

# On the distribution of the critical values of random spherical harmonics

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## Abstract

We study the limiting distribution of critical points and extrema of random spherical harmonics, in the high energy limit. In particular, we first derive the density functions of extrema and saddles; we then provide analytic expressions for the variances and we show that the empirical measures in the high-energy limits converge weakly to their expected values. Our arguments require a careful investigation of the validity of the Kac-Rice formula in nonstandard circumstances, entailing degeneracies of covariance matrices for first and second derivatives of the processes being analyzed.

- **Keywords and Phrases:** Spherical Harmonics, Critical Points, Kac-Rice Formula, Legendre Polynomials, Hilb's Asymptotics
- **AMS Classification:** 60G60, 60D05, 60B10, 33C55, 42C10

## 1 Introduction

### 1.1 Statement of the main results

The purpose of this paper is to investigate the asymptotic distribution of critical values for Gaussian spherical harmonics, in the high energy (Laplace eigenvalue) limit. Below we will find explicit expressions for the density of extreme values and saddles; more importantly, we will also find functional form for their variances; by means of the above we will study the convergence of the empirical distributions of extrema to their limiting expressions. Some motivating applications are discussed below; first let us introduce our models and results more formally.

It is well-known that the eigenvalues  $\lambda$  of the Laplace equation

$$\Delta_{\mathcal{S}^2} f + \lambda f = 0$$

on the two-dimensional sphere  $\mathcal{S}^2$ , are of the form  $\lambda = \lambda_\ell = \ell(\ell + 1)$  for some integer  $\ell \geq 1$ . For any given eigenvalue  $\lambda_\ell$ , the corresponding eigenspace is the  $(2\ell + 1)$ -dimensional space  $\mathcal{L}_\ell$  of spherical harmonics of degree  $\ell$ ; we can choose an arbitrary  $L^2$ -orthonormal basis  $\{Y_{\ell m}(\cdot)\}_{m=-\ell, \dots, \ell}$ , and consider random eigenfunctions of the form

$$f_\ell(x) = \frac{1}{\sqrt{2\ell + 1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x),$$

where the coefficients  $\{a_{\ell m}\}$  are independent, standard Gaussian variables; this is invariant w.r.t. the choice of  $\{Y_{\ell m}\}$ . The random fields  $\{f_\ell(x), x \in \mathcal{S}^2\}$  are isotropic, meaning that the probability laws of  $f_\ell(\cdot)$  and  $f_\ell^g(\cdot) := f_\ell(g \cdot)$  are the same for any rotation  $g \in SO(3)$ . Also,  $f_\ell$  are centred Gaussian, and from the addition theorem for spherical harmonics (see [4] theorem 9.6.3) the covariance function is given by,

$$\mathbb{E}[f_\ell(x)f_\ell(y)] = P_\ell(\cos d(x, y)),$$

where  $P_\ell$  are the usual Legendre polynomials,  $\cos d(x, y) = \cos \theta_x \cos \theta_y + \sin \theta_x \sin \theta_y \cos(\varphi_x - \varphi_y)$  is the spherical geodesic distance between  $x$  and  $y$ ,  $\theta \in [0, \pi]$ ,  $\varphi \in [0, 2\pi)$  are standard spherical coordinates and  $(\theta_x, \varphi_x)$ ,  $(\theta_y, \varphi_y)$  are the spherical coordinates of  $x$  and  $y$  respectively.

Let  $I \subseteq \mathbb{R}$  be any interval in the real line; we are interested in the number of critical points, extrema and saddles of  $f_\ell$  with value in  $I$ :

$$\begin{aligned} \mathcal{N}_I^c(f_\ell) &= \mathcal{N}_I^c(f_\ell) = \#\{x \in \mathcal{S}^2 : f_\ell(x) \in I, \nabla f_\ell(x) = 0\}, \\ \mathcal{N}_I^e(f_\ell) &= \mathcal{N}_I^e(f_\ell) = \#\{x \in \mathcal{S}^2 : f_\ell(x) \in I, \nabla f_\ell(x) = 0, \det(\nabla^2 f_\ell(x)) > 0\}, \\ \mathcal{N}_I^s(f_\ell) &= \mathcal{N}_I^s(f_\ell) = \#\{x \in \mathcal{S}^2 : f_\ell(x) \in I, \nabla f_\ell(x) = 0, \det(\nabla^2 f_\ell(x)) < 0\}. \end{aligned}$$

We use  $a = c, e, s$  to denote critical points, extrema and saddles respectively; it is obvious that for all  $I$  we have a.s.

$$\mathcal{N}_I^c(f_\ell) = \mathcal{N}_I^e(f_\ell) + \mathcal{N}_I^s(f_\ell).$$

### 1.1.1 Expected number of critical points

For a nice domain  $\mathcal{D} \subseteq \mathcal{S}^2$  we introduce

$$\mathcal{N}^c(f_\ell; \mathcal{D}, I) = \#\{x \in \mathcal{D} : f_\ell(x) \in I, \nabla f_\ell(x) = 0\}.$$

Our first theorem gives the asymptotic behaviour for the expected number of critical points of  $f_\ell$  with values lying in  $I$ . Let us introduce the density functions

$$\begin{aligned}\pi_1^c(t) &= \frac{\sqrt{3}}{\sqrt{8\pi}}(2e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}}, \\ \pi_1^e(t) &= \frac{\sqrt{3}}{\sqrt{2\pi}}(e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}}, \\ \pi_1^s(t) &= \frac{\sqrt{3}}{\sqrt{2\pi}}e^{-\frac{3}{2}t^2}.\end{aligned}$$

We have the following:

**Proposition 1.1.** *For every interval  $I \subseteq \mathbb{R}$  we have as  $\ell \rightarrow \infty$*

$$\mathbb{E}[\mathcal{N}_I^c(f_\ell)] = \frac{2}{\sqrt{3}}\ell^2 \int_I \pi_1^c(t) dt + O(1),$$

and

$$\mathbb{E}[\mathcal{N}_I^a(f_\ell)] = \frac{\ell^2}{\sqrt{3}} \int_I \pi_1^a(t) dt + O(1),$$

for  $a = e, s$ . The constant in the  $O(\cdot)$  term is universal.

It is immediate from Proposition 1.1 that we have

$$\mathbb{E}[\mathcal{N}_{\mathbb{R}}^c(f_\ell)] = \frac{2}{\sqrt{3}}\ell^2 + O(1), \quad \mathbb{E}[\mathcal{N}_{\mathbb{R}}^e(f_\ell)] = \frac{\ell^2}{\sqrt{3}} + O(1), \quad \mathbb{E}[\mathcal{N}_{\mathbb{R}}^s(f_\ell)] = \frac{\ell^2}{\sqrt{3}} + O(1),$$

(the special case  $I = \mathbb{R}$ ) addressed in [24] Theorem 2.3. The results above were confirmed with great accuracy by numerical simulations [16] to be published. A recent article by Feng and Zelditch has worked out the expected density of critical values in the complex analytic context, for any Kaehler manifold and metric [17].

The density functions for critical points, extrema and saddles are plotted in Figure 1 and Figure 2. The distribution of saddle points is Gaussian with zero mean, whereas the extrema are bimodal; since for  $f_\ell$  all the maxima (resp., minima) are necessarily positive, this also holds in the limit; the unique peak of their density is located approximately at  $\pm 1.685 \dots$

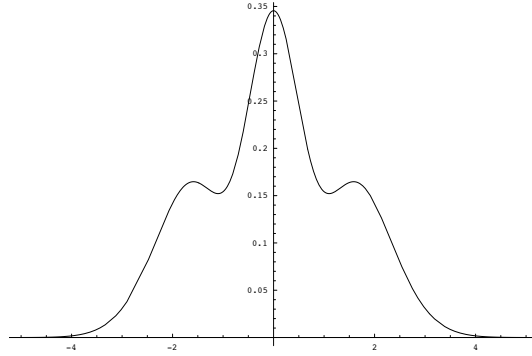


Figure 1: Limiting probability density for critical points:  $\frac{\sqrt{3}}{\sqrt{8\pi}}(2e^{-t^2} + t^2 - 1)e^{-\frac{t^2}{2}}$ .

### 1.1.2 Asymptotic fluctuations of critical values

The question of asymptotic fluctuations of critical values around the expected number is more challenging. Here we write

$$p_1^c(t) = \frac{4}{\sqrt{3}}\pi_1^c(t), \quad p_1^e(t) = \frac{2}{\sqrt{3}}\pi_1^e(t), \quad p_1^s(t) = \frac{2}{\sqrt{3}}\pi_1^s(t),$$

and introduce the functions

$$\begin{aligned}p_2^c(t) &= \frac{\sqrt{2}}{\sqrt{\pi}} \left[ -4 + t^2 + t^4 + e^{-t^2} 2(4 + 3t^2) \right] e^{-\frac{t^2}{2}}, \\ p_2^e(t) &= \frac{\sqrt{2}}{\sqrt{\pi}} \left[ -4 + t^2 + t^4 + e^{-t^2} (4 + 3t^2) \right] e^{-\frac{t^2}{2}}, \\ p_2^s(t) &= \frac{\sqrt{2}}{\sqrt{\pi}} (4 + 3t^2) e^{-\frac{3}{2}t^2},\end{aligned}$$

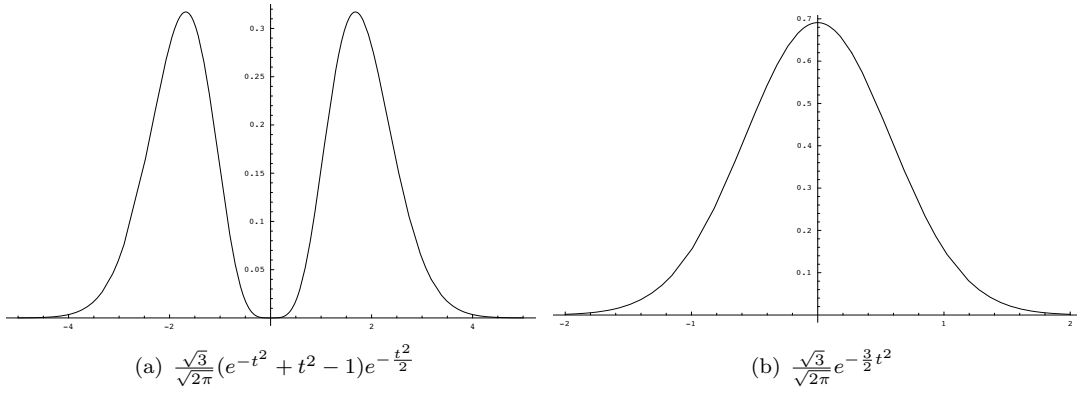


Figure 2: Limiting probability density for extrema (a) and saddles (b).

and

$$p_3^a(t) = \frac{5}{4}p_1^a(t) - \frac{1}{4}p_2^a(t)$$

for  $a = c, e, s$ . Explicitly,

$$p_3^c(t) = \frac{1}{\sqrt{8\pi}}e^{-\frac{3}{2}t^2} \left[ 2 - 6t^2 - e^{t^2}(1 - 4t^2 + t^4) \right],$$

$$p_3^e(t) = \frac{1}{\sqrt{8\pi}}e^{-\frac{3}{2}t^2} \left[ 1 - 3t^2 - e^{t^2}(1 - 4t^2 + t^4) \right],$$

$$p_3^s(t) = \frac{1}{\sqrt{8\pi}}(1 - 3t^2)e^{-\frac{3}{2}t^2}.$$

Finally, for  $a = c, e, s$  we denote

$$\nu^a(I) = \left[ \int_I p_3^a(t) dt \right]^2.$$

Our principal result concerns the asymptotic behaviour of the variance.

**Theorem 1.2.** For every interval  $I \subseteq \mathbb{R}$  as  $\ell \rightarrow \infty$

$$\text{Var}(\mathcal{N}_I^a(f_\ell)) = \ell^3 \nu^a(I) + O(\ell^{5/2}),$$

$a = c, e, s$ , where the constant in the  $O(\cdot)$  term is universal.

Note that for simplicity all our results are formulated for intervals  $I$ , however they can be easily extended to more general cases, for instance Borel subsets of  $\mathbb{R}$ . The plots for the kernel of these variances are given in Figure 3.

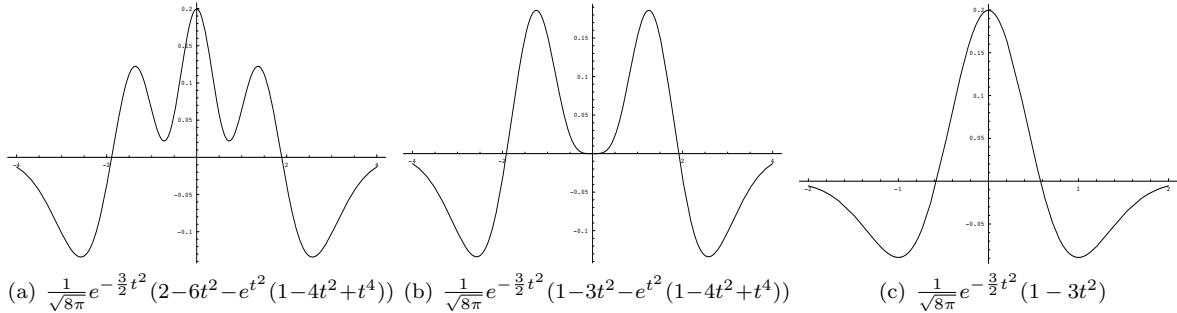


Figure 3:  $p_3^a$  for critical points (a), extrema (b) and saddles (c).

**Remark 1.3.** It is straightforward to evaluate the leading terms for critical points, extrema and saddles for any given interval  $[a, b]$ , as an explicit function of  $a$  and  $b$ . We have

$$\nu^c([a, b]) = \left[ \frac{-ae^{-\frac{3}{2}a^2}(2 + (a^2 - 1)e^{a^2}) + be^{-\frac{3}{2}b^2}(2 + (b^2 - 1)e^{b^2})}{\sqrt{8\pi}} \right]^2,$$

$$\nu^e([a, b]) = \left[ \frac{-ae^{-\frac{3}{2}a^2}(1 + (a^2 - 1)e^{a^2}) + be^{-\frac{3}{2}b^2}(1 + (b^2 - 1)e^{b^2})}{\sqrt{8\pi}} \right]^2,$$

$$\nu^s([a, b]) = \left[ \frac{-ae^{-\frac{3}{2}a^2} + be^{-\frac{3}{2}b^2}}{\sqrt{8\pi}} \right]^2.$$

More discussion on the behaviour of the leading constant  $\nu^a(I)$  for symmetric intervals  $I$  around the origin is reported in the next subsection.

**Remark 1.4.** In Figure 4 we illustrate the behaviour of the variances for the excursion sets  $I = [u, \infty)$ , i.e., we plot

$$\begin{aligned}\nu^c([u, \infty)) &= \frac{1}{8\pi} e^{-3u^2} u^2 (2 + e^{u^2} (u^2 - 1))^2, \\ \nu^e([u, \infty)) &= \frac{1}{8\pi} e^{-3u^2} u^2 (1 + e^{u^2} (u^2 - 1))^2, \\ \nu^s([u, \infty)) &= \frac{1}{8\pi} e^{-3u^2} u^2.\end{aligned}$$

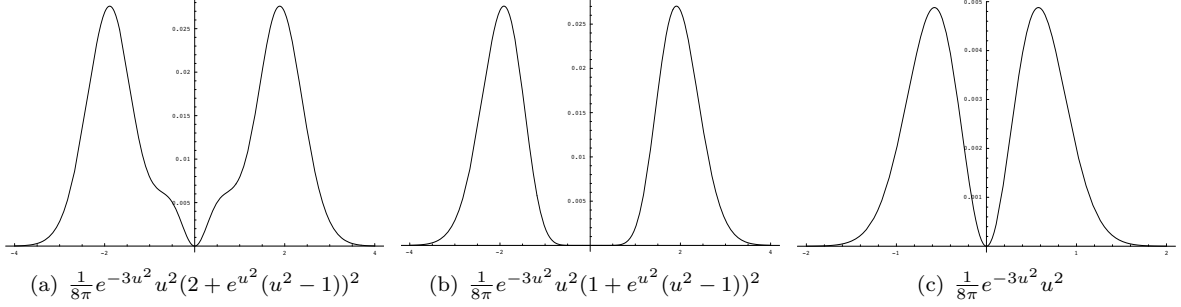


Figure 4:  $\nu^a([u, \infty))$  for critical points (a), extrema (b) and saddles (c).

In our view, the asymptotic law we proved for the variances are of independent interest; they also imply the convergence of empirical measures of critical points and extrema to their theoretical limit. More precisely, let

$$F_\ell(z) = \frac{\mathcal{N}^c(f_\ell; (-\infty, z))}{\mathbb{E}[\mathcal{N}^c(f_\ell; \mathbb{R})]}, \quad F_\ell^*(z) = \frac{\mathcal{N}^e(f_\ell; (-\infty, z))}{\mathcal{N}^e(f_\ell; \mathbb{R})},$$

be the empirical distribution function of critical points for  $f_\ell$  under deterministic and random normalizations, respectively. Now define the distribution functions  $F_\infty$  as

$$F_\infty(z) = \lim_{\ell \rightarrow \infty} \mathbb{E}[F_\ell(z)] = \int_{-\infty}^z \pi_1^c(t) dt = \int_{-\infty}^z \frac{\sqrt{3}}{\sqrt{8\pi}} (2e^{-t^2} + t^2 - 1) e^{-\frac{t^2}{2}} dt.$$

Our next result concerns the uniform convergence of the empirical distribution function to  $F_\infty(z)$ .

**Corollary 1.5.** *For all  $\varepsilon > 0$ , as  $\ell \rightarrow \infty$ , we have*

$$\mathbb{P}\{\sup_z |F_\ell^*(z) - F_\infty(z)| \geq \varepsilon\}, \quad \mathbb{P}\{\sup_z |F_\ell(z) - F_\infty(z)| \geq \varepsilon\} \rightarrow 0.$$

In practice, loosely speaking, the latter result shows that for each random realization of a high degree spherical harmonic the same empirical density of critical values will be observed, up to asymptotically negligible fluctuations.

## 1.2 On Berry cancellation

An interesting phenomenon occurs when we consider the extrema variance with values falling into  $I$ , an infinitesimally small neighbourhood of the origin, or for a fixed interval  $I$  with vanishing leading constant  $\nu^a(I)$  (more details are given below). In related circumstances, it is known [28] that the nodal length variance for random eigenfunctions on the torus and on the sphere is of lower order than for other level curves; on  $S^2$  the nodal length variance is proportional to  $\log \ell$  [27], whereas for generic level curves the variance is proportional to  $\ell$ . This behaviour was discovered by Michael Berry in [11], and thereupon is referred to as *Berry's cancellation phenomenon*.

From Theorem 1.2, it is easy to obtain, by a simple evaluation of the integral, that the variance of the number of extrema for a generic interval  $I = [-\varepsilon/2 + x_0, \varepsilon/2 + x_0]$ , is asymptotic to

$$\lim_{\varepsilon \rightarrow 0} \lim_{\ell \rightarrow \infty} \frac{\text{Var}(\mathcal{N}^e(f_\ell; [-\varepsilon/2 + x_0, \varepsilon/2 + x_0]))}{\varepsilon^2 \ell^3} = [p_3^\varepsilon(x_0)]^2,$$

where  $[p_3^\varepsilon(x_0)]^2 > 0$ , almost everywhere, see Figure 3. In contrast, from Theorem 1.2, we may also deduce the behaviour of the extrema variance in a vanishing interval  $I = [-\varepsilon, \varepsilon]$  around the origin:

**Corollary 1.6.** *As  $\varepsilon \rightarrow 0$*

$$\lim_{\varepsilon \rightarrow 0} \frac{\nu^e([-\varepsilon, \varepsilon])}{\varepsilon^{10}} = \frac{1}{8\pi}. \quad (1.1)$$

*Proof.* The statement follows immediately by evaluating

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{10}} \left[ \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{8\pi}} e^{-\frac{3}{2}t^2} \left[ 1 - 3t^2 - e^{t^2} (1 - 4t^2 + t^4) \right] dt \right]^2 = \lim_{\varepsilon \rightarrow 0} \frac{e^{-3\varepsilon^2} \left[ 1 + e^{\varepsilon^2} (-1 + \varepsilon^2) \right]^2}{2\pi \varepsilon^8} = \frac{1}{8\pi}.$$

□

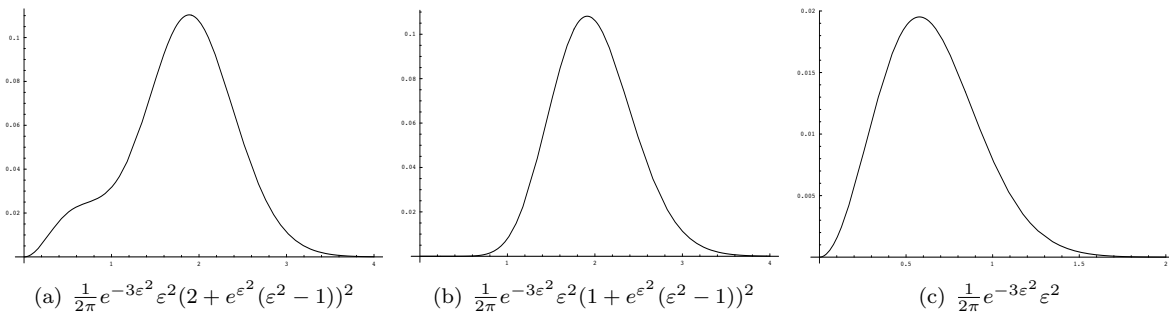


Figure 5:  $\nu^\alpha([-\varepsilon, \varepsilon])$  for critical points (a), extrema (b) and saddles (c).

Figure 5 illustrates the behaviour of these functions for symmetric intervals around the origin.

We note also that for some specifically chosen (but fixed) intervals  $I \subseteq \mathbb{R}$  with  $\nu^\alpha(I) = 0$ , such as, for example  $I = \mathbb{R}$  (for the latter case the unrestricted total number of critical points, extrema or saddles is counted), the order of magnitude of the variance is lower than  $\ell^3$ . In this case Theorem 1.2 reads:

$$\text{Var}(\mathcal{N}_{\mathbb{R}}^c(f_\ell)), \text{Var}(\mathcal{N}_{\mathbb{R}}^e(f_\ell)), \text{Var}(\mathcal{N}_{\mathbb{R}}^s(f_\ell)) = O(\ell^{\frac{5}{2}}).$$

as  $\ell \rightarrow \infty$ . It seems though that by our present methods we may sharpen the latter bound to the next term of order  $O(\ell^2 \log \ell)$ , and, unless further cancellation occurs, it may be the true asymptotic behaviour of each of the three quantities above. However, in a recent numerical simulation by D. Belyaev [8] the observed fluctuations were far too small for the latter to hold.

### 1.3 Overview of the proof

Our proof below is technically demanding, and we present here its main conceptual steps for the variance result (Theorem 1.2). Our argument is based on a suitably modified version of the Kac-Rice formula for the number of zeroes of the gradient of  $f_\ell$ . The first technical difficulty is related to the fact that the 6-dimensional vector  $(f_\ell(x), \nabla f_\ell(x), \nabla^2 f_\ell(x))$  is always degenerate, as the level field  $f_\ell$  is a linear combination of gradient and second order derivatives. However, this issue is relatively easily mended by reducing the dimension of the problem to take this degeneracy into account.

A much trickier issue arises when considering the two-point correlation function needed for the evaluation of the variance. Here we have to cope with the 4-dimensional Gaussian random vectors of the form

$$(\nabla f_\ell(x), \nabla f_\ell(y)), \quad (x, y) \in \mathcal{S}^2,$$

imposing suitable conditions to ensure that  $(f_\ell(x), f_\ell(y)) \in I$ . *A priori* there is no certainty that this random vector is nondegenerate, a condition that guarantees the applicability of the standard Kac-Rice. Our basic idea is to split the range  $\mathcal{S}^2$  of the integration of the Kac-Rice integral into two parts: the short range regime  $d(x, y) < C/\ell$ ,  $d(x, y) = \arccos(\langle x, y \rangle)$  denoting the usual spherical distance and  $C$  a sufficiently big positive constant, and the long range regime  $d(x, y) > C/\ell$ . In the short range regime the Kac-Rice formula holds only approximately, and we can prove by a partitioning argument inspired from [25] that the corresponding contribution is of order  $O(\ell^2)$ . The proof of the latter requires a precise Taylor analysis of the behaviour of Legendre polynomials and their derivatives around the origin, and related analytic functions.

The main term comes from the long range regime. Here the asymptotic analysis is based on the properties of multivariate conditional Gaussian variables, and an asymptotic study of the tail decay of the Legendre polynomials and their derivatives. In this regime, Kac-Rice formula holds exactly and we shall exploit the fact that a Gaussian expectation is an analytic function with respect to the parameters of the corresponding covariance matrix outside its singularities. It is then possible to compute the Taylor expansion of these expected values around the origin with respect to the vanishing entries of the covariance matrix; a small finite number of these (depending on the interval  $I$ ) make an asymptotically significant contribution to the variance, whereas the rest are negligible.

## 1.4 Background and motivation

### 1.4.1 Cosmology and CMB

Our main motivation for this paper is given by cosmological and astrophysical applications. Indeed, it is well-known that random spherical harmonics are the Fourier components of square integrable isotropic fields on the sphere, i.e., for every centred Gaussian spherical random field  $f(x)$  the following spectral representation holds [20]:

$$f(x) = \sum_{\ell=1}^{\infty} f_\ell(x) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{C_\ell} a_{\ell m} Y_{\ell m}(x),$$

where equality holds in the  $L^2$  sense and the sequence  $\{C_\ell\}_{\ell=1,2,\dots}$  denotes the so-called angular power spectrum, which fully characterizes the dependence structure of  $f(x)$ . The analysis of spherical random fields is now at the heart of observational cosmology, for instance for experiments handling Cosmic Microwave Background radiation data, see e.g., [1]

and [10]. In summary, we can represent CMB observations as a realization of the isotropic, Gaussian random function, which we denote by  $\hat{f}(x)$ ; realizations of the random spherical harmonic components  $\hat{f}_\ell(x)$  are then obtained by standard Fourier analysis, i.e.

$$\hat{f}_\ell(x) := \sum_{m=-\ell}^{\ell} \hat{a}_{\ell m} Y_{\ell m}(x), \quad \hat{a}_{\ell m} := \int_{\mathcal{S}^2} \hat{f}(x) \bar{Y}_{\ell m}(x) dx, \quad (1.2)$$

the bar denoting as usual complex conjugation. It is to be noted that in many experimental circumstances the realizations of these random fields are observed only on subsets of the sphere, and this can make the inverse Fourier transform in (1.2) unfeasible: however, very recently some more sophisticated statistical techniques have indeed led to the reconstruction of full sky data maps, see [12], and in this setting the empirical derivation of  $\hat{f}_\ell$  has become possible. A natural question is whether these observed CMB maps are indeed consistent with the starting assumptions of Gaussianity and isotropy; departures from these assumptions could signal either spurious features introduced by the algorithms to produce the maps, or physically motivated deviations from standard cosmological models. Examples of the former are, for instance, astrophysical components which have not been properly removed from CMB maps, such as so-called point-sources (galaxies and other astrophysical objects unrelated to CMB).

Our results can be exploited in this setting by means of the implementation of a number of Gaussianity and isotropy tests. For instance, it is possible to compare the actual number of maxima above a given threshold  $u$  for an observed component  $\hat{f}_\ell$  with its expected value and standard deviation which we reported in the previous subsection; i.e., for any given threshold value  $u$ , we may construct statistics such as

$$Z_\ell(u) = \frac{\mathcal{N}_\ell(f_\ell; [u, \infty)) - \mathbb{E}[\mathcal{N}_\ell(f_\ell; [u, \infty))]}{\sqrt{\text{Var}(\mathcal{N}_\ell(f_\ell; [u, \infty))}}.$$

By the results on expected values and variances provided in this paper, the previous statistic can be computed explicitly for any value of  $u$ . It is natural to expect that convergence to a standard Gaussian limit will hold in the high-energy regime, under the null assumption that  $\{f_\ell\}$  is a pure Gaussian field; on the contrary, nonGaussian features such as the previously mentioned point sources will show up as a higher number of observed maxima than predicted under Gaussian circumstances; therefore high values of  $Z_\ell$  will signal the presence of spurious components. Extensions to cover joint tests on multiple threshold  $u_1, \dots, u_p$  are straightforward. Of course, the actual implementation of these procedures on real data will require further work, which we delay to future research (see [16]).

#### 1.4.2 Nodal domains of Laplace eigenfunctions

The *nodal components* of  $f_\ell$  are the connected components of the *nodal line*  $f_\ell^{-1}(0)$ , and the nodal domains of  $f_\ell$  are the connected components of its complement  $\mathcal{S}^2 \setminus f_\ell^{-1}(0)$ . The extrema density function vanishing at the origin (Figure 2 (a)) supports the stability concept of nodal domains as established by [21, 22, 23], i.e., the fact that small perturbations of the spherical harmonics do not affect significantly the nodal portraits. This conclusion is strengthened by our results on the behaviour of the critical points variance around the origin.

It was asserted that the nodal structure of  $f_\ell$  (or Laplace eigenfunctions, random or deterministic, on generic surfaces) could be modelled [13] by a bond percolation-like model which could be explained as follows.

Let  $\mathcal{L}^+$  and  $\mathcal{L}^-$  be the (random) sets of the local minima and maxima of  $f_\ell$  respectively. Under the percolation model  $\mathcal{L}^+$  and  $\mathcal{L}^-$  are thought of as mutually dual square grids with  $\approx \ell^2 = \ell \times \ell$  points ('sites'), each representing a maximum or minimum respectively. Each pair of adjacent (w.r.t. the grid) sites are connected by an 'open' bond in  $\mathcal{L}^+$  with probability 1/2 independent of other bonds, whence the dual bond in  $\mathcal{L}^-$  is 'closed' and vice versa. One can then study some aspects of the percolation process described, such as the number of clusters of  $\mathcal{L}^\pm$  (representing the number of the nodal domains of  $f_\ell$ ), their area distribution etc. It is important that there are only few low-lying extrema (see Proposition 1.1 and Figure 2 (a)), corresponding to nodal domains unstable under small perturbations of  $f_\ell$ .

Recent numerical studies revealed small but significant deviation from the percolation model (e.g. [9]); this deviation may be attributed [8] to the unsubstantiated rigidity assumption on the sites positions along  $\mathcal{L}^\pm$ , and it was suggested [8] that the rigidity of the sets  $\mathcal{L}^\pm$  should be relaxed. Theorem 1.2 then may be used to determine the measure of flexibility or rigidity expected from the sets  $\mathcal{L}^\pm$  to satisfy in a more sophisticated percolation-like model for the nodal structure of Laplace eigenfunctions.

#### 1.4.3 Persistence barcodes

Our results may also find natural applications in the rapidly growing areas of applied algebraic topology and topological data analysis, and in particular for the characterization of the stochastic properties of *persistence barcodes* and *persistence diagrams* (see e.g. [14] or [3]) for excursion sets of random spherical harmonics. Write

$$A_u(f_\ell) = \{x \in \mathcal{S}^2 : f_\ell(x) > u\}$$

for the excursion sets of  $f_\ell$ , and let us recall that a *barcode* for  $A_u(f_\ell)$  is a pair of graphs, each corresponding to one of the two homology groups for the corresponding excursion sets,  $H_k(A_u(f_\ell))$  where  $k = 0, 1$ . Loosely speaking,  $H_0(A_u(f_\ell))$  is generated by the elements that represent the connected components of the excursion sets, and  $H_1(A_u(f_\ell))$  is generated by elements that represent 1-dimensional "loops". Each of the two graphs in this barcode is a collection of bars; a bar in the graph representing  $H_0$ , starting at threshold  $u_{start}$  and ending at threshold  $u_{end}$ , corresponds to a generator of  $H_0(A_u)$  that "appeared" at level  $u_{start}$  and "disappeared" at level  $u_{end}$ ; if two connected components of  $A_u(f_\ell)$  merge, then only one of the two corresponding bars remains. An analogous meaning can be given to the bars in the second graph, see [14],[3]

for more details and discussion. Hence the number of bars in graph  $k$  at any level  $u$  equals the *Betti number*  $\beta_k(A_u(f_\ell))$  for the excursion region corresponding to this threshold.

A *persistence diagram* for  $H_k$ ,  $k = 0, 1$  is a set of pairs  $(u_{end}(k), u_{start}(k))$  corresponding to the starting and ending points of these bars. In [3], p. 107–108 it is explained that the starting points of the  $H_0$  bars correspond to the heights of local maxima, whereas the ending points of the  $H_1$  bars correspond to the heights of the local minima. Hence our results in this paper establish the density of  $u_{end}(1)$  and  $u_{start}(0)$  in the case of random spherical harmonics; the shape of our curves can be compared to the simulated results reported in [3], figure 6.2.2, which represent persistence diagrams of excursion sets from a Gaussian isotropic random field on the unit square.

## 1.5 Plan of the paper

The plan of this paper is as follows: in section 2 we establish the asymptotic density of critical points, extrema and saddles; in section 3 we discuss the approximate Kac-Rice formula instrumental for establishing our results; section 4 discusses the derivation of the two-point correlation function while section 5 is devoted to the proofs for the expressions of the variances reported in the introduction. Finally, section 6 provides the convergence results for the empirical measures of critical points and extrema. A number of auxiliary results of more technical nature facilitating the computations of covariance matrices and asymptotics for Legendre polynomials are collected in the appendix.

## 1.6 Acknowledgements

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# 2 Asymptotic density of critical values

## 2.1 On the Kac-Rice formula for the expected number of critical values

Let  $\mathcal{E} \subseteq \mathbb{R}^n$  be a nice Euclidian domain, and  $g : \mathcal{E} \rightarrow \mathbb{R}^n$  a centred Gaussian random field, a.s. smooth. The set  $g^{-1}(0)$  a.s. consists of finitely many zeros of  $g$ . One defines the zero density (also referred to as “first intensity”)  $K_1 = K_{1,g} : \mathcal{E} \rightarrow \mathbb{R}$  of  $g$  as

$$K_1(x) = \phi_{g(x)}(\mathbf{0}) \cdot \mathbb{E}[|\det J_g(x)| | g(x) = \mathbf{0}],$$

where  $\phi_{g(x)}$  is the (Gaussian) probability density of  $g(x) \in \mathbb{R}^n$  and  $J_g(x)$  is the Jacobian matrix of  $g$  at  $x$ . Under the assumption that for all  $x \in \mathcal{E}$  the distribution of  $g(x)$  is non-degenerate in  $\mathbb{R}^n$  (i.e. is not concentrated in a proper subspace of the latter), the expected number of zeros of  $g$  on  $\mathcal{E}$  is given by [6, 2]

$$\mathbb{E}[\#g^{-1}(\mathbf{0})] = \int_{\mathcal{E}} K_1(x) dx.$$

To apply the latter formula in our case we will work with spherical coordinates on  $\mathcal{S}^2$  and use an explicit orthonormal frame (see section 2.2); counting the critical points of  $f = f_\ell$  is equivalent to counting the zeros of the map  $[0, \pi]^2 \rightarrow \mathbb{R}^2$  given by  $x \mapsto \nabla f(x) = (f_1(x), f_2(x))$ , where  $(f_1, f_2)$  are the partial derivatives of  $f$ . Here we have

$$K_1(x) = K_{1,\ell}(x) = \phi_{\nabla f(x)}(0, 0) \cdot \mathbb{E}[|\det H_f(x)| | \nabla f(x) = 0],$$

where  $H_f(x)$  is the Hessian matrix of  $f$  at  $x$ . An explicit computation of the covariance matrix of  $\nabla f(x) \in \mathbb{R}^2$  shows that for  $\ell$  sufficiently big the distribution of the latter is non-degenerate so that [2], Theorem 11.2.1 yields that the expected total number of critical points of  $f$  is

$$\mathbb{E}[\mathcal{N}_{\mathbb{R}}^c(f)] = \int_{\mathcal{S}^2} K_1(x) dx;$$

the isotropy of  $f_\ell$  implies that  $K_1(x) \equiv K_1$  depends on  $\ell$  only, independent of  $x$ .

For counting the number of critical points with corresponding value lying in  $I \subseteq \mathbb{R}$ , we need to modify  $K_1(x)$  so that this time we define

$$K_1(x; I) = \phi_{\nabla f(x)}(0, 0) \cdot \mathbb{E}[|\det H(x)| \cdot \mathbf{1}_I(f(x)) | \nabla f(x) = 0], \quad (2.1)$$

where  $\mathbf{1}_I$  is the characteristic function of  $I$  on  $\mathbb{R}$ . In this case [2], Theorem 11.2.1 yields

$$\mathbb{E}[\mathcal{N}_I^c(f)] = \int_{\mathcal{S}^2} K_{1,I}(x) dx,$$

again, under the non-degeneracy assumption on  $\nabla f(x)$ . Note that in our case there is a linear dependency between the value  $f(x)$ , involved in the definition (2.1) of  $K_1(x; I)$ , and the Hessian  $H(x)$  (see (2.2) below); nevertheless the non-degeneracy of  $\nabla f(x)$  is sufficient for an application of [6], Theorem 6.3 or [2], Theorem 11.2.1; the linear dependency (2.2) allows us to reduce the dimension of the Gaussian distribution involved in the evaluation of  $K_1(x, I)$  from 6 to 5, a considerable technical simplification. It is easy to adapt the same approach to separate the critical points into extrema and saddles (see section 2.2, towards the end).

## 2.2 Application of Kac-Rice in coordinates

In this section we formulate the (precise) Kac-Rice formula to derive the expected value of the number of critical points, saddles and extrema with values in a given interval  $I \subset \mathbb{R}$ . To this aim, let us first introduce some notation.

Given  $x \in \mathcal{S}^2$ , consider a local orthogonal frame  $\{e_1^x, e_2^x\}$  defined in some neighbourhood of  $x$ , such that, for any regular function  $h : \mathcal{S}^2 \rightarrow \mathbb{R}$ , we have  $e_1^x e_2^x h = e_2^x e_1^x h$ . Via an isometry, for every  $x \in \mathcal{S}^2$ , it is possible to obtain a (local) identification

$$T_x(\mathcal{S}^2) \cong \mathbb{R}^2,$$

so that we do not have to work with probability densities defined on the tangent planes  $T_x(\mathcal{S}^2)$  which depend on the point  $x \in \mathcal{S}^2$ ; in particular, we shall work with the orthogonal frame

$$\left\{ e_1^x = \frac{\partial}{\partial \theta_x}, e_2^x = \frac{\partial}{\partial \varphi_x} \right\}.$$

Since the  $f_\ell$  are eigenfunctions of the spherical Laplacian, we have that the value of the spherical harmonic at every fixed point  $x \in \mathcal{S}^2$  is a linear combination of its first and second order derivatives at  $x$ . If the point  $x \in \mathcal{S}^2$  is also a critical point for  $f_\ell$  it follows that the value of the spherical harmonic at  $x$  is a linear combination of its second order derivatives, i.e.,

$$e_1^x e_1^x f_\ell(x) + e_2^x e_2^x f_\ell(x) = -\ell(\ell+1)f_\ell(x). \quad (2.2)$$

For  $x \in \mathcal{S}^2$  we define the random vectors:

$$Z_{\ell;x} = (\nabla f_\ell(x), \nabla^2 f_\ell(x)),$$

where

$$\nabla f_\ell(x) = (e_1^x f_\ell(x), e_2^x f_\ell(x)),$$

and  $\nabla^2 f_\ell$  is defined as

$$\nabla^2 f_\ell(x) = (e_1^x e_1^x f_\ell(x), e_1^x e_2^x f_\ell(x), e_2^x e_2^x f_\ell(x)).$$

We denote by

$$D_{\ell;x}(\xi_{x,1}, \xi_{x,2}, \zeta_{x,1}, \zeta_{x,2}, \zeta_{x,3}),$$

the probability density functions of  $Z_{\ell;x}$ ; the vectors  $Z_{\ell;x}$  are centered Gaussian in  $\mathbb{R}^5$ . By the isotropic property of  $f_\ell$  it is possible and indeed convenient to perform our computations along a specific geodesic; we constrain ourselves to the equatorial line  $\theta_x = \frac{\pi}{2}$ . With this choice the  $5 \times 5$  covariance matrix  $\sigma_\ell$  of  $Z_{\ell;x}$  is (see the computations in Appendix B)

$$\sigma_\ell = \begin{pmatrix} a_\ell & b_\ell \\ b_\ell^t & c_\ell \end{pmatrix},$$

where

$$a_\ell = \begin{pmatrix} \frac{\lambda_\ell}{2} & 0 \\ 0 & \frac{\lambda_\ell}{2} \end{pmatrix}, \quad b_\ell = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$c_\ell = \begin{pmatrix} \frac{\lambda_\ell}{8}[3\lambda_\ell - 2] & 0 & \frac{\lambda_\ell}{8}[\lambda_\ell + 2] \\ 0 & \frac{\lambda_\ell}{8}[\lambda_\ell - 2] & 0 \\ \frac{\lambda_\ell}{8}[\lambda_\ell + 2] & 0 & \frac{\lambda_\ell}{8}[3\lambda_\ell - 2] \end{pmatrix} = \frac{\lambda_\ell^2}{8} \begin{pmatrix} 3 - \frac{2}{\lambda_\ell} & 0 & 1 + \frac{2}{\lambda_\ell} \\ 0 & 1 - \frac{2}{\lambda_\ell} & 0 \\ 1 + \frac{2}{\lambda_\ell} & 0 & 3 - \frac{2}{\lambda_\ell} \end{pmatrix},$$

where  $\lambda_\ell = \ell(\ell+1)$ . From the isotropy the following result follows at once:

**Lemma 2.1** (Kac-Rice formula). *The expected value of  $\mathcal{N}_I^c(f_\ell)$  is given by*

$$\mathbb{E}[\mathcal{N}_I^c(f_\ell)] = 4\pi K_{1,\ell}(I),$$

where

$$K_{1,\ell}(I) = \int_{\mathbb{R}^3} |\zeta_1 \zeta_3 - \zeta_2^2| \mathbb{1}_{\left\{ \frac{\zeta_1 + \zeta_3}{\ell(\ell+1)} \in I \right\}} D_{\ell;x}(0, 0, \zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3.$$

*Proof.* First, from (2.2), we have

$$\begin{aligned} \mathcal{N}_I^c(f_\ell) &= \#\{x \in \mathcal{S}^2 : f_\ell(x) \in I, \nabla f_\ell(x) = 0\} \\ &= \#\left\{ x \in \mathcal{S}^2 : -\frac{e_1^x e_1^x f_\ell(x) + e_2^x e_2^x f_\ell(x)}{\ell(\ell+1)} \in I, \nabla f_\ell(x) = 0 \right\}. \end{aligned}$$

We can now apply Theorem 11.2.1 in [2], and get:

$$\begin{aligned} \mathbb{E}[\mathcal{N}_I^c(f_\ell)] &= \int_{\mathcal{S}^2} dx \int_{\mathbb{R}^3} |\zeta_{x,1} \zeta_{x,3} - \zeta_{x,2}^2| \mathbb{1}_{\left\{ -\frac{\zeta_{x,1} + \zeta_{x,3}}{\ell(\ell+1)} \in I \right\}} D_{\ell;x}(0, 0, \zeta_{x,1}, \zeta_{x,2}, \zeta_{x,3}) d\zeta_{x,1} d\zeta_{x,2} d\zeta_{x,3} \\ &= \int_{\mathcal{S}^2} dx \int_{\mathbb{R}^3} |\zeta_{x,1} \zeta_{x,3} - \zeta_{x,2}^2| \mathbb{1}_{\left\{ \frac{\zeta_{x,1} + \zeta_{x,3}}{\ell(\ell+1)} \in I \right\}} D_{\ell;x}(0, 0, \zeta_{x,1}, \zeta_{x,2}, \zeta_{x,3}) d\zeta_{x,1} d\zeta_{x,2} d\zeta_{x,3}. \end{aligned}$$

By the isotropic property the density

$$K_{1,\ell}(I) = \int_{\mathbb{R}^3} |\zeta_{x,1} \zeta_{x,3} - \zeta_{x,2}^2| \mathbb{1}_{\left\{ \frac{\zeta_{x,1} + \zeta_{x,3}}{\ell(\ell+1)} \in I \right\}} D_{\ell;x}(0, 0, \zeta_{x,1}, \zeta_{x,2}, \zeta_{x,3}) d\zeta_{x,1} d\zeta_{x,2} d\zeta_{x,3}$$

does not depend on  $x$ , and the result of the present lemma follows.  $\square$



**Remark 2.2.** For the critical points and the saddles we have the analogous result

$$\mathbb{E}[\mathcal{N}_I^a(f_\ell)] = 4\pi K_{1,\ell}^a(I),$$

for  $a = e, s$ , where, for example,

$$K_{1,\ell}^e(I) = \int_{\mathbb{R}^3} |\zeta_1 \zeta_3 - \zeta_2^2| \mathbb{1}_{\left\{\frac{\zeta_1 + \zeta_3}{\ell(\ell+1)} \in I\right\}} \mathbb{1}_{\{\zeta_1 \zeta_3 - \zeta_2^2 > 0\}} D_\ell(0, 0, \zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3.$$

### 2.3 Asymptotic density of critical points

We will now exploit the Kac-Rice formula and the degeneracy discussed above for spherical harmonics to prove our first result on the expected number of critical points and extrema of  $f_\ell$  with values lying in an interval  $I \subseteq \mathbb{R}$ .

**Lemma 2.3.** For  $\ell \rightarrow \infty$ , we have

$$\mathbb{E}[\mathcal{N}_I^c(f_\ell)] = \frac{\ell^2}{2} \int_I p_1^c(t) dt + O(1),$$

where

$$p_1^c(t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} |z_1 t \sqrt{8} - z_1^2 - z_2^2| \exp\left\{-\frac{3}{2}t^2\right\} \exp\left\{-\frac{1}{2}(z_1^2 + z_2^2 - \sqrt{8}tz_1)\right\} dz_1 dz_2.$$

*Proof.* From Lemma 2.1, we have

$$\mathbb{E}[\mathcal{N}_I^c(f_\ell)] = 4\pi K_{1,\ell}(I). \quad (2.3)$$

Since the first and the second order derivatives of  $f_\ell(x)$  are independent at every fixed point  $x \in \mathcal{S}^2$ , we can write

$$K_{1,\ell}(I) = \int_{\mathbb{R}^3} |\zeta_1 \zeta_3 - \zeta_2^2| \mathbb{1}_{\left\{\frac{\zeta_1 + \zeta_3}{\lambda_\ell} \in I\right\}} D_\ell(0, 0, \zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3$$

with

$$D_\ell(0, 0, \zeta_1, \zeta_2, \zeta_3) = D_{1,\ell}(0, 0) D_{2,\ell}(\zeta_1, \zeta_2, \zeta_3),$$

where  $D_{1,\ell}$  and  $D_{2,\ell}$  are the marginal densities of the random vectors  $\nabla f_\ell$  and  $\nabla^2 f_\ell$  respectively. Now, observing the matrices  $a_\ell$  and  $c_\ell$ , it follows immediately that

$$D_{1,\ell}(0, 0) = \frac{1}{2\pi} \frac{2}{\lambda_\ell},$$

and

$$\frac{\sqrt{8}}{\lambda_\ell} \nabla^2 f_\ell = (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) \sim N(0, \tilde{c}_\ell),$$

with

$$\tilde{c}_\ell = \frac{8}{\lambda_\ell^2} c_\ell = \begin{pmatrix} 3 - \frac{2}{\lambda_\ell} & 0 & 1 + \frac{2}{\lambda_\ell} \\ 0 & 1 - \frac{2}{\lambda_\ell} & 0 \\ 1 + \frac{2}{\lambda_\ell} & 0 & 3 - \frac{2}{\lambda_\ell} \end{pmatrix}. \quad (2.4)$$

It then follows that

$$\begin{aligned} K_{1,\ell}(I) &= \frac{1}{\pi \lambda_\ell} \int_{\mathbb{R}^3} |\zeta_1 \zeta_3 - \zeta_2^2| \mathbb{1}_{\left\{\frac{\zeta_1 + \zeta_3}{\lambda_\ell} \in I\right\}} D_{2,\ell}(\zeta_1, \zeta_2, \zeta_3) d\zeta_1 d\zeta_2 d\zeta_3 \\ &= \frac{1}{\pi \lambda_\ell} \frac{\lambda_\ell^2}{8} \int_{\mathbb{R}^3} |\tilde{z}_1 \tilde{z}_3 - \tilde{z}_2^2| \mathbb{1}_{\left\{\frac{\tilde{z}_1 + \tilde{z}_3}{\sqrt{8}} \in I\right\}} \frac{1}{(2\pi)^{3/2} \sqrt{\det \tilde{c}_\ell}} \exp\left\{-\frac{1}{2}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \tilde{c}_\ell^{-1} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix}\right\} d\tilde{z}_1 d\tilde{z}_2 d\tilde{z}_3. \end{aligned}$$

After the change of variables

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & \sqrt{8} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ t \end{pmatrix},$$

the latter expression is

$$\begin{aligned} K_{1,\ell}(I) &= \frac{1}{\pi \lambda_\ell} \frac{\lambda_\ell^2}{8} \int_{\mathbb{R}^3} |z_1(\sqrt{8}t - z_1) - z_2^2| \mathbb{1}_{\{t \in I\}} \\ &\quad \times \frac{1}{(2\pi)^{3/2} \sqrt{\det \tilde{c}_\ell}} \exp\left\{-\frac{1}{2}(z_1, z_2, \sqrt{8}t - z_1) \tilde{c}_\ell^{-1} \begin{pmatrix} z_1 \\ z_2 \\ \sqrt{8}t - z_1 \end{pmatrix}\right\} \sqrt{8} dz_1 dz_2 dt \\ &= \frac{1}{\pi} \frac{\lambda_\ell}{8} \int_I dt \int_{\mathbb{R}^2} |z_1(\sqrt{8}t - z_1) - z_2^2| \end{aligned}$$

$$\times \frac{1}{(2\pi)^{3/2} \sqrt{\det \tilde{c}_\ell}} \exp \left\{ -\frac{1}{2} (z_1, z_2, \sqrt{8t} - z_1) \tilde{c}_\ell^{-1} \begin{pmatrix} z_1 \\ z_2 \\ \sqrt{8t} - z_1 \end{pmatrix} \right\} \sqrt{8} dz_1 dz_2, \quad (2.5)$$

where

$$\frac{1}{\sqrt{\det \tilde{c}_\ell}} = \frac{\lambda_\ell}{\sqrt{8}(\lambda_\ell - 2)},$$

and

$$\exp \left\{ -\frac{1}{2} (z_1, z_2, \sqrt{8t} - z_1) \tilde{c}_\ell^{-1} \begin{pmatrix} z_1 \\ z_2 \\ \sqrt{8t} - z_1 \end{pmatrix} \right\} = \exp \left\{ -\frac{3\lambda_\ell - 2}{2(\lambda_\ell - 2)} t^2 \right\} \exp \left\{ -\frac{1}{2} \frac{\lambda_\ell}{\lambda_\ell - 2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\}$$

so that

$$\begin{aligned} K_{1,\ell}(I) &= \frac{1}{\pi} \frac{\lambda_\ell}{8} \frac{\lambda_\ell}{\lambda_\ell - 2} \int_I dt \int_{\mathbb{R}^2} |z_1(\sqrt{8t} - z_1) - z_2^2| \\ &\quad \times \frac{1}{(2\pi)^{3/2}} \exp \left\{ -\frac{3\lambda_\ell - 2}{2(\lambda_\ell - 2)} t^2 \right\} \exp \left\{ -\frac{1}{2} \frac{\lambda_\ell}{\lambda_\ell - 2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} dz_1 dz_2. \end{aligned}$$

Now let us write

$$\mathbb{E}[\mathcal{N}_I^c(f_\ell)] = \frac{\lambda_\ell}{2} \int_I g_{1,\ell}^c(t) dt, \quad (2.6)$$

where

$$g_{1,\ell}^c(t) = \frac{1}{(2\pi)^{3/2}} \frac{\lambda_\ell}{\lambda_\ell - 2} \int_{\mathbb{R}^2} |z_1 t \sqrt{8} - z_1^2 - z_2^2| \exp \left\{ -\frac{3\lambda_\ell - 2}{2(\lambda_\ell - 2)} t^2 \right\} \exp \left\{ -\frac{1}{2} \frac{\lambda_\ell}{\lambda_\ell - 2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} dz_1 dz_2. \quad (2.7)$$

Now consider the expansions

$$\frac{\lambda_\ell}{\lambda_\ell - 2} = 1 + O(\ell^{-2}), \quad \exp \left\{ -\frac{3\lambda_\ell - 2}{2(\lambda_\ell - 2)} t^2 \right\} = \exp \left\{ -\frac{3}{2} t^2 \right\} + t^2 \exp \left\{ -\frac{3}{2} t^2 \right\} O(\ell^{-2}),$$

and

$$\exp \left\{ -\frac{1}{2} \frac{\lambda_\ell}{\lambda_\ell - 2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} = \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} + (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} O(\ell^{-2});$$

we can observe that, for  $n, n', k, k' \in \mathbb{N}$ ,

$$\exp \left\{ -\frac{3}{2} t^2 \right\} \int_{\mathbb{R}} |z_2|^n z_2^{n'} t^k \exp \left\{ -\frac{z_2^2}{2} \right\} dz_2$$

and

$$\exp \left\{ -\frac{3}{2} t^2 \right\} \int_{\mathbb{R}} |z_1|^n z_1^{n'} |t|^k t^{k'} \exp \left\{ -\frac{1}{2} (z_1^2 - \sqrt{8} z_1 t) \right\} dz_1$$

are bounded by terms of the form  $\text{const} \times t^k e^{-\frac{3}{2} t^2}$  and  $\text{const} \times |t|^{k+k'} e^{-\frac{t^2}{2}}$  respectively; hence we have

$$\int_I g_{1,\ell}^c(t) dt = \int_I p_1^c(t) dt + O(\ell^{-2}), \quad (2.8)$$

where

$$p_1^c(t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} |z_1 t \sqrt{8} - z_1^2 - z_2^2| \exp \left\{ -\frac{3}{2} t^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} dz_1 dz_2.$$

We finally obtain the statement of the present lemma by substituting (2.8) into (2.6).  $\square$

**Remark 2.4.** Introducing the corresponding conditions on the Hessian and following the lines of the previous proof we get the analogous result for extrema and saddles, i.e.,

$$\mathbb{E}[\mathcal{N}_I^a(f_\ell)] = \frac{\ell^2}{2} \int_I p_1^a(t) dt + O(1),$$

where, for  $a = e, s$ , we have

$$p_1^e(t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} |z_1 t \sqrt{8} - z_1^2 - z_2^2| \mathbf{1}_{\{z_1 t \sqrt{8} - z_1^2 - z_2^2 > 0\}} \exp \left\{ -\frac{3}{2} t^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} dz_1 dz_2,$$

and

$$p_1^s(t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} |z_1 t \sqrt{8} - z_1^2 - z_2^2| \mathbf{1}_{\{z_1 t \sqrt{8} - z_1^2 - z_2^2 < 0\}} \exp \left\{ -\frac{3}{2} t^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8} z_1 t) \right\} dz_1 dz_2.$$

We can now prove Proposition 1.1 by deriving explicit expressions for  $p_1^c$ ,  $p_1^e$  and  $p_1^s$  (see also [5] for alternative techniques in a related setting). For this purpose, let  $Y = (Y_1, Y_2, Y_3)$  be a centered jointly Gaussian random vector with covariance matrix

$$\bar{c}_\infty = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Denote by  $\phi_{Y_1+Y_3}$  the probability density of  $Y_1 + Y_3$ . The proof of Proposition 1.1 is given below.

*Proof of Proposition 1.1.* From Lemma 2.3, and in particular from (2.5), we observe that

$$p_1^c(t) = \sqrt{8} \cdot \mathbb{E} \left[ |Y_1 Y_3 - Y_2^2| \middle| Y_1 + Y_3 = \sqrt{8t} \right] \cdot \phi_{Y_1+Y_3}(\sqrt{8t}).$$

Similarly we write

$$\begin{aligned} p_1^e(t) &= \sqrt{8} \cdot \mathbb{E} \left[ |Y_1 Y_3 - Y_2^2| \mathbf{1}_{\{Y_1 Y_3 - Y_2^2 > 0\}} \middle| Y_1 + Y_3 = \sqrt{8t} \right] \cdot \phi_{Y_1+Y_3}(\sqrt{8t}), \\ p_1^s(t) &= \sqrt{8} \cdot \mathbb{E} \left[ |Y_1 Y_3 - Y_2^2| \mathbf{1}_{\{Y_1 Y_3 - Y_2^2 < 0\}} \middle| Y_1 + Y_3 = \sqrt{8t} \right] \cdot \phi_{Y_1+Y_3}(\sqrt{8t}). \end{aligned}$$

Now consider the transformation  $W_1 = Y_1$ ,  $W_2 = Y_2$  and  $W_3 = Y_1 + Y_3$ , i.e. the vector  $W = (W_1, W_2, W_3)$  is given by

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} Y;$$

the covariance matrix  $\Sigma_W$  of  $W$  is

$$\Sigma_W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 8 \end{pmatrix}.$$

Under the obvious notation we write

$$\Sigma_{(W_1, W_2)} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma_{W_3} = 8,$$

so that the conditional distribution of  $(W_1, W_2) | W_3 = \sqrt{8t}$  is Gaussian with covariance matrix

$$\Sigma_{(W_1, W_2) | W_3} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and expectation

$$\mathbb{E}[(W_1, W_2) | W_3 = \sqrt{8t}] = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \frac{1}{8} \sqrt{8t} = \begin{pmatrix} \sqrt{2t} \\ 0 \end{pmatrix}.$$

Hence we have

$$\begin{aligned} \mathbb{E} \left[ |Y_1 Y_3 - Y_2^2| \middle| Y_1 + Y_3 = \sqrt{8t} \right] &= \mathbb{E} \left[ |W_1(W_3 - W_1) - W_2^2| \middle| W_3 = \sqrt{8t} \right] \\ &= \mathbb{E} \left[ |\sqrt{8t}W_1 - W_1^2 - W_2^2| \middle| W_3 = \sqrt{8t} \right] \\ &= \mathbb{E} \left[ |\sqrt{8t}(Z_1 + \sqrt{2t}) - (Z_1 + \sqrt{2t})^2 - Z_2^2| \right] \\ &= \mathbb{E} \left[ |-Z_1^2 - Z_2^2 + 2t^2| \right], \end{aligned}$$

where  $Z_1, Z_2$  denote standard independent Gaussian variables. We can then implement a further change of variable  $\zeta = Z_1^2 + Z_2^2$  with the probability density function  $f_\zeta(z) = \frac{1}{2}e^{-\frac{z}{2}}$ . Hence

$$\mathbb{E} \left[ |-Z_1^2 - Z_2^2 + 2t^2| \right] = \mathbb{E} \left[ |-\zeta + 2t^2| \right] = \frac{1}{2} \int_0^{2t^2} (2t^2 - z)e^{-\frac{z}{2}} dz + \frac{1}{2} \int_{2t^2}^\infty (z - 2t^2)e^{-\frac{z}{2}} dz,$$

where

$$\frac{1}{2} \int_0^{2t^2} (2t^2 - z)e^{-\frac{z}{2}} dz = 2(t^2 - 1) + 2e^{-t^2}.$$

Likewise, with the change of variable  $y = \frac{z-2t^2}{2}$ ,

$$\frac{1}{2} \int_{2t^2}^\infty (z - 2t^2)e^{-\frac{z}{2}} dz = 2e^{-t^2} \int_0^\infty ye^{-y} dy = 2e^{-t^2}.$$

So we have

$$\mathbb{E} [Y_1 Y_3 - Y_2^2 | Y_1 + Y_3 = \sqrt{8t}] = 2(2e^{-t^2} + t^2 - 1),$$

and

$$p_1^c(t) = \sqrt{8} \mathbb{E} [Y_1 Y_3 - Y_2^2 | Y_1 + Y_3 = \sqrt{8t}] \phi_{Y_1+Y_3}(\sqrt{8t}) = \frac{\sqrt{2}}{\sqrt{\pi}} (2e^{-t^2} + t^2 - 1) e^{-\frac{t^2}{2}},$$

in fact, since  $Y_1 + Y_3$  is a centered Gaussian with variance 8, we have

$$\phi_{Y_1+Y_3}(\sqrt{8t}) = \frac{1}{\sqrt{8}\sqrt{2\pi}} e^{-\frac{(\sqrt{8t})^2}{2 \cdot 8}} = \frac{1}{4\sqrt{\pi}} e^{-\frac{t^2}{2}}.$$

Similarly, for the extrema we obtain

$$p_1^e(t) = \sqrt{8} \mathbb{E} [|-Z_1^2 - Z_2^2 + 2t^2| \mathbf{1}_{\{-Z_1^2 - Z_2^2 + 2t^2 > 0\}}] \phi_{Y_1+Y_3}(\sqrt{8t}) = \frac{\sqrt{2}}{\sqrt{\pi}} (e^{-t^2} + t^2 - 1) e^{-\frac{t^2}{2}}.$$

□

**Remark 2.5.** From the expressions for  $p_1^c$  and  $p_1^e$  we immediately obtain an expression for  $p_1^s$ :

$$p_1^s(t) = p_1^c(t) - p_1^e(t) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{3}{2}t^2}.$$

**Remark 2.6.** As mentioned in the introduction, the distribution that we found cannot be viewed as a special case of the general result which has recently been established on the sphere by [15] for real-valued,  $C^2$  Gaussian random fields  $\{f(t), t \in T\}$ ,  $T \subseteq \mathbb{R}^N$ . This is because condition C3' on page 15 of [15] is not satisfied for random spherical harmonics. Indeed, following their notation let us write, for  $i, j, k, l = 1, \dots, N$ ,

$$f_i(t) = \frac{\partial f(t)}{\partial t_i}, \quad f_{i,j}(t) = \frac{\partial^2 f(t)}{\partial t_i \partial t_j},$$

and define  $C', C''$  such that

$$\mathbb{E}[f_i(t) f_j(t)] = C' \delta_{ij}, \quad \mathbb{E}[f_{i,j}(t) f_{k,l}(t)] = C'' (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + (C'' + C') \delta_{ij} \delta_{kl}.$$

Then Condition C3' in [15] states that  $C'' + C' - (C')^2 \geq 0$ ; on the other hand in our case of spherical harmonics we have

$$C'' = \frac{\lambda_\ell^2}{8} - \frac{\lambda_\ell}{4}, \quad C' = \frac{\lambda_\ell}{2},$$

so the quantity  $C'' + C' - (C')^2$  is in this case equal to  $(-\ell^4 - 2\ell^3 + \ell^2 + 2\ell)/8$  which is negative for  $\ell > 1$ . Hence the limiting distribution in [15] Theorem 3.10, which depends on the square root of  $C'' + C' - (C')^2$ , is not applicable in our setting.

## 3 Approximate Kac-Rice for variance computation

### 3.1 On the Kac-Rice formula for computing 2nd (factorial) moment

In the setting of section 2.1,  $\mathcal{E} \subseteq \mathbb{R}^n$  a nice Euclidian domains, and  $g : \mathcal{E} \rightarrow \mathbb{R}^n$  a centred Gaussian random field, a.s. smooth, define the 2-point correlation function of critical points (also referred to as “2nd intensity”)  $K_2 = K_{2;g} : \mathcal{E}^2 \rightarrow \mathbb{R}$

$$K_2(x, y) = \phi_{(g(x), g(y))}(\mathbf{0}, \mathbf{0}) \cdot \mathbb{E}[|\det J_g(x)| \cdot |\det J_g(y)| | g(x) = g(y) = \mathbf{0}].$$

By the virtue of [2], Theorem 11.2.1, the 2nd factorial moment of  $g^{-1}(0)$  is given by

$$\mathbb{E}[\#g^{-1}(0) \cdot (\#g^{-1}(0) - 1)] = \int_{\mathcal{E}^2} K_2(x, y) dx dy,$$

provided that the Gaussian distribution of  $(g(x), g(y)) \in \mathbb{R}^{2n}$  is non-degenerate for all  $(x, y) \in \mathcal{E}^2$  [6]. Moreover, for  $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{E}$  two nice *disjoint* domains, we have

$$\mathbb{E}[\#g^{-1}(0) \cap \mathcal{D}_1 \cdot (\#g^{-1}(0) \cap \mathcal{D}_2)] = \iint_{\mathcal{D}_1 \times \mathcal{D}_2} K_2(x, y) dx dy, \quad (3.1)$$

under the same non-degeneracy assumption for all  $(x, y) \in \mathcal{D}_1 \times \mathcal{D}_2$ .

For the critical points of  $f = f_\ell$  we have

$$K_2(x, y) = K_{2;\ell}(x, y) = \phi_{(\nabla f(x), \nabla f(y))}(\mathbf{0}, \mathbf{0}) \cdot \mathbb{E}[|\det H_f(x)| \cdot |\det H_f(y)| | \nabla f(x) = \nabla f(y) = \mathbf{0}], \quad x \neq \pm y; \quad (3.2)$$

by the isotropy  $K_2(x, y) = K_2(d(x, y))$  depends only on the (spherical) distance between  $x$  and  $y$ . Here [2], Theorem 11.2.1 would yield

$$\mathbb{E}[\mathcal{N}_{\mathbb{R}}^c(f) \cdot (\mathcal{N}_{\mathbb{R}}^c(f) - 1)] = \iint_{\mathcal{S}^2 \times \mathcal{S}^2} K_2(x, y) dx dy, \quad (3.3)$$

provided that for all  $x, y \in \mathcal{S}^2$ , the Gaussian distribution of  $(\nabla f(x), \nabla f(y)) \in \mathbb{R}^4$  is non-degenerate [6].

Unfortunately, we were not able to validate the non-degeneracy assumption due to the technical difficulty of dealing with  $4 \times 4$  matrices depending on both  $x$  and  $y$  (and  $\ell$ ). Instead, we will prove that the (precise) Kac-Rice formula (3.3) holds up to an admissible error, i.e. an *approximate Kac-Rice* (formula (3.5) below), an approach inspired from [25]; our argument is based on a partitioning of the integration domain of (3.3) and applying (3.1) on the valid slices, bounding the contribution of the rest. It is easy to adapt the definition of the 2-point correlation in (3.2) in order to count critical points with values lying in  $I$ , or separate the critical points into extrema and saddles (cf. (3.6) below).

### 3.2 Statement of the principal formula

In this section we shall formulate the approximate Kac-Rice formula which is instrumental for our main result. First we need to introduce some more notation; define the function

$$L_{2,\ell}(\phi; t_1, t_2) = \frac{1}{2} \sin^4 \phi [P_\ell''(\cos \phi)]^2 v_1(t_1, t_2) - \frac{32}{\ell^2} \sin^6 \phi [P_\ell'''(\cos \phi)]^2 v_2(t_1, t_2) + \frac{64}{\ell^4} \sin^8 \phi [P_\ell''''(\cos \phi)]^2 v_3(t_1, t_2), \quad (3.4)$$

where

$$v_1(t_1, t_2) = p_1^c(t_1) p_1^c(t_2),$$

$$v_2(t_1, t_2) = \frac{1}{8^2} \left[ -3p_1^c(t_1) p_1^c(t_2) + \frac{1}{2} p_2^c(t_1) p_1^c(t_2) + \frac{1}{2} p_1^c(t_1) p_2^c(t_2) \right],$$

and

$$v_3(t_1, t_2) = \frac{1}{8^2} \left[ \frac{3}{8} p_1^c(t_1) - \frac{1}{8} p_2^c(t_1) \right] \left[ \frac{3}{8} p_1^c(t_2) - \frac{1}{8} p_2^c(t_2) \right].$$

We are now in a position to formulate the *Approximate Kac-Rice formula*:

**Proposition 3.1.** *For any sufficiently big constant  $C > 0$ , the variance of the critical points number  $\mathcal{N}_I^c(f_\ell)$  satisfies*

$$\text{Var}(\mathcal{N}_I^c(f_\ell)) = \int_{C/\ell}^{\pi/2} \iint_{I \times I} L_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2 \sin \phi d\phi + O(\ell^{5/2}). \quad (3.5)$$

The rest of the present section is dedicated to proving formula (3.5).

### 3.3 Two-point correlation function

Here we formulate some auxiliary results instrumental for our main argument below; our aim is to write an approximate formula for the variance as an integral of the two-point correlation function  $K_{2,\ell}$  defined for  $x \neq \pm y$  by

$$K_{2,\ell}(x, y; t_1, t_2) = \mathbb{E} \left[ |\nabla^2 f_\ell(x)| \cdot |\nabla^2 f_\ell(y)| \mid \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}, f_\ell(x) = t_1, f_\ell(y) = t_2 \right] \cdot \varphi_{x,y,\ell}(t_1, t_2, \mathbf{0}, \mathbf{0}), \quad (3.6)$$

where  $\varphi_{x,y,\ell}(t_1, t_2, \mathbf{0}, \mathbf{0})$  denotes the density of the 6-dimensional vector

$$(f_\ell(x), f_\ell(y), \nabla f_\ell(x), \nabla f_\ell(y))$$

in  $f_\ell(x) = t_1, f_\ell(y) = t_2, \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}$ . Note that, by the isotropy, the function  $K_{2,\ell}$  depends on the points  $x, y$  only via their geodesic distance  $\phi = d(x, y)$ ; by abuse of notation we write

$$K_{2,\ell}(\phi; t_1, t_2) = K_{2,\ell}(x, y; t_1, t_2).$$

Also, we note that  $K_{2,\ell}(\phi; t_1, t_2)$  is everywhere nonnegative. We shall need several results:

**Lemma 3.2.** *There exists a constant  $C > 0$  sufficiently big, such that for every nice domains  $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{S}^2$  with distance  $C/\ell < d(\mathcal{D}_1, \mathcal{D}_2) < \pi - C/\ell$ , we have*

$$\text{Cov}(\mathcal{N}^c(f_\ell; \mathcal{D}_1, I), \mathcal{N}^c(f_\ell; \mathcal{D}_2, I)) = \iint_{\mathcal{D}_1 \times \mathcal{D}_2} \iint_{I \times I} K_{2,\ell}(x, y; t_1, t_2) dt_1 dt_2 dx dy - \mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{D}_1, I)] \mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{D}_2, I)].$$

**Proposition 3.3** (Long-range asymptotics of the 2-point correlation function). *There exists a constant  $C > 0$ , such that for  $C/\ell < d(x, y) < \pi - C/\ell$ , one has:*

$$16\pi^2 K_{2,\ell}(x, y; t_1, t_2) = L_{2,\ell}(x, y; t_1, t_2) + \frac{\ell^4}{4} p_1^c(t_1) p_1^c(t_2) + E_{2,\ell}(x, y; t_1, t_2),$$

where  $L_{2,\ell}$  is as in (3.4) and the error term  $E_{2,\ell}$  is such that

$$\iint_{C/\ell < d(x,y) < \pi - C/\ell} \iint_{\mathbb{R} \times \mathbb{R}} |E_{2,\ell}(x, y; t_1, t_2)| dt_1 dt_2 dx dy = O(\ell^{5/2}). \quad (3.7)$$

**Lemma 3.4.** For any constant  $C > 0$ , we have

$$\int_{\mathbb{R}^2} |L_{2,\ell}(x, y; t_1, t_2)| dt_1 dt_2 = O(\ell^4),$$

uniformly for  $\ell \geq 1$ ,  $d(x, y) > C/\ell$ .

**Proposition 3.5.** There exist a constant  $c > 0$  such that for every nice domain  $\mathcal{D} \subseteq \mathcal{S}^2$  contained in some spherical cap of radius  $c/\ell$ , one has

$$\mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{D}, I) (\mathcal{N}^c(f_\ell; \mathcal{D}, I) - 1)] = \iint_{\mathcal{D} \times \mathcal{D}} \iint_{I \times I} K_{2,\ell}(x, y; t_1, t_2) dt_1 dt_2 dx dy$$

**Lemma 3.6.** There exists a constant  $c > 0$  such that, for  $d(x, y) < c/\ell$ , one has

$$\int_{\mathbb{R}^2} K_{2,\ell}(x, y; t_1, t_2) dt_1 dt_2 = O(\ell^4),$$

where the constant involved in the  $O$ -notation is universal.

The proofs of all the results given in section 3.3 are deferred to section 4.

## 3.4 Proof of Proposition 3.1

### 3.4.1 Partition of the sphere into Voronoi cells

We introduce the following notation for the spherical caps on  $\mathcal{S}^2$ :

$$\mathcal{B}(a, \varepsilon) = \{x \in \mathcal{S}^2 : d(a, x) \leq \varepsilon\}.$$

For any  $\varepsilon > 0$ , we say that  $\Xi_\varepsilon = \{\xi_{1,\varepsilon}, \dots, \xi_{N,\varepsilon}\}$  is a maximal  $\varepsilon$ -net, if  $\xi_{1,\varepsilon}, \dots, \xi_{N,\varepsilon}$  are in  $\mathcal{S}^2$ ,  $\forall i \neq j$  we have  $d(\xi_{i,\varepsilon}, \xi_{j,\varepsilon}) > \varepsilon$  and

$$\forall x \in \mathcal{S}^2, \quad d(x, \Xi_\varepsilon) \leq \varepsilon, \quad \bigcup_{\xi_{i,\varepsilon} \in \Xi_\varepsilon} \mathcal{B}(\xi_{i,\varepsilon}, \varepsilon) = \mathcal{S}^2,$$

$$\forall i \neq j, \quad \mathcal{B}(\xi_{i,\varepsilon}, \varepsilon/2) \cap \mathcal{B}(\xi_{j,\varepsilon}, \varepsilon/2) = \emptyset.$$

Heuristically, an  $\varepsilon$ -net is a grid of point at a distance at least  $\varepsilon$  from each other, and such that any additional point should be within distance  $\varepsilon$  from a point on the grid, see [20]. The number  $N$  of points in a  $\varepsilon$ -net on the sphere is necessarily commensurable to  $1/\varepsilon^2$ ; more precisely we have the following:

$$\frac{4}{\varepsilon^2} \leq N \leq \frac{4}{\varepsilon^2} \pi^2,$$

see [7], Lemma 5. Given an  $\varepsilon$ -net it is natural to partition the sphere into its Voronoi cells, defined below, each associated to a single point on the net; they are disjoint save to boundary overlaps.

**Definition 3.7.** Let  $\Xi_\varepsilon$  be a maximal  $\varepsilon$ -net. For all  $\xi_{i,\varepsilon} \in \Xi_\varepsilon$ , the associated family of Voronoi cells is defined by

$$\mathcal{V}(\xi_{i,\varepsilon}, \varepsilon) = \{x \in \mathcal{S}^2 : \forall j \neq i, d(x, \xi_{i,\varepsilon}) \leq d(x, \xi_{j,\varepsilon})\}.$$

We recall [7] that  $\mathcal{B}(\xi_{i,\varepsilon}, \varepsilon/2) \subseteq \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon) \subseteq \mathcal{B}(\xi_{i,\varepsilon}, \varepsilon)$ , hence  $\text{Vol}(\mathcal{V}(\xi_{i,\varepsilon}, \varepsilon)) \approx \varepsilon^2$ . Let

$$\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon), I) = \#\{x \in \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon) : f_\ell(x) \in I, \nabla f_\ell(x) = 0\}.$$

Note that, almost surely, the summation of the critical points over the Voronoi cells equals the total number of critical points:

$$\mathcal{N}_I^c(f_\ell) = \sum_{\xi_{i,\varepsilon} \in \Xi_\varepsilon} \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon), I).$$

Therefore, we have that

$$\text{Var}(\mathcal{N}_I^c(f_\ell)) = \sum_{\xi_{i,\varepsilon}, \xi_{j,\varepsilon} \in \Xi_\varepsilon} \text{Cov}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon), I), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j,\varepsilon}, \varepsilon), I)). \quad (3.8)$$

### 3.4.2 Proof of Proposition 3.1

We divide the sum in (3.8) into terms with corresponding points at distance  $d(x, y) \in (C/\ell, \pi - C/\ell)$  and  $d(x, y) \in [0, C/\ell] \cup [\pi - C/\ell, \pi]$ . For the former we will exploit the precise Kac-Rice formula below, in contrast to the latter regime whose contribution is bounded; first we define

$$\varepsilon = c/\ell, \quad (3.9)$$

where  $c$  is a positive constant sufficiently small so that, we may apply Proposition 3.5 stating that for every  $i$ ,

$$\begin{aligned} \text{Var}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon, i}), I)) &= \iint_{\mathcal{V}(\xi_{\varepsilon, i}) \times \mathcal{V}(\xi_{\varepsilon, i})} \iint_{I \times I} K_{2, \ell}(x, y; t_1, t_2) dt_1 dt_2 dx dy \\ &\quad + \mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon, i}), I)] - (\mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon, i}), I)])^2; \end{aligned} \quad (3.10)$$

and, by Proposition 1.1 and (3.9),

$$\mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon, i}), I)] \leq \mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{B}(\xi_{\varepsilon, i}; \varepsilon), I)] \leq \pi \varepsilon^2 \ell^2 = O(1). \quad (3.11)$$

Note that in the proof of Proposition 3.5 we exploit the non-degeneracy of the covariance matrix for sufficiently close points  $x, y$  established in Appendix E, whence we can apply Kac-Rice as in formula (3.10). Moreover, by Lemma 3.6, we have

$$\iint_{\mathcal{V}(\xi_{\varepsilon, i}) \times \mathcal{V}(\xi_{\varepsilon, i})} \iint_{I \times I} K_2(x, y; t_1, t_2) dt_1 dt_2 dx dy \leq \ell^4 \cdot (\pi \varepsilon^2)^2 = O(1), \quad (3.12)$$

again by (3.9). Substituting the estimates (3.11) and (3.12) into (3.10) yields

$$\text{Var}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon, i}), I)) = O(1). \quad (3.13)$$

uniformly for all  $\ell, i$ . Using the latter with the Cauchy-Schwartz inequality we may bound each individual summand in the summation (3.8) as

$$|\text{Cov}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i, \varepsilon}), I), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j, \varepsilon}), I))| \leq \sqrt{\text{Var}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i, \varepsilon}), I))} \cdot \sqrt{\text{Var}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j, \varepsilon}), I))} = O(1). \quad (3.14)$$

As there are  $O(\ell^2)$  pairs of Voronoi cells at distance  $d(x, y) \in [0, C/\ell] \cup [\pi - C/\ell, \pi]$ , (3.14) implies that the contribution of this range to (3.8) is

$$\sum_{d(\mathcal{V}(\xi_{i, \varepsilon}), \mathcal{V}(\xi_{j, \varepsilon})) \in [0, C/\ell] \cup [\pi - C/\ell, \pi]} |\text{Cov}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i, \varepsilon}), I), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j, \varepsilon}), I))| = O(\ell^2).$$

For Voronoi cells that are at distance greater than  $C/\ell$  and smaller than  $\pi - C/\ell$ , we may use the standard Kac-Rice formula in Lemma 3.2. Applying Kac-Rice individually on each of the pairs  $(\mathcal{V}(\xi_{i, \varepsilon}), \mathcal{V}(\xi_{j, \varepsilon}))$  such that  $d(\mathcal{V}(\xi_{i, \varepsilon}), \mathcal{V}(\xi_{j, \varepsilon})) \in (C/\ell, \pi - C/\ell)$ , we have

$$\begin{aligned} &\sum_{d(\mathcal{V}(\xi_{i, \varepsilon}), \mathcal{V}(\xi_{j, \varepsilon})) \in (C/\ell, \pi - C/\ell)} [\text{Cov}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i, \varepsilon}), I), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j, \varepsilon}), I)) + \mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i, \varepsilon}), I)] \mathbb{E}[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j, \varepsilon}), I)]] \\ &= \int_{\mathcal{W}} \iint_{I \times I} K_{2, \ell}(x, y; t_1, t_2) dt_1 dt_2 dx dy \end{aligned} \quad (3.15)$$

where

$$\mathcal{W} = \bigcup_{d(\mathcal{V}(\xi_{i, \varepsilon}), \mathcal{V}(\xi_{j, \varepsilon})) \in (C/\ell, \pi - C/\ell)} \mathcal{V}(\xi_{i, \varepsilon}) \times \mathcal{V}(\xi_{j, \varepsilon}),$$

is the union of all tuples of points belonging to Voronoi cells far from degeneracy regions. Now with the help of Proposition 3.3, we may write this summation (3.15) as

$$\begin{aligned} &\sum_{d(\mathcal{V}(\xi_{i, \varepsilon}), \mathcal{V}(\xi_{j, \varepsilon})) \in (C/\ell, \pi - C/\ell)} \text{Cov}(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i, \varepsilon}), I), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j, \varepsilon}), I)) \\ &= \int_{C/\ell}^{\pi/2} \iint_{I \times I} L_{2, \ell}(\phi; t_1, t_2) dt_1 dt_2 \sin \phi d\phi + \int_{\mathcal{W}} \iint_{I \times I} E_{2, \ell}(x, y; t_1, t_2) dt_1 dt_2 dx dy \\ &= \int_{C/\ell}^{\pi/2} \iint_{I \times I} L_{2, \ell}(\phi; t_1, t_2) dt_1 dt_2 \sin \phi d\phi + O(\ell^{5/2}), \end{aligned}$$

as claimed; note that the integrand function  $L_{2, \ell}$  is even, which allows to take  $\pi/2$  rather than  $\pi - C/\ell$  as the integration extreme, due to symmetry.

## 4 Asymptotics of the two-point correlation function

Here we prove the auxiliary results in Section 3.3.

### 4.1 Long-range asymptotics for the two-point correlation function

In this section we prove Lemma 3.2, Lemma 3.4, and Proposition 3.3.

### 4.1.1 Conditional covariance matrix

For  $x, y \in \mathcal{S}^2$  we define the following random vector

$$Z_{\ell;x,y} = (\nabla f_\ell(x), \nabla f_\ell(y), \nabla^2 f_\ell(x), \nabla^2 f_\ell(y)).$$

To write the Kac-Rice formula in coordinate system, given  $x, y \in \mathcal{S}^2$ , we consider two local orthogonal frames  $\{e_1^x, e_2^x\}$  and  $\{e_1^y, e_2^y\}$  defined in some neighbourhood of  $x$  and  $y$  respectively. This gives rise to the (local) identifications

$$T_x(\mathcal{S}^2) \cong \mathbb{R}^2 \cong T_y(\mathcal{S}^2), \quad (4.1)$$

so that, as discussed earlier we do not have to work with probability densities defined on tangent planes which depend on the points  $x$  and  $y$  respectively. Under the identification (4.1) the random vector  $Z_{\ell;x,y}$  is a  $\mathbb{R}^{10}$  centered Gaussian random vector. By isotropy, it is convenient to perform our computations along a specific geodesic. In particular, we focus on the equatorial line  $x = (\pi/2, \phi)$ ,  $y = (\pi/2, 0)$ , and we work with the orthogonal frames

$$\left\{ e_1^x = \frac{\partial}{\partial \theta_x}, e_2^x = \frac{\partial}{\partial \varphi_x} \right\}, \quad \left\{ e_1^y = \frac{\partial}{\partial \theta_y}, e_2^y = \frac{\partial}{\partial \varphi_y} \right\}. \quad (4.2)$$

In Appendix A we compute the entries of the  $10 \times 10$  covariance matrix of  $Z_{\ell;x,y}$ , i.e.,

$$\Sigma_\ell(\phi) = \begin{pmatrix} A_\ell(\phi) & B_\ell(\phi) \\ B_\ell^t(\phi) & C_\ell(\phi) \end{pmatrix},$$

where  $A_\ell(\phi)$ ,  $B_\ell(\phi)$  and  $C_\ell(\phi)$  are the covariance matrices of the gradient terms, first and second order derivatives, and second order derivatives, respectively. In Appendix B we compute the conditional covariance matrix  $\Omega_\ell(\phi)$  of the random vector

$$(\nabla^2 f_\ell(x), \nabla^2 f_\ell(y) | \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}),$$

i.e.,

$$\Omega_\ell(\phi) = C_\ell(\phi) - B_\ell(\phi)^t A_\ell(\phi)^{-1} B_\ell(\phi). \quad (4.3)$$

After scaling, we obtain

$$\Delta_\ell(\phi) = \frac{8}{\lambda_\ell^2} \Omega_\ell(\phi) = \begin{pmatrix} \Delta_{1,\ell}(\phi) & \Delta_{2,\ell}(\phi) \\ \Delta_{2,\ell}(\phi) & \Delta_{1,\ell}(\phi) \end{pmatrix}, \quad (4.4)$$

where the entries  $a_{i,\ell}(\phi)$ ,  $i = 1, \dots, 8$ , of  $\Delta_{1,\ell}(\phi)$  and  $\Delta_{2,\ell}(\phi)$  are defined by

$$\begin{aligned} \Delta_{1,\ell}(\phi) &= \begin{pmatrix} 3 - \frac{16\beta_{2,\ell}^2(\phi)}{\lambda_\ell(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))} - \frac{2}{\lambda_\ell} & 0 & 1 - \frac{16\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{\lambda_\ell(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))} + \frac{2}{\lambda_\ell} \\ 0 & 1 - \frac{16\beta_{1,\ell}^2(\phi)}{\lambda_\ell(\lambda_\ell^2 - 4\alpha_{1,\ell}^2(\phi))} & 0 \\ 1 - \frac{16\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{\lambda_\ell(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))} + \frac{2}{\lambda_\ell} & 0 & 3 - \frac{16\beta_{3,\ell}^2(\phi)}{\lambda_\ell(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))} - \frac{2}{\lambda_\ell} \end{pmatrix} \\ &= \begin{pmatrix} 3 + a_{1,\ell}(\phi) & 0 & 1 + a_{4,\ell}(\phi) \\ 0 & 1 + a_{2,\ell}(\phi) & 0 \\ 1 + a_{4,\ell}(\phi) & 0 & 3 + a_{3,\ell}(\phi) \end{pmatrix}, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \Delta_{2,\ell}(\phi) &= \begin{pmatrix} 8 \frac{\gamma_{1,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta_{2,\ell}^2(\phi)}{4\alpha_{2,\ell}^2(\phi) - \lambda_\ell^2}}{\lambda_\ell^2} & 0 & 8 \frac{\gamma_{3,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{4\alpha_{2,\ell}^2(\phi) - \lambda_\ell^2}}{\lambda_\ell^2} \\ 0 & 8 \frac{\gamma_{2,\ell}(\phi) + \frac{4\alpha_{1,\ell}(\phi)\beta_{1,\ell}^2(\phi)}{4\alpha_{1,\ell}^2(\phi) - \lambda_\ell^2}}{\lambda_\ell^2} & 0 \\ 8 \frac{\gamma_{3,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{4\alpha_{2,\ell}^2(\phi) - \lambda_\ell^2}}{\lambda_\ell^2} & 0 & 8 \frac{\gamma_{4,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta_{3,\ell}^2(\phi)}{4\alpha_{2,\ell}^2(\phi) - \lambda_\ell^2}}{\lambda_\ell^2} \end{pmatrix} \\ &= \begin{pmatrix} a_{5,\ell}(\phi) & 0 & a_{8,\ell}(\phi) \\ 0 & a_{6,\ell}(\phi) & 0 \\ a_{8,\ell}(\phi) & 0 & a_{7,\ell}(\phi) \end{pmatrix}, \end{aligned} \quad (4.6)$$

with

$$\begin{aligned} \alpha_{1,\ell}(\phi) &= P'_\ell(\cos \phi), \\ \alpha_{2,\ell}(\phi) &= -\sin^2 \phi P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi), \end{aligned}$$

$$\begin{aligned} \beta_{1,\ell}(\phi) &= \sin \phi P''_\ell(\cos \phi), \\ \beta_{2,\ell}(\phi) &= \sin \phi \cos \phi P''_\ell(\cos \phi) + \sin \phi P'_\ell(\cos \phi), \\ \beta_{3,\ell}(\phi) &= -\sin^3 \phi P'''_\ell(\cos \phi) + 3 \sin \phi \cos \phi P''_\ell(\cos \phi) + \sin \phi P'_\ell(\cos \phi), \end{aligned}$$

and

$$\begin{aligned} \gamma_{1,\ell}(\phi) &= (2 + \cos^2 \phi) P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi), \\ \gamma_{2,\ell}(\phi) &= -\sin^2 \phi P'''_\ell(\cos \phi) + \cos \phi P''_\ell(\cos \phi), \\ \gamma_{3,\ell}(\phi) &= -\sin^2 \phi \cos \phi P'''_\ell(\cos \phi) + (-2 \sin^2 \phi + \cos^2 \phi) P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi), \\ \gamma_{4,\ell}(\phi) &= \sin^4 \phi P''''_\ell(\cos \phi) - 6 \sin^2 \phi \cos \phi P'''_\ell(\cos \phi) + (-4 \sin^2 \phi + 3 \cos^2 \phi) P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi). \end{aligned}$$



*Proof of Lemma 3.2.* Lemma 3.2 follows from Theorem 6.3 in [6] since for  $C > 0$  sufficiently big the covariance matrix  $A_\ell(\phi)$  is nonsingular.  $\square$

### 4.1.2 Proof of Proposition 3.3

First we recall that the two-point correlation function is given by (3.6), a Gaussian expectation related to a vector with covariance matrix  $\Sigma_\ell(\phi)$ . To understand the asymptotic behaviour of the function  $K_{2,\ell}$  we will have to provide a more explicit formula by using the orthogonal frames (4.2) chosen above. Then we will have a frame-dependent formula for the two-point correlation function depending on the geodesic distance  $\phi = d(x, y)$ . To define  $K_{2,\ell}(\phi; t_1, t_2)$  pointwise (almost everywhere), we write, for any two intervals  $I_1, I_2$ :

$$\begin{aligned} & \iint_{I_1 \times I_2} K_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2 \\ &= \frac{1}{(2\pi)^2 \sqrt{\det(A_\ell(\phi))}} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\zeta_{x,1} \zeta_{x,3} - \zeta_{x,2}^2| \cdot |\zeta_{y,1} \zeta_{y,3} - \zeta_{y,2}^2| \cdot \mathbb{1}_{\left\{ \frac{\zeta_{x,1} + \zeta_{x,3}}{\lambda_\ell} \in I_1 \right\}} \cdot \mathbb{1}_{\left\{ \frac{\zeta_{y,1} + \zeta_{y,3}}{\lambda_\ell} \in I_2 \right\}} \\ & \times \frac{1}{(2\pi)^3} \exp \left\{ -\frac{1}{2} (\zeta_{x,1}, \zeta_{x,2}, \zeta_{x,3}, \zeta_{y,1}, \zeta_{y,2}, \zeta_{y,3}) \Omega_\ell(\phi)^{-1} (\zeta_{x,1}, \zeta_{x,2}, \zeta_{x,3}, \zeta_{y,1}, \zeta_{y,2}, \zeta_{y,3})^t \right\} \\ & \times \frac{1}{\sqrt{\det(\Omega_\ell(\phi))}} d\zeta_{x,1} d\zeta_{x,2} d\zeta_{x,3} d\zeta_{y,1} d\zeta_{y,2} d\zeta_{y,3}. \end{aligned}$$

Here we exploited the linear dependence (2.2). Now we scale the variables: for  $i = 1, 2, 3$  introduce  $\tilde{\zeta}_{x,i}$  and  $\tilde{\zeta}_{y,i}$ :

$$\zeta_{x,i} = \frac{\lambda_\ell}{\sqrt{8}} \tilde{\zeta}_{x,i}, \quad \zeta_{y,i} = \frac{\lambda_\ell}{\sqrt{8}} \tilde{\zeta}_{y,i}.$$

With the new variables we have

$$\begin{aligned} & \iint_{I_1 \times I_2} K_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2 = \frac{1}{(2\pi)^2 \sqrt{\det(A_\ell(\phi))}} \frac{\lambda_\ell^4}{8^2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\tilde{\zeta}_{x,1} \tilde{\zeta}_{x,3} - \tilde{\zeta}_{x,2}^2| \cdot |\tilde{\zeta}_{y,1} \tilde{\zeta}_{y,3} - \tilde{\zeta}_{y,2}^2| \\ & \times \mathbb{1}_{\left\{ \frac{\tilde{\zeta}_{x,1} + \tilde{\zeta}_{x,3}}{\sqrt{8}} \in I_1 \right\}} \mathbb{1}_{\left\{ \frac{\tilde{\zeta}_{y,1} + \tilde{\zeta}_{y,3}}{\sqrt{8}} \in I_2 \right\}} \\ & \times \frac{1}{(2\pi)^3} \exp \left\{ -\frac{1}{2} (\tilde{\zeta}_{x,1}, \tilde{\zeta}_{x,2}, \tilde{\zeta}_{x,3}, \tilde{\zeta}_{y,1}, \tilde{\zeta}_{y,2}, \tilde{\zeta}_{y,3}) \Delta_\ell(\phi)^{-1} (\tilde{\zeta}_{x,1}, \tilde{\zeta}_{x,2}, \tilde{\zeta}_{x,3}, \tilde{\zeta}_{y,1}, \tilde{\zeta}_{y,2}, \tilde{\zeta}_{y,3})^t \right\} \\ & \times \frac{1}{\sqrt{\det(\Delta_\ell(\phi))}} d\tilde{\zeta}_{x,1} d\tilde{\zeta}_{x,2} d\tilde{\zeta}_{x,3} d\tilde{\zeta}_{y,1} d\tilde{\zeta}_{y,2} d\tilde{\zeta}_{y,3}. \end{aligned}$$

Making the substitutions

$$\begin{aligned} \begin{pmatrix} \tilde{\zeta}_{x,1} \\ \tilde{\zeta}_{x,2} \\ \tilde{\zeta}_{x,3} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & \sqrt{8} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ t_1 \end{pmatrix}, \\ \begin{pmatrix} \tilde{\zeta}_{y,1} \\ \tilde{\zeta}_{y,2} \\ \tilde{\zeta}_{y,3} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & \sqrt{8} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ t_2 \end{pmatrix}, \end{aligned}$$

we obtain

$$\begin{aligned} & K_{2,\ell}(\phi; t_1, t_2) \\ &= \frac{1}{(2\pi)^2 \sqrt{\det(A_\ell(\phi))}} \frac{\lambda_\ell^4}{8^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| z_1 (\sqrt{8}t_1 - z_1) - z_2^2 \right| \cdot \left| w_1 (\sqrt{8}t_2 - w_1) - w_2^2 \right| \\ & \times \frac{1}{(2\pi)^3} \exp \left\{ -\frac{1}{2} v_{t_1, t_2}(z_1, z_2, w_1, w_2) \Delta_\ell(\phi)^{-1} v_{t_1, t_2}(z_1, z_2, w_1, w_2)^t \right\} \frac{1}{\sqrt{\det(\Delta_\ell(\phi))}} 8 dz_1 dz_2 dw_1 dw_2, \end{aligned} \quad (4.7)$$

where

$$v_{t_1, t_2}(z_1, z_2, w_1, w_2) = \begin{pmatrix} z_1, z_2, \sqrt{8}t_1 - z_1, w_1, w_2, \sqrt{8}t_2 - w_1 \end{pmatrix}.$$

Now let us observe that for the determinant of  $A_\ell(\phi)$  we have

$$(2\pi)^2 \sqrt{\det(A_\ell(\phi))} = (2\pi)^2 \sqrt{\frac{1}{16} (\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi)) (\lambda_\ell^2 - 4\alpha_{1,\ell}^2(\phi))} = \pi^2 \sqrt{(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi)) (\lambda_\ell^2 - 4\alpha_{1,\ell}^2(\phi))}. \quad (4.8)$$

At this point we consider the 2-point correlation function (4.7) as a function of the perturbing elements  $\{a_{i,\ell}(\phi)\}$ ,  $i = 1, \dots, 8$  defined in (4.5) and (4.6); to this end it is convenient to collect the elements into a single vector:

$$\mathbf{a} = \mathbf{a}_\ell(\phi) = (a_{1,\ell}(\phi), a_{2,\ell}(\phi), a_{3,\ell}(\phi), a_{4,\ell}(\phi), a_{5,\ell}(\phi), a_{6,\ell}(\phi), a_{7,\ell}(\phi), a_{8,\ell}(\phi)),$$

and write

$$\Delta(\mathbf{a}) = \begin{pmatrix} \Delta_1(\mathbf{a}) & \Delta_2(\mathbf{a}) \\ \Delta_2(\mathbf{a}) & \Delta_1(\mathbf{a}) \end{pmatrix},$$

where

$$\Delta_1(\mathbf{a}) = \begin{pmatrix} 3 + a_1 & 0 & 1 + a_4 \\ 0 & 1 + a_2 & 0 \\ 1 + a_4 & 0 & 3 + a_3 \end{pmatrix} \quad \text{and} \quad \Delta_2(\mathbf{a}) = \begin{pmatrix} a_5 & 0 & a_8 \\ 0 & a_6 & 0 \\ a_8 & 0 & a_7 \end{pmatrix}.$$

With this slight abuse of notation it is evident that

$$\Delta_\ell(\phi) = \Delta(\mathbf{a}_\ell(\phi)). \quad (4.9)$$

We then introduce the functions

$$\hat{q}(\mathbf{a}; t_1, t_2; z_1, z_2, w_1, w_2) = \frac{1}{\sqrt{\det(\Delta(\mathbf{a}))}} \exp \left\{ -\frac{1}{2} v_{t_1, t_2}(z_1, z_2, w_1, w_2) \Delta(\mathbf{a})^{-1} v_{t_1, t_2}(z_1, z_2, w_1, w_2)^t \right\},$$

and

$$q(\mathbf{a}; t_1, t_2) = \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| z_1 \sqrt{8t_1} - z_1^2 - z_2^2 \right| \cdot \left| w_1 \sqrt{8t_2} - w_1^2 - w_2^2 \right| \\ \times \hat{q}(\mathbf{a}; t_1, t_2; z_1, z_2, w_1, w_2) dz_1 dz_2 dw_1 dw_2.$$

Bearing in mind (4.7), (4.8) and (4.9) we have

$$K_{2,\ell}(\phi; t_1, t_2) = \frac{\lambda_\ell^4}{8\pi^2 \sqrt{(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))(\lambda_\ell^2 - 4\alpha_{1,\ell}^2(\phi))}} q(\mathbf{a}_\ell(\phi); t_1, t_2). \quad (4.10)$$

**Remark 4.1.** We note that

$$\iint_{I \times I} q(\mathbf{0}; t_1, t_2) dt_1 dt_2 = \frac{1}{8} \frac{1}{(2\pi)^3} \left[ \int_I dt \int_{\mathbb{R}^2} \left| z_1 \sqrt{8t} - z_1^2 - z_2^2 \right| \exp \left\{ -\frac{3}{2} t^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8t} z_1) \right\} dz_1 dz_2 \right]^2 \\ = \frac{1}{8} \left[ \int_I p_1^c(t) dt \right]^2.$$

Our next step is to study the asymptotic behaviour of the functions  $q$ , by means of a Taylor expansion around the origin  $\mathbf{a} = \mathbf{0}$ .

### Taylor expansion of the two-point correlation function

To understand the behaviour of the two-point correlation function in the long-range regime, we have to investigate the high energy asymptotic behaviour of the integrals

$$8\pi^2 \int_{C/\ell}^{\pi - C/\ell} K_{2,\ell}(\phi; t_1, t_2) \sin \phi d\phi = \lambda_\ell^2 \int_{C/\ell}^{\pi - C/\ell} \frac{\sin \phi}{\sqrt{(1 - 4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1 - 4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}} q(\mathbf{a}; t_1, t_2) d\phi, \quad (4.11)$$

recalling (4.10). In the range  $\phi \in (C/\ell, \pi - C/\ell)$  the covariance matrices we shall deal with are perturbations of the values they would have under independence between values at the points  $x, y$ . We can hence exploit perturbation theory (see [18], Theorem 1.5) to yield that the Gaussian expectations are analytic functions of the covariance matrix elements. Hence  $q(\cdot; t_1, t_2)$  is a smooth function, defined on some neighbourhood of the origin (its arguments are uniformly small for  $\phi \in (C/\ell, \pi - C/\ell)$ ), and we can expand it into a finite Taylor polynomial around the origin, as follows:

$$q(\mathbf{a}; t_1, t_2) = q(\mathbf{0}; t_1, t_2) + \sum_{i=1}^8 a_i \frac{\partial}{\partial a_i} q(\mathbf{0}; t_1, t_2) + \sum_{i \neq j} a_i a_j \frac{\partial^2}{\partial a_i \partial a_j} q(\mathbf{0}; t_1, t_2) \\ + \frac{1}{2} \sum_{i=1}^8 a_i^2 \frac{\partial^2}{\partial a_i^2} q(\mathbf{0}; t_1, t_2) + O(w(t_1, t_2) \|\mathbf{a}\|^3), \quad \text{as } \|\mathbf{a}\| \rightarrow 0, \quad (4.12)$$

for some  $w(t_1, t_2) \geq 0$ . Below we will evaluate all the derivatives involved in (4.12), and show in addition that

$$w(\cdot, \cdot) \in L^1(\mathbb{R}^2), \quad (4.13)$$

important for integrating (4.12) w.r.t.  $t$ . We will see that the variance of critical points is dominated by three terms of order  $O(\ell^3)$  in (4.12).

### Asymptotic behaviour of the integrals

We shall now introduce the following notation: for  $i, j = 1, 2, \dots, 8$ ,

$$A_{0,\ell} = \int_{C/\ell}^{\pi/2} \frac{\sin \phi}{\sqrt{(1 - 4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1 - 4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}} d\phi,$$

$$A_{i,\ell} = \int_{C/\ell}^{\pi/2} \frac{a_{i,\ell}(\phi)}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}} \sin \phi \, d\phi,$$

$$A_{ij,\ell} = \int_{C/\ell}^{\pi/2} \frac{a_{i,\ell}(\phi)a_{j,\ell}(\phi)}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}} \sin \phi \, d\phi.$$

As a consequence, we may write

$$\begin{aligned} 8\pi^2 \int_{C/\ell}^{\pi-C/\ell} K_{2,\ell}(\phi; t_1, t_2) \sin \phi \, d\phi &= 2\lambda_\ell^2 \left\{ A_{0,\ell} q(\mathbf{0}; t_1, t_2) + \sum_{i=1}^8 A_{i,\ell} \left[ \frac{\partial}{\partial a_i} q(\mathbf{a}; t_1, t_2) \right]_{\mathbf{a}=\mathbf{0}} \right. \\ &\quad + \frac{1}{2} \sum_{i,j=1}^8 A_{ij,\ell} \left[ \frac{\partial^2}{\partial a_i \partial a_j} q(\mathbf{a}; t_1, t_2) \right]_{\mathbf{a}=\mathbf{0}} \\ &\quad \left. + \int_{C/\ell}^{\pi/2} \frac{\sin \phi}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}} O(\|\mathbf{a}_\ell(\phi)\|^3) \, d\phi \right\}. \end{aligned}$$

We shall now study the high frequency asymptotic behaviour of the terms  $A_{0,\ell}$ ,  $A_{i,\ell}$ , and  $A_{ij,\ell}$ , for  $i, j = 1, 2, \dots, 8$ . First we shall show that the first term in the expansion cancels out with the squared expectation. More precisely, we shall prove the following lemma:

**Lemma 4.2.** *As  $\ell \rightarrow \infty$ , we have*

$$2\lambda_\ell^2 A_{0,\ell} \iint_{I \times I} q(\mathbf{0}; t_1, t_2) dt_1 dt_2 - (\mathbb{E}[\mathcal{N}_I^c(f_\ell)])^2 = \frac{\ell^3}{4} \left[ \int_I p_1^c(t) dt \right]^2 + O(\ell^2 \log \ell).$$

We shall then show that all linear terms  $A_{i,\ell}$  with  $i \neq 3$  are indeed subdominant. The bound  $O(\ell^{-3/2})$  may not be optimal; it is probably possible to improve it to  $O(\log \ell / \ell^2)$  by working somewhat harder; we postpone this analysis to future research.

**Lemma 4.3.** *As  $\ell \rightarrow \infty$ , for all  $i = 1, \dots, 8$ , such that  $i \neq 3$ , we have*

$$A_{i,\ell} = O(\ell^{-3/2}),$$

whereas for  $i = 3$ , we get

$$A_{3,\ell} = -8\ell^{-1} + O(\ell^{-2} \log \ell).$$

Finally, in the next lemma we study the asymptotic behaviour of the second order terms  $A_{ij,\ell}$ , for  $i, j = 1, \dots, 8$ ; again they are all subdominant, but for the term with index (7, 7):

**Lemma 4.4.** *As  $\ell \rightarrow \infty$ , for  $(i, j) \neq (7, 7)$ , we have*

$$A_{ij,\ell} = O(\ell^{-2} \log \ell),$$

and for  $(i, j) = (7, 7)$  we have

$$A_{77,\ell} = 32\ell^{-1} + O(\ell^{-2} \log \ell).$$

The proofs of Lemma 4.2, Lemma 4.3 and Lemma 4.4 are in Appendix D. We have proved that

$$\begin{aligned} &8\pi^2 \int_{C/\ell}^{\pi-C/\ell} \iint_{I \times I} K_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2 \sin \phi \, d\phi - (\mathbb{E}[\mathcal{N}_I^c(f_\ell)])^2 \\ &= \ell^3 \left\{ \frac{1}{4} \left[ \int_I p_1^c(t) dt \right]^2 - 16 \iint_{I \times I} \left[ \frac{\partial}{\partial a_3} q(\mathbf{a}; t_1, t_2) \right]_{\mathbf{a}=\mathbf{0}} dt_1 dt_2 \right. \\ &\quad \left. + 32 \iint_{I \times I} \left[ \frac{\partial^2}{\partial a_7^2} q(\mathbf{a}; t_1, t_2) \right]_{\mathbf{a}=\mathbf{0}} dt_1 dt_2 \right\} + O(\ell^{5/2}). \end{aligned} \tag{4.14}$$

We may rewrite the latter result as

$$8\pi^2 \int_{C/\ell}^{\pi-C/\ell} \iint_{I \times I} K_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2 \sin \phi \, d\phi - (\mathbb{E}[\mathcal{N}_I^c(f_\ell)])^2 = \int_{C/\ell}^{\pi/2} \iint_{I \times I} L_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2 \sin \phi \, d\phi + O(\ell^{5/2}),$$

with  $L_{2,\ell}$  defined by (3.4). In fact, once we have isolated the dominant terms in (4.14) (see Appendix D), we can write them, as function of  $\phi$ , in the form given in (3.4). In Appendix D, the remainder  $E_{2,\ell}(\phi; t_1, t_2)$  is computed explicitly, and the bound (3.7) is also established. To prove the statement of Proposition 3.3 we now compute the values of the derivatives of  $\hat{q}$  to obtain  $v_2$  and  $v_3$  in (3.4).

### Derivatives of $\hat{q}$

The relevant derivatives can be evaluated explicitly as follows. Let

$$\mathbf{a}_i = (0, \dots, 0, a_i, 0, \dots, 0),$$

for  $i = 1, \dots, 8$ ; recall that  $\hat{q}(\mathbf{a}; t_1, t_2; z_1, z_2, w_1, w_2)$  is an analytic function of the elements of the vector  $\mathbf{a}$ , see [18], Theorem 1.5, so that we can write

$$\left[ \frac{\partial^j}{\partial a_i^j} \hat{q}(\mathbf{a}; t_1, t_2; z_1, z_2, w_1, w_2) \right]_{\mathbf{a}=\mathbf{0}} = \left[ \frac{\partial^j}{\partial a_i^j} \hat{q}_\ell(\mathbf{a}_i; t_1, t_2; z_1, z_2, w_1, w_2) \right]_{\mathbf{a}_i=\mathbf{0}},$$

for  $j = 1, 2$ . Using Leibnitz integral rule and some tedious but mechanical computations, we obtain

$$\begin{aligned} \left[ \frac{\partial}{\partial a_3} q(\mathbf{a}_3; t_1, t_2) \right]_{\mathbf{a}_3=\mathbf{0}} &= \frac{1}{2 \cdot 8^2} \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |z_1 \sqrt{8t_1 - z_1^2 - z_2^2}| \cdot |w_1 \sqrt{8t_2 - w_1^2 - w_2^2}| \\ &\quad \times \exp \left\{ -\frac{3}{2} t_1^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8t_1} z_1) \right\} \\ &\quad \times \exp \left\{ -\frac{3}{2} t_2^2 \right\} \exp \left\{ -\frac{1}{2} (w_1^2 + w_2^2 - \sqrt{8t_2} w_1) \right\} \\ &\quad \times \left[ -6 + (3t_1 - \sqrt{2} z_1)^2 + (3t_2 - \sqrt{2} w_1)^2 \right] dz_1 dz_2 dw_1 dw_2, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \left[ \frac{\partial^2}{\partial a_7^2} q(\mathbf{a}_7; t_1, t_2) \right]_{\mathbf{a}_7=\mathbf{0}} &= \frac{1}{8^3} \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |z_1 \sqrt{8t_1 - z_1^2 - z_2^2}| \cdot |w_1 \sqrt{8t_2 - w_1^2 - w_2^2}| \\ &\quad \times \exp \left\{ -\frac{3}{2} t_1^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8t_1} z_1) \right\} \\ &\quad \times \exp \left\{ -\frac{3}{2} t_2^2 \right\} \exp \left\{ -\frac{1}{2} (w_1^2 + w_2^2 - \sqrt{8t_2} w_1) \right\} \\ &\quad \times \left[ 3 - (3t_1 - \sqrt{2} z_1)^2 \right] \cdot \left[ 3 - (3t_2 - \sqrt{2} w_1)^2 \right] dz_1 dz_2 dw_1 dw_2. \end{aligned} \quad (4.16)$$

Performing similar computations reveals that on a sufficiently small neighborhood of  $\mathbf{a} = \mathbf{0}$  in  $\mathbb{R}^8$  the function  $w(t_1, t_2)$  appearing in (4.12) has Gaussian tails w.r.t.  $(t_1, t_2)$ , and hence (4.13), i.e.  $w(\cdot, \cdot)$  belongs to  $L^1(\mathbb{R}^2)$ . Indeed, the inverse matrix  $\Delta(\mathbf{a})^{-1}$  appearing in the definition of  $\hat{q}$  is a perturbation of the identity, whence a Gaussian term in  $(t_1, t_2)$  factors out from its derivatives of every order. This concludes the proof of Proposition 3.3.

### 4.1.3 Proof of Lemma 3.4

To prove Lemma 3.4, it is sufficient to show that for  $\phi > C/\ell$  we have

$$\sin^4 \phi [P_\ell''(\cos \phi)]^2, \quad \frac{1}{\ell^2} \sin^6 \phi [P_\ell'''(\cos \phi)]^2, \quad \frac{1}{\ell^4} \sin^8 \phi [P_\ell''''(\cos \phi)]^2 = O(\ell^4),$$

uniformly in  $\ell$ . To establish these bounds, we note that, from Lemma C.3,

$$\begin{aligned} \sin^4 \phi [P_\ell''(\cos \phi)]^2 &= \sin^4 \phi \frac{\ell^3}{\sin^5 \phi} + O(\ell^3) = O(\ell^4), \\ \frac{1}{\ell^2} \sin^6 \phi [P_\ell'''(\cos \phi)]^2 &= \frac{\sin^6 \phi}{\ell^2} \frac{\ell^5}{\sin^7 \phi} + O(\ell^3) = O(\ell^4), \\ \frac{1}{\ell^4} \sin^8 \phi [P_\ell''''(\cos \phi)]^2 &= \frac{\sin^8 \phi}{\ell^4} \frac{\ell^7}{\sin^9 \phi} + O(\ell^3) = O(\ell^4), \end{aligned}$$

as claimed.

## 4.2 Short-range application of Kac-Rice

In this section, we provide the proofs of Proposition 3.5 and Lemma 3.6.

### 4.2.1 Conditional covariance matrix

With the scaling

$$\phi = \psi/\ell,$$

the matrix  $\Sigma_\ell$  becomes

$$\Sigma_\ell(\psi) = \begin{pmatrix} A_\ell(\psi) & B_\ell(\psi) \\ B_\ell^t(\psi) & C_\ell(\psi) \end{pmatrix},$$

where

$$A_\ell(\psi)_{4 \times 4} = \begin{pmatrix} \frac{\ell+1}{2\ell} & 0 & \alpha_{1,\ell}(\psi) & 0 \\ 0 & \frac{\ell+1}{2\ell} & 0 & \alpha_{2,\ell}(\psi) \\ \alpha_{1,\ell}(\psi) & 0 & \frac{\ell+1}{2\ell} & 0 \\ 0 & \alpha_{2,\ell}(\psi) & 0 & \frac{\ell+1}{2\ell} \end{pmatrix},$$

and the elements of the off-diagonal  $2 \times 2$  terms of  $A$  are given by

$$\begin{aligned}\alpha_{1,\ell}(\psi) &= \frac{1}{\ell^2} P'_\ell(\cos(\psi/\ell)), \\ \alpha_{2,\ell}(\psi) &= -\frac{1}{\ell^2} \sin^2(\psi/\ell) P''_\ell(\cos(\psi/\ell)) + \frac{1}{\ell^2} \cos(\psi/\ell) P'_\ell(\cos(\psi/\ell)).\end{aligned}$$

The matrix  $B_\ell(\psi)$  is given by

$$B_\ell(\psi)_{4 \times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & \beta_{1,\ell}(\psi) & 0 \\ 0 & 0 & 0 & \beta_{2,\ell}(\psi) & 0 & \beta_{3,\ell}(\psi) \\ 0 & -\beta_{1,\ell}(\psi) & 0 & 0 & 0 & 0 \\ -\beta_{2,\ell}(\psi) & 0 & -\beta_{3,\ell}(\psi) & 0 & 0 & 0 \end{pmatrix},$$

with elements

$$\begin{aligned}\beta_{1,\ell}(\psi) &= \frac{1}{\ell^3} \sin(\psi/\ell) P''_\ell(\cos(\psi/\ell)), \\ \beta_{2,\ell}(\psi) &= \sin(\psi/\ell) \cos(\psi/\ell) \frac{1}{\ell^3} P''_\ell(\cos(\psi/\ell)) + \sin(\psi/\ell) \frac{1}{\ell^3} P'_\ell(\cos(\psi/\ell)), \\ \beta_{3,\ell}(\psi) &= -\sin^3(\psi/\ell) \frac{1}{\ell^3} P'''_\ell(\cos(\psi/\ell)) + 3 \sin(\psi/\ell) \cos(\psi/\ell) \frac{1}{\ell^3} P''_\ell(\cos(\psi/\ell)) + \sin(\psi/\ell) \frac{1}{\ell^3} P'_\ell(\cos(\psi/\ell)).\end{aligned}$$

Finally, for the matrix  $C_\ell(\psi)$ , we have

$$C_\ell(\psi)_{6 \times 6} = \begin{pmatrix} c_\ell(0) & c_\ell(\psi) \\ c_\ell(\psi) & c_\ell(0) \end{pmatrix},$$

where

$$c_\ell(0) = \begin{pmatrix} \frac{\ell(1+3\ell(\ell+2))-2}{8\ell^3} & 0 & \frac{(\ell+1)(\ell^2+\ell+2)}{8\ell^3} \\ 0 & \frac{(\ell+1)(\ell^2+\ell-2)}{8\ell^3} & 0 \\ \frac{(\ell+1)(\ell^2+\ell+2)}{8\ell^3} & 0 & \frac{\ell(1+3\ell(\ell+2))-2}{8\ell^3} \end{pmatrix}.$$

The conditional covariance matrix  $\Delta_\ell(\psi)$  is given by

$$\Delta_\ell(\psi) = C_\ell(\psi) - B_\ell^t(\psi) A_\ell^{-1}(\psi) B_\ell(\psi) = \begin{pmatrix} \Delta_{1,\ell}(\psi) & \Delta_{2,\ell}(\psi) \\ \Delta_{2,\ell}(\psi) & \Delta_{1,\ell}(\psi) \end{pmatrix};$$

we shall use below only the explicit expression only for  $\Delta_{1,\ell}(\psi)$  which is given by

$$\begin{aligned}\Delta_{1,\ell}(\psi) &= \begin{pmatrix} \frac{2\ell(\ell+1)\beta_{2,\ell}(\psi)^2}{4\ell^2\alpha_{2,\ell}(\psi)^2 - (\ell+1)^2} + \frac{\ell(3\ell(\ell+2)+1)-2}{8\ell^3} & 0 & \frac{(\ell+1)\left(\frac{16\beta_{2,\ell}(\psi)\beta_{3,\ell}(\psi)\ell^4}{4\ell^2\alpha_{2,\ell}(\psi)^2 - (\ell+1)^2} + \ell^2 + \ell + 2\right)}{8\ell^3} \\ 0 & \frac{2\ell(\ell+1)\beta_{1,\ell}(\psi)^2}{4\ell^2\alpha_{1,\ell}(\psi)^2 - (\ell+1)^2} + \frac{(\ell-1)(\ell+1)(\ell+2)}{8\ell^3} & 0 \\ \frac{(\ell+1)\left(\frac{16\beta_{2,\ell}(\psi)\beta_{3,\ell}(\psi)\ell^4}{4\ell^2\alpha_{2,\ell}(\psi)^2 - (\ell+1)^2} + \ell^2 + \ell + 2\right)}{8\ell^3} & 0 & \frac{2\ell(\ell+1)\beta_{3,\ell}(\psi)^2}{4\ell^2\alpha_{2,\ell}(\psi)^2 - (\ell+1)^2} + \frac{\ell(3\ell(\ell+2)+1)-2}{8\ell^3} \end{pmatrix}.\end{aligned}$$

*Proof of Proposition 3.5.* The statement of Proposition 3.5 is an application of Theorem 11.5.1 in [2], provided that we check that of the  $4 \times 4$  covariance matrix  $A_\ell(\psi)$  of the first and the second order derivatives of  $f_\ell$  is nonsingular for sufficiently close  $x, y$  satisfying  $d(x, y) < c/\ell$  with  $c > 0$  sufficiently small. The latter is shown in Appendix E, with the aid of specialized computer software, by Taylor expanding the relevant determinant around the diagonal  $x = y$ .  $\square$

## 4.2.2 Proof of Lemma 3.6

We now need to study the high energy asymptotic behaviour of the kernel

$$\iint_{\mathbb{R} \times \mathbb{R}} K_{2,\ell}(\phi; t_1, t_2) dt_1 dt_2$$

for  $\phi < c/\ell$ . In view of the equality (4.7) we may proceed directly to bounding

$$\frac{\ell^4}{\sqrt{\det(A_\ell(\psi))}} \iint_{\mathbb{R} \times \mathbb{R}} \rho_\ell(\psi; t_1, t_2) dt_1 dt_2, \quad (4.17)$$

where  $\psi = \ell\phi$ , and

$$\begin{aligned}\rho_\ell(\psi; t_1, t_2) &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| z_1 t_1 - z_1^2 - z_2^2 \right| \cdot \left| w_1 t_2 - w_1^2 - w_2^2 \right| \frac{1}{\sqrt{\det(\Delta_\ell(\psi))}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (z_1, z_2, t_1 - z_1, w_1, w_2, t_2 - w_1) \Delta_\ell(\psi)^{-1} (z_1, z_2, t_1 - z_1, w_1, w_2, t_2 - w_1)^t \right\} dz_1 dz_2 dw_1 dw_2.\end{aligned}$$

Let  $a_{i,\ell}(\psi)$  be the entries of the matrix  $\Delta_{1,\ell}(\psi)$  defined by

$$\Delta_{1,\ell}(\psi) = \begin{pmatrix} 3 + a_{1,\ell}(\psi) & 0 & 1 + a_{4,\ell}(\psi) \\ 0 & 1 + a_{2,\ell}(\psi) & 0 \\ 1 + a_{4,\ell}(\psi) & 0 & 3 + a_{3,\ell}(\psi) \end{pmatrix}. \quad (4.18)$$

The main technical difficulty of this proof is that, as Lemma 4.5 below shows, the term  $\sqrt{\det(A_\ell(\psi))}$  appearing in the denominator is of the order  $\psi^2$  around the origin. As a consequence, a very delicate bound must be established on  $\rho_\ell(\psi; t_1, t_2)$  to ensure the convergence and the boundedness of the integral with respect to  $\psi$ .

**Lemma 4.5.** *Uniformly in  $\ell$ , we have for  $\psi \leq c$*

$$\det(A_\ell(\psi)) \geq c\psi^4, \quad (4.19)$$

for some universal  $c > 0$ .

**Lemma 4.6.** *Uniformly in  $\ell$ , we have*

$$\iint_{\mathbb{R} \times \mathbb{R}} \rho_\ell(\psi; t_1, t_2) dt_1 dt_2 = O(\psi^2). \quad (4.20)$$

*Proof of Proposition 3.6 assuming Lemmas 4.5 and 4.6.* Since we have shown that the numerator is uniformly bounded by terms of order  $\psi^2$ , the statement of Lemma 3.6 follows at once upon substituting the estimates (4.19) and (4.20) into (4.17).  $\square$

*Proof of Lemma 4.5.* First we note that

$$\begin{aligned} \det(A_\ell(\psi)) &= \frac{1}{16\ell^4} ((\ell + 1)^2 - 4\ell^2 \alpha_{1,\ell}^2(\psi)) ((\ell + 1)^2 - 4\ell^2 \alpha_{2,\ell}^2(\psi)) \\ &\geq \frac{1}{16} (1 - 2\ell \alpha_{2,\ell}(\psi) / (\ell + 1)) (1 - 2\ell \alpha_{1,\ell}(\psi) / (\ell + 1)). \end{aligned}$$

By exploiting the Taylor expansions given in Appendix E, for  $\alpha_{1,\ell}(\psi)$  and  $\alpha_{2,\ell}(\psi)$ , we obtain

$$(1 - 2\ell \alpha_{1,\ell}(\psi) / (\ell + 1)) (1 - 2\ell \alpha_{2,\ell}(\psi) / (\ell + 1)) = \left( \frac{3}{64} + \frac{3}{32\ell} - \frac{5}{64\ell^2} - \frac{1}{8\ell^3} + \frac{1}{16\ell^4} \right) \psi^4 + O(\psi^6),$$

which certainly implies the statement of the present lemma.  $\square$

*Proof of Lemma 4.6.* We can bound (4.20) by

$$\iint_{\mathbb{R} \times \mathbb{R}} \rho_\ell(\psi; t_1, t_2) dt_1 dt_2 \leq \mathbb{E} \left[ |X_1 X_3| |Y_1 Y_3| + |X_1 X_3| Y_2^2 + |Y_1 Y_3| X_2^2 + Y_2^2 X_2^2 \right],$$

where the random vector  $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$  is a multivariate Gaussian with zero mean and covariance matrix  $\Delta_\ell(\psi)$ . Applying repeatedly Cauchy-Schwarz and recalling that for Gaussian random variables we have  $\mathbb{E}[X_1^4] = 3(\mathbb{E}[X_1^2])^2$ , we obtain

$$\begin{aligned} \mathbb{E}[|X_1 X_3| |Y_1 Y_3|] &\leq \mathbb{E}[|X_1| |X_3| |Y_1| |Y_3|] \leq (\mathbb{E}[X_1^2 X_3^2] \cdot \mathbb{E}[Y_1^2 Y_3^2])^{1/2} \leq (\mathbb{E}[X_1^4] \mathbb{E}[X_3^4] \mathbb{E}[Y_1^4] \mathbb{E}[Y_3^4])^{1/4} \\ &= 3(\mathbb{E}[X_1^2] \mathbb{E}[X_3^2] \mathbb{E}[Y_1^2] \mathbb{E}[Y_3^2])^{1/2} = 3(3 + a_{1,\ell}(\psi))(3 + a_{3,\ell}(\psi)), \end{aligned}$$

where in the last equality we write explicitly the variances by replacing the elements of the covariance matrix  $\Delta_{1,\ell}(\psi)$ . Analogously we get

$$\begin{aligned} \mathbb{E}[|X_1 X_3| Y_2^2] &\leq 3(\mathbb{E}[X_1^2] \mathbb{E}[X_3^2])^{1/2} \mathbb{E}[Y_2^2] = 3((3 + a_{1,\ell}(\psi))(3 + a_{3,\ell}(\psi/\ell)))^{1/2} (1 + a_{2,\ell}(\psi)), \\ \mathbb{E}[|Y_1 Y_3| X_2^2] &\leq 3(\mathbb{E}[Y_1^2] \mathbb{E}[Y_3^2])^{1/2} \mathbb{E}[X_2^2] = 3((3 + a_{1,\ell}(\psi))(3 + a_{3,\ell}(\psi)))^{1/2} (1 + a_{2,\ell}(\psi)), \\ \mathbb{E}[Y_2^2 X_2^2] &\leq 3\mathbb{E}[Y_2^2] \mathbb{E}[X_2^2] = 3(1 + a_{2,\ell}(\psi))^2. \end{aligned}$$

Collecting the previous results, and after some direct calculations, we obtain

$$\begin{aligned} &\mathbb{E}[|X_1 X_3| \cdot |Y_1 Y_3| + |X_1 X_3| \cdot Y_2^2 + |Y_1 Y_3| \cdot X_2^2 + Y_2^2 X_2^2] \\ &\leq 3(3 + a_{1,\ell}(\psi))(3 + a_{3,\ell}(\psi)) + 6((3 + a_{1,\ell}(\psi))(3 + a_{3,\ell}(\psi/\ell)))^{1/2} (1 + a_{2,\ell}(\psi)) + 3(1 + a_{2,\ell}(\psi))^2 \\ &= \frac{3(\ell - 2)(\ell - 1)(\ell + 1)^3(\ell + 2)(\ell + 3)(\ell^2 + \ell - 2)\psi^2}{128\ell^7(3\ell(\ell + 1) - 2)} + O(\psi^3). \end{aligned}$$

$\square$

*Proof of Proposition 3.6.* We have hence shown that the numerator is uniformly bounded by terms of order  $\psi^2$ . The statement of Lemma 3.6 follows at once upon substituting the estimates (4.19) and (4.20) into (4.17).  $\square$

**Remark 4.7.** The geometric intuition of the previous result can be explained as follows. We impose the condition that two critical points are at (scaled) distance  $\psi$ , and study the asymptotic behaviour of the Hessian for small values of  $\psi$ . In this regime, the two critical points collide, and hence the Hessian approaches zero with locally quadratic behaviour. This is exactly the term we were looking for to cancel the determinant term of order  $\psi^{-2}$  in Lemma 4.6.

## 5 Asymptotic expression for the variance

Here we find the analytic expression for the variance stated in Theorem 1.2.

*Proof of Theorem 1.2.* In view of (4.14), (4.15) and (4.16) we can write

$$\text{Var}(\mathcal{N}_I^c(f_\ell)) = \frac{\ell^3}{4} \left( \left[ \int_I p_1^c(t) dt \right]^2 - \iint_{I \times I} g_2^c(t_1, t_2) dt_1 dt_2 + 16 \left[ \int_I g_3^c(t) dt \right]^2 \right) + O(\ell^{5/2}),$$

where

$$\begin{aligned} g_2^c(t_1, t_2) &= \frac{1}{2} \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |z_1 \sqrt{8t_1} - z_1^2 - z_2^2| \exp \left\{ -\frac{3}{2} t_1^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8t_1} z_1) \right\} \\ &\quad \times |w_1 \sqrt{8t_2} - w_1^2 - w_2^2| \exp \left\{ -\frac{3}{2} t_2^2 \right\} \exp \left\{ -\frac{1}{2} (w_1^2 + w_2^2 - \sqrt{8t_2} w_1) \right\} \\ &\quad \times \left[ -6 + (3t_1 - \sqrt{2} z_1)^2 + (3t_2 - \sqrt{2} w_1)^2 \right] dz_1 dz_2 dw_1 dw_2, \end{aligned}$$

and

$$g_3^c(t) = \frac{1}{8} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} |z_1 \sqrt{8t} - z_1^2 - z_2^2| \exp \left\{ -\frac{3}{2} t^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8t} z_1) \right\} \left[ 3 - (3t - \sqrt{2} z_1)^2 \right] dz_1 dz_2.$$

Let

$$k(z_1, z_2, t) = |z_1 \sqrt{8t} - z_1^2 - z_2^2| \exp \left\{ -\frac{3}{2} t^2 \right\} \exp \left\{ -\frac{1}{2} (z_1^2 + z_2^2 - \sqrt{8t} z_1) \right\};$$

we note that

$$\begin{aligned} g_2^c(t_1, t_2) &= -\frac{6}{2} p_1^c(t_1) p_1^c(t_2) \\ &\quad + \frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} (3t_1 - \sqrt{2} z_1)^2 k(z_1, z_2, t_1) dz_1 dz_2 \int_{\mathbb{R}^2} k(w_1, w_2, t_2) dw_1 dw_2 \\ &\quad + \frac{1}{2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} k(z_1, z_2, t_1) dz_1 dz_2 \int_{\mathbb{R}^2} (3t_2 - \sqrt{2} w_1)^2 k(w_1, w_2, t_2) dw_1 dw_2 \\ &= -3 p_1^c(t_1) p_1^c(t_2) + \frac{1}{2} p_2^c(t_1) p_1^c(t_2) + \frac{1}{2} p_1^c(t_1) p_2^c(t_2), \end{aligned}$$

where

$$p_2^c(t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} (3t - \sqrt{2} z_1)^2 k(z_1, z_2, t) dz_1 dz_2.$$

Note also that

$$g_3^c(t) = \frac{1}{8} \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} k(z_1, z_2, t) \left[ 3 - (3t - \sqrt{2} z_1)^2 \right] dz_1 dz_2 = \frac{3}{8} p_1^c(t) - \frac{1}{8} p_2^c(t).$$

Hence

$$\begin{aligned} \left[ \int_I p_1^c(t) dt \right]^2 - \iint_{I \times I} g_2^c(t_1, t_2) dt_1 dt_2 + 16 \left[ \int_I g_3^c(t) dt \right]^2 &= \left[ \int_I p_1^c(t) dt \right]^2 + 3 \left[ \int_I p_1^c(t) dt \right]^2 - \int_I p_1^c(t) dt \int_I p_2^c(t) dt \\ &\quad + \frac{9}{4} \left[ \int_I p_1^c(t) dt \right]^2 + \frac{1}{4} \left[ \int_I p_2^c(t) dt \right]^2 - \frac{3}{2} \int_I p_1^c(t) dt \int_I p_2^c(t) dt = \frac{1}{4} \left[ 5 \int_I p_1^c(t) dt - \int_I p_2^c(t) dt \right]^2, \end{aligned}$$

i.e.

$$\text{Var}(\mathcal{N}_I^c(f_\ell)) = \frac{\ell^3}{16} \left[ 5 \int_I p_1^c(t) dt - \int_I p_2^c(t) dt \right]^2 + O(\ell^{5/2})$$

with

$$\begin{aligned} p_1^c(t) &= \sqrt{8} \mathbb{E} \left[ |Y_1 Y_3 - Y_2^2| \mid Y_1 + Y_3 = \sqrt{8t} \right] \phi_{Y_1 + Y_3}(\sqrt{8t}) = \frac{\sqrt{2}}{\sqrt{\pi}} (2e^{-t^2} + t^2 - 1) e^{-\frac{t^2}{2}}, \\ p_2^c(t) &= \sqrt{8} \mathbb{E} \left[ (3t - \sqrt{2} Y_1)^2 |Y_1 Y_3 - Y_2^2| \mid Y_1 + Y_3 = \sqrt{8t} \right] \phi_{Y_1 + Y_3}(\sqrt{8t}). \end{aligned}$$

We now derive an analytic expression for  $p_2^c(t)$ . As in the proof of Proposition 1.1 we first write  $p_2^c(t)$  as

$$\begin{aligned} p_2^c(t) &= \sqrt{8} \mathbb{E} \left[ \left( 3t - \sqrt{2} (Z_1 + \sqrt{2}t) \right)^2 \cdot \left| \sqrt{8t} (Z_1 + \sqrt{2}t) - (Z_1 + \sqrt{2}t)^2 - Z_2^2 \right| \right] \cdot \phi_{Y_1 + Y_3}(\sqrt{8t}) \\ &= \sqrt{8} \mathbb{E} \left[ \left( t - \sqrt{2} Z_1 \right)^2 \cdot \left| -Z_1^2 - Z_2^2 + 2t^2 \right| \right] \cdot \phi_{Y_1 + Y_3}(\sqrt{8t}), \end{aligned}$$

where  $Z_1, Z_2$  denote standard independent Gaussian variables. Now

$$\phi_{Y_1+Y_3}(\sqrt{8}t) = \frac{1}{4\sqrt{\pi}} e^{-\frac{t^2}{2}},$$

and we need to compute

$$\mathbb{E}[(t - \sqrt{2}Z_1)^2 | -Z_1^2 - Z_2^2 + 2t^2].$$

The joint density function of  $\xi = Z_1$  and  $\zeta = Z_1^2 + Z_2^2$  is given by

$$f_{(\xi, \zeta)}(u, v) = \frac{\partial^2}{\partial u \partial v} \mathbb{P}[\xi < u, \zeta < v] = \frac{\partial^2}{\partial u \partial v} \mathbb{P}[Z_1 < u, Z_1^2 + Z_2^2 < v] = \frac{1}{2\pi} \frac{\partial^2}{\partial u \partial v} \int_{\substack{z_1 < u \\ 0 \leq z_1^2 + z_2^2 < v}} e^{-\frac{z_1^2 + z_2^2}{2}} dz_1 dz_2,$$

i.e.,

$$f_{(\xi, \zeta)}(u, v) = \begin{cases} 0 & u \leq -\sqrt{v}, \\ \frac{1}{2\pi} \frac{\partial^2}{\partial u \partial v} \int_{-\sqrt{v}}^u dz_1 \int_{-\sqrt{v-z_1^2}}^{\sqrt{v-z_1^2}} e^{-\frac{z_1^2 + z_2^2}{2}} dz_2 & -\sqrt{v} < u \leq 0, \\ \frac{1}{2\pi} \frac{\partial^2}{\partial u \partial v} \int_{-\sqrt{v}}^u dz_1 \int_{-\sqrt{v-z_1^2}}^{\sqrt{v-z_1^2}} e^{-\frac{z_1^2 + z_2^2}{2}} dz_2 & 0 < u < \sqrt{v}, \\ 0 & u \geq \sqrt{v}, \end{cases}$$

$$= \frac{1}{2\pi} \frac{e^{-\frac{v}{2}}}{\sqrt{v-u^2}} \mathbf{1}_{\{v \geq 0, u \in (-\sqrt{v}, \sqrt{v})\}}.$$

Then

$$\begin{aligned} \mathbb{E}[(t - \sqrt{2}\xi)^2 | -\zeta + 2t^2] &= \int_0^\infty dv \int_{-\sqrt{v}}^{\sqrt{v}} (t - \sqrt{2}u)^2 \cdot |-v + 2t^2| \frac{1}{2\pi} \frac{e^{-\frac{v}{2}}}{\sqrt{v-u^2}} du \\ &= \int_0^{2t^2} dv \int_{-\sqrt{v}}^{\sqrt{v}} (t - \sqrt{2}u)^2 (-v + 2t^2) \frac{1}{2\pi} \frac{e^{-\frac{v}{2}}}{\sqrt{v-u^2}} du \\ &\quad + \int_{2t^2}^\infty dv \int_{-\sqrt{v}}^{\sqrt{v}} (t - \sqrt{2}u)^2 (v - 2t^2) \frac{1}{2\pi} \frac{e^{-\frac{v}{2}}}{\sqrt{v-u^2}} du. \end{aligned}$$

Now

$$\int \frac{(t - \sqrt{2}u)^2}{\sqrt{v-u^2}} du = t^2 \int \frac{1}{\sqrt{v-u^2}} du - 2\sqrt{2}t \int \frac{u}{\sqrt{v-u^2}} du + 2 \int \frac{u^2}{\sqrt{v-u^2}} du,$$

where for any symmetric interval

$$\int_{-\sqrt{v}}^{\sqrt{v}} \frac{u}{\sqrt{v-u^2}} du = 0,$$

while

$$\int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{v-u^2}} du = \left[ -\arctan\left(\frac{u\sqrt{v-u^2}}{u^2-v}\right) \right]_{-\sqrt{v}}^{\sqrt{v}} = \pi,$$

and

$$\int_{-\sqrt{v}}^{\sqrt{v}} \frac{u^2}{\sqrt{v-u^2}} du = \left[ \frac{1}{2} \left( -u\sqrt{v-u^2} + v \arctan\left(\frac{u}{v-u^2}\right) \right) \right]_{-\sqrt{v}}^{\sqrt{v}} = \frac{\pi v}{2}.$$

Hence we have

$$\begin{aligned} \mathbb{E}[(t - \sqrt{2}\xi)^2 | -\zeta + 2t^2] &= \frac{1}{2} \int_0^{2t^2} e^{-\frac{v}{2}} (-v + 2t^2)(v + t^2) dv + \frac{1}{2} \int_{2t^2}^\infty e^{-\frac{v}{2}} (v - 2t^2)(v + t^2) dv \\ &= 2[-4 + t^2 + t^4 + e^{-t^2}(4 + 3t^2)] + 2e^{-t^2}(4 + 3t^2), \end{aligned}$$

which leads to

$$p_2^c(t) = [-4 + t^2 + t^4 + e^{-t^2} 2(4 + 3t^2)] \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{t^2}{2}}.$$

Finally,

$$5p_1^c(t) - p_2^c(t) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{3}{2}t^2} [2 - 6t^2 - e^{t^2}(1 - 4t^2 + t^4)].$$



Similarly, for the extrema, we have

$$\begin{aligned} p_2^e(t) &= \sqrt{8}\mathbb{E}[(3t - \sqrt{2}Y_1)|Y_1Y_3 - Y_2^2|\mathbb{1}_{\{Y_1Y_3 - Y_2^2 > 0\}}|Y_1 + Y_3 = \sqrt{8}t]\phi_{Y_1+Y_3}(\sqrt{8}t) \\ &= \sqrt{8}\mathbb{E}[(t - \sqrt{2}\xi)^2|-\xi + 2t^2|\mathbb{1}_{\{-\xi + 2t^2 > 0\}}]\phi_{Y_1+Y_3}(\sqrt{8}t), \end{aligned}$$

where

$$\mathbb{E}[(t - \sqrt{2}\xi)^2|-\xi + 2t^2|\mathbb{1}_{\{-\xi + 2t^2 > 0\}}] = 2[-4 + t^2 + t^4 + e^{-t^2}(4 + 3t^2)],$$

so that

$$p_2^e(t) = \left[-4 + t^2 + t^4 + e^{-t^2}(4 + 3t^2)\right] \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{t^2}{2}}$$

and

$$5p_1^e(t) - p_2^e(t) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{3}{2}t^2} [1 - 3t^2 - e^{t^2}(1 - 4t^2 + t^4)].$$

Finally, applying the same methods for the saddles, we have

$$\begin{aligned} \mathbb{E}[(3t - \sqrt{2}Y_1)|Y_1Y_3 - Y_2^2|\mathbb{1}_{\{Y_1Y_3 - Y_2^2 < 0\}}|Y_1 + Y_3 = \sqrt{8}t] &= \frac{1}{2} \int_{2t^2}^{\infty} (t^2 + v)(v - 2t^2)e^{-\frac{v}{2}} dv \\ &= 2(4 + 3t^2)e^{-t^2}, \end{aligned}$$

yielding

$$p_2^s(t) = \frac{\sqrt{2}}{\sqrt{\pi}} (4 + 3t^2)e^{-\frac{3}{2}t^2},$$

and

$$5p_1^s(t) - p_2^s(t) = \frac{\sqrt{2}}{\sqrt{\pi}} (1 - 3t^2)e^{-\frac{3}{2}t^2},$$

as claimed. □

## 6 Convergence of empirical measures

The following auxiliary lemma shows that the empirical measures under random and deterministic normalizations are asymptotically equivalent, uniformly in  $z$ .

**Lemma 6.1.** *For all  $\varepsilon > 0$ , as  $\ell \rightarrow \infty$ ,*

$$\mathbb{P}\{\sup_z |F_\ell(z) - F_\ell^*(z)| \geq \varepsilon\} \rightarrow 0,$$

*Proof.* We first note that

$$|F_\ell(z) - F_\ell^*(z)| = F_\ell^*(z) \left| 1 - \frac{F_\ell(z)}{F_\ell^*(z)} \right| = F_\ell^*(z) \left| 1 - \frac{\mathcal{N}_{\mathbb{R}}^c(f_\ell)}{\mathbb{E}[\mathcal{N}_{\mathbb{R}}^c(f_\ell)]} \right| \leq \left| 1 - \frac{\mathcal{N}_{\mathbb{R}}^c(f_\ell)}{\mathbb{E}[\mathcal{N}_{\mathbb{R}}^c(f_\ell)]} \right|.$$

The statement of the present lemma follows by observing that from Proposition 1.1 and Theorem 1.2 we have that

$$\frac{\mathcal{N}_{\mathbb{R}}^c(f_\ell)}{\mathbb{E}[\mathcal{N}_{\mathbb{R}}^c(f_\ell)]}$$

is a random variable with unitary mean and variance  $O(\ell^{-1})$ . □

We can now provide the proof of Proposition 1.5.

*Proof of Proposition 1.5.* We first note that in view Lemma 6.1 proving Proposition 1.5 is equivalent to proving that for all  $\varepsilon > 0$  and  $\delta > 0$  there exists  $\ell_{\varepsilon, \delta}$  such that for all  $\ell > \ell_{\varepsilon, \delta}$  we have

$$\mathbb{P}\{\sup_z |F_\ell(z) - \Phi_\infty(z)| > \varepsilon\} \leq \delta.$$

Fix  $\varepsilon > 0$  and choose  $K_\varepsilon > 0$  sufficiently big such that  $1/K_\varepsilon < \varepsilon/2$ . Now we define the partitions

$$-\infty = x_1 \leq x_2 \leq \dots \leq x_{K_\varepsilon} = \infty,$$

such that

$$\Phi_\infty(x_{k+1}) - \Phi_\infty(x_k) < \frac{\varepsilon}{2}.$$

For every  $z$  there exist  $i_{K_\varepsilon}^-(z)$  and  $i_{K_\varepsilon}^+(z) \in \{x_1, x_2, \dots, x_{K_\varepsilon}\}$  such that

$$z \in (i_{K_\varepsilon}^-(z), i_{K_\varepsilon}^+(z)).$$

Then, since  $F_\ell(z)$  and  $\Phi_\infty(z)$  are both non decreasing in  $z$ , we have

$$\begin{aligned} F_\ell(z) - \Phi_\infty(z) &\leq F(i_{K_\varepsilon}^+(z); f_\ell) - \Phi_\infty(i_{K_\varepsilon}^+(z)) + \frac{\varepsilon}{2}, \\ F_\ell(z) - \Phi_\infty(z) &\geq F(i_{K_\varepsilon}^-(z); f_\ell) - \Phi_\infty(i_{K_\varepsilon}^-(z)) - \frac{\varepsilon}{2}, \end{aligned}$$

so that

$$\sup_z |F_\ell(z) - \Phi_\infty(z)| \leq \max_{k=1, \dots, K_\varepsilon} |F_\ell(x_k) - \Phi_\infty(x_k)| + \frac{\varepsilon}{2}.$$

Then

$$\mathbb{P}\{\sup_z |F_\ell(z) - \Phi_\infty(z)| > \varepsilon\} \leq \mathbb{P}\left\{\max_{k=1, \dots, K_\varepsilon} |F_\ell(x_k) - \Phi_\infty(x_k)| > \frac{\varepsilon}{2}\right\}$$

and, in view of Proposition 1.2, each of the  $K_\varepsilon$  random variables

$$|F_\ell(x_k) - \Phi_\infty(x_k)|,$$

converges in probability to zero.  $\square$

## Appendices

### A Evaluation of covariance matrices

In this section we compute the covariance matrix  $\Sigma_\ell(x, y)$  for the 10-dimensional random vector  $Z_{\ell; x, y}$ , which combines the gradient and the elements of the Hessian evaluated at  $x, y$ .  $\Sigma_\ell(x, y)$  depends only on the geodesic distance  $\phi = d(x, y)$ , so, abusing notation, we shall write  $\Sigma_\ell(x, y) = \Sigma_\ell(\phi)$  whenever convenient, and similarly for the other functions we shall deal with. The computations are quite lengthy, but they do not require sophisticated arguments, other than iterative derivations of Legendre polynomials. It is convenient to write these matrices in block-diagonal form, i.e.

$$\Sigma_\ell(\phi) = \begin{pmatrix} A_\ell(\phi) & B_\ell(\phi) \\ B_\ell^t(\phi) & C_\ell(\phi) \end{pmatrix}.$$

In particular the  $A_\ell$  component collects the variances of the gradient terms, and it is given by

$$\begin{aligned} A_\ell(x, y)_{4 \times 4} &= \mathbb{E}\left[\begin{pmatrix} \nabla f_\ell(\bar{x}) \\ \nabla f_\ell(\bar{y}) \end{pmatrix} \begin{pmatrix} \nabla f_\ell(x) & \nabla f_\ell(y) \end{pmatrix}\right] \Big|_{x=\bar{x}, y=\bar{y}} \\ &= \begin{pmatrix} a_\ell(x, x) & a_\ell(x, y) \\ a_\ell(y, x) & a_\ell(y, y) \end{pmatrix}, \end{aligned}$$

where

$$a_\ell(x, x) = \begin{pmatrix} e_1^{\bar{x}} e_1^x r_\ell(\bar{x}, x) & e_1^{\bar{x}} e_2^x r_\ell(\bar{x}, x) \\ e_2^{\bar{x}} e_1^x r_\ell(\bar{x}, x) & e_2^{\bar{x}} e_2^x r_\ell(\bar{x}, x) \end{pmatrix} \Big|_{x=\bar{x}}, \quad a_\ell(x, y) = \begin{pmatrix} e_1^{\bar{x}} e_1^y r_\ell(\bar{x}, y) & e_1^{\bar{x}} e_2^y r_\ell(\bar{x}, y) \\ e_2^{\bar{x}} e_1^y r_\ell(\bar{x}, y) & e_2^{\bar{x}} e_2^y r_\ell(\bar{x}, y) \end{pmatrix} \Big|_{x=\bar{x}},$$

and

$$r_\ell(x, y) = \mathbb{E}[f_\ell(x)f_\ell(y)] = P_\ell(\cos d(x, y)),$$

with  $h(x, y) = \cos d(x, y) = \cos \theta_x \cos \theta_y + \sin \theta_x \sin \theta_y \cos(\varphi_x - \varphi_y)$ . Then, for example, computing explicitly the derivatives, we have

$$e_1^{\bar{x}} e_1^x r_\ell(\bar{x}, x) = P_\ell''(h(\bar{x}, x)) \frac{\partial}{\partial \theta_x} h(\bar{x}, x) \frac{\partial}{\partial \theta_x} h(\bar{x}, x) + P_\ell'(h(\bar{x}, x)) \frac{\partial}{\partial \theta_x} \frac{\partial}{\partial \theta_x} h(\bar{x}, x),$$

where

$$\begin{aligned} \frac{\partial}{\partial \theta_x} h(\bar{x}, x) &= -\cos \theta_x \sin \theta_{\bar{x}} + \cos \theta_{\bar{x}} \sin \theta_x \cos(\varphi_x - \varphi_{\bar{x}}) \Big|_{x=\bar{x}=(\pi/2, \varphi_x)} = 0, \\ \frac{\partial}{\partial \theta_x} h(\bar{x}, x) &= -\cos \theta_{\bar{x}} \sin \theta_x + \cos \theta_x \sin \theta_{\bar{x}} \cos(\varphi_x - \varphi_{\bar{x}}) \Big|_{x=\bar{x}=(\pi/2, \varphi_x)} = 0, \\ \frac{\partial}{\partial \theta_x} \frac{\partial}{\partial \theta_x} h(\bar{x}, x) &= \sin \theta_{\bar{x}} \sin \theta_x + \cos \theta_x \cos \theta_{\bar{x}} \cos(\varphi_x - \varphi_{\bar{x}}) \Big|_{x=\bar{x}=(\pi/2, \varphi_x)} = 1. \end{aligned}$$

We write then

$$a_\ell(x, x) \Big|_{x=(\pi/2, \varphi_x)} = a_\ell(y, y) \Big|_{y=(\pi/2, 0)} = \begin{pmatrix} P_\ell'(1) & 0 \\ 0 & P_\ell'(1) \end{pmatrix},$$

and, again with some slight abuse of notation,

$$a_\ell(x, y)|_{x=(\pi/2, \varphi_x), y=(\pi/2, 0)} = a_\ell(y, x)|_{x=(\pi/2, \varphi_x), y=(\pi/2, 0)} = \begin{pmatrix} \alpha_{1, \ell}(\phi) & 0 \\ 0 & \alpha_{2, \ell}(\phi) \end{pmatrix},$$

where as we recalled before  $\phi = d(x, y)$  and

$$\begin{aligned} \alpha_{1, \ell}(\phi) &= P'_\ell(\cos \phi), \\ \alpha_{2, \ell}(\phi) &= -\sin^2 \phi P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi). \end{aligned}$$

Now recall that  $P'_\ell(1) = \frac{\ell(\ell+1)}{2}$ , for  $\lambda_\ell = \ell(\ell+1)$ ; hence we have

$$A_\ell(\phi) = \begin{pmatrix} \frac{\lambda_\ell}{2} & 0 & \alpha_{1, \ell}(\phi) & 0 \\ 0 & \frac{\lambda_\ell}{2} & 0 & \alpha_{2, \ell}(\phi) \\ \alpha_{1, \ell}(\phi) & 0 & \frac{\lambda_\ell}{2} & 0 \\ 0 & \alpha_{2, \ell}(\phi) & 0 & \frac{\lambda_\ell}{2} \end{pmatrix}.$$

The matrix  $B_\ell$  collects the covariances between first and second order derivatives, and is given by

$$\begin{aligned} B_\ell(x, y)_{4 \times 6} &= \mathbb{E} \left[ \begin{pmatrix} \nabla f_\ell(\bar{x})^t \\ \nabla f_\ell(\bar{y})^t \end{pmatrix} \begin{pmatrix} \nabla^2 f_\ell(x) & \nabla^2 f_\ell(y) \end{pmatrix} \right] \Big|_{x=\bar{x}, y=\bar{y}} \\ &= \begin{pmatrix} b_\ell(x, x) & b_\ell(x, y) \\ b_\ell(y, x) & b_\ell(y, y) \end{pmatrix}. \end{aligned}$$

It is well-known that for Gaussian isotropic processes, for  $i, j = 1, 2$ , the second derivatives  $e_i^x e_j^y f_\ell(x)$  are independent of  $e_i^x f_\ell(x)$  at every fixed point  $x \in \mathcal{S}^2$  see, e.g. [2] section 5.5; we have then

$$b_\ell(x, x)|_{x=(\pi/2, \varphi_x)} = b_\ell(y, y)|_{y=(\pi/2, 0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

while

$$\begin{aligned} b_\ell(x, y)|_{x=(\pi/2, \varphi_x), y=(\pi/2, 0)} &= \begin{pmatrix} 0 & \beta_{1, \ell}(\phi) & 0 \\ \beta_{2, \ell}(\phi) & 0 & \beta_{3, \ell}(\phi) \end{pmatrix} \\ &= -b_\ell(y, x)|_{x=(\pi/2, \varphi_x), y=(\pi/2, 0)}. \end{aligned}$$

Here we have introduced the functions

$$\begin{aligned} \beta_{1, \ell}(\phi) &= \sin \phi P''_\ell(\cos \phi), \\ \beta_{2, \ell}(\phi) &= \sin \phi \cos \phi P''_\ell(\cos \phi) + \sin \phi P'_\ell(\cos \phi), \\ \beta_{3, \ell}(\phi) &= -\sin^3 \phi P'''_\ell(\cos \phi) + 3 \sin \phi \cos \phi P''_\ell(\cos \phi) + \sin \phi P'_\ell(\cos \phi). \end{aligned}$$

Finally, the matrix  $C_\ell$  contains the variances of second-order derivatives, and we have

$$\begin{aligned} C_\ell(x, y)_{6 \times 6} &= \mathbb{E} \left[ \begin{pmatrix} \nabla^2 f_\ell(\bar{x})^t \\ \nabla^2 f_\ell(\bar{y})^t \end{pmatrix} \begin{pmatrix} \nabla^2 f_\ell(x) & \nabla^2 f_\ell(y) \end{pmatrix} \right] \Big|_{x=\bar{x}, y=\bar{y}} \\ &= \begin{pmatrix} c_\ell(x, x) & c_\ell(x, y) \\ c_\ell(y, x) & c_\ell(y, y) \end{pmatrix}. \end{aligned}$$

Direct calculations yield

$$\begin{aligned} c_\ell(x, y)|_{x=(\pi/2, \varphi_x), y=(\pi/2, 0)} &= c_\ell(y, x)|_{x=(\pi/2, \varphi_x), y=(\pi/2, 0)} \\ &= \begin{pmatrix} \gamma_{1, \ell}(\phi) & 0 & \gamma_{3, \ell}(\phi) \\ 0 & \gamma_{2, \ell}(\phi) & 0 \\ \gamma_{3, \ell}(\phi) & 0 & \gamma_{4, \ell}(\phi) \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} \gamma_{1, \ell}(\phi) &= (2 + \cos^2 \phi) P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi), \\ \gamma_{2, \ell}(\phi) &= -\sin^2 \phi P'''_\ell(\cos \phi) + \cos \phi P''_\ell(\cos \phi), \\ \gamma_{3, \ell}(\phi) &= -\sin^2 \phi \cos \phi P'''_\ell(\cos \phi) + (-2 \sin^2 \phi + \cos^2 \phi) P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi), \\ \gamma_{4, \ell}(\phi) &= \sin^4 \phi P''''_\ell(\cos \phi) - 6 \sin^2 \phi \cos \phi P'''_\ell(\cos \phi) + (-4 \sin^2 \phi + 3 \cos^2 \phi) P''_\ell(\cos \phi) + \cos \phi P'_\ell(\cos \phi). \end{aligned}$$

Since  $P''_\ell(1) = \frac{\lambda_\ell}{8}(\lambda_\ell - 2)$ , it immediately follows that

$$\begin{aligned} c_\ell(x, x)|_{x=(\pi/2, \varphi_x)} &= \begin{pmatrix} 3P''_\ell(1) + P'_\ell(1) & 0 & P''_\ell(1) + P'_\ell(1) \\ 0 & P''_\ell(1) & 0 \\ P''_\ell(1) + P'_\ell(1) & 0 & 3P''_\ell(1) + P'_\ell(1) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\lambda_\ell}{8}[3\lambda_\ell - 2] & 0 & \frac{\lambda_\ell}{8}[\lambda_\ell + 2] \\ 0 & \frac{\lambda_\ell}{8}[\lambda_\ell - 2] & 0 \\ \frac{\lambda_\ell}{8}[\lambda_\ell + 2] & 0 & \frac{\lambda_\ell}{8}[3\lambda_\ell - 2] \end{pmatrix} = c_\ell(y, y)|_{y=(\pi/2, 0)}. \end{aligned}$$

## B The conditional covariance matrix $\Delta_\ell(\phi)$

In this section we compute the conditional covariance matrices  $\Omega_\ell(\phi)$  and  $\Delta_\ell(\phi)$  (eqs. 4.3, 4.4). To simplify the notation we will write  $\alpha_i, \beta_i, \gamma_i$  for  $\alpha_{i,\ell}(\phi), \beta_{i,\ell}(\phi)$  and  $\gamma_{i,\ell}(\phi)$ ; likewise we will adopt the shorthand notation  $A, B, C, \Omega, \Delta$  for  $A_\ell(\phi), B_\ell(\phi), C_\ell(\phi), \Omega_\ell(\phi)$  and  $\Delta_\ell(\phi)$ , respectively.

Let us first compute explicitly the inverse matrix  $A^{-1}$ ; we write  $A$  as a block matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$$

where

$$a_1 = \begin{pmatrix} \frac{\lambda_\ell}{2} & 0 \\ 0 & \frac{\lambda_\ell}{2} \end{pmatrix}, \quad a_2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

and we evaluate the following components:

$$(a_1 - a_2 a_1^{-1} a_2)^{-1} = \begin{pmatrix} \frac{2\lambda_\ell}{\lambda_\ell^2 - 4\alpha_1^2} & 0 \\ 0 & \frac{2\lambda_\ell}{\lambda_\ell^2 - 4\alpha_2^2} \end{pmatrix},$$

and

$$a_1^{-1} a_2 = a_2 a_1^{-1} = \begin{pmatrix} \frac{2\alpha_1}{\lambda_\ell} & 0 \\ 0 & \frac{2\alpha_2}{\lambda_\ell} \end{pmatrix}.$$

Now, to invert blockwise  $A$ , we need to compute the main diagonal blocks

$$a_1^{-1} + a_1^{-1} a_2 (a_1 - a_2 a_1^{-1} a_2)^{-1} a_2 a_1^{-1} = (a_1 - a_2 a_1^{-1} a_2)^{-1},$$

and the off-diagonal blocks

$$-a_1^{-1} a_2 (a_1 - a_2 a_1^{-1} a_2)^{-1} = -(a_1 - a_2 a_1^{-1} a_2)^{-1} a_2 a_1^{-1} = - \begin{pmatrix} \frac{4\alpha_1}{\lambda_\ell^2 - 4\alpha_1^2} & 0 \\ 0 & \frac{4\alpha_2}{\lambda_\ell^2 - 4\alpha_2^2} \end{pmatrix}.$$

We have then

$$A^{-1} = \begin{pmatrix} \frac{2\lambda_\ell}{\lambda_\ell^2 - 4\alpha_1^2} & 0 & -\frac{4\alpha_1}{\lambda_\ell^2 - 4\alpha_1^2} & 0 \\ 0 & \frac{2\lambda_\ell}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & -\frac{4\alpha_2}{\lambda_\ell^2 - 4\alpha_2^2} \\ -\frac{4\alpha_1}{\lambda_\ell^2 - 4\alpha_1^2} & 0 & \frac{2\lambda_\ell}{\lambda_\ell^2 - 4\alpha_1^2} & 0 \\ 0 & -\frac{4\alpha_2}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{2\lambda_\ell}{\lambda_\ell^2 - 4\alpha_2^2} \end{pmatrix}.$$

We are now in the position to compute the matrix  $B^t A^{-1} B$ ; indeed we get:

$$\begin{aligned} B^t A^{-1} B &= \begin{pmatrix} 0 & 0 & 0 & -\beta_2 \\ 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & 0 & -\beta_3 \\ 0 & \beta_2 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & \beta_3 & 0 & 0 \end{pmatrix} A^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 & \beta_3 \\ 0 & -\beta_1 & 0 & 0 & 0 & 0 \\ -\beta_2 & 0 & -\beta_3 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2\lambda_\ell \beta_2^2}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{2\lambda_\ell \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} & \frac{4\alpha_2 \beta_2^2}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{4\alpha_2 \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} \\ 0 & \frac{2\lambda_\ell \beta_1^2}{\lambda_\ell^2 - 4\alpha_1^2} & 0 & 0 & \frac{4\alpha_1 \beta_1^2}{\lambda_\ell^2 - 4\alpha_1^2} & 0 \\ \frac{2\lambda_\ell \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{2\lambda_\ell \beta_3^2}{\lambda_\ell^2 - 4\alpha_2^2} & \frac{4\alpha_2 \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{4\alpha_2 \beta_3^2}{\lambda_\ell^2 - 4\alpha_2^2} \\ \frac{4\alpha_2 \beta_2^2}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{4\alpha_2 \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} & \frac{2\lambda_\ell \beta_2^2}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{2\lambda_\ell \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} \\ 0 & \frac{4\alpha_1 \beta_1^2}{\lambda_\ell^2 - 4\alpha_1^2} & 0 & 0 & \frac{2\lambda_\ell \beta_1^2}{\lambda_\ell^2 - 4\alpha_1^2} & 0 \\ \frac{4\alpha_2 \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{4\alpha_2 \beta_3^2}{\lambda_\ell^2 - 4\alpha_2^2} & \frac{2\lambda_\ell \beta_2 \beta_3}{\lambda_\ell^2 - 4\alpha_2^2} & 0 & \frac{2\lambda_\ell \beta_3^2}{\lambda_\ell^2 - 4\alpha_2^2} \end{pmatrix}, \end{aligned}$$

From section A in this appendix we have

$$C = \begin{pmatrix} 3 \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} & 0 & \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} & \gamma_1 & 0 & \gamma_3 \\ 0 & \frac{\lambda_\ell(\lambda_\ell - 2)}{8} & 0 & 0 & \gamma_2 & 0 \\ \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} & 0 & 3 \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} & \gamma_3 & 0 & \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} \\ \gamma_1 & 0 & \gamma_3 & 3 \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} & 0 & \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} \\ 0 & \gamma_2 & 0 & 0 & \frac{\lambda_\ell(\lambda_\ell - 2)}{8} & 0 \\ \gamma_3 & 0 & \gamma_4 & \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} & 0 & 3 \frac{\lambda_\ell(\lambda_\ell - 2)}{8} + \frac{\lambda_\ell}{2} \end{pmatrix};$$

The remaining computations to obtain  $\Omega$  and  $\Delta$  are straightforward.

## C Some estimates on Legendre polynomials

Let us first recall the following:

**Lemma C.1** (Hilb's asymptotics, [26], page 195, Theorem 8.21.6.). *For any  $\varepsilon > 0$  and any constant  $C > 0$ , we have*

$$P_\ell(\cos \phi) = \left( \frac{\phi}{\sin \phi} \right)^{1/2} J_0((\ell + 1/2)\phi) + \delta_\ell(\phi),$$

where  $J_\nu$  is the Bessel function of the first kind,  $P_\ell$  denotes Legendre polynomials, and the error term satisfies

$$\delta_\ell(\phi) \ll \begin{cases} \phi^2 O(1), & 0 < \phi < C/\ell, \\ \phi^{1/2} O(\ell^{-3/2}), & C/\ell \leq \phi, \end{cases}$$

uniformly w.r.t.  $\ell \geq 1$  and  $\phi \in [0, \pi - \varepsilon]$ .

**Lemma C.2.** *The following asymptotic representation for the Bessel functions of the first kind holds:*

$$\begin{aligned} J_0(x) &= \left( \frac{2}{\pi x} \right)^{1/2} \cos(x - \pi/4) \sum_{k=0}^{\infty} (-1)^k g(2k) (2x)^{-2k} \\ &\quad + \left( \frac{2}{\pi x} \right)^{1/2} \cos(x + \pi/4) \sum_{k=0}^{\infty} (-1)^k g(2k+1) (2x)^{-2k-1}, \end{aligned}$$

where  $\varepsilon > 0$ ,  $|\arg x| \leq \pi - \varepsilon$ ,  $g(0) = 1$  and  $g(k) = \frac{(-1)(-3^2)\cdots(-(2k-1)^2)}{2^{2k} k!} = (-1)^k \frac{[(2k)!]^2}{2^{2k} k!}$ .

For a proof of Lemma C.2 see [19], Section 5.11. In the rest of the paper we use the following notation:

$$\begin{aligned} R_1(\ell, \phi) &= O(\ell^{-1/2} \phi^{-5/2}), & R_2(\ell, \phi) &= O(\ell^{1/2} \phi^{-7/2}), \\ R_3(\ell, \phi) &= O(\ell^{1+1/2} \phi^{-9/2}), & R_4(\ell, \phi) &= O(\ell^{2+1/2} \phi^{-11/2}). \end{aligned}$$

**Lemma C.3.** *For any constant  $C > 0$ , we have, uniformly for  $\ell \geq 1$  and  $\phi \in [C/\ell, \pi/2]$ :*

$$P'_\ell(\cos \phi) = \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} \left[ \sin \psi_\ell^- - \frac{1}{8\ell\phi} \sin \psi_\ell^+ \right] + R_1(\ell, \phi), \quad (\text{C.1})$$

$$P''_\ell(\cos \phi) = \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{2+1/2} \phi} \left[ -\cos \psi_\ell^- + \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right] - \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{3+1/2} \phi} \left[ \cos \psi_{\ell-1}^+ + \frac{1}{8\ell\phi} \cos \psi_{\ell-1}^- \right] + R_2(\ell, \phi), \quad (\text{C.2})$$

$$\begin{aligned} P'''_\ell(\cos \phi) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{3-1/2}}{\sin^{3+1/2} \phi} \left[ \cos \psi_\ell^+ + \frac{1}{8\ell\phi} \cos \psi_\ell^- \right] \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{4+1/2} \phi} \left[ \frac{1}{2} (\cos \psi_{\ell+1}^- + 5 \cos \psi_{\ell-1}^-) - \frac{1}{8\ell\phi} \frac{1}{2} (\cos \psi_{\ell+1}^+ + 5 \cos \psi_{\ell-1}^+) \right] \\ &\quad - \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{5+1/2} \phi} \left[ 3 \cos \phi \sin \psi_{\ell-1}^- - \frac{1}{8\ell\phi} 3 \cos \phi \sin \psi_{\ell-1}^+ \right] + R_3(\ell, \phi), \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} P''''_\ell(\cos \phi) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{4-1/2}}{\sin^{4+1/2} \phi} \left[ \cos \psi_\ell^- - \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right] \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{\ell^{3-1/2}}{\sin^{5+1/2} \phi} \left[ -\frac{3}{2} (\sin \psi_{\ell+1}^- + 3 \sin \psi_{\ell-1}^-) + \frac{1}{8\ell\phi} \frac{3}{2} (\sin \psi_{\ell+1}^+ + 3 \sin \psi_{\ell-1}^+) \right] \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{6+1/2} \phi} \left[ -\frac{1}{2} (\cos \psi_{\ell+2}^- + 16 \cos \psi_\ell^- + 13 \cos \psi_{\ell-2}^-) + \frac{1}{8\ell\phi} \frac{1}{2} (\cos \psi_{\ell+2}^+ + 16 \cos \psi_\ell^+ + 13 \cos \psi_{\ell-2}^+) \right] \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{7+1/2} \phi} \left[ -3(5 - 4 \sin^2 \phi) \cos \psi_{\ell-1}^+ - \frac{1}{8\ell\phi} 3(5 - 4 \sin^2 \phi) \cos \psi_{\ell-1}^- \right] + R_4(\ell, \phi), \end{aligned} \quad (\text{C.4})$$

where  $\psi_{\ell+k}^\pm = (\ell + k + 1/2)\phi \pm \pi/4$ .

*Proof.* By applying the Hilb's asymptotics in Lemma C.1 and Lemma C.2, we obtain

$$\begin{aligned} P_{\ell+\alpha}(\cos \phi) &= \cos \psi_{\ell+\alpha}^- \sum_{k=0}^{\infty} h(2k) [s_{2k,r}(\ell, \phi) + \sigma_{2k,r}(\ell, \alpha, \phi)] \\ &\quad + \cos \psi_{\ell+\alpha}^+ \sum_{k=0}^{\infty} h(2k+1) [s_{2k+1,r}(\ell, \phi) + \sigma_{2k+1,r}(\ell, \alpha, \phi)] \\ &\quad + \phi^{1/2} O((\ell + \alpha)^{-3/2}). \end{aligned}$$

where  $r = 0, 1, 2 \dots$  and

$$h(2k) = \sqrt{\frac{2}{\pi}} (-1)^k g(2k) 2^{-2k}, \quad h(2k+1) = \sqrt{\frac{2}{\pi}} (-1)^k g(2k+1) 2^{-2k-1},$$

$$s_{2k,r}(\ell, \phi) = \frac{\phi^{-2k}}{\sqrt{\sin \phi}} \frac{1}{\ell^{2k+1/2}} \sum_{n=0}^r \binom{-2k-1/2}{n} \left( \frac{1/2+\alpha}{\ell} \right)^n,$$

$$s_{2k+1,r}(\ell, \phi) = \frac{\phi^{-2k-1}}{\sqrt{\sin \phi}} \frac{1}{\ell^{2k+3/2}} \sum_{n=0}^r \binom{-2k-3/2}{n} \left( \frac{1/2+\alpha}{\ell} \right)^n,$$

$$\sigma_{2k,r}(\ell, \alpha, \phi) = \frac{\phi^{-2k}}{\sqrt{\sin \phi}} \frac{1}{\ell^{2k+1/2}} \sum_{n=r+1}^{\infty} \binom{-2k-1/2}{n} \left( \frac{1/2+\alpha}{\ell} \right)^n,$$

$$\sigma_{2k+1,r}(\ell, \alpha, \phi) = \frac{\phi^{-2k-1}}{\sqrt{\sin \phi}} \frac{1}{\ell^{2k+3/2}} \sum_{n=r+1}^{\infty} \binom{-2k-3/2}{n} \left( \frac{1/2+\alpha}{\ell} \right)^n.$$

To obtain the asymptotic behaviour of the first derivative in (C.1) we first note that

$$P'_\ell(\cos \phi) = \frac{\ell+1}{\sin^2 \phi} [\cos \phi P_\ell(\cos \phi) - P_{\ell+1}(\cos \phi)]$$

where

$$\begin{aligned} \cos \phi P_\ell(\cos \phi) - P_{\ell+1}(\cos \phi) &= \cos \phi \cos \psi_\ell^- \sum_{k=0}^{\infty} h(2k) [s_{2k,0}(\ell, \phi) + \sigma_{2k,0}(\ell, 0, \phi)] \\ &\quad + \cos \phi \cos \psi_\ell^+ \sum_{k=0}^{\infty} h(2k+1) [s_{2k+1,0}(\ell, \phi) + \sigma_{2k+1,0}(\ell, 0, \phi)] \\ &\quad + \cos \phi \phi^{1/2} O(\ell^{-3/2}) \\ &\quad - \cos \psi_{\ell+1}^- \sum_{k=0}^{\infty} h(2k) [s_{2k,0}(\ell, \phi) + \sigma_{2k,0}(\ell, 1, \phi)] \\ &\quad - \cos \psi_{\ell+1}^+ \sum_{k=0}^{\infty} h(2k+1) [s_{2k+1,0}(\ell, \phi) + \sigma_{2k+1,0}(\ell, 1, \phi)] \\ &\quad + \phi^{1/2} O((\ell+1)^{-3/2}). \end{aligned}$$

Now observe that

$$\cos \phi \cos \psi_\ell^\pm - \cos \psi_{\ell+1}^\pm = \sin \phi \sin \psi_\ell^\pm;$$

we obtain

$$\cos \phi P_\ell(\cos \phi) - P_{\ell+1}(\cos \phi) = \sin \phi \sin \psi_\ell^- h(0) s_{0,0}(\ell, \phi) + \sin \phi \sin \psi_\ell^+ h(1) s_{1,0}(\ell, \phi) + R'_1(\ell, \phi),$$

where

$$R'_1(\ell, \phi) = \ell^{-3/2} \phi^{-1/2}.$$

Then (C.1) easily follows, since we get

$$P'_\ell(\cos \phi) = \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} [\sin \psi_\ell^- - \frac{1}{8\ell\phi} \sin \psi_\ell^+] + O(\ell^{-1/2} \phi^{-5/2}).$$

To prove the asymptotic behaviour of the second derivative in (C.2) we start from

$$P''_\ell(\cos \phi) = \frac{\ell+1}{\sin^4 \phi} [(1+2\cos^2 \phi + \ell \cos^2 \phi) P_\ell(\cos \phi) - (5+2\ell) \cos \phi P_{\ell+1}(\cos \phi) + (\ell+2) P_{\ell+2}(\cos \phi)], \quad (\text{C.5})$$

and we note that

$$\begin{aligned} \cos^2 \phi \cos \psi_\ell^\pm - 2 \cos \phi \cos \psi_{\ell+1}^\pm + \cos \psi_{\ell+2}^\pm &= -\sin^2 \phi \cos \psi_\ell^\pm, \\ (1+2\cos^2 \phi) \cos \psi_\ell^\pm - 5 \cos \phi \cos \psi_{\ell+1}^\pm + 2 \cos \psi_{\ell+2}^\pm &= \pm \sin \phi \cos \psi_{\ell-1}^\mp. \end{aligned}$$

Then we obtain

$$(1+2\cos^2 \phi + \ell \cos^2 \phi) P_\ell(\cos \phi) - (5+2\ell) \cos \phi P_{\ell+1}(\cos \phi) + (\ell+2) P_{\ell+2}(\cos \phi)$$

$$\begin{aligned}
&= -\ell \sin^2 \phi \cos \psi_\ell^- h(0) s_{0,0}(\ell, \phi) - \ell \sin^2 \phi \cos \psi_\ell^+ h(1) s_{1,0}(\ell, \phi) \\
&\quad - \sin \phi \cos \psi_{\ell-1}^+ h(0) s_{0,0}(\ell, \phi) + \sin \phi \cos \psi_{\ell-1}^- h(1) s_{1,0}(\ell, \phi) + R'_2(\ell, \phi)
\end{aligned} \tag{C.6}$$

where

$$R'_2(\ell, \phi) = \ell^{-3/2} \phi^{-1/2}.$$

Plugging (C.6) in (C.5), we finally have

$$\begin{aligned}
P_\ell''(\cos \phi) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{2+1/2} \phi} \left\{ -\cos \psi_\ell^- + \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right\} + \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{3+1/2} \phi} \left\{ -\cos \psi_{\ell-1}^+ - \frac{1}{8\ell\phi} \cos \psi_{\ell-1}^- \right\} \\
&\quad + O(\ell^{1/2} \phi^{-9/2}).
\end{aligned}$$

To obtain the asymptotic behaviour of the third derivative in (C.3), we write

$$\begin{aligned}
P_\ell'''(x) &= \frac{\ell^2(\ell+1)}{(x^2-1)^3} [-x^3 P_\ell(x) + 3x^2 P_{\ell+1}(x) - 3x P_{\ell+2}(x) + P_{\ell+3}(x)] \\
&\quad + \frac{\ell(\ell+1)}{(x^2-1)^3} [-(3x+5x^3)P_\ell(x) + (3+18x^2)P_{\ell+1}(x) - 18x P_{\ell+2}(x) + 5P_{\ell+3}(x)] \\
&\quad + \frac{\ell+1}{(x^2-1)^3} [-(9x+6x^3)P_\ell(x) + (6+27x^2)P_{\ell+1}(x) - 24x P_{\ell+2}(x) + 6P_{\ell+3}(x)].
\end{aligned}$$

Now, for  $r=1$ , we obtain, for example, that

$$\begin{aligned}
&-\cos^3 \phi P_\ell(\cos \phi) + 3 \cos^2 \phi P_{\ell+1}(\cos \phi) - 3 \cos \phi P_{\ell+2}(\cos \phi) + P_{\ell+3}(\cos \phi) \\
&= -\cos^3 \phi \left[ \cos \psi_\ell^- \sum_{k=0}^{\infty} [h(2k) s_{2k,1}(\ell, \phi) + \sigma_{2k,1}(\ell, 0, \phi)] + \cos \psi_\ell^+ \sum_{k=0}^{\infty} [h(2k+1) s_{2k+1,1}(\ell, \phi) + \sigma_{2k+1,1}(\ell, 0, \phi)] \right] \\
&\quad + 3 \cos^2 \phi \left[ \cos \psi_{\ell+1}^- \sum_{k=0}^{\infty} [h(2k) s_{2k,1}(\ell, \phi) + \sigma_{2k,1}(\ell, 1, \phi)] + \cos \psi_{\ell+1}^+ \sum_{k=0}^{\infty} [h(2k+1) s_{2k+1,1}(\ell, \phi) + \sigma_{2k+1,1}(\ell, 1, \phi)] \right] \\
&\quad - 3 \cos \phi \left[ \cos \psi_{\ell+2}^- \sum_{k=0}^{\infty} [h(2k) s_{2k,1}(\ell, \phi) + \sigma_{2k,1}(\ell, 2, \phi)] + \cos \psi_{\ell+2}^+ \sum_{k=0}^{\infty} [h(2k+1) s_{2k+1,1}(\ell, \phi) + \sigma_{2k+1,1}(\ell, 2, \phi)] \right] \\
&\quad + \cos \psi_{\ell+3}^- \sum_{k=0}^{\infty} [h(2k) s_{2k,1}(\ell, \phi) + \sigma_{2k,1}(\ell, 3, \phi)] + \cos \psi_{\ell+3}^+ \sum_{k=0}^{\infty} [h(2k+1) s_{2k+1,1}(\ell, \phi) + \sigma_{2k+1,1}(\ell, 3, \phi)];
\end{aligned}$$

now exploiting the identities

$$\begin{aligned}
&-\cos^3 \phi \cos \psi_\ell^\pm + 3 \cos^2 \phi \cos \psi_{\ell+1}^\pm - 3 \cos \phi \cos \psi_{\ell+2}^\pm + \cos \psi_{\ell+3}^\pm = \pm \sin^3 \phi \cos \psi_\ell^\mp, \\
&3 \cos^2 \phi \cos \psi_{\ell+1}^\pm - 3 \cdot 2 \cos \phi \cos \psi_{\ell+2}^\pm + 3 \cos \psi_{\ell+3}^\pm = \pm 3 \sin^2 \phi \cos \psi_{\ell+1}^\mp,
\end{aligned}$$

we get

$$\begin{aligned}
&-\cos^3 \phi P_\ell(\cos \phi) + 3 \cos^2 \phi P_{\ell+1}(\cos \phi) - 3 \cos \phi P_{\ell+2}(\cos \phi) + P_{\ell+3}(\cos \phi) \\
&= -\sin^3 \phi \cos \psi_\ell^+ h(0) s_{0,0}(\ell, \phi) + \sin^3 \phi \cos \psi_\ell^- h(1) s_{1,0}(\ell, \phi) + R'_3(\ell, \phi).
\end{aligned}$$

For the other terms we need to apply the following equalities:

$$\begin{aligned}
&-(3 \cos \phi + 5 \cos^3 \phi) \cos \psi_\ell^\pm + (3 + 18 \cos^2 \phi) \cos \psi_{\ell+1}^\pm - 18 \cos \phi \cos \psi_{\ell+2}^\pm + 5 \cos \psi_{\ell+3}^\pm = \frac{1}{2} \sin^2 \phi [\cos \psi_{\ell+1}^\pm + 5 \cos \psi_{\ell-1}^\pm], \\
&(-9 \cos \phi - 6 \cos^3 \phi) \cos \psi_\ell^\pm + (6 + 27 \cos^2 \phi) \cos \psi_{\ell+1}^\pm - 24 \cos \phi \cos \psi_{\ell+2}^\pm + 6 \cos \psi_{\ell+3}^\pm = -3 \sin \phi \cos \phi \sin \psi_{\ell-1}^\pm,
\end{aligned}$$

and we get

$$\begin{aligned}
P_\ell'''(\cos \phi) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{3-1/2}}{\sin^{3+1/2} \phi} \left[ \cos \psi_\ell^+ + \frac{1}{8\ell\phi} \cos \psi_\ell^- \right] \\
&\quad - \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{4+1/2} \phi} \left[ \frac{1}{2} (\cos \psi_{\ell+1}^- + 5 \cos \psi_{\ell-1}^-) - \frac{1}{8\ell\phi} \frac{1}{2} (\cos \psi_{\ell+1}^+ + 5 \cos \psi_{\ell-1}^+) \right] \\
&\quad + \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{5+1/2} \phi} \left[ 3 \cos \phi \sin \psi_{\ell-1}^- - \frac{1}{8\ell\phi} 3 \cos \phi \sin \psi_{\ell-1}^+ \right] + R_3(\ell, \phi).
\end{aligned}$$

Finally, to prove (C.4), we start from

$$\begin{aligned}
&P_\ell''''(x) \\
&= \frac{\ell^3(\ell+1)}{(x^2-1)^4} [x^4 P_\ell(x) - 4x^3 P_{\ell+1}(x) + 6x^2 P_{\ell+2}(x) - 4x P_{\ell+3}(x) + P_{\ell+4}(x)]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\ell^2(\ell+1)}{(x^2-1)^4} [(9x^4+6x^2)P_\ell(x) - (42x^3+12x)P_{\ell+1}(x) + (66x^2+6)P_{\ell+2}(x) - 42xP_{\ell+3}(x) + 9P_{\ell+4}(x)] \\
& + \frac{\ell(\ell+1)}{(x^2-1)^4} [(26x^4+42x^2+3)P_\ell(x) - (146x^3+78x)P_{\ell+1}(x) + (231x^2+30)P_{\ell+2}(x) - 134xP_{\ell+3}(x) + 26P_{\ell+4}(x)] \\
& + \frac{\ell+1}{(x^2-1)^4} [(24x^4+72x^2+9)P_\ell(x) - (168x^3+111x)P_{\ell+1}(x) + (246x^2+36)P_{\ell+2}(x) - 132xP_{\ell+3}(x) + 24P_{\ell+4}(x)].
\end{aligned}$$

Let us introduce the further identities

$$\begin{aligned}
& \cos^4 \phi \cos \psi_\ell^\pm - 4 \cos^3 \phi \cos \psi_{\ell+1}^\pm + 6 \cos^2 \phi \cos \psi_{\ell+2}^\pm - 4 \cos \phi \cos \psi_{\ell+3}^\pm + \cos \psi_{\ell+4}^\pm = \sin^4 \phi \cos \psi_\ell^\pm, \\
& \cos^4 \phi - 4 \cos^3 \phi + 6 \cos^2 \phi - 4 \cos \phi + 1 = (\cos \phi - 1)^4,
\end{aligned}$$

$$\begin{aligned}
& (6 \cos^2 \phi + 9 \cos^4 \phi) \cos \psi_\ell^\pm - (12 \cos \phi + 42 \cos^3 \phi) \cos \psi_{\ell+1}^\pm + (6 + 66 \cos^2 \phi) \cos \psi_{\ell+2}^\pm - 42 \cos \phi \cos \psi_{\ell+3}^\pm \\
& + 9 \cos \psi_{\ell+4}^\pm = -\frac{3}{2} \sin^3 \phi [\sin \psi_{\ell+1}^\pm + 3 \sin \psi_{\ell-1}^\pm], \\
& (6 \cos^2 \phi + 9 \cos^4 \phi) - (12 \cos \phi + 42 \cos^3 \phi) + (6 + 66 \cos^2 \phi) - 42 \cos \phi + 9 = 3(\cos \phi - 1)^3 (3 \cos \phi - 5),
\end{aligned}$$

$$\begin{aligned}
& (3 + 42 \cos^2 \phi + 26 \cos^4 \phi) \cos \psi_\ell^\pm + (-78 \cos \phi - 146 \cos^3 \phi) \cos \psi_{\ell+1}^\pm + (30 + 231 \cos^2 \phi) \cos \psi_{\ell+2}^\pm \\
& - 134 \cos \phi \cos \psi_{\ell+3}^\pm + 26 \cos \psi_{\ell+4}^\pm = -\frac{1}{2} \sin^2 \phi [\cos \psi_{\ell+2}^\pm + 16 \cos \psi_\ell^\pm + 13 \cos \psi_{\ell-2}^\pm], \\
& (3 + 42 \cos^2 \phi + 26 \cos^4 \phi) + (-78 \cos \phi - 146 \cos^3 \phi) + (30 + 231 \cos^2 \phi) \\
& - 134 \cos \phi + 26 = (\cos \phi - 1)^2 (59 - 94 \cos \phi + 26 \cos^2 \phi),
\end{aligned}$$

$$\begin{aligned}
& (9 + 72 \cos^2 \phi + 24 \cos^4 \phi) \cos \psi_\ell^\pm + (-111 \cos \phi - 168 \cos^3 \phi) \cos \psi_{\ell+1}^\pm + (36 + 246 \cos^2 \phi) \cos \psi_{\ell+2}^\pm \\
& - 132 \cos \phi \cos \psi_{\ell+3}^\pm + 24 \cos \psi_{\ell+4}^\pm = \pm 3 \sin \phi (5 - 4 \sin^2 \phi) \cos \psi_{\ell-1}^\pm,
\end{aligned}$$

from which we obtain the dominant terms in (C.4). The analysis of the remainder terms is omitted for brevity's sake.  $\square$

In what follows we write  $f_\ell(\phi) \simeq g_\ell(\phi)$ , if there exists a constant  $c_0 > 0$  such that

$$-c_0 g_\ell(\phi) \leq f_\ell(\phi) \leq c_0 g_\ell(\phi),$$

for all  $\ell \geq 1$  and  $\phi \in [C/\ell, \pi/2]$ . From Lemma C.3 it follows immediately that:

**Lemma C.4.** *Uniformly in  $\ell \geq 1$  and  $\phi \in [C/\ell, \pi/2]$ , we have*

$$\begin{aligned}
P'_\ell(\cos \phi) &\simeq \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} + R_1(\ell, \phi), & P''_\ell(\cos \phi) &\simeq \sum_{i=0}^1 \frac{\ell^{2-i-1/2}}{\sin^{2+i+1/2} \phi} + R_2(\ell, \phi), \\
P'''_\ell(\cos \phi) &\simeq \sum_{i=0}^2 \frac{\ell^{3-i-1/2}}{\sin^{3+i+1/2} \phi} + R_3(\ell, \phi), & P''''_\ell(\cos \phi) &\simeq \sum_{i=0}^3 \frac{\ell^{4-i-1/2}}{\sin^{4+i+1/2} \phi} + R_4(\ell, \phi).
\end{aligned}$$

*Proof.* From (C.1) and since  $\phi \in [C/\ell, \pi/2]$ , we obtain

$$P'_\ell(\cos \phi) \leq \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} \left[1 + \frac{1}{8\ell\phi}\right] + R_1(\ell, \phi) \leq \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} \left[1 + \frac{1}{8C}\right] + R_1(\ell, \phi),$$

and

$$P'_\ell(\cos \phi) \geq \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} \left[-1 - \frac{1}{8\ell\phi}\right] + R_1(\ell, \phi) \geq -\sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2} \phi} \left[1 + \frac{1}{8C}\right] + R_1(\ell, \phi).$$

The proof is analogous in the other cases.  $\square$

From Lemma C.4 we obtain the following asymptotics for the elements of the covariance matrix  $\Sigma$ .

**Lemma C.5.** *For every  $\ell \geq 1$  and  $\phi \in [C/\ell, \pi/2]$ , we have*

$$\begin{aligned}
\frac{\alpha_{1,\ell}(\phi)}{\ell^2} &\simeq \frac{1}{\ell^{1+1/2} \sin^{1+1/2} \phi} + O(\ell^{-5/2} \phi^{-5/2}) \\
\frac{\alpha_{2,\ell}(\phi)}{\ell^2} &\simeq \sum_{i=0}^1 \frac{1}{\ell^{i+1/2} \sin^{i+1/2} \phi} + O(\ell^{-3/2} \phi^{-3/2}) \\
\frac{\beta_{1,\ell}(\phi)}{\ell^3} &\simeq \sum_{i=0}^1 \frac{1}{\ell^{1+i+1/2} \sin^{1+i+1/2} \phi} + O(\ell^{-5/2} \phi^{-5/2})
\end{aligned}$$



$$\begin{aligned}
\frac{\beta_{2,\ell}(\phi)}{\ell^3} &\simeq \sum_{i=0}^1 \frac{1}{\ell^{1+i+1/2} \sin^{1+i+1/2} \phi} + \frac{1}{\ell^{2+1/2} \sin^{1/2} \phi} + O(\ell^{-5/2} \phi^{-5/2}) \\
\frac{\beta_{3,\ell}(\phi)}{\ell^3} &\simeq \sum_{i=0}^2 \frac{1}{\ell^{i+1/2} \sin^{i+1/2} \phi} + \frac{1}{\ell^{2+1/2} \sin^{1/2} \phi} + O(\ell^{-3/2} \phi^{-3/2}) \\
\frac{\gamma_{1,\ell}(\phi)}{\ell^4} &\simeq \sum_{i=0}^1 \frac{1}{\ell^{2+i+1/2} \sin^{2+i+1/2} \phi} + \frac{1}{\ell^{3+1/2} \sin^{1+1/2} \phi} + O(\ell^{-7/2} \phi^{-7/2}) \\
\frac{\gamma_{2,\ell}(\phi)}{\ell^4} &\simeq \sum_{i=0}^1 \frac{1}{\ell^{1+i+1/2} \sin^{1+i+1/2} \phi} + \frac{1}{\ell^{2+1/2} \sin^{1/2} \phi} + O(\ell^{-5/2} \phi^{-5/2}) \\
\frac{\gamma_{3,\ell}(\phi)}{\ell^4} &\simeq \sum_{i=0}^3 \frac{1}{\ell^{1+i+1/2} \sin^{1+i+1/2} \phi} + \sum_{i=0}^3 \frac{1}{\ell^{2+i+1/2} \sin^{i+1/2} \phi} + O(\ell^{-5/2} \phi^{-5/2}) \\
\frac{\gamma_{4,\ell}(\phi)}{\ell^4} &\simeq \sum_{i=0}^3 \frac{1}{\ell^{i+1/2} \sin^{i+1/2} \phi} + \sum_{i=0}^1 \frac{1}{\ell^{2+i+1/2} \sin^{i+1/2} \phi} + O(\ell^{-3/2} \phi^{-3/2}).
\end{aligned}$$

**Lemma C.6.** For  $k = 0, 1, 2, \dots$  and  $n = 1, 2, 3, \dots$ , we have

$$\frac{1}{\ell^k} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^n \phi} \sin \phi d\phi = \begin{cases} O(\ell^{-k}) & \text{for } n = 1, \\ O(\ell^{-k} \log \ell) & \text{for } n = 2, \\ O(\ell^{n-k-2}) & \text{for } n \geq 3. \end{cases}$$

**Lemma C.7.** For  $n = 0, 1, 2, \dots$ , we have

$$\frac{1}{\ell^{n+1/2}} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{n+1/2} \phi} \sin \phi d\phi = \begin{cases} O(\ell^{-1/2}) & \text{for } n = 0, \\ O(\ell^{-1-1/2}) & \text{for } n = 1, \\ O(\ell^{-2}) & \text{for } n \geq 2. \end{cases}$$

**Lemma C.8.** For  $k, n = 1, 2, 3, \dots$ , we have

$$\frac{1}{\ell^k} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^n \phi} \frac{1}{\phi^k} \sin \phi d\phi = \begin{cases} O(\ell^{-1} \log \ell) & \text{for } n + k = 2, \\ O(\ell^{n-2}) & \text{for } n + k \geq 3. \end{cases}$$

*Proof.* We first note that

$$\frac{1}{\ell^k} \int_{C/\ell}^{\pi/2} \frac{1}{\phi^{n+k-1}} d\phi \leq \frac{1}{\ell^k} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^n \phi} \frac{1}{\phi^k} \sin \phi d\phi \leq \frac{1}{\ell^k} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{n+k-1} \phi} d\phi,$$

then the conclusion follows from Lemma C.6 and by observing that

$$\frac{1}{\ell^k} \int_{C/\ell}^{\pi/2} \frac{1}{\phi^{n+k-1}} d\phi = \begin{cases} O(\ell^{-1} \log \ell) & \text{for } n + k = 2, \\ O(\ell^{n-2}) & \text{for } n + k \geq 3. \end{cases}$$

□

## D Bounds for the terms $A_{0,\ell}$ , $A_{i,\ell}$ and $A_{ij,\ell}$

*Proof of Lemma 4.2.* By expanding the denominator in  $A_{0,\ell}$ , we write

$$\begin{aligned}
A_{0,\ell} &= \int_{C/\ell}^{\pi/2} \left[ 1 + 2 \frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\right) \right] \left[ 1 + 2 \frac{\alpha_{1,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{1,\ell}^4(\phi)}{\lambda_\ell^4}\right) \right] \sin \phi d\phi \\
&= \cos\left(\frac{C}{\ell}\right) + \int_{C/\ell}^{\pi/2} \left[ 2 \frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\right) + 2 \frac{\alpha_{1,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{1,\ell}^4(\phi)}{\lambda_\ell^4}\right) \right. \\
&\quad \left. + \left( 2 \frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\right) \right) \left( 2 \frac{\alpha_{1,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{1,\ell}^4(\phi)}{\lambda_\ell^4}\right) \right) \right] \sin \phi d\phi.
\end{aligned}$$

The idea is that the leading term in this expansion produces the cancellation of the component  $(\mathbb{E}[\mathcal{N}_I^c(f_\ell)])^2$ . Indeed, in view of Remark 4.1, we have

$$2\lambda_\ell^2 \cos\left(\frac{C}{\ell}\right) \iint_{I \times I} q(\mathbf{0}; t_1, t_2) dt_1 dt_2 - \frac{\lambda_\ell^2}{4} \left[ \int_I p_1^c(t) dt \right]^2 = O(\ell^2).$$

We consider now the rate of the terms

$$2\lambda_\ell^2 \int_{C/\ell}^{\pi/2} \left[ 2 \frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2} + 2 \frac{\alpha_{1,\ell}^2(\phi)}{\lambda_\ell^2} + 4 \frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2} \frac{\alpha_{1,\ell}^2(\phi)}{\lambda_\ell^2} + O\left(\frac{\alpha_{1,\ell}^4(\phi)}{\lambda_\ell^4}\right) + O\left(\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\right) \right] \sin \phi d\phi \iint_{I \times I} q(\mathbf{0}; t_1, t_2) dt_1 dt_2.$$

We apply here Lemma C.5 and Lemma C.6 to identify the rate of the dominant term. In fact, from Lemma C.5, we obtain the asymptotic behaviour of each addend of the integrand function, then Lemma C.6 gives the asymptotic behaviour of their integrals. We immediately see that

$$\frac{1}{\lambda_\ell^2} \int_{C/\ell}^{\frac{\pi}{2}} \alpha_{2,\ell}^2(\phi) \sin \phi d\phi = O(\ell^{-1}). \quad (\text{D.1})$$

To obtain the multiplicative constant of the leading term (D.1) note that, from Lemma C.3 and applying again Lemma C.6, to determine the non-dominant terms, we can write

$$\begin{aligned} \frac{2}{\lambda_\ell^2} \int_{C/\ell}^{\frac{\pi}{2}} \alpha_{2,\ell}^2(\phi) \sin \phi d\phi &= \frac{2}{\lambda_\ell^2} \int_{C/\ell}^{\frac{\pi}{2}} [-\sin^2 \phi P_\ell''(\cos \phi) + \cos \phi P_\ell'(\cos \phi)]^2 \sin \phi d\phi \\ &= \frac{2}{\lambda_\ell^2} \int_{C/\ell}^{\frac{\pi}{2}} \left\{ -\sin^2 \phi \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{2+1/2} \phi} \left[ -\cos \psi_\ell^- + \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right] \right\}^2 \sin \phi d\phi + O(\ell^{-2} \log \ell) \\ &= \frac{4}{\pi} \int_{C/\ell}^{\frac{\pi}{2}} \frac{1}{\ell \sin \phi} \left[ -\cos \psi_\ell^- + \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right]^2 \sin \phi d\phi + O(\ell^{-2} \log \ell), \end{aligned} \quad (\text{D.2})$$

where  $\psi_\ell^\pm = (\ell + 1/2)\phi \pm \pi/4$ . Applying Lemma C.8 to get the asymptotic behaviour of the integral (D.2), we have

$$\begin{aligned} \frac{2}{\lambda_\ell^2} \int_{C/\ell}^{\frac{\pi}{2}} \alpha_{2,\ell}^2(\phi) \sin \phi d\phi &= \frac{4}{\pi} \frac{1}{\ell} \int_{C/\ell}^{\frac{\pi}{2}} \cos^2 \psi_\ell^- d\phi + O(\ell^{-2} \log \ell) \\ &= \frac{4}{\pi} \frac{1}{\ell} \int_{C/\ell}^{\frac{\pi}{2}} \left[ \frac{1}{2} + \frac{1}{2} \cos(2\psi_\ell^-) \right] d\phi + O(\ell^{-2} \log \ell) \\ &= \frac{2}{\pi} \frac{1}{\ell} \int_{C/\ell}^{\frac{\pi}{2}} d\phi + \frac{2}{\pi\ell} \int_{C/\ell}^{\frac{\pi}{2}} \cos(2\psi_\ell^-) d\phi + O(\ell^{-2} \log \ell) \\ &= \frac{2}{\pi} \frac{1}{\ell} \left( \frac{\pi}{2} + \frac{C}{\ell} \right) + \frac{2}{\pi\ell} \frac{\cos(C(2 + 1/\ell)) + \sin(\ell\pi)}{1 + 2\ell} + O(\ell^{-2} \log \ell) \\ &= \ell^{-1} + O(\ell^{-2} \log \ell). \end{aligned}$$

□

*Proof of Lemma 4.3.* We start by observing that the terms  $A_{i,\ell}$  can be written in the form

$$A_{i,\ell} = \int_{C/\ell}^{\pi/2} \frac{N_{i,\ell}(\phi)}{(1 - 4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)^{m/2} (1 - 4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)^{n/2}} \sin \phi d\phi$$

for a suitable function  $N_{i,\ell}(\phi)$  and  $n, m = 1, 2, 3$ . By expanding in power series around the origin the ratio

$$\frac{1}{(1 - 4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)^{m/2} (1 - 4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)^{n/2}}$$

and with computations analogous to those performed in the proof of Lemma 4.2, it follows that the dominant terms of  $A_{i,\ell}$  are all of the form

$$\int_{C/\ell}^{\pi/2} N_{i,\ell}(\phi) \sin \phi d\phi. \quad (\text{D.3})$$

We study now the asymptotic behaviour of (D.3), for  $i = 1, \dots, 8$ :

- To obtain the asymptotic behaviour of  $A_{1,\ell}(\phi)$ , we note that

$$\int_{C/\ell}^{\pi/2} N_{1,\ell}(\phi) \sin \phi d\phi = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \beta_{2,\ell}^2(\phi) \sin \phi d\phi.$$

Now Lemma C.5 gives the asymptotic behaviour of the terms of the integrand function and Lemma C.6 gives the asymptotic behaviour of the integrand of each term, so that we get:

$$\int_{C/\ell}^{\pi/2} \left( \frac{\beta_{2,\ell}(\phi)}{\ell^3} \right)^2 \sin \phi d\phi = O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^3 \sin^3 \phi} \sin \phi d\phi \right) = O(\ell^{-2}).$$

- For the term  $A_{2,\ell}(\phi)$  we note that

$$\int_{C/\ell}^{\pi/2} N_{2,\ell}(\phi) \sin \phi d\phi = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \beta_{1,\ell}^2(\phi) \sin \phi d\phi$$

and, applying again Lemma C.5 and Lemma C.6, we have

$$\int_{C/\ell}^{\pi/2} \left( \frac{\beta_{1,\ell}(\phi)}{\ell^3} \right)^2 \sin \phi d\phi = O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^3 \sin^3 \phi} \sin \phi d\phi \right) = O(\ell^{-2}).$$

- The term  $A_{3,\ell}(\phi)$  leads to

$$\int_{C/\ell}^{\pi/2} N_{3,\ell}(\phi) \sin \phi d\phi = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \beta_{3,\ell}^2(\phi) \sin \phi d\phi$$

and, applying Lemma C.5 and Lemma C.6,

$$\int_{C/\ell}^{\pi/2} \left( \frac{\beta_{3,\ell}(\phi)}{\ell^3} \right)^2 \sin \phi d\phi = O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell \sin \phi} \sin \phi d\phi \right) = O(\ell^{-1}).$$

Since it is a dominant term, we compute now the leading constant of the term  $A_{3,\ell}(\phi)$ . Recalling the definition of  $\beta_{3,\ell}$  and Lemma C.3, we get

$$\begin{aligned} & -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \beta_{3,\ell}^2(\phi) \sin \phi d\phi \\ &= -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \left[ -\sin^3 \phi P_\ell'''(\cos \phi) + 3 \sin \phi \cos \phi P_\ell''(\cos \phi) + \sin \phi P_\ell'(\cos \phi) \right]^2 \sin \phi d\phi \\ &= -\frac{16}{\ell^6} \int_{C/\ell}^{\pi/2} \left[ -\sin^3 \phi \sqrt{\frac{2}{\pi}} \frac{\ell^{3-1/2}}{\sin^{3+1/2} \phi} \cos \psi_\ell^+ \right]^2 \sin \phi d\phi + O(\ell^{-2} \log \ell), \end{aligned}$$

where we have also applied Lemma C.6, Lemma C.7 and Lemma C.8 to identify the leading term. Now computing explicitly the integral, we have

$$\begin{aligned} -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \beta_{3,\ell}^2(\phi) \sin \phi d\phi &= -16 \frac{2}{\pi} \frac{1}{\ell} \int_{C/\ell}^{\pi/2} \cos^2 \psi_\ell^+ d\phi + O(\ell^{-2} \log \ell) \\ &= -16 \frac{2}{\pi} \frac{1}{\ell} \int_{C/\ell}^{\pi/2} \cos^2[(\ell + 1/2)\phi + \pi/4] d\phi + O(\ell^{-2} \log \ell) \\ &= -16 \frac{2}{\pi} \frac{1}{\ell} \frac{\pi}{4} + O(\ell^{-2} \log \ell) \\ &= -8\ell^{-1} + O(\ell^{-2} \log \ell). \end{aligned}$$

- The term  $A_{4,\ell}(\phi)$  leads to

$$\int_{C/\ell}^{\pi/2} N_{4,\ell}(\phi) \sin \phi d\phi = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi/2} \beta_{2,\ell}(\phi) \beta_{3,\ell}(\phi) \sin \phi d\phi$$

again, by Lemma C.5 and Lemma C.6, we have

$$\int_{C/\ell}^{\pi/2} \frac{\beta_{2,\ell}(\phi)}{\ell^3} \frac{\beta_{3,\ell}(\phi)}{\ell^3} \sin \phi d\phi = O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^2 \sin^2 \phi} \sin \phi d\phi \right) = O(\ell^{-2} \log \ell).$$

- For  $A_{5,\ell}(\phi)$  we have

$$\int_{C/\ell}^{\pi/2} N_{5,\ell}(\phi) \sin \phi d\phi = \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi/2} \gamma_{1,\ell}(\phi) \sin \phi d\phi + \frac{8 \cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi/2} \alpha_{2,\ell}(\phi) \beta_{2,\ell}^2(\phi) \sin \phi d\phi$$

and then, by Lemma C.5, and Lemma C.7,

$$\begin{aligned} \int_{C/\ell}^{\pi/2} \frac{\gamma_{1,\ell}(\phi)}{\ell^4} \sin \phi d\phi &= O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^{2+1/2} \sin^{2+1/2} \phi} \sin \phi d\phi \right) = O(\ell^{-2}), \\ \int_{C/\ell}^{\pi/2} \left( \frac{\beta_{2,\ell}(\phi)}{\ell^3} \right)^2 \frac{\alpha_{2,\ell}(\phi)}{\ell^2} \sin \phi d\phi &= O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^{3+1/2} \sin^{3+1/2} \phi} \sin \phi d\phi \right) = O(\ell^{-2}). \end{aligned}$$

- The term  $A_{6,\ell}(\phi)$  leads to

$$\int_{C/\ell}^{\pi/2} N_{6,\ell}(\phi) \sin \phi d\phi = \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi/2} \gamma_{2,\ell}(\phi) \sin \phi d\phi + \frac{8 \cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi/2} \alpha_{1,\ell}(\phi) \beta_{1,\ell}^2(\phi) \sin \phi d\phi$$

and applying again Lemma C.5 and Lemma C.7, we have

$$\begin{aligned} \int_{C/\ell}^{\pi/2} \frac{\gamma_{2,\ell}(\phi)}{\ell^4} \sin \phi d\phi &= O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^{1+1/2} \sin^{1+1/2} \phi} \sin \phi d\phi \right) = O(\ell^{-1-1/2}), \\ \int_{C/\ell}^{\pi/2} \left( \frac{\beta_{1,\ell}(\phi)}{\ell^3} \right)^2 \frac{\alpha_{1,\ell}(\phi)}{\ell^2} \sin \phi d\phi &= O\left( \int_{C/\ell}^{\pi/2} \frac{1}{\ell^{4+1/2} \sin^{4+1/2} \phi} \sin \phi d\phi \right) = O(\ell^{-2}). \end{aligned}$$

- For  $A_{7,\ell}(\phi)$ , we get

$$\int_{C/\ell}^{\pi/2} N_{7,\ell}(\phi) \sin \phi \, d\phi = \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi/2} \gamma_{4,\ell}(\phi) \sin \phi \, d\phi + \frac{8 \cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi/2} \alpha_{2,\ell}(\phi) \beta_{3,\ell}^2(\phi) \sin \phi \, d\phi$$

and, by Lemma C.5 and Lemma C.7,

$$\begin{aligned} \int_{C/\ell}^{\pi/2} \frac{\gamma_{4,\ell}(\phi)}{\ell^4} \sin \phi \, d\phi &= O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^{1/2} \sin^{1/2} \phi} \sin \phi \, d\phi\right) = O(\ell^{-1/2}), \\ \int_{C/\ell}^{\pi/2} \left(\frac{\beta_{3,\ell}(\phi)}{\ell^3}\right)^2 \frac{\alpha_{2,\ell}(\phi)}{\ell^2} \sin \phi \, d\phi &= O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^{1+1/2} \sin^{1+1/2} \phi} \sin \phi \, d\phi\right) = O(\ell^{-1-1/2}). \end{aligned} \quad (\text{D.4})$$

Since the leading term in  $\gamma_{4,\ell}(\phi)$  is oscillatory, we can get a sharper bound for the term in (D.4), by observing that

$$\begin{aligned} &\int_{C/\ell}^{\pi/2} \frac{\gamma_{4,\ell}(\phi)}{\ell^4} \sin \phi \, d\phi \\ &= \frac{1}{\ell^4} \int_{C/\ell}^{\pi/2} [\sin^4 \phi P_\ell''''(\cos \phi) - 6 \sin^2 \phi \cos \phi P_\ell'''(\cos \phi) + (-4 \sin^2 \phi + 3 \cos^2 \phi) P_\ell''(\cos \phi) + \cos \phi P_\ell'(\cos \phi)] \sin \phi \, d\phi \\ &= \frac{1}{\sqrt{\ell}} \int_{C/\ell}^{\pi/2} \frac{1}{\sin^{1/2} \phi} [\cos \psi_\ell^- - \frac{1}{8\ell\phi} \cos \psi_\ell^+] \sin \phi \, d\phi + O(\ell^{-1-1/2}) \\ &\leq \frac{1}{\sqrt{\ell}} \int_{C/\ell}^{\pi/2} \cos \psi_\ell^- \, d\phi + \frac{1}{\sqrt{\ell}} \int_{C/\ell}^{\pi/2} \frac{1}{8\ell\phi} \, d\phi + O(\ell^{-1-1/2}) \\ &= \frac{1}{\sqrt{\ell}} \frac{2}{1+2\ell} \left[ \sin\left(\frac{\ell\pi}{2}\right) + \sin\left(\frac{\pi}{4} - \frac{C}{2\ell} - C\right) \right] + O(\ell^{-1-1/2}) \\ &= O(\ell^{-1-1/2}). \end{aligned}$$

- Finally for  $A_{8,\ell}(\phi)$  we have

$$\int_{C/\ell}^{\pi/2} N_{8,\ell}(\phi) \sin \phi \, d\phi = \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi/2} \gamma_{3,\ell}(\phi) \sin \phi \, d\phi + \frac{8 \cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi/2} \alpha_{2,\ell}(\phi) \beta_{2,\ell}(\phi) \beta_{3,\ell}(\phi) \sin \phi \, d\phi$$

where, from Lemma C.5 and Lemma C.7, we have

$$\begin{aligned} \int_{C/\ell}^{\pi/2} \frac{\gamma_{3,\ell}(\phi)}{\ell^4} \sin \phi \, d\phi &= O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^{1+1/2} \sin^{1+1/2} \phi} \sin \phi \, d\phi\right) = O(\ell^{-1-1/2}), \\ \int_{C/\ell}^{\pi/2} \frac{\beta_{2,\ell}(\phi)}{\ell^3} \frac{\beta_{3,\ell}(\phi)}{\ell^3} \frac{\alpha_{2,\ell}(\phi)}{\ell^2} \sin \phi \, d\phi &= O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^{2+1/2} \sin^{2+1/2} \phi} \sin \phi \, d\phi\right) = O(\ell^{-2}). \end{aligned}$$

□

*Proof of Lemma 4.4.* We note that the second order terms are all of the form

$$\int_{C/\ell}^{\pi/2} \frac{a_{i,\ell}(\phi) a_{j,\ell}(\phi)}{\sqrt{(1 - 4\alpha_{2,\ell}^2/\lambda_\ell^2)(1 - 4\alpha_{1,\ell}^2/\lambda_\ell^2)}} \sin \phi \, d\phi$$

with  $i, j = 1, 2, \dots, 8$ . Applying Lemma C.6, Lemma C.7 and Lemma C.8 we identify, as in the proof of the previous lemma, that the product of two terms  $a_{i,\ell}(\phi) a_{j,\ell}(\phi)$  such that at least one is non dominant produces a non dominant term, so that, if  $(i, j) \neq (7, 7)$ , we immediately see that

$$\int_{C/\ell}^{\pi/2} \frac{a_{i,\ell}(\phi) a_{j,\ell}(\phi)}{\sqrt{(1 - 4\alpha_{2,\ell}^2/\lambda_\ell^2)(1 - 4\alpha_{1,\ell}^2/\lambda_\ell^2)}} \sin \phi \, d\phi = O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^2 \sin^2 \phi} \sin \phi \, d\phi\right) = O(\ell^{-2} \log \ell).$$

Instead, for  $(i, j) = (7, 7)$ , we have the square of the integrand function in (D.4), that in view of Lemma C.6, is immediately seen to be dominant, i.e.,

$$\int_{C/\ell}^{\pi/2} \frac{a_{7,\ell}^2(\phi)}{\sqrt{(1 - 4\alpha_{2,\ell}^2/\lambda_\ell^2)(1 - 4\alpha_{1,\ell}^2/\lambda_\ell^2)}} \sin \phi \, d\phi = O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell \sin \phi} \sin \phi \, d\phi\right) = O(\ell^{-1}).$$

Since we need the multiplicative constant of the leading terms we first note that:

$$a_{7,\ell}^2(\phi) = \frac{8^2}{\lambda_\ell^4} \left[ \gamma_{4,\ell}^2(\phi) + \frac{16}{(1 - 4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)^2} \frac{\alpha_{2,\ell}^2(\phi) \beta_{3,\ell}^4(\phi)}{\lambda_\ell^4} - \frac{8}{1 - 4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2} \frac{\gamma_{4,\ell}(\phi) \alpha_{2,\ell}(\phi) \beta_{3,\ell}^2(\phi)}{\lambda_\ell^2} \right],$$

then, by isolating the leading integral terms with the aid of Lemma C.6, Lemma C.7 and Lemma C.8, and finally by computing the integral, we get

$$\begin{aligned}
& \int_{C/\ell}^{\pi/2} a_{7,\ell}^2(\phi) \sin \phi \, d\phi \\
&= 8^2 \int_{C/\ell}^{\pi/2} \frac{\gamma_{4,\ell}^2(\phi)}{\lambda_\ell^4} \sin \phi \, d\phi + O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^2 \sin^2 \phi} \sin \phi \, d\phi\right) \\
&= 8^2 \int_{C/\ell}^{\pi/2} \left\{ \sqrt{\frac{\pi}{2}} \frac{1}{\ell^{1/2} \sin^{1/2} \phi} \left[ \cos \psi_\ell^- - \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right] + O\left(\frac{1}{\ell^{1+1/2} \sin^{1+1/2} \phi}\right) \right\}^2 \sin \phi \, d\phi + O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^2 \sin^2 \phi} \sin \phi \, d\phi\right) \\
&= 8^2 \int_{C/\ell}^{\pi/2} \left\{ \sqrt{\frac{\pi}{2}} \frac{1}{\ell^{1/2} \sin^{1/2} \phi} \left[ \cos \psi_\ell^- - \frac{1}{8\ell\phi} \cos \psi_\ell^+ \right] \right\}^2 \sin \phi \, d\phi + O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^2 \sin^2 \phi} \sin \phi \, d\phi\right) \\
&= 8^2 \int_{C/\ell}^{\pi/2} \left\{ \frac{2}{\pi \ell \sin \phi} \left[ \frac{1}{2} + \frac{1}{2} \cos(2\psi_\ell^-) + \frac{1}{64\ell^2 \phi^2} \cos^2 \psi_\ell^+ - \frac{1}{4\ell\phi} \cos \psi_\ell^- \cos \psi_\ell^+ \right] \right\} \sin \phi \, d\phi + O\left(\int_{C/\ell}^{\pi/2} \frac{1}{\ell^2 \sin^2 \phi} \sin \phi \, d\phi\right) \\
&= 32\ell^{-1} + O(\ell^{-2} \log \ell).
\end{aligned}$$

□

## E Nonsingularity of the covariance matrix for $\phi < c/\ell$

We only need to show that, after scaling, the determinant of the matrix  $A_\ell(\psi)$ , evaluated for points on the equatorial line  $x = (\pi/2, \phi)$ ,  $y = (\pi/2, 0)$ , is strictly positive for  $c > \psi > 0$ ; for points outside the equator the covariance matrix is obtained by a change of basis: the corresponding matrix does not depend on  $\ell$  and can be easily shown to be full rank.

By expanding the terms of the matrix up to order 4 around  $\psi = 0$ , for  $\lambda_\ell = \ell(\ell + 1)$  we have

$$\begin{aligned}
\alpha_{1,\ell}(\psi) &= \frac{\ell + 1}{2\ell} - \frac{\lambda_\ell(\lambda_\ell - 2)\psi^2}{16\ell^4} + \frac{(\ell - 1)(\ell + 1)(\ell + 2)(\lambda_\ell - 4)\psi^4}{2^7 3 \ell^5} + O(\psi^6) \\
\alpha_{2,\ell}(\psi) &= \frac{\ell + 1}{2\ell} - \frac{\lambda_\ell(3\lambda_\ell - 2)\psi^2}{2^4 \ell^4} + \frac{\lambda_\ell(5(\ell - 1)\ell(\ell + 1)(\ell + 2) + 2^3)\psi^4}{2^7 3 \ell^6} + O(\psi^6)
\end{aligned}$$

It should be noted that, as  $\ell \rightarrow \infty$ , all coefficients converge to constants; more importantly, the constants involved in the  $O$ -notation for all the  $O(\psi^6)$  terms are universal. A computer-oriented computation yields the following Taylor expansion for the determinant:

$$\begin{aligned}
\det(A_\ell(\psi)) &= \frac{(\ell - 1)(\ell + 1)^4(\ell + 2)(3\ell(\ell + 1) - 2)\psi^4}{2^8 \ell^8} + O(\psi^6) \\
&= \psi^4 \left( \frac{3}{2^8} + O(\ell^{-1}) \right) + O(\psi^6) > 0
\end{aligned}$$

the inequality holding for  $c$  sufficiently small, because by the above, the  $O(\psi^6)$  term is universal.

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