On the Geometric Interpretation of the Polynomial Lie Bracket for nonlinear time-delay systems

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***Abstract*— Time–delay systems are infinite dimensional, thus standard differential geometric tools can not be applied in a straightforward way. Though, thanks to a suitable extended Lie Bracket – or Polynomial Lie Bracket – which has been introduced recently, it is still possible to build up a geometric framework to tackle the analysis and synthesis problems for nonlinear time delay systems. The major contribution herein is to show that those geometric generalizations are not just formal, but are interpreted in terms of successive forward and backward flows similarly to the Lie Bracket of delay free vector fields.**

I. Introduction

Nonlinear time–delay models have to be considered in many control applications. For instance they are used in control network systems [7] whenever the communication time can not be neglected with respect of the dynamics of individual systems; biological systems are also known to involve time–delay models [12].

Whereas major advances were obtained for the stabilization of this class of systems, fundamental structural problems such as controllability/accessibility or the observer design remain open. The theory of delay–free nonlinear control systems went through a dramatic success story in the 1980's in solving such structural problems thanks to the differential geometric approach. Apart some attempts for its extension to time–delay nonlinear systems [5], [9], it still remains in a pioneering stage. For time-delay systems affected by constant commensurate delays, a new approach was proposed in [2] based on the differential representation of the given dynamics. Such an approach has allowed to obtain interesting results either on accessibility of a class of time delay systems [4] and observer design [3], through the use of a new operator, the extended Lie Bracket operator, defined on a finite dimensional system which can be associated to the given time-delay system. Such an approach has been further developed in [1] with the introduction of a more general tool: the Polynomial Lie Bracket. The main idea beyond the extended Lie bracket operator was the consideration that even though time delay systems are infinite dimensional systems, when affected by constant commensurate delays, they could be approximated by some finite dimensional system of appropriate dimension linked to the delay affecting the system. The Polynomial Lie bracket does not require this kind of approximation, an its use fits well for the comprehension of the structural properties of the given system.

In order to show both the peculiarities of time delay systems with respect to delay free ones and the importance of the Polynomial Lie bracket, in the present paper we will consider the class of nonlinear single input time–delay driftless systems affected by constant commensurate delay, which coversthe case of constant multiple commensurate delays as well [6]. Within such a class of systems we will focus our attention on the simple case of system described by the differential equation

$$\dot{x}(t) \quad = \quad g(x(t), x(t-\tau))u(t), \tag{1}$$

where τ is a constant commensurate delay and the function $g(x(t), x(t-\tau))$, is analytic in its arguments. Already in this particular case it is possible to understand the role of the delay in the accessibility problem. As well know, in fact, in the delay free case, a single input system of dimension $n > 1$ is never accessible, whereas in [4] it was shown that single input delay systems may happen to be accessible. A flavour of what happens when multiple delays are present is also given.

The outline is as follows. Section II is devoted to notations which are used throughout the paper, as well as to recalls on the Generalized Lie Bracket and related results. The main contributions on the Lie Bracket interpretation are given in Section III III. Conclusions and perspectives are provided in Section IV.

II. Preliminaries

A. An introductory example

Consider system (1) with $g(\cdot) = \begin{pmatrix} x_2(t-\tau) \\ 1 \end{pmatrix}$:

$$\dot{x}(t) = \begin{pmatrix} x_2(t-\tau) \\ 1 \end{pmatrix} u(t) \tag{2}$$

In Figure 1 below, the trajectory of the system is shown for a switching sequence of the input signal. The input switches from 1 to -1 and includes four such forward and backward cycles. Differently from what would happen in the delay free case when the input switches, the trajectory does not stay on the same integral manifold of one single vector field. A new direction is taken in the motion, which shows that the delay adds some additional freedom for the control direction and yields accessibility of the example under consideration. This is a surprising property of single input driftless nonlinear time–delay systems and contradicts pre-conceived ideas as it could not happen for delay free systems. As it will be argued in Section III the motion in the x_1 direction of the final point of each cycle has to be interpreted as the motion along the

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nonzero Lie Bracket of the delayed control vector field with itself. These general intuitive considerations are formalized through formal and precise definitions in the paper.

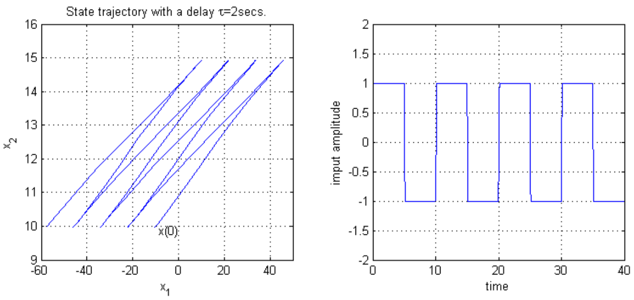

Fig. 1. Forward and backward integration yields a motion in a specific direction

B. Notations

Consider the class of nonlinear time-delay systems (1). General notations valid throughout the paper are as follows.

- $\mathbf{x}_{[s]}^T = (x^T(t), \cdots x^T(t-s\tau)) \in I\!R^{(s+1)n}$, denotes the vector consisting of the first $(s+1)n$ components of the state of the infinite dimensional system (1). $\mathbf{x}_{[0]} = [x_{1,[0]}, \cdots, x_{n,[0]}]^T = x(t) \in I\!R^n$, will denote the instantaneous values of the state variable.
- $\mathbf{x}_{[s]}^T(-i) = (x^T(t-i\tau), \cdots x^T(t-s\tau-i\tau))$. Accordingly, $x_{j,[0]}(-i) := x_j(t-i\tau)$ denotes the j–th component of the instantaneous values of the state variable delayed by $D = i\tau$. When no confusion is possible the subindex will be omitted so that $\mathbf{x}$ will stand for $\mathbf{x}_{[s]}$.
- $\mathcal{K}$ denotes the field of causal meromorphic functions $f(\mathbf{x}_{[s]}, \mathbf{u}_{[j]})$, with $s, j \in I\!N$.
- Given a function $f(\mathbf{x}_{[s]}, \mathbf{u}_{[j]})$, we will denote by $f(-l) = f(\mathbf{x}_{[s]}(-l), \mathbf{u}_{[j]}(-l))$;
- d is the standard differential operator;
- δ represents the backward time-shift operator: for $a(\cdot), f(\cdot) \in \mathcal{K}$: $\delta[\,a\,df] = a(-1)\delta df = a(-1)df(-1)$;
- $\mathcal{K}(\delta]$ is the (left) ring of polynomials in δ with coefficients in $\mathcal{K}$.

As it was already recalled, the proposed approach starts by considering the differential representation of the given dynamics. Thus one gets that, using the notation just introduced, such an infinitesimal representation is given by

$$d\dot{\mathbf{x}}_{[0]} = f(\mathbf{x}, \mathbf{u}, \delta)d\mathbf{x}_{[0]} + \hat{g}(\mathbf{x}, \delta)d\mathbf{u}_{[0]} \qquad (3)$$

where

$$\begin{aligned} f(\mathbf{x}, \mathbf{u}, \delta) &= \frac{\partial g(\mathbf{x}_{[0]}, \mathbf{x}_{[0]}(-1))}{\partial \mathbf{x}_{[0]}} + \frac{\partial g(\mathbf{x}_{[0]}, \mathbf{x}_{[0]}(-1))}{\partial \mathbf{x}_{[0]}(-1)}\delta \\ \hat{g}(\mathbf{x}, \delta) &= g(\mathbf{x}_{[0]}, \mathbf{x}_{[0]}(-1)) \end{aligned}$$

C. Generalized Lie Derivative and Generalized Lie Bracket

The notions defined next, as the polynomial Lie bracket, make sense for time–delay control systems (3). The definitions of Generalized Lie Derivative and Generalized Lie Bracket are recalled now in the case of causal functions and submodule elements. The more general case can be found in [1].

More precisely, a general submodule element $r(\mathbf{x}, \delta) = \sum_{j=0}^{s} r^j(\mathbf{x})\delta^j$ gives rise to a series of Lie Derivatives, called Generalized Lie Derivatives and defined as follows.

Definition 1: Given the function $\varphi(\mathbf{x}_{[s]})$ and the submodule element $r(\mathbf{x}, \delta) = \sum_{j=0}^{s} r^j(\mathbf{x})\delta^j \in \mathcal{K}^n(\delta]$, the Generalized Lie derivative $L_{r^\mu(\mathbf{x})}\varphi(\mathbf{x}_{[s]})$ is defined for $\mu = 0, ..., s$ as follows

$$L_{r^\mu(\mathbf{x})}\varphi(\mathbf{x}_{[s]}) = \sum_{l=0}^{\mu} \frac{\partial \varphi(\mathbf{x}_{[s]})}{\partial \mathbf{x}_{[0]}(-l)} r^{\mu-l}(\mathbf{x}(-l)). \qquad (4)$$

Generalized Lie derivatives according to Definition 1 are standard Lie derivatives of $\varphi(\mathbf{x}_{[s]})$ along the following extended vector fields.

$$\begin{pmatrix} \mathbf{r}^0(\mathbf{x}) & \cdots & \mathbf{r}^s(\mathbf{x}) \\ 0 & \ddots & \vdots \\ 0 & \cdots & \mathbf{r}^0(\mathbf{x}(-s)) \end{pmatrix}$$

Starting from two polynomial submodule elements, yields again a series vector fields named Generalized Lie Brackets, and defined as follows.

Definition 2: Let $r_q(\mathbf{x}, \delta) = \sum_{j=0}^{s} r_q^j(\mathbf{x})\delta^j \in \mathcal{K}^n(\delta]$, $q = 1, 2$. For any $k, l \geq 0$, the Generalized Lie bracket $[r_1^k(\cdot), r_2^l(\cdot)]_{E_i}$, on $I\!R^{(i+1)n}$, $i \geq 0$, is defined as

$$\left[r_1^k(\cdot), r_2^l(\cdot)\right]_{E_i} = \sum_{j=0}^{i} \Big([r_1^{k-j}, r_2^{l-j}]_E \Big)^T_{(\mathbf{x}(-j))} \frac{\partial}{\partial \mathbf{x}_{[0]}(-j)}, (5)$$

where

$$\left[r_1^k(\cdot), r_2^l(\cdot)\right]_E = \Big(L_{r_1^k(\mathbf{x})} r_2^l(\mathbf{x}) - L_{r_2^l(\mathbf{x})} r_1^k(\mathbf{x}) \Big). \qquad (6)$$

The Generalized Lie brackets (5) have shown to characterize the integrability conditions, that is when the $\Delta^\perp(\delta]$ is generated by $d\lambda_\mu(\mathbf{x}) = \Lambda_\mu(\mathbf{x}, \delta)d\mathbf{x}_{[0]}$, $\mu \in [1, n-j]$ [2]. Conditions in terms of $\Delta(\delta]$ have instead been given in [8].

Let us finally recall the definitions of Lie bracket for time-delay systems and polynomial Lie bracket, introduced in [1] where they have been effectively used to address the integrability problem of any (not necessarily causal) submodule $\Delta^\perp(\delta]$ and characterize in a complete way the accessibility of a given time-delay system.

Definition 3: Given $r_i(\mathbf{x}_{[s]}, \delta) \in \mathcal{K}^{*n}(\delta]$, $i = 1, 2$, the Lie Bracket $[r_1(\mathbf{x}_{[s]}, \delta), r_2(\mathbf{x}_{[s]}, \delta)]$, is a $(4s+1)$-uple of polynomial vectors $r_{12,j}(\mathbf{x}, \delta)$, defined as

$$r_{12,j}(\mathbf{x}, \delta) = \sum_{\ell=0}^{2s} [r_1^{\ell-j}, r_2^{\ell}]_{E_0}\delta^\ell_, \; j \in [-2s, 2s]. \qquad (7)$$

Recalling that a polynomial vector $r_1(\mathbf{x}_{[s]}, \delta)$ acts on a function $\epsilon(t)$ and denoting its image as $\mathbf{R}_1(\mathbf{x}_{[s]}, \epsilon) :=$

$\sum_{j=0}^{s} r_1^j(\mathbf{x})\epsilon(-j)$, the polynomial Lie Bracket is then defined as follows:

Definition 4: Given $r_i(\mathbf{x}_{[s]},\delta) \in \mathcal{K}^n(\delta]$, $i = 1,2$, the polynomial Lie Bracket $[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)]$ is defined as

$$[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)] := ad_{\mathbf{R}_1(\mathbf{x}_{[s]},\epsilon)} r_2(\mathbf{x}_{[s]},\delta) =$$
$$\dot{r}_2(\mathbf{x},\delta)|_{\dot{x}_{[0]}=\mathbf{R}_1(\mathbf{x},\epsilon)} - \sum_{k=0}^{s} \frac{\partial \mathbf{R}_1(\mathbf{x}_{[s]},\epsilon)}{\partial \mathbf{x}(-k)}\delta^k r_2(\mathbf{x},\delta).$$

With some abuse, the Polynomial Lie Bracket and the standard Lie bracket are both denoted by $[.,.]$. No confusion is possible, since in the Polynomial Lie bracket, some $\epsilon(i)$ will always be present inside the brackets.

we end the present section by highlighting the relations between the Lie bracket, the Generalized Lie bracket and the Polynomial Lie bracket. More precisely

- The link between the Lie bracket (7) and the Generalized Lie bracket (5) can be easily established by noting that
$$r_{12,j}(\mathbf{x},\delta) = (I_n\delta^{2s},\cdots,I_n\delta,I_n)\left([r_1^{2s-j}, r_2^{2s}]_{E_{2s}}|_{\mathbf{x}(2s)}\right)$$
- Standard computations on the Polynomial Lie bracket show that
$$[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)] = \sum_{j=-2s}^{2s} r_{12,j}(\mathbf{x},\delta)\epsilon(j). \quad (8)$$
- If the given vectors are independent of δ and of the delay, one recovers (up to $\epsilon(0)$), the standard Lie bracket since
$$[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)] = [r_1^0(x)\epsilon(0), r_2^0(x)] = [r_1^0, r_2^0]\epsilon(0).$$
Instead, if delays are present, $[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)]$ immediately enlightens some important differences with respect to the delay–free case, such as the loss of validity of the Straightening Theorem. In fact, since the term depending on δ undergoes a different kind of operation with respect to the term depending on ϵ, starting from $r(\mathbf{x},\delta)$ and its corresponding image $\mathbf{R}(\mathbf{x},\epsilon)$, in general
$$\dot{r}(\mathbf{x},\delta)|_{\dot{x}_{[0]}=\mathbf{R}(\mathbf{x},\epsilon)} \neq \sum_{k=0}^{s} \frac{\partial \mathbf{R}(\mathbf{x}_{[s]},\epsilon)}{\partial \mathbf{x}(-k)}\delta^k r(\mathbf{x},\delta)$$
which shows that in general $[\mathbf{R}(\mathbf{x},\epsilon), r(\mathbf{x},\delta)] \neq 0$. For instance, in example (2) taking $r(\mathbf{x},\delta) = \hat{g}(\mathbf{x},\delta)$, one has $r(\mathbf{x},\delta) = \begin{pmatrix} x_2(-1) \\ 1 \end{pmatrix}$. Then $\mathbf{R}(\mathbf{x},\epsilon) = \begin{pmatrix} x_2(-1) \\ 1 \end{pmatrix}\epsilon(0)$ and
$$\begin{aligned}[\mathbf{R}(\mathbf{x},\epsilon), r(\mathbf{x},\delta)] &= \begin{pmatrix} \epsilon(-1)-\epsilon(0)\delta \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}(\epsilon(-1)-\epsilon(0)\delta) \neq 0.\end{aligned}$$

III. The Polynomial Lie Bracket interpretation

To clarify the role of the polynomial Lie Bracket, recall as a preliminary that in the delay–free case the geometric interpretation of the Lie bracket can be easily obtained by considering a simple example of a two input driftless system. As reported in [11] if one considers the dynamic system

$$\dot{x}(t) = g_1(x(t))u_1(t) + g_2(x(t))u_2(t)$$

and applies the sequence $[(0,1),(1,0),(0,-1),(-1,0)]$, where each control acts exactly for a time h, then the first order derivative of the flow in the origin is zero, while the second order derivative of the flow in the origin is exactly twice the bracket $[g_2, g_1]$. Of course if one refers to a one input system then using a constant control allows to move forward or backward on a unique integral manifold of the considered control vector field.

Based on this consideration, let us go back to the time delay system (1) and consider the dynamics over four steps applying the control sequence $[1,0,-1,0]$. Then one gets that

$$\begin{aligned}\dot{x}(t) &= g(x(t),x(t-\tau))u(t) \\ \dot{x}(t-\tau) &= g(x(t-\tau),x(t-2\tau))u(t-\tau) \\ \dot{x}(t-2\tau) &= g(x(t-2\tau),x(t-3\tau))u(t-2\tau) \\ \dot{x}(t-3\tau) &= g(x(t-3\tau),x(t-4\tau))u(t-3\tau)\end{aligned} \quad (9)$$

Such a system can be rewritten in the form

$$\dot{z}(t) = g_1(z(t))u_1(t) + g_2(z(t))u_2(t) \quad (10)$$

where $z_1(t) = x(t)$, $z_2(t) = x(t-\tau)$, $z_3(t) = x(t-2\tau)$, $z_4(t) = x(t-3\tau)$, $u_1(t) = u(t-\tau) = -u(t-3\tau)$ and $u_2(t) = u(t) = -u(t-2\tau)$. In (10)

$$g_1(z) = \begin{pmatrix} 0 \\ g(z_2,z_3) \\ 0 \\ g(z_4,c_0) \end{pmatrix}, g_2(z) = \begin{pmatrix} g(z_1,z_2) \\ 0 \\ g(z_3,z_4) \\ 0 \end{pmatrix}.$$

with c_0 the initial condition of x on the interval $[-4\tau,-3\tau)$. Of course not all the trajectories of $z_1(t)$ in (10) will be trajectories of $x(t)$ in (9), whereas all the trajectories of $x(t)$ for $t \in [0,4\tau)$ in (9) can be recovered as trajectories of $z_1(t)$ in (10) for $t \in [0,4\tau)$, whenever the system is initialized with constant initial conditions.

The sequence $[(0,1),(1,0),(0,-1),(-1,0)]$ for system (10), can then be recovered by applying the sequence $[1,0,-1,0]$ to $u(t)$ with the initialization $u = 0$ on the interval $[-\tau,0)$ and the switching applies exactly after τ. Such an example shows immediately that the second order

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derivative in 0 is characterized by

$$[g_1, g_2] \quad = \quad \left[\begin{pmatrix} 0 \\ g(z_2, z_3) \\ 0 \\ g(z_4, c_0) \end{pmatrix}, \begin{pmatrix} g(z_1, z_2) \\ 0 \\ g(z_3, z_4) \\ 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} 0 \\ g(x(t-\tau), x(t-2\tau)) \\ 0 \\ g(x(t-3\tau), x(t-4\tau)) \end{pmatrix}, \begin{pmatrix} g(x(t), x(t-\tau)) \\ 0 \\ g(x(t-2\tau), x(t-3\tau)) \\ 0 \end{pmatrix}\right].$$

It is immediately seen that the $\frac{\partial}{\partial x(t)}$ component is given by $\frac{\partial g(x(t),x(t-\tau))}{\partial x(t-\tau)} g(x(t-\tau), x(t-2\tau))$ which in general is nonzero.

While the previous discussion allows to mimic what happens in the delay free case by using a specific sequence, in the general case one may refer to other kinds of sequences. Before going into the technical details, consider again the dynamics

$$\dot{x}(t) = \begin{pmatrix} x_2(t-\tau) \\ 1 \end{pmatrix} u(t)$$

with the piecewise control which varies from 1 to -1 every 5 seconds. In the next figures the role of the delay and the role of the duration of the control are shown through simulations.

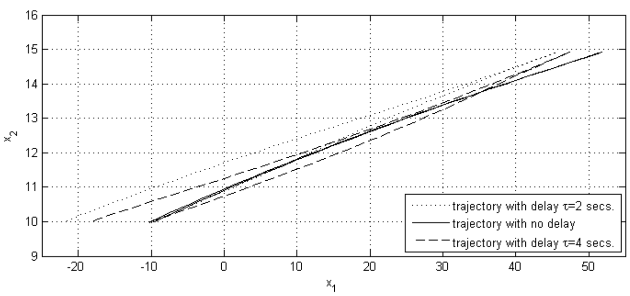

Fig. 2. Trajectory of the dynamics with the same input signal and different delays

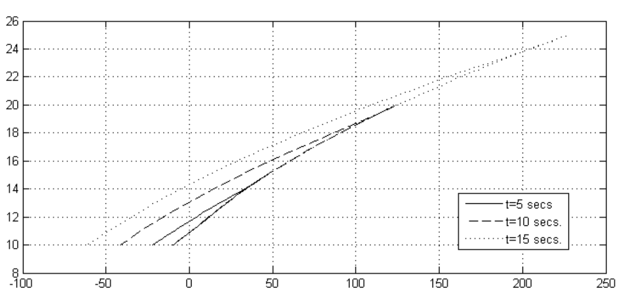

Fig. 3. Trajectory of the dynamics with the same delay by input signals of different duration

More precisely, in Figure 2, one cycle of the same switching signal is considered, for different delays. The behaviour of the system trajectory easily shows that the delay itself could be used as an additional “control”. It is worth noting that in the particular case in which the control varies from 1, to -1 with a period equal to an integer multiple of the delay, the trajectory would remain on one unique integral manifold as in the delay–free case.

In Figure 3 the input signal still switches from 1 to -1, but the duration changes. It is easily seen that the end point are all on a curve which is interpreted exactly as the integral manifold of the Polynomial Lie bracket.

Let us now go through the technical details, and consider the given dynamics (1), with its differential representation (3). Starting from $\hat{g}(\mathbf{x}, \delta) = g(x(t), x(t-\tau))$, let $\hat{\mathbf{G}}(\mathbf{x}, \epsilon) = g(x(t), x(t-\tau))\epsilon(0)$ one can thus compute the associated Polynomial Lie Bracket

$$\begin{aligned}[\hat{\mathbf{G}}(x, \epsilon), \hat{g}(x, \delta)] \quad &= \quad \dot{g}(\mathbf{x}(0), \mathbf{x}(-\tau)|_{\dot{\mathbf{x}}_{[0]} = g(\mathbf{x}(0), \mathbf{x}(-\tau))\epsilon(0)} \\ &\quad -\epsilon(0)\frac{\partial g(\mathbf{x}(0), \mathbf{x}(-\tau))}{\partial \mathbf{x}(0)} g(\mathbf{x}(0), \mathbf{x}(-\tau)) \\ &\quad -\epsilon(0)\frac{\partial g(\mathbf{x}(0), \mathbf{x}(-\tau))}{\partial \mathbf{x}(-\tau)} \delta g(\mathbf{x}(0), \mathbf{x}(-\tau)) \\ &= \frac{\partial g(\mathbf{x}(0), \mathbf{x}(-\tau))}{\partial \mathbf{x}(-\tau)} g(\mathbf{x}(-\tau), \mathbf{x}(-2\tau))(\epsilon(-\tau) - \epsilon(0)\delta)\end{aligned}$$

which is thus different from zero. In the example considered throughout the paper, we have already shown that

$$[\hat{\mathbf{G}}(x, \epsilon), \hat{g}(x, \delta)] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\epsilon(-1) - \epsilon(0)\delta)$$

Consider the dynamics (1) and assume th control changes over 2 steps, $t_2 = t_1 = t$. The flow is thus given by

$$\begin{aligned} x(t_1, t_2) \quad &= \quad x(t_1 + t_2) = \phi_{u_2}^{t_2}(\phi_{u_1}^{t_1}(x(0)) \\ x(t_1, t_2, \tau) \quad &= \quad x(t_1 + t_2 - \tau) = \phi_{\bar{u}_2}^{t_2}(\phi_{\bar{u}_1}^{t_1}(x(-\tau)).\end{aligned}$$

Note that for the retarded trajectory $x(t_1, t_2, \tau)$ the control $\bar{u}$ is no more constant over the interval $(t_1 - \tau, t_2 - \tau)$, but it is piecewise constant.

The first order derivative with respect to time of the flow, x', is then given by

$$x'(t_1, t_2) \quad = \quad \frac{\partial \phi_{u_2}^{t_2}(x(t_1))}{\partial t_2}\frac{\partial t_2}{\partial t} + \frac{\partial \phi_{u_2}^{t_2}(\phi_{u_1}^{t_1}(x(0))}{\partial t_1}\frac{\partial t_1}{\partial t}$$

Accordingly

$$\begin{aligned} x'(t_1, t_2) \quad &= \quad g(x(\bar{t}_2), x(\bar{t}_2 - \tau))u_2(\bar{t}_2) \\ &\quad + \frac{\partial \phi_{u_2}^{t_2}}{\partial x(\bar{t}_1)} g(x(\bar{t}_1), x(\bar{t}_1 - \tau))u_1(\bar{t}_1)\end{aligned}$$

Setting now $u_2 = -u_1$ yields

$$\begin{aligned} x'(t_1, t_2) \quad &= \\ &\left(g(x(\bar{t}_2), x(\bar{t}_2 - \tau)) - \frac{\partial \phi_{u_2}^{t_2}}{\partial x(\bar{t}_1)} g(x(t_1), x(t_1 - \tau))\right) u_2(\bar{t}_2) \\ x'(t_1) \quad &= \quad g(x(t_1), x(t_1 - \tau))u_1(\bar{t}_1)\end{aligned}$$

Recalling that $t_1 = t_2 = t$, let $t \to 0$, so that

$$\begin{aligned} x'(0,0) &= 0 \\ x'(0) &= g(x(0), x(-\tau))u_1(\bar{t}_1) \end{aligned}$$

One can repeat the same reasoning on $x(t_1, t_2, \tau)$, taking as initial point $x(-\tau)$. One thus has that

$$\begin{aligned} x'(t_1,t_2,\tau) &= \frac{\partial \phi_{u_2}^{t_2}(x(t_1-\tau))}{\partial t_2}\frac{\partial t_2}{\partial t} + \frac{\partial \phi_{u_2}^{t_2}\phi_{u_1}^{t_1}(x(-\tau))}{\partial t_1}\frac{\partial t_1}{\partial t} \\ &= g(x(\bar{t}_2-\tau), x(\bar{t}_2-2\tau))\bar{u}_2(\bar{t}_2-\tau) \\ &+\frac{\partial \phi_{\bar{u}_2}^{t_2}}{\partial x(\bar{t}_1-\tau)} g(x(\bar{t}_1-\tau), x(\bar{t}_1-2\tau))\bar{u}_1(\bar{t}_1-\tau) \end{aligned}$$

Since $u_2 = -u_1$, one immediately gets that $\bar{u}_2 = -\bar{u}_1$ so that

$$\begin{aligned} x'(t_1,t_2,\tau) &= \Bigg(g(x(\bar{t}_2-\tau), x(\bar{t}_2-2\tau)) \\ &- \frac{\partial \phi_{\bar{u}_2}^{t_2}}{\partial x(\bar{t}_1-\tau)} g(x(\bar{t}_1-\tau), x(\bar{t}_1-2\tau))\Bigg)\bar{u}_2(\bar{t}_2-\tau) \\ x'(t_1,\tau) &= g(x(t_1-\tau), x(t_1-2\tau))\bar{u}_1(\bar{t}_1-\tau) \end{aligned}$$

and $x'(0,0,\tau) = 0$ representing the first order derivative, compute starting from $x(-\tau)$.

Let us now consider the second order derivative of $x(t_1, t_2)$, taking as initial point $x(0)$. In this case, through standard computations one has that

$$\begin{aligned} x''(t_1,t_2) &= \frac{\partial^2 \phi_{u_2}^{t_2}(x(t_1))}{\partial t_2^2} + 2\frac{\partial^2 \phi_{u_2}^{t_2}(\phi_{u_1}^{t_1}(x(0))}{\partial t_1 \partial t_2} \\ &+ \frac{\partial^2 \phi_{u_2}^{t_2}(\phi_{u_1}^{t_1}(x(0))}{\partial t_1^2} \end{aligned}$$

Through standard computations one gets that for $t_1 = t_2 = t$ which goes to zero

$$\begin{aligned} x''(0,0) &= u_2(\bar{t}_2)\frac{\partial g(x(0), x(-\tau))}{\partial x(0)} x'(0,0) \\ &+ u_2(\bar{t}_2)\frac{\partial g(x(\bar{0}), x(-\tau))}{\partial x(-\tau)} \bar{x}'(0,0,\tau) \end{aligned} \tag{11}$$

where $\bar{x}'(0,0,\tau)$ is $x'(t_1 + t_2 - \tau)$ computed starting from $x(0)$ and for $t_1 = t_2 = t$ which goes to zero.

$$\bar{x}'(0,0,\tau) = x'(0,0,\tau) - g(x(-\tau), x(-2\tau))u_1(-\tau)$$

By considering that $x'(0,0) = 0$, substituting the previous expression into (11) one thus gets for $t \to 0$

$$\begin{aligned} &x''(0,0) = \\ &-\tfrac{\partial g(x(0), x(-\tau))}{\partial x(-\tau)} g(x(-\tau), x(-2\tau))u(\bar{t}_1)\bar{u}(t_1-\tau) \end{aligned}$$

Let us finally compute the differential of $x''(0,0)$ with respect to the control variable.

We get that

$$\begin{aligned} dx''(0,0) &= \\ &-\frac{\partial g(x(0), x(-\tau))}{\partial x(-\tau)} g(x(-\tau), x(-2\tau))\,(u(-\tau) + u(0)\delta)\,du \end{aligned}$$

which coincides with our expression whenever we set $\epsilon(-k) = -u(-k\tau)$.

IV. Some concluding remarks

The case discussed in this paper already enlightens some peculiarities of time–delay systems with respect to the delay free case. In the particular case considered, the Extended Lie Bracket and the Polynomial Lie Bracket , end up with the same result. in general, as already underlined in the remarks, the polynomial Lie bracket ends up with a collection of extended Lie bracket ,which actually define the directions that can be used to move. we end this discussion by proposing another example which highlight this point.

For instance, let us consider the dynamic system

$$\dot{x}(t) = \begin{pmatrix} x_3(t-\tau) \\ x_3(t-2\tau) \\ 1 \end{pmatrix} u(t)$$

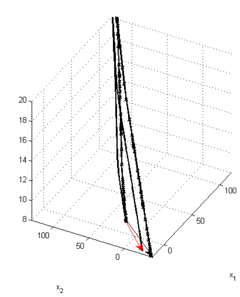

Fig. 4. Trajectories obtained through two different input sequences

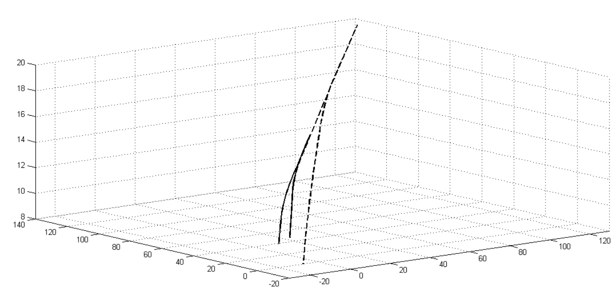

Fig. 5. Trajectories obtained through the same input sequence but with different duration

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Using the definition of Polynomial Lie Bracket, one gets that

$$r(\mathbf{x},\delta) = \begin{pmatrix} x_3(t-\tau) \\ x_3(t-2\tau) \\ 1 \end{pmatrix}, \quad \mathbf{R}(\mathbf{x},\epsilon) = \begin{pmatrix} x_3(t-\tau) \\ x_3(t-2\tau) \\ 1 \end{pmatrix} \epsilon(0).$$

Accordingly

$$[\mathbf{R}(\mathbf{x},\epsilon), r(\mathbf{x},\delta)] \quad = \quad \begin{pmatrix} \epsilon(-1) - \epsilon(0)\delta \\ \epsilon(-2) - \epsilon(0)\delta^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (\epsilon(-1) - \epsilon(0)\delta) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \epsilon(-2) - \epsilon(0)\delta^2$$

so that we find two directions $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ which span the plane (x_1, x_2).

In figure 4 two different sequences for the control are used which show that one can get two different directions in the plane (x_1, x_2), while in figure 5 the same control sequence is used but with different duration.

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