_+(\vartheta))\,.\end{aligned} \tag{38}$$

Hence, $\lim_{\vartheta\searrow 0}I'_+(\vartheta)=-\log\varphi_+(\lambda_c)$. By similar arguments and definitions we get that I_- is smooth on $(-1/\alpha_-,0)$ where it holds

$$\lim_{\vartheta\nearrow 0} I'_-(\vartheta)=\lim_{\vartheta\nearrow 0}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\big(\lambda_-(\vartheta)+\vartheta\log\varphi_-(\lambda_-(\vartheta))\big)=\log\varphi_-(\lambda_c)\,.$$

To conclude the proof of (35) it remains to show that $\log\varphi_-(\lambda_c)=-\log\varphi_+(\lambda_c)$. To this aim we observe that

$$\log\varphi_-(\lambda_c)+\log\varphi_+(\lambda_c)=\log[\varphi_+(\lambda_c)\varphi_-(\lambda_c)]=\log\left(\frac{1}{4f_+(\lambda_c)f_-(\lambda_c)}\right)=0$$

due to (29), its analogous version for $\varphi_-(\lambda_c)$ and since, by definition, λ_c is the unique solution of $4f_-(\lambda)f_+(\lambda) = 1$ This concludes the proof of (35) and that I is smooth on $(-1/\alpha_-, 1/\alpha_+) \setminus \{0\}$. Due to (35) one easily gets that I is differentiable at 0 and $I'(0)$ equals the limits in (35). This implies that I is C^1 on $(-1/\alpha_-, 1/\alpha_+)$. □

Combining Lemmas 5.1, 5.5 and Proposition 5.3 we are finally able to prove Theorem 7 and therefore also Theorem 2 due to Lemma 4.1.

Proof of Theorem 7. Below the labeling of items is as in Theorem 2.

The fact that I is finite and C^1 on $\big(-\frac{1}{\alpha_-}, \frac{1}{\alpha_+}\big)$ and infinite outside $[-\frac{1}{\alpha_-}, \frac{1}{\alpha_+}]$ follows from (20) and Proposition 5.3. This proves Item (i).

To prove Item (ii) note that if $\mathbb{P}(T_1 = \alpha_+) > 0$ then $\alpha_+ > 0$. Hence, by (30) we get

$$\lim_{\vartheta \nearrow \frac{1}{\alpha_+}} I(\vartheta) = \lim_{\vartheta \nearrow \frac{1}{\alpha_+}} \vartheta J_+\big(\frac{1}{\vartheta}\big) = \frac{J_+(\alpha_+)}{\alpha_+} < \infty$$

and the last term equals $I(1/\alpha_+)$ by definition of I. If, on the other hand, $\mathbb{P}(T_1 = \alpha_+) = 0$, then by (30) we get

$$\lim_{\vartheta \nearrow \frac{1}{\alpha_+}} I(\vartheta) = \lim_{u \searrow \alpha_+} \frac{J_+(u)}{u} = +\infty \,.$$

The correspondent statements for $\vartheta \searrow -1/\alpha_-$ are obtained in the same way, and this concludes the proof of Item (ii).

To see (iii) recall that $I'(\vartheta) = -\log \varphi_+(\lambda_+(\vartheta))$ for $\lambda \in (0, 1/\alpha_+)$ (see (38)). Observe now that $\lim_{\vartheta \nearrow \frac{1}{\alpha_+}} \lambda_+(\vartheta) = -\infty$ (due to Lemma 5.1–(ii) and (36)). This implies that

$$\lim_{\vartheta \nearrow \frac{1}{\alpha_+}} (-\log \varphi_+(\lambda_+(\vartheta))) = -\log \varphi_+(-\infty) = +\infty \,.$$

Similarly one sees that

$$\lim_{\vartheta \searrow -\frac{1}{\alpha_-}} I'(\vartheta) = \lim_{\vartheta \searrow -\frac{1}{\alpha_-}} \log \varphi_-(\lambda_-(\vartheta)) = \log \varphi_-(-\infty) = -\infty \,.$$

We now consider Item (iv). We observe that I_+, I_- are l.s.c. because they can be expressed as pointwise suprema of continuous functions, and by (34) they attach in 0 in a continuous fashion. We now prove that I is convex. Being suprema of families of linear functions, I_+ and I_- are convex. Therefore I is convex on $(0, +\infty)$ and $(-\infty, 0)$ separately. To prove the convexity on all $\mathbb{R}$ it remains to show that I is also convex in $\vartheta = 0$. Since the left and right branches of I are differentiable, it suffices to show that the left derivative at $\vartheta = 0$ is non greater than the right derivative. In fact, they are equal due to (35). Let us now prove that I is strictly convex on the closure of $(-1/\alpha_-, 1/\alpha_+)$. We know that $I'(\vartheta) = \log \varphi_-(\lambda_-(\vartheta))$ on $(-1/\alpha_-, 0]$ and $I'(\vartheta) = -\log \varphi_+(\lambda_+(\vartheta))$ on $[0, 1/\alpha_+)$ (see the proof of Lemma 5.5). By Lemma 5.1–(ii) $\log \varphi_\pm$ is strictly increasing with positive derivative, while we know that λ_+ is a strictly decreasing function on $(0, 1/\alpha_+)$ and λ_- is a strictly increasing on $(-1/\alpha_-, 0)$. Using also that I' is continuous at 0 we conclude that I' is strictly increasing on $(-1/\alpha_-, 1/\alpha_+)$, hence I is strictly convex on $(-1/\alpha_-, 1/\alpha_+)$.

We conclude with Item (v). We know that I' is strictly increasing on $(-1/\alpha_-, 1/\alpha_+)$. Due to Item (iii) it simple to conclude that there exists a unique minimum point $\vartheta_* \in (-1/\alpha_-, 1/\alpha_+)$ such that I is strictly decreasing on $(-1/\alpha_-, \vartheta_*)$ and strictly increasing on $(\vartheta_*, 1/\alpha_+)$. It remains to prove that $\vartheta_* = v$.

If $v > 0$ then by Prop. 5.3–(iv) $v = 1/\vartheta_c^+$ and so $I(v) = vJ_+\big(1/v\big) = (1/\vartheta_c^+)J_+(\vartheta_c^+) = 0$. If $v < 0$ then $v = -1/\vartheta_c^-$ and so $I(v) = -vJ_-\big(-1/v\big) = (1/\vartheta_c^-)J_-(\vartheta_c^-) = 0$. If, finally,

$v = 0$ then again by Prop. 5.3–(iv) we have $0 = \lambda_c = I(0) = I(v)$. In all cases $I(v) = 0$ and since I is non–negative we conclude that $v = \vartheta_*$. □

6. Proof of Theorem 6–(ii)

Below we show how one can deduce the LDP for the process Z itself from the LDP for the hitting times. Due to Theorem 4.1.11 in [11], the LDP for Z_t/t holds with rate function I and speed t if we show that

$$\lim_{\varepsilon\to 0}\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{Z_t}{t}\in(\vartheta-\varepsilon,\vartheta+\varepsilon)\right)\geq -I(\vartheta)\,, \tag{LB}$$

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{Z_t}{t}\in(\vartheta-\varepsilon,\vartheta+\varepsilon)\right)\leq -I(\vartheta)\,. \tag{UB}$$

6.1. **The lower bound.** Given $\vartheta \neq 0$ and $\delta, c \in (0,1)$ we define the events

$$A_t = A_t(\delta,\vartheta) := \{(1-\delta)t < T_{\lfloor \vartheta t\rfloor} < (1+\delta)t\}$$
$$B_t = B_t(\delta,c) := \{\nu(t+\delta t) - \nu(t-\delta t) \leq ct\}\,,$$

Lemma 6.1. *For any $\vartheta \neq 0$ and $\delta, c \in (0,1)$ there exists $\tilde{t} = \tilde{t}(c) > 0$ such that*

$$A_t \cap B_t \subseteq \{Z_t/t \in (\vartheta - 2c, \vartheta + 2c)\}$$

for all $t > \tilde{t}$.

Proof. Take any $t > 0$ and assume $A_t \cap B_t$ holds. Then, due to Assumption (A1),

$$\begin{aligned}|Z_t - \lfloor\vartheta t\rfloor| &= |Z_t - Z_{T_{\lfloor\vartheta t\rfloor}}| = \left|W_{\nu(t)} - W_{\nu(T_{\lfloor\vartheta t\rfloor})}\right| \\ &\leq \nu(t\vee T_{\lfloor\vartheta t\rfloor}) - \nu(t\wedge T_{\lfloor\vartheta t\rfloor}) \leq \nu(t+\delta t) - \nu(t-\delta t) \leq ct\,.\end{aligned}$$

Hence $Z_t \in [\lfloor\vartheta t\rfloor - ct, \lfloor\vartheta t\rfloor + ct]$, thus leading to the thesis. □

Lemma 6.2. *For any $\vartheta \neq 0$ and $\delta \in (0,1)$ it holds*

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}(A_t) \geq -I(\vartheta) = \begin{cases} -\vartheta J_+\big(\frac{1}{\vartheta}\big) & \text{if } \vartheta > 0\,, \\ \vartheta J_-\big(-\frac{1}{\vartheta}\big) & \text{if } \vartheta < 0\,. \end{cases}$$

Proof. We give the proof for $\vartheta > 0$, the one for $\vartheta < 0$ being the same. Note that, fixed $\varepsilon > 0$, for t large enough it holds

$$\frac{1}{t}\log\mathbb{P}(A_t) \geq \frac{1}{t}\log\mathbb{P}\left(\frac{1-\delta}{\vartheta}+\varepsilon < \frac{T_{\lfloor\vartheta t\rfloor}}{\lfloor\vartheta t\rfloor} < \frac{1+\delta}{\vartheta}\right).$$

Thanks to the LDP for the hitting times T_n this implies that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}(A_t) &\geq \vartheta\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{1-\delta}{\vartheta}+\varepsilon < \frac{T_n}{n} < \frac{1+\delta}{\vartheta}\right) \\ &\geq -\vartheta \inf_{\left(\frac{1-\delta}{\vartheta}+\varepsilon,\frac{1+\delta}{\vartheta}\right)} J_+ \geq -\vartheta J_+\big(\frac{1}{\vartheta}\big)\end{aligned}$$

as long as ε is chosen small enough so that $\frac{1}{\vartheta} \in \big(\frac{1-\delta}{\vartheta}+\varepsilon, \frac{1+\delta}{\vartheta}\big)$. □

Since τ_i's are positive i.i.d. random variables, for every $p_0 \in (0,1)$ we can find some $\eta > 0$ such that $p := \mathbb{P}(\tau_i \geq \eta) > p_0$. In particular, the i.i.d. random variables r_i's with $r_i := \mathbb{1}(\tau_i \geq \eta)$ are Bernoulli of parameter p. They are a useful tool to bound the probability of B_t^c:

Lemma 6.3. *For any $\vartheta \neq 0$ and any $c \in (0,1)$, there exists a constant $\delta_* = \delta_*(\theta, c) \in (0,1)$ depending only on θ, c such that, for all $\delta \in (0, \delta_*]$, it holds*

$$\lim_{t\to\infty} \frac{\mathbb{P}(B_t^c)}{\mathbb{P}(A_t)} = 0\,.$$

Proof. We restrict to the case $\vartheta > 0$, being the proof for $\vartheta < 0$ similar. We observe that the event B_t^c implies the event

$$\bigcup_{j=0}^{\infty} \Big\{ \nu(t-\delta t) = j\,,\ \sum_{\ell=1}^{\lceil ct\rceil -1} \tau_{j+1+\ell} \leq 2\delta t \Big\}\,.$$

Since the event $\{\nu(t-\delta t) = j\}$ depends only on $\tau_1, \tau_2, \dots, \tau_{j+1}$, by independence we get

$$\begin{aligned} \mathbb{P}(B_t^c) &\leq \sum_{j=0}^{\infty} \mathbb{P}(\nu(t-\delta t) = j) \mathbb{P}\Big(\sum_{\ell=1}^{\lceil ct\rceil -1} \tau_{j+1+\ell} \leq 2\delta t\Big) \\ &= \mathbb{P}\Big(\sum_{\ell=1}^{\lceil ct\rceil -1} \tau_\ell \leq 2\delta t\Big) \leq \mathbb{P}\Big(\sum_{\ell=1}^{\lceil ct\rceil -1} r_\ell \leq 2\delta t/\eta\Big)\,. \end{aligned} \tag{39}$$

Above we have used that $\tau_\ell \geq \eta\, r_\ell$. Now we use Cramér Theorem for sums of i.i.d. p–Bernoulli r.v.'s. The associated rate function is given by (cf. exercise 2.2.23 in [11])

$$\mathcal{I}_p(x) = \begin{cases} x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p} & \text{if } x \in [0,1]\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

with the convention that $0 \log 0 := 0$. Trivially, $\mathcal{I}_p$ is strictly decreasing on $[0,p]$ and strictly increasing on $[p,1]$, while $\mathcal{I}_p(p) = 0$. Let $t_* := \lceil ct \rceil - 1$. Writing

$$\frac{1}{t} \log \mathbb{P}\Big(\sum_{\ell=1}^{\lceil ct\rceil -1} r_\ell \leq 2\delta t/\eta\Big) = \frac{t_*}{t} \frac{1}{t_*} \log \mathbb{P}\Big(\frac{1}{t_*} \sum_{\ell=1}^{t_*} r_\ell \leq \frac{2\delta t}{\eta t_*}\Big)$$

and using that $2\delta t (\eta t_*)^{-1} \leq 3\delta (\eta c)^{-1}$ for t large enough, we get

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\sum_{\ell=1}^{\lceil ct\rceil -1} r_\ell \leq 2\delta t/\eta\Big) \leq -c \inf_{(-\infty, \frac{3\delta}{\eta c}]} \mathcal{I}_p\,. \tag{40}$$

Now we have to choose carefully the constants in order to win. Fix $\vartheta > 0$ and $c \in (0,1)$. The function $\mathcal{I}_p(0) = \log \frac{1}{1-p}$ is increasing in p and $\lim_{p\to 1} \mathcal{I}_p(0) = +\infty$. In particular, there exists $p_0 > 0$ such that $\mathcal{I}_p(0) > \vartheta J_+(1/\vartheta)/c$ for all $p \geq p_0$. We fix η such that $p := \mathbb{P}(\tau_i \geq \eta) > p_0$.

If $p = 1$ then $\tau_i \geq \eta$ a.s. In particular, equation (39) gives $\mathbb{P}(B_t^c) \leq \mathbb{1}\big(ct - 1 \leq \frac{2\delta t}{\eta}\big)$, so by setting $\delta_* = \eta c/4$ we have that for any $\delta \leq \delta_*$ and t large enough $\mathbb{P}(B_t^c) = 0$. This, combined with Lemma 6.2, gives the thesis.

Assume, on the other hand, that $p < 1$. Recall that $\mathcal{I}_p(0) > \vartheta J_+(1/\vartheta)/c$. Since $\lim_{\varepsilon \searrow 0} \mathcal{I}_p(\varepsilon) = \mathcal{I}_p(0)$ and $\mathcal{I}_p$ is decreasing near 0, we can fix $\varepsilon_0 > 0$ such that $\mathcal{I}_p(\varepsilon) > \vartheta J_+(1/\vartheta)/c$ for all $\varepsilon \in [0, \varepsilon_0]$. Note that the (now fixed) constants η, p, ε_0 depend only on ϑ, c. To conclude let $\delta_* = (\eta c \varepsilon_0/4) \wedge 1$. Then for each $\delta \in (0, \delta_*]$ we have $3\delta(\eta c)^{-1} \leq \varepsilon_0$ and therefore the last term in (40) is strictly bounded from above by $-\vartheta J_+(1/\vartheta)$. Coming back to (39) and (40) we conclude that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(B_t^c) < -\vartheta J_+(1/\vartheta)\,. \tag{41}$$

The above bound together with Lemma 6.2 implies the thesis. □

Combining Lemmas 6.2 and 6.3 we can prove the following key lower bound:

Lemma 6.4. *For any $\vartheta \neq 0$ and $\varepsilon \in (0,1/2)$ the following holds*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\frac{Z_t}{t} \in (\vartheta-\varepsilon, \vartheta+\varepsilon)\Big) \geq -I(\vartheta)\,. \tag{42}$$

Proof. Given $\varepsilon > 0$, take $c := \varepsilon/2$ and $\delta := \delta_*(\vartheta, c)$ in the definition of A_t, B_t given in Lemma 6.1, where the constant δ_* is as in Lemma 6.3. Due to Lemma 6.1 for t large enough we have

$$\mathbb{P}\Big(\frac{Z_t}{t} \in (\vartheta-\varepsilon, \vartheta+\varepsilon)\Big) \geq \mathbb{P}(A_t \cap B_t) \geq \mathbb{P}(A_t) - \mathbb{P}(B_t^c) = \mathbb{P}(A_t)\Big(1 - \frac{\mathbb{P}(B_t^c)}{\mathbb{P}(A_t)}\Big)$$

which implies

$$\frac{1}{t}\log \mathbb{P}\Big(\frac{Z_t}{t} \in (\vartheta-\varepsilon, \vartheta+\varepsilon)\Big) \geq \frac{1}{t}\log \mathbb{P}(A_t) + \frac{1}{t}\log\Big(1 - \frac{\mathbb{P}(B_t^c)}{\mathbb{P}(A_t)}\Big)\,.$$

Using Lemma 6.2 to control the first term in the r.h.s. and Lemma 6.3 to control the second term in the r.h.s. we get the thesis. □

Being (42) uniform in $\varepsilon \in (0,1/2)$, one can let $\varepsilon \to 0$ to conclude that the lower bound (LB) holds for all $\vartheta \neq 0$. If $\vartheta = 0$, take any $\varepsilon > 0$ and let $u = \varepsilon/2$. Then by Lemma 6.4 one has

$$\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}\Big(\frac{Z_t}{t} \in (-\varepsilon, +\varepsilon)\Big) \geq \liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}\Big(\frac{Z_t}{t} \in (u-\varepsilon/4, u+\varepsilon/4)\Big) \geq -I(u)\,.$$

Letting then $\varepsilon \to 0$ and therefore $u \to 0$ gives (recall Theorems 2, 7)

$$\lim_{\varepsilon\to 0}\liminf_{t\to\infty} \frac{1}{t}\log \mathbb{P}\Big(\frac{Z_t}{t} \in (-\varepsilon, \varepsilon)\Big) \geq -\lim_{u\to 0} I(u) = -I(0)\,.$$

This concludes the proof of (LB) for all $\vartheta \in \mathbb{R}$.

6.2. **The upper bound.** We now move to the proof of (UB). This is rather easy if the asymptotic velocity v vanishes.

Lemma 6.5. *If $v = 0$ then* (UB) *holds for all $\vartheta \in \mathbb{R}$.*

Proof. If $\vartheta = 0$ it is enough to observe that by the LLN $Z_t/t \to 0$ almost surely as $t \to \infty$, and therefore in probability. Since $I(0) = 0$ by Theorems 2, 7, we get the thesis.

To deal with the case $\vartheta \neq 0$, recall that $v = 0 \Leftrightarrow \lambda_c = 0 \Leftrightarrow J_\pm$ are strictly decreasing on $(\alpha_\pm, \infty)$ (see Prop. 5.3). Since $J_+ \equiv +\infty$ on $(-\infty, \alpha_+)$ and due to (30) we conclude that $J_+ : \mathbb{R} \to [0, +\infty]$ is a decreasing extended function. Fix now any $\vartheta > 0$ and $\varepsilon > 0$ such that $\vartheta - \varepsilon > 0$. Then, given any $\tilde{\varepsilon} > 0$, for t large it holds

$$\begin{aligned}\mathbb{P}\Big(\frac{Z_t}{t} \in (\vartheta-\varepsilon, \vartheta+\varepsilon)\Big) &\leq \mathbb{P}\Big(\frac{Z_t}{t} > \vartheta-\varepsilon\Big) \leq \mathbb{P}(Z_t \geq \lfloor(\vartheta-\varepsilon)t\rfloor) \\ &\leq \mathbb{P}(T_{\lfloor(\vartheta-\varepsilon)t\rfloor} \leq t) = \mathbb{P}\Big(\frac{T_{\lfloor(\vartheta-\varepsilon)t\rfloor}}{\lfloor(\vartheta-\varepsilon)t\rfloor} \leq \frac{1}{\vartheta-\varepsilon} + \tilde{\varepsilon}\Big)\,.\end{aligned}$$

Hence, using the LDP for the hitting times T_n and the fact that J_+ is decreasing,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big(\frac{Z_t}{t}\in(\vartheta-\varepsilon,\vartheta+\varepsilon)\Big)\\ &\le(\vartheta-\varepsilon)\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{T_n}{n}\le\frac{1}{\vartheta-\varepsilon}+\tilde\varepsilon\right)\\ &\le-(\vartheta-\varepsilon)\inf_{\left(-\infty,\frac{1}{\vartheta-\varepsilon}+\tilde\varepsilon\right)}J_+=-(\vartheta-\varepsilon)J_+\Big(\frac{1}{\vartheta-\varepsilon}+\tilde\varepsilon\Big)\,.\end{aligned}\tag{43}$$

Letting $\tilde\varepsilon\to0$ and using that J_+ is l.s.c. (see Prop. 5.3) we get that the first member of (43) is bounded from above by $-(\vartheta-\varepsilon)J_+\big(1/(\vartheta-\varepsilon)\big)$. Taking now the limit $\varepsilon\to0$ and using again that J_+ is l.s.c. we get the thesis for $\vartheta>0$. The proof of (UB) for $\vartheta<0$ follows by similar arguments. □

We now prove that (UB) holds for all $\vartheta\in\mathbb{R}$ assuming $v>0$. The case $v<0$ can be addressed in the same way. The proof we present is based on a method introduced in [10], that we re–adapt to our setting. The strategy consists in reducing the problem to proving the following:

Proposition 6.6. *Assume $v>0$ and define $S_t:=\inf\{s\ge t: Z_s\le0\}$. Then it holds*

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(S_t<\infty)\le-I(0)\,.\tag{44}$$

The fact that the above result implies that (UB) holds for all $\vartheta\in\mathbb{R}$ can be seen reasoning as in [10], page 1017, with minor modifications. For completeness we give a sketch of the proof in Appendix A.

A detailed proof of Proposition 6.6 is, on the other hand, given below. This choice is due to the presence of a small gap [20] in the proof presented in [10] (see formula (4.14) on page 1020 there), and to the fact that some additional arguments are necessary since our holding times can be in general arbitrarily small while in [10] they are bounded from below by 1.

Proof of Proposition 6.6. Due to Prop. 5.3, since $v>0$, $\lambda_c>0$ and the critical point $\vartheta_c^\pm$ of $J_\pm$ is finite and positive. Take any $u\in(0,1/\vartheta_c^+)$ and fix $c>1$ integer such that $c/u>\vartheta_c^-$. Let, in order to simplify the notation, $b_t:=\mathbb{P}(S_t<\infty)$ with the convention that $b_t=1$ if $t<0$. Recall that $T_{\lfloor tu\rfloor}$ is the hitting time of $\lfloor tu\rfloor$, and define

$$\tilde T_0:=\begin{cases}\inf\{s\ge T_{\lfloor tu\rfloor}:Z_s=0\}, & \text{if } T_{\lfloor tu\rfloor}<\infty\,,\\ +\infty & \text{otherwise}\,.\end{cases}$$

Then we have

$$\begin{aligned}b_t\le&\,\mathbb{P}(T_{\lfloor tu\rfloor}\ge t)+\mathbb{P}(T_{\lfloor tu\rfloor}<t,\,\tilde T_0-T_{\lfloor tu\rfloor}\ge ct,\,S_t<\infty)\\ &+\mathbb{P}(T_{\lfloor tu\rfloor}<t,\,\tilde T_0-T_{\lfloor tu\rfloor}<ct,\,S_t<\infty)\,.\end{aligned}\tag{45}$$

For the first term in the r.h.s. of (45) the LDP for the hitting times T_n, $n\to\infty$, implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{T_{\lfloor tu\rfloor}}{\lfloor tu\rfloor}\ge\frac{t}{\lfloor tu\rfloor}\right)\le\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(\frac{T_{\lfloor tu\rfloor}}{\lfloor tu\rfloor}\ge\frac{1}{u}\right)\le-uJ_+(1/u)=-I(u)\,.\tag{46}$$

Above we have used that J_+ is increasing on $(\vartheta_c^+,+\infty)$.

For the second term we apply the strong Markov property at time $T_{\lfloor tu \rfloor}$ (cf. Definition 2.1–(iv)) to get

$$\mathbb{P}(T_{\lfloor tu \rfloor} < t,\, \tilde{T}_0 - T_{\lfloor tu \rfloor} \geq ct,\, S_t < \infty) \leq \mathbb{P}(T_{-\lfloor tu \rfloor} \geq ct)\,.$$

Therefore, by the LDP for the hitting times T_{-n}, $n \to \infty$, and the fact that J_- is increasing on $(\vartheta_c^-, +\infty)$, we obtain

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T_{\lfloor tu \rfloor} < t,\, \tilde{T}_0 - T_{\lfloor tu \rfloor} \geq ct,\, S_t < \infty) &\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\frac{T_{-\lfloor tu \rfloor}}{\lfloor tu \rfloor} \geq \frac{c}{u}\Big) \\ &\leq -uJ_-(c/u) = -cI(-u/c)\,.\end{aligned} \tag{47}$$

For the third term in the r.h.s. of (45) one has to deal with the critical points of $J_\pm$, so the idea is to localize things. Fix $m \in \mathbb{N}$ positive. Fix $0 < u' < u$, hence $1/\lfloor tu \rfloor \leq 1/tu'$ for t large (as we assume). We take u' very near to u such that $1/u' > \vartheta_c^+$ and $c/u' > 1/\vartheta_c^-$. Then

$$\begin{aligned}&\mathbb{P}\Big(\frac{T_{\lfloor tu \rfloor}}{\lfloor tu \rfloor} < \frac{1}{u'},\, \frac{\tilde{T}_0 - T_{\lfloor tu \rfloor}}{\lfloor tu \rfloor} < \frac{c}{u'},\, S_t < \infty\Big) \\ &\quad= \sum_{k=1}^{m}\sum_{\ell=1}^{mc} \mathbb{P}\Big(\frac{T_{\lfloor tu \rfloor}}{\lfloor tu \rfloor} \in \Big[\frac{k-1}{mu'}, \frac{k}{mu'}\Big),\, \frac{\tilde{T}_0 - T_{\lfloor tu \rfloor}}{\lfloor tu \rfloor} \in \Big[\frac{\ell-1}{mu'}, \frac{\ell}{mu'}\Big),\, S_t < \infty\Big) \\ &\quad\leq \sum_{k=1}^{m}\sum_{\ell=1}^{mc} \mathbb{P}\Big(\frac{T_{\lfloor tu \rfloor}}{\lfloor tu \rfloor} \in \Big[\frac{k-1}{mu'}, \frac{k}{mu'}\Big]\Big)\mathbb{P}\Big(\frac{T_{-\lfloor tu \rfloor}}{\lfloor tu \rfloor} \in \Big[\frac{\ell-1}{mu'}, \frac{\ell}{mu'}\Big]\Big) b_{t-\frac{(k+\ell)t}{m}}\end{aligned} \tag{48}$$

where we have applied the strong Markov property at times $T_{\lfloor tu \rfloor}$ and $\tilde{T}_0$ and used that if $s \leq t$ then $b_s \geq b_t$ since $S_s \leq S_t$. Now we analyze each term separately. Define

$$\begin{aligned}w_+(r,\delta) &:= \max\{|J_+(s) - J_+(t)| : s,t \in [\vartheta_c^+, r], |s-t| \leq \delta\}\,, \\ w_-(r,\delta) &:= \max\{|J_-(s) - J_-(t)| : s,t \in [\vartheta_c^-, r], |s-t| \leq \delta\}\,,\end{aligned} \tag{49}$$

with the convention that $w_\pm(r,\delta) = 0$ if $r < \vartheta_c^\pm$. The LDP for the hitting times T_n then gives

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\frac{T_{\lfloor tu \rfloor}}{\lfloor tu \rfloor} \in \Big[\frac{k-1}{mu'}, \frac{k}{mu'}\Big]\Big) &\leq -u \inf_{\left[\frac{k-1}{mu'}, \frac{k}{mu'}\right]} J_+ \\ &\leq -uJ_+(k/mu') + uw_+\big(\frac{k}{mu'}, \frac{1}{mu'}\big) = -\frac{k}{m} I\big(\frac{u'm}{k}\big) + uw_+\big(\frac{k}{mu'}, \frac{1}{mu'}\big)\,.\end{aligned} \tag{50}$$

Similarly we get

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\frac{T_{-\lfloor tu \rfloor}}{\lfloor tu \rfloor} \in \Big[\frac{\ell-1}{mu'}, \frac{\ell}{mu'}\Big]\Big) &\leq -u \inf_{\left[\frac{\ell-1}{mu'}, \frac{\ell}{mu'}\right]} J_- \\ &\leq -uJ_-(\ell/mu') + uw_-\big(\frac{\ell}{mu'}, \frac{1}{mu'}\big) = -\frac{\ell}{m} I\big(-\frac{um'}{\ell}\big) + uw_-\big(\frac{\ell}{mu'}, \frac{1}{mu'}\big)\,.\end{aligned} \tag{51}$$

We set

$$\begin{aligned}W_{k,\ell} &:= w_+\big(\frac{k}{mu'}, \frac{1}{mu'}\big) + w_-\big(\frac{\ell}{mu'}, \frac{1}{mu'}\big)\,, \\ W &:= \max\Big\{w_+\big(\frac{1}{u'}, \frac{1}{mu'}\big)\,,\, w_-\Big(\frac{c}{u'}, \frac{1}{mu'}\Big)\Big\}\end{aligned}$$

The above inequalities (50) and (51), and the convexity of I, we have for any $\varepsilon > 0$ and t large enough that

$$\begin{aligned}\mathbb{P}\Big(\frac{T_{\lfloor tu\rfloor}}{\lfloor tu\rfloor} \in \Big[\frac{k-1}{mu'}, \frac{k}{mu'}\Big]\Big)\mathbb{P}\Big(\frac{T_{-\lfloor tu\rfloor}}{\lfloor tu\rfloor} \in \Big[\frac{\ell-1}{mu'}, \frac{\ell}{mu'}\Big]\Big)\\ \leq e^{t\varepsilon + u'tW_{k,\ell}} e^{-t\left[\frac{k}{m}I\left(\frac{u'm}{k}\right)+\frac{\ell}{m}I\left(-\frac{u'm}{\ell}\right)\right]}\\ \leq e^{t\varepsilon + u'tW_{k,\ell}} e^{-t\frac{(k+\ell)}{m}I(0)} \leq e^{t\varepsilon - t\frac{(k+\ell)}{m}\left(I(0)-\frac{W}{c_0}\right)},\end{aligned} \tag{52}$$

where $c_0 := \min\{\vartheta_c^+, \vartheta_c^-\}$. We explain the last bound. Note that $W_{k,\ell} = 0$ if $k \leq \vartheta_c^+ mu'$ and $\ell \leq \vartheta_c^- mu'$. On the other hand, since $\frac{k}{mu'} \leq \frac{1}{u'}$ and $\frac{\ell}{mu'} \leq \frac{c}{u'}$, we have $W_{k,\ell} \leq W$. Hence it holds

$$u'W_{k,\ell} \leq u'W\mathbb{1}(k+\ell > c_0 mu') \leq \frac{(k+\ell)}{c_0 m}W\,. \tag{53}$$

Let now

$$J := \min\{I(u),\, cI(-c/u),\, I(0)\} - W/c_0\,. \tag{54}$$

Then putting (45), (46), (47), (48) and (52) we have

$$b_t \leq e^{-tI(u)+t\varepsilon} + e^{-tcI(-u/c)+t\varepsilon} + \sum_{k=1}^{m}\sum_{\ell=1}^{mc} e^{t\varepsilon - t\frac{(k+\ell)}{m}J} b_{t-\frac{(k+\ell)t}{m}}\,.$$

Note that if $k+\ell > m$, then $b_{t-\frac{(k+\ell)t}{m}} = 1$. Hence we get

$$b_t \leq (2+m^2c)e^{-tJ+t\varepsilon} + e^{t\varepsilon} \sum_{\substack{(k,\ell):1\leq k\leq m,\ 1\leq \ell\leq mc\\ k+\ell\leq m}} e^{-t\frac{(k+\ell)}{m}J} b_{t-\frac{(k+\ell)t}{m}}\,. \tag{55}$$

Call $x := \limsup_{t\to\infty}\frac{1}{t}\log b_t \in [-\infty, 0]$. Since, given a finite family of functions $\{f_i(t)\}_{i\in I}$, it holds $\limsup_{t\to\infty}\frac{1}{t}\log\big(\sum_i f_i(t)\big) \leq \max_{i\in I}\limsup_{t\to\infty}\frac{1}{t}\log\big(f_i(t)\big)$ from (55) we get

$$x \leq \varepsilon + \max_{j:2\leq j\leq m}\left\{-\frac{jJ}{m} + \left(1-\frac{j}{m}\right)x\right\} = x + \varepsilon - \min_{j:2\leq j\leq m}\frac{j(J+x)}{m}\,.$$

The above bound holds for any $\varepsilon > 0$, hence we conclude that $0 \leq -\min_{j:2\leq j\leq m}\frac{j(J+x)}{m}$. This implies that $J + x \leq 0$, i.e.

$$\limsup_{t\to\infty}\frac{1}{t}\log b_t \leq -J\,. \tag{56}$$

Now, let $m \to \infty$ first, so that $W \to 0$ due to the fact that $J_\pm$ are even C^1 on $(\alpha_\pm, \infty)$ and $\alpha_\pm < \vartheta_c^\pm$ (see Prop. 5.3 and recall that $1/u' > \vartheta_c^+$, $c/u' > \vartheta_c^-$). Now we let $u \to 0$. By Theorems 2, 7 and since $v > 0$, $\min\{I(u),\, cI(-c/u),\, I(0)\}$ converges to $I(0)$ as $u \to 0$. This leads to the thesis. □

7. Proof of Theorem 8 (Gallavotti–Cohen type symmetry)

Due to the definition of I, $I(\vartheta) = I(-\vartheta) + c\vartheta$ for all $\vartheta \in \mathbb{R}$ if and only if

$$J_+(\vartheta) = J_-(\vartheta) + c\,, \qquad \forall \vartheta > 0\,. \tag{57}$$

We now prove that (57) and Item (iii) with $c = -\log C$ are equivalent. To this aim, assume that $\varphi_+(\lambda) = C\varphi_-(\lambda)$ for all $\lambda \leq \lambda_c$ and some $C > 0$. Then $\log\varphi_+(\lambda) = \log\varphi_-(\lambda) + \log C = \log\varphi_-(\lambda) - c$ for all $\lambda \in \mathbb{R}$. Hence, taking the Legendre transform and recalling the definition (6) of $J_\pm$ as Legendre transform of $\log\varphi_\pm$, we get (57).

On the other hand suppose that (57) holds. We claim that $J_+(\vartheta) = J_-(\vartheta) + c$ also for all $\vartheta \leq 0$. Indeed, since $\alpha_\pm \geq 0$, the claim follows from Proposition 5.3–(ii) for $\vartheta < 0$. If α_+, α_- are both positive then Proposition 5.3–(ii) implies the claim also for $\vartheta = 0$. If α_+, α_- are both zero, then (57) and the right continuity of $J_\pm$ at $\alpha_\pm$ (see Proposition 5.3–(v)) imply the claim for $\vartheta = 0$. We now show that α_- and α_+ must be either both positive or both zero, thus concluding the proof of our claim. Suppose for example that $\alpha_- = 0$ and $\alpha_+ > 0$. Then we would have $J_+(\vartheta) = \infty$ for $\vartheta \in (0, \alpha_+)$ (by Proposition 5.3–(ii)). This fact together with (57) implies that $J_-(\vartheta) = +\infty$ for $\vartheta \in (0, \alpha_+)$. Applying Proposition 5.3–(ii) to J_- we conclude that $\alpha_+ \leq \alpha_-$ thus getting a contradiction.

Due to (57) and the above claim we conclude that $J_+(\vartheta) = J_-(\vartheta) + c$ for all $\vartheta \in \mathbb{R}$. $J_+(\vartheta)$, $J_-(\vartheta) + c$ are the Legendre transforms of $\log \varphi_+, \log \varphi_- - c$, respectively, thought as extended functions from $\mathbb{R}$ to $(-\infty, +\infty]$. Due to Lemma 5.1 $\log \varphi_+, \log \varphi_- - c$ are convex, l.s.c. and not everywhere infinite. Hence, by the Fenchel–Moreau Theorem (cf. [6]) we conclude that $\log \varphi_+ = \log \varphi_- - c$, i.e. Item (iii) holds with $c = -\log C$.

We now prove that Item (ii) implies Item (iii). To this aim assume that τ_i and w_i are independent. Then $f_+(\lambda) = \mathbb{E}(e^{\lambda \tau_i})p$ and $f_-(\lambda) = \mathbb{E}(e^{\lambda \tau_i})q$ for all $\lambda \leq \lambda_c$. Combining this with (22) we get

$$\frac{\varphi_+(\lambda)}{\varphi_-(\lambda)} = \frac{f_+(\lambda)}{f_-(\lambda)} = \frac{p}{q} =: C\,, \qquad \forall \lambda \leq \lambda_c\,,$$

which is Item (iii).

Finally we prove that Item (iii) implies Item (ii). Hence assume $\varphi_+(\lambda) = C\varphi_-(\lambda)$ for all $\lambda \leq \lambda_c$. By (22) we have $\frac{\varphi_+(\lambda)}{\varphi_-(\lambda)} = \frac{f_+(\lambda)}{f_-(\lambda)} = C$. Moreover, taking $\lambda = 0$ in the previous identity, from the definition of $f_\pm$ we deduce that $C = p/q$.

In particular, given $\lambda, \gamma \leq 0$, we can write

$$\begin{aligned} \mathbb{E}\big(e^{\lambda \tau_i + \gamma w_i}\big) &= \mathbb{E}\big(e^{\gamma} e^{\lambda \tau_i} \mathbb{1}(w_i = 1)\big) + \mathbb{E}\big(e^{-\gamma} e^{\lambda \tau_i} \mathbb{1}(w_i = -1)\big) \\ &= e^{\gamma} f_+(\lambda) + e^{-\gamma} f_-(\lambda) = f_+(\lambda)\Big(e^{\gamma} + e^{-\gamma}\frac{q}{p}\Big)\,. \end{aligned}$$

On the other hand:

$$\mathbb{E}\big(e^{\lambda \tau_i}\big) = \mathbb{E}\big(e^{\lambda \tau_i} \mathbb{1}(w_i = 1)\big) + \mathbb{E}\big(e^{\lambda \tau_i} \mathbb{1}(w_i = -1)\big) = f_+(\lambda)\Big(1 + \frac{q}{p}\Big) = \frac{f_+(\lambda)}{p}$$

and

$$\mathbb{E}\big(e^{\gamma w_i}\big) = e^{\gamma} p + e^{-\gamma} q = p\Big(e^{\gamma} + e^{-\gamma}\frac{q}{p}\Big)\,.$$

Putting all together, we conclude that

$$\mathbb{E}\big(e^{\lambda \tau_i + \gamma w_i}\big) = f_+(\lambda)\Big(e^{\gamma} + e^{-\gamma}\frac{q}{p}\Big) = \Big(\frac{f_+(\lambda)}{p}\Big)\Big(p\big(e^{\gamma} + e^{-\gamma}\frac{q}{p}\big)\Big) = \mathbb{E}\big(e^{\lambda \tau_i}\big)\mathbb{E}\big(e^{\gamma w_i}\big)\,,$$

thus implying the independence of τ_i, w_i.

8. Proof of Theorem 3 (LDP via Gärtner–Ellis theorem for Markov rw's)

We introduce a $\mathbb{Z}$–valued process $(N_t)_{t \in \mathbb{R}_+}$ given by the cell number of X_t. More precisely, we set $N_t := n$ if $X_t = v_n$ for some $v \in V \setminus \{\overline{v}\}$. Note that the cell number process is in general not a Markovian process and that $|X_t^* - N_t| \leq 1$.

We introduce the constant κ defined as $\kappa := \max\{r(x) \,:\, x \in \mathcal{V}\}$, where $r(x) = \sum_{y:(x,y) \in \mathcal{E}} r(x,y)$. Due to the periodicity (2), κ is a well defined constant in $(0, +\infty)$.

Lemma 8.1. *For each $n \in \mathbb{Z} \setminus \{0\}$ and $t \in \mathbb{R}_+$ it holds $\mathbb{P}(N_t = n) \leq e^{\kappa t|n| - |n|\log|n|}$. In particular, for each $\lambda \in \mathbb{R}$ and $t \in \mathbb{R}_+$ it holds $\mathbb{E}(e^{\lambda N_t}) < +\infty$.*

Proof. The event $\{N_t = n\}$ implies that the r.w. X has performed at least $|n|$ jumps within time t. On the other hand, by definition of κ, the random walk X waits at each $x \in \mathcal{V}$ an exponential time of mean at least $1/\kappa$. Hence (by a coupling argument) $\mathbb{P}(N_t = n) \leq P(\mathcal{Z}_t \geq |n|)$, where $\mathcal{Z}_t$ is a Poisson random variable with intensity κt. Since $E\left(e^{a\mathcal{Z}_t}\right) = e^{\kappa t(e^a - 1)}$, by Chebyshev inequality with $a = \log|n|$ we get

$$\mathbb{P}(N_t = n) \leq P(\mathcal{Z}_t \geq |n|) \leq e^{-|n|\log|n|} E\left(e^{\mathcal{Z}_t \log|n|}\right) = e^{-|n|\log|n| + \kappa t(|n|-1)},$$

thus proving the bound on $\mathbb{P}(N_t = n)$. As a consequence, we obtain

$$\mathbb{E}(e^{\lambda N_t}) \leq 1 + 2\sum_{n=1}^{\infty} e^{\lambda|n| + \kappa t|n| - |n|\log|n|} < +\infty. \qquad \square$$

We now define a new function $F : (V \setminus \{\overline{v}\}) \times \mathbb{R} \times \mathbb{R}_+ \ni (v, \lambda, t) \to F(v, \lambda, t) \in \mathbb{R}_+$ as

$$F(v, \lambda, t) = \sum_{n \in \mathbb{Z}} e^{\lambda n} \mathbb{P}(X_t = v_n) = \mathbb{E}\left(e^{\lambda N_t} \mathbb{1}(X_t = v_n \text{ for some } n \in \mathbb{Z})\right). \tag{58}$$

Recall that, given $v \neq w$ in $V \setminus \{\overline{v}\}$, we have set

$$r(v) := r(v_n), \quad r_-(w, v) := r(w_{n-1}, v_n), \quad r_0(w, v) := r(w, v), \quad r_+(w, v) := r(w_{n+1}, v_n).$$

Lemma 8.2. *Given $\lambda \in \mathbb{R}$ consider the finite matrix $\mathcal{A}(\lambda)$ defined in (11). Consider the vector–valued function $\mathbb{R}_+ \ni t \mapsto F^{(\lambda)}(t) \in \mathbb{R}^{V \setminus \{\overline{v}\}}$ defined as $F^{(\lambda)}(t)_v := F(v, \lambda, t)$. Then $F^{(\lambda)}(\cdot)$ is C^1 in t and*

$$\partial_t F^{(\lambda)}(t) = \mathcal{A}(\lambda) F^{(\lambda)}(t). \tag{59}$$

Proof. Fixed v, λ, we write $F(v, \lambda, \cdot)$ as the function series $F(v, \lambda, t) = \sum_{n \in \mathbb{Z}} f_n(t)$, where $f_n(t) = e^{\lambda n} \mathbb{P}(X_t = v_n)$. By [37][Theorem 2.8.2], the function $\mathbb{R}_+ \ni t \mapsto \mathbb{P}(X_t = v_n) \in [0, 1]$ is differentiable and moreover

$$\begin{aligned}\partial_t \mathbb{P}(X_t = v_n) = -r(v_n)\mathbb{P}(X_t = v_n) + \sum_{w \in V \setminus \{\overline{v}, v\}} \Big[& r(w_{n-1}, v_n)\mathbb{P}(X_t = w_{n-1}) \\ & + r(w_n, v_n)\mathbb{P}(X_t = w_n) + r(w_{n+1}, v_n)\mathbb{P}(X_t = w_{n+1})\Big].\end{aligned} \tag{60}$$

Then, by Lemma 8.1, we conclude that, for $M > 0$ and $n \in \mathbb{Z}$ with $|n| \geq 2$, it holds

$$\begin{aligned}&\|f_n\|_{L^\infty[-M,M]} \leq e^{|\lambda| \cdot |n| + \kappa M - |n|\log|n|}, \\ &\|\partial_t f_n\|_{L^\infty[-M,M]} \leq 4\kappa|V| e^{|\lambda| \cdot |n| + \kappa(|n|+1)M - (|n|-1)\log(|n|-1)},\end{aligned}$$

The space $C^1[-M, M]$ (of functions C^1 on $(-M, M)$, such that they and their first derivates have continuous extensions to $[-M, M]$) is a Banach space endowed with the norm $\|f\| := \|f\|_{L^\infty[-M,M]} + \|\partial_t f\|_{L^\infty[-M,M]}$. We therefore conclude that $F(v, \lambda, t) = \sum_{n \in \mathbb{Z}} f_n(t)$ belongs to $C^1(\mathbb{R})$ and $\partial_t F(v, \lambda, t) = \sum_{n \in \mathbb{Z}} f_n'(t)$, i.e.

$$\begin{aligned}\partial_t F(v, \lambda, t) &= \sum_{n \in \mathbb{Z}} e^{\lambda n} \partial_t \mathbb{P}(X_t = v_n) \\ &= -r(v)F(v, \lambda, t) + \sum_{w \in V \setminus \{\overline{v}, v\}} \left[e^{\lambda} r_-(w, v) + r_0(w, v) + e^{-\lambda} r_+(w, v)\right] F(w, \lambda, t).\end{aligned}$$

This concludes the proof. $\square$

We can now conclude the proof of Theorem 3. Since $|X^*_t - N_t| \leq 1$, it is enough to prove the same LDP for N_t/t. Due to Lemma 8.2 we have $F^{(\lambda)}(t) = e^{(t-1)\mathcal{A}(\lambda)}F^{(\lambda)}(1)$. Since the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected, Definition (58) implies that the vector $F^{(\lambda)}(1)$ has strictly positive entries. In particular we can write

$$\mathbb{E}(e^{\lambda N_t}) = e^{-\kappa(t-1)} \sum_{v,v' \in V\setminus\{\overline{v}\}} \left[e^{[\mathcal{A}(\lambda)+\kappa](t-1)}\right]_{v,v'} F(v', \lambda, 1)\,,$$

where $\kappa := \max\{r(x) \,:\, x \in \mathcal{V}\}$, as above. Note that $\mathcal{A}(\lambda)+\kappa$ is an irreducible matrix with nonnegative entries and therefore, by Perron–Frobenius theorem, it has a simple positive eingevalue $\bar{\gamma}$ and an associated eigenvector with strictly positive entries $(a(v))_{v\in V\setminus\{\overline{v}\}}$, while any other eigenvalue $\bar{\gamma}'$ is such that $|\bar{\gamma}'| \leq \bar{\gamma}$ (in particular, $\mathcal{R}(\bar{\gamma}') < \mathcal{R}(\bar{\gamma})$). The above eigenvalue $\bar{\gamma}$ is the so called Perron–Frobenius eigenvalue and equals the spectral radius of $\mathcal{A}(\lambda) + \kappa$ (note that $\bar{\gamma} = \bar{\gamma}(\lambda)$). Call

$$\begin{cases} C(\lambda) := \max\{F(v, \lambda, 1)/a_v \,:\, v \in V \setminus \{\overline{v}\}\}\,, \\ c(\lambda) \;:= \min\{F(v, \lambda, 1)/a_v \,:\, v \in V \setminus \{\overline{v}\}\}\,. \end{cases}$$

Note that $C(\lambda), c(\lambda)$ are positive constants. Then

$$\mathbb{E}(e^{\lambda N_t}) \leq C(\lambda)e^{-\kappa(t-1)} \sum_{v,v' \in V\setminus\{\overline{v}\}} \left[e^{[\mathcal{A}(\lambda)+\kappa](t-1)}\right]_{v,v'} a_{v'} = e^{(\bar{\gamma}-\kappa)(t-1)}C(\lambda) \sum_{v\in V\setminus\{\overline{v}\}} a_v\,,$$

$$\mathbb{E}(e^{\lambda N_t}) \geq c(\lambda)e^{-\kappa(t-1)} \sum_{v,v' \in V\setminus\{\overline{v}\}} \left[e^{[\mathcal{A}(\lambda)+\kappa](t-1)}\right]_{v,v'} a_{v'} = e^{(\bar{\gamma}-\kappa)(t-1)}c(\lambda) \sum_{v\in V\setminus\{\overline{v}\}} a_v\,.$$

It then follows that the limit $\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}(e^{\lambda N_t})$ exists and equals $\bar{\gamma}(\lambda) - \kappa$, which corresponds to $\Lambda(\lambda)$ by the previous discussion.

By finite–dimensional perturbation theory [34], the Perron–Frobenius eigenvalue $\bar{\gamma} = \bar{\gamma}(\lambda)$ is differentiable in λ, thus implying that $\Lambda(\lambda)$ is differentiable in λ. At this point the thesis follows from Gärtier–Ellis theorem (cf. [22][Lemma V.4 and Theorem V.6]).

9. Proof of Theorem 5 (GC type symmetry for Markov rw's)

We start with a technical result, that is also useful in the applications for the computation of the functions $f_\pm(\lambda)$. Consider a generic stochastic process $(X_t)_{t\in\mathbb{R}_+}$ as in Definition 2.1. Define

$$J_1 := \inf \{t > 0 \,:\, X_t \in \{-1_*, 0_*, 1_*\},\ \exists s \in (0, t) \text{ with } X_s \neq X_0\}\ ,$$

and set

$$\begin{aligned} \tilde{f}_\pm(\lambda) &:= \mathbb{E}_{0_*}(e^{\lambda J_1}\mathbb{1}(X_{J_1} = \pm 1_*))\,, \\ \tilde{f}_0(\lambda) &:= \mathbb{E}_{0_*}(e^{\lambda J_1}\mathbb{1}(X_{J_1} = 0_*))\,. \end{aligned} \tag{61}$$

Lemma 9.1. *If* $\tilde{f}_0(\lambda) < 1$, *then*

$$f_+(\lambda) = \frac{\tilde{f}_+(\lambda)}{1 - \tilde{f}_0(\lambda)}\,, \qquad f_-(\lambda) = \frac{\tilde{f}_-(\lambda)}{1 - \tilde{f}_0(\lambda)}\,.$$

If $\tilde{f}_0(\lambda) \geq 1$, *then* $f_+(\lambda) = f_-(\lambda) = +\infty$.

Proof. We call J_k's the consecutive times at which the stochastic process $\big(X_t\big)_{t\geq 0}$ hits the states of type n_*:

$$\begin{cases} J_0 := 0 \\ J_k := \inf\{t > J_{k-1} \,:\, X_t \in \{-1_*, 0_*, 1_*\}\,,\ \exists s \in (J_{k-1}, t) \text{ with } X_s \neq X_{J_{k-1}}\} \quad k \geq 1\,. \end{cases}$$

We can write

$$S = \sum_{k=0}^{\infty} \mathbb{1}(X_{J_0} = \ldots = X_{J_k} = 0_*, X_{J_{k+1}} \in \{-1_*, 1_*\}) J_{k+1}\,. \tag{62}$$

Taking the exponential at both sides and multiplying by $\mathbb{1}(X_S = 1_*)$ we get

$$e^{\lambda S}\mathbb{1}(X_S = 1_*) = \sum_{k=0}^{\infty} \mathbb{1}(X_{J_0} = \ldots = X_{J_k} = 0_*, X_{J_{k+1}} = 1_*) e^{\lambda J_{k+1}}$$

Note that, by Definition 2.1, w.r.t. the the probability measure $\mathbb{P}_{0_*}(\cdot | X_{J_0} = \ldots = X_{J_k} = 0_*, X_{J_{k+1}} = 1_*)$, the random variables $e^{\lambda(J_i - J_{i-1})}$, $1 \leq i \leq k+1$, are independent with expectation $\mathbb{E}_{0_*}(e^{\lambda J_1} | X_{J_1} = 0_*)$ if $1 \leq i \leq k$ and $\mathbb{E}_{0_*}(e^{\lambda J_1} | X_{J_1} = 1_*)$ if $i = k+1$. Hence,

$$\begin{aligned} f_+(\lambda) &= \mathbb{E}_{0_*}(e^{\lambda S}\mathbb{1}(X_S = 1_*)) \\ &= \sum_{k=0}^{\infty} \mathbb{P}_{0_*}(X_{J_1} = 0_*)^k \mathbb{P}_{0_*}(X_{J_1} = 1_*) \mathbb{E}_{0_*}(e^{\lambda J_1} | X_{J_1} = 0_*)^k \mathbb{E}_{0_*}(e^{\lambda J_1} | X_{J_1} = 1_*) \\ &= \sum_{k=0}^{\infty} \tilde{f}_0(\lambda)^k \tilde{f}_+(\lambda)\,. \end{aligned}.$$

A similar expression holds for $f_-(\lambda)$. At this point it is immediate to derive the thesis. □

Let us now come back to the same context of Section 3.2: $(X_t)_{t\in\mathbb{R}_+}$ is a Markov random walk on the quasi 1d lattice $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with positive rates $r(x, y)$, $(x, y) \in \mathcal{E}$, such that (2) and (12) hold. For the rest of this section, we refer to Markov random walk without state explicitly that they are Markov. Due to (12) in the figures of $G, \mathcal{G}$ we draw only unoriented edges with the convention that for each unoriented edge $\{x, y\}$ the graph in consideration presents both the edge (x, y) and the edge (y, x).

Recall that given an edge $(u, v) \in E$ in the fundamental graph $G = (V, E)$, we have defined (cf. (13)) $r(u, v) = r(\pi(u), \pi(v))$ where π is the map $V \to \mathcal{V}$ such that $\pi(u) = u_0$ if $u \neq \overline{v}$ and $\pi(\overline{v}) = \underline{v}_1 = 1_*$. Given $v \in V$ we set

$$r(v) := \sum_{y : (\pi(v), y) \in \mathcal{E}} r(\pi(v), y)\,. \tag{63}$$

Note that $r(\underline{v}) = r(\overline{v})$. We point out that the map $\pi : V \to \mathcal{V}$ does not induce a graph embedding of G into $\mathcal{G}$. Indeed, problems come from the neighbors of $\underline{v}, \overline{v}$ in G. Consider for example the fundamental graph G in Figure 3. Then x_0, y_{-1}, z_{-1} are neighboring points of $\underline{v}_0$, while x_1, y_0, z_0 are neighboring points of $\underline{v}_1$. Despite this phenomenon, the map π induces an isomorphism between the family of paths $(x_0, x_1, \ldots, x_m)$ in G from $\underline{v}$ to $\overline{v}$ with interior points in $V \setminus \{\underline{v}, \overline{v}\}$ and the family of paths $(x'_0, x'_1, \ldots, x'_m)$ in $\mathcal{G}$ from $\underline{v}_0 = 0_*$ to $\underline{v}_1 = 1_*$ with interior points in $\mathcal{V} \setminus \{0_*, 1_*\}$, moreover it holds $r(x_i, x_{i+1}) = r\big(\pi(x_i), \pi(x_{i+1})\big)$ for $0 \leq i < m$ and $r(x_i) = r(\pi(x_i))$ for $0 \leq i \leq m$. This property will be used below.

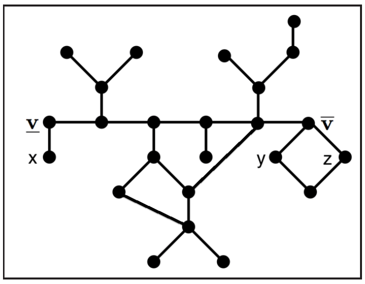

FIGURE 3. Example of fundamental graph G

By Theorem 8 the Gallavotti–Cohen type symmetry (14) is satisfied for some constant Δ if and only if $\varphi_+(\lambda)/\varphi_-(\lambda) = e^{\Delta}$ for all $\lambda \leq \lambda_c$. On the other hand, by (22) and the above Lemma 9.1, it holds

$$\frac{\varphi_+(\lambda)}{\varphi_-(\lambda)} = \frac{f_+(\lambda)}{f_-(\lambda)} = \frac{\tilde{f}_+(\lambda)}{\tilde{f}_-(\lambda)}, \qquad \forall \lambda \leq \lambda_c . \tag{64}$$

Given an integer $m \geq 1$, let $\mathcal{A}_m$ be the family of sequences $(x_0, x_1, \dots, x_m)$ such that $x_0 = \underline{v}$, $x_m = \overline{v}$, $(x_i, x_{i+1}) \in E$ for all $i : 0 \leq i < m$ and $x_i \in V \setminus \{\underline{v}, \overline{v}\}$ for all $0 < i < m$. We call $\mathcal{A}^*_m$ the family of sequences satisfying the same properties as above when exchanging the role of $\underline{v}$ and $\overline{v}$. Then we can write

$$\begin{aligned} \tilde{f}_+(\lambda) &= \sum_{m=1}^{\infty} \sum_{(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} \int_{\mathbb{R}_+^{m-1}} dt_1 dt_2 \dots dt_{m-1} e^{\sum_{i=0}^{m-1}(\lambda - r(x_i))t_i} \prod_{i=0}^{m-1} r(x_i, x_{i+1}) \\ &= \begin{cases} \sum_{m=1}^{\infty} \sum_{(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} \prod_{i=0}^{m-1} r(x_i,x_{i+1}) \prod_{i=0}^{m-1} \frac{1}{r(x_i)-\lambda} & \text{if } \lambda < \min_{x\in V} r(x) \\ +\infty & \text{otherwise} . \end{cases} \end{aligned} \tag{65}$$

Given $\gamma = (x_0, x_1, \dots, x_m)$ and given $e \in E$ we write $N_e(\gamma)$ for the number of indices $i : 0 \leq i \leq m-1$ such that $(x_i, x_{i+1}) = e$. Then the above formula can be rewritten as

$$\tilde{f}_+(\lambda) = \begin{cases} \sum_{m=1}^{\infty} \sum_{\gamma=(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} \prod_{e\in E} r(e)^{N_e(\gamma)} \prod_{i=0}^{m-1} \frac{1}{r(x_i)-\lambda} & \text{if } \lambda < \min_{x\in V} r(x) , \\ +\infty & \text{otherwise} . \end{cases} \tag{66}$$

A similar formula holds for $\tilde{f}_-(\lambda)$.

Suppose now that G is $(\underline{v}, \overline{v})$–minimal. We want to prove that (14) is satisfied with Δ given by (15). Call $G_1, G_2, \dots, G_k$ the subgraphs attached to the path $\gamma_* = (z_0, z_1, \dots, z_n)$ as in Def. 3.2, such that each G_a has exactly one point in common with $\{z_1, z_2, \dots, z_{n-1}\}$ (recall that $z_0 = \underline{v}$ and $z_n = \overline{v}$). Then given $(x_0, x_1, \dots, x_m) \in \mathcal{A}_m$ there exist indices

$$0 \leq i_1 < j_1 < i_2 < j_2 < \dots < i_{r-1} < j_{r-1} < i_r < j_r \leq m$$

such that for any $k : 1 \leq k \leq r$ the subpath (called *excursion*)

$$(x_{i_k}, x_{i_k+1}, \dots, x_{j_k})$$

satisfies: (i) $x_{i_k} = x_{j_k}$ and such a point belongs to γ, (ii) the points $x_{i_k+1}, \dots, x_{j_k-1}$ are in $G_a \setminus \{z_0, z_1, \dots, z_n\}$ for some $a : 1 \leq a \leq k$. When $r = 0$ then there is no excursion and the path $(x_0, x_1, \dots, x_m)$ has support in $\{z_0, z_1, \dots, z_n\}$. Call $(x_0, x_1, \dots, x_m)^*$ the new path obtained by inverting $(x_0, x_1, \dots, x_m)$ with the exception that the excursions inside are performed in their original orientation:

$$\begin{aligned}(x_0, x_1, \dots, x_m)^* := (&x_m, x_{m-1}, \dots, x_{j_r+1}, \underline{x_{i_r}, x_{i_r+1}, x_{i_r+2}, \dots, x_{j_r}}, x_{i_r-1}, x_{i_r-2}, \dots, \\ &x_{j_{r-1}+1}, \underline{x_{i_{r-1}}, x_{i_{r-1}+1}, \dots x_{j_{r-1}-1}, x_{j_{r-1}}}, x_{i_{r-1}-1}, \dots\dots, x_{j_1+1}, \\ &\underline{x_{i_1}, x_{i_1+1}, \dots, x_{j_1-1}, x_{j_1}}, x_{i_1-1}, x_{i_1-2}, \dots, x_2, x_1, x_0)\,.\end{aligned}$$

Above we have underlined the excursions (note they appear in their original orientation). We point out that the map $\mathcal{A}_m \ni (x_0, x_1, \dots, x_m) \to (x_0, x_1, \dots, x_m)^* \in \mathcal{A}_m^*$ is a bijection. Hence it holds

$$\tilde f_-(\lambda) = \begin{cases} \sum_{m=1}^\infty \sum_{\gamma=(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} \prod_{e\in E} r(e)^{N_e(\gamma^*)} \prod_{i=1}^m \frac{1}{r(x_i)-\lambda} & \text{if } \lambda < \min_{x\in V} r(x) \\ +\infty & \text{otherwise}\,. \end{cases} \tag{67}$$

Note that $\prod_{i=0}^{m-1} \frac{1}{r(x_i)-\lambda} = \prod_{i=1}^m \frac{1}{r(x_i)-\lambda}$ since $r(\underline{v}) = r(\overline{v})$. By construction $N_e(\gamma) = N_e(\gamma^*)$ if e is not of the form $(z_i, z_{i\pm1})$. On the other hand,

$$\begin{aligned} &\frac{\prod_{i=0}^{n-1} r(z_i, z_{i+1})^{N_{(z_i,z_{i+1})}(\gamma)} \prod_{i=0}^{n-1} r(z_{i+1}, z_i)^{N_{(z_{i+1},z_i)}(\gamma)}}{\prod_{i=0}^{n-1} r(z_i, z_{i+1})^{N_{(z_i,z_{i+1})}(\gamma_*)} \prod_{i=0}^{n-1} r(z_{i+1}, z_i)^{N_{(z_{i+1},z_i)}(\gamma_*)}} = \\ &\frac{\prod_{i=0}^{n-1} r(z_i, z_{i+1})^{N_{(z_i,z_{i+1})}(\gamma)} \prod_{i=0}^{n-1} r(z_{i+1}, z_i)^{N_{(z_{i+1},z_i)}(\gamma)}}{\prod_{i=0}^{n-1} r(z_i, z_{i+1})^{N_{(z_{i+1},z_i)}(\gamma)} \prod_{i=0}^{n-1} r(z_{i+1}, z_i)^{N_{(z_i,z_{i+1})}(\gamma)}} = \\ &\prod_{i=0}^{n-1} \left(\frac{r(z_i, z_{i+1})}{r(z_{i+1}, z_i)}\right)^{N_{(z_i,z_{i+1})}(\gamma) - N_{(z_{i+1},z_i)}(\gamma)} = \prod_{i=0}^{n-1} \frac{r(z_i, z_{i+1})}{r(z_{i+1}, z_i)}\,, \end{aligned} \tag{68}$$

since it must be $N_{(z_i,z_{i+1})}(\gamma) - N_{(z_{i+1},z_i)}(\gamma) = 1$ for any path $\gamma \in \mathcal{A}_m$ for some $m \geq 1$. Due to (66), (67) and the previous observations, we get that $\tilde f_+(\lambda)/\tilde f_-(\lambda) = e^\Delta$ with Δ given in (15). Due to (64) and Theorem 8 we get (14).

We now prove the reverse implication. Consider the oriented subgraph $\hat G = (\hat V, \hat E)$ consisting of the points in V and edges in E that appear in some path γ as γ varies in $\mathcal{A}_m$ and m varies in $\{1, 2, \dots\}$.

Proposition 9.2. *Suppose that the fundamental graph G is not $(\underline{v}, \overline{v})$–minimal. Fix $\big(r(e) : e \in E \setminus \hat E\big) \in (0, +\infty)^{E\setminus\hat E}$. Call $\mathcal{R} \subset (0, +\infty)^{\hat E}$ the family of vectors $\big(r(e) : e \in \hat E\big) \in (0, +\infty)^{\hat E}$ for which the random walk on $\mathcal{G}$ induced by $(r(e) : e \in E)$ satisfies the Gallavotti–Cohen type symmetry* (14) *for some constant Δ, depending on $(r(e) : e \in E)$. Then $\mathcal{R}$ has zero Lebesgue measure in $(0, +\infty)^{\hat E}$.*

Since $\hat E \neq \emptyset$ and by Fubini theorem, this would conclude the proof of Theorem 5. The proof is in part based on complex analysis.

9.1. **Proof of Proposition 9.2.** From now on $r(e)$, $e \in E\setminus\hat E$, are fixed positive constants. We first prove some preliminary results.

Lemma 9.3. *Define the open subset $\Omega \subset (-\infty, 0) \times (0, +\infty)^{\hat E}$ as the family of vectors $(\lambda, (r(e))_{e\in\hat E})$ with $r(e) > 0$ $\forall e \in \hat E$ and $-\lambda > 3\max_{v\in\hat V} r(v) + 1$, where $r(v)$ is the value defined in* (63) *for the random walk on $\mathcal{G}$ induced by $\big(r(e) : e \in E\big)$.*

Consider the positive function $h_\pm\Big(\lambda, \big(r(e)\big)_{e\in\hat E}\Big)$ defined on Ω as the function $\tilde f_\pm(\lambda)$ for the random walk on G induced by $(r(e) : e \in E)$. Then there exists an holomorphic *function* $h^*_\pm : \Omega_* \to \mathbb{C}$ *defined on an open subset* $\Omega_* \subset \mathbb{C}\times\mathbb{C}^{\hat E}$ *such that* $\Omega = \Omega_* \cap (\mathbb{R}\times\mathbb{R}^{\hat E})$ *and* $h_\pm$ *is the restriction to* Ω *of the function* $h^*_\pm$.

Proof. In what follows, to simplify the notation, we write $\underline{r}$ instead of $\big(r(e)\,:\,e\in\hat E\big)$. In general $\underline{r}$ will be an element of $\mathbb{C}^{\hat E}$. Given $v\in\hat V$ we define the map $\phi_v:\mathbb{C}^{\hat E}\to\mathbb{C}$ as

$$\phi_v(\bar r) := \begin{cases} \sum_{(v,y)\in\hat E} r(v,y) & \text{if } v\in\hat V\setminus\{\underline{v},\overline{v}\}\,,\\ \sum_{(v,y)\in\hat E} r(v,y) + \sum_{(\underline{v},y)\in E\setminus\hat E} r(\underline{v},y) + \sum_{(\overline{v},y)\in E\setminus\hat E} r(\overline{v},y) & \text{if } v=\underline{v},\overline{v}\,. \end{cases} \tag{69}$$

Recall that $\sum_{(\underline{v},y)\in E\setminus\hat E} r(\underline{v},y)$ and $\sum_{(\overline{v},y)\in E\setminus\hat E} r(\overline{v},y)$ are fixed positive constants since the values $r(e)$, $e\in E\setminus\hat E$, have been fixed once for all. Moreover note that for $\underline{r}\in(0,+\infty)^{\hat E}$ it holds $\phi_v(\underline{r}) = r(v)$, where $r(v)$ is the value defined in (63) for the random walk on $\mathcal{G}$ induced by $\big(r(e)\,:\,e\in E\big)$.

Given $\underline{r}\in\mathbb{C}^{\hat E}$ we define $\Re(\underline{r})\in\mathbb{R}^{\hat E}$ as the vector whose entries are the real part of the entries of $\underline{r}$, i.e.

$$\Re\left(\underline{r}\right)(e) := \Re\left(r(e)\right)\,, \qquad e\in\hat E\,.$$

We define $\Omega_*\subset\mathbb{C}\times\mathbb{C}^{\hat E}$ as the set of vectors $(\lambda,\underline{r})$ satisfying the following properties:

(i) $\Re(\underline{r})\in(0,+\infty)^{\hat E}$,
(ii) $|r(e)|\le 2\Re\left(r(e)\right)$ $\forall e\in\hat E$,
(iii) $-\Re(\lambda) > 3\phi_v\left(\Re\left(\underline{r}\right)\right)+1$ for all $v\in\hat V$.

Note that $\Omega_*\cap(\mathbb{R}\times\mathbb{R}^{\hat E}) = \Omega$.

For each $\gamma=(x_0,x_1,\dots,x_m)\in\mathcal{A}_m$, $m\ge 1$, we consider the holomorphic function (cf. [21]) $g_\gamma:\Omega_*\to\mathbb{C}$ defined as

$$g_\gamma(\lambda,\underline{r}) := \prod_{e\in\hat E} r(e)^{N_e(\gamma)} \prod_{i=0}^{m-1} \frac{1}{\phi_{x_i}(\underline{r})-\lambda}\,. \tag{70}$$

Recall that $N_e(\gamma)$ counts the number of times the edge e appears along the path γ and that $\phi_{x_i}(\underline{r})$ is an affine function of $\underline{r}$.

Fix $(\lambda^*,\underline{r}^*)\in\Omega_*$. Consider the open subset $U(\lambda^*,\underline{r}^*)\subset\Omega_*$ given by the vectors $(\lambda,\underline{r})\in\Omega_*$ such that $-\Re(\lambda^*)/\sqrt{2} < -\Re(\lambda) < -\sqrt{2}\Re(\lambda^*)$ and $\Re\left(\underline{r}^*\right)/\sqrt{2} < \Re\left(\underline{r}\right) < \sqrt{2}\Re\left(\underline{r}^*\right)$. Trivially, $(\lambda^*,\underline{r}^*)\in U(\lambda^*,\underline{r}^*)$.

If $\gamma\in\mathcal{A}_m$ and $(\lambda,\underline{r})\in U(\lambda^*,\underline{r}^*)$ we can bound

$$\begin{aligned} |g_\gamma(\lambda,\underline{r})| = \prod_{e\in\hat E}|r(e)|^{N_e(\gamma)}\prod_{i=0}^{m-1}\frac{1}{|\phi_{x_i}(\underline{r})-\lambda|} &\le 2^m\prod_{e\in\hat E}\Re\left(r(e)\right)^{N_e(\gamma)}\prod_{i=1}^{m}\frac{1}{\phi_{x_i}\left(\Re(\underline{r})\right)-\Re(\lambda)}\\ &\le 4^m\prod_{e\in\hat E}\Re\left(r^*(e)\right)^{N_e(\gamma)}\prod_{i=0}^{m-1}\frac{1}{\phi_{x_i}\left(\Re(\underline{r}^*)\right)-\Re(\lambda^*)}\\ &= \prod_{e\in\hat E}\Re\left(4r^*(e)\right)^{N_e(\gamma)}\prod_{i=0}^{m-1}\frac{1}{\phi_{x_i}\left(\Re(\underline{r}^*)\right)-\Re(\lambda^*)}\\ &\le \prod_{e\in\hat E}\Re\left(4r^*(e)\right)^{N_e(\gamma)}\prod_{i=0}^{m-1}\frac{1}{\phi_{x_i}\left(\Re(4\underline{r}^*)\right)+1}\,. \end{aligned} \tag{71}$$

Indeed the first bound follows from Assumptions (i) and (ii) in the definition of Ω_*, the second bound follows from the definition of $U(\lambda^*, \underline{r}^*)$, the last identity follows from the fact that all edges of γ are in $\hat{E}$, while the last bound follows from Assumption (iii) in the definition of Ω_* since we can bound

$$\phi_{x_i}\left(\Re(\underline{r}^*)\right) - \Re(\lambda^*) \geq 4\phi_{x_i}\left(\Re(\underline{r}^*)\right) + 1 \geq \phi_{x_i}\left(\Re(4\underline{r}^*)\right) + 1 \,. \tag{72}$$

We are now interested to the infinite series of holomorphic functions

$$\sum_{m=1}^{\infty} \sum_{\gamma=(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} g_\gamma(\lambda, \underline{r}) \,. \tag{73}$$

By (71) for any $(\lambda, \underline{r}) \in U(\lambda^*, \underline{r}^*)$ we have

$$\sum_{m=1}^{\infty} \sum_{\gamma=(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} \left|g_\gamma(\lambda, \underline{r})\right|$$

$$\leq \sum_{m=1}^{\infty} \sum_{\gamma=(x_0,x_1,\dots,x_m)\in\mathcal{A}_m} \prod_{e\in\hat{E}} \Re\left(4r^*(e)\right)^{N_e(\gamma)} \prod_{i=0}^{m-1} \frac{1}{\phi_{x_i}\left(\Re(4\underline{r}^*)\right) + 1} \,.$$

Comparing with (66), the above r.h.s. equals the function $\tilde{f}_+^*\left(\Re(\lambda^*)\right)$ with $\tilde{f}_+^*$ defined as the function $\tilde{f}_+$ referred to the random walk on $\mathcal{G}$ induced by weights

$$E \ni e \to \begin{cases} 4r^*(e) & \text{if } e \in \hat{E}\,, \\ r(e) & \text{if } e \in E \setminus \hat{E}\,. \end{cases}$$

Since $\Re(\lambda^*) < 0$ the value $\tilde{f}_+^*\left(\Re(\lambda^*)\right)$ is finite by definition of $\tilde{f}_+^*$.

Since each compact subset of Ω_* can be covered by the union of a finite family of sets of the form $U(\lambda^*, \underline{r}^*)$ we conclude that series (73) converges uniformly on compact subsets of Ω_*. By a classical theorem in complex analysis (see e.g. [35][Ch. I, Prop.2] or [21][Ch. I, Lemma 11]), we conclude that the limiting function h_+^* is holomorphic. Since by (66) the function h_+ in the main statement equals the series (73) on $\Omega = \Omega_* \cap (\mathbb{R} \times \mathbb{R}^{\hat{E}})$ we conclude that h_+ is the restriction of h_+^* on Ω. By similar arguments, h_- is the restriction of h_-^* on Ω, h_-^* being an holomorphic function on Ω_* whose definition is analogous to h_+^*. □

Since $h_- > 0$ on Ω, there exists an open subset $\Omega_{**} \subset \mathbb{C} \times \mathbb{C}^{\hat{E}}$ with $\Omega \subset \Omega_{**} \subset \Omega_*$ and such that $h_-^* \neq 0$ on Ω_{**}. At cost to restrict Ω_{**} we can assume that

$$\{\lambda \in \mathbb{C} \,:\, (\lambda, \underline{r}) \in \Omega_{**}\} \tag{74}$$

is connected for any fixed $\underline{r} \in (0, +\infty)^{\hat{E}}$.

Remark 9.4. *By definition of Ω, given $\underline{r} \in (0, +\infty)^{\hat{E}}$, it holds $(\lambda, \underline{r}) \in \Omega_{**}$ if λ is real and $-\lambda > 3\max_{v\in\hat{V}} r(v) + 1$.*

The function h_+^*/h_-^* is well defined and holomorphic on Ω_{**}. As a consequence, also the derivative $h := \partial_\lambda(h_+^*/h_-^*)$ is holomorphic (cf. [7][Sec. IV.2.2]). Note that, due to (64) and Theorem 8, the function $\frac{h_+^*}{h_-^*}(\lambda, \underline{r})$ restricted to Ω does not depend on λ if $\underline{r} \in \mathcal{R}$, the set defined in Proposition 9.2. In particular, $h(\lambda, \underline{r}) = 0$ if $(\lambda, \underline{r}) \in \Omega$ and $\underline{r} \in \mathcal{R}$. Consider the holomorphic function $\lambda \to h(\lambda, \underline{r})$, where $\underline{r} \in \mathcal{R}$ is fixed. This function is defined on the set $\{\lambda \in \mathbb{C} : (\lambda, \underline{r}) \in \Omega_{**}\}$. Since it has no isolated zeros and since (74) is connected, we get that $h(\lambda, \underline{r}) = 0$ for any $\underline{r} \in \mathcal{R}$ and any $\lambda \in \mathbb{C} : (\lambda, \underline{r}) \in \Omega_{**}$ (see [7]).

Suppose now, by contradiction, that the set $\mathcal{R}$ has positive Lebesgue measure (here and in what follows we refer to the $|\hat{E}|$–dimensional Lebesgue measure). Fix $\lambda < 0$ and define $\Omega_\lambda := \left\{\underline{r} \in \mathbb{R}^{\hat{E}} : (\lambda, \underline{r}) \in \Omega\right\}$ and the function $h_\lambda : \Omega_\lambda \to \mathbb{R}$ as $h_\lambda(\underline{r}) := h(\lambda, \underline{r})$. Note that Ω_λ is connected and that h_λ is a real analytic function (locally it admits a convergent power series expansion, since restriction of an holomorphic function). Since $\Omega_\lambda \subset \Omega_{\lambda'}$ if $\lambda' < \lambda$ and since $\cup_{\lambda<0}\Omega_\lambda = (0, +\infty)^{\hat{E}}$, we can find $\lambda_0 < 0$ such that $\Omega_\lambda \cap \mathcal{R}$ has positive Lebesgue measure for $\lambda \leq \lambda_0$. From now on we assume $\lambda \leq \lambda_0$. This implies that the set $\{h_\lambda = 0\}$ has positive Lebesgue measure. We claim that it must then be $h_\lambda \equiv 0$ on the entire connected set Ω_λ as a consequence of Weierstrass Preparation Theorem. Indeed, h_λ is the restriction to Ω_λ of the holomorphic function $h(\lambda, \cdot)$ defined on an open subset of $\mathbb{C}^{\hat{E}}$ containing Ω_λ. Then the thesis follows from this general fact:

Lemma 9.5. *Fix $n \geq 1$ integer. Let V be an open set of $\mathbb{C}^n$ such that $U := V \cap \mathbb{R}^n$ is connected. Let $f : V \to \mathbb{C}$ be an holomorphic function. Then either $f \equiv 0$ on U or the set $\{z \in U : f(z) = 0\}$ has zero n–dimensional Lebesgue measure.*

Proof. Note that U is open. Below Lebesgue measure is considered as n–dimensional. It is enough to prove the following claim:

Claim 9.6. *For any $z \in U$ there is a neighborhood B_z of z in U such that the set $\{y \in B_z : f(y) = 0\}$ has nonempty open part or has zero Lebesgue measure.*

Let us first assume the above claim and show how to conclude. If for all $z \in U$ the set $\{y \in B_z : f(y) = 0\}$ has zero Lebesgue measure, then each compact subset $K \subset U$ can be covered by a finite family B_{z_1}, B_{z_2},...,B_{z_r}, thus implying that $\{y \in K : f(y) = 0\}$ has zero Lebesgue measure. This trivially leads to the fact that $\{z \in U : f(y) = 0\}$ has zero Lebesgue measure. On the other hand, if for some $z \in U$ the set $\{y \in B_z : f(z) = 0\}$ has nonempty open part, then the analytic function given by f restricted to U is zero on a ball inside U and therefore is zero on all U (see [7][Ch. IV.2.3]).

At this point we only need to prove the above Claim 9.6. If $f(z) \neq 0$ then for B_z small the set $\{y \in B_z : f(z) = 0\}$ is empty and we are done. Suppose that $f(z) = 0$ and f not identically zero around z. By Weierstrass preparation theorem [21][Ch. II.B], there exists $\varepsilon > 0$ such that for all $y = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n$ with $|y_i - z_i| < \varepsilon$ for all i it holds

$$\begin{aligned} f(y) = h(y)\Big[&(y_n - z_n)^k + a_1(y_1, \dots, y_{n-1})(y_n - z_n)^{k-1} \\ &+ \dots + a_{k-1}(y_1, \dots, y_{n-1})(y_n - z_n) + a_k(y_1, \dots, y_{n-1})\Big], \end{aligned} \tag{75}$$

where $y = (y_1, \dots, y_n)$, k is a suitable integer, $a_1, \dots, a_k$ are holomorphic functions, and h is a never–zero holomorphic function. It then follows that, fixed $(y_1, \dots, y_{n-1})$ with $|y_i - z_i| < \varepsilon$, the set $\{y_n \in \mathbb{C} : |y_n - z_n| < \varepsilon\,,\ f(y_1, \dots, y_{n-1}, y_n) = 0\}$ has cardinality at most k (in particular, it has zero Lebesgue measure when intersected with $\mathbb{R}$). The thesis follows by taking $B_z := \{y \in \mathbb{R}^n : |y_i - z_i| < \varepsilon\}$ and applying Fubini theorem. □

Up to now, assuming that $\mathcal{R}$ has positive Lebesgue measure, we have proved that for each $\underline{r} \in (0, +\infty)^{\hat{E}}$ it holds $h(\lambda, \underline{r}) = 0$ for $\lambda < 0$ and $|\lambda|$ large enough: $\lambda < \lambda_0$ and $-\lambda > 3\max_{v\in\hat{V}} r(v) + 1$ (see Remark 9.4). In particular, it is simple to define an increasing function $\varphi : (0, +\infty) \to (0, +\infty)$ such that $h(\lambda, \underline{r}) = 0$ for all $\underline{r} \in (0, +\infty)^{\hat{E}}$ and $\lambda < -\varphi\left(\max_{e\in\hat{E}} r(e)\right)$. In particular we have proved the following fact:

Fact 9.7. *For each fixed* $\underline{r} \in (0,+\infty)^{\hat{E}}$*, the ratio* $h_+(\lambda,\underline{r})/h_-(\lambda,\underline{r})$ *is constant for* $\lambda < -\varphi\left(\max_{e\in\hat{E}} r(e)\right)$.

We now show that this is in contradiction with the assumption that the fundamental graph G is not $(\underline{v},\overline{v})$–minimal. Indeed, since G is not $(\underline{v},\overline{v})$–minimal, there exist at least two paths $\gamma^{(1)} = (z_0, z_1, \dots, z_M)$ and $\gamma^{(2)} = (z'_0, z'_1, \dots, z'_{M'})$ in $\mathcal{A}_M$ and $\mathcal{A}_{M'}$ respectively, such that the points z_i are all distinct, the points z'_i are all distinct, and for some non–negative integers κ_1, κ_2 with $\kappa_1 + \kappa_2 + 2 \leq M \wedge M'$ it holds

$$\begin{aligned}
& z_i = z'_i && \forall\, 0 \leq i \leq \kappa_1\,,\\
& z_{M-i} = z'_{M'-i} && \forall\, 0 \leq i \leq \kappa_2\,,\\
& \{z_{\kappa_1+1},\dots,z_{M-\kappa_2-1}\} \cap \{z'_0, z'_1, \dots, z'_{M'}\} = \emptyset\,,\\
& \{z_0, z_1, \dots, z_M\} \cap \{z'_{\kappa_1+1},\dots,z'_{M'-\kappa_2-1}\} = \emptyset\,.
\end{aligned}$$

In other words, $\gamma^{(1)}$ and $\gamma^{(2)}$ are linear chains, they have in common the first κ_1+1 points and the last κ_2+1 points, while they divide in their interior part.

Let $E_\star := \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 := \{(z_i, z_{i+1}), (z_{i+1}, z_i) : 0 \leq i < M\}\,, \qquad \Gamma_2 := \{(z'_j, z'_{j+1}), (z'_{j+1}, z'_j) : 0 \leq j < M'\}\,.$$

Note that $E_\star \subset \hat{E}$. Introduce a new connected fundamental graph $G' = (V', E')$, where $E' = (E \setminus \hat{E}) \cup E_\star$ and V' is given by the vertices appearing in the edges of E'. As marked vertices we take again $\underline{v}, \overline{v}$.

Let $\bar{r} = \big(r(e) \,:\, e \in E_\star\big) \in (0,+\infty)^{E_\star}$ and for $k \geq 1$ let $\underline{r}^{(k)} \in (0,+\infty)^{\hat{E}}$ be defined as $\bar{r}$ on $E_\star$ and as $1/k$ on $\hat{E} \setminus E_\star$. Then

$$\lim_{k\to\infty} h_\pm\left(\lambda, \underline{r}^{(k)}\right) = \tilde{f}^*_\pm(\lambda)\,, \qquad \lambda \leq -\varphi\Big(\max_{e\in E_\star} r(e)\Big)\,, \tag{76}$$

where $\tilde{f}^*_\pm$ refers to the rw on the quasi 1d lattice induced by $(G', \underline{v}, \overline{v})$ and by the weights $r(e)$ with $e \in E \setminus \hat{E}$ (that have been fixed once and for all) and the weights $r(e)$ with $e \in E_\star$. The limit (76) follows from the fact that, as $k \to \infty$, the probability to have a jump along an edge not in E' goes to zero (use the graphical construction for Markov chains)

Due to (76) and Fact 9.7 we have that the ratio $\tilde{f}^*_+(\lambda)/\tilde{f}^*_-(\lambda)$ is constant for $\lambda < 0$ with $|\lambda|$ large. At this point, to have a contradiction it is enough to prove that for a suitable choice of $\bar{r}$ the above assertion is impossible. Let $\mathcal{A}'_m$ be the analogous of $\mathcal{A}_m$ referred now to the graph $G' = (V', E')$. Then for each path γ in $\mathcal{A}'_m$ for some m going from $\underline{v}$ to $\overline{v}$, it holds either

$$\begin{aligned}
& N_{(z_i,z_{i+1})}(\gamma) - N_{(z_{i+1},z_i)}(\gamma) = 1 \quad \text{for all } i : 0 \leq i < m\,,\\
& N_{(x,y)}(\gamma) - N_{(y,x)}(\gamma) = 0 \quad \text{for all } (x,y) \in \Gamma_2 \setminus \Gamma_1\,,
\end{aligned} \tag{77}$$

or

$$\begin{aligned}
& N_{(z'_i,z'_{i+1})}(\gamma) - N_{(z'_{i+1},z'_i)}(\gamma) = 1 \quad \text{for all } i : 0 \leq i < m'\,,\\
& N_{(x,y)}(\gamma) - N_{(y,x)}(\gamma) = 0 \quad \text{for all } (x,y) \in \Gamma_1 \setminus \Gamma_2\,,
\end{aligned} \tag{78}$$

So we can define the two disjoint sets

$$\begin{aligned}
& \mathcal{P}_1 := \{ \text{ paths in } \cup_{m\geq 1} \mathcal{A}'_m \text{ such that (77) holds}\}\\
& \mathcal{P}_2 := \{ \text{ paths in } \cup_{m\geq 1} \mathcal{A}'_m \text{ such that (78) holds}\}
\end{aligned}$$

Inverting the role of $\underline{v}, \overline{v}$ and considering paths from $\overline{v}$ to $\underline{v}$, one can define $(\mathcal{A}'_m)^*, \mathcal{P}_1^*, \mathcal{P}_2^*$ analogous of $\mathcal{A}'_m$, $\mathcal{P}_1$, $\mathcal{P}_2$, respectively. For example, $\mathcal{P}_1^*$ is given by the paths γ in $\cup_{m\geq 1}(\mathcal{A}'_m)^*$ such that

$$\begin{aligned} &N_{(z_{i+1},z_i)}(\gamma) - N_{(z_i,z_{i+1})}(\gamma) = 1 \quad \text{for all } i : 0 \leq i < m\,, \\ &N_{(x,y)}(\gamma) - N_{(y,x)}(\gamma) = 0 \quad \text{for all } (x,y) \in \Gamma_2 \setminus \Gamma_1\,, \end{aligned} \tag{79}$$

Given a path $\gamma = (x_0, x_1, \dots, x_m)$ we define the reversed path $\gamma^* = (x_m, x_1, \dots, x_0)$. Note that if $\gamma \in \mathcal{P}_i$ then $\gamma^* \in \mathcal{P}_i^*$. Using formulas similar to (66) referred now to G' we have

$$\frac{\tilde f_+(\lambda)}{\tilde f_-(\lambda)} = \frac{\tilde f_{1,+}(\lambda) + \tilde f_{2,+}(\lambda)}{\tilde f_{1,-}(\lambda) + \tilde f_{2,-}(\lambda)} \tag{80}$$

where, for $s = 1, 2$,

$$\begin{aligned} \tilde f_{s,+}(\lambda) &= \sum_{m=1}^{\infty} \sum_{\gamma \in \mathcal{P}_s \cap \mathcal{A}'_m} \prod_{e \in E_\star} r(e)^{N_e(\gamma)} \prod_{i=0}^{m-1} \frac{1}{r(x_i) - \lambda}\,, \\ \tilde f_{s,-}(\lambda) &= \sum_{m=1}^{\infty} \sum_{\gamma \in \mathcal{P}^*_s \cap (\mathcal{A}'_m)^*} \prod_{e \in E_\star} r(e)^{N_e(\gamma)} \prod_{i=0}^{m-1} \frac{1}{r(x_i) - \lambda}\,. \end{aligned}$$

Note that $r(x)$ is now referred to the fundamental graph G' with weights $r(e)$ with $e \in E\setminus\hat E$ (that have been fixed once and for all) and the weights $r(e)$ with $e \in E_\star$. Simply forget G.

Due to (77) and (79) and similar formulas, for $s = 1, 2$ we get

$$\tilde f_{s,+}(\lambda) = \tilde f_{s,-}(\lambda) \cdot \Delta_s\,, \qquad \Delta_1 := \prod_{i=0}^{M-1} \frac{r(z_i, z_{i+1})}{r(z_{i+1}, z_i)}\,, \qquad \Delta_2 := \prod_{i=0}^{M'-1} \frac{r(z'_i, z'_{i+1})}{r(z'_{i+1}, z'_i)}\,. \tag{81}$$

Combining (80) and (81) we have

$$\frac{\tilde f_+(\lambda)}{\tilde f_-(\lambda)} = \frac{\tilde f_{1,+}(\lambda) + \tilde f_{2,+}(\lambda)}{\Delta_1^{-1}\tilde f_{1,+}(\lambda) + \Delta_2^{-1}\tilde f_{2,+}(\lambda)}\,. \tag{82}$$

If $\Delta_1 = \Delta_2$ then the ratio $\tilde f_+(\lambda)/\tilde f_-(\lambda)$ is independent of λ, but this happens for a set of rates of Lebesgue measure 0. Assume now $\Delta_1 \neq \Delta_2$. Then dividing by $\tilde f_{1,+}(\lambda)$ we conclude that the l.h.s. of (82) does not depend on λ for $\lambda < 0$ with $|\lambda|$ large if and only if the same holds for the ratio $\tilde f_{2,+}(\lambda)/\tilde f_{1,+}(\lambda)$.

We point out that each path in $\mathcal{P}_1$ belongs to $\mathcal{A}'_m$ for some $m \geq M$. The only path in $\mathcal{P}_1$ belonging to $\mathcal{A}'_M$ is $\gamma^{(1)}$, while there is no path in $\mathcal{P}_1$ belonging to $\mathcal{A}'_{M+1}$. Moreover, $|\mathcal{A}'_m| \leq 3^m$ since, when constructing a path $\gamma \in \mathcal{A}'_m$ vertex by vertex, at each step we can choose only among 2 or 3 neighbors. Hence

$$\Big| \sum_{m=M+2}^{\infty} \sum_{\gamma \in \mathcal{P}_1 \cap \mathcal{A}'_m} \prod_{e \in E_\star} r(e)^{N_e(\gamma)} \prod_{i=0}^{m-1} \frac{1}{r(x_i) - \lambda} \Big| \leq \sum_{m=M+2}^{\infty} \frac{c^m}{|\lambda|^m} = \frac{(c/|\lambda|)^{M+2}}{1 - c/|\lambda|}\,, \tag{83}$$

where $c := 3\max\{r(e) : e \in E_*\}$. In particular, separating the contribution of $\gamma^{(1)}$ from the other paths in the definition of $\tilde f_{1,+}(\lambda)$, we have that

$$\tilde f_{1,+}(\lambda) = c_1 \prod_{i=0}^{M-1} \frac{1}{r(z_i) - \lambda} + O\left(\frac{1}{|\lambda|^{M+2}}\right)\,, \qquad c_1 := \prod_{i=0}^{M-1} r(z_i, z_{i+1})$$

Above $O\left(\frac{1}{|\lambda|^{M+2}}\right)$ means that the term in consideration is bounded in modulus by $C/|\lambda|^{M+2}$. Note that for $\lambda < 0$ with $|\lambda|$ large we have

$$\frac{1}{r(z_i)-\lambda} = \frac{1}{|\lambda|}\frac{1}{1+r(z_i)/|\lambda|} = \frac{1}{|\lambda|}\left(1-\frac{r(z_i)}{|\lambda|}+\mathcal{E}_i(\lambda)\right)$$

where $\lim_{\lambda\to-\infty}|\lambda|\mathcal{E}_i(\lambda)=0$. The same arguments hold for $\tilde f_{2,+}$ where $c_2 := \prod_{i=0}^{M'-1} r(z'_i, z'_{i+1})$. In conclusion we have

$$\tilde f_{1,+}(\lambda) = \frac{c_1}{|\lambda|^M} - \frac{c_1}{|\lambda|^{M+1}}\sum_{i=0}^{M} r(z_i) + o\left(\frac{1}{|\lambda|^{M+1}}\right),$$

$$\tilde f_{2,+}(\lambda) = \frac{c_2}{|\lambda|^{M'}} - \frac{c_2}{|\lambda|^{M'+1}}\sum_{i=0}^{M'} r(z'_i) + o\left(\frac{1}{|\lambda|^{M'+1}}\right).$$

Since $\tilde f_{1,+}(\lambda), \tilde f_{2,+}(\lambda)$ are proportional for $\lambda<0$ with $|\lambda|$ large, it must be $M=M'$ and $\sum_{i=0}^M r(z_i) = \sum_{i=0}^M r(z'_i)$. These identities cannot be true in general. If $M\neq M'$ trivially we have a contradiction. Otherwise take $r(e)=1$ for all $e\in\Gamma_1$ and $r(e)=a>0$ for all $a\in\Gamma_2\setminus\Gamma_1$. If a is large then the identity $\sum_{i=0}^M r(z_i)=\sum_{i=0}^M r(z'_i)$ fails. □

Appendix A. Proposition 6.6 implies (UB)

For completeness, following similar arguments as in [10], we explain how one can deduce from Proposition 6.6 the upper bound (UB) for all $\vartheta\in\mathbb{R}$, assuming $v>0$. Recall that $S_t := \inf\{s\ge t : Z_s\le 0\}$, and observe that for all $u>0$ it holds

$$\mathbb{P}\Big(\inf_{s\ge t} Z_s \le ut\Big) \le q^{-ut}\mathbb{P}(S_t<\infty), \qquad q := \mathbb{P}(w_1=-1). \tag{84}$$

To prove the above bound observe that one possible way of realizing the event $\big\{\inf_{s\ge t} Z_s \le 0\big\}$ is the following. If $Z_t > \lfloor ut\rfloor$ then the process hits $\lfloor ut\rfloor$ after time t and then performs $\lfloor ut\rfloor$ consecutive steps to the left. If $Z_t\le\lfloor ut\rfloor$ then after time t the process performs $\lfloor ut\rfloor$ consecutive steps to the left. In particular we get

$$\mathbb{P}(S_t<\infty) = \mathbb{P}\Big(\inf_{s\ge t} Z_s\le 0\Big) \ge \mathbb{P}\Big(\inf_{s\ge t} Z_s \le \lfloor ut\rfloor\Big)q^{\lfloor ut\rfloor} \ge \mathbb{P}\Big(\inf_{s\ge t} Z_s\le ut\Big)q^{ut}.$$

From (84) and Proposition 6.6 we readily get (UB) for $\vartheta=0$:

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big(\frac{Z_t}{t}\in(-\varepsilon,\varepsilon)\Big) \le \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big(\inf_{s\ge t} Z_s\le\varepsilon t\Big) \le -\varepsilon - I(0) \xrightarrow{\varepsilon\to 0} -I(0).$$

Fix, now, any $\vartheta>0$ and take ε small enough so that $u:=\vartheta-\varepsilon>0$ and fix $u'\in(0,u)$. Let m be any positive integer. Then we have for t large (as we assume)

$$\begin{aligned}\mathbb{P}\left(\frac{Z_t}{t}\in(\vartheta-\varepsilon,\vartheta+\varepsilon)\right) &\le \mathbb{P}(Z_t\in[ut,ut+2\varepsilon t]) = \mathbb{P}(T_{\lfloor ut\rfloor}\le t,\, Z_t\in[ut,ut+2\varepsilon t])\\ &\le \mathbb{P}\Big(\frac{T_{\lfloor ut\rfloor}}{\lfloor ut\rfloor}\le\frac{1}{u'},\, \inf_{s\ge t} Z_s\le ut+2\varepsilon t\Big)\\ &\le \sum_{k=1}^m \mathbb{P}\Big(\frac{T_{\lfloor ut\rfloor}}{\lfloor ut\rfloor}\in\Big[\frac{(k-1)}{u'm},\frac{k}{u'm}\Big]\Big)\mathbb{P}\left(\inf_{s\ge t-\frac{kt}{m}} Z_s\le 2\varepsilon t\right)\\ &\le q^{-2\varepsilon t}\sum_{k=1}^m \mathbb{P}\Big(\frac{T_{\lfloor ut\rfloor}}{\lfloor ut\rfloor}\in\Big[\frac{(k-1)}{u'm},\frac{k}{u'm}\Big]\Big)\mathbb{P}\big(S_{t-\frac{kt}{m}}<\infty\big).\end{aligned}$$

We point out that the third inequality above follows from the strong Markov property applied at time $T_{\lfloor ut \rfloor}$ and the fact that the probability $\mathbb{P}\left(\inf_{s\geq t-a} Z_s \leq 2\varepsilon t\right)$ is increasing in a. The last inequality follows from (84).

Reasoning as in (50) and using Proposition 6.6, we get for $1 \leq k < m$

$$\mathbb{P}\Big(\frac{T_{\lfloor ut \rfloor}}{\lfloor ut \rfloor} \in \Big[\frac{(k-1)}{u'm}, \frac{k}{u'm}\Big]\Big)\mathbb{P}\big(S_{t-\frac{kt}{m}} < \infty\big) \leq e^{t\varepsilon + tuw_+\left(\frac{1}{u'}, \frac{1}{mu'}\right)} e^{-t\left[\frac{k}{m} I\left(\frac{mu'}{k}\right) + \left(1-\frac{k}{m}\right) I(0)\right]}$$
$$\leq e^{t\varepsilon + tuw_+\left(\frac{1}{u'}, \frac{1}{mu'}\right)} e^{-tI(u')},$$

where t is taken large enough and w_+ is defined as in (49). Note that the last inequality follows from the convexity of I. When $k = m$, on the other hand, $\mathbb{P}\big(S_{t-\frac{kt}{m}} < \infty\big) = \mathbb{P}\big(S_0 < \infty\big) = 1$ and, as in (50), for t large we have

$$\mathbb{P}\Big(\frac{T_{\lfloor ut \rfloor}}{\lfloor ut \rfloor} \in \Big[\frac{(m-1)}{u'm}, \frac{1}{u'}\Big]\Big) \leq e^{t\varepsilon + tuw_+\left(\frac{1}{u'}, \frac{1}{mu'}\right)} e^{-tI(u')}.$$

Putting all together, we have shown that for any ε small and t large enough it holds

$$\mathbb{P}\Big(\frac{Z_t}{t} \in (\vartheta - \varepsilon, \vartheta + \varepsilon)\Big) \leq m \cdot q^{-2\varepsilon t} \cdot e^{t\varepsilon + tuw_+\left(\frac{1}{u'}, \frac{1}{mu'}\right)} e^{-tI(u')}$$

with $u = \vartheta - \varepsilon$, and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\frac{Z_t}{t} \in (\vartheta - \varepsilon, \vartheta + \varepsilon)\Big) \leq -2\varepsilon \log q + \varepsilon + uw_+\big(\frac{1}{u'}, \frac{1}{mu'}\big) - I(u').$$

Letting, now, $m \to \infty$ and then $\varepsilon \to 0$ (so that also $u \to \vartheta$) and taking $u' \to \vartheta$ gives (UB).

The proof of the same bound for $\vartheta < 0$ follows by similar arguments.

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