Sapienza Università di Roma

Dottorato di Ricerca in Statistica Metodologica

Tesi di Dottorato XXVI Ciclo – anno 2013

Dipartimento di Scienze Statistiche

Partial differential equations related to time-changed processes and pseudoprocesses

Bruno Toaldo

Contents

Ringraziamenti

Ovviamente non ho fatto tutto da solo.

Un grazie in particolare è dovuto

alla mia famiglia

per avermi reso capace di arrivare fin *qui*,

a Enzo e al suo gruppo

per aver dato un senso a *qui*,

ai ragazzi del dottorato in statistica metodologica

per aver reso *qui* un posto più divertente,

e a Natalia

per aver reso tutto più semplice.

Introduction

The aim of the present work is to present the interplay between time-changed stochastic processes (and pseudoprocesses) and partial differential equations. With a certain abuse of language we refer to the time-change of the process $X(t)$, $t > 0$, as

$$X\left(Y(t)\right), \qquad t > 0. \tag{1}$$

A well-known example of time-changed process is the iterated Brownian motion $B_1\left(B_2(t)\right)$, $t > 0$, (see Burdzy (1993a,b, 1998)) where B_i, $i = 1, 2$ are independent. In DeBlassie (2004) it has been shown that the distribution $q(x, t)$ of the iterated Brownian motion solves the fourth-order equation

$$\frac{\partial}{\partial t} q(x,t) = \frac{1}{2^3} \frac{\partial^4}{\partial x^4} + \frac{1}{2\sqrt{2\pi t}} \frac{d^2}{dx^2} \delta(x), \qquad x \in \mathbb{R}, t > 0. \tag{2}$$

The study of the iterated Brownian motion has been stimulated by the fact that it is able to model diffusions in cracks (Burdzy (1998), Chudnovsky and Kunin (1987)). In the present work we choose $Y(t)$, $t > 0$, with non-decreasing paths and in particular we will focus on subordinators and their inverses. This restriction is crucial.

Subordination, a brief overview

A subordinator $\sigma^f(t)$ is a stochastic process with independent and stationary increments and non-decreasing paths. Furthermore a subordinator is a Lévy process and thus one have (see Itô (1942))

$$\sigma^f(t) = bt + \sum_{0 \leq s \leq t} \Delta_s, \qquad \text{a.s. } \forall t > 0, \tag{3}$$

where Δ_s, $s > 0$, is a Poisson point process with characteristic measure $\nu(ds) + a\delta_\infty$, δ_∞ is the Dirac point mass at infinity and ν, a, b, are known as the Lévy triplet and are such that

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-sx}\right) \nu(ds), \qquad \int_0^\infty (s \wedge 1)\nu(ds) < \infty. \tag{4}$$

We call $X\left(\sigma^f(t)\right)$, a subordinate process. The transition probabilities $\mu_t(B) = \Pr\left\{\sigma^f(t) \in B\right\}$, $B \subseteq [0, \infty)$, of subordinators are convolution semigroups supported on $[0, \infty)$, for which

$$\int_0^\infty e^{-\lambda s} \mu_t(ds) = e^{-tf(\lambda)}. \tag{5}$$

We recall that in general a family of sub-probability measures p_t, $t \geq 0$, on $\mathbb{R}^n$ is said to be a convolution semigroup if

1. $p_t\left(\mathbb{R}^n\right) \leq 1$, $\forall t \geq 0$,
2. $p_t * p_s = p_{t+s}$, $\forall s, t \geq 0$,
3. $p_t \to \delta_0$, vaguely as $t \to 0$.

The concept of subordination has been introduced by Bochner (1949, 1955) and is related to C_0-semigroups. A bounded linear operator T_t acting on a function $u \in \mathfrak{B}$, where $(\mathfrak{B}, \|\cdot\|)$ is a Banach space, is said to be a C_0-semigroup if $\forall u \in \mathfrak{B}$

1. $T_0 u = u$,
2. $T_t T_s u = T_{t+s} u$, $\forall u \in \mathfrak{B}$, $s, t \geq 0$,
3. $\lim_{t \to 0} \|T_t u - u\| = 0$, (strong continuity).

The operator

$$T^f u = \int_0^\infty T_s u \, \mu_t(ds), \qquad u \in \mathfrak{B}, \tag{6}$$

where the integral must be meant in the Bochner sense, is said to be a subordinate semigroup in the sense of Bochner and is again a C_0-semigroup. A classical result due to Phillips (1952) state that the infinitesimal generator of $T^f u$ is

$$A^f = -f\left(-A\right) u = -a + bAu + \int_0^\infty \left(T_s u - u\right) \nu(ds). \tag{7}$$

The fractional case

If

$$\nu(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds, \qquad \alpha \in (0,1) \tag{8}$$

one gets

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-sx}\right) \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds = x^\alpha \tag{9}$$

which is the Laplace exponent of the α-stable subordinator. Consider the process

$$L^\alpha(t) = \inf \{ s > 0 : \sigma^\alpha(s) \geq t \}. \tag{10}$$

Such process is known in literature as the inverse of the α-stable subordinator and has non-decreasing, non-independent and non-stationary increments (see Meerschaert and Sikorskii (2012)). In Baeumer and Meerschaert (2001) it has been shown that a Lévy process time-changed with $L^\alpha(t)$, $t > 0$, solves the equation

$$\frac{{}^R\partial^\alpha}{\partial t^\alpha} u - u_0 = Lu \tag{11}$$

subjecto to suitable initial conditions. The time-derivative appearing in (11) is the Riemann-Liouville fractional derivative defined for $\alpha \in (0,1)$ as

$$\frac{{}^R d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds, \qquad \alpha \in (0,1). \tag{12}$$

Anomalous diffusions

The study of fractional equations has gained considerable popularity during the past four decades. This is also due to the fact that a lot of applied scientists have recognized the importance of fractional equations as a powerful tool able to describe the reality. This is the case, for example, of Anomalous Diffusions (AD). It is well-known that the mean-square displacement of a Brownian Motion (BM) is linear in time and equal to $2t$. However this is the way as the heat spreads over a homogeneous media. When a diffusion process is obstructed or take place in a non-homogenous media the mean-squared displacement is often non linear in time and equal to t^α. Such kind of diffusions are described by fractional equations.

Chapter 1

Space-time fractional telegraph equations

Articles: D'Ovidio et al. (2012). Time-changed processes related to space-time fractional telegraph equations.

D'Ovidio et al. (2014). Fractional telegraph-type equations and hyperbolic Brownian motion.

Summary

In this work we construct compositions of vector processes of the form $\boldsymbol{S}_n^{2\beta}\left(c^2 \mathcal{L}^\nu(t)\right)$, $t > 0$, $\nu \in \left(0, \frac{1}{2}\right]$, $\beta \in (0, 1]$, $n \in \mathbb{N}$, whose distribution is related to space-time fractional n-dimensional telegraph equations. We present within a unifying framework the pde connections of n-dimensional isotropic stable processes $\boldsymbol{S}_n^{2\beta}$ whose random time is represented by the inverse $\mathcal{L}^\nu(t)$, $t > 0$, of the superposition of independent positively-skewed stable processes, $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$, $t > 0$, ($H_1^{2\nu}$, H_2^ν, independent stable subordinators). As special cases for $n = 1$, $\nu = \frac{1}{2}$ and $\beta = 1$ we examine the telegraph process T at Brownian time $|B|$ (Orsingher and Beghin (2004)) and establish the equality in distribution $B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right) \overset{\text{law}}{=} T\left(|B(t)|\right)$, $t > 0$. Furthermore the iterated Brownian motion (Allouba and Zheng (2001)) and the two-dimensional motion at finite velocity with a random time are investigated. For all these processes we present their counterparts as Brownian motion at delayed stable-distributed time. The last section of the paper is devoted to the interplay between time-fractional hyperbolic equations and processes defined on the n-dimensional Poincaré half-space.

1.1 Introduction and preliminaries

1.1.1 Introduction

The study of the interplay between fractional equations and stochastic processes has began in the middle of the Eighties with the analysis of simple time-fractional diffusion equations (see Fujita (1990) for a rigorous work on this field, or more recently Allouba and Nane (2012), where the compositions of Brownian sheets with Brownian motions are considered). In some papers the connection between fractional diffusion equations and stable processes is explored (see for example Orsingher and Beghin (2009), Zolotarev (1986)). The iterated Brownian motion has distribution satisfying the following fractional equation

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} u(x,t) = \frac{1}{2^{\frac{3}{2}}} \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, t > 0, \tag{1.1}$$

(see for example Allouba and Zheng (2001)) and also the fourth-order equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{2^3} \frac{\partial^4}{\partial x^4} u(x,t) + \frac{1}{2\sqrt{2\pi t}} \frac{d^2}{dx^2} \delta(x), \qquad x \in \mathbb{R}, t > 0, \tag{1.2}$$

see DeBlassie (2004) (also for an interpretation of the iterated Brownian motion to model the motion of a gas in a crack). Zaslavsky (1994) has studied the fractional kinetic equation (derivatives are meant in the sense of Riemann-Liouville)

$$\frac{\partial^\beta}{\partial t^\beta} g(x,t) = Lg(x,t) + p_0(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}, \qquad 0 < \beta < 1, x \in \mathbb{R}, \tag{1.3}$$

where $p_0 \in C^\infty(\mathbb{R}^1)$ is the initial condition and

$$Lf = -a_2 \frac{df}{dx} + Dq \frac{d^\alpha f}{d(-x)^\alpha} + Dp \frac{d^\alpha f}{dx^\alpha}. \tag{1.4}$$

For $p = q = 1/2$, the differential operator (1.4) is symmetric and Saichev and Zaslavsky (1997) have given the solution to (1.3) in the form $g(x,t) = \int p_0(x-y)h(y,t)dy$ where

$$f(x,t) = \frac{t}{\beta} \int_0^\infty p(x,\xi)\, h_\beta \left(\frac{t}{\xi^{\frac{1}{\beta}}} \right) \xi^{-\frac{1}{\beta}-1} d\xi, \tag{1.5}$$

where $p(x,\xi)$ is the fundamental soution to

$$\frac{\partial p}{\partial t} = Lp \tag{1.6}$$

and the function h_β appearing in (1.5) is the law of a positively skewed stable r.v. with Laplace transform

$$\int_0^\infty e^{-st} h_\beta(t) dt = e^{-s^\beta}. \tag{1.7}$$

Clearly $l_\beta(\xi,t) = \frac{t}{\beta} h_\beta \left(\left(\frac{t}{\xi^{\frac{1}{\beta}}} \right) \right) \xi^{-\frac{1}{\beta}-1}$ is the density of the inverse L^β of H^β since

$$\Pr\left\{H^\beta(t) > \xi\right\} = \Pr\left\{L^\beta(\xi) < t\right\}. \tag{1.8}$$

Therefore the use of the inverse of subordinators in the solution of fractional equations with one time-fractional derivative can be traced back in the papers mentioned above and in Baeumer and Meerschaert (2001).

When the fractional equation has a telegraph structure, with more than one time-fractional derivative involved, that is for $\nu \in (0,1]$

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu} \right) u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, t > 0, \lambda > 0, c > 0, \tag{1.9}$$

the relationship of its solution with the time-changed telegraph processes is examined and established in Orsingher and Beghin (2004) . The space-fractional telegraph equation (with M. Riesz space derivatives) has been considered in Orsingher and Zhao (2003), while the connection between space-fractional equations and asymmetric stable processes has been established in Feller (1952).

Fractional telegraph equations from the analytic point of view have been studied by many authors (see Saxena et al. (2006) for equations with n time derivatives). For their solutions have been worked out also numerical techniques (see, for example, Momani (2005)). Telegraph equations have an extraordinary importance in electrodynamics (the scalar Maxwell equations are of this type), in the theory of damped vibrations and in probability because they are connected with finite velocity random motions.

In this paper we consider various types of processes obtained by composing symmetric stable processes $\boldsymbol{S}_n^{2\beta}(t)$, $t > 0$, $0 < \beta \leq 1$, with the inverse of the sum of two independent stable subordinators (instead of one as in Baeumer and Meerschaert (2001)) say $\mathcal{L}^\nu(t)$, $t > 0$, $0 < \nu \leq \frac{1}{2}$. These time-changed processes, $\boldsymbol{W}_n(t) = \boldsymbol{S}_n^{2\beta}\left(c^2 \mathcal{L}^\nu(t)\right)$, $t > 0$, have distributions, $w_\nu^\beta(\boldsymbol{x},t)$, $\boldsymbol{x} \in \mathbb{R}^n$, $t > 0$, which satisfy telegraph-type space-time fractional equations of the form

$$\left(\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{{}^C\partial^\nu}{\partial t^\nu} \right) w_\nu^\beta(\boldsymbol{x},t) = -c^2 (-\Delta)^\beta w_\nu^\beta(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0, c > 0, \lambda > 0, \tag{1.10}$$

where $0 < \beta \leq 1$, $0 < \nu \leq \frac{1}{2}$, subject to the initial condition

$$w_\nu^\beta(\boldsymbol{x},0) = \delta(\boldsymbol{x}). \tag{1.11}$$

The fractional Laplacian $(-\Delta)^\beta$, appearing in (1.10), is defined and analyzed in Section 1.3 below. The fractional derivatives appearing in (1.10) are meant in the

Dzerbayshan-Caputo sense, that is, for an absolutely continuous function $f \in L^1(\mathbb{R})$ (for fractional calculus consult Kilbas et al. (2006)),

$$\frac{{}^C\partial^\nu}{\partial t^\nu} f(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{\frac{d^m}{ds^m} f(s)}{(t-s)^{\nu+1-m}} \, ds, \qquad m-1<\nu<m, m \in \mathbb{N}. \tag{1.12}$$

Equation (1.10) includes as particular cases all fractional equations studied so far (including diffusion equations) and also the main equations of mathematical physics as limit cases. Thus the distribution of the composed process $\boldsymbol{S}_n^{2\beta}(\mathcal{L}^\nu(t))$, $t>0$, represents the fundamental solution of the most general n-dimensional time-space fractional telegraph equation. We give the general Fourier transform of the solution to (1.10) with initial condition (1.11) as

$$\begin{aligned} &\mathbb{E} e^{i\boldsymbol{\xi}\cdot \boldsymbol{S}_n^{2\beta}(c^2\mathcal{L}^\nu(t))} = \\ &= \frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}}\right) E_{\nu,1}(r_1 t^\nu) + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}}\right) E_{\nu,1}(r_2 t^\nu)\right], \end{aligned} \tag{1.13}$$

where

$$r_1 = -\lambda+\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}, \qquad r_2 = -\lambda-\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}. \tag{1.14}$$

and

$$E_{\nu,\psi}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k+\psi)}, \qquad \nu,\psi>0, \tag{1.15}$$

is the two-parameters Mittag-Leffler function (see, for example, Haubold, Mathai and Saxena (2011) for a general overview on the Mittag-Leffler functions). Our result therefore includes all previous results in a unique framework and sheds an additional insight into the literature in this field.

An important role in our analysis is played by the time change based on the process $\mathcal{L}^\nu(t)$, $t>0$. We consider first the sum of two independent positively skewed stable r.v.'s $H_1^{2\nu}(t)$ and $H_2^\nu(t)$, $t>0$, $0<\nu\leq\frac{1}{2}$,

$$\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t), \qquad t>0, \tag{1.16}$$

whose distribution $\hbar_\nu(x,t)$ is governed by the space fractional equation

$$\frac{\partial}{\partial t}\hbar_\nu(x,t) = -\left(\frac{\partial^{2\nu}}{\partial x^{2\nu}} + 2\lambda\frac{\partial^\nu}{\partial x^\nu}\right)\hbar_\nu(x,t), \qquad x\geq 0, t>0, 0<\nu\leq\frac{1}{2}. \tag{1.17}$$

In (1.17) the fractional derivatives must be meant in the Riemann-Liouville sense which, for a function $f\in L^1(\mathbb{R})$, is defined as

$$\frac{\partial^\nu}{\partial x^\nu} f(x) = \frac{1}{\Gamma(m-\nu)}\frac{d^m}{dx^m}\int_0^x \frac{f(s)}{(x-s)^{\nu+1-m}}\,ds, \qquad m-1<\nu<m, m\in\mathbb{N}. \tag{1.18}$$

We then take the inverse $\mathcal{L}^\nu(t)$, $t>0$, to the process $\mathcal{H}^\nu(t)$, $t>0$, defined as

$$\mathcal{L}^\nu(t) \;=\; \inf\left\{s>0: H_1^{2\nu}(s)+(2\lambda)^{\frac{1}{\nu}}\, H_2^\nu(s)\geq t\right\}, \qquad t>0, \tag{1.19}$$

whose distribution is related to that of $\mathcal{H}^\nu(t)$, $t>0$, by means of the formula

$$\Pr\left\{\mathcal{L}^\nu(t)<x\right\} \;=\; \Pr\left\{\mathcal{H}^\nu(x)>t\right\}. \tag{1.20}$$

The distribution $\ell_\nu(x,t)$ of $\mathcal{L}^\nu(t)$, $t>0$, satisfies the time-fractional telegraph equation

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}}+2\lambda\frac{\partial^\nu}{\partial t^\nu}\right)\ell_\nu(x,t) \;=\; -\frac{\partial}{\partial x}\ell_\nu(x,t), \qquad x\geq 0, t>0, 0<\nu\leq\frac{1}{2}, \tag{1.21}$$

where the fractional derivatives appearing in (1.21) are again in the Riemann-Liouville sense. We are able to give explicit forms of the Laplace transforms of $h_\nu(x,t)$ and $\ell_\nu(x,t)$ in terms of Mittag-Leffler functions for all values of $0<\nu\leq\frac{1}{2}$. For example, for the distribution $\ell_\nu(x,t)$ of $\mathcal{L}^\nu(t)$ we have that, for $\gamma<\lambda^2$,

$$\int_0^\infty e^{-\gamma x}\ell_\nu(x,t)\,dx \;= \\ =\frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right)E_{\nu,1}\left(r_1t^\nu\right)+\left(1-\frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right)E_{\nu,1}\left(r_2t^\nu\right)\right], \tag{1.22}$$

where

$$r_1 \;=\; -\lambda+\sqrt{\lambda^2-\gamma}, \qquad r_2 \;=\; -\lambda-\sqrt{\lambda^2-\gamma}. \tag{1.23}$$

The distribution $\ell_\nu(x,t)$ of $\mathcal{L}^\nu(t)$, $t>0$, has the general form

$$\ell_\nu(x,t) \;=\; \int_0^t l_{2\nu}\left(x,s\right)\, h_\nu(t-s,2\lambda x)\,ds+2\lambda\int_0^t l_\nu(2\lambda x,s)\,h_{2\nu}\left(t-s,x\right)\,ds, \tag{1.24}$$

where the distributions of $H^{2\nu}$, H^ν, and that of their inverse processes $L^{2\nu}$ and L^ν appear. For our analysis it is relevant to obtain the distributions of $\mathcal{H}^{\frac{1}{2}}(t)$, $t>0$, and $\mathcal{L}^{\frac{1}{2}}(t)$, $t>0$. We also obtain explicitely the distributions of $H^{\frac{1}{3}}(t)$ and $H^{\frac{2}{3}}(t)$, $t>0$, and also of their inverses $L^{\frac{1}{3}}(t)$ and $L^{\frac{2}{3}}(t)$, $t>0$, in terms of Airy functions. By means of the convolutions of these distributions we arrive at the following cumbersome density of the random time $\mathcal{L}^{\frac{1}{3}}(t)$, $t>0$,

$$\Pr\left\{\mathcal{L}^{\frac{1}{3}}(t)\in dx\right\} \;=\frac{2\lambda}{\sqrt{\pi}}\int_0^t ds\int_0^\infty dw\, e^{-w}w^{-\frac{1}{6}}\,\mathrm{Ai}\left(-x\sqrt[3]{\frac{2^2w}{3(t-s)^2}}\right)\,\mathrm{Ai}\left(\frac{2\lambda x}{\sqrt[3]{3s}}\right)\cdot \\ \cdot\frac{3}{\sqrt[3]{3s}}\sqrt[3]{\frac{2^2}{3(t-s)^2}}\left[\frac{x}{2s}+\frac{s}{t-s}\right]\,dx. \tag{1.25}$$

For $n = 1$, $\beta = 1$ and $\nu = 1$ in (1.10), we get the telegraph equation which is satisfied by the distribution of the one-dimensional telegraph process

$$T(t) = V(0) \int_0^t (-1)^{N(s)}\, ds, \qquad t > 0, \tag{1.26}$$

where $N(t)$, $t > 0$ is an homogeneous Poisson process, with parameter $\lambda > 0$, independent from the symmetric r.v. $V(0)$ (with values $\pm c$). Properties of this process (including first-passage time distributions) are studied in Foong and Kanno (1994) and a telegraph process with random velocities has been recently considered by Stadje and Zacks (2004).

For $n = 1$, $\beta = 1$ and $\nu = \frac{1}{2}$ the special equation

$$\begin{cases} \left(\frac{\partial}{\partial t} + 2\lambda \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) w_{\frac{1}{2}}^1(x,t) = c^2 \frac{\partial^2}{\partial x^2} w_{\frac{1}{2}}^1(x,t), & x \in \mathbb{R}, t > 0, \\ w_{\frac{1}{2}}^1(x,0) = \delta(x), \end{cases} \tag{1.27}$$

has solution coinciding with the distribution of $T(|B(t)|)$, $t > 0$, where $|B(t)|$, $t > 0$, is a reflecting Brownian motion independent from T (see Orsingher and Beghin (2004)). For $\lambda \to \infty$, $c \to \infty$, in such a way that $\frac{c^2}{\lambda} \to 1$ the fractional diffusion equation (1.1) is obtained from (1.27) and the composition $T(|B(t)|)$, $t > 0$, converges in distribution to the iterated Brownian motion. Our result, specialized to this particular case gives the following unexpected equality in distribution

$$T(|B(t)|) \overset{\text{law}}{=} B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t) \right), \qquad t > 0, \tag{1.28}$$

where

$$\Pr\left\{ B\left(c^2 \mathcal{L}^\nu(t)\right) \in dx \right\} = \frac{\lambda\, dx}{c\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} e^{-\frac{x^2}{4c^2 s} - \frac{\lambda^2 s^2}{t-s}} \left(\frac{s}{2(t-s)} + 1 \right) ds, \tag{1.29}$$

and

$$\Pr\{T(|B(t)|) \in dx\} = \int_0^\infty \Pr\{T(s) \in dx\} \Pr\{|B(t)| \in ds\}. \tag{1.30}$$

The absolutely continuous component of the distribution of the telegraph process $T(t)$, $t > 0$, reads

$$\Pr\{T(s) \in dx\} = \frac{dx\, e^{-\lambda t}}{2c} \left\{ \lambda\, I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2} \right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2} \right) \right\}, \tag{1.31}$$

where $|x| < ct$, $t > 0$, $c > 0$, and

$$I_0(x) = \sum_{k=0}^\infty \left(\frac{x}{2} \right)^{2k} \frac{1}{(k!)^2}. \tag{1.32}$$

For $n = 2$, $\beta = 1$ and $\nu = 1$, equation (1.10) coincides with that of damped planar vibrations (we call it planar telegraph equation) and governs the vertical oscillations of thin deformable structures. The solution to

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t}\right) r(x,y,t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r(x,y,t), & x^2+y^2 < c^2t^2, t > 0, \\ r(x,y,0) = \delta(x,y), \\ r_t(x,y,0) = 0, \end{cases} \tag{1.33}$$

corresponds to the distribution $r(x,y,t)$ of the vector $\boldsymbol{T}(t) = (X(t), Y(t))$ related to a planar motion described in Orsingher and De Gregorio (2007). This random motion $\boldsymbol{T}(t)$, $t > 0$, is performed at finite velocity c, possesses sample paths composed by segments whose orientation is uniform in $(0, 2\pi)$, and with changes of direction at Poisson times. The distribution $r(x,y,t)$ of $\boldsymbol{T}(t)$, $t > 0$, is concentrated inside a circle C_{ct} of radius ct and has an absolutely continuous component which reads

$$r(x,y,t) = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}}, \qquad (x,y) \in C_{ct}, t > 0. \tag{1.34}$$

If no Poisson event occurs, the moving particle reaches the boundary ∂C_{ct} of C_{ct} with probability $e^{-\lambda t}$. The vector process $\boldsymbol{T}(t)$, $t > 0$, taken at a random time represented by a reflecting Brownian motion, $|B(t)|$, has distribution

$$q(x,y,t) = \int_0^\infty \Pr\{X(t) \in ds, Y(t) \in ds\} \Pr\{|B(t)| \in ds\} \tag{1.35}$$

which satisfies the fractional equation

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\right) q(x,y,t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q(x,y,t), \qquad (x,y) \in \mathbb{R}^2, t > 0. \tag{1.36}$$

However, the distribution of $\boldsymbol{B}_2\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right)$, $t > 0$, does not coincide with (1.35) ($\boldsymbol{B}_2$ is a two dimensional Brownian motion). In this case the role of $T(t)$, $t > 0$, in (1.28) is here played by a process which is a slight modification of $\boldsymbol{T}(t)$, $t > 0$. We take the planar process with law

$$\mathfrak{r}(x,y,t) = \frac{\lambda\, e^{-\lambda t}}{2\pi c} \left[\frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}} + e^{-\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}}\right], \quad x^2+y^2 < c^2t^2, t > 0, \tag{1.37}$$

which also solves equation (1.33). The process with distribution

$$\mathfrak{q}(x,y,t) = \int_0^\infty \mathfrak{r}(x,y,s) \left[\Pr\{|B(t)| \in ds\} + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \Pr\{|B(t)| \in ds\}\right]$$

$$= \int_0^\infty \left(\mathfrak{r}(x,y,s) + \frac{\partial}{\partial s}\mathfrak{r}(x,y,s) \right) \Pr\{|B(t)| \in ds\}, \tag{1.38}$$

has the same law of a planar Brownian motion at the time $\mathcal{L}^{\frac{1}{2}}(t)$, $t > 0$. The process $\mathfrak{T}(t)$, $t > 0$, possessing distribution (1.37) is obtained from $\boldsymbol{T}(t)$, $t > 0$, by disregarding displacements started off by even-order Poisson events.

The last section of the paper is concerned with random motions on the hyperbolic Poincaré half-space, $\mathbb{H}^n = \{\boldsymbol{x}, y : \boldsymbol{x} \in \mathbb{R}^{n-1}, y > 0\}$, whose distributions are governed by fractional equations of the form

$$\begin{cases} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu}\right) p_n^\nu(\eta,t) = \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\left(\frac{1}{\sinh^{n-1}\eta} p_n^\nu(\eta,t)\right)\right), & \eta > 0, t > 0 \\ p_n^\nu(\eta,0) = \delta(\eta), \end{cases} \tag{1.39}$$

for $0 < \nu \le \frac{1}{2}$ and $n \in \mathbb{N}$. The corresponding kernel

$$\kappa_n^\nu(\eta,t) = \frac{1}{\sinh^{n-1}\eta} p_n^\nu(\eta,t), \qquad \eta > 0, t > 0, \tag{1.40}$$

solves instead the fractional equations

$$\begin{cases} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu}\right) \kappa_n^\nu(\eta,t) = \frac{1}{\sinh^{n-1}\eta}\frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\kappa_n^\nu(\eta,t)\right), & \eta > 0, t > 0, \\ \kappa_n^\nu(\eta,0) = \delta(\eta). \end{cases} \tag{1.41}$$

The process $\mathcal{T}_n^\nu(t)$, $t > 0$, in $\mathbb{H}^n$ which possesses distribution $p_n^\nu(x,t)$ solving (1.39) is obtained by means of the composition

$$\mathcal{T}_n^\nu(t) = B_n^{hp}(\mathcal{L}^\nu(t)), \qquad t > 0, \tag{1.42}$$

where B_n^{hp} is the hyperbolic Brownian motion in $\mathbb{H}^n$. The hyperbolic Brownian motion has been introduced in the plane by Gertsenshtein and Vasiliev (1959) and in $\mathbb{H}^3$ by Karpelevich, Tutubalin and Shur (1959), in 1959. In successive papers many properties of the hyperbolic Brownian motions have been explored (see for example Getoor (1961), Gruet (1996), Lao and Orsingher (2007), Matsumoto and Yor (2005)). The relationship between kernels in $\mathbb{H}^2$ and $\mathbb{H}^3$ and kernels in higher-order spaces is represented by Millson formula

$$k_{n+2}(\eta,t) = -\frac{e^{-nt}}{2\pi \sinh\eta}\frac{\partial}{\partial\eta}k_n(\eta,t), \qquad \eta > 0, t > 0, n \in \mathbb{N}. \tag{1.43}$$

Since p_3^{hp} and k_3 are considerably simpler than p_2^{hp} and k_2 we give explicit expressions for the distribution

$$p_3^{\frac{1}{2}}(\eta,t) = \frac{\lambda\eta\sinh\eta}{2\pi}\int_0^t \frac{e^{-s}}{s^{\frac{3}{2}}\sqrt{t-s}} e^{-\frac{\lambda^2 s^2}{t-s} - \frac{\eta^2}{4s}} \left(\frac{s}{t-s} + 2\right) ds, \tag{1.44}$$

where $\eta > 0$ and $t > 0$. This distribution solves the fractional-hyperbolic telegraph equation (1.39), for $\nu = \frac{1}{2}$ and $n = 3$.

1.1.2 Notations

For the reader convenience we list below the main notations used throughout the paper.

- $\boldsymbol{S}_n^{2\beta}(t) = \left(S_1^{2\beta}(t), S_2^{2\beta}(t), \cdots, S_n^{2\beta}(t)\right)$, $t>0$, $0<\beta\le 1$, $n\in\mathbb{N}$ is a isotropic stable n-dimensional process with law $v_\beta(\boldsymbol{x},t)$, $\boldsymbol{x}\in\mathbb{R}^n$, $t>0$.
- $H^\nu(t)$, $t>0$, $0<\nu<1$, is a totally positively-skewed stable process (stable subordinator), with law $h_\nu(x,t)$, $x\ge 0$, $t>0$.
- $L^\nu(t)$, $t>0$, is the inverse of $H^\nu(t)$, $t>0$, and has law $l_\nu(x,t)$, $x\ge 0$, $t>0$.
- $\mathscr{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$, $t>0$, is the sum of two independent stable subordinators and has law $\hbar_\nu(x,t)$, $x\ge 0$, $t>0$.
- $\mathscr{L}^\nu(t)$, $t>0$, is the inverse of $\mathscr{H}^\nu(t)$, $t>0$ and possesses distribution $\mathscr{l}_\nu(x,t)$, $x\ge 0$, $t>0$.
- $T(t)$, $t>0$, is a telegraph process with parameters $c>0$ and $\lambda>0$ and law $p_T(x,t)$, $-ct<x<ct$, $t>0$.
- $\boldsymbol{W}_n(t) = \boldsymbol{S}_n^{2\beta}\left(c^2\mathscr{L}^\nu(t)\right)$, $t>0$, has law $w_\nu^\beta(\boldsymbol{x},t)$, $\boldsymbol{x}\in\mathbb{R}^n$, $t>0$.
- $\mathscr{W}(t) = T\left(|B(t)|\right)$, $t>0$, has distribution $\mathscr{w}(x,t)$, $x\in\mathbb{R}$, $t>0$.
- $\boldsymbol{T}(t)$, $t>0$, is the planar process with infinite directions, parameters $c,\lambda>0$ and law $r(x,y,t)$, $(x,y)\in C_{ct} = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<c^2t^2\}$, $t>0$.
- $\mathfrak{T}(t)$, $t>0$, is the planar process with infinite directions, parameters $c,\lambda>0$ and law $\mathfrak{r}(x,y,t)$, $(x,y)\in C_{ct} = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<c^2t^2\}$, $t>0$, constructed by disregading displacements started off only by even-labelled Poisson events.
- $\boldsymbol{Q}(t) = \boldsymbol{T}\left(|B(t)|\right)$, $t>0$, has law $q(x,y,t)$, $(x,y)\in\mathbb{R}^2$, $t>0$.
- $B_n^{hp}(t)$, $t>0$, is the n-dimensional hyperbolic Brownian motion in $\mathbb{H}^n = \{(\boldsymbol{x},y) : \boldsymbol{x}\in\mathbb{R}^{n-1}, y>0\}$ and has law $p_n^{hp}(\eta,t)$, $\eta>0$, $t>0$ with kernel $k_n(\eta,t)$, $\eta>0$, $t>0$.
- $\mathscr{T}_n^\nu(t) = B_n^{hp}\left(\mathscr{L}^\nu(t)\right)$, $t>0$, has distribution $\mathscr{p}_n^\nu(\eta,t)$, $\eta>0$, $t>0$ and kernel $\kappa_n^\nu(\eta,t)$, $\eta>0$, $t>0$.
- By $\widetilde{f}$ we denote the Laplace transform of the function f and by $\widehat{f}$ we denote its Fourier transform.

1.1.3 Preliminaries

Let us consider a stable process $S^\nu(t)$, $t > 0$, $0 < \nu \le 2$, $\nu \neq 1$, with characteristic function

$$\mathbb{E}e^{i\xi S^\nu(t)} = e^{-\sigma|\xi|^\nu t\left(1-i\theta\,\mathrm{sign}(\xi)\,\tan\frac{\nu\pi}{2}\right)} \tag{1.45}$$

where $\theta \in [-1,1]$ is the skewness parameter and

$$\sigma = \cos\frac{\pi\nu}{2}. \tag{1.46}$$

For $\theta = 1$ the distribution corresponding to (1.45) is totally positively skewed and for $\theta = -1$ is totally negatively skewed. The stable process with stationary and independent increments, totally positively skewed will be denoted as $H^\nu(t)$, $t > 0$. We note that the density $h_\nu(x,t)$, of $H^\nu(t)$, is zero at $x = 0$ as the following calculation show

$$\begin{aligned}
h_\nu(0,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}e^{i\xi H^\nu(t)}\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\sigma|\xi|^\nu t\left(1-i\tan\frac{\nu\pi}{2}\right)}d\xi \\
&= \frac{1}{2\pi}\left[\int_0^\infty e^{-\sigma|\xi|^\nu t\left(1-i\tan\frac{\nu\pi}{2}\right)}d\xi + \int_{-\infty}^0 e^{-\sigma|\xi|^\nu t\left(1+i\tan\frac{\nu\pi}{2}\right)}d\xi\right] \\
&= \frac{1}{2\pi}\left[\int_0^\infty e^{-|\xi|^\nu t e^{-\frac{i\nu\pi}{2}}}\,d\xi + \int_0^\infty e^{-|\xi|^\nu t e^{\frac{i\nu\pi}{2}}}\,d\xi\right] \\
&= \frac{1}{2\pi}\left[\int_0^\infty e^{-z}\left(\frac{z}{t}\right)^{\frac{1}{\nu}-1}e^{\frac{i\pi}{2}}dz + \int_0^\infty e^{-z}\left(\frac{z}{t}\right)^{\frac{1}{\nu}-1}\frac{1}{t}e^{-\frac{i\pi}{2}}dz\right] \\
&= \frac{\cos\frac{\pi}{2}}{\pi}\int_0^\infty e^{-z}\left(\frac{z}{t}\right)^{\frac{1}{\nu}-1}\frac{1}{t}\,dz = 0.
\end{aligned} \tag{1.47}$$

The positively skewed stable r.v. $H^\nu(t)$ has x-Laplace transform

$$\widetilde{h_\nu}(\mu,t) = \mathbb{E}e^{-\mu H^\nu(t)} = e^{-t\mu^\nu}, \qquad 0 < \nu < 1, \tag{1.48}$$

and therefore Fourier transform

$$\begin{aligned}
\widehat{h_\nu}(\xi,t) &= \mathbb{E}e^{i\xi H^\nu(t)} = \mathbb{E}\left(e^{-(-i\xi)H^\nu(t)}\right) = e^{-t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^\nu} \\
&= e^{-t|\xi|^\nu\cos\frac{\pi\nu}{2}\left(1-i\,\mathrm{sign}(\xi)\tan\frac{\pi\nu}{2}\right)}.
\end{aligned} \tag{1.49}$$

This shows once again that the skeweness parameter is $\theta = 1$.

The probability law $h_\nu(x,t)$, of $H^\nu(t)$, $t > 0$, solves the boundary-initial problem

$$\begin{cases}
\left(\frac{\partial}{\partial t} + \frac{\partial^\nu}{\partial x^\nu}\right)h_\nu(x,t) = 0, & x > 0, t > 0, 0 < \nu < 1, \\
h_\nu(0,t) = 0, \\
h_\nu(x,0) = \delta(x).
\end{cases} \tag{1.50}$$

By taking the x-Laplace transform of the Riemann-Liouville fractional derivative appearing in (1.50) we have that

$$\begin{aligned}
&\mathcal{L}\left[\frac{\partial^\nu}{\partial x^\nu} h_\nu(x,t)\right](\mu) = \int_0^\infty e^{-\mu x}\frac{\partial^\nu}{\partial x^\nu}h_\nu(x,t)\,dx \\
&= \int_0^\infty e^{-\mu x}\left[\frac{1}{\Gamma(1-\nu)}\frac{d}{dx}\int_0^x \frac{h_\nu(z,t)}{(x-z)^\nu}dz\right]dx \\
&= \int_0^\infty e^{-\mu x}\left[\frac{1}{\Gamma(1-\nu)}\int_0^x \frac{d}{dx}\frac{h_\nu(x-z,t)}{z^\nu}dz + \frac{h_\nu(0,t)}{\Gamma(1-\nu)\,x^\nu}\right]dx \\
&= \frac{h_\nu(0,t)}{\Gamma(1-\nu)}\int_0^\infty e^{-\mu x}x^{1-\nu-1}\,dx + \frac{1}{\Gamma(1-\nu)}\int_0^\infty \frac{dz}{z^\nu}\int_z^\infty dx\, e^{-\mu x}\frac{d}{dx}h_\nu(x-z,t) \\
&= h_\nu(0,t)\mu^{\nu-1} + \frac{1}{\Gamma(1-\nu)}\int_0^\infty e^{-\mu z}z^{-\nu}dz\int_0^\infty e^{-\mu x}\frac{d}{dx}h_\nu(x,t)dx \\
&= h_\nu(0,t)\mu^{\nu-1} + \left[\int_0^\infty e^{-\mu x}h_\nu(x,t)dx\right]\mu\,\frac{1}{\mu^{1-\nu}} - \mu^{\nu-1}h_\nu(0,t) = \mu^\nu \widetilde{h}_\nu(\mu,t).
\end{aligned} \tag{1.51}$$

Therefore

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{h}_\nu(\mu,t) + \mu^\nu \widetilde{h}_\nu(\mu,t) = 0, & \mu>0, t>0, \\ \widetilde{h}_\nu(\mu,0) = 1, \end{cases} \tag{1.52}$$

so that

$$\widetilde{h}_\nu(\mu,t) = e^{-\mu^\nu t}. \tag{1.53}$$

In other words the density of a positively skewed stable r.v. solves the space-fractional problem (1.50).

We will also deal with the inverse process of $H^\nu(t)$, $t>0$, say $L^\nu(t)$, $t>0$, for which

$$\Pr\{H^\nu(x)>t\} = \Pr\{L^\nu(t)<x\}, \qquad x>0, t>0. \tag{1.54}$$

Such a process has non-negative, non-stationary and non-independent increments. Furthemore we recall that the law $l_\nu(x,t)$ of $L^\nu(t)$, can be written as

$$l_\nu(x,t) = \frac{1}{t^\nu}W_{-\nu,1-\nu}\left(-\frac{x}{t^\nu}\right), \qquad x\ge 0, t>0, \tag{1.55}$$

where

$$W_{a,b}(x) = \sum_{k=0}^\infty \frac{x^k}{k!\,\Gamma(ak+b)}, \qquad x\in\mathbb{R}, a>-1, b\in\mathbb{C}, \tag{1.56}$$

is the Wright function, and has Laplace transform

$$\widetilde{l}_\nu(x,\mu) = \int_0^\infty e^{-\mu t}l_\nu(x,t)dt = \int_0^\infty e^{-\mu t}\frac{1}{t^\nu}W_{-\nu,1-\nu}\left(-\frac{x}{t^\nu}\right)dt = \mu^{\nu-1}e^{-x\mu^\nu}. \tag{1.57}$$

1.2 Sum of stable subordinators, $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$

For the construction of the vector process $\boldsymbol{W}_n(t) = \boldsymbol{S}_n^{2\beta}(c^2 \mathcal{L}^\nu(t))$, $t > 0$, whose distribution is driven by the general space-time fractional telegraph equation (1.10), we need the sum $\mathcal{H}^\nu(t)$, $t > 0$, of two independent positively skewed processes. The second step consists in constructing the process $\mathcal{L}^\nu(t)$, $t > 0$, inverse to $\mathcal{H}^\nu(t)$, $t > 0$. We now start by considering the following sum

$$\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t), \qquad t > 0, 0 < \nu \le \frac{1}{2}, \tag{1.58}$$

with $H_1^{2\nu}$, H_2^ν, independent, positively-skewed, stable random variables, $\lambda > 0$. The distribution of $\mathcal{H}^\nu(t)$ can be written as

$$\hbar_\nu(x,t) = \int_0^x h_{2\nu}(y,t)\, h_\nu(x-y, 2\lambda t)\, dy. \tag{1.59}$$

Taking the double Laplace transform of (1.59), with respect to t and x, we get

$$\begin{aligned}\widetilde{\widetilde{\hbar}}_\nu(\gamma,\mu) &= \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\gamma x} \hbar_\nu(x,t)\, dx\, dt = \int_0^\infty e^{-\mu t - t\gamma^{2\nu} - 2\lambda t \gamma^\nu} dt \\ &= \frac{1}{\gamma^{2\nu} + 2\lambda\gamma^\nu + \mu} = \left[\frac{1}{\gamma^\nu - r_2} - \frac{1}{\gamma^\nu - r_1}\right] \frac{1}{r_2 - r_1}\end{aligned} \tag{1.60}$$

where, for $0 < \mu < \lambda^2$,

$$\begin{cases} r_1 = -\lambda - \sqrt{\lambda^2 - \mu}, \\ r_2 = -\lambda + \sqrt{\lambda^2 - \mu}. \end{cases} \tag{1.61}$$

By means of formula

$$\int_0^\infty e^{-\gamma x} x^{\alpha - 1} E_{\alpha,\alpha}(\eta x^\alpha)\, dx = \frac{1}{\gamma^\alpha - \eta}, \tag{1.62}$$

where $E_{\nu,\nu}(z)$ is the Mittag-Leffler function defined in (1.15), we can invert the x-Laplace transform in (1.60) obtaining, for $\mu < \lambda^2$,

$$\begin{aligned}&\widetilde{\hbar}_\nu(x,\mu) = \\ &= \frac{x^{\nu-1}}{2\sqrt{\lambda^2-\mu}} \left[E_{\nu,\nu}\left(\left(-\lambda + \sqrt{\lambda^2-\mu}\right) x^\nu\right) - E_{\nu,\nu}\left(\left(-\lambda - \sqrt{\lambda^2-\mu}\right) x^\nu\right)\right] \\ &= \frac{1}{2\sqrt{\lambda^2-\mu}} \left[\frac{1}{-\lambda + \sqrt{\lambda^2-\mu}} \frac{\partial}{\partial x} E_{\nu,1}\left(\left(-\lambda + \sqrt{\lambda^2-\mu}\right) x^\nu\right)\right. \\ &\quad \left. - \frac{1}{-\lambda - \sqrt{\lambda^2-\mu}} \frac{\partial}{\partial x} E_{\nu,1}\left(\left(-\lambda - \sqrt{\lambda^2-\mu}\right) x^\nu\right)\right].\end{aligned} \tag{1.63}$$

Formula (1.63) gives the explicit form of the t-Laplace transform of $\hbar_\nu(x,t)$ in terms of Mittag-Leffler functions. In view of formula

$$E_{\nu,1}(-\lambda t^\nu) = \frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda^{\frac{1}{\nu}}tx} x^{\nu-1}\sin\pi\nu}{x^{2\nu}+1+2x^\nu\cos\pi\nu}\,dx, \qquad 0<\nu<1, \tag{1.64}$$

we have that

$$\begin{aligned}
\widetilde{\hbar}_\nu(x,\mu) &= \frac{1}{2\sqrt{\lambda^2-\mu}}\left[\frac{1}{-\lambda+\sqrt{\lambda^2-\mu}}\frac{\partial}{\partial x}\int_0^\infty \frac{e^{-xy\left(\lambda-\sqrt{\lambda^2-\mu}\right)^{\frac{1}{\nu}}} y^{\nu-1}\sin\pi\nu\,dy}{\pi\left(y^{2\nu}+1+2y^\nu\cos\pi\nu\right)}\right. \\
&\quad \left.+\frac{1}{\lambda+\sqrt{\lambda^2-\mu}}\frac{\partial}{\partial x}\int_0^\infty \frac{e^{-xy\left(\lambda+\sqrt{\lambda^2-\mu}\right)^{\frac{1}{\nu}}} y^{\nu-1}\sin\pi\nu\,dy}{\pi\left(y^{2\nu}+1+2y^\nu\cos\pi\nu\right)}\right] \\
&= \int_0^\infty \frac{dy\; y^\nu\sin\pi\nu}{\pi\left(y^{2\nu}+1+2y^\nu\cos\pi\nu\right)}\frac{1}{2\sqrt{\lambda^2-\mu}}\left[\left(\lambda-\sqrt{\lambda^2-\mu}\right)^{\frac{1}{\nu}-1}\cdot\right. \\
&\quad \left.\cdot\, e^{-xy\left(\lambda-\sqrt{\lambda^2-\mu}\right)^{\frac{1}{\nu}}} - \left(\lambda+\sqrt{\lambda^2-\mu}\right)^{\frac{1}{\nu}-1} e^{-xy\left(\lambda+\sqrt{\lambda^2-\mu}\right)^{\frac{1}{\nu}}}\right] \\
&= \mathbb{E}\left\{\frac{\mathcal{U}^\nu}{2\sqrt{\lambda^2-\mu}}\left[(-r_2)^{\frac{1}{\nu}-1} e^{-x\mathcal{U}^\nu(-r_2)^{\frac{1}{\nu}}} - (-r_1)^{\frac{1}{\nu}-1} e^{-x\mathcal{U}^\nu(-r_1)^{\frac{1}{\nu}}}\right]\right\} \\
&= \frac{1}{r_2-r_1}\frac{\partial}{\partial x}\mathbb{E}\left[\frac{e^{-x\mathcal{U}^\nu(-r_2)^{\frac{1}{\nu}}}}{r_2} - \frac{e^{-x\mathcal{U}^\nu(-r_1)^{\frac{1}{\nu}}}}{r_1}\right],
\end{aligned} \tag{1.65}$$

where $\mathcal{U}^\nu$ is the Lamperti distribution with density

$$\frac{\Pr\{\mathcal{U}^\nu \in du\}}{du} = \frac{\sin\pi\nu}{\pi}\frac{u^{\nu-1}}{1+u^{2\nu}+2u^\nu\cos\pi\nu}, \qquad u>0, \tag{1.66}$$

and represents the law of the ratio of two independent stable r.v.'s of the same order ν.

Theorem 1.2.1. *The law $\hbar_\nu(x,t)$ of the process $\mathcal{H}^\nu(t) = H_1^{2\nu}(t) + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(t)$ solves the fractional problem*

$$\begin{cases}
\frac{\partial}{\partial t}\hbar_\nu(x,t) = -\left(\frac{\partial^{2\nu}}{\partial x^{2\nu}}+2\lambda\frac{\partial^\nu}{\partial x^\nu}\right)\hbar_\nu(x,t), & x>0, t>0, 0<\nu<\frac{1}{2}, \\
\hbar_\nu(0,t) = 0, \\
\hbar_\nu(x,0) = \delta(x).
\end{cases} \tag{1.67}$$

The fractional derivatives appearing in (1.67) are intended in the Riemann-Liouville sense.

Proof. By considering (1.49), we have that the Fourier transform of $\hbar_\nu(x,t)$ is written as

$$\widehat{\hbar}_\nu(\xi,t) = \mathbb{E}e^{i\xi\mathcal{H}^\nu(t)} = \mathbb{E}e^{i\xi\left[H^{2\nu}(t)+(2\lambda)^{\frac{1}{\nu}}H^\nu(t)\right]} = \mathbb{E}e^{i\xi H^{2\nu}(t)}e^{i\xi H^\nu(2\lambda t)}$$

$$= e^{-t|\xi|^{2\nu}\cos\pi\nu(1-i\,\mathrm{sign}(\xi)\tan\pi\nu) - 2\lambda t|\xi|^{\nu}\cos\frac{\pi\nu}{2}\left(1-i\,\mathrm{sign}(\xi)\tan\frac{\pi\nu}{2}\right)}$$
$$= e^{-t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{2\nu} - 2\lambda t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{\nu}}, \tag{1.68}$$

and thus

$$\frac{\partial}{\partial t}\widehat{\mathfrak{h}}_{\nu}(\xi,t) = \left[-\left(|\xi|\,e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{2\nu} - 2\lambda\left(|\xi|\,e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{\nu}\right]\cdot$$
$$\cdot\, e^{-t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{2\nu} - 2\lambda t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{\nu}} \tag{1.69}$$

In view of the relationship

$$|\xi|\,e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)} = -i\xi \tag{1.70}$$

we have that formula (1.69) can be rewritten as

$$\frac{\partial}{\partial t}\widehat{\mathfrak{h}}_{\nu}(\xi,t) = \left[-(-i\xi)^{2\nu} - 2\lambda(-i\xi)^{\nu}\right]e^{-t(-i\xi)^{2\nu} - 2\lambda t(-i\xi)^{\nu}}. \tag{1.71}$$

In (1.51) we have shown that

$$\mathcal{L}\left[\frac{\partial^{\nu}}{\partial x^{\nu}}h_{\nu}(x,t)\right](\mu) = \int_0^{\infty} e^{-\mu x}\frac{\partial^{\nu}}{\partial x^{\nu}}h_{\nu}(x,t)\,dx = \mu^{\nu}\widetilde{h}_{\nu}(\mu,t) \tag{1.72}$$

and thus for a sufficiently good function f we have the following Fourier transform

$$\mathcal{F}\left[\frac{\partial^{\nu}}{\partial x^{\nu}}f(x)\right](\xi) = \int_0^{\infty} e^{-(-i\xi)x}\frac{\partial^{\nu}}{\partial x^{\nu}}f(x)\,dx = (-i\xi)^{\nu}\,\widehat{f}(\xi). \tag{1.73}$$

In view of (1.73) we have that the Fourier transform of the right-hand side of the equation (1.67), equipped with the boundary conditions, is written as

$$-\mathcal{F}\left[\frac{\partial^{2\nu}}{\partial x^{2\nu}}\mathfrak{h}_{\nu}(x,t) + 2\lambda\frac{\partial^{\nu}}{\partial x^{\nu}}\mathfrak{h}_{\nu}(x,t)\right](\xi) =$$
$$= -\int_0^{\infty} e^{-(-i\xi)x}\frac{\partial^{2\nu}}{\partial x^{2\nu}}\mathfrak{h}_{\nu}(x,t)\,dx - 2\lambda\int_0^{\infty} e^{-(-i\xi)x}\frac{\partial^{\nu}}{\partial x^{\nu}}\mathfrak{h}_{\nu}(x,t)\,dx$$
$$= -\left((-i\xi)^{2\nu} + 2\lambda(-i\xi)^{\nu}\right)\widehat{\mathfrak{h}}_{\nu}(\xi,t)$$
$$= -\left((-i\xi)^{2\nu} + 2\lambda(-i\xi)^{\nu}\right)e^{-t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{2\nu} - 2\lambda t\left(|\xi|e^{-\frac{i\pi}{2}\mathrm{sign}(\xi)}\right)^{\nu}}$$
$$= -\left((-i\xi)^{2\nu} + 2\lambda(-i\xi)^{\nu}\right)e^{-t(-i\xi)^{2\nu} - 2\lambda t(-i\xi)^{\nu}}, \tag{1.74}$$

which coincides with formula (1.71). This is tantamount to saying that the Fourier transform $\widehat{\mathfrak{h}}_{\nu}(\xi,t)$ is the solution to

$$\begin{cases} \frac{\partial}{\partial t}\widehat{\mathfrak{h}}_{\nu}(\xi,t) = -\left((-i\xi)^{2\nu} + 2\lambda(-i\xi)^{\nu}\right)\widehat{\mathfrak{h}}_{\nu}(\xi,t), & \xi\in\mathbb{R}, t>0, \\ \widehat{\mathfrak{h}}_{\nu}(\xi,0) = 1, \end{cases} \tag{1.75}$$

and this completes the proof. □

1.2.1 The inverse process $\mathcal{L}^\nu(t)$

Let $\mathcal{L}^\nu(t)$, $t>0$, be the inverse process of $\mathcal{H}^\nu(t)$, $t>0$, as defined in (1.19) for which

$$\Pr\{\mathcal{L}^\nu(t)<x\} = \Pr\{\mathcal{H}^\nu(x)>t\}, \qquad x,t>0, \tag{1.76}$$

and let $l_\nu(x,t)$ be the law of $\mathcal{L}^\nu(t)$, $t>0$. We have the following result.

Theorem 1.2.2. *The law $l_\nu(x,t)$ of the process $\mathcal{L}^\nu(t)$, $t>0$, solves the time-fractional boundary-initial problem*

$$\begin{cases} \left(\frac{\partial^{2\nu}}{\partial t^{2\nu}}+2\lambda\frac{\partial^\nu}{\partial t^\nu}\right) l_\nu(x,t) = -\frac{\partial}{\partial x} l_\nu(x,t), & x>0, t>0, 0<\nu<\frac{1}{2}, \\ l_\nu(x,0) = \delta(x), \\ l_\nu(0,t) = \frac{t^{-2\nu}}{\Gamma(1-2\nu)}+2\lambda\frac{t^{-\nu}}{\Gamma(1-\nu)}, \end{cases} \tag{1.77}$$

and has x-Laplace transform which reads, for $0<\gamma<\lambda^2$,

$$\widetilde{l_\nu}(\gamma,t) = \frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right)E_{\nu,1}(r_1t^\nu)+\left(1-\frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right)E_{\nu,1}(r_2t^\nu)\right], \tag{1.78}$$

where

$$r_1 = -\lambda+\sqrt{\lambda^2-\gamma}, \qquad r_2 = -\lambda-\sqrt{\lambda^2-\gamma}. \tag{1.79}$$

The fractional derivatives appearing in (1.77) are intended in the Riemann-Liouville sense.

Proof. We first show that the analytical solution to the problem (1.77) has double Laplace transform $\widetilde{\widetilde{l_\nu}}(\gamma,\mu)$ written as

$$\widetilde{\widetilde{l_\nu}}(\gamma,\mu) = \frac{\mu^{2\nu-1}+2\lambda\mu^{\nu-1}}{\mu^{2\nu}+2\lambda\mu^\nu+\gamma}. \tag{1.80}$$

By taking the t-Laplace transform of the equation in (1.77) we have that

$$\mu^{2\nu}\widetilde{l_\nu}(x,\mu)+2\lambda\mu^\nu\widetilde{l_\nu}(x,\mu) = -\frac{\partial}{\partial x}\widetilde{l_\nu}(x,\mu). \tag{1.81}$$

By taking into account the boundary condition and performing the x-Laplace transform of (1.81) we have that

$$\left(\mu^{2\nu}+2\lambda\mu^\nu\right)\widetilde{\widetilde{l_\nu}}(\gamma,\mu) = \widetilde{l_\nu}(0,\mu)-\gamma\,\widetilde{\widetilde{l_\nu}}(\gamma,\mu). \tag{1.82}$$

Now, by considering the boundary condition, we get that

$$\widetilde{l_\nu}(0,\mu) = \int_0^\infty dt\, e^{-\mu t} l_\nu(0,t) = \int_0^\infty dt\, e^{-\mu t}\left[\frac{t^{-2\nu}}{\Gamma(1-2\nu)}+2\lambda\frac{t^{-\nu}}{\Gamma(1-\nu)}\right]$$

$$= \mu^{2\nu-1} + 2\lambda\mu^{\nu-1}, \tag{1.83}$$

and thus

$$\widetilde{\widetilde{\ell_\nu}}(\gamma, \mu) = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma}. \tag{1.84}$$

Now we show that the double Laplace transform of the law $\ell_\nu(x, t)$ coincides with (1.80). We first recall that

$$\begin{aligned}\widetilde{h_\nu}(\mu, x) &= \int_0^\infty dt\, e^{-\mu t} h_\nu(t, x) = \mathbb{E}e^{-\mu \mathcal{H}^\nu(x)} = \mathbb{E}e^{-\mu H^{2\nu}(x)} \mathbb{E}e^{-\mu H^\nu(2\lambda x)} \\ &= \widetilde{h_{2\nu}}(\mu, x)\, \widetilde{h_\nu}(\mu, 2\lambda x) = e^{-x\mu^{2\nu} - x2\lambda\mu^\nu}, \qquad x > 0,\end{aligned} \tag{1.85}$$

where we used result (1.48). By considering the construction of the process $\mathcal{L}^\nu(t)$, $t > 0$, as the inverse process of $\mathcal{H}^\nu(t)$, $t > 0$, as stated in (1.76), we get

$$\ell_\nu(x, t) = \frac{\Pr\{\mathcal{L}^\nu(t) \in dx\}}{dx} = -\frac{\partial}{\partial x} \Pr\{\mathcal{H}^\nu(x) < t\} = -\frac{\partial}{\partial x} \int_0^t h_\nu(s, x)\, ds. \tag{1.86}$$

In view of (1.86), the double Laplace transform of $\ell_\nu(x, t)$ can be obtained observing that

$$\begin{aligned}\widetilde{\widetilde{\ell_\nu}}(\gamma, \mu) &= \int_0^\infty dx\, e^{-\gamma x} \int_0^\infty dt\, e^{-\mu t} \left[-\frac{\partial}{\partial x} \int_0^t h_\nu(s, x)\, ds \right] \\ &= -\int_0^\infty dx\, e^{-\gamma x} \frac{\partial}{\partial x} \int_0^\infty dt\, e^{-\mu t} \int_0^t h_\nu(s, x)\, ds \\ &= -\frac{1}{\mu} \int_0^\infty dx\, e^{-\gamma x} \frac{\partial}{\partial x} \widetilde{h_\nu}(x, \mu) = -\frac{1}{\mu} \int_0^\infty dx\, e^{-\gamma x} \left[\frac{\partial}{\partial x} e^{-x\mu^{2\nu} - 2\lambda x\mu^\nu} \right] \\ &= \left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right) \int_0^\infty dx\, e^{-\gamma x - x\mu^{2\nu} - 2\lambda x\mu^\nu} = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma},\end{aligned} \tag{1.87}$$

which coincides with (1.80). Now we pass to the derivation of the x-Laplace transform of $\ell_\nu(x, t)$. We can write

$$\begin{aligned}\widetilde{\widetilde{\ell_\nu}}(\gamma, \mu) &= \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^\nu + \gamma} = \frac{\mu^{\nu-1}}{\mu^\nu - r_1} + \frac{\mu^{\nu-1}}{\mu^\nu - r_2} - \frac{\mu^{2\nu-1}}{(\mu^\nu - r_1)(\mu^\nu - r_2)} \\ &= \frac{\mu^{\nu-1}}{\mu^\nu - r_1} + \frac{\mu^{\nu-1}}{\mu^\nu - r_2} - \left[\frac{\mu^{\nu-(1-\nu)}}{\mu^\nu - r_1} - \frac{\mu^{\nu-(1-\nu)}}{\mu^\nu - r_2} \right] \frac{1}{2\sqrt{\lambda^2 - \gamma}},\end{aligned} \tag{1.88}$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - \gamma}, \qquad r_2 = -\lambda - \sqrt{\lambda^2 - \gamma}. \tag{1.89}$$

Now we need the following results

$$\int_0^\infty e^{-\mu t} E_{\nu,1}(r_j t^\nu)\, dt = \frac{\mu^{\nu-1}}{\mu^\nu - r_j}, \qquad j = 1, 2,$$

$$\int_0^\infty e^{-\mu t} t^{(1-\nu)-1} E_{\nu,1-\nu}\left(r_j t^\nu\right) dt \;=\; \frac{\mu^{2\nu-1}}{\mu^\nu - r_j}. \tag{1.90}$$

Therefore

$$\widetilde{\mathfrak{l}}_\nu\left(\gamma,t\right) \;=\; E_{\nu,1}\left(r_1 t^\nu\right) + E_{\nu,1}\left(r_2 t^\nu\right) - \frac{t^{-\nu}}{2\sqrt{\lambda^2-\gamma}}\left[E_{\nu,1-\nu}\left(r_1 t^\nu\right) - E_{\nu,1-\nu}\left(r_2 t^\nu\right)\right]. \tag{1.91}$$

Since

$$E_{\nu,1-\nu}\left(z\right) \;=\; zE_{\nu,1}(z) + \frac{1}{\Gamma\left(1-\nu\right)} \tag{1.92}$$

we have that

$$\begin{aligned}
\widetilde{\mathfrak{l}}_\nu\left(\gamma,t\right) \;&=\; E_{\nu,1}\left(r_1 t^\nu\right) + E_{\nu,1}\left(r_2 t^\nu\right) - \frac{t^{-\nu}}{2\sqrt{\lambda^2-\gamma}}\left[r_1 t^\nu E_{\nu,1}\left(r_1 t^\nu\right) - r_2 t^\nu E_{\nu,1}\left(r_2 t^\nu\right)\right] \\
&=\; \left(1 - \frac{-\lambda+\sqrt{\lambda^2-\gamma}}{2\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_1 t^\nu\right) + \left(1 - \frac{\lambda+\sqrt{\lambda^2-\gamma}}{2\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_2 t^\nu\right) \\
&=\; \frac{1}{2}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_1 t^\nu\right) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_2 t^\nu\right)\right],
\end{aligned} \tag{1.93}$$

which coincides with (1.78).

Now we check that the Laplace transform (1.93) solves the fractional equation

$$\begin{aligned}
\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^\nu}{\partial t^\nu}\right) \widetilde{\mathfrak{l}}_\nu\left(\gamma,t\right) \;&=\; -\gamma\, \widetilde{\mathfrak{l}}_\nu\left(\gamma,t\right) + \mathfrak{l}_\nu\left(0,t\right) \\
&=\; -\gamma\, \widetilde{\mathfrak{l}}_\nu\left(\gamma,t\right) + \frac{t^{-2\nu}}{\Gamma(1-2\nu)} + 2\lambda \frac{t^{-\nu}}{\Gamma\left(1-\nu\right)}
\end{aligned} \tag{1.94}$$

which is the x-Laplace transform of the equation appearing in (1.77). Since

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}} \widetilde{\mathfrak{l}}_\nu(\gamma,t) - \frac{t^{-2\nu}}{\Gamma\left(1-2\nu\right)} \;=\; \frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} \widetilde{\mathfrak{l}}_\nu(\gamma,t) \tag{1.95}$$

$$\frac{\partial^{\nu}}{\partial t^{\nu}} \widetilde{\mathfrak{l}}_\nu(\gamma,t) - \frac{t^{-\nu}}{\Gamma\left(1-\nu\right)} \;=\; \frac{{}^C\partial^{\nu}}{\partial t^{\nu}} \widetilde{\mathfrak{l}}_\nu(\gamma,t) \tag{1.96}$$

we therefore need to show that

$$\left(\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{{}^C\partial^{\nu}}{\partial t^{\nu}}\right) \widetilde{\mathfrak{l}}_\nu(\gamma,t) \;=\; -\gamma \widetilde{\mathfrak{l}}_\nu(\gamma,t). \tag{1.97}$$

In light of

$$\frac{{}^C\partial^{\nu}}{\partial t^{\nu}} E_{\nu,1}\left(r_j t^\nu\right) \;=\; r_j E_{\nu,1}\left(r_j t^\nu\right), \qquad j \;=\; 1,2, \tag{1.98}$$

$$\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}\left(r_j t^\nu\right) = r_j^2\, E_{\nu,1}\left(r_j t^\nu\right) + \frac{t^{-\nu} r_j}{\Gamma\left(1-\nu\right)}, \tag{1.99}$$

we are able to show that (1.78) solves (1.94). We first check result (1.99) as follows, for $0 < 2\nu < 1$

$$\begin{aligned}
&\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}\left(r_j t^\nu\right) = \sum_{k=0}^{\infty} \frac{r_j^k}{\Gamma\left(\nu k+1\right)} \frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} t^{\nu k} \\
&= \sum_{k=1}^{\infty} \frac{r_j^k}{\Gamma\left(\nu k+1\right)} \frac{\nu k}{\Gamma\left(1-2\nu\right)} \int_0^t s^{\nu k-1} \left(t-s\right)^{-2\nu} ds \\
&= \sum_{k=1}^{\infty} \frac{r_j^k\, t^{\nu k-2\nu}}{\Gamma\left(\nu k\right)} \frac{1}{\Gamma\left(1-2\nu\right)} \int_0^1 s^{\nu k-1} \left(1-s\right)^{1-2\nu-1} ds \\
&= \sum_{k=1}^{\infty} \frac{r_j^k\, t^{\nu k-2\nu}}{\Gamma\left(\nu k-2\nu+1\right)} = \sum_{k=0}^{\infty} \frac{r_j^{k+1}\, t^{\nu k-\nu}}{\Gamma\left(\nu k-\nu+1\right)} \\
&= r_j t^{-\nu} \left[\sum_{k=1}^{\infty} \frac{\left(r_j t^\nu\right)^k}{\Gamma\left(\nu k-\nu+1\right)} + \frac{1}{\Gamma\left(1-\nu\right)} \right] = r_j^2\, E_{\nu,1}\left(r_j t^\nu\right) + \frac{t^{-\nu} r_j}{\Gamma\left(1-\nu\right)}.
\end{aligned} \tag{1.100}$$

Therefore

$$\begin{aligned}
&\left(\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{{}^C\partial^{\nu}}{\partial t^{\nu}} \right) \widetilde{l_\nu}\left(\gamma, t\right) = \\
&= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}\left(r_1 t^\nu\right) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} E_{\nu,1}\left(r_2 t^\nu\right) \right] \\
&\quad + 2\lambda \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \frac{{}^C\partial^{\nu}}{\partial t^{\nu}} E_{\nu,1}\left(r_1 t^\nu\right) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \frac{{}^C\partial^{\nu}}{\partial t^{\nu}} E_{\nu,1}\left(r_2 t^\nu\right) \right] \\
&= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \left(r_1^2\, E_{\nu,1}\left(r_1 t^\nu\right) + \frac{t^{-\nu} r_1}{\Gamma\left(1-\nu\right)} \right) \right. \\
&\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \left(r_2^2\, E_{\nu,1}\left(r_2 t^\nu\right) + \frac{t^{-\nu} r_2}{\Gamma\left(1-\nu\right)} \right) \right] \\
&\quad + 2\lambda \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \left(r_1 E_{\nu,1}\left(r_1 t^\nu\right)\right) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) r_2 E_{\nu,1}\left(r_2 t^\nu\right) \right] \\
&= \frac{1}{2} \left[r_1 \left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_1 t^\nu\right)\left(r_1 + 2\lambda\right) + r_2 \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) \cdot \right. \\
&\quad \left. \cdot\, E_{\nu,1}\left(r_2 t^\nu\right)\left(r_2 + 2\lambda\right) \right] \\
&= -\frac{\gamma}{2} \frac{\lambda + \sqrt{\lambda^2-\gamma}}{\sqrt{\lambda^2-\gamma}} E_{\nu,1}\left(r_1 t^\nu\right) - \frac{\gamma}{2} \frac{\sqrt{\lambda^2-\gamma} - \lambda}{\sqrt{\lambda^2-\gamma}} E_{\nu,1}\left(r_2 t^\nu\right) \\
&= -\gamma \left[\frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_1 t^\nu\right) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right) E_{\nu,1}\left(r_2 t^\nu\right) \right] \right]
\end{aligned}$$

$$= -\gamma \, \widetilde{\ell_\nu} \, (\gamma, t) \, . \tag{1.101}$$

In the last steps we used the fact that

$$\left(1+\frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right)\frac{r_1\, t^{-\nu}}{\Gamma\,(1-\nu)}+\left(1-\frac{\lambda}{\sqrt{\lambda^2-\gamma}}\right)\frac{r_2\, t^{-\nu}}{\Gamma\,(1-\nu)} \; = \; 0, \tag{1.102}$$

and

$$r_1 + 2\lambda \; = \; -r_2, \qquad r_2 + 2\lambda \; = \; -r_1, \qquad r_1 r_2 \; = \; \gamma. \tag{1.103}$$

□

Remark 1.2.3. The derivation of result (1.78) suggests an alternative proof for the Fourier transform (Theorem 2.2 in Orsingher and Beghin (2004)) of the law of the time-fractional telegraph process.

Remark 1.2.4. From (1.88) we get the time Laplace transform of $\ell_\nu(x,t)$, for $x > 0, \mu > 0, 0 < \nu < \frac{1}{2}$, as

$$\widetilde{\ell_\nu} \, (x, \mu) \; = \; \mu^{2\nu - 1} e^{-x\mu^{2\nu}} e^{-2\lambda x \mu^\nu} + 2\lambda \mu^{\nu-1} e^{-2\lambda x \mu^\nu} e^{-x\mu^{2\nu}}. \tag{1.104}$$

Since (see formulas (1.55) and (1.57))

$$\widetilde{l_\nu}(x, \mu) \; = \; \int_0^\infty e^{-\mu t} \frac{1}{t^\nu} W_{-\nu, 1-\nu}\left(-\frac{x}{t^\nu}\right) \, dt \; = \; \mu^{\nu-1} e^{-x\mu^\nu} \tag{1.105}$$

and (see formula (1.53))

$$\widetilde{h_\nu}(\mu, t) \; = \; \int_0^\infty e^{-\mu x} h_\nu(x,t) \, dx \; = \; e^{-t\mu^\nu}, \tag{1.106}$$

we are able to invert (1.104) and we obtain the explicit distribution of the process $\mathcal{L}^\nu(t)$, $t > 0$, which reads

$$\begin{aligned} \ell_\nu(x,t) \; &= \frac{\Pr\left\{\mathcal{L}^\nu(t) \in dx\right\}}{dx} \\ &= \int_0^t l_{2\nu}\,(x,s)\; h_\nu(t-s, 2\lambda x)\, ds + 2\lambda \int_0^t l_\nu(2\lambda x, s)\, h_{2\nu}\,(t-s,x)\; ds \\ &= \int_0^t \frac{1}{s^{2\nu}} \, W_{-2\nu, 1-2\nu}\left(-\frac{x}{s^{2\nu}}\right)\, h_\nu(t-s, 2\lambda x)\, ds \\ &\quad + 2\lambda \int_0^t \frac{1}{s^\nu} \, W_{-\nu, 1-\nu}\left(-\frac{2\lambda x}{s^\nu}\right)\, h_{2\nu}\,(t-s, x)\; ds. \end{aligned} \tag{1.107}$$

The densities h_ν and $h_{2\nu}$ can be written down in terms of series expansion of stable laws (see pag. 245 of Orsingher and Beghin (2009)).

1.3 n-dimensional stable laws and fractional Laplacian

Let

$$\boldsymbol{S}_n^{2\beta}(t) = \left(S_1^{2\beta}(t), S_2^{2\beta}(t), \cdots, S_n^{2\beta}(t)\right), \qquad t > 0, \beta \in (0,1], \tag{1.108}$$

be the isotropic stable n-dimensional process with joint characteristic function

$$\begin{aligned} \widehat{v_n^{2\beta}}(\boldsymbol{\xi}, t) &= \widehat{v_n^{2\beta}}(\xi_1, \xi_2, \cdots, \xi_n, t) = \mathbb{E}e^{i\boldsymbol{\xi}\cdot\boldsymbol{S}_n^{2\beta}(t)} = e^{-t\left(\sqrt{\xi_1^2+\xi_2^2+\cdots+\xi_n^2}\right)^{2\beta}} \\ &= e^{-t\|\boldsymbol{\xi}\|^{2\beta}}. \end{aligned} \tag{1.109}$$

The density corresponding to the characteristic function $\widehat{v_n^{2\beta}}(\boldsymbol{\xi}, t)$ is given by

$$v_n^{2\beta}(\boldsymbol{x}, t) = v_n^{2\beta}(x_1, x_2, \cdots, x_n, t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} e^{-t\|\boldsymbol{\xi}\|^{2\beta}} d\boldsymbol{\xi}. \tag{1.110}$$

The equation governing the distribution $v_n^{2\beta}(\boldsymbol{x}, t)$ of the vector process $\boldsymbol{S}_n^{2\beta}(t)$, $t > 0$, is

$$\left(\frac{\partial}{\partial t} + (-\Delta)^\beta\right) v_n^{2\beta}(\boldsymbol{x}, t) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0, \tag{1.111}$$

where the fractional negative Laplacian is related to the classical Laplacian by means of the following relationships (Bochner representation, see for example Balakrishnan (1960), Bochner (1949))

$$\begin{aligned} &\frac{\sin \pi\beta}{\pi}\int_0^\infty d\lambda\, \lambda^{\beta-1}(\lambda-\Delta)^{-1}\,\Delta = \frac{\sin\pi\beta}{\pi}\int_0^\infty \lambda^{\beta-1}\left(\int_0^\infty e^{-w(\lambda-\Delta)}dw\right)\Delta\, d\lambda \\ &= \frac{\sin\pi\beta}{\pi}\Delta\,\Gamma(\beta)\int_0^\infty w^{1-\beta-1}e^{-w(-\Delta)}dw = \frac{\Delta}{\Gamma(1-\beta)}\int_0^\infty w^{1-\beta-1}e^{-w(-\Delta)}dw \\ &= -(-\Delta)^\beta. \end{aligned} \tag{1.112}$$

A definition of the fractional negative Laplacian can be given in the space of the Fourier transforms as follows

$$-(-\Delta)^\beta u(\boldsymbol{x}) = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}\left(\xi_1^2+\xi_2^2+\cdots+\xi_n^2\right)^\beta \widehat{u}(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.113}$$

where

$$\mathrm{Dom}\,(-\Delta)^\beta = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n}|\widehat{u}(\boldsymbol{\xi})|^2\left(1+\|\boldsymbol{\xi}\|^{2\beta}\right)d\boldsymbol{\xi} < \infty\right\}. \tag{1.114}$$

An equivalent alternative definition of the n-dimensional fractional Laplacian is

$$(-\Delta)^\beta u(\boldsymbol{x}) = c(\beta, n)\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(\boldsymbol{x})-u(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|^{n+2\beta}}dy, \tag{1.115}$$

where the multiplicative constant $c(\beta, n)$ must be evaluated in such a way that

$$\int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(-\Delta\right)^{\beta} u(\boldsymbol{x})\, d\boldsymbol{x} = \|\boldsymbol{\xi}\|^{2\beta} \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} u(\boldsymbol{x})\, d\boldsymbol{x}. \tag{1.116}$$

Let us focus our attention on the one-dimensional case of (1.115). In this case we have that, for $0 < 2\beta < 1$,

$$\begin{aligned}
&\left(-\frac{\partial^2}{\partial x^2}\right)^{\beta} u(x) = c(\beta,1)\, \text{P.V.} \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2\beta}} \\
&= c(\beta,1) \lim_{\epsilon\to 0} \left[\int_{-\infty}^{0-\epsilon} \frac{u(x)-u(x-z)}{|z|^{1+2\beta}} dz + \int_{0+\epsilon}^{\infty} \frac{u(x)-u(x-z)}{|z|^{1+2\beta}} dz \right] \\
&= c(\beta,1) \lim_{\epsilon\to 0} \left[\int_{0+\epsilon}^{\infty} \frac{u(x)-u(x+z)}{z^{1+2\beta}} dz + \int_{0+\epsilon}^{\infty} \frac{u(x)-u(x-z)}{z^{1+2\beta}} dz \right] \\
&= \frac{\Gamma(1-2\beta)}{2\beta} c(\beta,1) \left[\frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \left(\int_{-\infty}^{x} \frac{u(z)\, dz}{(x-z)^{2\beta}} - \int_{x}^{\infty} \frac{u(z)\, dz}{(z-x)^{2\beta}} \right) \right],
\end{aligned} \tag{1.117}$$

where in the intermediate steps, we considered the relation between the Marchaud and the Weyl fractional derivatives. By setting

$$c(\beta,1) = \frac{2\beta}{2\,\Gamma(1-2\beta)\, \cos\beta\pi}, \tag{1.118}$$

we have that, for $0 < 2\beta < 1$,

$$\begin{aligned}
&-\left(-\frac{\partial^2}{\partial x^2}\right)^{\beta} u(x) = \\
&= -\frac{1}{2\cos\beta\pi} \left[\frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_{-\infty}^{x} \frac{u(z)\, dz}{(x-z)^{2\beta}} - \frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_{x}^{\infty} \frac{u(z)\, dz}{(z-x)^{2\beta}} \right] \\
&= -\frac{1}{2\cos\beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{u(z)}{|x-z|^{2\beta}} dz = \frac{\partial^{2\beta}}{\partial|x|^{2\beta}} u(x),
\end{aligned} \tag{1.119}$$

where $\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}$ represents the Riesz operator.

Remark 1.3.1. We notice that, for $0 < 2\beta < 1$,

$$\mathcal{F}\left[\frac{\partial^{2\beta}}{\partial|x|^{2\beta}} u(x)\right](\xi) = -|\xi|^{2\beta}\, \widehat{u}(\xi). \tag{1.120}$$

This is due to the calculation

$$\begin{aligned}
&\mathcal{F}\left[\frac{\partial^{2\beta}}{\partial|x|^{2\beta}} u(x)\right](\xi) = \\
&= -\frac{1}{2\cos\beta\pi} \frac{1}{\Gamma(1-2\beta)} \left[\int_{-\infty}^{\infty} dx\, e^{i\xi x} \left(\frac{d}{dx} \int_{-\infty}^{x} \frac{u(z)\, dz}{(x-z)^{2\beta}} - \frac{d}{dx} \int_{x}^{\infty} \frac{u(z)\, dz}{(z-x)^{2\beta}} \right) \right] \\
&= \frac{i\xi}{2\cos\beta\pi} \frac{1}{\Gamma(1-2\beta)} \left[\int_{-\infty}^{\infty} dx\, e^{i\xi x} \left(\int_{-\infty}^{x} \frac{u(z)\, dz}{(x-z)^{2\beta}} - \int_{x}^{\infty} \frac{u(z)\, dz}{(z-x)^{2\beta}} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i\xi}{2\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\left[\int_{-\infty}^{\infty} dz\, u(z)\left(\int_z^{\infty}\frac{e^{i\xi x}\,dx}{(x-z)^{2\beta}} - \int_{-\infty}^{z}\frac{e^{i\xi x}\,dx}{(z-x)^{2\beta}}\right)\right] \\
&= \frac{i\xi}{2\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\left[\int_{-\infty}^{\infty} e^{i\xi z}u(z)\,dz\left(\int_0^{\infty}\frac{e^{i\xi y}}{y^{2\beta}}\,dy - \int_0^{\infty}\frac{e^{-i\xi y}}{y^{2\beta}}\,dy\right)\right] \\
&= -\frac{2\xi}{2\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\int_{-\infty}^{\infty} e^{i\xi z}u(z)\,dz\int_0^{\infty}\frac{\sin\xi y}{y^{2\beta}}\,dy \\
&= -\frac{\xi}{\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\frac{\widehat{u}(\xi)}{\Gamma(2\beta)}\int_0^{\infty}\int_0^{\infty}\sin\xi y\, e^{-wy}w^{2\beta-1}\,dw\,dy \\
&= -\frac{\xi}{\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\frac{\widehat{u}(\xi)}{\Gamma(2\beta)}\int_0^{\infty}dw\,w^{2\beta-1}\int_0^{\infty}dy\,e^{-wy}\left(\frac{e^{i\xi y}-e^{-i\xi y}}{2i}\right) \\
&= -\frac{\xi^2}{\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\frac{\widehat{u}(\xi)}{\Gamma(2\beta)}\int_0^{\infty}dw\frac{w^{2\beta-1}}{w^2+\xi^2} \\
&= -\frac{\xi^2}{\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\frac{\widehat{u}(\xi)}{\Gamma(2\beta)}\int_0^{\infty}dw\,w^{2\beta-1}\int_0^{\infty}dy\,e^{-y\left(w^2+\xi^2\right)} \\
&= -\frac{\xi^2}{2\cos\beta\pi}\frac{1}{\Gamma(1-2\beta)}\frac{\widehat{u}(\xi)}{\Gamma(2\beta)}\frac{\Gamma(\beta)\,\Gamma(1-\beta)}{|\xi|^{2-2\beta}} = -|\xi|^{2\beta}\,\widehat{u}(\xi).
\end{aligned}
\tag{1.121}
$$

This concludes the proof of (1.120).

1.4 Space-time fractional telegraph equation

We consider now the composition of an isotropic vector of stable processes $\boldsymbol{S}_n^{2\beta}(t)$, $t>0$, defined in (1.108), with the positively-valued process, defined in (1.76),

$$
\mathcal{L}^{\nu}(t) = \inf\left\{s>0 : \mathcal{H}^{\nu}(s) = H_1^{2\nu}(s) + (2\lambda)^{\frac{1}{\nu}}H_2^{\nu}(s) \geq t\right\}, \qquad t>0, \tag{1.122}
$$

where $H_1^{2\nu}$, H_2^{ν} are independent positively skewed stable processes of order 2ν and ν, respectively. The distribution $w_\nu^\beta(\boldsymbol{x},t)$ of the process $\boldsymbol{S}_n^{2\beta}\left(c^2\mathcal{L}^{\nu}(t)\right)$, $t>0$, $\beta\in(0,1]$, is the fundamental solution to the space-time fractional telegraph equation

$$
\left(\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda\frac{{}^C\partial^{\nu}}{\partial t^{\nu}}\right) w_\nu^\beta(\boldsymbol{x},t) = -c^2\left(-\Delta\right)^{\beta} w_\nu^\beta(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\mathbb{R}^n, t>0. \tag{1.123}
$$

In our view the next theorem generalizes some previous results because we here have fractionality in space and time and the equation (1.123) is defined in $\mathbb{R}^n$.

Theorem 1.4.1. *For $\nu\in\left(0,\frac{1}{2}\right]$, $\beta\in(0,1]$ and $c>0$ the solution to the Cauchy problem for the space-time fractional n-dimensional telegraph equation*

$$
\begin{cases}
\left(\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda\frac{{}^C\partial^{\nu}}{\partial t^{\nu}}\right) w_\nu^\beta(\boldsymbol{x},t) = -c^2\left(-\Delta\right)^{\beta} w_\nu^\beta(\boldsymbol{x},t), & \boldsymbol{x}\in\mathbb{R}^n, t>0 \\
w_\nu^\beta(\boldsymbol{x},0) = \delta(\boldsymbol{x}),
\end{cases} \tag{1.124}
$$

coincides with the probability law of the vector process

$$\boldsymbol{W}_n(t) \;=\; \boldsymbol{S}_n^{2\beta}\left(c^2\mathcal{L}^{\nu}(t)\right), \qquad t>0, \tag{1.125}$$

and has Fourier transform which reads

$$\widehat{w_\nu^\beta}\left(\boldsymbol{\xi},t\right) \;= \\ = \frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}}}\right)E_{\nu,1}\left(r_1t^\nu\right)+\left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}}}\right)E_{\nu,1}\left(r_2t^\nu\right)\right], \tag{1.126}$$

where

$$r_1 \;=\; -\lambda+\sqrt{\lambda^2-c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}}, \qquad r_2 \;=\; -\lambda-\sqrt{\lambda^2-c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}}. \tag{1.127}$$

The time derivatives appearing in (1.124) *must be meant in the Dzerbayshan-Caputo sense. The fractional Laplacian is defined in* (1.113)*.*

Proof. By taking the Laplace transform of (1.124) we have

$$\mu^{2\nu}\widetilde{w_\nu^\beta}\left(\boldsymbol{x},\mu\right)-\mu^{2\nu-1}\delta(\boldsymbol{x})+2\lambda\left[\mu^\nu\widetilde{w_\nu^\beta}\left(\boldsymbol{x},\mu\right)-\mu^{\nu-1}\delta(\boldsymbol{x})\right] \;=\; -c^2\left(-\Delta\right)^\beta\widetilde{w_\nu^\beta}\left(\boldsymbol{x},\mu\right), \tag{1.128}$$

where we used the fact that (see Kilbas et al. (2006) page 98, Lemma 2.24)

$$\mathcal{L}\left[\frac{{}^C\partial^\nu}{\partial t^\nu}w_\nu^\beta\left(\boldsymbol{x},t\right)\right] \;=\; \mu^\nu\widetilde{w_\nu^\beta}(\boldsymbol{x},\mu)-\mu^{\nu-1}w_\nu^\beta(\boldsymbol{x},0). \tag{1.129}$$

Now the Fourier transform of (1.128) yields

$$\left(\mu^{2\nu}+2\lambda\mu^\nu\right)\widehat{\widetilde{w_\nu^\beta}}\left(\boldsymbol{\xi},\mu\right)-\left(\mu^{2\nu-1}+2\lambda\mu^{\nu-1}\right) \;=\; -c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}\widehat{\widetilde{w_\nu^\beta}}\left(\boldsymbol{\xi},\mu\right), \tag{1.130}$$

and thus

$$\widehat{\widetilde{w_\nu^\beta}}\left(\boldsymbol{\xi},\mu\right) \;=\; \frac{\mu^{2\nu-1}+2\lambda\mu^{\nu-1}}{\mu^{2\nu}+2\lambda\mu^\nu+c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}}, \qquad \mu>0, \boldsymbol{\xi}\in\mathbb{R}^n. \tag{1.131}$$

The probability density of the process $\boldsymbol{W}_n(t)$, $t>0$, defined in (1.125), can be written as

$$w_\nu^\beta\left(\boldsymbol{x},t\right) \;=\; \int_0^\infty v_\beta\left(\boldsymbol{x},c^2s\right)\,l_\nu\left(s,t\right)\,ds, \tag{1.132}$$

and has Fourier transform equal to

$$\int_{\mathbb{R}^n}e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}\,w_\nu^\beta\left(\boldsymbol{x},t\right)\,d\boldsymbol{x} \;=\; \int_0^\infty e^{-c^2s\left\|\boldsymbol{\xi}\right\|^{2\beta}}\,l_\nu(s,t)\,ds. \tag{1.133}$$

In order to show that the Laplace transform of (1.133) concides with (1.131), we have to derive the Laplace transform of $\ell_\nu(x,t)$, with respect to the time t. Since

$$\Pr\{\mathcal{L}^\nu(t)<x\} = \Pr\{\mathcal{H}^\nu(x)>t\} \tag{1.134}$$

we have that

$$\begin{aligned}
&\widetilde{\ell_\nu}(x,\mu) = \\
&= \int_0^\infty e^{-\mu t}\frac{\partial}{\partial x}\int_t^\infty \Pr\{\mathcal{H}^\nu(x)\in ds\}\,dt = \int_0^\infty e^{-\mu t}\left(-\frac{\partial}{\partial x}\int_0^t \hbar_\nu(s,x)\,ds\right)dt \\
&= -\frac{\partial}{\partial x}\frac{e^{-x\mu^{2\nu}-2\lambda x\mu^\nu}}{\mu} = \left(\mu^{2\nu-1}+2\lambda\mu^{\nu-1}\right)e^{-x\mu^{2\nu}-2\lambda x\mu^\nu},
\end{aligned} \tag{1.135}$$

where we used result (1.85). Now we can complete the proof by taking the Laplace transform of (1.133) so that, in view of (1.135), we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}dt\int_0^\infty e^{-c^2 s\|\boldsymbol{\xi}\|^{2\beta}}\,\ell_\nu(s,t)\,ds = \\
&= \left(\mu^{2\nu-1}+2\lambda\mu^{\nu-1}\right)\int_0^\infty e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}-s\mu^{2\nu}-2\lambda s\mu^\nu}\,ds = \frac{\mu^{2\nu-1}+2\lambda\mu^{\nu-1}}{\mu^{2\nu}+2\lambda\mu^\nu+c^2\|\boldsymbol{\xi}\|^{2\beta}},
\end{aligned} \tag{1.136}$$

which coincides with (1.131). The unicity of Fourier-Laplace transform proves that the claimed result holds. The proof that the Fourier transform of $w_\nu^\beta(\boldsymbol{x},t)$ has the form (1.126) can be carried out by means of the calculation performed in Theorem 1.2.2. We have that

$$\begin{aligned}
\widehat{\widetilde{w_\nu^\beta}}(\boldsymbol{\xi},\mu) &= \frac{\mu^{2\nu-1}+2\lambda\mu^{\nu-1}}{\mu^{2\nu}+2\lambda\mu^\nu+c^2\|\boldsymbol{\xi}\|^{2\beta}} = \frac{\mu^{\nu-1}}{\mu^\nu-r_1}+\frac{\mu^{\nu-1}}{\mu^\nu-r_2}-\frac{\mu^{2\nu-1}}{(\mu^\nu-r_1)(\mu^\nu-r_2)} \\
&= \frac{\mu^{\nu-1}}{\mu^\nu-r_1}+\frac{\mu^{\nu-1}}{\mu^\nu-r_2}-\left[\frac{\mu^{\nu-(1-\nu)}}{\mu^\nu-r_1}-\frac{\mu^{\nu-(1-\nu)}}{\mu^\nu-r_2}\right]\frac{1}{2\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}},
\end{aligned} \tag{1.137}$$

where

$$r_1 = -\lambda+\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}, \qquad r_2 = -\lambda-\sqrt{\lambda^2-c^2\|\boldsymbol{\xi}\|^{2\beta}}. \tag{1.138}$$

and thus by inverting (1.137) by means of (1.90), we obtain result (1.126). An alternative derivation of (1.126) can be carried out as follows

$$\begin{aligned}
\widehat{w_\nu^\beta}(\boldsymbol{\xi},t) &= \int_{-\infty}^\infty e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}d\boldsymbol{x}\int_0^\infty \Pr\left\{\boldsymbol{S}_n^{2\beta}\left(c^2 s\right)\in d\boldsymbol{x}\right\}\Pr\{\mathcal{L}^\nu(t)\in ds\} \\
&= \int_0^\infty e^{-c^2 s\|\boldsymbol{\xi}\|^{2\beta}}\Pr\{\mathcal{L}^\nu(t)\in ds\} = (1.126)
\end{aligned} \tag{1.139}$$

because of Theorem 1.2.2. □

1.4.1 The case $\nu = \frac{1}{2}$, subordinator with drift

The fractional equation (1.123), for $n = 1$, $\nu = \frac{1}{2}$, reads

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) w^{\beta}_{\frac{1}{2}}(x,t) \;=\; c^2 \left(\frac{\partial^{2\beta}}{\partial |x|^{2\beta}} \right) w^{\beta}_{\frac{1}{2}}(x,t), \qquad 0 < \beta < 1, \tag{1.140}$$

where $\frac{\partial^{2\beta}}{\partial |x|^{2\beta}}$ is the Riesz operator defined in (1.119). For $\beta = 1$ we have the special case

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) w^{1}_{\frac{1}{2}}(x,t) \;=\; c^2 \frac{\partial^2}{\partial x^2} w^{1}_{\frac{1}{2}}(x,t) \tag{1.141}$$

dealt with in Orsingher and Beghin (2004). The construction of the composition related to equation (1.140) involves the subordinator

$$\mathcal{H}^{\frac{1}{2}}(t) \;=\; t + (2\lambda)^2 H^{\frac{1}{2}}(t), \qquad t > 0, \tag{1.142}$$

where $H^{\frac{1}{2}}(t)$, $t > 0$, is a positively-skewed stable process and has the same law as the first-passage time of a Brownian motion through level $\frac{t}{\sqrt{2}}$. We note that $\mathcal{H}^{\frac{1}{2}}(t)$, $t > 0$, has distribution with support $[t, \infty)$ and thus differs from $\mathcal{H}^{\nu}(t)$, $t > 0$, $0 < \nu < \frac{1}{2}$, which instead has support $[0, \infty)$. The distribution of (1.142) writes

$$\Pr\left\{ \mathcal{H}^{\frac{1}{2}}(t) < x \right\} \;=\; \int_0^{\frac{x-t}{(2\lambda)^2}} \frac{t}{\sqrt{2}} \frac{e^{-\frac{t^2}{4z}}}{\sqrt{2\pi z^3}} \, dz, \qquad x > t > 0. \tag{1.143}$$

The inverse process

$$\mathcal{L}^{\frac{1}{2}}(t) \;=\; \inf\left\{ s : s + (2\lambda)^2 H^{\frac{1}{2}}(s) \geq t \right\} \;=\; \inf\left\{ s : \mathcal{H}^{\frac{1}{2}}(s) \geq t \right\} \tag{1.144}$$

is related to (1.142) by means of the relationship

$$\Pr\left\{ \mathcal{L}^{\frac{1}{2}}(t) < x \right\} \;=\; \Pr\left\{ \mathcal{H}^{\frac{1}{2}}(x) > t \right\} \;=\; \int_{\frac{t-x}{(2\lambda)^2}}^{\infty} \frac{x}{\sqrt{2}} \frac{e^{-\frac{x^2}{4z}}}{\sqrt{2\pi z^3}} \, dz. \tag{1.145}$$

From (1.145) we can extract the distributon of $\mathcal{L}^{\frac{1}{2}}(t)$, $t > 0$, in the following manner

$$\begin{aligned} l_{\frac{1}{2}}(x,t) \;&=\; \frac{\Pr\left\{ \mathcal{L}^{\frac{1}{2}}(t) \in dx \right\}}{dx} \;=\; \frac{\partial}{\partial x} \int_{\frac{t-x}{(2\lambda)^2}}^{\infty} \frac{x\, e^{-\frac{x^2}{4z}}}{\sqrt{4\pi z^3}} \, dz \\ &=\; \frac{2\lambda x \, e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{4\pi\,(t-x)^3}} + 2\lambda \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{\pi (t-x)}}, \qquad 0 < x < t. \end{aligned} \tag{1.146}$$

Remark 1.4.2. The distribution (1.146) can be also obtained from the general case (1.107) which for $\nu = \frac{1}{2}$ becomes, for $0 < x < t$,

$$l_{\frac{1}{2}}(x,t) \;=\; \int_0^t \delta(s-x)\, h_{\frac{1}{2}}(t-s, 2\lambda x) \, ds + 2\lambda \int_0^t l_{\frac{1}{2}}(2\lambda x, s)\, \delta\left(x - (t-s)\right) \, ds$$

$$= h_{\frac{1}{2}}(t-x, 2\lambda x) + 2\lambda\, l_{\frac{1}{2}}(2\lambda x, t-x)$$
$$= \frac{2\lambda x\, e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{4\pi\,(t-x)^3}} + 2\lambda \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{\pi(t-x)}}. \tag{1.147}$$

In the last step we used the fact that

$$L^{\frac{1}{2}}(t) \stackrel{law}{=} |B(t)|\,, \qquad t>0, \tag{1.148}$$

where $L^{\frac{1}{2}}(t)$, $t>0$, dealt with in section 1.1.3, is the inverse of the totally positively-skewed stable process $H^{\frac{1}{2}}(t)$, $t>0$.

The t-Laplace transform of (1.146) becomes

$$\begin{aligned}
\widetilde{\ell_{\frac{1}{2}}}(x,\mu) \;&=\; \int_x^\infty e^{-\mu t}\, \ell_{\frac{1}{2}}(x,t)\, dt \;= \\
&= \frac{2\lambda x}{\sqrt{2}} \int_x^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{2\pi(t-x)^3}}\, dt + 2\lambda \int_x^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4(t-x)}}}{\sqrt{\pi(t-x)}}\, dt \\
&= \frac{2\lambda x}{\sqrt{2}} e^{-\mu x} \int_0^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4t}}}{\sqrt{2\pi t^3}} dt + 2\lambda e^{-\mu x} \int_0^\infty e^{-\mu t} \frac{e^{-\frac{(2\lambda x)^2}{4t}}}{\sqrt{\pi t}}\, dt \\
&= e^{-\mu x} e^{-2\lambda x\sqrt{\mu}} + 2\lambda\, \mu^{-\frac{1}{2}}\, e^{-\mu x} e^{-2\lambda x \sqrt{\mu}}.
\end{aligned} \tag{1.149}$$

Finally the x-Laplace transform of (1.149) becomes

$$\begin{aligned}
\widetilde{\widetilde{\ell_{\frac{1}{2}}}}(\gamma,\mu) \;&=\; \int_0^\infty e^{-\gamma x} \left(\int_x^\infty e^{-\mu t}\, \ell_{\frac{1}{2}}(x,t)\, dt \right) dx \\
&= \frac{1}{\mu+\gamma+2\lambda\sqrt{\mu}} + \frac{2\lambda}{\sqrt{\mu}} \frac{1}{\mu+\gamma+2\lambda\sqrt{\mu}} \;=\; \frac{1+2\lambda\mu^{-\frac{1}{2}}}{\mu+\gamma+2\lambda\sqrt{\mu}},
\end{aligned} \tag{1.150}$$

which coincides with (1.88), for $\nu = \frac{1}{2}$. Let us now consider the process $\boldsymbol{W}_n(t) = \boldsymbol{S}_n^{2\beta}\left(c^2 \mathcal{L}^\nu(t)\right)$, $t>0$, dealt with in Theorem 1.4.1. For $\beta = 1$, $n=1$ and $\nu = \frac{1}{2}$ this process becomes

$$W_1(t) \;=\; S_1^2\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right) \;=\; B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right), \qquad t>0 \tag{1.151}$$

where B represents a standard Brownian motion and $\mathcal{L}^{\frac{1}{2}}(t)$, $t>0$, is the process defined in (1.144). With

$$p_{|B|}(x,t) \;=\; \frac{e^{-\frac{x^2}{4t}}}{\sqrt{\pi t}}, \qquad x>0, t>0, \tag{1.152}$$

we denote the law of the process $|B(t)|$, $t>0$. In view of the previous results we are able to prove the following theorem.

Theorem 1.4.3. *The law of* (1.151) *coincides with the law of the composition*

$$\mathcal{W}(t) = T\left(|B(t)|\right), \qquad t > 0, \tag{1.153}$$

where T is the telegraph process (1.26) *with parameters $c > 0$, $\lambda > 0$ and law $p_T(x,t)$ which has characteristic function*

$$\widehat{p_T}\left(\xi, t\right) = = \frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right)e^{-\lambda t+t\sqrt{\lambda^2-c^2\xi^2}} + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right)e^{-\lambda t-t\sqrt{\lambda^2-c^2\xi^2}}\right]. \tag{1.154}$$

In other words we have the following equality in distribution

$$B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right) \overset{law}{=} T\left(|B(t)|\right), \qquad t > 0. \tag{1.155}$$

Proof. First we show that the Fourier-Laplace transform of the law $w^1_{\frac{1}{2}}(x,t)$ of the process $W_1(t) = S_1^2\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right) = B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right)$, $t > 0$, is written as in (1.136) for $\nu = \frac{1}{2}$, $\beta = 1$, $n = 1$, and reads

$$\widehat{\widetilde{w^1_{\frac{1}{2}}}}\left(\xi, \mu\right) = \frac{1+2\lambda\mu^{-\frac{1}{2}}}{\mu+2\lambda\sqrt{\mu}+c^2\xi^2}. \tag{1.156}$$

We have that

$$\begin{aligned}
\widetilde{w^1_{\frac{1}{2}}}(x,\mu) &= \int_0^\infty e^{-\mu t}\left(\int_0^t p_B\left(x, c^2 s\right)\, \ell_{\frac{1}{2}}(s,t)\, ds\right) dt \\
&= \int_0^\infty p_B(x,c^2 s)\, ds \int_s^\infty e^{-\mu t}\ell_{\frac{1}{2}}(s,t)\, dt \\
&= \int_0^\infty p_B(x,c^2 s)\, ds \left[\int_s^\infty e^{-\mu t}\left(\frac{2\lambda s\, e^{-\frac{(2\lambda s)^2}{4(t-s)}}}{\sqrt{4\pi\left(t-s\right)^3}} + 2\lambda\frac{e^{-\frac{(2\lambda s)^2}{4(t-s)}}}{\sqrt{\pi(t-s)}}\right) dt\right] \\
&= \int_0^\infty p_B\left(x, c^2 s\right)\left(e^{-s\left(\mu+2\lambda\sqrt{\mu}\right)} + 2\lambda\sqrt{\mu}e^{-s\left(\mu+2\lambda\sqrt{\mu}\right)}\right) ds \\
&= \int_0^\infty \frac{e^{-\frac{x^2}{4c^2 s}}}{\sqrt{4\pi c^2 s}}e^{-s\left(\mu+2\lambda\sqrt{\mu}\right)}\, ds + 2\lambda\mu^{-\frac{1}{2}}\int_0^\infty \frac{e^{-\frac{x^2}{4c^2 s}}}{\sqrt{4\pi c^2 s}}e^{-s\left(\mu+2\lambda\sqrt{\mu}\right)}\, ds,
\end{aligned} \tag{1.157}$$

and thus taking the Fourier transform we get

$$\begin{aligned}
\widehat{\widetilde{w^1_{\frac{1}{2}}}}\left(\xi, \mu\right) &= \int_0^\infty e^{-sc^2\xi^2}e^{-s\left(\mu+2\lambda\sqrt{\mu}\right)}\, ds + 2\lambda\mu^{-\frac{1}{2}}\int_0^\infty e^{-sc^2\xi^2}e^{-s\left(\mu+2\lambda\sqrt{\mu}\right)}\, ds \\
&= \frac{1+2\lambda\mu^{-\frac{1}{2}}}{\mu+2\lambda\sqrt{\mu}+c^2\xi^2}.
\end{aligned} \tag{1.158}$$

Now we are going to prove that the law $w(x,t)$ of the process $\mathcal{W}(t)$, $t>0$, has Fourier-Laplace transform that coincides with (1.156). We have that

$$w(x,t) = \int_0^\infty p_T(x,s)\, p_{|B|}(s,t)\, ds, \tag{1.159}$$

and thus the Fourier transform of $w(x,t)$ reads

$$\begin{aligned}
\widehat{w}(\xi,t) &= \int_{-\infty}^{\infty} e^{i\xi x} dx \int_0^\infty p_T(x,s)\; p_{|B|}(s,t)\, ds \\
&= \frac{1}{2}\int_0^\infty \left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right) e^{-\lambda s + s\sqrt{\lambda^2-c^2\xi^2}} \right. \\
&\quad \left. + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right) e^{-\lambda s - s\sqrt{\lambda^2-c^2\xi^2}}\right] p_{|B|}(s,t)\, ds.
\end{aligned} \tag{1.160}$$

Passing now to the Laplace transform we have

$$\begin{aligned}
\widetilde{\widehat{w}}(\xi,\mu) &= \frac{1}{2}\int_0^\infty e^{-\mu t}\, dt \int_0^\infty \left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right) e^{-\lambda s + s\sqrt{\lambda^2-c^2\xi^2}} \right. \\
&\quad \left. + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right) e^{-\lambda s - s\sqrt{\lambda^2-c^2\xi^2}}\right] \frac{e^{-\frac{s^2}{4t}}}{\sqrt{\pi t}}\, ds \\
&= \frac{1}{2}\int_0^\infty \left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right) e^{-\lambda s + s\sqrt{\lambda^2-c^2\xi^2}} \right. \\
&\quad \left. + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right) e^{-\lambda s - s\sqrt{\lambda^2-c^2\xi^2}}\right] \frac{e^{-s\sqrt{\mu}}}{\sqrt{\mu}}\, ds \\
&= \frac{1}{2\sqrt{\mu}}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right)\left(\frac{1}{\lambda+\sqrt{\mu}-\sqrt{\lambda^2-c^2\xi^2}}\right) \right. \\
&\quad \left. + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\xi^2}}\right)\left(\frac{1}{\lambda+\sqrt{\mu}+\sqrt{\lambda^2-c^2\xi^2}}\right)\right] \\
&= \frac{\left(\lambda+\sqrt{\lambda^2-c^2\xi^2}\right)\left(\lambda+\sqrt{\mu}+\sqrt{\lambda^2-c^2\xi^2}\right)}{\left(2\sqrt{\mu}\sqrt{\lambda^2-c^2\xi^2}\right)\left(\mu+2\lambda\sqrt{\mu}+c^2\xi^2\right)} \\
&\quad + \frac{\left(\sqrt{\lambda^2-c^2\xi^2}-\lambda\right)\left(\lambda+\sqrt{\mu}-\sqrt{\lambda^2-c^2\xi^2}\right)}{\left(2\sqrt{\mu}\sqrt{\lambda^2-c^2\xi^2}\right)\left(\mu+2\lambda\sqrt{\mu}+c^2\xi^2\right)} \\
&= \frac{1+2\lambda\mu^{-\frac{1}{2}}}{\mu+2\lambda\sqrt{\mu}+c^2\xi^2},
\end{aligned} \tag{1.161}$$

which coincides with (1.156). □

This shows that for each t we have the following equality in distribution

$$T\left(|B(t)|\right) \overset{\text{law}}{=} B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right), \qquad t>0, \tag{1.162}$$

where the role of the Brownian motion is interchanged in the two members of (1.162). Thus, by suitably slowing down the time in (1.162), we obtain the same distributional effect of a telegraph process taken at a Brownian time.

Remark 1.4.4. The probability distribution of the process

$$W_1(t) = B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right), \qquad t > 0, \tag{1.163}$$

can be written as

$$\begin{aligned} w_{\frac{1}{2}}^1(x,t) &= \frac{\lambda}{c\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} e^{-\frac{x^2}{4c^2 s} - \frac{\lambda^2 s^2}{t-s}} \left[\frac{s}{2(t-s)} + 1 \right] ds \\ &= \frac{\lambda}{c\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} e^{-\frac{x^2}{4c^2 s} - \frac{\lambda^2 s^2}{t-s}} \left[\frac{1}{2} \left(1 + \frac{t}{t-s} \right) \right] ds \\ &\overset{y=\lambda s}{=} \frac{\sqrt{\lambda}}{c\pi} \int_0^{\lambda t} e^{-\frac{\lambda x^2}{4c^2 y}} e^{-\frac{y^2}{t-\frac{y}{\lambda}}} \frac{1}{\sqrt{y}\sqrt{t-\frac{y}{\lambda}}} \left[\frac{1}{2} \left(1 + \frac{t}{t-\frac{y}{\lambda}} \right) \right] dy. \end{aligned} \tag{1.164}$$

Taking the limit for $c \to \infty$, $\lambda \to \infty$, $\frac{c^2}{\lambda} \to 1$, formula (1.164) becomes

$$\lim_{\substack{\lambda, c \to \infty \\ \frac{c^2}{\lambda} \to 1}} y_{\frac{1}{2}}^1(x,t) = 2 \int_0^\infty \frac{e^{-\frac{x^2}{4y}}}{\sqrt{4\pi y}} \frac{e^{-\frac{y^2}{t}}}{\sqrt{\pi t}} dy \tag{1.165}$$

which coincides with the distribution of an iterated Brownian motion $B_1(|B_2(t)|)$, $t > 0$, with $B_j, j = 1, 2$, independent Brownian motions. From (1.164) we can see that the distribution of $W_1(t)$, $t > 0$, has a bell-shaped structure.

Finally we show that the density $w_{\frac{1}{2}}^1(x,t)$ integrates to unity in force of the calculation

$$\begin{aligned} \int_{-\infty}^{\infty} w_{\frac{1}{2}}^1(x,t)\, dx &= \int_{-\infty}^{\infty} dx \int_0^t ds\, \frac{e^{-\frac{x^2}{4s}}}{\sqrt{4\pi s}} \ell_{\frac{1}{2}}(s,t) = \int_0^t ds \left(\frac{\partial}{\partial s} \int_{\frac{t-s}{(2\lambda)^2}}^{\infty} \frac{s\, e^{-\frac{s^2}{4z}}}{\sqrt{4\pi z^3}} dz \right) \\ &= \left[\int_{\frac{t-s}{(2\lambda)^2}}^{\infty} \frac{s\, e^{-\frac{s^2}{4z}}}{\sqrt{4\pi z^3}} dz \right]_{s=0}^{s=t} = \int_0^\infty \frac{t e^{-\frac{t^2}{4z}}}{\sqrt{4\pi z^3}} dz = 1. \end{aligned} \tag{1.166}$$

In the intermediate step, formula (1.146) has been applied.

Remark 1.4.5. The characteristic function of the process $T^{2\beta}(t)$, $t > 0$, whose distribution satisfies

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) p_T^{2\beta}(x,t) = c^2 \frac{\partial^{2\beta}}{\partial |x|^{2\beta}} p_T^{2\beta}(x,t), & 0 < \beta < 1, \beta \neq \frac{1}{2} \\ p_T^{2\beta}(x,0) = \delta(x), \\ \frac{\partial}{\partial t} p_T^{2\beta}(x,t) \Big|_{t=0} = 0, \end{cases} \tag{1.167}$$

reads

$$\mathbb{E}e^{i\xi T^{2\beta}(t)} = \\ = \frac{e^{-\lambda t}}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2|\xi|^{2\beta}}}\right)e^{t\sqrt{\lambda^2-c^2|\xi|^{2\beta}}}+\left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2|\xi|^{2\beta}}}\right)e^{-t\sqrt{\lambda^2-c^2|\xi|^{2\beta}}}\right] \tag{1.168}$$

see Orsingher and Zhao (2003). Therefore by performing the same steps as in theorem (1.4.3) we prove that

$$S_1^{2\beta}\left(\mathcal{L}^{\frac{1}{2}}(t)\right) \overset{\text{law}}{=} T^{2\beta}\left(|B(t)|\right), \qquad t>0. \tag{1.169}$$

1.4.2 The case $\nu = \frac{1}{3}$, convolutions of Airy functions

We first recall that the totally positively-skewed stable process $H^{\frac{1}{3}}(t)$, $t>0$ has law

$$\Pr\left\{H^{\frac{1}{3}}(t)\in dx\right\} = \frac{t}{x\sqrt[3]{3x}}\text{Ai}\left(\frac{t}{\sqrt[3]{3x}}\right)dx, \qquad x>0, t>0, \tag{1.170}$$

where $\text{Ai}(\cdot)$ is the Airy function. Result (1.170) can be obtained from the general series expansion of the stable law of order $\frac{1}{3}$ (see Orsingher and Beghin (2009) page 245) which reads

$$\begin{aligned} h_{\frac{1}{3}}(x,1) &= \frac{1}{3\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(\frac{k+1}{3}\right)}{k!}x^{-\frac{1}{3}(k+1)-1}\sin\left(\frac{\pi}{3}(k+1)\right) \\ &= \frac{1}{3\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(\frac{k+1}{3}\right)}{k!}x^{-\frac{k+1}{3}-1}(-1)^k\sin\left(\frac{2\pi(k+1)}{3}\right) \\ &= \frac{1}{3}\frac{3^{\frac{2}{3}}}{x\sqrt[3]{x}}\text{Ai}\left(\frac{1}{\sqrt[3]{3x}}\right) = \frac{1}{x\sqrt[3]{3x}}\text{Ai}\left(\frac{1}{\sqrt[3]{3x}}\right), \end{aligned} \tag{1.171}$$

where we used formula (4.10) of Orsingher and Beghin (2009), which reads

$$\text{Ai}(w) = \frac{3^{-\frac{2}{3}}}{\pi}\sum_{k=0}^{\infty}\left(3^{\frac{1}{3}}w\right)^k\frac{\sin\left(\frac{2\pi(k+1)}{3}\right)}{k!}\Gamma\left(\frac{k+1}{3}\right). \tag{1.172}$$

Since

$$H^{\frac{1}{3}}(t) \overset{\text{law}}{=} t^3 H^{\frac{1}{3}}(1), \tag{1.173}$$

we have result (1.170). From the relatioship between $H^{\frac{1}{3}}(t)$, $t>0$, and the inverse process $L^{\frac{1}{3}}(t)$, $t>0$,

$$\Pr\left\{H^{\frac{1}{3}}(t)<x\right\} = \Pr\left\{L^{\frac{1}{3}}(x)>t\right\} \tag{1.174}$$

we extract the density of $L^{\frac{1}{3}}(x)$, $x > 0$,

$$\frac{\Pr\left\{L^{\frac{1}{3}}(x) \in dt\right\}}{dt} = -\frac{\partial}{\partial t}\int_0^x \frac{t}{s}\frac{1}{\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds$$
$$= -\int_0^x \frac{1}{s\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds - \int_0^x \frac{t}{s\sqrt[3]{3s}}\mathrm{Ai}'\left(\frac{t}{\sqrt[3]{3s}}\right)\frac{ds}{\sqrt[3]{3s}}. \tag{1.175}$$

Since

$$\frac{\partial}{\partial s}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) = -\frac{t}{3s\sqrt[3]{3s}}\mathrm{Ai}'\left(\frac{t}{\sqrt[3]{3s}}\right) \tag{1.176}$$

we conclude that, for $x > 0$, $t > 0$,

$$l_{\frac{1}{3}}(t,x) = \frac{\Pr\left\{L^{\frac{1}{3}}(x) \in dt\right\}}{dt}$$
$$= \int_0^x \frac{-1}{s\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds + \int_0^x \frac{3}{\sqrt[3]{3s}}\frac{\partial}{\partial s}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds$$
$$= \int_0^x \frac{-1}{s\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) ds + \left[\frac{3}{\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right)\right]_{s=0}^{s=x} + \int_0^x \frac{ds}{s\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right)$$
$$= \frac{3}{\sqrt[3]{3x}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3x}}\right). \tag{1.177}$$

In the last step we took into account the asymptotic expansion 7.2.19 of Bleistein and Handelsman (1986).

With similar calculation we obtain the law $h_{\frac{2}{3}}(x,t)$ of the process $H^{\frac{2}{3}}(t)$, $t > 0$, which is expressed in terms of Airy function. From the general series expression of the stable law (see Orsingher and Beghin (2009)) we have that,

$$h_{\frac{2}{3}}(x,1) =$$
$$= \frac{2}{3\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(\frac{2}{3}(k+1)\right)}{k!}x^{-\frac{2}{3}(k+1)-1}\sin\left(\frac{2\pi}{3}(k+1)\right)$$
$$= \frac{2}{3\pi\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{x^{-\frac{2}{3}(k+1)-1}}{2^{1-\frac{2}{3}(k+1)}}\Gamma\left(\frac{k+1}{3}\right)\sin\left(\frac{2\pi}{3}(k+1)\right)\int_0^\infty dw\, e^{-w}w^{\frac{k+1}{3}+\frac{1}{2}-1}$$
$$= \frac{1}{x}\sqrt[3]{\frac{2^2}{3x^2}}\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-w}w^{-\frac{1}{6}}\,\mathrm{Ai}\left(-\sqrt[3]{\frac{2^2 w}{3x^2}}\right) dw, \tag{1.178}$$

and thus, in force of the fact that $H^{\frac{2}{3}}(t) \overset{\text{law}}{=} t^{\frac{3}{2}}H^{\frac{2}{3}}(1)$,

$$h_{\frac{2}{3}}(x,t) = \frac{t}{\sqrt{\pi}}\frac{1}{x}\int_0^\infty dw\, e^{-w}w^{-\frac{1}{6}}\sqrt[3]{\frac{2^2}{3x^2}}\,\mathrm{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3x^2}}\right). \tag{1.179}$$

Remark 1.4.6. We check that the distribution (1.179) integrates to unity. We have that

$$\begin{aligned}
&\int_0^\infty h_{\frac{2}{3}}(x,t)\,dx \; = \\
&= \frac{t}{\sqrt{\pi}}\int_0^\infty dw\, e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3}} \int_0^\infty dx\, x^{-\frac{2}{3}-1} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3x^2}}\right) \\
&\overset{y=x^{-\frac{2}{3}}t\sqrt[3]{\frac{2^2w}{3}}}{=} \frac{t}{\sqrt{\pi}}\int_0^\infty dw\, e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3}}\frac{3}{2}\left(t\sqrt[3]{\frac{2^2 w}{3}}\right)^{-1}\int_0^\infty dy\,\operatorname{Ai}(-y) \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty dw\, e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3}}\left(\sqrt[3]{\frac{2^2 w}{3}}\right)^{-1} \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty dw\, e^{-w} w^{-\frac{1}{6}-\frac{1}{3}} \; = \; \frac{1}{\sqrt{\pi}}\int_0^\infty dw\, e^{-w} w^{\frac{1}{2}-1} \; = \; 1,
\end{aligned} \tag{1.180}$$

where we used the fact that

$$\int_0^\infty dy\,\operatorname{Ai}(-y) \; = \; \frac{2}{3}. \tag{1.181}$$

For the law of the process $L^{\frac{2}{3}}(x)$, $x>0$, we therefore have that

$$\begin{aligned}
\Pr\left\{L^{\frac{2}{3}}(x) < t\right\} \; &= \; \Pr\left\{H^{\frac{2}{3}}(t) > x\right\} \\
&= \int_0^\infty \int_x^\infty \frac{t}{\sqrt{\pi}}\frac{1}{z}\sqrt[3]{\frac{2^2}{3z^2}} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) e^{-w} w^{-\frac{1}{6}}\, dw\, dz
\end{aligned} \tag{1.182}$$

and thus

$$\begin{aligned}
l_{\frac{2}{3}}(t,x) \; &= \int_0^\infty \int_x^\infty \frac{dw\,dz}{z\sqrt{\pi}}\sqrt[3]{\frac{2^2}{3z^2}} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) e^{-w} w^{-\frac{1}{6}} \\
&\quad - \int_0^\infty \int_x^\infty \frac{t}{z\sqrt{\pi}}\sqrt[3]{\frac{2^2}{3z^2}}\sqrt[3]{\frac{2^2 w}{3z^2}} \operatorname{Ai}'\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) dz\, dw \\
&= \int_0^\infty \int_x^\infty \frac{dw\,dz}{z\sqrt{\pi}}\sqrt[3]{\frac{2^2}{3z^2}} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) e^{-w} w^{-\frac{1}{6}} \\
&\quad - \frac{3}{2}\int_x^\infty \int_0^\infty \frac{1}{\sqrt{\pi}}\sqrt[3]{\frac{2^2}{3z^2}} e^{-w} w^{-\frac{1}{6}} \frac{\partial}{\partial z}\operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) dw\, dz \\
&= \int_0^\infty \int_x^\infty \frac{dw\,dz}{z\sqrt{\pi}}\sqrt[3]{\frac{2^2}{3z^2}} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) e^{-w} w^{-\frac{1}{6}} \\
&\quad - \left[\frac{3}{2\sqrt{\pi}}\int_0^\infty dw \sqrt[3]{\frac{2^2}{3z^2}} e^{-w} w^{-\frac{1}{6}} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right)\right]_{z=x}^{z=\infty} \\
&\quad - \int_0^\infty \int_x^\infty \frac{dw\,dz}{z\sqrt{\pi}}\sqrt[3]{\frac{2^2}{3z^2}} \operatorname{Ai}\left(-t\sqrt[3]{\frac{2^2 w}{3z^2}}\right) e^{-w} w^{-\frac{1}{6}}
\end{aligned}$$

$$= \frac{3}{2\sqrt{\pi}} \int_0^\infty \sqrt[3]{\frac{2^2}{3x^2}} e^{-w} w^{-\frac{1}{6}} \operatorname{Ai}\left(-t \sqrt[3]{\frac{2^2 w}{3x^2}} \right) dw. \tag{1.183}$$

For checking that (1.183) integrates to unity one can perform calculation similar to that of Remark 1.4.6.

Now we have all the information to get the distribution of the process $\mathcal{L}^{\frac{1}{3}}(t)$, $t > 0$, by means of formula (1.107). We have that

$$\begin{aligned} \ell_{\frac{1}{3}}(x,t) &= \frac{\Pr\left\{ \mathcal{L}^{\frac{1}{3}}(t) \in dx \right\}}{dx} \\ &= \int_0^t l_{\frac{2}{3}}(x, t-s)\, h_{\frac{1}{3}}(s, 2\lambda x) ds + 2\lambda \int_0^t l_{\frac{1}{3}}(2\lambda x, s)\, h_{\frac{2}{3}}(t-s, x)\, ds \\ &= \int_0^t ds \left[\frac{3}{2\sqrt{\pi}} \int_0^\infty dw \sqrt[3]{\frac{2^2}{3(t-s)^2}} e^{-w} w^{-\frac{1}{6}} \operatorname{Ai}\left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) dw \right] \cdot \\ &\quad \cdot \frac{2\lambda x}{s\sqrt[3]{3s}} \operatorname{Ai}\left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) + 2\lambda \int_0^t ds \frac{3}{\sqrt[3]{3s}} \operatorname{Ai}\left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) \cdot \\ &\quad \cdot \frac{s}{\sqrt{\pi}(t-s)} \int_0^\infty dw\, e^{-w} w^{-\frac{1}{6}} \sqrt[3]{\frac{2^2}{3(t-s)^2}} \operatorname{Ai}\left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) \\ &= \frac{2\lambda}{\sqrt{\pi}} \int_0^t ds \int_0^\infty dw\, e^{-w} w^{-\frac{1}{6}} \operatorname{Ai}\left(-x \sqrt[3]{\frac{2^2 w}{3(t-s)^2}} \right) \operatorname{Ai}\left(\frac{2\lambda x}{\sqrt[3]{3s}} \right) \cdot \\ &\quad \cdot \frac{3}{\sqrt[3]{3s}} \sqrt[3]{\frac{2^2}{3(t-s)^2}} \left[\frac{x}{2s} + \frac{s}{t-s} \right]. \end{aligned} \tag{1.184}$$

Result (1.184) permits us to write explicitly the solution of the fractional telegraph equation (1.10) for $\nu = \frac{1}{3}$, $\beta = 1$ and $n = 1$, as

$$w_{\frac{1}{3}}^1(x,t) = \int_0^\infty \frac{e^{-\frac{x^2}{4c^2 s}}}{\sqrt{4\pi c^2 s}} \ell_{\frac{1}{3}}(s,t)\, ds, \qquad x \in \mathbb{R}, t > 0. \tag{1.185}$$

1.4.3 The planar case

Let us consider the planar process

$$\boldsymbol{T}(t) = (X(t), Y(t)), \qquad t > 0, \tag{1.186}$$

with infinite directions and finite velocity c, investigated in Orsingher and De Gregorio (2007), which has probability law (see formula 1.2 therein)

$$r(x,y,t) = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - (x^2+y^2)}}}{\sqrt{c^2t^2 - (x^2+y^2)}}, \qquad x^2 + y^2 < c^2t^2, t > 0, \tag{1.187}$$

which satisfies the telegraph equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t}\right) r(x, y, t) \ = \ c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) r(x, y, t). \tag{1.188}$$

The distribution of $\boldsymbol{T}(t)$, $t > 0$, has a singular component uniformly distributed on the circle $\partial C_{ct} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = c^2 t^2\}$ with probability mass equal to $e^{-\lambda t}$. The process $\boldsymbol{T}(t)$, $t > 0$, describes a random motion where directions change at Poisson paced times and the orientation of each segment of the sample paths is uniform in $[0, 2\pi)$.

Let $q(x, y, t)$ be the distribution obtained by means of the composition of the process $\boldsymbol{T}(t)$ with a reflecting Brownian motion with law

$$p_{|B|}(s, t) \ = \ \frac{e^{\frac{-s^2}{4t}}}{\sqrt{\pi t}}, \qquad t > 0, s > 0, \tag{1.189}$$

which satisfies the equation

$$\frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s, t) \ = \ -\frac{\partial}{\partial s} p_{|B|}(s, t) \tag{1.190}$$

and also

$$\frac{\partial}{\partial t} p_{|B|}(s, t) \ = \ \frac{\partial^2}{\partial s^2} p_{|B|}(s, t) \tag{1.191}$$

We have the following theorem.

Theorem 1.4.7. *The law of the composition*

$$\boldsymbol{Q}(t) \ = \ \boldsymbol{T}\left(|B(t)|\right), \qquad t > 0 \tag{1.192}$$

written as

$$q(x, y, t) = \int_0^\infty r(x, y, s)\, p_{|B|}(s, t)\, ds, \tag{1.193}$$

satisfies the 2*-dimensional time-fractional equation*

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\right) q\left(x, y, t\right) \ = \ c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q\left(x, y, t\right), \qquad x, y \in \mathbb{R}, t > 0, \tag{1.194}$$

subject to the initial condition

$$q(x, y, 0) \ = \ \delta(x, y). \tag{1.195}$$

Proof. By considering (1.193) and (1.190) we can write

$$\begin{aligned} \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} q(x,y,t) &= \int_0^\infty r(x,y,s)\, \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s,t)\, ds \\ &= \int_0^\infty r(x,y,s) \left(-\frac{\partial}{\partial s} p_{|B|}(s,t) \right) ds \\ &= \left[-p_{|B|}(s,t)\, r(x,y,s) \right]_{s=0}^{s=\infty} + \int_0^\infty p_{|B|}(s,t) \frac{\partial}{\partial s} r(x,y,s)\, ds. \end{aligned} \tag{1.196}$$

In the previous step it must be taken into account that the boundary ∂C_{cs} is excluded. From (1.193) and (1.191) we have that

$$\begin{aligned} \frac{\partial}{\partial t} q(x,y,t) &= \int_0^\infty r(x,y,s)\, \frac{\partial}{\partial t} p_{|B|}(s,t)\, ds = \int_0^\infty r(x,y,s)\, \frac{\partial^2}{\partial s^2} p_{|B|}(s,t)\, ds \\ &= \left[r(x,y,s)\, \frac{\partial}{\partial s} p_{|B|}(s,t) \right]_{s=0}^{s=\infty} - \int_0^\infty \frac{\partial}{\partial s} r(x,y,s)\, \frac{\partial}{\partial s} p_{|B|}(s,t)\, ds \\ &= -\left[p_{|B|}(s,t)\, \frac{\partial}{\partial s} r(x,y,s) \right]_{s=0}^{s=\infty} + \int_0^\infty p_{|B|}(s,t)\, \frac{\partial^2}{\partial s^2} r(x,y,s)\, ds. \end{aligned} \tag{1.197}$$

Thus, by looking at (1.188), (1.196) and (1.197) we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} q(x,y,t) + 2\lambda \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} q(x,y,t) = \\ &= \int_0^\infty p_{|B|}(s,t) \left[\frac{\partial^2}{\partial s^2} r(x,y,s) + 2\lambda \frac{\partial}{\partial s} r(x,y,s) \right] ds \\ &= \int_0^\infty p_{|B|}(s,t)\, c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) r(x,y,s)\, ds = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) q(x,y,t). \end{aligned} \tag{1.198}$$

which means that $q(x,y,t)$ satisfies equation (1.194). □

It is easy to show that the process $\boldsymbol{Q}(t) = \boldsymbol{T}(|B(t)|)$, $t > 0$, has not the same law of the process $\boldsymbol{W}_2(t) = \boldsymbol{B}_2\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right)$, $t > 0$. However it is possible to construct a planar process, say $\boldsymbol{\mathfrak{T}}(t)$, $t > 0$ (which is a slightly different version of $\boldsymbol{T}(t)$, $t > 0$) composed with a suitable "time process" which has the same distribution as $\boldsymbol{W}_2(t)$, $t > 0$. The planar random motion $\boldsymbol{\mathfrak{T}}(t)$, $t > 0$, with distribution

$$\mathfrak{r}(x,y,t) = \frac{\lambda\, e^{-\lambda t}}{2\pi c} \left[\frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}} + e^{-\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}} \right], \tag{1.199}$$

where $(x,y) \in C_{ct} = \{(x,y) : x^2+y^2 < c^2t^2\}$, can be constructed starting from the model dealt with in Orsingher and De Gregorio (2007). The distribution is based on the solution to the planar telegraph equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) \mathfrak{r}(x,y,t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \mathfrak{r}(x,y,t), \tag{1.200}$$

namely

$$\mathfrak{r}\,(x,y,t) \;=\; \frac{e^{-\lambda t}}{\sqrt{c^2t^2-(x^2+y^2)}}\left[Ae^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}+Be^{-\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}\right], \quad (1.201)$$

with $A = B = \frac{\lambda}{2\pi c}$ and thus we can easily check that

$$\iint_{C_{ct}} dx\,dy\,\mathfrak{r}\,(x,y,t) \;=\; 1-e^{-2\lambda t}. \quad (1.202)$$

We take a particle starting from the origin, moving at finite velocity c, and changing direction (chosen with uniform distribution) at Poisson times and neglect displacements started off by even-labelled times. The sample paths of this motion are constructed by piecing together only odd-order displacements of the planar motion $\boldsymbol{T}(t)$, $t > 0$. The process just described has distribution (1.199) as shown below

$$\begin{aligned}
&\mathfrak{r}(x,y,t) = \\
&= \frac{\Pr\{\boldsymbol{\mathfrak{T}}(t)\in d\boldsymbol{x}\}}{d\boldsymbol{x}} = \frac{\lambda\,e^{-\lambda t}}{2\pi c}\left[\frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}+e^{-\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}}\right] \\
&= \frac{\lambda^2}{c^2}\frac{1}{\pi}e^{-\lambda t}\left[\sum_{k=0}^{\infty}\left(\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}\right)^{2k-1}\frac{1}{(2k)!}\right] \\
&= \frac{\lambda^2}{c^2}\frac{1}{\pi}\sum_{k=0}^{\infty}\left(\frac{\lambda}{c}\right)^{2k-1}(2k+1)\left(c^2t^2-\left(x^2+y^2\right)\right)^{k-\frac{1}{2}}\frac{e^{-\lambda t}}{(2k)!(2k+1)}\frac{(\lambda t)^{2k+1}}{(\lambda t)^{2k+1}} \\
&= 2\sum_{k=0}^{\infty}\Pr\{X(t)\in dx, Y(t)\in dy|N(t)=2k+1\}\;e^{-\lambda t}\frac{(\lambda t)^{2k+1}}{(2k+1)!} \\
&= 2\sum_{k=0}^{\infty}\Pr\{\boldsymbol{T}(t)\in d\boldsymbol{x}|N(t)=2k+1\}\;e^{-\lambda t}\frac{(\lambda t)^{2k+1}}{(2k+1)!},
\end{aligned} \quad (1.203)$$

where, for $x^2+y^2 < c^2t^2$ (see Orsingher and De Gregorio (2007)),

$$\frac{\Pr\{X(t)\in dx, Y(t)\in dy|N(t)=n\}}{dx\,dy} \;=\; \frac{n}{2n(ct)^n}\left(c^2t^2-\left(x^2+y^2\right)\right)^{\frac{n}{2}-1}, \quad (1.204)$$

and

$$2e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^{2k+1}}{(2k+1)!} \;=\; \sum_{k=0}^{\infty}2\Pr\{N(t)\;=\;2k+1\}\;=\;1-e^{-2\lambda t}. \quad (1.205)$$

The factor 2 appearing in (1.203) and (1.205) can be interpreted as follows. The displacements generated by an even number of Poisson events are disregarded and replaced by displacements produced by an odd number of deviations. Therefore, odd-order Poisson events ignite twice the displacements considered in (1.203).

Theorem 1.4.8. *The composition with distribution*

$$\mathfrak{q}(x,y,t) \;=\; \int_0^\infty ds\, \mathfrak{r}\,(x,y,s) \left[p_{|B|}\,(s,t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}\,(s,t) \right], \tag{1.206}$$

which satisfies the time-fractional equation

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{{}^C\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) \mathfrak{q}(x,y,t) \;=\; c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \mathfrak{q}(x,y,t), \tag{1.207}$$

has the same law of the process $\boldsymbol{W}_2(t) = \boldsymbol{B}_2\left(c^2 \mathcal{L}^{\frac{1}{2}}(t) \right)$.

Proof. We begin by evaluating the Fourier-Laplace transform of (1.206).

$$\begin{aligned}
&\widehat{\widetilde{\mathfrak{q}}}(\xi,\alpha,\mu) \\
&= \int_0^\infty ds \int_0^\infty dt\, e^{-\mu t} \int_{C_{ct}} dx\, dy\, e^{i\xi x + i\alpha y} \mathfrak{r}(x,y,s) \left[p_{|B|}(s,t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s,t) \right] \\
&= \frac{2\lambda + \sqrt{\mu}}{2\lambda\sqrt{\mu}} \int_0^\infty ds \int_{C_{ct}} dx\, dy\, e^{i\xi x + i\alpha y}\, \mathfrak{r}\,(x,y,s)\; e^{-s\sqrt{\mu}}.
\end{aligned} \tag{1.208}$$

Now we need the Fourier transform of the law $\mathfrak{r}(x,y,t)$ of the process $\mathfrak{T}(t)$, $t>0$, which reads

$$\begin{aligned}
&\widehat{\mathfrak{r}}(\xi,\alpha,t) = \\
&= \frac{\lambda\, e^{-\lambda t}}{2\pi c} \iint_{C_{ct}} e^{i\xi x + i\alpha y} \left[\frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}} + e^{-\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}} \right] dx\, dy \\
&= \frac{\lambda\, e^{-\lambda t}}{2\pi c} \int_0^{2\pi} d\theta \int_0^{ct} d\rho\, \rho e^{i\rho(\xi\cos\theta + \alpha\sin\theta)} \frac{\lambda}{c} \frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2}} + e^{-\frac{\lambda}{c}}\sqrt{c^2t^2-\rho^2}}{\sqrt{c^2t^2-\rho^2}} \\
&= \frac{2\lambda^2 e^{-\lambda t}}{c^2} \int_0^{ct} \rho \sum_{m=0}^\infty \left(\frac{\lambda}{c}\sqrt{c^2t^2-\rho^2} \right)^{2m-1} \frac{1}{(2m)!} J_0\left(\rho\sqrt{\xi^2+\alpha^2} \right) d\rho \\
&= \frac{2\lambda e^{-\lambda t}}{c} \sum_{m=0}^\infty \frac{\left(\frac{\lambda}{c}\right)^{2m}}{(2m)!} \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\sqrt{\xi^2+\alpha^2}}{2} \right)^{2k}}{(k!)^2} \cdot \int_0^{ct} \left(c^2t^2 - \rho^2 \right)^{m-\frac{1}{2}} \rho^{2k+1}\, d\rho \\
&= \frac{2\lambda e^{-\lambda t}}{c} \sum_{m=0}^\infty \frac{\left(\frac{\lambda}{c}\right)^{2m}}{(2m)!} \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{\sqrt{\xi^2+\alpha^2}}{2} \right)^{2k}}{2(k!)^2\,(ct)^{-(2m+2k+1)}} \int_0^1 y^k\,(1-y)^{m-\frac{1}{2}}\, dy \\
&= \frac{\lambda}{c} e^{-\lambda t} \sum_{m=0}^\infty \left(\frac{\lambda}{c} \right)^{2m} \frac{1}{(2m)!} \sum_{k=0}^\infty (-1)^k \left(\frac{\sqrt{\xi^2+\alpha^2}}{2} \right)^{2k} \frac{(ct)^{2m+2k+1}\Gamma\left(m+\frac{1}{2}\right)}{k!\Gamma\left(k+m+1+\frac{1}{2}\right)}.
\end{aligned} \tag{1.209}$$

Thus, from (1.208), we have that

$$\widetilde{\widehat{\mathfrak{q}}}(\xi,\alpha,\mu) = \frac{2\lambda+\sqrt{\mu}}{2\lambda\sqrt{\mu}}\int_0^\infty ds\,\widehat{\mathfrak{r}}(\xi,\alpha,t)\,e^{-s\sqrt{\mu}} = \frac{1+2\lambda\mu^{-\frac{1}{2}}}{\mu+2\lambda\sqrt{\mu}+c^2(\xi^2+\alpha^2)} \tag{1.210}$$

in force of the calculation

$$\begin{aligned}
&\int_0^\infty ds\,\widehat{\mathfrak{r}}(\xi,\alpha,s)\,e^{-s\sqrt{\mu}} = \\
&= \frac{\lambda}{c}\int_0^\infty ds\,e^{-\lambda s}\sum_{m=0}^\infty \frac{\lambda^{2m}}{c^{2m}(2m)!}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{\sqrt{\xi^2+\alpha^2}}{2}\right)^{2k}\Gamma\left(m+\frac{1}{2}\right)}{k!\,(cs)^{-(2m+2k+1)}\,\Gamma\left(k+m+1+\frac{1}{2}\right)}e^{-s\sqrt{\mu}} \\
&= \lambda\sum_{m=0}^\infty \frac{\sqrt{\pi}2^{1-2m}\Gamma(2m)}{\lambda^{-2m}(2m)!\Gamma(m)}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{\sqrt{\xi^2+\alpha^2}}{2}\right)^{2k}c^{2k}}{k!\Gamma\left(k+m+1+\frac{1}{2}\right)}\int_0^\infty e^{-s\left(\lambda+\sqrt{\mu}\right)}s^{2m+2k+1}ds \\
&= \frac{\lambda}{2\left(\lambda+\sqrt{\mu}\right)^2}\sum_{m=0}^\infty \frac{\lambda^{2m}\sqrt{\pi}2^{1-2m}}{m!\left(\lambda+\sqrt{\mu}\right)^{2m}}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{\sqrt{\xi^2+\alpha^2}}{2}\right)^{2k}}{k!\left(\lambda+\sqrt{\mu}\right)^{2k}c^{-2k}}\frac{\Gamma(2k+2m+2)}{\Gamma\left(k+m+1+\frac{1}{2}\right)} \\
&= \frac{\sqrt{\pi}\lambda}{2\left(\lambda+\sqrt{\mu}\right)^2}\sum_{m=0}^\infty \frac{\lambda^{2m}2^{1-2m}}{m!\left(\lambda+\sqrt{\mu}\right)^{2m}}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{\sqrt{\xi^2+\alpha^2}}{2}\right)^{2k}}{k!\left(\lambda+\sqrt{\mu}\right)^{2k}c^{-2k}}\frac{\Gamma(k+m+1)}{2^{1-2(k+m+1)}\sqrt{\pi}} \\
&= \frac{2\lambda}{\left(\lambda+\sqrt{\mu}\right)^2}\sum_{m=0}^\infty \frac{\lambda^{2m}}{m!\left(\lambda+\sqrt{\mu}\right)^{2m}}\sum_{k=0}^\infty \frac{(-1)^k\left(\sqrt{\xi^2+\alpha^2}\right)^{2k}}{k!\left(\lambda+\sqrt{\mu}\right)^{2k}c^{-2k}}\int_0^\infty e^{-u}u^{k+m}\,du \\
&= \frac{2\lambda}{\left(\lambda+\sqrt{\mu}\right)^2}\int_0^\infty du\,e^{u\frac{\lambda^2}{\left(\lambda+\sqrt{\mu}\right)^2}-u\frac{c^2\left(\xi^2+\alpha^2\right)}{\left(\lambda+\sqrt{\mu}\right)^2}-u} = \frac{\frac{2\lambda}{\left(\lambda+\sqrt{\mu}\right)^2}}{1-\frac{\lambda^2}{\left(\lambda+\sqrt{\mu}\right)^2}+\frac{c^2(\xi^2+\alpha^2)}{\left(\lambda+\sqrt{\mu}\right)^2}} \\
&= \frac{2\lambda}{\left(\lambda+\sqrt{\mu}\right)^2-\lambda^2+c^2\left(\xi^2+\alpha^2\right)} = \frac{2\lambda}{\mu+2\lambda\sqrt{\mu}+c^2\left(\xi^2+\alpha^2\right)}.
\end{aligned} \tag{1.211}$$

The Fourier-Laplace transform of the law of the process $\boldsymbol{B}_2\left(c^2\mathcal{L}^{\frac{1}{2}}(t)\right)$ is written as in (1.136) for $n=2$, $\beta=1$ and $\nu=\frac{1}{2}$ as the following calculation shows

$$\begin{aligned}
\widetilde{\widehat{w_{\frac{1}{2}}^1}}(\xi,\alpha,t) &= \int_0^\infty \widehat{p_{\boldsymbol{B}}}\left(\xi,\alpha,c^2s\right)\widetilde{\ell_{\frac{1}{2}}}(s,\mu)\,ds \\
&= \left(1+2\lambda\mu^{-\frac{1}{2}}\right)\int_0^\infty e^{-\mu s-\left(\xi^2+\alpha^2\right)c^2s}\left[e^{-2\lambda s\sqrt{\mu}}+2\lambda\frac{e^{-2\lambda s\sqrt{\mu}}}{\sqrt{\mu}}\right]ds \\
&= \frac{1+2\lambda\mu^{-\frac{1}{2}}}{2\lambda\sqrt{\mu}+\mu+c^2\left(\xi^2+\alpha^2\right)}.
\end{aligned} \tag{1.212}$$

In the previous calculation we use the Laplace transform of $\ell_{\frac{1}{2}}(x,t)$ obtained in (1.149). The proof is complete since (1.212), coincides with (1.210) and with the Fourier-Laplace transform of (1.207). □

Remark 1.4.9. Since for the first passage time $\tau_{\frac{s}{\sqrt{2}}} = \inf\left\{z : B(z) = \frac{s}{\sqrt{2}}\right\}$ of a Brownian motion through level $\frac{s}{\sqrt{2}}$ we have that

$$\int_0^\infty e^{-\mu t} \Pr\left\{\tau_{\frac{s}{\sqrt{2}}} \in dt\right\} = e^{-s\sqrt{\mu}}, \tag{1.213}$$

and

$$\int_0^\infty e^{-\mu t} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s,t)\, dt = e^{-s\sqrt{\mu}} \tag{1.214}$$

we can write

$$\begin{aligned}
&\int_0^\infty \mathfrak{r}(x,y,s) \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s,t)\, ds = \int_0^\infty \mathfrak{r}(x,y,s) \frac{s}{\sqrt{2}} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{2\pi t^3}}\, ds \\
&= \int_0^\infty \frac{\partial}{\partial s} \mathfrak{r}(x,y,s) \frac{e^{-\frac{s^2}{4t}}}{\sqrt{\pi t}} ds = \int_0^\infty \frac{\partial}{\partial s} \mathfrak{r}(x,y,s)\, p_{|B|}(s,t)\, ds.
\end{aligned} \tag{1.215}$$

This representation of the second term of (1.206) is extremely interesting because by integrating (1.215) in C_{ct} we get

$$\int_0^\infty \frac{\partial}{\partial s}(1 - e^{-2\lambda s}) p_{|B|}(s,t)\, ds = 2\lambda \int_0^\infty e^{-2\lambda s} p_{|B|}(s,t)\, ds \tag{1.216}$$

and yields the missing probability of the first term of (1.206).

Remark 1.4.10. We check that the law

$$\mathfrak{q}(x,y,t) = \int_0^\infty \mathfrak{r}(x,y,s) \left[p_{|B|}(s,t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s,t) \right] ds \tag{1.217}$$

integrates to unity. By taking the t-Laplace transform, the integral with respect to (x,y) becomes

$$\begin{aligned}
&\iint_{C_{ct}} dx\, dy \int_0^\infty dt\, e^{-\mu t} \mathfrak{q}(x,y,t) \\
&= \int_0^\infty \left(1 - e^{-2\lambda s}\right) \left[\int_0^\infty e^{-\mu t} \left(p_{|B|}(s,t) + \frac{1}{2\lambda} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p_{|B|}(s,t) \right) dt \right] ds \\
&= \int_0^\infty \left(1 - e^{-2\lambda s}\right) \left[\frac{e^{-s\sqrt{\mu}}}{\sqrt{\mu}} + \frac{e^{-s\sqrt{\mu}}}{2\lambda} \right] ds \\
&= \left(\frac{1}{\sqrt{\mu}} + \frac{1}{2\lambda} \right) \left[\int_0^\infty e^{-s\sqrt{\mu}} ds - \int_0^\infty e^{-s\left(2\lambda + \sqrt{\mu}\right)} ds \right] \\
&= \frac{2\lambda + \sqrt{\mu}}{2\lambda\sqrt{\mu}} \left(\frac{1}{\sqrt{\mu}} - \frac{1}{2\lambda + \sqrt{\mu}} \right) = \frac{1}{\mu} = \int_0^\infty e^{-\mu t} dt.
\end{aligned} \tag{1.218}$$

The same check can be done directly by taking into account formulas (1.215) and (1.216).

Relationships similar to $B\left(c^2\mathcal{L}^{\frac{1}{2}}(t)\right) \stackrel{\text{law}}{=} T(|B(t)|)$, $t>0$, and the analogous one in the plane, cannot be established in spaces of dimension $n \geq 3$, because random motions governed by telegraph equations in such spaces have not been constructed. Random flights in $\mathbb{R}^n$ have been studied (Orsingher and De Gregorio (2007)) but their distributions are not related to higher-dimensional telegraph equations.

1.5 Hyperbolic fractional telegraph equations

The Hyperbolic Brownian motion is a diffusion on the Poincaré half-space

$$\mathbb{H}^n = \left\{(\boldsymbol{x}, y) : \boldsymbol{x} \in \mathbb{R}^{n-1}, y > 0\right\}, \tag{1.219}$$

with generator, written in cartesian coordinates,

$$\mathfrak{H}_n = \frac{1}{2}\left[y^2 \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} + (2-n)y\frac{\partial}{\partial y}\right]. \tag{1.220}$$

In the half-plane $\mathbb{H}^2$ the hyperbolic Brownian motion was introduced by Gertsenshtein and Vasiliev (1959) while in $\mathbb{H}^3$ it was introduced by Karpelevich, Tutubalin and Shur (1959). The reader can also consult, for more details, Getoor (1961), Gruet (1996), Lao and Orsingher (2007), Matsumoto and Yor (2005). The hyperbolic Poincaré half-space is equipped with the metric

$$ds^2 = \frac{\sum_{j=1}^{n-1} dx_j^2 + dy^2}{y^2}, \tag{1.221}$$

and thus the hyperbolic distance in $\mathbb{H}^n$ is given by the formula

$$\cosh \eta(z', z) = 1 + \frac{\|z'-z\|^2}{2yy'}, \qquad z, z' \in \mathbb{H}^n, \tag{1.222}$$

where $\|\cdot\|$ is the usual euclidean norm. We define the operator $\mathfrak{H}_2$ as the governing operator of the planar hyperbolic Brownian motion $B_2^{hp}(t)$, $t>0$, which is written as

$$\mathfrak{H}_2 = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.223}$$

in Cartesian coordinates and takes the form

$$\mathfrak{H}_2^{hp} = \mathcal{G}_2 + \frac{1}{\sinh^2\eta}\frac{\partial^2}{\partial\alpha^2} \tag{1.224}$$

in hyperbolic coordinates, where

$$\mathcal{G}_2 = \frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\left(\sinh\eta\frac{\partial}{\partial\eta}\right). \tag{1.225}$$

Note that we disregard the factor $\frac{1}{2}$ in $\mathfrak{H}_2^{hp}$ in the forthcoming calculation as in the pioneering work by Gertsenshtein and Vasiliev (1959). The problem involving the radial part of (1.224) which is written as

$$\begin{cases} \frac{\partial}{\partial t} k_2(\eta, t) = \mathcal{G}_2 k_2(\eta, t), & \eta > 0, t > 0, \\ k_2(\eta, 0) = \delta(\eta), \end{cases} \tag{1.226}$$

has the following solution

$$k_2(\eta, t) = \frac{e^{-\frac{t}{4}}}{2^{\frac{3}{2}} \sqrt{\pi t^3}} \int_\eta^\infty \frac{\varphi e^{-\frac{\varphi^2}{4t}}}{\sqrt{\cosh \varphi - \cosh \eta}} d\varphi \tag{1.227}$$

to which we refer as the kernel of the law of $B_2^{hp}(t)$, $t > 0$. The law of $B_2^{hp}(t)$, $t > 0$ is therefore written as

$$p_2^{hp}(\eta, t) = \sinh \eta \, k_2(\eta, t), \qquad \eta > 0, t > 0. \tag{1.228}$$

The three-dimensional hyperbolic Brownian motion $B_3^{hp}(t)$, $t > 0$ is driven by the operator

$$\mathfrak{H}_3 = z^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - z \frac{\partial}{\partial z} \tag{1.229}$$

written in Cartesian coordinates. We are interested in the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} k_3(\eta, t) = \mathcal{G}_3 k_3(\eta, t), & \eta > 0, t > 0, \\ k_3(\eta, 0) = \delta(\eta). \end{cases} \tag{1.230}$$

where

$$\mathcal{G}_3 = \frac{1}{\sinh^2 \eta} \frac{\partial}{\partial \eta} \left(\sinh^2 \eta \frac{\partial}{\partial \eta} \right) \tag{1.231}$$

represents the radial part of $\mathfrak{H}_3^{hp}$ which coincides with $\mathfrak{H}_3$ in hyperbolic coordinates. The solution to (1.230) is given by

$$k_3(\eta, t) = \frac{e^{-t}}{2\sqrt{\pi t^3}} \frac{\eta e^{-\frac{\eta^2}{4t}}}{\sinh \eta}, \tag{1.232}$$

and thus the probability law of $B_3^{hp}(t)$, $t > 0$, reads

$$p_3^{hp}(\eta, t) = \sinh^2 \eta \, k_3(\eta, t). \tag{1.233}$$

In general, the law of a n-dimensional hyperbolic Brownian motion is written as

$$p_n^{hp}(\eta, t) = \sinh^{n-1} \eta \, k_n(\eta, t), \tag{1.234}$$

and solves the heat equation

$$\frac{\partial}{\partial t} p_n^{hp}(\eta, t) = \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\left(\frac{1}{\sinh^{n-1}\eta} p_n^{hp}(\eta, t)\right)\right) \tag{1.235}$$

where

$$\mathcal{G}_n^\star = \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\left(\frac{1}{\sinh^{n-1}\eta}\right)\right), \qquad n \in \mathbb{N}, \tag{1.236}$$

is the adjoint of

$$\mathcal{G}_n = \frac{1}{\sinh^{n-1}\eta}\frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\right), \tag{1.237}$$

in the sense that

$$\langle \mathcal{G}_n k_n,\, p_n\rangle = \langle k_n,\, \mathcal{G}_n^\star p_n\rangle, \qquad n \in \mathbb{N}. \tag{1.238}$$

Thus the n-dimensional kernel satisfies

$$\frac{\partial}{\partial t} k_n(\eta, t) = \mathcal{G}_n\, k_n(\eta, t). \tag{1.239}$$

The kernels for $n > 3$ can be obtained from k_2 and k_3 by means of Millson recursive formula (see Debiard, Gaveau and Mazet (1976))

$$k_{n+2}(\eta, t) = -\frac{e^{-nt}}{2\pi \sinh \eta}\frac{\partial}{\partial \eta} k_n(\eta, t). \tag{1.240}$$

By working out the derivatives we obtain a more explicit version of Millson formula

$$\begin{cases} k_{2j+1}(\eta, t) = \frac{e^{-\left(j^2-1\right)t}}{(2\pi)^{j-1}}\left(-\frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\right)^{j-1} k_3(\eta, t), & j \geq 1,\ n = 2j+1, \\ k_{2j+2}(\eta, t) = \frac{e^{-\left(j^2+j\right)t}}{(2\pi)^{j}}\left(-\frac{1}{\sinh\eta\frac{\partial}{\partial\eta}}\right)^{j} k_2(\eta, t), & j \geq 0,\ n = 2j+2. \end{cases}$$

Theorem 1.5.1. *The distribution of the composition*

$$\mathcal{T}_n^\nu(t) = B_n^{hp}\left(\mathcal{L}^\nu(t)\right), \qquad \nu \in \left(0, \frac{1}{2}\right], t > 0, \tag{1.241}$$

where B_n^{hp} is the n-dimensional hyperbolic Brownian motion in the Poincaré hyperbolic half-space $\mathbb{H}^n$, satisfies the fractional hyperbolic telegraph equation for $\nu \in \left(0, \frac{1}{2}\right]$,

$$\begin{cases} \left(\frac{{}^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda\frac{{}^C\partial^{\nu}}{\partial t^{\nu}}\right) p_n^\nu(\eta, t) = \frac{\partial}{\partial\eta}\left(\sinh^{n-1}\frac{\partial}{\partial\eta}\left(\frac{1}{\sinh^{n-1}} p_n^\nu(\eta, t)\right)\right), & \eta > 0, \\ p_n^\nu(\eta, 0) = \delta(\eta), \end{cases}$$

and thus the kernel

$$\kappa_n^\nu(\eta, t) = \frac{1}{\sinh^{n-1}\eta} p_n^\nu(\eta, t) \tag{1.242}$$

satisfies, for $\nu \in \left(0, \frac{1}{2}\right]$,

$$\begin{cases} \left(\frac{^C\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{^C\partial^{\nu}}{\partial t^{\nu}}\right) \kappa_n^{\nu}(\eta, t) = \frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta} \kappa_n^{\nu}(\eta, t)\right), & \eta > 0, \\ \kappa_n^{\nu}(\eta, 0) = \delta(\eta), \end{cases} \tag{1.243}$$

Proof. It is convenient to consider the Laplace transform, $\widetilde{\kappa_n^{\nu}}$ of the kernel κ_n^{ν}. We have that

$$\widetilde{\kappa_n^{\nu}}(\eta, \mu) = \int_0^{\infty} dt\, e^{-\mu t} \kappa_n^{\nu}(\eta, t) = \int_0^{\infty} dt\, e^{-\mu t} \int_0^{\infty} ds\, k_n(\eta, s)\, \mathcal{l}_{\nu}(s, t) \tag{1.244}$$

$$= \int_0^{\infty} k_n(\eta, s) \left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right) e^{-s\left(\mu^{2\nu}+2\lambda\mu^{\nu}\right)} ds. \tag{1.245}$$

Now we show that (1.245) satisfies the Laplace transform of (1.243) written as

$$\left(\mu^{2\nu} + 2\lambda\mu^{\nu}\right) \widetilde{\kappa_n^{\nu}}(\eta, \mu) = \frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta} \widetilde{\kappa_n^{\nu}}(\eta, \mu)\right). \tag{1.246}$$

By considering (1.239) and that $\kappa_n^{\nu}(\eta, 0) = \delta(\eta)$ we have, for $\eta > 0$

$$\begin{aligned}
&\frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta} \widetilde{\kappa_n^{\nu}}(\eta, \mu)\right) \\
&= \int_0^{\infty} \frac{ds}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta} k_n(\eta, s)\right) \left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right) e^{-s\left(\mu^{2\nu}+2\lambda\mu^{\nu}\right)} \\
&= \int_0^{\infty} ds \frac{\partial}{\partial s} k_n(\eta, s) \left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right) e^{-s\left(\mu^{2\nu}+2\lambda\mu^{\nu}\right)} \\
&= \left[k_n(\eta, s) \left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right) e^{-s\left(\mu^{2\nu}+2\lambda\mu^{\nu}\right)}\right]_{s=0}^{s=\infty} \\
&\quad + \left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right)\left(\mu^{2\nu} + 2\lambda\mu^{\nu}\right) \int_0^{\infty} k_n(\eta, s)\, e^{-s\left(\mu^{2\nu}+2\lambda\mu^{\nu}\right)} ds \\
&= \left(\mu^{2\nu} + 2\lambda\mu^{\nu}\right) \widetilde{\kappa_n^{\nu}}(\eta, \mu).
\end{aligned} \tag{1.247}$$

□

Remark 1.5.2. By taking profit of the simple structure of $p_3^{hp}(\eta, t)$ we can give, for $n = 3$, an alternative direct proof of the result of theorem 1.5.1. We first evaluate the Laplace transform $\widetilde{\kappa_3^{\nu}}(\eta, \mu)$, as follows

$$\begin{aligned}
\widetilde{\kappa_3^{\nu}}(\eta, \mu) &= \int_0^{\infty} k_3(\eta, s) \int_0^{\infty} e^{-\mu t} \mathcal{l}_{\nu}(s, t)\, dt\, ds = \int_0^{\infty} k_3(\eta, s)\, \widetilde{\mathcal{l}_{\nu}}(s, \mu)\, ds \\
&= \frac{\eta\left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right)}{2\sqrt{\pi}\sinh\eta} \int_0^{\infty} e^{-s\left(1+\mu^{2\nu}+2\lambda\mu^{\nu}\right)} \frac{e^{-\frac{\eta^2}{4s}}}{\sqrt{s^3}} ds \\
&= \frac{\left(\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}\right)}{\sinh\eta} e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^{\nu}}}.
\end{aligned} \tag{1.248}$$

Now we show that (1.248) solves the Laplace transform of (1.243) for $n = 3$. We have that

$$
\begin{aligned}
&\frac{1}{\sinh^2\eta}\frac{\partial}{\partial\eta}\left(\sinh^2\eta\frac{\partial}{\partial\eta}\widetilde{\kappa_3^\nu}(\eta,\mu)\right)\\
&=\frac{1}{\sinh^2\eta}\frac{\partial}{\partial\eta}\left(\sinh^2\eta\frac{\partial}{\partial\eta}\frac{(\mu^{2\nu-1}+2\lambda\mu^{\nu-1})}{\sinh\eta}e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}}\right)\\
&=-\frac{(\mu^{2\nu-1}+2\lambda\mu^{\nu-1})}{\sinh^2\eta}\frac{\partial}{\partial\eta}\left[e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}}\left(\sinh\eta\,\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}+\cosh\eta\right)\right]\\
&=\frac{(\mu^{2\nu-1}+2\lambda\mu^{\nu-1})\,e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}}}{\sinh^2\eta}\left[\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}\cdot\right.\\
&\left.\cdot\left(\sinh\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}+\cosh\eta\right)-\left(\cosh\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}+\sinh\eta\right)\right]\\
&=\frac{(\mu^{2\nu-1}+2\lambda\mu^{\nu-1})\,e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}}}{\sinh^2\eta}\left[\sinh\eta\,\left(1+\mu^{2\nu}+2\lambda\mu^\nu\right)-\sinh\eta\right]\\
&=\frac{(\mu^{2\nu-1}+2\lambda\mu^{\nu-1})}{\sinh\eta}e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}}\left(\left(1+\mu^{2\nu}+2\lambda\mu^\nu\right)-1\right)\\
&=\left(\mu^{2\nu}+2\lambda\mu^\nu\right)\frac{(\mu^{2\nu-1}+2\lambda\mu^{\nu-1})}{\sinh\eta}e^{-\eta\sqrt{1+\mu^{2\nu}+2\lambda\mu^\nu}}=\left(\mu^{2\nu}+2\lambda\mu^\nu\right)\widetilde{\kappa_3^\nu}(\eta,\mu).
\end{aligned}
\tag{1.249}
$$

Remark 1.5.3. For $\nu = \frac{1}{2}$ we know the explicit law of the process $\mathcal{L}^\nu(t)$, $t > 0$, which is written as in (1.146). Thus we have an explicit representation for the law of the process

$$
\mathcal{T}_3^{\frac{1}{2}}(t) = B_3^{hp}\left(\mathcal{L}^{\frac{1}{2}}(t)\right), \qquad t > 0 \tag{1.250}
$$

which reads

$$
\begin{aligned}
p_3^{\frac{1}{2}}(\eta,t) &= \sinh^2\eta\int_0^t\frac{e^{-s}}{2\sqrt{\pi s^3}}\frac{\eta\,e^{-\frac{\eta^2}{4s}}}{\sinh\eta}\left[\frac{\lambda s e^{-\frac{\lambda^2 s^2}{t-s}}}{\sqrt{\pi(t-s)^3}}+\frac{2\lambda\,e^{-\frac{\lambda^2 s^2}{t-s}}}{\sqrt{\pi(t-s)}}\right]ds\\
&=\frac{\lambda\eta\sinh\eta}{2\pi}\int_0^t\frac{e^{-s}}{s^{\frac{3}{2}}\sqrt{t-s}}e^{-\frac{\lambda^2 s^2}{t-s}-\frac{\eta^2}{4s}}\left(\frac{s}{t-s}+2\right)ds.
\end{aligned}
\tag{1.251}
$$

Chapter 2

Generalized space-time fractional equations

Article: Orsingher and Toaldo (2012). Space-time fractional equations and the related stable processes at random time.

Summary

In this paper we consider the general space-time fractional equation of the form $\sum_{j=1}^{m} \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} w(x_1, \cdots, x_n; t) = -c^2 \left(-\Delta\right)^{\beta} w(x_1, \cdots, x_n; t)$, for $\nu_j \in (0,1]$, $\beta \in (0,1]$ with initial condition $w(x_1, \cdots, x_n; 0) = \prod_{j=1}^{n} \delta(x_j)$. We show that the solution of the Cauchy problem above coincides with the distribution of the n-dimensional vector process $\boldsymbol{S}_n^{2\beta}\left(c^2 \mathcal{L}^{\nu_1, \cdots, \nu_m}(t)\right)$, $t > 0$, where $\boldsymbol{S}_n^{2\beta}$ is an isotropic stable process independent from $\mathcal{L}^{\nu_1, \cdots, \nu_m}(t)$ which is the inverse of $\mathcal{H}^{\nu_1, \cdots, \nu_m}(t) = \sum_{j=1}^{m} \lambda_j^{1/\nu_j} H^{\nu_j}(t)$, $t > 0$, with $H^{\nu_j}(t)$ independent, positively-skewed stable r.v.'s of order ν_j. The problem considered includes the fractional telegraph equation as a special case as well as the governing equation of stable processes. The composition $\boldsymbol{S}_n^{2\beta}\left(c^2 \mathcal{L}^{\nu_1, \cdots, \nu_m}(t)\right)$, $t > 0$, supplies a probabilistic representation for the solutions of the fractional equations above and coincides for $\beta = 1$ with the n-dimensional Brownian motion at the random time $\mathcal{L}^{\nu_1, \cdots, \nu_m}(t)$, $t > 0$. The iterated process $\mathfrak{L}_r^{\nu_1, \cdots, \nu_m}(t)$, $t > 0$, inverse to $\mathfrak{H}_r^{\nu_1, \cdots, \nu_m}(t) = \sum_{j=1}^{m} \lambda_j^{1/\nu_j} \, {}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots {}_rH^{\nu_j}(t) \cdots\right)\right)\right)$, $t > 0$, permits us to construct the process $\boldsymbol{S}_n^{2\beta}\left(c^2 \mathfrak{L}_r^{\nu_1, \cdots, \nu_m}(t)\right)$, $t > 0$, the distribution of which solves a space-fractional equation of the form of the generalized fractional telegraph equation. For $r \to \infty$ and $\beta = 1$ we obtain a distribution, independent from t, which represents the multidimensional generalisation of the Gauss-Laplace law and solves the equation $\sum_{j=1}^{m} \lambda_j w(x_1, \cdots, x_n) = c^2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} w(x_1, \cdots, x_n)$. Our analysis represents a general framework of the interplay between fractional differential equations

and composition of processes of which the iterated Brownian motion is a very particular case.

2.1 Introduction and preliminaries

2.1.1 Introduction

The study of the relationships between fractional differential equations and stochastic processes has gained considerable popularity during the past three decades. In pioneering works simple time-fractional diffusion equations have been considered (see for example Fujita (1990)) and its connection with stable processes has been established (see Orsingher and Beghin (2009); the reader can also consult Zolotarev (1986) for details on stable laws). In such papers the authors have shown that the compositions of processes have distributions satisfying fractional equations of different form. The iterated Brownian motion $B_1(|B_2(t)|)$, $t > 0$, (with B_1 and B_2 independent Brownian motions) has distribution solving the fractional equation (see Allouba and Zheng (2001))

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} u(x,t) = \frac{1}{2^{\frac{3}{2}}} \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, t > 0.$$

as well as the fourth-order equation (see DeBlassie (2004))

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{2^3} \frac{\partial^4}{\partial x^4} u(x,t) + \frac{1}{2\sqrt{2\pi t}} \frac{d^2}{dx^2} \delta(x), \qquad x \in \mathbb{R}, t > 0.$$

It has been shown by different authors (see Benachour et al. (1999)) that the solution to the biquadratic heat-equation

$$\frac{\partial}{\partial t} u(x,t) = -\frac{1}{2^3} \frac{\partial^4}{\partial x^4} u(x,t), \qquad x \in \mathbb{R}, t > 0,$$

coincides with

$$u(x,t) = \mathbb{E}\left\{ \frac{1}{\sqrt{2\pi\,|B(t)|}} \cos\left(\frac{x^2}{2\,|B(t)| - \frac{\pi}{4}} \right) \right\}$$

and appears as the distribution of the composition of the Fresnel pseudoprocess with an independent Brownian motion (see Orsingher and D'Ovidio (2011)).

When the fractional telegraph equation

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^{\nu}}{\partial t^{\nu}} \right) u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \qquad x \in \mathbb{R}, t > 0, \tag{2.1}$$

for $\nu \in (0,1]$, $\lambda > 0$, $c > 0$, is considered, the solution of problem (2.1) for $\nu = \frac{1}{2}$ has been proved to coincide with the distribution of $T(|B(t)|)$, $t > 0$, where $T(t)$, $t > 0$, is a telegraph process independent from the Brownian motion $B(t)$, $t > 0$ (see Orsingher and Beghin (2004)). From the analytical point of view, equations similar to (2.1) have been studied in the form

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) + a\frac{\partial^\beta}{\partial t^\beta}u(x,t) \; = \; c^2\frac{\partial^\gamma}{\partial x^\gamma}u(x,t) + \xi^2 u(x,t) + \varphi(x,t), \quad x \in \mathbb{R}, t > 0,$$

for $\alpha \in [0,1]$, $\beta \in [0,1]$, $\gamma > 0$, by Saxena et al. (2006). These authors have provided the Fourier transform of solutions of fractional equations of the form

$$a_1\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}u(x,t) + \cdots + a_{n+1}\frac{\partial^{\alpha_{n+1}}}{\partial t^{\alpha_{n+1}}}u(x,t) \; = \; c^2\frac{\partial^\beta}{\partial x^\beta}u(x,t) + \xi^2 u(x,t) + \varphi(x,t)$$

for $\alpha_1, \cdots, \alpha_{n+1} \in (0,1)$ and $\beta > 0$, (see Saxena et al. (2007)) in terms of generalized Mittag-Leffler functions (but no probabilistc interpretation has been given to these solutions). Telegraph equations emerge in electrodynamics, in the study of damped vibrations, in the analysis of the telegraph process. Its multidimensional version appears in studying vibrations of membranes and other structures subject to friction. Equations with many fractional derivatives emerge in the study of anomalous diffusions as pointed out by Saxena et al. (2006, 2007).

The symmetric stable laws have distribution satisfying the space-fractional equation

$$\frac{\partial}{\partial t}u(x,t) \; = \; \frac{\partial^\nu}{\partial |x|^\nu}u(x,t), \qquad x \in \mathbb{R}, t > 0,$$

where $\frac{\partial^\nu}{\partial |x|^\nu}$ is the Riesz fractional derivative. For asymmetric stable laws the connection with fractional equations has been established by Feller (1952). The connection between fractional telegraph equations and stable laws has been established in a recent paper by D'Ovidio et al. (2012), in which the authors considered the multidimensional space-fractional extension of (2.1)

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda\frac{\partial^\nu}{\partial t^\nu}\right)u(\boldsymbol{x},t) \; = \; -c^2\left(-\Delta\right)^\beta u(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0, \tag{2.2}$$

for $\nu \in \left(0,\frac{1}{2}\right]$, $\beta \in (0,1]$. The solution to (2.2) subject to the initial condition $u(\boldsymbol{x},0) = \delta(\boldsymbol{x})$ is given by the law of the composition of the form $\boldsymbol{S}_n^{2\beta}\left(c^2\mathcal{L}^\nu(t)\right)$, $t > 0$, where $\boldsymbol{S}_n^{2\beta}(t)$, $t > 0$, is a n-dimensional isotropic stable vector process and

$$\mathcal{L}^\nu(t) = \inf\left\{s : H_1^{2\nu}(s) + (2\lambda)^{\frac{1}{\nu}}H_2^\nu(s) \geq t\right\}$$

where $H_1^{2\nu}(t)$ and $H_2^\nu(t)$, $t > 0$, are independent positively-skewed stable processes, with $\nu \in \left(0,\frac{1}{2}\right]$. For $\beta = 1$ the composition above takes the form of a Brownian motion at the delayed time $\mathcal{L}^\nu(t)$, $t > 0$. For $\nu = \frac{1}{2}$ and $n = 1$ this establishes the fine distributional relationship

$$T\left(|B(t)|\right) \stackrel{\text{law}}{=} B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right), \qquad t > 0,$$

see D'Ovidio et al. (2012).

In the present paper we consider the further generalization of the space-time fractional equation with an arbirtrary number of time-fractional derivatives

$$\begin{cases} \sum_{j=1}^m \lambda_j \frac{{}^C\partial^{\nu_j}}{\partial t^{\nu_j}} w^\beta_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t) \; = \; -c^2 \left(-\Delta\right)^\beta w^\beta_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ w^\beta_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},0) \; = \; \delta(\boldsymbol{x}), \end{cases} \tag{2.3}$$

for $\nu_j \in (0,1]$, $\beta \in (0,1]$, $\lambda_j > 0$, $j = 1,\cdots,m$. The symbol $\frac{{}^C\partial^\nu}{\partial t^\nu}$ stands for the Dzerbayshan-Caputo fractional derivative which is defined as

$$\frac{{}^C\partial^\nu}{\partial t^\nu} f(t) \; = \; \frac{1}{\Gamma\left(m-\nu\right)} \int_0^t \frac{\frac{d^m}{ds^m} f(s)}{(t-s)^{\nu+1-m}} ds, \qquad m-1 < \nu < m, m \in \mathbb{N},$$

for an absolutely continuous function f (for fractional calculus the reader can consult Kilbas et al. (2006)). The fractional Laplacian $(-\Delta)^\beta$, $\beta \in (0,1)$ is defined and explored in Section 2.1.2 below. We show that the solution to (2.3) is given by the law of the process $\boldsymbol{S}^{2\beta}_n\left(c^2 \mathcal{L}^{\nu_1,\cdots,\nu_m}(t)\right)$, $t > 0$, where

$$\mathcal{L}^{\nu_1,\cdots,\nu_m}(t) \; = \; \inf\left\{s > 0 : \mathcal{H}^{\nu_1,\cdots,\nu_m}(s) > t\right\} \tag{2.4}$$

and

$$\mathcal{H}^{\nu_1,\cdots,\nu_m}(t) \; = \; \sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(t), \qquad t > 0, \tag{2.5}$$

for $H_j^{\nu_j}$, $j = 1,\cdots,m$, totally positively-skewed stable processes (stable subordinators), of order ν_j. In other words we show that the solution of a general space-time fractional equation (which includes reaction-diffusion equations, telegraph equations, diffusion equations as very special cases) coincides with the distribution of a stable vector process taken at a random time $\mathcal{L}^\nu(t)$, $t > 0$, constructed as the inverse of the combination of independent stable subordinators. For the classical Laplacian ($\beta = 1$) we have that the solution to (2.3) is the distribution of a Brownian motion at time $\mathcal{L}^\nu(t)$, $t > 0$.

We also prove that the law of the processes (2.4) and (2.5) are solutions of fractional differential equations. In particular we show that

$$h_{\nu_1,\cdots,\nu_m}(x,t) \; = \; \frac{\Pr\left\{\mathcal{H}^{\nu_1,\cdots,\nu_m}(t) \in dx\right\}}{dx},$$

is the solution to the space-fractional problem for $\nu_j \in (0,1)$

$$\begin{cases} \frac{\partial}{\partial t} \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t) = \sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j}}{\partial x^{\nu_j}} \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t), & x>0, t>0 \\ \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,0) = \delta(x), \\ \mathpzc{h}_{\nu_1,\cdots,\nu_m}(0,t) = 0, \end{cases} \tag{2.6}$$

while the law of $\mathpzc{L}^{\nu_2,\cdots,\nu_m}(t)$ solves

$$\begin{cases} \sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} \mathpzc{l}_{\nu_1,\cdots,\nu_m}(x,t) = -\frac{\partial}{\partial x} \mathpzc{l}_{\nu_1,\cdots,\nu_m}(x,t), & x>0, t>0, \\ \mathpzc{l}_{\nu_1,\cdots,\nu_m}(0,t) = \sum_{j=1}^m \lambda_j \frac{t^{\nu_j}}{\Gamma(1-\nu_j)}, \end{cases} \tag{2.7}$$

for $\nu_j \in (0,1)$. In (2.6) and (2.7) the fractional derivatives must be meant in the Riemann-Liouville sense that is, for an absolutely continuous function f,

$$\frac{\partial^\nu}{\partial x^\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \frac{d}{dx} \int_0^x \frac{f(s)}{(x-s)^{\nu+1-m}} ds, \qquad m-1<\nu<m, m \in \mathbb{N}.$$

A section is devoted to the case of the fractional equation with two time derivatives of order $\alpha \in (0,1]$ and $\nu \in (0,1]$ with $\alpha \neq \nu$,

$$\begin{cases} \left(\frac{{}^C\partial^\alpha}{\partial t^\alpha} + 2\lambda \frac{{}^C\partial^\nu}{\partial t^\nu} \right) w_{\alpha,\nu}^\beta(\boldsymbol{x},t) = -c^2 (-\Delta)^\beta w_{\alpha,\nu}^\beta(\boldsymbol{x},t), & \boldsymbol{x} \in \mathbb{R}^n, t>0, \\ w_{\alpha,\nu}^\beta(\boldsymbol{x},0) = \delta(\boldsymbol{x}), \end{cases} \tag{2.8}$$

which takes a telegraph-type structure for $\alpha = k\nu$, $k \in \mathbb{N}$, $k\nu \leq 1$. The Fourier-Laplace transform of the solution of (2.8) for $\alpha = k\nu$ reads

$$\int_0^\infty dt\, e^{-\mu t} \int_{\mathbb{R}^n} d\boldsymbol{x}\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_{k\nu,\nu}^\beta(\boldsymbol{x},t) = \frac{\mu^{k\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{k\nu} + 2\lambda\mu^\nu + c^2 \|\boldsymbol{\xi}\|^{2\beta}}, \tag{2.9}$$

where $\|\cdot\|$ is the usual euclidean norm. For $k=2$, $n=1$, $\beta=1$, we have the classical fractional telegraph equation studied in Orsingher and Beghin (2004). The Fourier transform of $w_{2\nu,\nu}(x,t)$ reads

$$\widehat{w}_{2\nu,\nu}(\xi,t) = \frac{1}{2}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}}\right) E_{\nu,1}(-\eta_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}}\right) E_{\nu,1}(-\eta_2 t^\nu) \right] \tag{2.10}$$

where η_1 and η_2 are the solutions to $\mu^{2\nu} + 2\lambda\mu^\nu + c^2\xi^2 = 0$ and

$$E_{\psi,\vartheta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\psi k + \vartheta)}, \qquad \psi, \vartheta > 0, z \in \mathbb{R},$$

is the two-parameter Mittag-Leffler function. For $\nu = 1$, (2.10) coincides with the characteristic function of the telegraph process. For $k=3$ and $\nu \leq \frac{1}{3}$ in (2.9) we obtain explicitely the Fourier transform of the solutions in terms of Mittag-Leffler functions and the Cardano roots A, B and C of the third order algebraic equations $\mu^{3\nu} + 2\lambda\mu^{\nu} + c^2 \|\boldsymbol{\xi}\|^{2\beta} = 0$. For $k > 3$ we can write

$$\widetilde{w}^{\beta}_{k\nu,\nu}(\boldsymbol{\xi},\mu) = \frac{\mu^{k\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{k\nu} + 2\lambda\mu^{\nu} + c^2 \|\boldsymbol{\xi}\|^{2\beta}} = \mu^{\nu-1}\prod_{i=1}^{k}\frac{\mu^{\nu-1}}{\mu^{\nu} - Z_i} + 2\lambda\mu^{\nu-1}\prod_{i=1}^{k}\frac{1}{\mu^{\nu} - Z_i}$$

but the explict evaluation of Z_i is, in general, impossible.

In Orsingher and Beghin (2009) n-times iterated Brownian motion

$$\mathfrak{I}_n(t) = B_1\left(|B_2\left(|B_3 \cdots \left(|B_{n+1}(t)|\right)\cdots|\right)|\right), \qquad t > 0,$$

is considered and its connection with the fractional diffusion equation

$$\frac{\partial^{\frac{1}{2^n}}}{\partial t^{\frac{1}{2^n}}}u(x,t) = 2^{\frac{1}{2^n}-2}\frac{\partial^2}{\partial x^2}u(x,t)$$

investigated. Here we consider first the n-times iterated positively-skewed stable process ${}_jH^{\nu_j}$ with weights $\lambda_j > 0$, $j = 1, \cdots, m$,

$$\mathfrak{H}^{\nu_1,\cdots,\nu_m}_r(t) = \sum_{j=1}^{m}\lambda_j^{\frac{1}{\nu_j}}\,{}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots{}_rH^{\nu_j}(t)\cdots\right)\right)\right), \qquad t > 0, \tag{2.11}$$

We construct the inverse of the process (2.11) as follows

$$\mathfrak{L}^{\nu_1,\cdots,\nu_m}_r(t) = \inf\left\{s > 0 : \mathfrak{H}^{\nu_1,\cdots,\nu_m}_r(s) \geq t\right\}, \qquad t > 0.$$

We show that the distribution of the composition

$$\boldsymbol{B}_n\left(c^2\mathfrak{L}^{\nu_1,\cdots,\nu_m}_r(t)\right), \qquad t > 0,$$

where $\boldsymbol{B}_n$ represents the n-dimensional Brownian motion, is the solution to the Cauchy problem for $\nu_j \in (0,1]$, $r \in \mathbb{N}$,

$$\begin{cases}\sum_{j=1}^{m}\lambda_j\frac{{}^C\partial^{\nu_j^r}}{\partial t^{\nu_j^r}}\mathfrak{w}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t) = c^2\Delta\mathfrak{w}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t), & \boldsymbol{x}\in\mathbb{R}^n, t > 0,\\ \mathfrak{w}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},0) = \delta(\boldsymbol{x}).\end{cases}$$

We show that for the number r of iterations tending to infinity

$$\boldsymbol{B}_n\left(c^2\mathfrak{L}^{\nu_1,\cdots,\nu_m}_r(t)\right) \underset{r\to\infty}{\overset{\text{law}}{\Longrightarrow}} X_{m,n},$$

where $X_{m,n}$ is a r.v. independent from t and possesses density equal to

$$\frac{\Pr\{X_{m,n}\in d\boldsymbol{x}\}}{d\boldsymbol{x}} = \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^{m}\lambda_j}}{c}\right)^{\frac{n+2}{2}}\|\boldsymbol{x}\|^{-\frac{n-2}{2}}K_{\frac{n-2}{2}}\left(\frac{\sqrt{\sum_{j=1}^{m}\lambda_j}}{c}\|\boldsymbol{x}\|\right), \tag{2.12}$$

where $K_\nu(x)$ is the modified Bessel function. For $n = 1$ the distribution (2.12) becomes the Gauss-Laplace law

$$\mathfrak{w}_m(x) = \frac{\sqrt{\sum_{j=1}^m \lambda_j}}{2c} e^{-\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}|x|}. \tag{2.13}$$

Result (2.13) was obtained also in Orsingher and Beghin (2009) and by a different approach for $\lambda_1 = 1$, $\lambda_j = 0$ for $j \geq 2$, $c = \frac{1}{2}$, was derived by Turban (2004) as the limit of iterated random walks.

2.1.2 Preliminaries

One dimensional stable laws

Let us consider a stable process, say $S^\nu(t)$, $t > 0$, $\nu \in (0, 2]$, $\nu \neq 1$, for which, in general,

$$\mathbb{E}e^{i\xi S^\nu(t)} = e^{-\sigma|\xi|^\nu t\left(1-i\theta \text{sign}(\xi) \tan \frac{\nu\pi}{2}\right)} \tag{2.14}$$

where $\theta \in [-1, 1]$ is the skewness parameter and $\sigma = \cos \frac{\pi\nu}{2}$. In this paper we consider positively skewed processes ($\theta = 1$) say $H^\nu(t)$, $t > 0$, whose characteristic function writes

$$\widehat{h}_\nu(\xi, t) = \mathbb{E}e^{i\xi H^\nu(t)} = e^{-t|\xi|^\nu \cos \frac{\pi\nu}{2}\left(1-i\,\text{sign}(\xi) \tan \frac{\pi\nu}{2}\right)} = e^{-t\left(|\xi|e^{-\frac{i\pi}{2}\text{sign}(\xi)}\right)^\nu} = e^{-t(-i\xi)^\nu} \tag{2.15}$$

where we used the fact that $|\xi|\, e^{-i\frac{\pi}{2}\text{sign}(\xi)} = -i\xi$. The process $H^\nu(t)$, $t > 0$, has the important property of having non-negative, stationary and independent increments, and thus it is suitable to play the role of a random time. The law $h_\nu(x, t)$, $x \geq 0$, of $H^\nu(t)$, $t > 0$, with Fourier transform $\widehat{h}_\nu(\xi, t)$ and Laplace transform

$$\widetilde{h}_\nu(\mu, t) = e^{-t\mu^\nu}, \tag{2.16}$$

solves the fractional diffusion equation, for $\nu \in (0, 1]$,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial^\nu}{\partial x^\nu}\right) h_\nu(x, t) = 0, & x > 0, t > 0, \\ h_\nu(x, 0) = \delta(x), \\ h_\nu(0, t) = 0, \end{cases}$$

where the fractional derivatives are intended in the Riemann-Liouville sense. We notice that the process given by the composition of $r \in \mathbb{N}$ independent stable subordinators of the same order $\nu \in (0, 1)$, say ${}_1H^\nu\left({}_2H^\nu\left(\cdots {}_r H^\nu(t)\cdots\right)\right)$, $t > 0$ has law which reads

$$\frac{\Pr\left\{{}_1H^\nu\left({}_2H^\nu\left(\cdots {}_r H^\nu(t)\cdots\right)\right) \in dx\right\}}{dx} =$$

$$= \int_0^\infty ds_1 \, {}_1h_\nu(x, s_1) \int_0^\infty ds_2 \, {}_2h_\nu(s_1, s_2) \int_0^\infty ds_3 \, {}_3h_\nu(s_2, s_3) \cdots \int_0^\infty ds_r \, {}_rh_\nu(s_r, t). \tag{2.17}$$

In view of (2.15) and (2.16) we can easily write the Laplace and Fourier transforms of (2.17). For example the Laplace transform reads

$$\begin{aligned} \mathbb{E}e^{-\mu \, {}_1H^\nu({}_2H^\nu(\cdots {}_rH^\nu(t)\cdots))} &= \int_0^\infty dx \, e^{-\mu x} \Pr\left\{ {}_1H^\nu\left({}_2H^\nu\left(\cdots {}_r H^\nu(t)\cdots\right)\right) \in dx \right\} \\ = \int_0^\infty ds_1 \, e^{-s_1\mu^\nu} \int_0^\infty ds_2 \, {}_2h_\nu(s_1, s_2) & \int_0^\infty ds_3 \, {}_3h_\nu(s_2, s_3) \cdots \int_0^\infty ds_r \, {}_rh_\nu(s_r, t) \\ = \int_0^\infty ds_2 e^{-s_2\mu^{\nu^2}} \int_0^\infty ds_3 \, {}_3h_\nu(s_2, s_3) \cdots & \int_0^\infty ds_r \, {}_rh_\nu(s_r, t) \; = \; e^{-t\mu^{\nu^r}} \end{aligned} \tag{2.18}$$

and therefore we have the following Fourier transform

$$\mathbb{E}e^{-i\xi \, {}_1H^\nu({}_2H^\nu(\cdots {}_rH^\nu(t)\cdots))} \; = \; e^{-t(-i\xi)^{\nu^r}} \tag{2.19}$$

Multidimensional stable laws and fractional Laplacian

Let us consider the isotropic n-dimensional process $\boldsymbol{S}_n^{2\beta}(t)$, $t > 0$, $\beta \in (0, 1]$, with density

$$v_\beta\left(\boldsymbol{x}, t\right) \; = \; \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\boldsymbol{\xi} \, e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \, e^{-t\|\boldsymbol{\xi}\|^{2\beta}}, \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0, \tag{2.20}$$

and therefore characteristic function

$$\widehat{v}_\beta\left(\boldsymbol{\xi}, t\right) \; = \; \mathbb{E}e^{i\boldsymbol{\xi}\cdot\boldsymbol{S}_n^{2\beta}(t)} \; = \; e^{-t\|\boldsymbol{\xi}\|^{2\beta}},$$

where the symbol $\|\cdot\|$ stands for the usual Euclidean norm. The law (2.20) is the solution to the fractional Cauchy problem, for $\beta \in (0, 1]$

$$\begin{cases} \left(\frac{\partial}{\partial t} + (-\Delta)^\beta\right) v_\beta\left(\boldsymbol{x}, t\right) \; = \; 0, & \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ v_\beta(\boldsymbol{x}, 0) \; = \; \delta(\boldsymbol{x}). \end{cases} \tag{2.21}$$

The fractional negative Laplacian appearing in (2.21) has been considered by many authors (see for example Balakrishnan (1960), Bochner (1949)). The Bochner representation of the fractional Laplacian reads

$$-\left(-\Delta\right)^\beta \; = \; \frac{\sin \pi\beta}{\pi} \int_0^\infty d\lambda \, \lambda^{\beta-1} \left(\lambda - \Delta\right)^{-1} \Delta.$$

Equivalently, an alternative useful definition can be given in the space of the Fourier transforms, as

$$-\left(-\Delta\right)^\beta u(\boldsymbol{x}) \; = \; \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \, \|\boldsymbol{\xi}\|^{2\beta} \, \widehat{u}\left(\boldsymbol{\xi}\right) \, d\boldsymbol{\xi}$$

where

$$\mathrm{Dom}\,(-\Delta)^{\beta} = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{u}(\boldsymbol{\xi})|^2 \left(1 + \|\boldsymbol{\xi}\|^{2\beta}\right) d\boldsymbol{\xi} < \infty \right\}.$$

In the one-dimensional case and for $0 < 2\beta < 1$ we have that (see for example D'Ovidio et al. (2012) for details on this point),

$$\left(-\frac{\partial^2}{\partial x^2}\right)^{\beta} u(x) = \frac{\partial^{2\beta}}{\partial |x|^{2\beta}} u(x),$$

where $\frac{\partial^{2\beta}}{\partial |x|^{2\beta}}$ is the Riesz operator usually defined as

$$\frac{\partial^{2\beta}}{\partial |x|^{2\beta}} u(x) = -\frac{1}{2\cos\beta\pi} \frac{1}{\Gamma(1-2\beta)} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{u(z)}{|x-z|^{2\beta}} dz$$

and for which the Fourier transform becomes

$$\mathcal{F}\left[\frac{\partial^{2\beta}}{\partial |x|^{2\beta}} u(x)\right] = -|\xi|^{2\beta}\, \widehat{u}(\xi).$$

2.2 Generalized fractional equations

2.2.1 Linear combination of stable processes

In this section we start by considering processes of the form

$$\mathcal{H}^{\nu_1,\cdots,\nu_m}(t) = \sum_{j=1}^{m} \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(t), \qquad t > 0, \nu_j \in (0,1), j = 1, \cdots, m, \tag{2.22}$$

where $H_j^{\nu_j}(t)$, $t > 0$, are independent stable subordinators of order $\nu_j \in (0,1]$ introduced in section 2.1.2. Furthermore we will deal with the inverse process of $\mathcal{H}^{\nu_1,\cdots,\nu_m}$, say $\mathcal{L}^{\nu_1,\cdots,\nu_m}(t)$, $t > 0$, which can be defined as the hitting time of $\mathcal{H}^{\nu_1,\cdots,\nu_m}$ as

$$\mathcal{L}^{\nu_1,\cdots,\nu_m}(t) = \inf\left\{ s > 0 : \mathcal{H}^{\nu_1,\cdots,\nu_m}(s) = \sum_{j=1}^{m} \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(s) \geq t \right\}, \qquad t > 0. \tag{2.23}$$

The definition (2.23) of the process $\mathcal{L}^{\nu_1,\cdots,\nu_m}$ permits us to write

$$\Pr\left\{\mathcal{L}^{\nu_1,\cdots,\nu_m}(t) < x\right\} = \Pr\left\{\mathcal{H}^{\nu_1,\cdots,\nu_m}(x) > t\right\}. \tag{2.24}$$

We present the following two results.

Theorem 2.2.1. *We have that*

i) The solution to the problem for $\nu_j \in (0,1)$, $j = 1, \cdots, m$,

$$\begin{cases} \frac{\partial}{\partial t} \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t) = -\sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j}}{\partial x^{\nu_j}} \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t), & x>0, t>0, \\ \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,0) = \delta(x), \\ \mathpzc{h}_{\nu_1,\cdots,\nu_m}(0,t) = 0. \end{cases} \tag{2.25}$$

is given by the law of the process $\mathcal{H}^{\nu_1,\cdots,\nu_m}(t)$, $t>0$, defined in (2.22).

ii) The solution to the problem for $\nu_j \in (0,1)$, $j = 1, \cdots, m$,

$$\begin{cases} \sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} \mathpzc{l}_{\nu_1,\cdots,\nu_m}(x,t) = -\frac{\partial}{\partial x} \mathpzc{l}_{\nu_1,\cdots,\nu_m}(x,t), & x>0, t>0, \\ \mathpzc{l}_{\nu_1,\cdots,\nu_m}(0,t) = \sum_{j=1}^m \lambda_j \frac{t^{-\nu_j}}{\Gamma(1-\nu_j)}, \end{cases} \tag{2.26}$$

is given by the law of the process $\mathcal{L}^{\nu_1,\cdots,\nu_m}(t)$, $t>0$, defined in (2.23).

The fractional derivatives appearing in (2.25) and (2.26) must be intended in the Riemann-Liouville sense.

Proof of i). Since for the Riemann-Liouville fractional derivative we have that

$$\mathcal{F}\left[\frac{\partial^\nu}{\partial x^\nu} u(x)\right](\xi) = (-i\xi)^\nu \, \widehat{u}(x) \tag{2.27}$$

we can write the Fourier transform of the problem (2.25) as

$$\frac{\partial}{\partial t} \widehat{\mathpzc{h}}_{\nu_1,\cdots,\nu_m}(\xi,t) = -\mathcal{F}\left[\sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j}}{\partial x^{\nu_j}} \mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t)\right](\xi) = \sum_{j=1}^m \lambda_j (-i\xi)^{\nu_j} \widehat{\mathpzc{h}}_{\nu_1,\cdots,\nu_m}(\xi,t),$$

and therefore we have that

$$\begin{cases} \frac{\partial}{\partial t} \widehat{h}_{\nu_1,\cdots,\nu_m}(\xi,t) = \sum_{j=1}^m \lambda_j (-i\xi)^{\nu_j} \widehat{\mathpzc{h}}_{\nu_1,\cdots,\nu_m}(\xi,t) \\ \widehat{h}_{\nu_1,\cdots,\nu_m}(\xi,0) = 1. \end{cases} \tag{2.28}$$

The Fourier transform of the law $\mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t)$ of the process (2.22) is written as

$$\mathbb{E} e^{i\xi \mathcal{H}^{\nu_1,\cdots,\nu_m}(t)} = \mathbb{E} e^{i\xi \sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(t)} \overset{(2.15)}{=} e^{-t\sum_{j=1}^m \lambda_j (-i\xi)^{\nu_j}}. \tag{2.29}$$

for which

$$\frac{\partial}{\partial t} \mathbb{E} e^{i\xi \mathcal{H}^{\nu_1,\cdots,\nu_m}(t)} = \sum_{j=1}^m \lambda_j (-i\xi)^{\nu_j} e^{-t\sum_{j=1}^m \lambda_j (-i\xi)^{\nu_j}}.$$

This is tantamount to saying that the Fourier transform of $\mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t)$ is the solution to the problem (2.28) and thus $\mathpzc{h}_{\nu_1,\cdots,\nu_m}(x,t)$ is the solution to (2.25). □

Proof of ii). In this proof we will make use of the Laplace transform of the Riemann-Liouville fractional derivative which, in view of (2.27), can be written as

$$\mathcal{L}\left[\frac{\partial^\nu}{\partial t^\nu}u(t)\right](\mu) = \mu^\nu \widetilde{u}(\mu).$$

Taking the Laplace transform of (2.26) with respect to t we get

$$\sum_{j=1}^m \lambda_j \mu^{\nu_j} \widetilde{\ell}_{\nu_1,\cdots,\nu_m}(x,\mu) = -\frac{\partial}{\partial x}\widetilde{\ell}_{\nu_1,\cdots,\nu_m}(x,\mu), \tag{2.30}$$

and performing the x-Laplace transform of (2.30) we arrive at

$$\sum_{j=1}^m \lambda_j \mu^{\nu_j} \widetilde{\widetilde{\ell}}_{\nu_1,\cdots,\nu_m}(\gamma,\mu) = \widetilde{\ell}_{\nu_1,\cdots,\nu_m}(0,\mu) - \gamma\widetilde{\widetilde{\ell}}_{\nu_1,\cdots,\nu_m}(\gamma,\mu). \tag{2.31}$$

The boundary condition appearing in (2.31) can be derived from (2.26) as

$$\widetilde{\ell}_{\nu_1,\cdots,\nu_m}(0,\mu) = \int_0^\infty dt\, e^{-\mu t}\sum_{j=1}^m \lambda_j \frac{t^{-\nu_j}}{\Gamma(1-\nu_j)} = \sum_{j=1}^m \lambda_j \mu^{\nu_j - 1}$$

and thus from (2.31) we have that

$$\widetilde{\widetilde{\ell}}_{\nu_1,\cdots,\nu_m}(\gamma,\mu) = \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j-1}}{\sum_{j=1}^m \lambda_j \mu^{\nu_j} + \gamma}. \tag{2.32}$$

Now we show that the Fourier-Laplace transform of the law of the process $\mathcal{L}^{\nu_1,\cdots,\nu_m}(t)$, $t > 0$, coincides with (2.32). By taking into account the property (2.24) of the law of $\mathcal{L}^{\nu_1,\cdots,\nu_m}$, we obtain

$$\begin{aligned}
\widetilde{\widetilde{\ell}}_{\nu_1,\cdots,\nu_m}(\gamma,\mu) &= \int_0^\infty dt\, e^{-\mu t}\int_0^\infty dx\, e^{-\gamma x}\ell_{\nu_1,\cdots,\nu_m}(x,t) \\
&= \int_0^\infty dt\, e^{-\mu t}\int_0^\infty dx\, e^{-\gamma x}\left[-\frac{\partial}{\partial x}\int_0^t dz\, h_{\nu_1,\cdots,\nu_m}(z,x)\right] \\
&= -\frac{1}{\mu}\int_0^\infty dx\, e^{-\gamma x}\left[\frac{\partial}{\partial x}\widetilde{h}_{\nu_1,\cdots,\nu_m}(\mu,x)\right] \\
&= -\frac{1}{\mu}\int_0^\infty dx\, e^{-\gamma x}\left[\frac{\partial}{\partial x}\mathbb{E}e^{-\mu\sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(x)}\right] \\
&\overset{(2.16)}{=} -\frac{1}{\mu}\int_0^\infty dx\, e^{-\gamma x}\left[\frac{\partial}{\partial x}e^{-x\sum_{j=1}^m \lambda_j\mu^{\nu_j}}\right] = \frac{\sum_{j=1}^m \lambda_j\mu^{\nu_j-1}}{\sum_{j=1}^m \lambda_j\mu^{\nu_j}+\gamma},
\end{aligned}$$

which coincides with (2.32). The proof of Theorem 2.2.1 is thus concluded. □

2.2.2 Generalized fractional telegraph-type equations

In this section we study equations of the form

$$\sum_{j=1}^{m} \lambda_j \frac{{}^C\partial^{\nu_j}}{\partial t^{\nu_j}} w^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t) \;=\; -c^2\left(-\Delta\right)^{\beta} w^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\mathbb{R}^n, t>0, \quad (2.33)$$

for $\nu_j \in (0,1]$, $j=1,\cdots,m$, $\beta\in(0,1]$, $c>0$, $\lambda>0$. The symbol $\frac{{}^C\partial^{\nu}}{\partial t^{\nu}}$ stands for the Dzerbayshan-Caputo fractional derivative. Equation (2.33) generalizes the telegraph equation in that an arbitrary number m of time-fractional derivatives appears and the n-dimensional fractional Laplacian governs the space fluctuations. Concerning the equation (2.33) we present the following result.

Theorem 2.2.2. *The solution to the problem for $\nu_j \in (0,1]$, $j=1,\cdots,m$, $\beta\in(0,1]$,*

$$\begin{cases} \sum_{j=1}^{m} \lambda_j \frac{{}^C\partial^{\nu_j}}{\partial t^{\nu_j}} w^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t) \;=\; -c^2\left(-\Delta\right)^{\beta} w^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t), & x\in\mathbb{R}^n, t>0, \\ w^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},0) \;=\; \delta\left(\boldsymbol{x}\right). \end{cases} \quad (2.34)$$

is given by the law of the process

$$\boldsymbol{W}_n^{\nu_1,\cdots,\nu_m}(t) \;=\; \boldsymbol{S}_n^{2\beta}\left(c^2 \mathcal{L}^{\nu_1,\cdots,\nu_m}(t)\right), \qquad t>0, \quad (2.35)$$

where $\boldsymbol{S}_n^{2\beta}$ is the isotropic vector process dealt with in section 2.1.2 and $\mathcal{L}^{\nu_1,\cdots,\nu_m}(t)$, $t>0$ is the process defined in (2.23).

Proof. Since for the Dzerbayshan-Caputo fractional derivative we have that,

$$\mathcal{L}\left[\frac{{}^C\partial^{\nu}}{\partial t^{\nu}}u(t)\right](\mu) \;=\; \mu^{\nu}\widetilde{u}(\mu) - \mu^{\nu-1}u(0), \qquad \nu\in(0,1),$$

we can write the Laplace transform of (2.34) as

$$\sum_{j=1}^{m} \lambda_j \mu^{\nu_j} \widetilde{w}^{\beta}_{\nu_1,\cdots,\nu_m}(x,\mu) - \sum_{j=1}^{m} \lambda_j \mu^{\nu_j-1}\delta(\boldsymbol{x}) \;=\; -c^2\left(-\Delta\right)^{\beta} \widetilde{w}^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},\mu).$$

The Fourier-Laplace transform of (2.34) is therefore written as

$$\widehat{\widetilde{w}}^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{\xi},\mu) \;=\; \frac{\sum_{j=1}^{m} \lambda_j \mu^{\nu_j-1}}{\sum_{j=1}^{m} \lambda_j \mu^{\nu_j} + c^2 \left\|\boldsymbol{\xi}\right\|^{2\beta}}. \quad (2.36)$$

Considering (2.24) we can derive the Fourier-Laplace transform of the process (2.35). We have that

$$\widehat{\widetilde{w}}^{\beta}_{\nu_1,\cdots,\nu_m}(\boldsymbol{\xi},\mu) \;=\; \int_{\mathbb{R}^n} d\boldsymbol{x}\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \int_0^{\infty} dt\, e^{-\mu t} \int_0^{\infty} ds\, v_{\beta}\left(\boldsymbol{x}, c^2 s\right)\, l_{\nu_1,\cdots,\nu_m}(s,t)$$

$$
\begin{aligned}
&= \int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}} \int_0^\infty dt\, e^{-\mu t} \left[-\frac{\partial}{\partial s} \int_0^t \mathscr{h}_{\nu_1,\cdots,\nu_m}(z,s)\, dz \right] \\
&= -\frac{1}{\mu} \int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}} \left(\frac{\partial}{\partial s} \widetilde{\mathscr{h}}_{\nu_1,\cdots,\nu_m}(\mu,s) \right) \\
&= -\frac{1}{\mu} \int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}} \left(\frac{\partial}{\partial s} \mathbb{E} e^{-\mu \sum_{j=1}^m \lambda_j H_j^{\nu_j}(s)} \right) \\
&\overset{(2.16)}{=} -\frac{1}{\mu} \int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}} \left(\frac{\partial}{\partial s} e^{-s\sum_{j=1}^m \lambda_j \mu^{\nu_j}} \right) \\
&= \sum_{j=1}^m \lambda_j \mu^{\nu_j-1} \int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta} - s\sum_{j=1}^m \lambda_j\mu^{\nu_j}} = \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j-1}}{\sum_{j=1}^m \lambda_j \mu^{\nu_j} + c^2 \|\boldsymbol{\xi}\|^{2\beta}} \\
&= (2.36).
\end{aligned}
$$

Since the Fourier-Laplace transform of the problem (2.34) coincides with that of the law of the process $\boldsymbol{S}_n^{2\beta}\left(c^2 \mathscr{L}^{\nu_1,\cdots,\nu_m}(t)\right)$, $t > 0$, the proof is complete. □

2.2.3 Telegraph-type equations with two time-fractional derivatives

When in the equation (2.33) only two time derivatives appear we can rewrite the problem, for $\alpha, \nu \in (0, 1]$ as

$$
\begin{cases}
\left(\frac{^C\partial^\alpha}{\partial t^\alpha} + 2\lambda \frac{^C\partial^\nu}{\partial t^\nu} \right) w_{\alpha,\nu}^\beta(\boldsymbol{x},t) = -c^2 (-\Delta)^\beta w_{\alpha,\nu}^\beta(\boldsymbol{x},t), & \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\
w_{\alpha,\nu}^\beta(\boldsymbol{x},0) = \delta(\boldsymbol{x}).
\end{cases}
\tag{2.37}
$$

For $\alpha = 2\nu$, $\nu \in \left(0, \frac{1}{2}\right]$ the reader can recongnize in (2.37) the standard form of the classical fractional telegraph equation, investigated from a probabilistic point of view in Orsingher and Beghin (2004) (for $n = 1$ and $\beta = 1$) and in D'Ovidio et al. (2012) (for $n \in \mathbb{N}$ and $\beta \in (0, 1)$). In view of Theorem 2.2.2 is it not difficult to prove the following result.

Corollary 2.2.3. *The solution of the fractional Cauchy problem (2.37) is given by the law of the process*

$$
\boldsymbol{W}_n^{\alpha,\nu}(t) = \boldsymbol{S}_n^{2\beta}\left(c^2 \mathscr{L}^{\alpha,\nu}(t)\right), \qquad t > 0. \tag{2.38}
$$

where

$$
\mathscr{L}^{\alpha,\nu}(t) = \inf\left\{ s > 0 : \mathscr{H}^{\alpha,\nu}(s) = H_1^\alpha + (2\lambda)^{\frac{1}{\nu}} H_2^\nu(s) > t \right\},
$$

for H_1^α and H_2^α independent stable subordinators.

Proof. The proof of this result can be carried out by repeating the arguments of Theorem 2.2.2 and will not be reported here. It is sufficient to assume that $\lambda_1 = 1$, $\lambda_2 = 2\lambda$, $\lambda > 0$ and $\lambda_j = 0$ for $j > 2$. □

2.2.4 The case $\alpha = k\nu$

Let us consider $\alpha = k\nu$, $\nu \in \left(0, \frac{1}{k}\right]$, $k \in \mathbb{N}$, in (2.37). The problem becomes

$$\begin{cases} \left(\frac{{}^C\partial^{k\nu}}{\partial t^{k\nu}} + 2\lambda \frac{{}^C\partial^{\nu}}{\partial t^{\nu}}\right) w^{\beta}_{k\nu,\nu}(\boldsymbol{x},t) = -c^2 \left(-\Delta\right)^{\beta} w^{\beta}_{k\nu,\nu}(\boldsymbol{x},t), & \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ w^{\beta}_{k\nu,\nu}(\boldsymbol{x},0) = \delta(\boldsymbol{x}). \end{cases} \tag{2.39}$$

In view of Corollary 2.2.3 the solution to (2.39) is given by the law of the process $\boldsymbol{S}^{2\beta}_n\left(c^2 \mathcal{L}^{k\nu}(t)\right)$, $t > 0$. The Fourier-Laplace transform of $w^{\beta}_{k\nu,\nu}(\boldsymbol{x},t)$ can be now written as

$$\widetilde{\widehat{w}}^{\beta}_{k\nu,\nu}\left(\boldsymbol{\xi},\mu\right) = \frac{\mu^{k\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{n\nu} + 2\lambda\mu^{\nu} + c^2 \left\|\boldsymbol{\xi}\right\|^{2\beta}} = \mu^{n-1} \prod_{i=1}^{k} \frac{\mu^{\nu-1}}{\mu^{\nu} - Z_i} + 2\lambda\mu^{\nu-1} \prod_{i=1}^{k} \frac{1}{\mu^{\nu} - Z_i}$$

where Z_i are the roots of $\mu^{k\nu} + 2\lambda\mu^{\nu} + c^2 \left\|\boldsymbol{\xi}\right\|^{2\beta} = 0$.

For $k = 3$ we get

$$\widetilde{\widehat{w}}^{\beta}_{3\nu,\nu}\left(\boldsymbol{\xi},\mu\right) = \frac{\mu^{3\nu-1}}{\mu^{\nu} - A} \frac{1}{\mu^{\nu} - B} \frac{1}{\mu^{\nu} - C} + 2\lambda \frac{\mu^{\nu-1}}{\mu^{\nu} - A} \frac{1}{\mu^{\nu} - B} \frac{1}{\mu^{\nu} - C}, \tag{2.40}$$

where A, B and C are the solutions to $\mu^{3\nu} + 2\lambda\mu^{\nu} + c^2 \left\|\boldsymbol{\xi}\right\|^{2\beta} = 0$. Formula (2.40) can be rewritten as

$$\begin{aligned} \widetilde{\widehat{w}}^{\beta}_{3\nu,\nu}\left(\boldsymbol{\xi},\mu\right) &= \frac{\left(\mu^{3\nu-1} + 2\lambda\mu^{\nu-1}\right)}{\mu^{\nu} - A} \left[\left(\frac{1}{\mu^{\nu} - B} - \frac{1}{\mu^{\nu} - C}\right) \frac{1}{B - C}\right] \\ &= \left(\mu^{3\nu-1} + 2\lambda\mu^{\nu-1}\right) \left[\left(\frac{1}{\mu^{\nu} - A} - \frac{1}{\mu^{\nu} - B}\right) \frac{1}{\left(A - B\right)\left(B - C\right)} \right. \\ &\quad \left. - \left(\frac{1}{\mu^{\nu} - A} - \frac{1}{\mu^{\nu} - C}\right) \frac{1}{\left(A - C\right)\left(B - C\right)}\right] \\ &= \left(\mu^{3\nu-1} + 2\lambda\mu^{\nu-1}\right) \left[\frac{1}{\mu^{\nu} - A} \frac{1}{\left(B - A\right)\left(C - A\right)} + \frac{1}{\mu^{\nu} - B} \frac{1}{\left(A - B\right)\left(C - B\right)} \right. \\ &\quad \left. + \frac{1}{\mu^{\nu} - C} \frac{1}{\left(A - C\right)\left(B - C\right)}\right]. \end{aligned} \tag{2.41}$$

By considering now the relationship

$$\int_0^{\infty} e^{-\mu t} t^{(1-2\nu)-1} E_{\nu,1-2\nu}\left(C t^{\nu}\right) dt = \frac{\mu^{\nu-(1-2\nu)}}{\mu^{\nu} - C}$$

we can invert (2.41) with respect to μ. Thus we can explicitely write the characteristic function of the process $\boldsymbol{S}^{2\beta}_n\left(c^2 \mathcal{L}^{3\nu,\nu}(t)\right)$, $t > 0$, as

$$\begin{aligned} \mathbb{E} e^{i\boldsymbol{\xi}\cdot \boldsymbol{S}^{2\beta}_n\left(c^2 \mathcal{L}^{3\nu,\nu}(t)\right)} = & \frac{t^{-2\nu} E_{\nu,1-2\nu}\left(A t^{\nu}\right) + 2\lambda E_{\nu,1}\left(A t^{\nu}\right)}{\left(B - A\right)\left(C - A\right)} + \frac{t^{-2\nu} E_{\nu,1-\nu}\left(B t^{\nu}\right) + 2\lambda E_{\nu,1}\left(B t^{\nu}\right)}{\left(A - B\right)\left(C - B\right)} \\ & + \frac{t^{-2\nu} E_{\nu,1-2\nu}\left(C t^{\nu}\right) + 2\lambda E_{\nu,1}\left(C t^{\nu}\right)}{\left(A - C\right)\left(B - C\right)}. \end{aligned}$$

2.3 Multidimensional Gauss-Laplace distributions and infinite compositions

In Orsingher and Beghin (2009) the authors have shown that the process

$$\mathfrak{I}_n(t) = B_1\left(|B_2\left(|B_3 \cdots (|B_{n+1}(t)|) \cdots |)|\right)\right), \qquad t > 0,$$

converges in distribution for $n \to \infty$ to a Gauss-Laplace (or bilateral exponential) random variable independent from $t > 0$. In this section we show that the process $\boldsymbol{B}_n\left(\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)\right)$, $t > 0$, converges in distribution, for $r \to \infty$, to a multidimensional version of the Gauss-Laplace r.v. and solves the equation, for $\nu_j \in (0,1)$, $r \in \mathbb{N}$,

$$\sum_{j=1}^m \lambda_j \frac{{}^C\partial^{\nu_j^r}}{\partial t^{\nu_j^r}} \mathfrak{w}_{\nu_1,\cdots,\nu_m}^{\beta,r}(\boldsymbol{x},t) = c^2 \Delta \mathfrak{w}_{\nu_1,\cdots,\nu_m}^{\beta,r}(\boldsymbol{x},t), \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0.$$

The process $\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)$, $t > 0$, is defined as

$$\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t) = \inf\left\{s > 0 : \mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(s) \geq t\right\}, \qquad t > 0$$

where

$$\mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(t) = \sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} \, {}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots {}_rH^{\nu_j}(t)\cdots\right)\right)\right), \qquad t > 0.$$

We start by presenting the following results.

Corollary 2.3.1. *We have that*

i) The solution to the problem for $\nu_j \in (0,1)$, $j = 1, \cdots, m$, $r \in \mathbb{N}$,

$$\begin{cases} \frac{\partial}{\partial t} \mathfrak{h}_{\nu_1,\cdots,\nu_m}^r(x,t) = -\sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j^r}}{\partial x^{\nu_j^r}} \mathfrak{h}_{\nu_1,\cdots,\nu_m}^r(x,t), & x > 0, t > 0, \\ \mathfrak{h}_{\nu_1,\cdots,\nu_m}^r(x,0) = \delta(x), \\ \mathfrak{h}_{\nu_1,\cdots,\nu_m}^r(0,t) = 0, \end{cases} \tag{2.42}$$

is given by the law of the process

$$\mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(t) = \sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} \, {}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots {}_rH^{\nu_j}(t)\cdots\right)\right)\right), \qquad t > 0.$$

ii) The solution to the problem for $\nu_j \in (0,1)$, $j = \cdots, m$, $r \in \mathbb{N}$,

$$\begin{cases} \sum_{j=1}^m \lambda_j \frac{\partial^{\nu_j^r}}{\partial t^{\nu_j^r}} \mathfrak{l}_{\nu_1,\cdots,\nu_m}^r(x,t) = -\frac{\partial}{\partial x} \mathfrak{l}_{\nu_1,\cdots,\nu_m}^r(x,t), & x > 0, t > 0, \\ \mathfrak{l}_{\nu_1,\cdots,\nu_m}^r(0,t) = \sum_{j=1}^m \lambda_j \frac{t^{\nu_j^r}}{\Gamma\left(1-\nu_j^r\right)} \end{cases} \tag{2.43}$$

is given by the law of the process

$$\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t) = \inf\left\{s > 0 : \mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(s) \geq t\right\}, \qquad t > 0. \tag{2.44}$$

Proof of i). The proof is carried out in the same spirit of Theorem 2.2.1, thus by considering the Fourier transform of (2.42) we get

$$\begin{cases} \frac{\partial}{\partial t}\widehat{\mathfrak{h}}^r_{\nu_1,\cdots,\nu_m}(\xi,t) = -\sum_{j=1}^m \lambda_j \left(-i\xi\right)^{\nu_j^r} \widehat{\mathfrak{h}}^r_{\nu_1,\cdots,\nu_m}(\xi,t) \\ \widehat{\mathfrak{h}}^r_{\nu_1,\cdots,\nu_m}(\xi,0) = 1. \end{cases} \tag{2.45}$$

The proof is completed by observing that the solution to (2.45) is given by the Fourier transform of the law of the process $\mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(t)$, $t>0$, which can be obtained by means of the calculation

$$\mathbb{E}e^{i\xi\mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(t)} = \mathbb{E}e^{i\xi\sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}}\, {}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots {}_rH^{\nu_j}(t)\cdots\right)\right)\right)} \overset{(2.19)}{=} e^{-t\sum_{j=1}^m \lambda_j(-i\xi)^{\nu_j^r}},$$

that is the solution to (2.45). □

Proof of ii). By considering the double Laplace transform of (2.43) we have that

$$\sum_{j=1}^m \lambda_j \mu^{\nu_j^r} \widetilde{\widetilde{\mathfrak{l}}}^r_{\nu_1,\cdots,\nu_m}(\gamma,\mu) = \widetilde{\mathfrak{l}}^r_{\nu_1,\cdots,\nu_m}(0,\mu) - \gamma \widetilde{\widetilde{\mathfrak{l}}}^r_{\nu_1,\cdots,\nu_m}(\gamma,\mu),$$

where the boundary condition is given by

$$\int_0^\infty dt\, e^{-\mu t} \sum_{j=1}^m \lambda_j \frac{t^{\nu_j^r}}{\Gamma\left(1-\nu_j^r\right)} = \sum_{j=1}^m \lambda_j \mu^{\nu_j^r - 1},$$

and thus

$$\widetilde{\widetilde{\mathfrak{l}}}^r_{\nu_1,\cdots,\nu_m}(\gamma,\mu) = \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j^r-1}}{\sum_{j=1}^m \lambda_j \mu^{\nu_j^r} + \gamma} \tag{2.46}$$

The definition (2.44) permits us to state that the processes $\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)$, $t>0$, and $\mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(t)$, $t>0$, are related by the fact that

$$\Pr\left\{\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t) < x\right\} = \Pr\left\{\mathfrak{H}_r^{\nu_1,\cdots,\nu_m}(x) > t\right\},$$

and thus we can perform manipulations similar to those of Theorem 2.2.1. We have that the double Laplace transform of the law $\mathfrak{l}^r_{\nu_1,\cdots,\nu_m}(x,t)$ is then given by

$$\begin{aligned} \widetilde{\widetilde{\mathfrak{l}}}^r_{\nu_1,\cdots,\nu_m}(\gamma,\mu) &= \int_0^\infty dt\, e^{-\mu t}\int_0^\infty dx\, e^{-\gamma x}\left[-\frac{\partial}{\partial x}\int_0^t dz\, \mathfrak{h}^r_{\nu_1,\cdots,\nu_m}(z,x)\right] \\ &= -\frac{1}{\mu}\int_0^\infty dx\, e^{-\gamma x}\left[\frac{\partial}{\partial x}\widetilde{\mathfrak{h}}^r_{\nu_1,\cdots,\nu_m}(\mu,x)\right] \\ &= -\frac{1}{\mu}\int_0^\infty dx\, e^{-\gamma x}\left[\frac{\partial}{\partial x}\mathbb{E}e^{-\mu\sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}}\, {}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots {}_rH^{\nu_j}(t)\cdots\right)\right)\right)}\right] \\ &\overset{(2.19)}{=} -\frac{1}{\mu}\int_0^\infty dx\, e^{-\gamma x}\left[\frac{\partial}{\partial x}e^{-x\sum_{j=1}^m \lambda_j \mu^{\nu_j^r}}\right] = \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j^r-1}}{\sum_{j=1}^m \lambda_j \mu^{\nu_j^r}+\gamma}, \end{aligned}$$

and coincides with (2.46). □

Theorem 2.3.2. *The solution to the problem for* $\nu_j \in (0,1]$, $\beta \in (0,1]$, $j = 1, \cdots, m$, $r \in \mathbb{N}$,

$$\begin{cases} \sum_{j=1}^m \lambda_j \frac{{}^C\partial^{\nu_j^r}}{\partial t^{\nu_j^r}} \mathfrak{w}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t) = -c^2(-\Delta)^\beta \mathfrak{w}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},t), & \boldsymbol{x} \in \mathbb{R}^n, t>0, \\ \mathfrak{w}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{x},0) = \delta(\boldsymbol{x}), \end{cases} \tag{2.47}$$

is given by the law of the process

$$\mathfrak{W}_n(t) = \boldsymbol{S}_n^{2\beta}\left(c^2 \mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)\right), \qquad t>0. \tag{2.48}$$

where the process $\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)$, $t>0$, *is defined in* (2.44). *For* $\beta = 1$, *the process* (2.48) *becomes the subordinated Brownian motion* $\boldsymbol{B}_n(c^2\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t))$, $t>0$.

Proof. The Fourier-Laplace transform of (2.47) can be easily derived as in Theorem 2.2.2 and reads

$$\widetilde{\widehat{\mathfrak{w}}}^{\beta,r}_{\nu_1,\cdots,\nu_m}(\boldsymbol{\xi},\mu) = \frac{\sum_{j=1}^j \lambda_j \mu^{\nu_j^r - 1}}{\sum_{j=1}^m \lambda_j \mu^{\nu_j^r} + c^2\|\boldsymbol{\xi}\|^{2\beta}}. \tag{2.49}$$

By considering the law of the process $\boldsymbol{S}_n^{2\beta}(c^2\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t))$ we have that

$$\begin{aligned} &\int_0^\infty dt\, e^{-\mu t}\mathbb{E}e^{i\boldsymbol{\xi}\cdot \boldsymbol{S}_n^{2\beta}\left(c^2\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)\right)} \\ &= \int_{\mathbb{R}^n} d\boldsymbol{x}\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \int_0^\infty dt\, e^{-\mu t}\int_0^\infty ds\, v_\beta\left(\boldsymbol{x}, c^2 s\right)\, \mathfrak{l}^r_{\nu_1,\cdots,\nu_m}(s,t) \\ &= \int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}}\int_0^\infty dt\, e^{-\mu t}\left[-\frac{\partial}{\partial s}\int_0^t \mathfrak{h}^r_{\nu_1,\cdots,\nu_m}(z,s)\,dz\right] \\ &= -\frac{1}{\mu}\int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}}\left(\frac{\partial}{\partial s}\widetilde{\mathfrak{h}}^r_{\nu_1,\cdots,\nu_m}(\mu,s)\right) \\ &= -\frac{1}{\mu}\int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}}\left(\frac{\partial}{\partial s}\mathbb{E}e^{-\mu\sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}}\, {}_1H^{\nu_j}\left({}_2H^{\nu_j}\left({}_3H^{\nu_j}\left(\cdots {}_rH^{\nu_j}(t)\cdots\right)\right)\right)}\right) \\ &\overset{(2.18)}{=} -\frac{1}{\mu}\int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta}}\left(\frac{\partial}{\partial s}e^{-s\sum_{j=1}^m \lambda_j \mu^{\nu_j^r}}\right) \\ &= \sum_{j=1}^m \lambda_j\mu^{\nu_j^r-1}\int_0^\infty ds\, e^{-sc^2\|\boldsymbol{\xi}\|^{2\beta} - s\sum_{j=1}^m \lambda_j\mu^{\nu_j^r}} = \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j^r-1}}{\sum_{j=1}^m \lambda_j\mu^{\nu_j^r} + c^2\|\boldsymbol{\xi}\|^{2\beta}} \end{aligned}$$

which coincides with (2.49). □

We now consider the limiting case for $r \to \infty$ where the iteration of the process $\boldsymbol{S}_n^{2\beta}(c^2\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t))$, $t>0$, is infinitely extended. In the next theorem we have that the limiting law of

$$\lim_{r\to\infty} \boldsymbol{S}_n^{2\beta}\left(c^2\mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)\right), \qquad t>0,$$

is, for $\beta = 1$, a generalization to $\mathbb{R}^n$ of the Gauss-Laplace probability density. This result represents an extension to the n-dimensional case of the infinitely iterated Brownian motion.

Theorem 2.3.3. *The distribution of the limiting process*

$$\lim_{r\to\infty} \boldsymbol{B}_n\left(c^2 \mathfrak{L}_r^{\nu_1,\cdots,\nu_m}(t)\right) \overset{law}{=} X_{m,n}$$

does not depend on t and reads

$$\mathfrak{w}_m(\boldsymbol{x}) = \frac{\Pr\{X_{m,n} \in d\boldsymbol{x}\}}{d\boldsymbol{x}} = \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}\right)^{\frac{n+2}{2}} \|\boldsymbol{x}\|^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}\left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}\|\boldsymbol{x}\|\right). \tag{2.50}$$

The density (2.50) solves the equation

$$\left(\sum_{j=1}^m \lambda_j\right)\mathfrak{w}_m(x_1,\cdots,x_n) = c^2\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\mathfrak{w}_m(x_1,\cdots,x_n),$$

which is obtained from (2.47) by letting $r\to\infty$.

Proof. By assuming

$$A = \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}\right)^{\frac{n+2}{2}}, \qquad B = \frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c},$$

the density

$$\mathfrak{w}_m(\boldsymbol{x}) = A\left(\sum_{j=1}^n x_j^2\right)^{-\frac{n-2}{4}} K_{\frac{n-2}{2}}\left(B\sqrt{\sum_{j=1}^n x_j^2}\right)$$

has first-order derivative which reads

$$\frac{\partial}{\partial x_j}\mathfrak{w}_m(\boldsymbol{x}) =$$

$$= AB\frac{x_j}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}}} K'_{\frac{n-2}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right) - A\frac{\left(\frac{n-2}{2}\right) x_j\, K_{\frac{n-2}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right)}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}+\frac{1}{2}}}$$

$$= -ABx_j\left(\sum_{j=1}^n x_j^2\right)^{-\frac{n}{4}} K_{\frac{n}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right). \tag{2.51}$$

In the last step we applied the relationship

$$\frac{d}{dz}K_\nu(z) = \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z) \tag{2.52}$$

of Lebedev (1965), page 110. The second-order derivative now becomes

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_j^2}\mathfrak{w}_m\left(x_1,\cdots,x_n\right) = \\
&= - AB\frac{1}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}}}K_{\frac{n}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right) + \frac{n}{2}AB\frac{x_j^2\, K_{\frac{n}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right)}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}+1}} \\
&\quad - AB^2\frac{x_j^2}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}+\frac{1}{2}}}K'_{\frac{n}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right) \\
&= AB^2\frac{x_j^2}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}+\frac{1}{2}}}K_{\frac{n}{2}+1}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right) - AB\frac{K_{\frac{n}{2}}\left(B\left(x_1^2+\cdots+x_n^2\right)^{\frac{1}{2}}\right)}{\left(\sum_{j=1}^n x_j^2\right)^{\frac{n}{4}}}.
\end{aligned} \tag{2.53}
$$

By considering the relationship

$$K_{\nu+1}(z) = K_{\nu-1}(z) + 2\frac{\nu}{z}K_\nu(z)$$

of Lebedev (1965), page 110, the derivative (2.53) takes the form

$$\frac{\partial^2}{\partial x_j^2}\mathfrak{w}_m\left(x_1,\cdots,x_n\right) = AB^2x_j^2\left(\sum_{j=1}^n x_j^2\right)^{-\frac{n}{4}-\frac{1}{2}}K_{\frac{n}{2}-1}\left(B\left(x_1^2+\cdots+x_n^2\right)\right).$$

The Laplacian of $\mathfrak{w}_m(x_1,\cdots,x_m)$ therefore becomes

$$\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2}\mathfrak{w}_m\left(\boldsymbol{x}\right) = AB^2\left(\sum_{j=1}^n x_j^2\right)^{-\frac{n+2}{4}}K_{\frac{n-2}{2}}\left(B\left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}}\right)$$

and thus taking A and B explicitely we obtain the desired result

$$c^2\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2}\mathfrak{w}_m(x_1,\cdots,x_n) = \sum_{j=1}^m\lambda_j\mathfrak{w}_m(x_1,\cdots,x_n).$$

□

Remark 2.3.4. For $r\to\infty$ the Fourier-Laplace transform (2.49) becomes

$$\widehat{\widetilde{\mathfrak{w}}}^\beta_m(\boldsymbol{\xi},\mu) = \frac{1}{\mu}\frac{\sum_{j=1}^m\lambda_j}{\sum_{j=1}^m\lambda_j + c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}},$$

and thus the Fourier transform takes the form

$$\widehat{\mathfrak{w}}^\beta_m\left(\boldsymbol{\xi}\right) = \frac{\sum_{j=1}^m\lambda_j}{\sum_{j=1}^m\lambda_j + c^2\left\|\boldsymbol{\xi}\right\|^{2\beta}}. \tag{2.54}$$

The inversion of the Fourier transform (2.54) can be carried out by means of the hyperspherical coordinates. Thus we have that

$$\mathfrak{w}_m^\beta(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \frac{\sum_{j=1}^m \lambda_j}{\sum_{j=1}^m \lambda_j + c^2 \|\boldsymbol{\xi}\|^{2\beta}} d\boldsymbol{\xi}$$

$$= \frac{1}{(2\pi)^n}\int_0^\infty d\rho\, \rho^{n-1} \frac{\sum_{j=1}^m \lambda_j}{\sum_{j=1}^m \lambda_j + c^2\rho^{2\beta}} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{n-2}$$

$$\int_0^{2\pi} d\phi\, e^{-i\rho[x_1 \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2}\sin\phi + x_2 \sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\phi]}$$

$$e^{-i\rho[x_3\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-3}\cos\theta_{n-2}+\,\cdots\,+x_{n-1}\sin\theta_1\cos\theta_2 + x_n\cos\theta_1]} \sin^{n-2}\theta_1 \;\cdots\; \sin\theta_{n-2}$$

$$= \frac{1}{(2\pi)^{n-1}}\int_0^\infty \rho^{n-1} \frac{\sum_{j=1}^m \lambda_j}{\sum_{j=1}^m \lambda_j + c^2\rho^{2\beta}} d\rho \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{n-2}\, \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}$$

$$e^{-i\rho[x_3\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-3}\cos\theta_{n-2}+\,\cdots\,+x_{n-1}\sin\theta_1\cos\theta_2 + x_n\cos\theta_1]}$$

$$J_0\left(\rho\sqrt{x_1^2+x_2^2}\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\right)$$

We now evaluate the integrals with respect to θ_j by means of formula 6.688 page 727 of Gradshteyn and Ryzhik (2007), which reads

$$\int_0^{\frac{\pi}{2}} \sin^{\nu+1} x \cos(\beta\cos x)\, J_\nu(\alpha \sin x)\, dx = \sqrt{\frac{\pi}{2}} \frac{\alpha^\nu}{(\alpha^2+\beta^2)^{\frac{\nu}{2}+\frac{1}{4}}} J_{\nu+\frac{1}{2}}\left(\sqrt{\alpha^2+\beta^2}\right).$$

valid for $\Re(\nu) > -1$. We start with the integral with respect to θ_{n-2}

$$\int_0^\pi d\theta_{n-2}\, e^{-i\rho x_3 \sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2}} \sin\theta_{n-2}\, J_0\left(\rho\sqrt{x_1^2+x_2^2}\sin\theta_1\cdots\sin\theta_{n-2}\right)$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta_{n-2}\cos(\rho x_3\sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2})\sin\theta_{n-2} J_0\left(\rho\sqrt{x_1^2+x_2^2}\,\sin\theta_1\cdots\sin\theta_{n-2}\right)$$

$$= \sqrt{2\pi}\left(\rho\sin\theta_1\cdots\sin\theta_{n-3}\sqrt{x_1^2+x_2^2+x_3^2}\right)^{-\frac{1}{2}} J_{\frac{1}{2}}\left(\rho\sqrt{x_1^2+x_2^2+x_3^2}\,\sin\theta_1\cdots\sin\theta_{n-3}\right)$$

and thus the integral with respect to θ_{n-3} becomes

$$\sqrt{2\pi}\int_0^\pi d\theta_{n-3} e^{-i\rho x_4\sin\theta_1\cdots\sin\theta_{n-4}\cos\theta_{n-3}} \sin^2\theta_{n-3}$$

$$\left(\rho\sin\theta_1\cdots\sin\theta_{n-3}\sqrt{x_1^2+x_2^2+x_3^2}\right)^{-\frac{1}{2}} J_{\frac{1}{2}}\left(\rho\sqrt{x_1^2+x_2^2+x_3^2}\sin\theta_1\cdots\sin\theta_{n-3}\right)$$

$$= 2\sqrt{2\pi}\left(\rho\sin\theta_{n-1}\cdots\sin\theta_{n-4}\sqrt{x_1^2+x_2^2+x_3^2}\right)^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} d\theta_{n-3}\, \sin^{\frac{3}{2}}\theta_{n-3}$$

$$\cos(\rho x_4 \sin\theta_1\cdots\sin\theta_{n-4}\cos\theta_{n-3})\; J_{\frac{1}{2}}\left(\rho\sqrt{x_1^2+x_2^2+x_3^2}\,\sin\theta_1\cdots\sin\theta_{n-3}\right)$$

$$= \left(\sqrt{2\pi}\right)^2 \left(\rho^2\sin^2\theta_1\cdots\sin^2\theta_{n-4}\left(x_1^2+x_2^2+x_3^2+x_4^2\right)\right)^{-\frac{1}{2}}$$

$$J_1\left(\rho\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-4}\right).$$

After $n-2$ integrations we arrive at the integral with respect to ρ which reads

$$\mathfrak{w}_m^\beta(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}}\int_0^\infty d\rho\,\frac{\sum_{j=1}^m\lambda_j}{\sum_{j=1}^m\lambda_j+c^2\rho^{2\beta}}J_{\frac{n-2}{2}}\left(\rho\sqrt{\sum_{i=j}^n x_j^2}\right)\frac{\rho^{\frac{n}{2}}}{\left(\sqrt{\sum_{i=j}^n x_j^2}\right)^{\frac{n-2}{2}}} \tag{2.55}$$

which, for $\beta=1$ and after the change of variable $\rho^2c^2=y^2$, becomes

$$\begin{aligned}\mathfrak{w}_m(\boldsymbol{x}) &= (2\pi)^{-\frac{n}{2}}\frac{1}{c}\int_0^\infty dy\,\frac{\sum_{j=1}^m\lambda_j}{\sum_{j=1}^m\lambda_j+y^2}J_{\frac{n-2}{2}}\left(\frac{y}{c}\sqrt{\sum_{j=1}^n x_j^2}\right)\frac{\left(\frac{y}{c}\right)^{\frac{n}{2}}}{\left(\sqrt{\sum_{j=1}^n x_j^2}\right)^{\frac{n-2}{2}}}\\
&\overset{\text{for } n<5}{=} \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\right)^{\frac{n+2}{2}}\left(\sqrt{\sum_{j=1}^n x_j^2}\right)^{-\frac{n-2}{2}}K_{\frac{n-2}{2}}\left(\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\sqrt{\sum_{j=1}^n x_j^2}\right)\\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\right)^{\frac{n+2}{2}}\|\boldsymbol{x}\|^{-\frac{n-2}{2}}K_{\frac{n-2}{2}}\left(\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\|\boldsymbol{x}\|\right),\end{aligned}$$

where we used formula 6.566 page 679 of Gradshteyn and Ryzhik (2007), which reads

$$\int_0^\infty dx\,x^{\nu+1}J_\nu(ax)\frac{1}{x^2+b^2} = b^\nu K_\nu(ab), \qquad a>0,\ \Re(b)>0,\ -1<\Re(\nu)<\frac{3}{2}.$$

Remark 2.3.5. We can check that (2.50) for all $n\in\mathbb{N}$ is a true probability density.

$$\begin{aligned}\int_{\mathbb{R}^n}\mathfrak{w}_m(\boldsymbol{x})\,d\boldsymbol{x} &= \frac{\operatorname{area}(S_1^n)}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\right)^{\frac{n+2}{2}}\int_0^\infty\rho^{n-1-\frac{n-2}{2}}K_{\frac{n}{2}-1}\left(\rho\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\right)d\rho\\
&= \frac{(2\pi)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\right)^{\frac{n+2}{2}}\int_0^\infty\rho^{\frac{n}{2}}K_{\frac{n}{2}-1}\left(\rho\frac{\sqrt{\sum_{j=1}^m\lambda_j}}{c}\right)d\rho = 1\end{aligned}$$

in force of formula 6.561(16) of Gradshteyn and Ryzhik (2007) page 676

$$\int_0^\infty x^\mu K_\nu(ax) = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right), \tag{2.56}$$

valid for $\Re(\mu+1\pm\nu)>0$ and $\Re(a)>0$. The non-negativity of (2.50) is shown by the following integral representation

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt$$

valid for $|\arg(z)| < \frac{\pi}{2}$ (see Gradshteyn and Ryzhik (2007) page 917 formula 8.432).

By considering that

$$K_{-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \tag{2.57}$$

from (2.50) we derive the following probability density for $\boldsymbol{x} \in \mathbb{R}^3$,

$$\begin{aligned} \mathfrak{w}_m(x_1, x_2, x_3) &= \\ = \frac{\sum_{j=1}^m \lambda_j}{(2c)^2 \pi \sum_{j=1}^n x_j^2} e^{-\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c} \sqrt{\sum_{j=1}^n x_j^2}} &= \frac{\sum_{j=1}^m \lambda_j}{(2c)^2 \pi \left\|\boldsymbol{x}\right\|^2} e^{-\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c} \|\boldsymbol{x}\|} \end{aligned}$$

In the two dimensional case the distribution (2.50) has a simple structure which reads

$$\mathfrak{w}_m(x_1, x_2) = \frac{1}{2\pi} \frac{\sum_{j=1}^m \lambda_j}{c^2} K_0 \left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c} \|\boldsymbol{x}\| \right).$$

In view of (2.57) it is also easy to show that the distribution (2.50) coincides for $n = 1$ with the classical Gauss-Laplace distribution. We have that for $n = 1$ (2.50) becomes

$$\begin{aligned} \mathfrak{w}_m(x) &= \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c} \right)^{\frac{3}{2}} \sqrt{|x|} \sqrt{\frac{\pi c}{2\sqrt{\sum_{j=1}^m \lambda_j}\,|x|}} e^{-\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}|x|} \\ &= \frac{\sqrt{\sum_{j=1}^m \lambda_j}}{2c} e^{-\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}|x|} \end{aligned} \tag{2.58}$$

Furthermore, for $\lambda_1 = 1$, $\lambda_2 = 2\lambda$, $\lambda > 0$ and $\lambda_j = 0$ for $j = 3, \cdots, m$, we note that (2.58) coincides with formula (3.18) of Orsingher and Beghin (2009).

Remark 2.3.6. By considering the iterated random walk

$$Y_n(k) = S_1 \left(S_2 \left(\cdots \left(S_n(k) \right) \cdots \right) \right), \qquad k \in \mathbb{N},$$

with S_j, $j = 1, \cdots, n$, independent random walks, Turban (2004) has shown that for $n \to \infty$, $Y_n(k)$ converges to a stationary r.v. (independent from k) which possesses Gauss-Laplace distribution, in accord with result (2.50) of the present work and with (3.12) of Orsingher and Beghin (2009).

Chapter 3

On the subordinate Poisson process

Article: Orsingher and Toaldo (2013). Counting processes with Bernstein intertimes and random jumps.

Summary

We consider here point processes $N^f(t)$, $t > 0$, with independent increments and integer-valued jumps whose distribution is expressed in terms of Bernštein functions f with Lévy measure ν. We obtain the general expression of the probability generating functions G^f of N^f, the equations governing the state probabilities p_k^f of N^f, and their corresponding explicit form. We give also the distribution of the first-passage times T_k^f of N^f, and the related governing equation. We study in detail the cases of the fractional Poisson process, the relativistic Poisson process and the Gamma Poisson process whose state probabilities have the form of negative binomial. The distribution of the times $\tau_j^{l_j}$ of jumps with height l_j ($\sum_{j=1}^r l_j = k$) under the condition $N(t) = k$ for all these special processes is investigated in detail.

3.1 Introduction

In this paper we consider a class of point processes with stationary independent integer-valued increments of arbitrary range. These processes can be regarded as generalizations of the Poisson process where jumps can take any positive value. Furthermore we shall show that these processes $N^f(t)$, $t > 0$, can be viewed as time-changed Poisson processes $N\left(H^f(t)\right)$ where $H^f(t)$ are subordinators, independent from N, associated with the Bernštein function f. The probabilistic behaviour of the processes $\mathcal{P}^f(t)$, with related counting process $N^f(t)$, is described by the following properties.

i) $\mathcal{P}^f(t)$ has independent increments;

ii)

$$\Pr\left\{N^f[t,t+dt)=k\right\} = \begin{cases} dt\frac{\lambda^k}{k!}\int_0^\infty e^{-\lambda s}s^k\nu(ds)+o(dt), & k\geq 1,\\ 1-dt\int_0^\infty\left(1-e^{-\lambda s}\right)\nu(ds)+o(dt), & k=0, \end{cases} \tag{3.1}$$

where

$$f(\lambda) = \int_0^\infty \left(1-e^{-\lambda s}\right)\nu(ds) \tag{3.2}$$

is the integral representation of the Bernštein functions. By ν we denote a non-negative Lévy measure on the positive half-line such that

$$\int_0^\infty (s\wedge 1)\,\nu(ds) < \infty. \tag{3.3}$$

We often speak of $\mathcal{P}^f(t)$, $t>0$, as generalized Poisson processes performing integer-valued jumps of arbitrary height. These processes can be used to model many different concrete and real phenomena. For example, if we consider the car accidents in the time interval $[0,t)$, the number of injured people in each clash can take any positive number. Analogously in floods or earthquakes, the number of destroyed buildings in each event can be clearly of arbitrary magnitude and thus can be represented by $\mathcal{P}^f(t)$, $t>0$, with suitably chosen Bernštein function f and Lévy measure ν.

We observe that for

$$\nu(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)}ds, \qquad \alpha\in(0,1), \tag{3.4}$$

we obtain the space-fractional Poisson process studied in Orsingher and Polito (2012). In this case the subordinator corresponding to the space-fractional Poisson is a stable process of order α and positively skewed. If the Lévy measure is the Dirac point mass at one, then the corresponding subordinated Poisson process is

$$N_1\left(N_2(t)\right), \qquad t>0, \tag{3.5}$$

where N_i, $i = 1, 2$, are independent homogeneous Poisson process with rate $\lambda > 0$. Such process has been investigate in Orsingher and Polito (2012). The subordinator H^f has Laplace transform

$$\mathbb{E}e^{-\mu H^f(t)} = e^{-tf(\mu)} = e^{-t\int_0^\infty \left(1-e^{-s\mu}\right)\nu(ds)} \tag{3.6}$$

and thus in the case of the space-fractional Poisson process, $f(\mu) = \mu^\alpha$. The probability distributions $p_k^f(t) = \Pr\left\{N^f(t) = k\right\}$ are governed by difference-differential equations of the form

$$\frac{d}{dt}p_k^f(t) = -f(\lambda)p_k^f(t) + \sum_{m=1}^{k} \frac{\lambda^m}{m!}p_{k-m}^f(t)\int_0^\infty e^{-s\lambda}s^m \nu(ds), \qquad k \geq 0, t > 0, \tag{3.7}$$

with the usual initial conditions. From (3.7) we extract the probability generating function $G^f(u,t)$ of $N^f(t)$ as

$$G^f(u,t) = e^{-tf(\lambda(1-u))} = e^{-t\int_0^\infty \left(1-e^{-s\lambda(1-u)}\right)\nu(ds)}. \tag{3.8}$$

We prove also that

$$\mathbb{E}u^{N\left(H^f(t)\right)} = e^{-tf(\lambda(1-u))} \tag{3.9}$$

and thus we show that

$$N^f(t) \overset{\text{law}}{=} N\left(H^f(t)\right). \tag{3.10}$$

By means of the shift operator $B^m p_k^f(t) = p_{k-m}^f(t)$ we can rewrite equation (3.7) as

$$\frac{d}{dt}p_k^f(t) = -f\left(\lambda\left(I-B\right)\right)p_k^f(t), \qquad t > 0, k \geq 0, \tag{3.11}$$

which for $f(x) = x^\alpha$ coincides with the equation (2.4) of Orsingher and Polito (2012). We also present a further representation of the generalized Poisson process $\mathcal{P}^f(t)$, $t > 0$, as the scale limit of a continuous-time random walk with steps X_j having distribution

$$\Pr\left\{X_j = k\right\} = \frac{1}{u(n)}\int_0^\infty \Pr\left\{N(s) = k\right\}\mathbb{I}_{[k\geq n]}\,\nu(ds), \qquad k \in \mathbb{N}, \tag{3.12}$$

where

$$u(n) = \int_0^\infty \Pr\left\{N(s) \geq n\right\}\nu(ds). \tag{3.13}$$

For example, for the space-fractional Poisson process the distribution (3.12) becomes

$$\Pr\left\{X_j = k\right\} = \frac{\Gamma(k-\alpha)/k!}{\sum_{j=n}^\infty \Gamma(j-\alpha)/j!}, \qquad k \geq n. \tag{3.14}$$

For the hitting-times

$$T_k^f = \inf \left\{ t > 0 : N^f(t) \geq k \right\} \tag{3.15}$$

we show that

$$\Pr\left\{ T_k^f \in ds \right\} / ds = -\frac{d}{ds} \sum_{l=0}^{k-1} \frac{(-\lambda)^l}{l!} \frac{d^l}{d\lambda^l} e^{-sf(\lambda)}. \tag{3.16}$$

We note that for $f(\lambda) = \lambda$ (case of the homogeneous Poisson process) formula (3.16) yields the Erlang distribution

$$\Pr\{T_k \in ds\} = \lambda^k e^{-\lambda s} \frac{s^{k-1}}{(k-1)!} ds, \qquad k \geq 1, s > 0. \tag{3.17}$$

In some special cases it is possible to write down the distribution $p_k^f(t)$, $t > 0$, $k \geq 0$, and to analyse many related random variables. When the Lévy measure is $\nu(ds) = \alpha s^{-\alpha-1}/\Gamma(1-\alpha)$, and therefore $f(\lambda) = \lambda^\alpha$, we have the space-fractional Poisson process whose distribution is written in many alternative forms as

$$\begin{aligned} p_k^\alpha(t) &= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t)}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)} \\ &= \frac{(-1)^k}{k!} \frac{d^k}{du^k} e^{-t\lambda^\alpha u^\alpha} \Big|_{u=1} \\ &= \frac{e^{-\lambda^\alpha t}}{k!} \left[c_{k,k} t^k + c_{k-1,k} t^{k-1} + \cdots + c_{1,k} t \right], \end{aligned} \tag{3.18}$$

where $c_{k,j}$, $j = 1, \ldots, n$, are suitable coefficients. In this case the conditional distributions of the instants of occurrence of jumps of $N^\alpha(t)$ is analyzed. The possibility of multiple jumps makes the form of the conditional distributions

$$\Pr\left\{ \bigcap_{j=1}^{r} \left\{ \tau_j^{l_j} \in dt_j \right\} \Big| N^\alpha(t) = k \right\} \tag{3.19}$$

rather complicated and can be given in closed form for small values of k, only. By $\tau_j^{l_j}$ we mean the instant of occurrence of the j-th jump of length l_j. The space-fractional Poisson process has the drawback of having infinite mean values as emerges from the form

$$G^\alpha(u,t) = e^{-t\lambda^\alpha (1-u)^\alpha}, \qquad |u| \leq 1, \alpha \in (0,1), \tag{3.20}$$

of the p.g.f.. This defect is circumvented when the Poisson process with relativistic stable subordinator is considered, that is for

$$\nu(ds) = \frac{\alpha s^{-\alpha-1} e^{-\theta s}}{\Gamma(1-\alpha)}, \qquad \theta > 0, 0 < \alpha < 1, s > 0, \tag{3.21}$$

with corresponding Bernštein function

$$f(\mu) = (\theta+\mu)^{\alpha} - \theta^{\alpha}. \tag{3.22}$$

In this case the probability distribution of $N^{\alpha,\theta}(t)$, $t>0$, writes

$$p_m^{\alpha,\theta}(t) = \frac{(-1)^m}{m!} \frac{\lambda^m e^{\theta^\alpha}}{(\theta+\lambda)^m} \sum_{k=0}^{\infty} \frac{(-t(\lambda+\theta)^\alpha)^k}{k!} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-m)} \tag{3.23}$$

and has p.g.f.

$$G^{\alpha,\theta}(u,t) = e^{-t\{[\theta+\lambda(1-u)]^\alpha - \theta^\alpha\}}. \tag{3.24}$$

Clearly for $\theta = 0$, $N^{\alpha,\theta}(t)$, $t>0$, coincides with the space-fractional Poisson process and (3.23) coincides with (3.18). From (3.24) we easily see that the moments of $N^{\alpha,\theta}(t)$ are finite and

$$\begin{aligned} \mathbb{E}N^{\alpha,\theta}(t) &= \lambda\theta^{\alpha-1}t, \\ \mathrm{Var}N^{\alpha,\theta}(t) &= \lambda t\theta^{\alpha-2}\left(\lambda(1-\alpha)+\theta\right). \end{aligned} \tag{3.25}$$

The most attractive subordinated Poisson process emerging in our analysis corresponds to the Lévy measure

$$\nu(ds) = \frac{e^{-s}}{s} ds, \qquad s>0, \tag{3.26}$$

and thus as Bernštein function

$$f(x) = \log(1+x). \tag{3.27}$$

We call this process Poisson with Gamma subordinator or simply Gamma Poisson process and we will denote it by $N^{\Gamma}(t)$, $t>0$. It is well-known that for $\lambda = (1-p)/p$, $p \in (0,1)$ we obtain the negative binomial process studied in Brix (1999), Kozubowski and Podgórski (2009) and fractionalized in Beghin (2013). The distribution of $N^\Gamma(t)$ has the following form

$$\Pr\left\{N^\Gamma(t) = k\right\} = \frac{\lambda^k \Gamma(k+t)}{k!\Gamma(t)\left(\lambda+1\right)^{k+t}}, \qquad k \geq 0 \tag{3.28}$$

with p.g.f.

$$G^\Gamma(u,t) = \left(1+\lambda(1-u)\right)^{-t}, \qquad |u|<1. \tag{3.29}$$

The independence of increments and the structure of the distribution (3.28) permits us to obtain a number of interesting distributions related to the Gamma process. For example, we have that

$$\Pr\left\{\bigcap_{j=1}^{r}\left\{\tau_j^{l_j} \in dt_j\right\} \middle| N^\Gamma(t) = k\right\} = \frac{k!\Gamma(t)}{\Gamma(t+k)} \prod_{j=1}^{r} \frac{dt_j}{l_j} \tag{3.30}$$

for $0 < t_1 < \cdots < t_j < \cdots < t_r < t$, $r \leq k$. In (3.29) $\tau_j^{l_j}$ denotes the instant of the occurrence of the j-th event of amplitude l_j. Furthermore

$$\Pr\left\{N^\Gamma(s) = r | N^\Gamma(t) = k\right\} = \mathbb{E}\left[\binom{k}{r} X^r (1-x)^{k-r}\right] \tag{3.31}$$

where X is a Beta r.v. with parameter s and $t-s$. The result (3.31) generalizes a fine feature of the homogeneous Poisson processes which relates it with the Bernoulli r.v.. The correlation function of $N^\Gamma(t)$, $t>0$, has the form

$$\text{Cov}\left[N^\Gamma(s), N^\Gamma(t)\right] = \lambda(\lambda+1)\min(s,t) \tag{3.32}$$

and

$$\text{Cov}\left[N^\Gamma(s), N^\Gamma(w) | N^\Gamma(t) = k\right] = \frac{k}{t(t+1)}\left(1+\frac{k}{t}\right)\min(s,t)\min(t-s, t-w), \tag{3.33}$$

for $s, w \in (0,t)$. We also study the distribution of $N_1^\Gamma(t) - N_2^\Gamma(t)$, with N_1^Γ and N_2^Γ independent and establish the relationship with the Skellam distribution of $N_1(t) - N_2(t)$ for the homogeneous Poisson process.

3.2 General result

We now examine in detail the main properties of the process $N^f(t)$, $t>0$, with independent increments outlined in the introduction. Our first result is the difference-differential equations governing their state probabilities

$$p_k^f(t) = \Pr\left\{N^f(t) = k\right\}, \qquad k \geq 0. \tag{3.34}$$

Theorem 3.2.1. *The probabilities* $p_k^f(t) = \Pr\left\{N^f(t) = k\right\}$, $k \geq 0$, *are solutions to the equations*

$$\frac{d}{dt}p_k^f(t) = -f(\lambda)p_k^f(t) + \sum_{m=1}^{k}\frac{\lambda^m}{m!}p_{k-m}^f(t)\int_0^\infty e^{-s\lambda}s^m\nu(ds), \qquad k \geq 0, t>0, \tag{3.35}$$

with initial conditions

$$p_k^f(0) = \begin{cases} 1, & k=0 \\ 0, & k \geq 1. \end{cases} \tag{3.36}$$

The p.g.f. $G^f(u,t) = \mathbb{E}u^{N^f(t)}$, $|u|<1$, *satisfies the linear, homogeneous equation*

$$\begin{cases} \frac{\partial}{\partial t}G^f(u,t) = -f\left(\lambda(1-u)\right)G^f(u,t) \\ G^f(u,0) = 1, \end{cases} \tag{3.37}$$

and has the form

$$G^f(u,t) = e^{-tf(\lambda(1-u))} = e^{-t\int_0^\infty \left(1-e^{-s\lambda(1-u)}\right)\nu(ds)} \tag{3.38}$$

Proof. Since $N^f(t)$ has independent increments and the distribution of jumps is given by (3.1) we can write

$$\begin{aligned} p_k(t+dt) &= \Pr\left\{N^f[t+dt) = k\right\} = \Pr\left\{\bigcup_{j=0}^{k}\left\{N^f(t)=j, N^f[t,t+dt) = k-j\right\}\right\} \\ &= \sum_{j=0}^{k-1}\Pr\left\{N^f(t)=j\right\} dt \frac{\lambda^{k-j}}{(k-j)!}\int_0^\infty e^{-\lambda s}s^{k-j}\nu(ds) \\ &\quad + \Pr\left\{N^f(t)=k\right\}\left(1 - dt\int_0^\infty \left(1-e^{-\lambda s}\right)\nu(ds)\right). \end{aligned} \tag{3.39}$$

A simple expansion permits us to obtain, in the limit, equation (3.35). From equation (3.35) we have that

$$\begin{aligned} \frac{\partial}{\partial t}G^f(u,t) &= \sum_{k=0}^\infty u^k \frac{d}{dt}p_k(t) \\ &= -f(\lambda)\sum_{k=0}^\infty u^k p_k^f(t) + \sum_{k=1}^\infty u^k \sum_{m=1}^k \frac{\lambda^m}{m!}p^f_{k-m}(t)\int_0^\infty e^{-s\lambda}s^m\nu(ds) \\ &= -f(\lambda)G^f(u,t) + \sum_{m=1}^\infty \frac{\lambda^m}{m!}\int_0^\infty e^{-s\lambda}s^m\nu(ds)\sum_{k=m}^\infty u^k p^f_{k-m}(t) \\ &= -f(\lambda)G^f(u,t) + G^f(u,t)\int_0^\infty \left(e^{-s\lambda(1-u)} - e^{-s\lambda}\right)\nu(ds) \\ &= -G^f(u,t)\int_0^\infty \left(1-e^{-s\lambda(1-u)}\right)\nu(ds) \\ &= -G^f(u,t)\, f(\lambda(1-u)). \end{aligned} \tag{3.40}$$

In the last step we take into account the representation (3.2) of the Bernštein functions. □

Remark 3.2.2. *The appearence of $p_{k-j}(t)$, $k \geq j \geq 2$, in (3.35) makes the master equation of the state probabilities $p_k^f(t)$, substantially different from the case of the classical Poisson process. This fact is related to the possibility of jumps of arbitrary height. We also observe that*

$$N^f(t) \overset{\text{law}}{=} N\left(H^f(t)\right) \tag{3.41}$$

where H^f is the subordinator with Laplace transform (3.6). This can be ascertained by evaluating the p.g.f. of $N\left(H^f(t)\right)$, $t>0$, as follows

$$\mathbb{E}u^{N\left(H^f(t)\right)} = \sum_{k=0}^\infty u^k\int_0^\infty \Pr\{N(s)=k\}\Pr\left\{H^f(t)\in ds\right\}$$

$$= \int_0^\infty e^{-s\lambda(1-u)} \Pr\left\{H^f(t) \in ds\right\}$$
$$= G^f(u,t). \tag{3.42}$$

In view of (3.41) we can write the distribution of $N^f(t)$ as

$$\begin{aligned} \Pr\left\{N^f(t) = k\right\} &= \Pr\left\{N\left(H^f(t)\right) = k\right\} = \int_0^\infty e^{-\lambda s} \frac{(\lambda s)^k}{k!} \Pr\left\{H^f(t) \in ds\right\} \\ &= \frac{(-1)^k}{k!} \frac{d^k}{du^k} \int_0^\infty e^{-\lambda s u} \Pr\left\{H^f(t) \in ds\right\} \Big|_{u=1} \\ &= \frac{(-1)^k}{k!} \frac{d^k}{du^k} e^{-tf(\lambda u)} \Big|_{u=1}. \end{aligned} \tag{3.43}$$

Remark 3.2.3. *The equation (3.35) can alternatively be written as*

$$\frac{d}{dt} p_k^f(t) = -f\left(\lambda\left(I - B\right)\right) p_k^f(t), \qquad t > 0, k \geq 0, \tag{3.44}$$

where B is the shift operator such that $Bp_k^f(t) = p_{k-1}^f(t)$. This can be shown as follows

$$\begin{aligned} & - f\left(\lambda(I - B)\right) p_k^f(t) \\ = & - \int_0^\infty \left(1 - e^{-\lambda s(I-B)}\right) \nu(ds)\, p_k^f(t) \\ = & - \int_0^\infty \left(1 - e^{-\lambda s} \sum_{m=0}^\infty \frac{(\lambda s B)^m}{m!}\right) \nu(ds)\, p_k^f(t) \\ = & - \int_0^\infty \left(p_k^f(t) - e^{-\lambda s} \sum_{m=0}^k \frac{(\lambda s)^m}{m!} p_{k-m}^f(t)\right) \nu(ds) \\ = & - \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds) p_k^f(t) + \sum_{m=1}^k \frac{\lambda^m}{m!} p_{k-m}^f(t) \int_0^\infty e^{-\lambda s} s^m \nu(ds) \\ = & - f(\lambda) p_k^f(t) + \sum_{m=1}^k \frac{\lambda^m}{m!} p_{k-m}^f(t) \int_0^\infty e^{-\lambda s} s^m \nu(ds). \end{aligned} \tag{3.45}$$

Clearly (3.45) coincides the with right-hand member of (3.35).

A further representation of $N^f(t)$, $t > 0$, can be obtained as the limit of a suitable compound Poisson process.

Theorem 3.2.4. *Let*

$$u(n) = \int_0^\infty \Pr\left\{N(s) \geq n\right\} \nu(ds), \qquad n \in \mathbb{N}, \tag{3.46}$$

where $N(s)$, $s > 0$, is a homogeneous Poisson process with rate $\lambda > 0$. The compound Poisson process

$$Z_n(t) = \sum_{j=1}^{N\left(\frac{t}{\lambda} u(n)\right)} X_j, \qquad t > 0, \tag{3.47}$$

where X_j, $j = 1, 2, \dots,$ are discrete i.i.d. r.v.'s with probability law

$$\Pr\{X_j = k\} = \frac{1}{u(n)} \int_0^\infty \Pr\{N(s) = k\} \; \nu(ds), \qquad k \geq n \in \mathbb{N}, \forall j = 1, 2, \dots, \tag{3.48}$$

converges in distribution to the subordinated Poisson process $N^f(t)$ as $n \to 0$. In other words

$$N^f(t) \overset{\text{law}}{=} N\left(H^f(t)\right) \overset{\text{law}}{=} \lim_{n\to 0} Z_n(t). \tag{3.49}$$

Proof. The p.g.f. of $Z_n(t)$ writes

$$\begin{aligned}
\mathbb{E}u^{Z_n(t)} &= e^{-tu(n)\left(1-\mathbb{E}u^X\right)} \\
&= \exp\left\{-tu(n)\sum_{k=0}^{\infty}\left(1-u^k\right)\Pr\{X=k\}\right\} \\
&\overset{(3.48)}{=} \exp\left\{-tu(n)\sum_{k=0}^{\infty}\left(1-u^k\right)\frac{1}{u(n)}\int_0^\infty \Pr\{N(s)=k\}\,\mathbb{I}_{[k\geq n]}\,\nu(ds)\right\} \\
&= \exp\left\{-t\int_0^\infty\sum_{k=n}^{\infty}\left(1-u^k\right)\Pr\{N(s)=k\}\,\nu(ds)\right\}.
\end{aligned} \tag{3.50}$$

By taking the limit for $n \to 0$ of (3.50) we have that

$$\begin{aligned}
\lim_{n\to 0}\mathbb{E}u^{Z_n(t)} &= \exp\left\{-t\int_0^\infty\sum_{k=0}^{\infty}\left(1-u^k\right)\Pr\{N(s)=k\}\,\nu(ds)\right\} \\
&= \exp\left\{-t\int_0^\infty\left(1-e^{-\lambda s(1-u)}\right)\nu(ds)\right\} \\
&= e^{-tf(\lambda(1-u))}.
\end{aligned} \tag{3.51}$$

□

Remark 3.2.5. *If we take into account processes whose probabilities satisfy the time-fractional equation*

$$\frac{d^\nu}{\partial t^\nu}p_k^f(t) = -f(\lambda)p_k^f(t) + \sum_{m=1}^{k}\frac{\lambda^m}{m!}p_{k-m}^f(t)\int_0^\infty e^{-s\lambda}s^m\nu(ds), \qquad k \geq 0, t > 0, \tag{3.52}$$

for $\nu \in (0,1)$ the corresponding p.g.f. has the form

$$G_\nu^f(u,t) = E_{\nu,1}\left(-t^\nu \int_0^\infty \left(1-e^{-s\lambda(1-u)}\right)\nu(ds)\right) \tag{3.53}$$

where $E_{\nu,1}(x)$ is the Mittag-Leffler function and the fractional derivative appearing in (3.52) must be understood in the Caputo sense. For the space fractional Poisson process $f(\lambda)=\lambda^\alpha$, $0<\alpha<1$, the distribution of the process related to (3.53) is explicitely given by formula (2.29) of Orsingher and Polito (2012). The processes whose distribution is governed by (3.52) admits the following representation

$$B\left(H^f\left(L^\nu(t)\right)\right), \qquad t>0, \tag{3.54}$$

where L^ν and the stable subordinator H^ν are related by

$$\Pr\{L^\nu(t)>x\} = \Pr\{H^\nu(x)<t\}. \tag{3.55}$$

3.3 Hitting-times of the subordinated Poisson process

In this section we study the hitting-times

$$T_k^f = \inf\left\{t>0 : N^f(t)\geq k\right\}, \tag{3.56}$$

of the subordinated Poisson processes. The fact that $N^f(t)$ performs jumps of random length makes T_k^f substantially different from the Erlang process related to the homogeneous Poisson process. Indeed, the law of T_k^f can be written down as follows

$$\begin{aligned}
&\Pr\left\{T_k^f \in ds\right\} \\
&= \Pr\left\{\bigcup_{j=1}^k \left\{N^f(s)=k-j,\, N^f[s,s+ds)\geq j\right\}\right\} \\
&= ds\sum_{j=1}^k \Pr\left\{N^f(s)=k-j\right\}\sum_{m=j}^\infty \Pr\left\{N^f[s,s+ds)=m\right\} \\
&= ds\sum_{j=1}^k \int_0^\infty \Pr\{N(z)=k-j\}\Pr\left\{H^f(s)\in dz\right\}\sum_{m=j}^\infty \frac{\lambda^m}{m!}\int_0^\infty e^{-\lambda u}u^m\,\nu(du) \\
&= ds\sum_{j=1}^k \int_0^\infty \frac{(\lambda z)^{k-j}}{(k-j)!}e^{-\lambda z}\Pr\left\{H^f(s)\in dz\right\}\int_0^\infty \Pr\{N(u)\geq j\}\,\nu(du)
\end{aligned}$$

$$= ds \sum_{j=1}^{k} \frac{(-\lambda)^{k-j}}{(k-j)!} \int_0^\infty \frac{d^{k-j}}{d\lambda^{k-j}} e^{-\lambda z} \Pr\left\{H^f(s) \in dz\right\} \int_0^\infty \Pr\left\{N(u) \geq j\right\} \nu(du)$$

$$= ds \sum_{j=1}^{k} \frac{(-\lambda)^{k-j}}{(k-j)!} \frac{d^{k-j}}{d\lambda^{k-j}} e^{-sf(\lambda)} \int_0^\infty \Pr\left\{N(u) \geq j\right\} \nu(du)$$

$$= ds \sum_{l=0}^{k-1} \frac{(-\lambda)^{l}}{l!} \frac{d^{l}}{d\lambda^{l}} e^{-sf(\lambda)} \int_0^\infty \left(1 - \sum_{r=0}^{k-l-1} \frac{(\lambda u)^r}{r!} e^{-\lambda u}\right) \nu(du). \tag{3.57}$$

The distribution of T_k^f can be also obtained by observing that

$$\begin{aligned} \Pr\left\{T_k^f < s\right\} &= \Pr\left\{N^f(s) \geq k\right\} \\ &= \sum_{j=k}^{\infty} \int_0^\infty e^{-\lambda z} \frac{(\lambda z)^j}{j!} \Pr\left\{H^f(s) \in dz\right\} \end{aligned} \tag{3.58}$$

and thus

$$\begin{aligned} \Pr\left\{T_k^f \in ds\right\} \Big/ ds &= \frac{d}{ds} \sum_{j=k}^{\infty} \int_0^\infty e^{-\lambda z} \frac{(\lambda z)^j}{j!} \Pr\left\{H^f(s) \in dz\right\} \\ &= \frac{d}{ds} \int_0^\infty \Pr\left\{N(z) \geq k\right\} \Pr\left\{H^f(s) \in dz\right\} \\ &= \frac{d}{ds} \int_0^\infty \left(1 - \sum_{l=0}^{k-1} \frac{(\lambda z)^l}{l!} e^{-\lambda z}\right) \Pr\left\{H^f(s) \in dz\right\} \\ &= -\frac{d}{ds} \sum_{l=0}^{k-1} \frac{(-\lambda)^l}{l!} \int_0^\infty \frac{d^l}{d\lambda^l} e^{-\lambda z} \Pr\left\{H^f(s) \in dz\right\} \\ &= -\frac{d}{ds} \sum_{l=0}^{k-1} \frac{(-\lambda)^l}{l!} \frac{dl^l}{d\lambda^l} e^{-sf(\lambda)}, \qquad s > 0. \end{aligned} \tag{3.59}$$

Remark 3.3.1. *In particular, we observe that from (3.57) and (3.59) we have that*

$$\Pr\left\{T_1^f \in ds\right\} = f(\lambda) e^{-sf(\lambda)}\, ds, \qquad s > 0, \tag{3.60}$$

This proves that the waiting time of the first event for all subordinated Poisson processes is exponential. Instead

$$\Pr\left\{T_2^f \in ds\right\} = e^{-sf(\lambda)} \left(f(\lambda) - \lambda f'(\lambda) + \lambda s f'(\lambda) f(\lambda)\right) ds, \qquad s > 0, \tag{3.61}$$

and for $f(\lambda) = \lambda$ (ordinary Poisson case) we recover the Gamma distribution with parameters $(2, \lambda)$. Result (3.61) can also be obtained from (3.57). For $f(\lambda) = \lambda^\alpha$ (space-fractional Poisson process) we have that

$$\Pr\left\{T_2^\alpha \in ds\right\} = ds \lambda^\alpha e^{-s\lambda^\alpha} \left(1 - \alpha + \lambda^\alpha s\right), \qquad s > 0. \tag{3.62}$$

Clearly (3.61) *cannot be the distribution of the sum of exponantial r.v.'s* (3.60) *because the second event can also be obtained as a jump of magnitude equal to two. Finally we observe that*

$$\Pr\left\{T_k^f \in ds\right\} = \Pr\left\{T_{k-1}^f \in ds\right\} - \frac{(-\lambda)^{k-1}}{(k-1)!}\frac{d}{ds}\frac{d^{k-1}}{d\lambda^{k-1}}e^{-sf(\lambda)}ds \qquad s \in (0,\infty), \tag{3.63}$$

so that the distributions of T_k can be derived successively.

Here we derive the equation governing the distribution of T_k. First we note that

$$\mathcal{G}^f(u,s) = \sum_{k=1}^{\infty} u^k \frac{\Pr\{T_k \in ds\}}{ds} = \frac{u}{1-u} f\left(\lambda(1-u)\right) e^{-sf(\lambda(1-u))}. \tag{3.64}$$

This can be proved as follows

$$\begin{aligned}
\mathcal{G}^f(u,s) &= \sum_{k=1}^{\infty} u^k \frac{\Pr\{T_k \in ds\}}{ds} \\
&= \frac{d}{ds}\sum_{j=1}^{\infty}\sum_{k=1}^{j} u^k \int_0^\infty e^{-\lambda z}\frac{(\lambda z)^j}{j!}\Pr\left\{H^f(s) \in dz\right\} \\
&= \frac{d}{ds}\sum_{j=1}^{\infty}\frac{u^{j+1}-u}{u-1}\int_0^\infty e^{-\lambda z}\frac{(\lambda z)^j}{j!}\Pr\left\{H^f(s) \in dz\right\} \\
&= \frac{d}{ds}\int_0^\infty \frac{u}{u-1}\left(e^{-\lambda z(1-u)}-1\right)\Pr\left\{H^f(s) \in dz\right\} \\
&= \frac{u}{1-u} f(\lambda(1-u))e^{-sf(\lambda(1-u))}
\end{aligned} \tag{3.65}$$

Theorem 3.3.2. *The probability density*

$$q_k^f(t) = \Pr\left\{T_k^f \in dt\right\} \Big/ dt \tag{3.66}$$

solves the equation

$$f(\lambda(1-B))q_k^f(t) = -\frac{d}{dt}q_k^f(t). \tag{3.67}$$

Proof. Since

$$f(\lambda(1-B))q_k^f(t) = f(\lambda)q_k^f(t) - \sum_{m=1}^{k-1}\int_0^\infty e^{-\lambda s}\frac{(\lambda s)^m}{m!}\nu(ds)q_{k-m}^f(t) \tag{3.68}$$

we can write, since $q_0 = 0$,

$$\sum_{k=1}^{\infty} u^k f(\lambda(1-B))q_k^f(t)$$

$$
\begin{aligned}
&= f(\lambda)\mathcal{G}^f(u,t) - \sum_{k=1}^{\infty} u^k \sum_{m=1}^{k} \int_0^{\infty} e^{-\lambda s} \frac{(\lambda s)^m}{m!} \nu(ds) q^f_{k-m}(t) \\
&= f(\lambda)\mathcal{G}^f(u,t) - \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} \left(\int_0^{\infty} e^{-\lambda s} \frac{(\lambda s)^m}{m!} \nu(ds) \right) u^k q^f_{k-m}(t) \\
&= f(\lambda)\mathcal{G}^f(u,t) - \mathcal{G}^f(u,t) \int_0^{\infty} e^{-\lambda s} \left(e^{u\lambda s} - 1 \right) \nu(ds) \\
&= f\left(\lambda(1-u)\right) \mathcal{G}^f(u,t). \qquad (3.69)
\end{aligned}
$$

From (3.65) we get

$$
f\left(\lambda(1-u)\right) \mathcal{G}^f(u,t) = -\frac{d}{dt}\mathcal{G}^f(u,t), \qquad (3.70)
$$

which completes the proof. □

3.4 Some particular cases

In this section we specialize the function f in order to analyse some particular cases of $N^f(t)$, $t > 0$.

3.4.1 The space-fractional Poisson process

If

$$
\nu(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds, \qquad \alpha \in (0,1), \qquad (3.71)
$$

we obtain the space-fractional Poisson process $N^\alpha(t)$, $t > 0$, studied in Orsingher and Polito (2012). The distributions of jumps (3.1) and (3.2) specialize to

$$
\Pr\left\{N^\alpha[t,t+dt) = k\right\} = \begin{cases} \frac{(-1)^{k+1}\lambda^\alpha}{k!} \alpha(\alpha-1)\cdots(\alpha-k+1)dt + o(dt), & k > 0 \\ 1 - \lambda^\alpha dt + o(dt), & k = 0, \end{cases} \qquad (3.72)
$$

since $f(\lambda) = \lambda^\alpha$. The distribution of N^α can be written in three different ways as

$$
\begin{aligned}
p_k^\alpha(t) = \Pr\left\{N^\alpha(t) = k\right\} &= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-k)} \\
&= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t)^r}{r!} (\alpha r)(\alpha r-1)\cdots(\alpha r-k+1) \\
&= \frac{(-1)^k}{k!} \frac{d^k}{du^k} e^{-t\lambda^\alpha u^\alpha} \bigg|_{u=1} \qquad (3.73)
\end{aligned}
$$

and we note that the probabilities (3.72) can be obtained directly from (3.73).

Remark 3.4.1. *In light of* (3.73) *the distribution of the space-fractional Poisson process has the following alternative form*

$$p_k^\alpha(t) = \frac{e^{-\lambda^\alpha t}}{k!}\left[c_{k,k}t^k + c_{k-1,k}t^{k-1} + \cdots + c_{2,k}t^2 + c_{1,k}t\right] \tag{3.74}$$

where the coefficients $c_{j,k}$*,* $j = 1,\ldots k$*, can be computed by means of successive derivatives. In particular, we have that*

$$\begin{aligned} c_{k,k} &= (\alpha\lambda^\alpha t)^k, & c_{k-1,k} &= \alpha^{k-1}(1-\alpha)\frac{k(k-1)}{2}(\lambda^\alpha t)^{k-1}, \\ c_{2,k} &= (\lambda^\alpha t)^2\alpha^2\prod_{j=1}^{k-2}(j-\alpha)\frac{k(k-1)}{2}, & c_{1,k} &= \alpha t\lambda^\alpha\prod_{j=1}^{k-1}(j-\alpha). \end{aligned} \tag{3.75}$$

For $\alpha = 1$ *all the coefficients* $c_{j,k}$*,* $j = 1,\ldots,k-1$*, are equal to zero and we recover from* (3.74) *the distribution of the homogeneous Poisson process. The coefficients* (3.75) *are sufficient to obtain* $p_j^\alpha(t)$*,* $1 \le j \le 4$ *as*

$$\begin{cases} p_2^\alpha(t) = \frac{e^{-\lambda^\alpha t}}{2}\left[(\lambda^\alpha\alpha t)^2 + \alpha(1-\alpha)\lambda^\alpha t\right] \\ p_3^\alpha(t) = \frac{e^{-\lambda^\alpha t}}{3!}\left[(\lambda^\alpha\alpha t)^3 + 3(\lambda^\alpha\alpha t)^2(1-\alpha) + (\lambda^\alpha\alpha t)(1-\alpha)(2-\alpha)\right] \\ p_4^\alpha(t) = \frac{e^{-\lambda^\alpha t}}{4!}\left[(\alpha\lambda^\alpha t)^4 + 6(\lambda^\alpha\alpha t)^3(1-\alpha) + 6(\alpha\lambda^\alpha t)^2(1-\alpha)(2-\alpha)\right. \\ \qquad \left. +\lambda^\alpha\alpha t(1-\alpha)(2-\alpha)(3-\alpha)\right] \end{cases} \tag{3.76}$$

Remark 3.4.2. *From* (3.73) *we obtain that, for* $k \ge 1$*,*

$$\begin{aligned} \Pr\{N^\alpha[t,t+dt) = k\} &= \frac{(-1)^{k+1}}{k!}\lambda^\alpha\alpha(\alpha-1)\cdots(\alpha-k+1)\,dt \\ &= \lambda^\alpha dt\frac{(-1)^{k+1}\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)}. \end{aligned} \tag{3.77}$$

Since

$$\begin{aligned} \sum_{k=1}^\infty \Pr\{N^\alpha[t,t+dt) = k\} &= \lambda^\alpha dt\,\Gamma(\alpha+1)\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k!\Gamma(\alpha+1-k)} \\ &= \frac{\lambda^\alpha\,dt\,\Gamma(\alpha+1)\sin\pi\alpha}{\pi}\sum_{k=1}^\infty\frac{\Gamma(k-\alpha)}{k!} \\ &= \frac{\lambda^\alpha dt\Gamma(\alpha+1)\sin\pi\alpha}{\pi}\int_0^\infty e^{-w}\sum_{k=1}^\infty\frac{w^{k-\alpha-1}}{k!}dw \\ &= \frac{\lambda^\alpha\,dt\,\Gamma(\alpha+1)\sin\pi\alpha}{\pi}\int_0^\infty e^{-w}\left(e^w-1\right)w^{-\alpha-1}dw \\ &= \frac{\lambda^\alpha dt\sin\pi\alpha\Gamma(\alpha+1)\Gamma(1-\alpha)}{\alpha\pi} = \lambda^\alpha dt \end{aligned} \tag{3.78}$$

we get

$$\Pr\{N^\alpha[t,t+dt) = 0\} = 1 - \lambda^\alpha dt \tag{3.79}$$

Remark 3.4.3. *In light of the independence of increments for the space-fractional Poisson process we have that*

$$\begin{aligned}\Pr\{N^\alpha(s)=r|N^\alpha(t)=k\} &= \frac{\Pr\{N^\alpha(s)=r\}\Pr\{N^\alpha(t-s)=k-r\}}{\Pr\{N^\alpha(t)=k\}}\\ &= \binom{k}{r}\left.\frac{\frac{d^r}{du^r}e^{-s\lambda^\alpha u^\alpha}\,\frac{d^{k-r}}{du^{k-r}}e^{-(t-s)\lambda^\alpha u^\alpha}}{\frac{d^k}{du^k}e^{-\lambda^\alpha t u^\alpha}}\right|_{u=1}\\ &= \binom{k}{r}\frac{\sum_{j=1}^r c_{j,r}s^j\,\sum_{n=1}^{k-r}c_{n,k-r}(t-s)^n}{\sum_{l=1}^k c_{l,k}t^l}, \qquad (3.80)\end{aligned}$$

where we used (3.74). For $\alpha=1$ we get that $c_{r,r}, c_{k-r,k-r}, c_{k,k}\neq 0$ and $c_{j,r}=c_{n,k-r}=c_{l,k}=0$, for $j<r$, $n<k-r$, $l<k$ and thus we recover from (3.80) the binomial distribution.

In the time interval $[0,t]$ the instants of occurences of the upward jumps are denoted by $\tau_j^{l_j}$, $1\le j\le r$, $l_j\ge 1$, where r is the number of jumps in $[0,t]$ and l_j is the height of the j-jumps. We can write the following distribution

$$\Pr\left\{\bigcap_{j=1}^r\left\{\tau_j^{l_j}\in dt_j\right\}\middle|N^\alpha(t)=k\right\} = \frac{k!\,(\lambda^\alpha\Gamma(\alpha+1))^r\,(-1)^{k+r}\prod_{j=1}^r\frac{dt_j}{l_j!\Gamma(\alpha+1-l_j)}}{\sum_{n=1}^k c_{n,k}t^n} \qquad (3.81)$$

for $0<t_1<\dots<t_r<t$, where we used the independence of the increments and (3.72). If $N^\alpha(t)=k$, and $l_j=1$, $\forall j$, we have that

$$\Pr\left\{\bigcap_{j=1}^k\{\tau_j^1\in dt_j\}\middle|N^\alpha(t)=k\right\} = \frac{k!\,(\alpha\lambda^\alpha)^k}{\sum_{j=1}^k c_{j,k}t^j} \qquad (3.82)$$

on the simplex

$$S_t=\{t_i, i=1,\dots,k: 0<t_1<t_2<\dots<t_k<t\}. \qquad (3.83)$$

Clearly, for $\alpha=1$, we retrive from (3.82) the uniform distribution on the set S_t. Since the coefficients $c_{j,k}$ can be calculated in some specific cases, the distribution can be written down explicitely for small values of k. For example, for $k=2$ we have that

$$\Pr\left\{\bigcap_{j=1}^2\{\tau_j^1\in dt_j\}\middle|N^\alpha(t)=2\right\} = \frac{(\alpha\lambda^\alpha)^2\,dt_1dt_2}{(\alpha\lambda^\alpha t)^2+\alpha(1-\alpha)\lambda^\alpha t}, \qquad 0<t_1<t_2<t, \qquad (3.84)$$

$$\Pr\{\tau_1^2\in dt_1|N^\alpha(t)=2\} = \frac{\frac12\alpha(1-\alpha)\lambda^\alpha dt_1}{(\alpha\lambda^\alpha t)^2+\alpha(1-\alpha)\lambda^\alpha t}, \qquad 0<t_1<t. \qquad (3.85)$$

3.4.2 Poisson process with relativistic (tempered) stable subordinator

In the case the Lévy measure has the form

$$\nu(ds) = \frac{\alpha s^{-\alpha-1}e^{-\theta s}}{\Gamma(1-\alpha)}ds, \qquad \theta > 0, 0 < \alpha < 1, \tag{3.86}$$

we obtain an extension of the space-fractional Poisson process. This new Poisson process has the form $N^{\alpha,\theta}(t) = N\left(H^{\alpha,\theta}(t)\right)$ where $H^{\alpha,\theta}$ is the relativistic or tempered stable subordinator. Such process is calles relativistic since it appeared in the study of the stability of the relativistic matter (see Lieb (1990)). From (3.38) we obtain the p.g.f. as

$$\begin{aligned}
G^{\alpha,\theta}(u,t) &= \mathbb{E}u^{N^{\alpha,\theta}(t)} \\
&= \exp\left\{-t\int_0^\infty \left(1-e^{-\lambda(1-u)s}\right)\frac{\alpha s^{-\alpha-1}e^{-\theta s}}{\Gamma(1-\alpha)}ds\right\} \\
&= e^{-t\{[\theta+\lambda(1-u)]^\alpha-\theta^\alpha\}} \\
&= e^{\theta^\alpha t}\sum_{k=0}^\infty \frac{[-t\left(\theta+\lambda(1-u)\right)]^\alpha}{k!} \\
&= e^{\theta^\alpha t}\sum_{k=0}^\infty \frac{[-t\left(\theta+\lambda\right)^\alpha]^k}{k!}\left(1-\frac{\lambda u}{\theta+\lambda}\right)^{\alpha k} \\
&= e^{\theta^\alpha t}\sum_{k=0}^\infty \frac{(-t(\theta+\lambda))^k}{k!}\sum_{m=0}^\infty \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-m)m!}\left(-\frac{\lambda u}{\theta+\lambda}\right)^m \\
&= \sum_{m=0}^\infty u^m\left[\frac{(-1)^m}{m!}\frac{\lambda^m e^{\theta^\alpha t}}{(\theta+\lambda)^m}\sum_{k=0}^\infty \frac{[-t(\theta+\lambda)^\alpha]^k}{k!}\frac{\Gamma\left(\alpha k+1\right)}{\Gamma\left(\alpha k+1-m\right)}\right].
\end{aligned} \tag{3.87}$$

From (3.87) we extract the distribution of $N^{\alpha,\theta}$ as follows

$$\Pr\left\{N^{\alpha,\theta}(t)=m\right\} = \frac{(-1)^m}{m!}\frac{\lambda^m e^{\theta^\alpha t}}{(\theta+\lambda)^m}\sum_{k=0}^\infty \frac{\left(-t\left(\lambda+\theta\right)^\alpha\right)^k}{k!}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-m)}, \qquad m \geq 0. \tag{3.88}$$

For $\theta = 0$, formula (3.88) yields the distribution of the space-fractional Poisson process (see formula (1.2) of Orsingher and Polito (2012)). An alternative form of (3.88) is

$$\Pr\left\{N^{\alpha,\theta}(t)=m\right\} = \frac{(-1)^m}{m!}\left(\frac{\lambda}{\lambda+\theta}\right)^m e^{\theta^\alpha t}\frac{d^m}{du^m}e^{-tu^\alpha(\theta+\lambda)^\alpha}\bigg|_{u=1} \tag{3.89}$$

and can be derived either from (3.88) or from (3.43). From (3.89) (and also from (3.1)) we have that, for $m \geq 1$,

$$\Pr\left\{N^{\alpha,\theta}[t,t+dt)=m\right\} = \frac{(-1)^{m+1}\left(\frac{\lambda}{\lambda+\theta}\right)^m}{m!}(\lambda+\theta)^\alpha \alpha(\alpha-1)\cdots(\alpha-m+1)dt \tag{3.90}$$

and this represents the distribution of the jumps during $[t,t+dt)$. Formula (3.90) shows that high jumps have less probability to occur than in the space-fractional Poisson process. Since

$$\begin{aligned}
&\sum_{m=1}^\infty \Pr\left\{N^{\alpha,\theta}[t,t+dt)=m\right\} \\
&= \sum_{m=1}^\infty \frac{(-1)^m}{m!}\left(\frac{\lambda}{\lambda+\theta}\right)^m (\lambda+\theta)^\alpha \alpha(\alpha-1)\cdots(\alpha-m+1)\,dt \\
&= dt\,(\theta+\lambda)^\alpha \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m!}\left(\frac{\lambda}{\lambda+\theta}\right)^m \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-m)} \\
&= dt\,(\theta+\lambda)^\alpha \frac{\Gamma(\alpha+1)\sin\pi\alpha}{\pi}\sum_{m=1}^\infty \left(\frac{\lambda}{\lambda+\theta}\right)^m \frac{1}{m!}\Gamma(m-\alpha) \\
&= dt\,(\theta+\lambda)^\alpha \frac{\Gamma(\alpha+1)\sin\pi\alpha}{\pi}\int_0^\infty w^{-\alpha-1}e^{-w}\sum_{m=1}^\infty \left(\frac{w\lambda}{\lambda+\theta}\right)^m \frac{1}{m!}dw \\
&= dt\,(\theta+\lambda)^\alpha \frac{\Gamma(\alpha+1)\sin\pi\alpha}{\pi}\int_0^\infty w^{-\alpha-1}\left[e^{-\frac{\theta w}{\lambda+\theta}}-e^{-w}\right]dw \\
&= dt\,(\theta+\lambda^\alpha)\frac{\sin\pi\alpha}{\pi}\left[\Gamma(1-\alpha)\Gamma(\alpha)-\left(\frac{\theta}{\lambda+\theta}\right)^\alpha\Gamma(1-\alpha)\Gamma(\alpha)\right] \\
&= dt\left[(\theta+\lambda)^\alpha-\theta^\alpha\right]
\end{aligned} \tag{3.91}$$

we get

$$\Pr\left\{N^{\alpha,\theta}[t,t+dt)=0\right\} = 1-dt\left[(\theta+\lambda)^\alpha-\theta^\alpha\right] \tag{3.92}$$

Remark 3.4.4. *We notice that*

$$\begin{aligned}
\mathbb{E}N^{\alpha,\theta}(t) &= \lambda\theta^{\alpha-1}t, \\
\mathrm{Var}\left[N^{\alpha,\theta}(t)\right] &= \lambda t\theta^{\alpha-2}\left(\lambda(1-\alpha)+\theta\right), \\
\mathrm{Cov}\left[N^{\alpha,\theta}(t)N^{\alpha,\theta}(s)\right] &= \lambda s\theta^{\alpha-2}\left(\lambda(1-\alpha)+\theta\right)(s\wedge t).
\end{aligned} \tag{3.93}$$

From (3.93) it is apparent that in the space-fractional Poisson process ($\theta=0$) the mean values diverge.

3.4.3 Poisson process with Gamma subordinator

For the Lévy measure

$$\nu(ds) = \frac{e^{-s}}{s}ds, \qquad s > 0, \tag{3.94}$$

the distribution of the related Poisson process has a particularly simple and interesting form. We note that the Bernštein function corresponding to the Lévy measure $\nu(ds) = \frac{e^{-s}}{s}ds$ is

$$f(x) = \int_0^\infty \left(1 - e^{-sx}\right) \frac{e^{-s}}{s}ds = \log(1+x). \tag{3.95}$$

Therefore the probability generating function (3.38) reduces to the form

$$G^\Gamma(u,t) = e^{-t\log(1+\lambda(1-u))} = (1+\lambda(1-u))^{-t}, \tag{3.96}$$

and thus the intertime T between successive clusters of events has law

$$\Pr\{T > t\} = \frac{1}{(1+\lambda)^t}. \tag{3.97}$$

Formula (3.96) is clearly the p.g.f. of $N^\Gamma(t) \overset{\text{law}}{=} N\left(H^\Gamma(t)\right)$ where H^Γ is the Gamma subordinator with Laplace transform

$$\mathbb{E}e^{-\mu t H^f(t)} = (1+\mu)^{-t}. \tag{3.98}$$

The distribution of $N^\Gamma(t)$, $t > 0$, can be extracted from (3.96) as the next Theorem shows.

Theorem 3.4.5. *The process $N^\Gamma(t)$, $t > 0$, has the following distribution*

$$\begin{aligned}\Pr\left\{N^\Gamma(t) = k\right\} &= \begin{cases} \frac{\lambda^k t(t+1)\cdots(t+k-1)}{k!}\frac{1}{(\lambda+1)^{t+k}}, & k \geq 1 \\ \frac{1}{(1+\lambda)^t}, & k = 0, \end{cases} \\ &= \frac{\lambda^k\Gamma(k+t)}{\Gamma(t)k!\,(\lambda+1)^{t+k}}. \end{aligned} \tag{3.99}$$

Proof. From (3.96) we have that

$$\begin{aligned} G^\Gamma(u,t) &= (1+\lambda(1-u))^{-t} \\ &= \left(1 - \frac{\lambda u}{1+\lambda}\right)^{-t}(1+\lambda)^{-t} \\ &= (1+\lambda)^{-t}\sum_{k=0}^\infty \frac{\Gamma(-t+1)}{k!\Gamma(-t+1-k)}\left(-\frac{\lambda u}{1+\lambda}\right)^k \end{aligned}$$

$$
\begin{aligned}
&= (1+\lambda)^{-t} \sum_{k=0}^{\infty} u^k \left(-\frac{\lambda}{1+\lambda}\right)^k \frac{\Gamma(t+k)}{k!\Gamma(t)}(-1)^k \\
&= \sum_{k=0}^{\infty} u^k \left[\frac{\lambda^k \Gamma(t+k)}{k!\Gamma(t)} \frac{1}{(1+\lambda)^{t+k}}\right]. \qquad (3.100)
\end{aligned}
$$

A second, alternative derivation of (3.99) proceeds as follows

$$
\begin{aligned}
\Pr\left\{N^{\Gamma}(t)=k\right\} &= \frac{1}{k!}\frac{\partial^k}{\partial u^k} G^{\Gamma}(u,t)\Big|_{u=0} \\
&= \frac{(-1)^k \lambda^k (-t)(-t-1)\cdots(-t-k+1)}{k!} (1+\lambda(1-u))^{-t-k}\Big|_{u=0} \\
&= \frac{\lambda^k t(t+1)\cdots(t+k-1)}{k!} \frac{1}{(\lambda+1)^{t+k}}. \qquad (3.101)
\end{aligned}
$$

The probability of zero events is therefore

$$
\Pr\left\{N^{\Gamma}(t)=0\right\} = G^{\Gamma}(0,t) = \frac{1}{(1+\lambda)^t}. \qquad (3.102)
$$

□

Remark 3.4.6. *The distribution (3.101) of $N^{\Gamma}(t)$ can also be written as*

$$
\Pr\left\{N^{\Gamma}(t)=k\right\} = \frac{\lambda^k}{(1+\lambda)^{k+t}} \frac{\Gamma(k+t)}{\Gamma(t)} \frac{1}{k!} = \mathbb{E}\Pr\left\{N(\mathcal{T})=k\right\} \qquad (3.103)
$$

where $\mathcal{T}$ is gamma distributed with parameters $(1,t)$ (that is the distribution of H^{Γ}) and N is a homogeneous Poisson process with parameter λ, independent from $\mathcal{T}$. Furthermore (3.99) can be regarded as an extension of the negative binomial $\mathcal{B}^i$ where

$$
\Pr\left\{\mathcal{B}^i = k\right\} = \frac{\Gamma(i+k)}{\Gamma(i)\Gamma(k+1)} p^i q^k \qquad (3.104)
$$

for $i=t$, $p=1/(1+\lambda)$, $q=\lambda/(1+\lambda)$ (see also Kozubowski and Podgórski (2009)).

Corollary 3.4.7. *The distribution of jumps in this case has the form*

$$
\Pr\left\{N^{\Gamma}[t,t+dt)=k\right\} = \begin{cases} \left(\frac{\lambda}{\lambda+1}\right)^k \frac{1}{k} dt, & k \geq 1, \\ 1-\log(1+\lambda)dt, & k=0, \end{cases} \qquad (3.105)
$$

as can be inferred from (3.1) and also from (3.99). The jumps possess logarithmic distribution.

Remark 3.4.8. *We observe that, for $s<t$, $r \leq k$,*

$$
\Pr\left\{N^{\Gamma}(s)=r | N^{\Gamma}(t)=k\right\} = \binom{k}{r} \frac{\Gamma(t)}{\Gamma(t-s)\Gamma(s)} \frac{\Gamma(s+r)\Gamma(t-s+k-r)}{\Gamma(k+t)}
$$

$$= \binom{k}{r} \frac{B(s+r, t-s+k-r)}{B(s,t-s)}. \qquad (3.106)$$

Furthermore from (3.106) *we can write, for* $0 \le r \le k$,

$$\begin{aligned} \Pr\left\{N^{\Gamma}(s) = r | N^{\Gamma}(t) = k\right\} &= \binom{k}{r} \frac{\int_0^1 x^{s+r-1}(1-x)^{t-s+k-r-1}dx}{B(s,t-s)} \\ &= \mathbb{E}\left[\binom{k}{r} X^r (1-X)^{k-r}\right] \end{aligned} \qquad (3.107)$$

where X *is a r.v. with Beta distribution with parameter* s *and* $t-s$, *that is*

$$\Pr\left\{X \in dx\right\} = \frac{x^{s-1}(1-x)^{t-s-1}}{B(s,t-s)} dx. \qquad (3.108)$$

Formula (3.107) *shows that in the Gamma Poisson process the conditional number of events at time* $s < t$ *is a randomized Bernoulli if* $N(t) = k$.

Remark 3.4.9. *In view of* (3.99), (3.105), *and the independence of the increments of the Gamma Poisson process we have that*

$$\Pr\left\{\bigcap_{j=1}^{r}\left\{\tau_j^{l_j} \in dt_j\right\} \middle| N^{\Gamma}(t) = k\right\} = \frac{k!\Gamma(t)}{\Gamma(t+k)} \prod_{j=1}^{r} \frac{dt_j}{l_j} \qquad (3.109)$$

on the simplex $0 < t_1 < t_2 < \cdots < t_r < t$ *and* $\sum_{j=1}^{r} l_j = k$. *Some special cases of* (3.109) *are*

i) $l_j = 1$, $\forall j = 1, \ldots, r$, *and thus* $r = k$. *In this case we have that*

$$\Pr\left\{\bigcap_{j=1}^{k}\left\{\tau_j^{1} \in dt_1\right\} \middle| N^{\Gamma}(t) = k\right\} = \frac{k!\Gamma(t)}{\Gamma(t+k)} \prod_{j=1}^{k} dt_j, \qquad 0 < t_1 < \cdots < t_k < t; \qquad (3.110)$$

ii) $l_1 = k$ *and thus* $r = 1$ *(unique jump of length* k*). Here we get*

$$\Pr\left\{\tau_1^k \in dt_1 | N^{\Gamma}(t) = k\right\} = \frac{dt_1}{k} \frac{k!\Gamma(t)}{\Gamma(t+k)}, \qquad 0 < t_1 < t; \qquad (3.111)$$

iii) $k = 2m$, $l_j = 2$, $\forall j$, *and therefore* $r = m$. *We have that*

$$\Pr\left\{\bigcap_{j=1}^{m}\left\{\tau_j^{2} \in dt_j\right\} \middle| N^{\Gamma}(t) = 2m\right\} = \frac{(2m)!\,\Gamma(t)}{2^m\Gamma(t+2m)} \prod_{j=1}^{m} dt_j, \qquad (3.112)$$

for $0 < t_1 < \cdots < t_m < t$.

Remark 3.4.10. *From* (3.96) *we obtain the* n*-th factorial moment of* $N^\Gamma(t)$*,* $t>0$*, as*

$$\mathbb{E}\left[N^\Gamma(t)\left(N^\Gamma(t)-1\right)\cdots\left(N^\Gamma(t)-r+1\right)\right] = \lambda^r t(t+1)\cdots(t+r-1). \quad (3.113)$$

While $\mathbb{E}N^\Gamma(t)=\lambda t$*, the variance becomes* $\mathrm{Var}N^\Gamma(t)=\lambda t(\lambda+1)$ *and*

$$\mathrm{Cov}\left[N^\Gamma(t),N^\Gamma(s)\right] = \lambda(\lambda+1)(s\wedge t). \quad (3.114)$$

Furthermore we have that

$$\begin{aligned}\mathbb{E}\left[\int_0^t N^\Gamma(s)ds\right] &= \lambda t^2/2 \\ \mathrm{Var}\left[\int_0^t N^\Gamma(s)ds\right] &= \lambda(\lambda+1)t^3/3\end{aligned} \quad (3.115)$$

Remark 3.4.11. *We can write also the following conditional mean values*

$$\mathbb{E}\left[N^\Gamma(s)|N^\Gamma(t)=k\right] = \frac{ks}{t}, \qquad 0<s<t, \quad (3.116)$$

$$\begin{aligned}&\mathbb{E}\left[N^\Gamma(s)N^\Gamma(w)|N^\Gamma(t)=k\right] \\ &= \frac{ks}{t}+k(k-1)\frac{(s(s+1)}{t(t+1)}+k(k-1)\frac{s(w-s)}{t(t+1)}, \quad \text{for } 0<s<w<t\end{aligned} \quad (3.117)$$

$$\begin{aligned}&\mathrm{Cov}\left[N^\Gamma(s),N^\Gamma(w)|N^\Gamma(t)=k\right] \\ &= \frac{k}{t(t+1)}\left(1+\frac{k}{t}\right)\min(s,t)\,\min(t-s,t-w).\end{aligned} \quad (3.118)$$

As a special case we extract from (3.118) *the conditional variance as*

$$\mathrm{Var}\left[N^\Gamma(s)|N^\Gamma(t)=k\right] = \frac{sk(t-s)}{t(t+1)}\left(1+\frac{k}{t}\right), \qquad 0<s<t, \quad (3.119)$$

and from (3.117)

$$\mathbb{E}\left[\left(N^\Gamma(s)\right)^2|N^\Gamma(t)=k\right] = \frac{s}{t}k+k(k-1)\frac{s}{t}\frac{s+1}{t+1}. \quad (3.120)$$

As a check we observe that

$$\begin{aligned}\mathrm{Var}N^\Gamma(s) &= \mathbb{E}\left[\mathrm{Var}N^\Gamma(s)|N^\Gamma(t)\right]+\mathrm{Var}\left[\mathbb{E}\left[N^\Gamma(s)|N^\Gamma(t)\right]\right] \\ &= \frac{s(t-s)}{t(t+1)}\mathbb{E}N^\Gamma(t)+\frac{s(t-s)}{t^2(t+1)}\mathbb{E}\left(N^\Gamma\right)^2(t)+\frac{s^2}{t^2}\mathrm{Var}N(t) \\ &= \frac{s}{t}\frac{t-s}{t+1}\lambda t+\frac{s}{t^2}\left(\lambda(\lambda+1)t+\lambda^2t^2\right)+\frac{s^2}{t^2}\lambda(\lambda+1)t \\ &= \lambda s(1+\lambda).\end{aligned} \quad (3.121)$$

Remark 3.4.12. *We consider here the distribution of* $N_1^\Gamma(t) - N_2^\Gamma(t)$, $t > 0$, *where* N_j^Γ, $j = 1, 2$, *are independent Gamma Poisson processes. This leads to a generalization of the Skellam law of the difference of independent homogenous Poisson processes. We have that*

$$\begin{aligned}
&\Pr\left\{N_1^\Gamma(t) - N_2^\Gamma(t) = r\right\} \\
=&\sum_{k=0}^{\infty} \frac{\lambda^k \Gamma(k+t) \lambda^{k+r} \Gamma(k+r+t)}{(1+\lambda)^{k+t}\Gamma(t) k! (1+\lambda)^{k+r+t}(k+r)!\Gamma(t)} \\
=&\frac{1}{(1+\lambda)^{2t}\Gamma^2(t)} \sum_{k=0}^{\infty} \frac{\lambda^{2k+r}}{(1+\lambda)^{2k+r} k!(k+r)!} \int_0^\infty dw \int_0^\infty dz \, e^{-w-z} w^{k+t-1} z^{k+r+t-1} \\
=&\frac{1}{(1+\lambda)^{2t}\Gamma^2(t)} \int_0^\infty \int_0^\infty dw \, dz \, e^{-w-z} w^{t-\frac{r}{2}-1} z^{\frac{r}{2}+t-1} \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda}{1+\lambda}\sqrt{wz}\right)^{2k+r}}{k!(k+r)!} \\
=&\frac{1}{(1+\lambda)^{2t}\Gamma^2(t)} \int_0^\infty \int_0^\infty e^{-w-z} w^{t-\frac{r}{2}-1} z^{\frac{r}{2}+t-1} I_r\left(\frac{2\lambda\sqrt{wz}}{1+\lambda}\right) \\
=&\int_0^\infty \int_0^\infty \Pr\left\{N_1^u(1) - N_2^y(1) = r\right\} e^{-\frac{u+y}{\lambda}} \frac{(uy)^{t-1} du dy}{\lambda^{2t}\Gamma^2(t)} \\
=&\mathbb{E} \Pr\left\{N_1^U(1) - N_2^Y(1) = r\right\} \\
=&\mathbb{E} \Pr\left\{N_1^1(U) - N_2^1(Y) = r\right\}
\end{aligned} \tag{3.122}$$

where U *and* Y *are independent Gamma r.v.'s with parameter* 1 *and* t, *and* $I_0(x)$ *is a Bessel function. For the reader's convenience we recall that the Skellam distribution reads*

$$\Pr\left\{N_1^\lambda(t) - N_2^\beta(t) = r\right\} = e^{-(\beta+\lambda)t} \left(\frac{\lambda}{\beta}\right)^{\frac{r}{2}} I_{|r|}\left(2t\sqrt{\lambda\beta}\right), \qquad r \in \mathbb{Z}, \tag{3.123}$$

for independent Poisson processes N_1^λ, N_2^β, *with rate* λ, β.

Chapter 4

Convolution-type derivatives and time-changed C_0-semigroups

Article: Toaldo (2013). Convolution-type derivatives, hitting-times of subordinators and time-changed C_0-semigroups.

Summary

This paper takes under consideration subordinators and their inverse processes (hitting-times). The governing equations of such processes is presented by means of convolution-type integro-differential operators similar to the fractional derivatives. Furthermore the concept of time-changed C_0-semigroup is discussed in case the time-change is performed by means of the hitting-time of a subordinator. Such time-change gives rise to bounded linear operators governed by integro-differential time-operators. Because these operators are non-local the presence of long-range dependence is investigated.

4.1 Introduction

The study of subordinators and their hitting-times has attracted the attention of many researchers since the Fourties. In particular a great effort has been dedicated to the study of the relationships between Bochner subordination and Cauchy problems (Bochner (1949, 1955)). See Feller (1966), Jacob (2001), Schilling et al. (2010) and the references therein for more information on Bochner subordination. A subordinator ${}^f\sigma(t)$, $t > 0$, is a Lévy process with stationary and independent increments and non-decreasing paths for which $\mathbb{E}e^{-\lambda\, {}^f\sigma(t)} = e^{-tf(\lambda)}$ where f is a Bernstein function (see Bertoin (1996, 1997) for more details on subordinators). Its inverse process is defined as

$$ {}^{f}L(t) \;=\; \inf\left\{ s > 0 : {}^{f}\sigma(s) > t \right\} \tag{4.1} $$

and is the hitting-time of ${}^{f}\sigma$. When the function f is $f(\lambda) = \lambda^{\alpha}$, $\alpha \in (0,1)$, the related subordinator is called the *α-stable subordinator* and the inverse process $L^{\alpha}(t) = \inf\{s > 0 : \sigma^{\alpha}(s) > t\}$ is called the *inverse stable subordinator* (see Meerschaert and Sikorskii (2012), Meerschaert and Straka (2013), Samorodnitsky and Taqqu (1944) for more information on the stable subordinator and its inverse process). The relationships between such processes and partial differential equations have been object of intense study in the past three decades and have gained considerable popularity together with the study of fractional calculus (for fractional calculus the reader can consult Kilbas et al. (2006)). As pointed out in Orsingher and Beghin (2004, 2009), fractional PDEs are indeed related to time-changed processes while the relationships between time-fractional Cauchy problems and the inverse of the stable subordinator was explored for the first time by Baeumer and Meerschaert (2001), Meerschaert et al. (2009), Saichev and Zaslavsky (1997), Zaslavsky (1994). Equations of fractional order appear in a lot of physical phenomena (Meerschaert and Sikorskii (2012)) and in particular for modeling anomalous diffusions (see for example Benson et al. (2001), D'Ovidio (2012)).

In the present paper we deal with the inverse processes ${}^{f}L(t)$, $t > 0$, of subordinators ${}^{f}\sigma(t)$, $t > 0$, with Laplace exponent the Bernstein function f having the following representation

$$ f(x) \;=\; a + bx + \int_0^{\infty} \left(1 - e^{-sx}\right) \bar{\nu}(ds) \tag{4.2} $$

for a non-negative measure $\bar{\nu}$ on $(0,\infty)$ (Bernstein (1929), Schilling et al. (2010)). We consider the case in which the tail $s \to \nu(s) = a + \bar{\nu}(s,\infty)$ is absolutely continuous on $(0,\infty)$ and we define integro-differential operators similar to the fractional derivatives. In particular we show how the operator

$$ {}^{f}\mathfrak{D}_t u(t) \;=\; b\frac{d}{dt}u(t) + \int_0^t \frac{\partial}{\partial t}u(t-s)\,\nu(s)\,ds \tag{4.3} $$

allows us to write the governing equations of

$$ \mathcal{T}_t u \;=\; \int_0^{\infty} T_s u\, l_t(ds), \qquad u \in \mathfrak{B}, \tag{4.4} $$

where $l_t(B) = \Pr\left\{ {}^{f}L(t) \in B \right\}$ are the transition probabilities of ${}^{f}L$ and T_s is a C_0-semigroup on the Banach space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$. We call the operator $\mathcal{T}_t$ a time-changed

C_0-semigroup. In fact the main result of the present paper shows that $\mathcal{T}_t u$, $u \in \mathfrak{B}$, is a bounded strongly continuous linear operator on $\mathfrak{B}$ and solves the problem

$$\begin{cases} {}^f\mathfrak{D}_t q(t) = Aq(t), & 0 < t < \infty, \\ q(0) = u \in \mathrm{Dom}\,(A)\,. \end{cases} \tag{4.5}$$

where A is the infinitesimal generator of the C_0-semigroup $T_t u$, $u \in \mathfrak{B}$.

A central role in our analysis is played by the tail $\nu(s)$ of the Lévy measure $\bar{\nu}$ since it emerges through all the results of the paper. It appears in the definitions of convolution-type derivatives of the form (4.3) we will discuss in Section 4.2. Furthermore we prove the following convergence in distribution

$$\lim_{\gamma \to 0} \left(bt + \sum_{j=1}^{N(t\,\nu(\gamma))} Y_j \right) \overset{\text{law}}{=} {}^f\sigma(t), \qquad t > 0, \tag{4.6}$$

where Y_j are i.i.d. random variables with distribution

$$\Pr\{Y_j \in dy\} = \frac{1}{\nu(\gamma)} \left(\bar{\nu}(dy) + a\delta_\infty\right) \mathbb{I}_{y > \gamma}, \qquad \gamma > 0, \forall j = 1, \ldots, n, \tag{4.7}$$

and $N(t)$, $t > 0$, is a homogeneous Poisson process. The symbol δ_∞ stands for the Dirac point mass at infinity.

List of symbols

Here is a list of the most important notations adopted in the paper.

- With $\mathcal{L}\left[u(\cdot)\right](\lambda) = \widetilde{u}(\lambda)$ we denote the Laplace transform of the function u.
- $\mathcal{F}\left[u(\cdot)\right](\xi) = \widehat{u}(\xi)$ indicates the Fourier transform of the function u.
- With ${}^f\sigma(t)$, $t > 0$, we denote the subordinator with Laplace exponent f.
- $\mu_t(B) = \Pr\left\{ {}^f\sigma(t) \in B \right\}$ indicates the convolution semigroup (transition probabilities) associated with the subordinator ${}^f\sigma(t)$, $t > 0$. When the measure μ_t has a density we adopt the abuse of notation $\mu_t(ds) = \mu_t(s)ds$ where $\mu_t(s)$ indicates the density of μ_t.
- ${}^fL(t)$, $t > 0$, indicates the inverse of the subordinator ${}^f\sigma(t)$, $t > 0$.
- The symbol $l_t(B) = \Pr\left\{ {}^fL(t) \in B \right\}$ indicates the transition probabilities of ${}^fL(t)$, $t > 0$. In case l_t has a density we denote it, by abuse of notation, as $l_t(s)$.
- With A we denote the infinitesimal generator of the semigroup $T_t u$ for $u \in \mathcal{B}$ ($\mathcal{B}$ is a Banach space).

4.2 Convolution-type derivatives

In this section we define convolution-type operators similar to the fractional derivatives. The logic of our definitions starts from the observation of the fractional derivative of order $\alpha \in (0,1)$ (in the Riemann-Liouville sense) to be considered the first-order derivative of the Laplace convolution $u(t) * t^{-\alpha}/\Gamma(1-\alpha)$ (see Kilbas et al. (2006))

$$\frac{d^\alpha}{dt^\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{u(s)}{(t-s)^\alpha}\,ds. \tag{4.8}$$

Formula (4.8) can be formally viewed as $\left(\frac{d}{dt}\right)^\alpha$ for $\alpha \in (0,1)$. Here we generalize this idea respect to a Bernstein function (Bernstein (1929)). A Bernstein function is a function $f(x) : (0,\infty) \to \mathbb{R}$ of class C^∞, $f(x) \geq 0$, $\forall x > 0$ for which

$$(-1)^k f^{(k)}(x) \leq 0, \qquad \forall x > 0 \text{ and } k \in \mathbb{N}. \tag{4.9}$$

A function f is said to be a Bernstein function if, and only if, admits the representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-sx}\right)\bar{\nu}(ds), \qquad x > 0, \tag{4.10}$$

where $a, b \geq 0$ and $\bar{\nu}(ds)$ is a non-negative measure on $(0,\infty)$ satisfying the integrability condition

$$\int_0^\infty (z \wedge 1)\,\bar{\nu}(dz) < \infty. \tag{4.11}$$

According to the literature we refer to the measure $\bar{\nu}$ and to the triplet $(a, b, \bar{\nu})$ as the *Lévy measure* and the *Lévy triplet* of the Bernstein function f. The representation (4.10) is called the *Lévy-Khintchine* representation of f.

The Bernstein functions are closely related to the so-called completely monotone functions (see more on Bernstein function in Jacob (2001), Schilling et al. (2010)). The function $g(x) : (0,\infty) \to \mathbb{R}$ is completely monotone if has derivatives of all order satisfying

$$(-1)^k g^{(k)}(x) \geq 0, \qquad \forall x > 0 \text{ and } k \in \{0\} \cup \mathbb{N}. \tag{4.12}$$

By Bernstein Theorem (see Bernstein (1929)) the function g is completely monotone if and only if

$$g(x) = \int_0^\infty e^{-sx} m(ds), \qquad x > 0, \tag{4.13}$$

when the above integral converges $\forall x > 0$ and where $m(ds)$ is a non-negative measure on $[0,\infty)$. Here and all throughout the paper the following symbology and definitions will be the same. We call $f(x)$, $x > 0$, the Bernstein function with representation (4.10) and we consider the completely monotone function

$$g(x) = \frac{f(x)}{x}, \qquad x > 0, \tag{4.14}$$

with representation

$$g(x) = b + \int_0^\infty e^{-sx}\, \nu(s) ds, \tag{4.15}$$

where $\nu(s)$ is the tail of the Lévy measure appearing in (4.10)

$$\nu(s)ds = (a + \bar{\nu}(s, \infty))\, ds. \tag{4.16}$$

The representations (4.14) and (4.15) define a completely monotone function and are valid for every Bernstein function f (see for example Schilling et al. (2010) Corollary 3.7 *(iv)*). We observe that $\nu(s)$ is in general a right-continuous and non-increasing function for which

$$\int_0^1 (a + \bar{\nu}(s, \infty))\, ds = \int_0^1 \nu(s)\, ds < \infty. \tag{4.17}$$

Furthermore we note that

$$\bar{\nu}(s, \infty) < \infty, \text{ for all } s > 0. \tag{4.18}$$

In order to justify (4.18) we recall the inequality

$$\left(1 - e^{-1}\right)(t \wedge 1) \leq 1 - e^{-t}, \qquad t \geq 0, \tag{4.19}$$

which can be extended as

$$\left(1 - e^{-\epsilon}\right)(t \wedge \epsilon) \leq \left(1 - e^{-t}\right), \quad \text{for all } 0 < \epsilon \leq 1,\, t \geq 0. \tag{4.20}$$

By taking into account (4.20) we can rewrite for all $0 < \epsilon \leq 1$ the integrability condition (4.11) as

$$\int_0^\infty (t \wedge \epsilon)\, \bar{\nu}(dt) < \infty, \qquad \text{for all } 0 < \epsilon \leq 1, \tag{4.21}$$

since

$$\int_0^\infty (t \wedge \epsilon)\, \bar{\nu}(dt) \leq \frac{e^\epsilon}{e^\epsilon - 1} \int_0^\infty \left(1 - e^{-t}\right) \bar{\nu}(dt) = \frac{e^\epsilon}{e^\epsilon - 1} f(1) < \infty \tag{4.22}$$

and this implies (4.18). When the Lévy measure has finite mass, that is

$$\bar{\nu}(0, \infty) < \infty, \tag{4.23}$$

and if $b = 0$, the corresponding Bernstein function f is bounded.

4.2.1 Convolution-type derivatives on the positive half-axis

In this section we define a generalization, respect to a Bernstein function f, of the classical Riemann-Liouville fractional derivative and we discuss some of its fundamental properties. Here is the first definition.

Definition 4.2.1. *Let $u(t) \in AC([c,d])$, $0 < c \leq t \leq d < \infty$ that is the space of absolutely continuous function on $[c,d]$. Let f be a Bernstein function with representation (4.10) and let $\bar{\nu}$ be the corresponding Lévy measure with tail $\nu(s) = a + \bar{\nu}(s,\infty)$. Assume that $s \to \nu(s)$ is absolutely continuous on $(0,\infty)$. We define the generalized Riemann-Liouville derivative according to the Bernstein function f as*

$$ {}^f\mathcal{D}_t^{(c,d)} u(t) := \frac{d}{dt}\left[bu(t) + \int_0^{t-c} u(t-s)\,\nu(s)ds \right]. \tag{4.24}$$

The representation (4.24) can be extenended for defining the derivative on the half-axis $\mathbb{R}^+$ as it is done for the classical Riemann-Liouville fractional derivative (see Kilbas et al. (2006) page 79). Hence we write

$$ {}^f\mathcal{D}_t^{(0,\infty)} u(t) := \frac{d}{dt}\left[bu(t) + \int_0^{t} u(t-s)\,\nu(s)ds \right]. \tag{4.25}$$

Lemma 4.2.2. *Let ${}^f\mathcal{D}_t^{(c,\infty)}u(t)$, $t \geq c \geq 0$, be as in Definition 4.2.1 and let $|u(t)| \leq Me^{\lambda_0 t}$ for some $\lambda_0, M > 0$. We have the following result*

$$ \mathcal{L}\left[{}^f\mathcal{D}_t^{(c,+\infty)} u(t) \right](\lambda) = f(\lambda)\,\widetilde{u}(\lambda) - be^{-\lambda c}u(c), \qquad \Re\lambda > \lambda_0. \tag{4.26}$$

Proof. The Laplace transform can be evaluated explicitely as follows

$$\begin{aligned} \mathcal{L}\left[{}^f\mathcal{D}_t^{(c,+\infty)} u(t) \right](\lambda) &= b\lambda\widetilde{u}(\lambda) - be^{-\lambda c}u(c) + \mathcal{L}\left[\frac{d}{dt}\int_0^{t-c} u(t-s)\,\nu(s)ds \right](\lambda) \\ &= b\lambda\widetilde{u}(\lambda) - be^{-\lambda c}u(c) + \lambda\mathcal{L}\left[\int_0^{t-c} u(t-s)\,\nu(s)ds \right](\lambda) \\ &= b\lambda\widetilde{u}(\lambda) - be^{-\lambda c}u(c) + \lambda\int_0^{\infty}\int_{s+c}^{\infty} e^{-\lambda t}u(t-s)\,\nu(s)dt\,ds \\ &= \lambda g(\lambda)\widetilde{u}(\lambda) - be^{-\lambda c}u(c) \\ &= f(\lambda)\,\widetilde{u}(\lambda) - be^{-\lambda c}u(c). \end{aligned} \tag{4.27}$$

In the last steps we used (4.14) and (4.15). □

In view of the previous Lemma we note that our definition is consistent and generalize the Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$ in a reasonable way.

Remark 4.2.3. *Let the function f of Definition 4.2.1 be $f(x) = x^\alpha$, $x > 0$, $\alpha \in (0,1)$, for which (4.10) becomes*

$$x^\alpha = \int_0^\infty \left(1 - e^{-sx}\right) \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds, \tag{4.28}$$

that is to say $a = 0$ and $b = 0$ and

$$\bar{\nu}(ds) = \frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds \tag{4.29}$$

and therefore

$$\nu(s)ds = ds \int_s^\infty \frac{\alpha z^{-\alpha-1}}{\Gamma(1-\alpha)} dz = \frac{s^{-\alpha} ds}{\Gamma(1-\alpha)}. \tag{4.30}$$

By performing these substitutions in Definition 4.2.1 it is easy to show that

$$^f\mathcal{D}_t^{(0,+\infty)} u(t) = \frac{^R d^\alpha}{dt^\alpha} u(t) \tag{4.31}$$

where

$$\frac{^R d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds \tag{4.32}$$

is the Riemann-Liouville fractional derivative.

By following the logic inspiring the fractional Dzerbayshan-Caputo derivative (see Kilbas et al. (2006)) defined, for an absolutely continuous function $u(t)$, $t > 0$, as

$$\frac{^C d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds, \tag{4.33}$$

we can give the following alternative definition of generalized derivative respect to a Bernstein function.

Definition 4.2.4. *Let f and ν be as in Definition 4.2.1. Let $u(t) \in AC([c,d])$, $0 < c \leq t \leq d < \infty$. We define the generalized Dzerbayshan-Caputo derivative according to the Bernstein function f as*

$$^f\mathfrak{D}_t^{(c,d)} u(t) := b \frac{d}{dt} u(t) + \int_0^{t-c} \frac{\partial}{\partial t} u(t-s)\, \nu(s) ds. \tag{4.34}$$

As already done for the classical Dzerbayshan-Caputo derivative we can extend (4.34) on the half-axis $\mathbb{R}^+$ (see for example Kilbas et al. (2006) page 97) as

$$^f\mathfrak{D}_t^{(0,\infty)} u(t) := b \frac{d}{dt} u(t) + \int_0^t \frac{\partial}{\partial t} u(t-s)\, \nu(s) ds. \tag{4.35}$$

Throughout the paper we will write for the sake of simplicity $^f\mathfrak{D}_t$ instead of $^f\mathfrak{D}_t^{(0,\infty)}$.

Lemma 4.2.5. *Let ${}^f\mathfrak{D}_t$ be as in (4.35) and let $|u(t)| \leq Me^{\lambda_0 t}$, for some λ_0, $M > 0$. We obtain*

$$\mathcal{L}\left[{}^f\mathfrak{D}_t u(t)\right](\lambda) \;=\; f(\lambda)\widetilde{u}(\lambda) - \frac{f(\lambda)}{\lambda}u(0), \qquad \Re\lambda > \lambda_0. \tag{4.36}$$

Proof. By evaluating explicitely the Laplace transform we obtain

$$\begin{aligned}
\mathcal{L}\left[{}^f\mathfrak{D}_t u(t)\right](\lambda) &= b\lambda\widetilde{u}(\lambda) - bu(0) + \int_0^\infty e^{-\lambda t}\int_0^t \frac{d}{dt}u(t-s)\nu(s)ds\,dt \\
&= b\lambda\widetilde{u}(\lambda) - bu(0) + \int_0^\infty \int_s^\infty e^{-\lambda t}\frac{d}{dt}u(t-s)\,\nu(s)\,dt\,ds \\
&= b\lambda\widetilde{u}(\lambda) - bu(0) + \int_0^\infty e^{-\lambda s}\nu(s)ds\,(\lambda\widetilde{u}(\lambda) - u(0)) \\
&= \lambda g(\lambda)\widetilde{u}(\lambda) - g(\lambda)u(0) \\
&= f(\lambda)\widetilde{u}(\lambda) - \frac{f(\lambda)}{\lambda}u(0)
\end{aligned} \tag{4.37}$$

where we used the relationships (4.14) and (4.15). □

Remark 4.2.6. *By performing the same substitutions of Remark 4.2.3 it is easy to show that*

$$^f\mathfrak{D}_t u(t) \;=\; \frac{{}^C d^\alpha}{dt^\alpha}u(t) \tag{4.38}$$

where $\frac{{}^C d^\alpha}{dt^\alpha}$ is the Dzerbayshan-Caputo derivative defined in (4.33).

It is well known that the Riemann-Liouville fractional derivative of a function $u \in AC([c,d])$ exist almost everywhere in $[c,d]$ and can be written as (see Kilbas et al. (2006) page 73)

$$\frac{{}^R d^\alpha}{dt^\alpha}u(t) \;=\; \frac{{}^C d^\alpha}{dt^\alpha}u(t) + \frac{(t-c)^{-\alpha}}{\Gamma(1-\alpha)}u(c). \tag{4.39}$$

Here is a more general result.

Proposition 4.2.7. *Let ${}^f\mathcal{D}_t^{(c,d)}$ and ${}^f\mathfrak{D}_t^{(c,d)}$ be respectively as in Definitions 4.2.1 and 4.2.4. We have that ${}^f\mathcal{D}_t^{(c,d)}u(t)$ exists almost everywhere in $[c,d]$ and can be written as*

$$^f\mathcal{D}_t^{(c,d)}u(t) \;=\; {}^f\mathfrak{D}_t^{(c,d)}u(t) + \nu(t-c)u(c). \tag{4.40}$$

Proof. Let $V(s) = \int \nu(s)ds$ and

$$\varpi(s) \;=\; \int_0^s \nu(z)dz, \qquad 0 < s < \infty, \tag{4.41}$$

such that

$$\int \nu(t-s)ds \;=\; -\varpi(t-s) - V(0). \tag{4.42}$$

Since $u \in AC([c,d])$ we have for $c < s < d$

$$u(s) = \int_c^s u'(z)dz + u(c) \tag{4.43}$$

and therefore we can rewrite ${}^f\mathcal{D}_t^{(c,d)}u(t)$ as

$$\begin{aligned}
&{}^f\mathcal{D}_t^{(c,d)}u(t) \\
&= b\frac{d}{dt}u(t) + \frac{d}{dt}\int_c^t \left(\int_c^s u'(z)dz + u(c)\right)\nu(t-s)\,ds \\
&= -u'(t)V(0) + \nu(t-c)\,u(t) + \frac{d}{dt}\int_c^t u'(s)\varpi(t-s)\,ds + \frac{d}{dt}\int_c^t u'(s)V(0)ds \\
&= \nu(t-c)u(c) + \int_c^t u'(s)\nu(t-s)\,ds \\
&= \nu(t-c)u(c) + \int_0^{t-c} u'(t-s)\,\nu(s)\,ds.
\end{aligned} \tag{4.44}$$

In the second step we performed an integration by parts. □

4.2.2 Convolution-type derivatives on the whole real axis

In this section we develop a generalized space-derivative respect to a Bernstein function f with domain on the whole real axis $\mathbb{R}$, by following the logic inspiring the Weyl derivatives.

Definition 4.2.8. *Let f and $\nu(s)$ be as in Definition 4.2.1. We define the generalized Weyl derivative, according to the Bernstein function f, on the whole real axis as*

$${}^f\mathcal{D}_x^+u(x) \;:=\; \left[b\frac{d}{dx}u(x) + \int_0^\infty \frac{\partial}{\partial x}u(x-s)\,\nu(s)ds\right], \tag{4.45}$$

and

$${}^f\mathcal{D}_x^-u(x) \;:=\; -\left[b\frac{d}{dx}u(x) + \int_0^\infty \frac{\partial}{\partial x}u(x+s)\,\nu(s)ds\right]. \tag{4.46}$$

Some remarks on the domain of definition of (4.45) and (4.46) are stated in Section 4.5.1.

Lemma 4.2.9. *Let ${}^f\mathcal{D}_x^{\pm}$ be as in Definition 4.2.8. We have that*

$$\mathcal{F}\left[{}^f\mathcal{D}_x^+u(x)\right](\xi) \;=\; f(-i\xi)\widehat{u}(\xi) \tag{4.47}$$

and

$$\mathcal{F}\left[{}^f\mathcal{D}_x^- u(x)\right](\xi) \;=\; f(i\xi)\widehat{u}(\xi). \tag{4.48}$$

Proof. By evaluating the first Fourier transform explicitely, we obtain

$$\begin{aligned}
\mathcal{F}\left[{}^f\mathcal{D}_x^+ u(x)\right](\xi) \;&=\; -\, bi\xi\widehat{u}(\xi) - i\xi\, \mathcal{F}\left[\int_0^\infty u(x-s)\nu(s)ds\right](\xi) \\
&=\; -\, bi\xi\widehat{u}(\xi) - i\xi \int_0^\infty \int_{\mathbb{R}} e^{i\xi z + i\xi s} u(z)\, dz\, \nu(s)ds \\
&=\; -\, bi\xi\widehat{u}(\xi) - i\xi \int_0^\infty ds\, e^{i\xi s} \left(a + \int_s^\infty \bar{\nu}(dz)\right)\widehat{u}(\xi)
\end{aligned} \tag{4.49}$$

and by integrating by parts we get that

$$\begin{aligned}
\mathcal{F}\left[{}^f\mathcal{D}_x^+ u(x)\right](\xi) \;&= a\widehat{u}(\xi) - bi\xi\widehat{u}(\xi) + \int_0^\infty \left(1 - e^{i\xi s}\right)\bar{\nu}(ds)\,\widehat{u}(\xi) \\
&= f(-i\xi)\,\widehat{u}(\xi).
\end{aligned} \tag{4.50}$$

By repeating the same calculation one can easily prove (4.48). □

Remark 4.2.10. *Definitions (4.45) and (4.46) are consistent with the Weyl definition of fractional derivatives on the whole real axis which are, for $\alpha \in (0,1)$ and $x \in \mathbb{R}$, (see Kilbas et al. (2006))*

$$\frac{{}^+d^\alpha}{dx^\alpha}u(x) \;=\; \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^x \frac{u(s)}{(x-s)^\alpha}ds, \quad \textit{right derivative}, \tag{4.51}$$

and

$$\frac{{}^-d^\alpha}{dx^\alpha}u(x) \;=\; -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^\infty \frac{u(s)}{(s-x)^\alpha}ds, \quad \textit{left derivative}. \tag{4.52}$$

We have

$${}^f\mathcal{D}_x^\pm u(x) \;=\; \frac{{}^\pm d^\alpha}{dx^\alpha}u(x), \qquad x \in \mathbb{R}. \tag{4.53}$$

We resort to the fact that (see Kilbas et al. (2006) page 90)

$$\mathcal{F}\left[\frac{{}^\pm\partial^\alpha}{\partial x^\alpha}u(x)\right](\xi) \;=\; (\mp i\xi)^\alpha\widehat{u}(\xi) \tag{4.54}$$

and thus by combining (4.54) with Lemma 4.2.9 the proof of (4.53) is complete. The reader can also check the result by performing the substitution $b = 0$ and

$$\nu(s)ds \;=\; \frac{s^{-\alpha}}{\Gamma(1-\alpha)}ds \tag{4.55}$$

in (4.45) and (4.46) which yields (4.51) and (4.52) with a change of variable.

4.3 Subordinators, hitting-times and continuous time random walks

A subordinator ${}^f\sigma(t)$, $t > 0$, is a stochastic process in continuous time with independent and homogeneous increments and non-decreasing paths (see more on subordinators in Bertoin (1996, 1997)). The transition probabilities of subordinators $\mu_t(B) = \Pr\left\{ {}^f\sigma(t) \in B\right\}$, $B \subset [0, \infty)$ Borel, $t > 0$, are convolution semigroups of sub-probability measure with the following property concerning the Laplace transform

$$\mathcal{L}\left[\mu_t\right](\lambda) = e^{-tf(\lambda)} \tag{4.56}$$

where f is a Bernstein function having representation (4.10). A family μ_t, $t > 0$, of sub-probability measures on $\mathbb{R}^n$ is called a convolution semigroup on $\mathbb{R}^n$ if it satisfies the conditions

- $\mu_t\left(\mathbb{R}^n\right) \leq 1$, $\forall t \geq 0$;
- $\mu_s * \mu_t = \mu_{t+s}$, $\forall s, t \geq 0$, and $\mu_0 = \delta_0$;
- $\mu_t \to \delta_0$, vaguely as $t \to 0$,

where we denoted by δ_0 the Dirac point mass at zero. The fact that the tail function $s \to \nu(s)$ of the Lévy measure $\bar{\nu}$ is absolutely continuous on $(0, \infty)$ and that $\bar{\nu}(0, \infty) = \infty$ is a sufficient condition for saying the transition probabilities of the corresponding subordinator are absolutely continuous (see Sato (1999), Theorem 27.7). We recall that a measure μ on $\mathcal{B}\left(\mathbb{R}^d\right)$ is said to be absolutely continuous if given $B \in \mathcal{B}\left(\mathbb{R}^d\right)$ satisfying $\mathrm{Leb}\left(B\right) = 0$ then $\mu\left(B\right) = 0$ ($\mathcal{B}$ indicates the Borel σ-algebra).

It has been shown that any subordinator has a Laplace exponent as in (4.56) and that any Bernstein function with representation (4.10) is the Laplace exponent of a subordinator (see for example Bertoin (1997)). A subordinator is a step process if its associated Bernstein function f is bounded. Looking at the representation (4.10) we see that a Bernstein function is bounded if $\bar{\nu}(0, \infty) < \infty$ and $b = 0$. If these conditions are not fulfilled (and thus $b > 0$ and $\bar{\nu}(0, \infty) = \infty$) the subordinator is a strictly increasing process.

The inverse process of a subordinator is defined as

$${}^fL(t) = \inf\left\{s > 0 : {}^f\sigma(s) > t\right\}, \qquad s, t > 0, \tag{4.57}$$

and thus ${}^fL$ is the hitting-time of ${}^f\sigma$ since ${}^f\sigma$ has non-decreasing paths (see Bertoin (1996, 1997)). With this in hand we note that ${}^fL$ is again a non-decreasing process but in general it has non-stationary and non-independent increments. In what follows we develop some properties of the transition probabilities of ${}^fL(t)$, $t > 0$, denoted by $l_t(B) = \Pr\left\{ {}^fL(t) \in B \right\}$.

Lemma 4.3.1. *Let ${}^f\sigma(t)$, $t > 0$, and ${}^fL(t)$, $t > 0$, be respectively a subordinator and its inverse. Let f be the Laplace exponent of ${}^f\sigma$ represented as in (4.10) for $a, b \geq 0$. Let $\nu(s)$ be the tail of the Lévy measure $\bar{\nu}$ and $l_t(B)$ the transition probabilities of ${}^fL$. We have that*

$$\mathcal{L}\left[l_\bullet(s,\infty)\right](\lambda) = \frac{1}{\lambda} e^{-sf(\lambda)}. \tag{4.58}$$

Proof. We resort to the fact that ${}^f\sigma$ has non-decreasing paths and thus, in view of the construction (4.57) of ${}^fL$ we have

$$\Pr\left\{ {}^fL(t) > s \right\} = \Pr\left\{ {}^f\sigma(s) < t \right\}. \tag{4.59}$$

In view of (4.59) we observe that

$$\int_0^\infty e^{-\lambda t} l_t(s,\infty]\, dt = \int_0^\infty e^{-\lambda t} \mu_s[0,t) dt \tag{4.60}$$

and thus

$$\int_0^\infty e^{-\lambda t} l_t[s,\infty) dt = \int_0^\infty e^{-\lambda t} \int_0^t \mu_s(dz)\, dt = \frac{1}{\lambda} e^{-sf(\lambda)}. \tag{4.61}$$

□

Proposition 4.3.2. *Let ${}^f\sigma(t)$, $t > 0$, be the subordinator with Laplace exponent f represented by (4.10) for $a \geq 0$, $b \geq 0$. Let ν be the tail of the Lévy measure $\bar{\nu}$. Let assume that $\bar{\nu}(0,\infty) = \infty$ and that $s \to \nu(s) = a + \bar{\nu}(s,\infty)$ is absolutely continuous on $(0,\infty)$. Let ${}^fL(t)$, $t > 0$, be the inverse of ${}^f\sigma$, in the sense of (4.57), with transition probabilities $l_t(B) = \Pr\left\{ {}^fL(t) \in B \right\}$. We have the following results.*

1. *The transition probabilities l_t have a density such that $l_t(ds) = l_t(s)ds$ and $l_t(s) = b\mu_s(t) + (\nu(t) * \mu_s(t))$ where with abuse of notation we denoted with $l_t(s)$ and $\mu_s(t)$ respectively the density of $l_t(ds)$ and $\mu_s(dt)$ and the symbol $*$ stands for the Laplace convolution $\int_0^t \mu_s(t-z)\nu(z)dz$. Furthermore $\mathcal{L}\left[l_\bullet(s)\right](\lambda) = \frac{f(\lambda)}{\lambda} e^{-sf(\lambda)}$.*

2. *$\lim_{h\to 0} l_{t+h} = l_t$ $\forall t \geq 0$ and $\lim_{t\to 0} l_t[0,\infty) = \delta_0[0,\infty)$.*

3. *$l_t(0) = \nu(t)$, $\forall t > 0$.*

4. *$l_t[0,\infty) = 1$, $\forall a, b \geq 0$.*

Proof. 1. Since we assume $\bar{\nu}(0,\infty) = \infty$ and $s \to \nu(s)$ absolutely continuous on $(0,\infty)$, we have that from Theorem 27.7 in Sato (1999) the transition probabilities $\mu_t(dx)$ are absolutely continuous and therefore have a density $\mu_t(x)$. Thus we write

$$\begin{aligned}\mathcal{L}\left[b\mu_s(\cdot) + (\mu_s(\cdot) * \nu(\cdot))\right](\lambda) &= be^{-sf(\lambda)} + \int_0^\infty e^{-\lambda t} \int_0^t \mu_s(t-z)\nu(z)dz\, dt \\ &= be^{-sf(\lambda)} + \int_0^\infty dz \int_z^\infty dt\, e^{-\lambda t}\, \mu_s(t-z)\nu(z) \\ &= \frac{f(\lambda)}{\lambda} e^{-sf(\lambda)}, \end{aligned} \tag{4.62}$$

where we used (4.15). From (4.62) we get

$$\int_s^\infty \mathcal{L}\left[b\mu_w(\cdot) + (\mu_w(\cdot) * \nu(\cdot))\right](\lambda)\, dw \;=\; \int_s^\infty \frac{f(\lambda)}{\lambda} e^{-wf(\lambda)} dw \;=\; \frac{1}{\lambda} e^{-sf(\lambda)}. \tag{4.63}$$

Since (4.63) coincides with (4.61) we can write

$$\int_s^\infty \left(b\mu_w(t) + (\mu_w(t) * \nu(t))\right) dw \;=\; l_t(s,\infty) \tag{4.64}$$

which completes the proof.

2. We have

$$\begin{aligned}\lim_{h\to 0} l_{t+h}[s,\infty) \;&=\; \lim_{h\to 0} \int_s^\infty \left(b\mu_s(t+h) + \int_0^{t+h} \mu_s(t+h-z)\nu(z)dz \right) ds \\ &= l_t[s,\infty) \end{aligned} \tag{4.65}$$

since $\mu_s(t)$ is a density. Furthermore

$$\lim_{t\downarrow 0} l_t[0,\infty) \;=\; \lim_{t\downarrow 0} \int_0^\infty \left(b\mu_s(t) + \int_0^t \mu_s(t-z)\nu(z)dz \right) ds \;=\; \delta_0[0,\infty). \tag{4.66}$$

3. This is obvious since for $t > 0$, $l_t(0) \;=\; b\mu_0(t) + \nu(t) * \mu_0(t) = \nu(t)$.

4. The proof of this can be carried out by observing that

$$\int_0^\infty e^{-\lambda t} l_t[0,\infty)\, dt \;=\; \int_0^\infty e^{-\lambda t} l_t[s,\infty)\, dt \Big|_{s=0} \;=\; \frac{1}{\lambda} e^{-sf(\lambda)} \Big|_{s=0} \;=\; \frac{1}{\lambda}. \tag{4.67}$$

□

Subordinators are related to Continuous Time Random Walks (CTRWs). The CTRWs (introduced in Montroll and Weiss (1965)) are processes in continuous time in which the number of jumps performed in a certain amount of time t is a random variable, as well as the jump's length. For example, the stable subordinator can be viewed (in distribution) as the limit of a CTRW performing a Poissonian number of power-law jumps (see for example Meerschaert and Sikorskii (2012)). In Meerschaert and Scheffer (2004), among other things, the authors pointed out that the limit process of a CTRW with infinite-mean waiting times converge to a Lévy motion time-changed by means of the hitting-time $L^\alpha(t)$, $t > 0$, of the stable subordinator $\sigma^\alpha(t)$, $t > 0$. Since subordinators are also Lévy processes they can be decomposed according to the Lévy-Itô decomposition (Itô (1942)). By following the logic of the Lévy-Itô decomposition we derive a CTRW converging (in distribution) to a subordinator with laplace exponent f and having a hitting-time converging to its inverse. Our CTRW is therefore the sum of a pure drift and a compound Poisson. The distribution of the jumps' length need some attention. In particular we define i.i.d. random variables Y_j representing the random length of the jump, with distribution

$$\Pr\{Y_j \in dy\} = \frac{1}{\nu(\gamma)} \left(\bar{\nu}(dy) + a\,\delta_\infty\right) \mathbb{I}_{y>\gamma}, \qquad \gamma > 0,\ \forall j = 1, \cdots, n, \tag{4.68}$$

where δ_∞ indicates the Dirac point mass at ∞ and $a \geq 0$. In (4.68) $\bar{\nu}$ and ν are respectively the Lévy measure and its tail as defined in equations from (4.10) to (4.16) and upon which the definitions of convolution-type derivatives of previous section are based. The parameter $a \geq 0$ is that in (4.10) and it is known in literature as the killing rate of the subordinator. The distribution (4.68) can be taken as follows. The probability of a jump of length $y > \gamma > 0$ is given by the normalized Lévy measure when $a = 0$. When $a > 0$ the probability of a jump of infinite length increases since $\bar{\nu}(y) \overset{y\to\infty}{\longrightarrow} 0$ and thus $\Pr\{Y \in dy\}/dy \overset{y\to\infty}{\longrightarrow} a/\nu(\gamma)$. When constructing a CTRW with Poisson waiting times and jump length's distribution (4.68) by choosing $a > 0$ we obtain a limit process (for $\gamma \to 0$) assuming value $+\infty$ from a certain time $\zeta < \infty$. Usually ζ is called the lifetime of the process (see Bertoin (1997)). The case $a > 0$ in (4.68) therefore gives rise to the so-called killed subordinators. A killed subordinator ${}^f\widehat{\sigma}_t$, is defined as

$${}^f\widehat{\sigma}_t = \begin{cases} {}^f\sigma_t, & t < \zeta, \\ +\infty, & t \geq \zeta, \end{cases} \tag{4.69}$$

where

$$\zeta = \inf\left\{t > 0 : {}^f\sigma(t) = \infty\right\}. \tag{4.70}$$

Obviously $a = 0$ implies $\zeta = \infty$. For simplicity we will use the notation ${}^f\sigma_t$ both for killed and non-killed subordinators when no confusion arises. We are ready to prove the following convergences in distribution inspired by the Lévy-Ito decomposition and usefull in order to understand the role of the Lévy measure $\bar{\nu}$ and its tail $\nu(s)$.

Proposition 4.3.3. *Let $N(t)$, $t>0$, be a homogeneous Poisson process with parameter $\theta=1$ independent from the i.i.d. random variables Y_j with distribution (4.68). Let f be the Bernstein function with representation (4.10) Laplace exponent of the subordinator ${}^f\sigma(t)$, $t>0$, and let ${}^fL(t)$, $t>0$ be the inverse of ${}^f\sigma$ as in (4.57). Let $\nu(s)$ be the tail of the Lévy measure $\bar{\nu}$. The following convergences in distribution are true.*

1.
$$\lim_{\gamma\to 0}\left(bt+\sum_{j=0}^{N(t\,\nu(\gamma))}Y_j\right) \overset{law}{=} {}^f\sigma(t), \tag{4.71}$$

2.
$$\lim_{\gamma\to 0}\inf\left\{s>0: bs+\sum_{j=0}^{N(s\nu(\gamma))}Y_j>t\right\} \overset{law}{=} {}^fL(t). \tag{4.72}$$

Proof. In order to prove (1) we consider the following Laplace transform

$$\begin{aligned}
&\mathbb{E}\exp\left\{-\lambda bt-\lambda\sum_{j=0}^{N(t\,\nu(\gamma))}Y_j\right\}\\
&=e^{-\lambda bt}\mathbb{E}\left[\mathbb{E}\left(e^{-\lambda Y}\right)^{N(t\nu(\gamma))}\right]\\
&=\exp\left\{-\lambda bte^{-t\nu(\gamma)\left(1-\mathbb{E}e^{-\lambda Y}\right)}\right\}\\
&=\exp\left\{-t\left(b\lambda+\nu(\gamma)\int_\gamma^\infty\left(1-e^{-\lambda y}\right)\Pr\{Y\in dy\}\right)\right\},
\end{aligned}\tag{4.73}$$

where $\Pr\{Y\in dy\}$ is the one in (4.68). In the previous steps we used the independence of the random variables Y_j and the fact that

$$\mathbb{E}e^{-\lambda N(t\nu(\gamma))}=e^{-t\nu(\gamma)\left(1-e^{-\lambda}\right)}. \tag{4.74}$$

By performing the limit for $\gamma\to 0$ in (4.73) we obtain

$$\begin{aligned}
&\lim_{\gamma\to 0}\mathbb{E}\exp\left\{-\lambda bt-\lambda\sum_{j=0}^{N(t\,\nu(\gamma))}Y_j\right\}\\
&=\exp\left\{-t\left(a+b\lambda+\int_0^\infty\left(1-e^{-\lambda y}\right)\bar{\nu}(dy)\right)\right\}\\
&=e^{-tf(\lambda)},
\end{aligned}\tag{4.75}$$

and this proves (1).

Now we prove (2). Let $Z(t) = \inf\left\{s > 0 : bs + \sum_{j=0}^{N(s\nu(\gamma))} Y_j > t\right\}$. By definition we have that

$$\Pr\{Z(t) > s\} = \Pr\left\{bs + \sum_{j=0}^{N(s\,\nu(\gamma))} Y_j < t\right\} \tag{4.76}$$

and thus

$$\mathcal{L}\left[\Pr\{Z(\cdot) > s\}\right](\lambda) = \mathcal{L}\left[\Pr\left\{bs + \sum_{j=0}^{N(s\,\nu(\gamma))} Y_j < \cdot\right\}\right](\lambda). \tag{4.77}$$

By taking profit of calculation (4.73) we obtain

$$\mathcal{L}\left[\Pr\{Z(\cdot) > s\}\right](\lambda) = \frac{1}{\lambda}\exp\left\{-s\left(a + b\lambda + \int_0^\infty \left(1 - e^{-\lambda y}\right)\bar{\nu}(dy)\right)\right\} \tag{4.78}$$

and by performing the limit for $\gamma \to 0$ we arrive at

$$\lim_{\gamma\to 0}\mathcal{L}\left[\Pr\{Z(\cdot) > s\}\right](\lambda) = \frac{1}{\lambda}e^{-sf(\lambda)}. \tag{4.79}$$

Since (4.79) coincides with (4.61) the proof is complete. □

Remark 4.3.4. *For $f(x) = x^\alpha$, $\alpha \in (0,1)$ result (4.71) becomes*

$$\lim_{\gamma\to 0}\sum_{j=0}^{N\left(t\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)} Y_j \overset{law}{=} \sigma^\alpha(t), \tag{4.80}$$

where $\sigma^\alpha(t)$, $t > 0$, is the stable subordinator of order $\alpha \in (0,1)$ and the i.i.d. random variables Y_j have power-law distribution

$$\Pr\{Y \in dy\}/dy = \alpha\gamma^\alpha y^{-\alpha-1}\mathbb{I}_{y>\gamma}, \qquad \gamma > 0, \tag{4.81}$$

which can be obtained from (4.68) by performing the substitutions

$$\bar{\nu}(y) = \frac{\alpha y^{-\alpha-1}}{\Gamma(1-\alpha)}dy, \qquad \text{and} \qquad \nu(\gamma) = \frac{\gamma^{-\alpha}}{\Gamma(1-\alpha)}, \tag{4.82}$$

due to the fact that $f(x) = x^\alpha =$ (4.28) ($a = 0$, $b = 0$). The result (4.80) is well-known (see for example Meerschaert and Sikorskii (2012)) and represents the convergence in distribution of a CTRW with power-law distributed jumps to the stable subordinator.

4.4 Densities and related governing equations

In this section we present in a unifying framework the governing equations of the densities of subordinators and their inverses, by making use of the operators defined in Section 4.2.

Theorem 4.4.1. *Let* ${}^f\sigma(t)$, $t>0$, *and* ${}^fL(t)$, $t>0$, *be respectively a subordinator and its inverse. Let* $\bar{\nu}$ *be the Lévy measure such that* $\bar{\nu}(0,\infty)=\infty$ *and let* $\nu(s)=a+\bar{\nu}(s,\infty)$. *Assume* $s\to\nu(s)$ *is absolutely continuous on* $(0,\infty)$. *Let* $\zeta=\inf\left\{t>0:\,{}^f\sigma(t)=\infty\right\}$.

1. *The probability density* $\mu_t(x)$ *of the subordinator* ${}^f\sigma$ *is the solution to the problem*

$$\begin{cases} \frac{\partial}{\partial t}\mu_t(x) \,=\, -\,{}^f\mathcal{D}_x^{(bt,+\infty)}\mu_t(x), & x>bt, 0<t<\zeta, b\geq 0,\\ \mu_t(bt) \,=\, 0, & t<\zeta,\\ \mu_0(x) \,=\, \delta(x), \\ \mu_\zeta(x) \,=\, \delta(x-\infty). \end{cases} \tag{4.83}$$

2. *The probability density* $l_t(x)$ *of* ${}^fL(t)$, $t>0$, *is the solution to the equation*

$${}^f\mathcal{D}_t^{(0,\infty)}l_t(x) \,=\, -\frac{\partial}{\partial x}l_t(x), \qquad t>0,\;\; \textit{and} \begin{cases} 0<x<\frac{t}{b}<\zeta, & \textit{if } b>0,\\ 0<x<\zeta, & \textit{if } b=0, \end{cases} \tag{4.84}$$

subject to

$$\begin{cases} l_t(t/b)=0,\\ l_t(0)=\nu(t),\\ l_0(x)=\delta(x). \end{cases} \tag{4.85}$$

The operator ${}^f\mathcal{D}_x^{(bt,+\infty)}$ *is the one of Definition 4.2.1.*

Proof. As already pointed out the conditions assumed on $\bar{\nu}$ and $\nu(s)$ ensure that $\mu_t(B)$ and $l_t(B)$ are absolutely continuous and therefore have densities we denote again by $\mu_t(x)$ and $l_t(x)$.

1. First we note that $\mu_t(x)=0$ for $x\leq bt$, $b\geq 0$, indeed from Proposition 4.3.3

$$\Pr\left\{{}^f\sigma(t)>bt\right\} \,=\, \lim_{\gamma\to 0}\Pr\left\{bt+\sum_{j=0}^{N(t\nu(\gamma))}Y_j>bt\right\}$$

$$= \lim_{\gamma \to 0} \Pr \left\{ \sum_{j=0}^{N(t\nu(\gamma))} Y_j > 0 \right\} = 1. \quad (4.86)$$

The Laplace transform of $\mu_t(x)$ is $\mathcal{L}[\mu_t(\cdot)](\phi) = e^{-tf(\phi)}$ and therefore $\mathcal{L}[\widetilde{\mu}_\bullet(\phi)](\lambda) = 1/(f(\lambda)+\phi)$. In view of Lemma 4.2.2 the Laplace transform of (4.83) with respect to x is

$$\frac{\partial}{\partial t}\widetilde{\mu}_t(\phi) = -f(\phi)\widetilde{\mu}_t(\phi) + be^{-bt\phi}\mu_t(bt) \quad (4.87)$$

and therefore by performing the Laplace transform with respect to t we obtain

$$\widetilde{\widetilde{\mu}}_\lambda(\phi) = \frac{1}{f(\lambda)+\phi} \quad (4.88)$$

where we used the facts that $\widetilde{\mu}_0(\phi) = 1$ and $\mu_t(bt) = 0$. This completes the proof of (1).

2. First we show that $l_t(x) = 0$ for $x \geq \frac{t}{b}$ when $b > 0$. By considering Proposition 4.3.3 we have

$$\Pr\left\{ {}^f L(t) < \frac{t}{b} \right\} = \Pr\left\{ {}^f\sigma\left(\frac{t}{b}\right) > t \right\}$$
$$= \lim_{\gamma \to 0} \Pr \left\{ t + \sum_{j=0}^{N\left(\frac{t}{b}\nu(\gamma)\right)} Y_j > t \right\} = 1. \quad (4.89)$$

The double Laplace transform of $l_t(x)$ reads

$$\mathcal{L}\left[\mathcal{L}\left[l_t(x)\right](\phi)\right](\lambda) = \frac{f(\lambda)/\lambda}{\phi + f(\lambda)}, \quad (4.90)$$

where we used Proposition 4.3.2. From this point we temporary assume that $b > 0$. We consider the Laplace transform with respect to x of (4.84) and we obtain

$$ {}^f\mathcal{D}_t^{(0,\infty)} \widetilde{l}_t(\phi) = -\phi \widetilde{l}_t(\phi) + l_t(0) - e^{-\phi(t/b)} l_t(t/b). \quad (4.91)$$

Considering the Laplace transform with respect to t of (4.91) and by taking into account (4.85) we get

$$f(\lambda)\widetilde{\widetilde{l}}_\lambda(\phi) - b\widetilde{l}_0(\phi) = -\phi \widetilde{\widetilde{l}}_\lambda(\phi) + \frac{f(\lambda)}{\lambda} - b \quad (4.92)$$

where we used the fact that

$$\int_0^\infty e^{-\lambda t}\nu(t)dt = \frac{f(\lambda)}{\lambda} - b \quad (4.93)$$

and Lemma 4.2.2. The conditions (4.85) imply $\widetilde{l}_0(\phi) = 1$ and thus

$$\widetilde{\widetilde{l}}_\lambda(\phi) = \frac{f(\lambda)/\lambda}{\phi + f(\lambda)}. \quad (4.94)$$

The proof for $b = 0$ can be carried out equivalently.

□

4.4.1 Some remarks on the long-range correlation

The operators ${}^f\mathfrak{D}_t$ and ${}^f\mathcal{D}_t^{(c,\infty)}$ are non-local and govern processes with different memory properties. The presence of long-range correlation can be detected in several ways (see for example Samorodnitsky (2006)). Here we will explore the rate by which the correlation of the inverses of subordinators decays (a similar approach can be found in Leonenko et al. (2013) applied to a fractional Pearson diffusion). In Veillette and Taqqu (2010) the authors derive an explicit formula for the moments of the inverse processes of subordinators. Such formula reads in our notation

$$\begin{aligned}&\mathbb{E}\left[{}^fL(t_1)^{m_1}\cdots{}^fL(t_n)^{m_n}\right]\\&=\int_0^{t_{\min}}\sum_{i=1}^n m_iU\left(t_1-\tau,\dots,t_n-\tau,m_1,\dots,m_{i-1},m_i-1,m_{i+1},\dots,m_n\right)U(d\tau)\end{aligned}\tag{4.95}$$

where $t_{\min}=\min(t_1,\cdots,t_n)$ and

$$U(x)=\mathbb{E}\left[{}^fL(x)\right]=\mathbb{E}\left[\int_0^\infty \mathbb{I}_{\left\{{}^f\sigma(t)\le x\right\}}dx\right]\tag{4.96}$$

is known as the renewal function and is the distribution function of the renewal measure $U(dx)$. The renewal measure is the potential measure of a subordinator and it is given by

$$U(B)=\mathbb{E}\int_0^\infty \mathbb{I}_{[{}^f\sigma(t)\in B]}\,dt=\int_0^\infty\mu_t(B)\,dt,\qquad\text{for } B\subseteq[0,\infty),\tag{4.97}$$

the reader can consults Song and Vondraček (2009) for further information. We recall the renewal function is subadditive that is

$$U(x+y)\le U(x)+U(y),\qquad\forall x,y,\ge 0\tag{4.98}$$

and that

$$\int_0^\infty e^{-\lambda x}U(dx)=\frac{1}{f(\lambda)}\qquad\int_0^\infty e^{-\lambda x}U(x)\,dx=\frac{1}{\lambda f(\lambda)}.\tag{4.99}$$

Furthermore it is well-known (see, for example, Bertoin (1997), Proposition 1.4) that there exist positive constants c and c' such that

$$c\,U(x)\le\frac{1}{f\left(\frac{1}{x}\right)}\le c'\,U(x).\tag{4.100}$$

By applying (4.95) we write

$$\mathbb{E}\,{}^fL(s)\,{}^fL(t)=\int_0^{s\wedge t}\left(U(s-\tau)+U(t-\tau)\right)U(d\tau)\tag{4.101}$$

which can be interpreted as a long-range dependency property. We can write for $w > 0$,

$$\begin{aligned} \mathbb{E}\left({}^f L(t)\,{}^f L(t+s)\right) &= \int_0^{t\wedge(t+s)} \left(U(t-\tau)+U(t+s-\tau)\right)U(d\tau) \\ &\geq \int_0^t U(s+2t-2\tau)U(d\tau) \\ &\geq \int_0^t \frac{1}{c' f\left(\frac{1}{s+2t-2\tau}\right)} U(d\tau) \end{aligned} \tag{4.102}$$

where we applied (4.98) and (4.100). We recall that $1/f$ is monotone and thus we can write

$$\lim_{s\to\infty} \int_0^t \frac{1}{c' f\left(\frac{1}{s+2t-2\tau}\right)} U(d\tau) = \int_0^t \lim_{s\to\infty} \frac{1}{c' f\left(\frac{1}{s+2t-2\tau}\right)} U(d\tau) > 0 \tag{4.103}$$

since $\lim_{z\to 0} f(z) \geq 0$. Fix $w, t > 0$ and use formula (4.103), we have

$$\int_w^\infty \mathbb{E}\,{}^f L(t)\,{}^f L(t+s)\,ds = +\infty, \qquad \forall w, t > 0. \tag{4.104}$$

4.5 On the governing equations of time-changed C_0-semigroups

In this section we discuss the concept of time-changed C_0-semigroups on a Banach space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ (see more on semigroup theory in Engel and Nagel (2000), Jacob (2001)) which we define as the Bochner integral

$$\mathcal{T}_t u = \int_0^\infty T_s u\, l_t(ds) \tag{4.105}$$

where T_s is a C_0-semigroup and l_t are the transition probabilites of the inverse ${}^f L(t)$, $t > 0$ of ${}^f\sigma(t)$, $t > 0$. We recall that a C_0-semigroup of operators on $\mathfrak{B}$ is a family of linear operators T_t (bounded and linear) which maps $\mathfrak{B}$ into itself and is strongly continuous that is

$$\lim_{t\to 0} \|T_t u - u\|_{\mathfrak{B}} = 0, \qquad \forall u \in \mathfrak{B}. \tag{4.106}$$

In other words a bounded linear operator T_t acting on a function $u \in \mathfrak{B}$ is said to be a C_0-semigroup if, $\forall u \in \mathfrak{B}$,

- $T_0 u = u$ (is the identity operator),
- $T_t T_s u = T_s T_t u = T_{t+s} u$, $\forall s, t \geq 0$,

- $\lim_{t\to 0} \|T_t u - u\|_{\mathfrak{B}} = 0$.

The infinitesimal generator of a C_0-semigroup is the operator

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t}, \tag{4.107}$$

for which

$$\text{Dom}\,(A) := \left\{ u \in \mathfrak{B} : \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as strong limit} \right\}. \tag{4.108}$$

The aim of this section is to write the initial value problem associated with $\mathcal{T}_t$ by making use of the convolution-type time-derivatives of Definition 4.2.4.

Theorem 4.5.1. *Let ${}^fL(t)$, $t > 0$, be the inverse process of a subordinator with Laplace exponent f and let l_t be the transition probabilities of ${}^fL$. Let $\bar{\nu}(0,\infty) = \infty$ and $s \to \nu(s) = a + \bar{\nu}(s,\infty)$ be absolutely continuous on $(0,\infty)$. Let $T_t u$, $u \in \mathfrak{B}$, be a (strongly continuous) C_0-semigroup on the Banach space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ such that $\|T_t u\|_{\mathfrak{B}} \leq \|u\|_{\mathfrak{B}}$. Let $(A, Dom\,(A))$ be the generator of $T_t u$. The operator defined by the Bochner integral*

$$\mathcal{T}_t u = \int_0^\infty T_s u \, l_t(ds) \tag{4.109}$$

acting on a function $u \in \mathfrak{B}$ is such that

1. *$\mathcal{T}_t u$ is a uniformly bounded linear operator on $\mathfrak{B}$,*

2. *$\mathcal{T}_t u$ is strongly continuous $\forall u \in \mathfrak{B}$,*

3. *$\mathcal{T}_t u$ solves the problem*

$$\begin{cases} {}^f\mathfrak{D}_t q(t) = Aq(t), & 0 < t < \infty, \\ q(0) = u \in Dom\,(A) \end{cases} \tag{4.110}$$

where the time-operator ${}^f\mathfrak{D}_t$ is the one appearing in Definition 4.2.4.

Proof. Now we prove the Theorem for $b > 0$ which is the case requiring some additional attention. The proof for $b = 0$ can be carried out equivalently and therefore is a particular case.

1. At first we show that the operator $\mathcal{T}_t u$ is uniformly bounded on $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$. From the hypothesys we have

$$\|T_t\| \leq 1, \qquad t \geq 0, \tag{4.111}$$

In view of (4.111) we can write

$$\begin{aligned} \|\mathcal{T}_t u\|_{\mathfrak{B}} &= \left\| \int_0^\infty T_s u \, l_t(ds) \right\|_{\mathfrak{B}} \\ &\leq \int_0^\infty \|T_s u\|_{\mathfrak{B}} \, l_t(ds) \leq \|u\|_{\mathfrak{B}} , \end{aligned} \tag{4.112}$$

since $l_t[0,\infty) = 1$, $\forall t \geq 0$, as showed in Proposition 4.3.2.

2. The strong continuity follows from the fact that

$$\begin{aligned} \lim_{h\to 0} \|\mathcal{T}_h u - u\|_{\mathfrak{B}} &= \left\| \int_0^\infty T_s u \, l_h(ds) - u \right\|_{\mathfrak{B}} \\ &\leq \int_0^\infty \|T_s u - u\|_{\mathfrak{B}} \, l_h(ds) \overset{h\to 0}{\longrightarrow} 0, \end{aligned} \tag{4.113}$$

since $l_h \to \delta_0$ as $h \to 0$ and T_s is strongly continuous.

3. Since T_t is a C_0-semigroup generated by $(A, \mathrm{Dom} A)$ we have

$$\frac{d}{dt} T_t u = A T_t u = T_t A u, \qquad \forall u \in \mathrm{Dom}\,(A) . \tag{4.114}$$

Now let

$$A_s = \frac{T_s u - u}{s}. \tag{4.115}$$

We note that

$$\begin{aligned} A_s \mathcal{T}_t u &= A_s \int_0^\infty T_z u \, l_t(dz) \\ &= \int_0^\infty \frac{T_{z+s} u - T_z u}{s} \, l_t(dz) \\ &= \int_0^\infty T_z \left(\frac{T_s u - u}{s} \right) \, l_t(dz) \end{aligned} \tag{4.116}$$

and since for $u \in \mathrm{Dom}\,(A)$ the limit for $s \to 0$ on the right-hand side exists we have that $\mathcal{T}_t$ maps $\mathrm{Dom}\,(A)$ into itself.

By using Lemma 4.2.5 we note that the Laplace transform of (4.110) becomes

$$\begin{cases} f(\lambda)\widetilde{q}(x,\lambda) - \frac{f(\lambda)}{\lambda} q(x,0) = A\widetilde{q}(x,\lambda) \\ q(x,0) = u(x). \end{cases} \tag{4.117}$$

Now define the operator

$${}^f R_{\lambda,A} := \int_0^\infty e^{-\lambda t} \mathcal{T}_t dt = \frac{f(\lambda)}{\lambda} R_{f(\lambda),A} \tag{4.118}$$

where

$$R_{f(\lambda),A} = \int_0^\infty e^{-tf(\lambda)} T_t dt. \tag{4.119}$$

We recall that since we assume $(A, \mathrm{Dom}\,(A))$ generate a C_0-semigroup for which $\|T_t u\|_{\mathfrak{B}} \leq \|u\|_{\mathfrak{B}}$, we necessarily have that A is closed and densely defined. Furthermore for all $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ we must have that $\lambda \in \rho(A)$ and $\|R_{\lambda,A}\| \leq \frac{1}{\Re\lambda}$, where

$$R_{\lambda,A} = \int_0^\infty e^{-\lambda t} T_t \, dt \tag{4.120}$$

is the resolvent operator and $\rho(A)$ is the resolvent set of A. The integral (4.119) is justified since every Bernstein function has an extension onto the right complex half-plane $\mathbb{H} = \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ which satisfies (see Schilling et al. (2010), Proposition 3.5)

$$\Re f(\lambda) = a + b\Re\lambda + \int_0^\infty \left(1 - e^{-s\Re\lambda} \cos \Im\lambda\right) \bar{\nu}(ds) > 0. \tag{4.121}$$

By computing we can evaluate the following Laplace transform

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\, {}^f\mathfrak{D}_t \mathcal{T}_t u \, dt = \\
&= \left[b \int_0^\infty e^{-\lambda t} \lim_{h\to 0} \frac{\mathcal{T}_{t+h} - \mathcal{T}_t}{h} u \, dt + \int_0^\infty e^{-\lambda t} \int_0^t \lim_{h\to 0} \frac{\mathcal{T}_{t+h-s}u - \mathcal{T}_{t-s}u}{h} \nu(s) ds \, dt \right] \\
&= \left[\lim_{h\to 0} b \frac{e^{\lambda h}}{h} \int_h^\infty e^{-\lambda t} \mathcal{T}_t u \, dt - b \lim_{h\to 0} \frac{1}{h} \int_0^\infty e^{-\lambda t} \mathcal{T}_t u \, dt \right. \\
&\quad \left. + \int_0^\infty ds\, \nu(s) \int_s^\infty e^{-\lambda t} \lim_{h\to 0} \frac{\mathcal{T}_{t+h-s}u - \mathcal{T}_{t-s}u}{h} \right] \\
&= \left[b \lim_{h\to 0} \frac{e^{\lambda h} - 1}{h}\, {}^f R_\lambda u \; - b \lim_{h\to 0} \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} \mathcal{T}_t u \, dt \right. \\
&\quad \left. + \left(\frac{f(\lambda)}{\lambda} - b \right) \left(\lim_{h\to 0} \frac{e^{\lambda h} - 1}{h}\, {}^f R_\lambda u - \lim_{h\to 0} \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} \mathcal{T}_t u \, dt \right) \right] \\
&= \left[\frac{f(\lambda)}{\lambda} \left(\lim_{h\to 0} \frac{e^{\lambda h} - 1}{h}\, {}^f R_\lambda u - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} \mathcal{T}_t u \, dt \right) \right] \\
&= f(\lambda)\, {}^f R_\lambda u - \frac{f(\lambda)}{\lambda} u,
\end{aligned} \tag{4.122}$$

where in the third step we used (4.15).

With this in hand we note that ${}^f R_{\lambda,A}$ satisfies

$$\left\| {}^f R_\lambda \right\| = \frac{\Re f(\lambda)}{\Re\lambda} \left\| R_{f(\lambda)} \right\| \leq \frac{\Re f(\lambda)}{\Re\lambda} \frac{1}{\Re f(\lambda)} = \frac{1}{\Re\lambda}. \tag{4.123}$$

Furthermore we can formally write

$$\int_0^\infty e^{-\lambda t} \mathcal{T}_t dt = \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} T_s \, ds = \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-s(f(\lambda) - A)} ds$$

$$= \frac{f(\lambda)}{\lambda} \frac{1}{f(\lambda) - A}, \tag{4.124}$$

where we used Proposition 4.3.2 to state that $\mathcal{L}\left[l_\bullet(s)\right](\lambda) = \frac{f(\lambda)}{\lambda} e^{-sf(\lambda)}$ and $l_t(s)$ represents by abuse of notation the density of $l_t(ds)$. In (4.124) we used the exponential representation $T_t = e^{tA}$. Since we do not assume that A is bounded the symbol e^{tA} should be intended as $e^{tA}u = \text{strong-}\lim_{\lambda\to\infty} e^{tA_\lambda}u$ (Yosida approximation) where $A_\lambda := \lambda A R_\lambda$.

Now we have to prove that $\forall u \in \text{Dom}(A)$ we must have ${}^f R_\lambda u \in \text{Dom}(A)$ and

$$\left(f(\lambda) - A\right) {}^f R_\lambda u \, = \, {}^f R_\lambda \left(f(\lambda) - A\right) u \, = \, \frac{f(\lambda)}{\lambda} u. \tag{4.125}$$

Now by the definition

$$A_h \, = \, \frac{1}{h} \left(T_h u - u\right) \tag{4.126}$$

for which $\lim_{h\to 0} A_h = A$, we find

$$\begin{aligned}
A_h \, {}^f R_\lambda u \, &= \frac{T_h - I}{h} \int_0^\infty e^{-\lambda t} \int_0^\infty T_s u \, l_t(ds) \, dt \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{T_{s+h}u - T_s u}{h} l_t(ds) \, dt \\
&= \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} \frac{T_{s+h}u - T_s u}{h} \, ds \\
&= \frac{e^{hf(\lambda)}}{h} \frac{f(\lambda)}{\lambda} \int_h^\infty e^{-sf(\lambda)} T_z u \, dz \, - \frac{1}{h} \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} T_s u \, ds \\
&= \frac{f(\lambda)}{\lambda} \frac{e^{hf(\lambda)} - 1}{h\lambda} \int_0^\infty e^{-zf(\lambda)} T_z u \, dz \, - \frac{f(\lambda)}{\lambda} \frac{1}{h} \int_0^h e^{-sf(\lambda)} T_s u \, ds \\
&\overset{h\to 0}{\longrightarrow} f(\lambda) \, {}^f R_\lambda u \, - \, \frac{f(\lambda)}{\lambda} u.
\end{aligned} \tag{4.127}$$

This proves that ${}^f R_\lambda u \in \text{Dom}\,(A)$ and that $\left(f(\lambda) - A\right) {}^f R_\lambda u = \frac{f(\lambda)}{\lambda} u$. Furthemore we find

$$\begin{aligned}
{}^f R_\lambda A u \, &= \int_0^\infty e^{-\lambda t} \mathcal{T}_t A u \, dt \, = \int_0^\infty e^{-\lambda t} \int_0^\infty T_s A u \, l_t(ds) \, dt \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{\partial}{\partial s} T_s u \, l_t(ds) \\
&= \frac{f(\lambda)}{\lambda} \int_0^\infty e^{-sf(\lambda)} \frac{\partial}{\partial s} T_s u \, ds \\
&= - \frac{f(\lambda)}{\lambda} u + f(\lambda) \, {}^f R_\lambda u,
\end{aligned} \tag{4.128}$$

which completes the proof.

□

4.5.1 Convolution-type space-derivatives and Phillips' formula

Let T_t be a C_0-semigroup acting on functions $u \in \mathfrak{B}$, where $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is Banach space. Let μ_t be a convolution semigroup of sub-probability measures on $[0,\infty)$ such that $\mathcal{L}[\mu_t] = e^{-tf}$ where f is a Bernstein function. The operator defined by the Bochner integral

$$ {}^fT_t u \;=\; \int_0^\infty T_s u\, \mu_t(ds), \qquad u \in \mathfrak{B}, \tag{4.129} $$

is called a subordinate semigroup in the sense of Bochner. A classical result due to Phillips (1952) state that the infinitesimal generator $\left({}^fA, \mathrm{Dom}\left({}^fA\right)\right)$ of the subordinate semigroup ${}^fT$ on $u \in \mathfrak{B}$ is written as

$$ {}^fAu \;=\; -f\left(-A\right)u \;=\; -au + bAu + \int_0^\infty \left(T_s u - u\right) \bar{\nu}(ds), \tag{4.130} $$

with $\mathrm{Dom}\left(A\right) \subseteq \mathrm{Dom}\left({}^fA\right)$.

In Definition 4.2.8 we developed the convolution-type space-derivatives ${}^f\mathcal{D}_x^{\pm}$ defined on the whole real axis. We have shown that they becomes, for $f(x) = x^\alpha$, $\alpha \in (0,1)$, the Weyl space-fractional derivatives defined in (4.51) and (4.52). In this section we show that $-\,{}^f\mathcal{D}_x^-$ can be viewed as the infinitesimal generator of the subordinate semigroup in the sense of Bochner

$$ Q_t u(x) \;=\; \int_0^\infty T_s^l u(x) \mu_t(ds) \tag{4.131} $$

where $T_t^l u(x) = u(x+t)$, $u \in L^p\left(\mathbb{R}\right)$, is the left translation semigroup.

Remark 4.5.2. *We recall that the left translation operator $T_t^l u = u(x+t)$, $t > 0$, $u \in L^p\left(\mathbb{R}\right)$, defines a strongly continuous C_0-semigroup on $L^p\left(\mathbb{R}\right)$ (see for example Engel and Nagel (2000) page 66) and has infinitesimal generator $A = \frac{\partial}{\partial x}$ with $\mathrm{Dom}\left(A\right) = W^{1,p}$, $1 \leq p < \infty$, where*

$$ W^{1,p}\left(\mathbb{R}\right) \;=\; \left\{u \in L^p\left(\mathbb{R}\right) : u \text{ absolutely continuous and } u' \in L^p\left(\mathbb{R}\right)\right\}. \tag{4.132} $$

This implies that $-\,{}^f\mathcal{D}_x^-$ have to coincide with Phillips' representation (4.130) with $A = \frac{\partial}{\partial x}$.

Proposition 4.5.3. *Let ${}^f\sigma(t)$ be a subordinator with Laplace exponent f and transition probabilities μ_t. Let $\zeta = \inf\left\{t > 0 : {}^f\sigma(t) = +\infty\right\}$. The solution to the initial value problem*

$$ \begin{cases} \frac{\partial}{\partial t} q(x,t) \;=\; -\,{}^f\mathcal{D}_x^-\, q(x,t), & x \in \mathbb{R}, 0 < t < \zeta, \\ q(x,0) \;=\; u(x) \in W^{1,p}\left(\mathbb{R}\right), \end{cases} \tag{4.133} $$

is given by the contractive strongly continuous semigroup of operators on $L^p(\mathbb{R})$

$$Q_t u(x) = \int_0^\infty u(x+y)\,\mu_t(dy), \qquad t < \zeta \tag{4.134}$$

which is the subordinate translation semigroup $T_t^l u(x) = u(x+t)$*, in the sense of Bochner. The operator* ${}^f\mathcal{D}_x^-$ *is that of Definition 4.2.8 and* $W^{1,p}$ *is defined in (4.132).*

Proof. Since $Q_t u$ is a subordinate semigroup in the sense of Bochner, it defines again a C_0-semigroup on $L^p(\mathbb{R})$. By applying Phillips' result (Phillips (1952)) we know that the infinitesimal generator of $Q_t u$ is written as

$$-f\left(-\frac{\partial}{\partial x}\right)u(x) = -au(x) + b\frac{\partial}{\partial x}u(x) + \int_0^\infty \left(T_s^l u(x) - u(x)\right)\bar{\nu}(ds). \tag{4.135}$$

Since

$$\left\|-f\left(-\frac{\partial}{\partial x}\right)u(x)\right\|_p \leq a\left\|u(x)\right\|_p + b\left\|\frac{\partial}{\partial x}u(x)\right\|_p + \int_0^\infty \left\|T_s^l u(x) - u(x)\right\|_p \bar{\nu}(ds) \tag{4.136}$$

by applying the well-known inequality (see for example Jacob (2001))

$$\left\|T_t u(x) - u(x)\right\| \leq \left(t\left\|Au(x)\right\| \wedge 2\left\|u(x)\right\|\right), \qquad u \in \operatorname{Dom}(A) \tag{4.137}$$

which is valid in general for a strongly continuous semigroup $T_t u(x)$ on a Banach space $(\mathfrak{B}, \|\cdot\|)$ and infinitesimal generator $(A, \operatorname{Dom}(A))$, we can write

$$\begin{aligned}\left\|-f\left(-\frac{\partial}{\partial x}\right)\right\|_p \leq{}& a\left\|u(x)\right\|_p + b\left\|\frac{\partial}{\partial x}u(x)\right\|_p + \int_0^\epsilon z\bar{\nu}(dz)\left\|\frac{\partial}{\partial x}u(x)\right\|_p \\ &+ 2\int_\epsilon^\infty \bar{\nu}(dz)\left\|u(x)\right\|_p.\end{aligned} \tag{4.138}$$

This shows that

$$\operatorname{Dom}\left(-f\left(-\frac{\partial}{\partial x}\right)\right) = \begin{cases} W^{1,p}(\mathbb{R}), & \text{if } b > 0, \\ W^{1,p}(\mathbb{R}), & \text{if } b = 0 \text{ and } \bar{\nu}(0,\infty) = \infty, \\ L^p(\mathbb{R}), & \text{if } b = 0 \text{ and } \bar{\nu}(0,\infty) < \infty. \end{cases} \tag{4.139}$$

since for $\bar{\nu}(0,\infty) < \infty$ we can choose $\epsilon = 0$ in (4.138).

The Definition 4.2.8 of ${}^f\mathcal{D}_x^-$

$$-{}^f\mathcal{D}_x^- u(x) = b\frac{\partial}{\partial x}u(x) + \int_0^\infty \frac{\partial}{\partial x}u(x+s)\nu(s)ds \tag{4.140}$$

for $u \in \mathrm{Dom}\left(\frac{\partial}{\partial x}\right) = W^{1,p}(\mathbb{R})$ can be rewritten as

$$\begin{aligned}
-{}^f\mathcal{D}_x^- u(x) &= b\frac{\partial}{\partial x}u(x) + \int_0^\infty \frac{\partial}{\partial s}u(x+s)\left(a + \int_s^\infty \bar{\nu}(dz)\right) ds \\
&= -au(x) + b\frac{\partial}{\partial x}u(x) + \int_0^\infty \bar{\nu}(dz) \int_0^z \frac{\partial}{\partial s}T_s^l u(x)\, ds \\
&= -au(x) + b\frac{\partial}{\partial x}u(x) + \int_0^\infty \bar{\nu}(dz)\left(T_z^l u(x) - u(x)\right)
\end{aligned} \tag{4.141}$$

which coincides with (4.135). This shows that $\mathrm{Dom}\left(-{}^f\mathcal{D}_x^- u(x)\right) = W^{1,p}(\mathbb{R})$. □

4.6 Example: the tempered stable subordinator

By setting the Bernstein function considered in previous sections to be $f(x) = x^\alpha$, $\alpha \in (0,1)$, we retrive the stable subordinator $\sigma^\alpha(t)$, $t > 0$, for which $\mathbb{E}e^{-\lambda\sigma^\alpha(t)} = e^{-t\lambda^\alpha}$, and its inverse process $L^\alpha(t)$, $t > 0$. Therefore by performing the substitution $f(x) = x^\alpha$ all throughout the paper we retrive the results related to fractional calculus. In this section we take as example the Bernstein function

$$f(x) = (x+\vartheta)^\alpha - \vartheta^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-xs}\right) e^{-\vartheta s} s^{-1-\alpha}\, ds, \tag{4.142}$$

where $\vartheta > 0$, $\alpha \in (0,1)$. The Bernstein function (4.142) is the Laplace exponent of the subordinator ${}^\vartheta\sigma^\alpha(t)$ such that

$$\mathbb{E}e^{-\lambda\, {}^\vartheta\sigma^\alpha(t)} = e^{-t((\lambda+\vartheta)^\alpha - \vartheta^\alpha)}. \tag{4.143}$$

The process ${}^\vartheta\sigma^\alpha(t)$, $t > 0$, is known in literature as the relativistic stable subordinator since it appears in the study of the stability of the relativistic matter (Lieb (1990)) but it is also known as the *tempered stable subordinator* (see for example Meerschaert and Sikorskii (2012) page 207, Rosiński (2007) or Zolotarev (1986), Lemma 2.2.1). From (4.142) we know that the Lévy measure has the explicit representation

$$\bar{\nu}(s)ds = \frac{\alpha e^{-\vartheta s} s^{-\alpha-1}}{\Gamma(1-\alpha)} ds, \tag{4.144}$$

and has infinite mass ($f(x)$ is not bounded). Furthermore its tail becomes

$$\nu(s) = \left(\frac{\alpha\vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)}\right), \tag{4.145}$$

where

$$\Gamma(-\alpha, s) = \int_s^\infty e^{-z} z^{-\alpha-1}\, dz \tag{4.146}$$

is the incomplete Gamma function. It is well-known that the governing equations of ${}^{\vartheta}\sigma^{\alpha}(t)$, $t>0$, is written by using the so-called *tempered fractional derivative*

$$\partial_x^{\vartheta,\alpha} u(x) \;=\; e^{-\vartheta x} \frac{{}^R\partial^\alpha}{\partial x^\alpha} \left[e^{\vartheta x}\, u(x) \right] - \vartheta^\alpha u(x), \qquad \alpha \in (0,1), \tag{4.147}$$

as

$$\frac{\partial}{\partial t}\, \mu_t^{\vartheta,\alpha}(x) \;=\; -\partial_x^{\vartheta,\alpha}\, \mu_t^{\vartheta,\alpha}(x), \qquad x>0, t>0, \tag{4.148}$$

see Meerschaert and Sikorskii (2012) page 209 and the references therein. According to Theorem 4.83 we must have

$$\frac{\partial}{\partial t}\, \mu_t^{\vartheta,\alpha}(x) \;=\; -\, {}^f\mathcal{D}_x^{(0,\infty)}\, \mu_t^{\vartheta,\alpha}(x), \qquad x>0, t>0, \tag{4.149}$$

and indeed it is easy to show that if $f(\lambda) = (\lambda+\vartheta)^\alpha - \vartheta^\alpha$

$${}^f\mathcal{D}_x^{(0,\infty)} u(x) \;=\; \frac{d}{dx} \int_0^x u(x-s) \left(\frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right) ds \;=\; \partial_x^{\vartheta,\alpha} u(x). \tag{4.150}$$

This can be done for example by observing that

$$\mathcal{L} \left[\frac{d}{dx} \int_0^x u(x-s) \left(\frac{\alpha \vartheta^\alpha \Gamma(-\alpha, s)}{\Gamma(1-\alpha)} \right) ds \right] (\lambda) \;=\; \mathcal{L} \left[\partial_x^{\vartheta,\alpha} u(x) \right] (\lambda). \tag{4.151}$$

The time operator ${}^f\mathfrak{D}_t$ governing the density of

$${}^{\vartheta}L^{\alpha}(t) \;=\; \inf \left\{ s>0 : \, {}^{\vartheta}\sigma^{\alpha}(s) > t \right\}, \tag{4.152}$$

becomes in this case

$${}^f\mathcal{D}_t^{(0,\infty)} l_t^{\vartheta,\alpha}(x) \;=\; \frac{\partial}{\partial t} \int_0^t l_{t-s}^{\vartheta,\alpha}(x) \left(\frac{\alpha \vartheta^\alpha \Gamma(-\alpha,\, s)}{\Gamma(1-\alpha)} \right) ds, \tag{4.153}$$

and therefore $l_t^{\vartheta,\alpha}(x)$, $t>0$, is the solution to

$$\begin{cases} \frac{\partial}{\partial t} \int_0^t l_{t-s}^{\vartheta,\alpha}(x) \left(\frac{\alpha\vartheta^\alpha \Gamma(-\alpha,s)}{\Gamma(1-\alpha)} \right) ds \;=\; -\frac{\partial}{\partial x}\, l_t^{\vartheta,\alpha}(x), & t>0, x>0, \\ l_t^{\vartheta,\alpha}(0) \;=\; \frac{\alpha\vartheta^\alpha \Gamma(-\alpha,t)}{\Gamma(1-\alpha)}, & t>0, \\ l_0^{\vartheta,\alpha}(x) \;=\; \delta(x). \end{cases} \tag{4.154}$$

Finally, in view of Proposition 4.3.3, we are able to write the CTRW converging in distribution to ${}^{\vartheta}\sigma^{\alpha}(t)$, $t>0$. We have

$$\lim_{\gamma \to 0} \sum_{j=0}^{N\left(t\left(\frac{\alpha\vartheta^\alpha \Gamma(-\alpha,\gamma)}{\Gamma(1-\alpha)}\right)\right)} Y_j \;\overset{\text{law}}{\longrightarrow}\; {}^{\vartheta}\sigma^{\alpha}(t) \tag{4.155}$$

where Y_j are i.i.d. random variables with distribution

$$\Pr\left\{ Y_j \in dy \right\}/dy \;=\; \frac{e^{-\vartheta y} y^{-\alpha-1}}{\vartheta^\alpha \Gamma\left(-\alpha, \gamma\right)} \mathbb{I}_{[y>\gamma]}, \qquad \gamma>0, \, \forall j = 1,\dots,n, \tag{4.156}$$

and $N(t)$, $t>0$, is a homogeneous Poisson process with parameter $\theta=1$.

Chapter 5

Subordinate pseudoprocesses

Article: Orsingher and Toaldo (2013). Pseudoprocesses related to space-fractional higher-order heat-type equations.

Summary

In this paper we construct pseudo random walks (symmetric and asymmetric) which converge in law to compositions of pseudoprocesses stopped at stable subordinators. We find the higher-order space-fractional heat-type equations whose fundamental solutions coincide with the law of the limiting pseudoprocesses. The fractional equations involve either Riesz operators or their Feller asymmetric counterparts. The main result of this paper is the derivation of pseudoprocesses whose law is governed by heat-type equations of real-valued order $\gamma > 2$. The classical pseudoprocesses are very special cases of those investigated here.

5.1 Introduction

In this paper we consider pseudoprocesses related to different types of fractional higher-order heat-type equations. Our starting point is the set of higher-order equations of the form

$$\frac{\partial}{\partial t}u_m(x,t) = \kappa_m \frac{\partial^m}{\partial x^m}u_m(x,t), \qquad x \in \mathbb{R}, t > 0, m \in \mathbb{N} > 2, \tag{5.1}$$

whose solutions have been investigated by many outstanding mathematicians such as Bernstein (1919), Lévy (1923), Pòlya (1923) and also, more recently, by means of the steepest descent method, by Li and Wong (1993). In (5.1) the constant κ_m is usually chosen in the form

$$\kappa_m = \begin{cases} \pm 1, & m = 2n+1, \\ (-1)^{n+1}, & m = 2n. \end{cases} \tag{5.2}$$

In our investigations we assume throughout that $\kappa_m = (-1)^n$ when $m = 2n+1$. Pseudoprocesses related to (5.1) have been constructed in the same way as for the Wiener process by Daletsky (1969), Daletsky and Fomin (1965), Krylov (1960), Ladohin (1962), Miyamoto (1966). More recently pseudoprocesses related to (5.1) have been considered by Debbi (2006), Lachal (2003, 2013), Mazzucchi (2013). For equations of the form

$$\frac{\partial}{\partial t} u_\gamma(x,t) = \frac{\partial^\gamma}{\partial |x|^\gamma} u_\gamma(x,t), \qquad x \in \mathbb{R}, t > 0, \tag{5.3}$$

where $0 < \gamma \leq 2$, and $\frac{\partial^\gamma}{\partial |x|^\gamma}$ is the Riesz operator, the fundamental solution has the form of the density of a symmetric stable process as Riesz himself has shown. For $\gamma > 2$ the equation (5.3) was studied by Debbi (see Debbi (2006, 2007)) who proved the sign-varying character of the corresponding solutions.

For asymmetric fractional operators of the form

$$^F D^{\gamma,\theta} = -\left[\frac{\sin\frac{\pi}{2}(\gamma-\theta)}{\sin \pi\gamma} \frac{{}^+\partial^\gamma}{\partial x^\gamma} + \frac{\sin\frac{\pi}{2}(\gamma+\theta)}{\sin \pi\gamma} \frac{{}^-\partial^\gamma}{\partial x^\gamma} \right] \tag{5.4}$$

the equation

$$\frac{\partial}{\partial t} u_{\gamma,\theta}(x,t) = {}^F D^{\gamma,\theta} u_{\gamma,\theta}(x,t), \qquad x \in \mathbb{R}, t > 0, 0 < \gamma \leq 2, \tag{5.5}$$

was studied by Feller (1952) who proved that the fundamental solution to (5.5) is the law of an asymmetric stable process of order γ. The fractional derivatives appearing in (5.4) are the Weyl fractional derivatives defined as

$$\begin{aligned} \frac{{}^+\partial^\gamma}{\partial x^\gamma} u(x) &= \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dx^m} \int_{-\infty}^x \frac{u(y)}{(x-y)^{\gamma+1-m}} dy \\ \frac{{}^-\partial^\gamma}{\partial x^\gamma} u(x) &= \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dx^m} \int_x^{\infty} \frac{u(y)}{(y-x)^{\gamma+1-m}} dy \end{aligned} \tag{5.6}$$

where $m-1 < \gamma < m$. The Riesz fractional derivatives appearing in (5.3) are combinations of the Weyl's derivatives (5.6) and are defined as

$$\frac{\partial^\gamma}{\partial |x|^\gamma} = -\frac{1}{2\cos\frac{\pi\gamma}{2}} \left[\frac{{}^+\partial^\gamma}{\partial x^\gamma} + \frac{{}^-\partial^\gamma}{\partial x^\gamma} \right]. \tag{5.7}$$

This paper is devoted to pseudoprocesses related to fractional equations of the form (5.3) and (5.5) when $\gamma > 2$. Of course, this implies that Weyl's fractional derivatives (5.6) are considered in the case $\gamma > 2$. The fundamental solutions of these equations are sign-varying as in the case of higher-order heat-type equations (5.1) studied in the literature (compare with Debbi (2006)).

Fractional equations arise, for example, in the study of thermal diffusion in fractal and porous media (Nigmatullin (1986), Saichev and Zaslavsky (1997)). Other fields of application of fractional equations can be found in Debbi (2006). Higher-order equations emerge in many contexts as in trimolecular chemical reactions (Gardiner (1985) page 295) and in the linear approximation of the Korteweg De Vries equation (see Beghin et al. (2007)).

In our paper we study pseudo random walks (for the definitions and properties of pseudo random walks and variables see Lachal (2013)) of the form

$$W^{\gamma,2k\beta}(t) = \sum_{j=1}^{N\left(t\gamma^{-2k\beta}\right)} U_j^{2k}(1) Q_j^{\gamma,2k\beta} \tag{5.8}$$

where the r.v.'s $Q_j^{\gamma,2k\beta}$ are independent from the Poisson process N, from the pseudo r.v.'s $U_j^{2k}(1)$ and from each other and have distribution for $0 < \beta < 1$, $\gamma > 0$, $k \in \mathbb{N}$,

$$\Pr\left\{Q_j^{\gamma,2k\beta} > w\right\} = \begin{cases} 1, & w < \gamma, \\ \left(\frac{\gamma}{w}\right)^{2k\beta}, & w \geq \gamma. \end{cases} \tag{5.9}$$

The $U_j^{2k}(1)$ are independent pseudo r.v.'s with law $u_{2k}(x,1)$ with Fourier transform

$$\int_{-\infty}^{\infty} e^{i\xi x} u_{2k}(x,1)\,dx = e^{-|\xi|^{2k}}. \tag{5.10}$$

The Poisson process N appearing in (5.8) is homogeneous and has rate $\lambda = \frac{1}{\Gamma(1-\beta)}$. We prove that

$$\lim_{\gamma\to 0} W^{\gamma,2k\beta}(t) \overset{\text{law}}{=} U^{2k}\left(H^\beta(t)\right) \tag{5.11}$$

where U^{2k} is the pseudoprocess of order $2k$ related to the heat-type equation (5.1) for $m = 2k$ and H^β is a stable subordinator of order $\beta \in (0,1)$ independent from U^{2k}. We show that the law of (5.11) is the fundamental solution to

$$\frac{\partial}{\partial t} v_{2k\beta}(x,t) = \frac{\partial^{2k\beta}}{\partial |x|^{2k\beta}} v_{2k\beta}(x,t), \qquad x \in \mathbb{R}, t > 0, \beta \in (0,1), k \in \mathbb{N}. \tag{5.12}$$

In other words, we are able to construct pseudoprocesses of order $\gamma > 2$ in the form of integer-valued pseudoprocesses stopped at stable distributed times as the limit of suitable pseudo random walks. We consider also pseudo random walks of the form

$$\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)} \epsilon_j U_j^{2k+1}(1) Q_j^{\gamma,\beta(2k+1)} \tag{5.13}$$

where the $Q_j^{\gamma,\beta(2k+1)}$ have distribution (5.9) (suitably adjusted), $U_j^{2k+1}(1)$ is an odd-order pseudo random variable with law $u_{2k+1}(x,1)$ and Fourier transform

$$\int_{-\infty}^{\infty} e^{i\xi x} u_{2k+1}(x,1)\,dx \;=\; e^{-i\xi^{2k+1}} \tag{5.14}$$

and the ϵ_j's are random variables which take values ± 1 with probability p and q. All the variables in (5.13) are independent from each other and also independent from the Poisson process N with rate $\lambda = \frac{1}{\Gamma(1-\beta)}$. In this case we are able to show that

$$\lim_{\gamma\to 0} W^{\gamma,(2k+1)\beta}(t) \stackrel{\text{law}}{=} U_1^{2k+1}\left(H_1^\beta(pt)\right) - U_2^{2k+1}\left(H_2^\beta(qt)\right) \tag{5.15}$$

where H_j^β, $j=1,2$, are independent stable subordinators independent also from the pseudoprocesses U_1, U_2. We prove that the law of (5.15) satisfies the higher-order fractional equation

$$\frac{\partial}{\partial t} w_{\beta(2k+1)}(x,t) \;=\; \mathfrak{R} w_{\beta(2k+1)}(x,t), \qquad x\in\mathbb{R}, t>0, \tag{5.16}$$

where

$$\mathfrak{R} \;=\; -\frac{1}{\cos\frac{\beta\pi}{2}}\left[p e^{i\pi\beta k} \frac{{}^{+}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} + q e^{-i\pi\beta k} \frac{{}^{-}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} \right]. \tag{5.17}$$

The Fourier transform of the fundamental solution of (5.16) reads

$$\widehat{w}_{\beta(2k+1)}(\xi,t) \;=\; e^{-t|\xi|^{\beta(2k+1)}\left(1-i\,\mathrm{sign}(\xi)\,(p-q)\tan\frac{\beta\pi}{2}\right)} \tag{5.18}$$

We note that (5.18) corresponds to the Fourier transform of the law of (5.15) with a suitable change of the time-scale that is

$$\mathbb{E}\exp\left\{ i\xi\left[U_1^{2k+1}\left(H_1^\beta\left(\frac{pt}{\cos\frac{\beta\pi}{2}}\right)\right) - U_2^{2k+1}\left(H_2^\beta\left(\frac{qt}{\cos\frac{\beta\pi}{2}}\right)\right)\right]\right\}$$
$$= e^{-t|\xi|^{\beta(2k+1)}(1-i\,\mathrm{sign}(\xi)\,(p-q)\,\tan\frac{\beta\pi}{2})} \tag{5.19}$$

The mean value here and below must be understood with respect to the signed measure of the pseudoprocess (see for example Debbi (2006)). We study also the pseudoprocesses governed by the equation

$$\frac{\partial}{\partial t} z_{\beta(2k+1),\theta}(x,t) \;=\; {}^{F}D^{\beta(2k+1),\theta} z_{\beta(2k+1),\theta}(x,t) \tag{5.20}$$

where ${}^{F}D^{\beta(2k+1),\theta}$ is the operator defined in (5.4) with γ replaced by $\beta(2k+1)$. Also in this case we study continuous-time random walks whose limit has Fourier transform equal to

$$\mathbb{E}e^{i\xi Z^{\beta(2k+1),\theta}} \;=\; e^{-t|\xi|^{\beta(2k+1)} e^{\frac{i\pi\theta}{2}\,\mathrm{sign}(\xi)}}, \qquad \beta\in(0,1), k\geq 1, -\beta<\theta<\beta. \tag{5.21}$$

When we take into account pseudo random walks constructed by means of even-order pseudo random variables we arrive at limits $Z^{2\beta k,\theta}(t)$, $t > 0$, with Fourier transform

$$\mathbb{E}e^{i\xi Z^{2\beta k,\theta}(t)} = e^{-t|\xi|^{2k\beta}\frac{\cos\frac{\pi}{2}\theta}{\cos\frac{\pi}{2}\beta}} \tag{5.22}$$

which shows the symmetric structure of the limiting pseudoprocess.

5.1.1 List of symbols

For the reader's convenience we give a short list of the most important symbols and definitions appearing in the paper.

- The right Weyl fractional derivative for $m-1 < \gamma < m$, $m \in \mathbb{N}$, $x \in \mathbb{R}$

$$\frac{{}^{+}\partial^\gamma}{\partial x^\gamma}u(x,t) = \frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_{-\infty}^{x}\frac{u(y,t)}{(x-y)^{\gamma+1-m}}dy \tag{5.23}$$

- The left Weyl fractional derivative for $m-1 < \gamma < m$, $m \in \mathbb{N}$, $x \in \mathbb{R}$,

$$\frac{{}^{-}\partial^\gamma}{\partial x^\gamma}u(x,t) = \frac{(-1)^m}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_{x}^{\infty}\frac{u(y,t)}{(y-x)^{\gamma+1-m}}dy \tag{5.24}$$

- The Riesz fractional derivative for $m-1 < \gamma < m$, $m \in \mathbb{N}$, $x \in \mathbb{R}$,

$$\frac{\partial^\gamma}{\partial|x|^\gamma} = -\frac{1}{2\cos\frac{\gamma\pi}{2}}\left[\frac{{}^{+}\partial^\gamma}{\partial x^\gamma} + \frac{{}^{-}\partial^\gamma}{\partial x^\gamma}\right] \tag{5.25}$$

- We introduce the operator $\mathfrak{R}$, for $\beta \in (0,1)$, $k \in \mathbb{N}$, $p,q \in [0,1] : p+q = 1$, $x \in \mathbb{R}$,

$$\mathfrak{R} = -\frac{1}{\cos\frac{\beta\pi}{2}}\left[pe^{i\pi\beta k}\frac{{}^{+}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} + qe^{-i\pi\beta k}\frac{{}^{-}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}}\right] \tag{5.26}$$

- The Feller derivative for $m-1 < \gamma < m$, $m \in \mathbb{N}$, $\theta > 0$, $x \in \mathbb{R}$,

$$ {}^{F}D^{\gamma,\theta} = -\left[\frac{\sin\frac{\pi}{2}(\gamma-\theta)}{\sin\pi\gamma}\frac{{}^{+}\partial^\gamma}{\partial x^\gamma} + \frac{\sin\frac{\pi}{2}(\gamma+\theta)}{\sin\pi\gamma}\frac{{}^{-}\partial^\gamma}{\partial x^\gamma}\right] \tag{5.27}$$

- $U^m(t)$, $t > 0$ is a pseudoprocess of order $m \in \mathbb{N}$ with law $u_m(x,t)$, $x \in \mathbb{R}$, $t > 0$, governed by (5.1)

- $H^\beta(t)$ is a stable subordinator of order $\beta \in (0,1)$ with probability density $h_\beta(x,t)$, $x \geq 0$, $t \geq 0$.

5.2 Preliminaries and auxiliary results

In this paper we consider higher-order heat-type equations where the space derivative is fractional in different ways.

5.2.1 Weyl fractional derivatives.

First of all we consider equations of the form

$$\frac{\partial}{\partial t}u_\gamma(x,t) = \frac{{}^{\pm}\partial^\gamma}{\partial x^\gamma}u_\gamma(x,t), \qquad x \in \mathbb{R}, t>0, \gamma>0, \tag{5.28}$$

where $\frac{{}^{\pm}\partial^\gamma}{\partial x^\gamma}$ are the space-fractional Weyl derivatives defined as

$$\frac{{}^{+}\partial^\gamma}{\partial x^\gamma}u_\gamma(x,t) = \frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_{-\infty}^{x}\frac{u(z,t)\,dz}{(x-z)^{\gamma-m+1}}, \quad m-1<\gamma<1, m\in\mathbb{N}, \tag{5.29}$$

$$\frac{{}^{-}\partial^\gamma}{\partial x^\gamma}u_\gamma(x,t) = \frac{(-1)^m}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_{x}^{\infty}\frac{u(z,t)\,dz}{(z-x)^{\gamma-m+1}}, \quad m-1<\gamma<m, m\in\mathbb{N}. \tag{5.30}$$

In our analysis the following result on the Fourier transforms of Weyl derivatives is very important.

Theorem 5.2.1 (Samko et al. (1993), page 137)**.** *The Fourier transforms of* (5.29) *and* (5.30) *read*

$$\int_{-\infty}^{\infty} dx\, e^{i\xi x}\frac{{}^{+}\partial^\gamma}{\partial x^\gamma}u(x,t) = (-i\xi)^\gamma\,\widehat{u}(\xi,t) = |\xi|^\gamma e^{-\frac{i\pi\gamma}{2}\ sign(\xi)}\,\widehat{u}(\xi,t), \tag{5.31}$$

$$\int_{-\infty}^{\infty} dx\, e^{i\xi x}\frac{{}^{-}\partial^\gamma}{\partial x^\gamma}u(x,t) = (i\xi)^\gamma\,\widehat{u}(\xi,t) = |\xi|^\gamma e^{\frac{i\pi\gamma}{2}\ sign(\xi)}\,\widehat{u}(\xi,t). \tag{5.32}$$

Clearly $\widehat{u}(\xi,t)$ is the x-Fourier transform of $u(x,t)$.

Proof. We give a sketch of the proof of (5.31) with some details.

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, e^{i\xi x}\frac{{}^{+}\partial^\gamma}{\partial x^\gamma}u(x,t) &= \int_{-\infty}^{\infty} dx\, e^{i\xi x}\left[\frac{1}{\Gamma(m-\gamma)}\frac{\partial^m}{\partial x^m}\int_{-\infty}^{x}dz\frac{u(z,t)}{(x-z)^{\gamma-m+1}}\right]\\
&= \int_{-\infty}^{\infty} dx\, e^{i\xi x}\left[\frac{1}{\Gamma(m-\gamma)}\int_0^\infty dz\frac{\partial^m}{\partial x^m}\frac{u(x-z,t)}{z^{\gamma-m+1}}\right]\\
&= \int_{-\infty}^{\infty} dw\, e^{i\xi w}\frac{\partial^m}{\partial w^m}u(w,t)\,\frac{1}{\Gamma(m-\gamma)}\int_0^\infty dz\, e^{i\xi z}z^{m-\gamma-1}\\
&= (-i\xi)^m\int_{-\infty}^{\infty} e^{i\xi w}u(w,t)\,dw\,\frac{1}{\Gamma(m-\gamma)}\int_0^\infty dz\, e^{i\xi z}z^{m-\gamma-1}.
\end{aligned} \tag{5.33}$$

The result
$$\frac{(-i\xi)^m}{\Gamma(m-\gamma)}\int_0^\infty dz\, e^{i\xi z}\, z^{m-\gamma-1} \;=\; |\xi|^\gamma e^{-\frac{i\pi}{2}\,\mathrm{sign}(\xi)} \tag{5.34}$$
can be obtained for example by applying the Cauchy integral Theorem (see Samko et al. (1993) page 138). □

5.2.2 Riesz fractional derivatives

By means of the Weyl fractional derivatives we arrive at the Riesz fractional derivative, for $m-1<\gamma<m$, $m\in\mathbb{N}$,

$$\begin{aligned}\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t) &= -\frac{\frac{\partial^m}{\partial x^m}}{2\cos\frac{\pi\gamma}{2}\Gamma(m-\gamma)}\left[\int_{-\infty}^x \frac{u(y,t)\,dy}{(x-y)^{\gamma-m+1}} + \int_x^\infty \frac{(-1)^m\,u(y,t)\,dy}{(y-x)^{\gamma-m+1}}\right]\\ &= -\frac{1}{2\cos\frac{\pi\gamma}{2}}\left(\frac{{}^+\partial^\gamma}{\partial x^\gamma}+\frac{{}^-\partial^\gamma}{\partial x^\gamma}\right)u(x,t)\end{aligned} \tag{5.35}$$

In view of Theorem 5.2.1 we have that, for $\gamma>0$, $\gamma\notin\mathbb{N}$,

$$\begin{aligned}\int_{-\infty}^\infty dx\, e^{i\xi x}\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t) \;&=\; -\frac{1}{2\cos\frac{\pi\gamma}{2}}\left[|\xi|^\gamma e^{-\frac{i\pi\gamma}{2}\,\mathrm{sign}(\xi)} + |\xi|^\gamma e^{\frac{i\pi\gamma}{2}\,\mathrm{sign}(\xi)}\right]\widehat{u}(\xi,t)\\ &=\; -\,|\xi|^\gamma\,\widehat{u}(\xi,t).\end{aligned} \tag{5.36}$$

Remark 5.2.2. *The general fractional higher-order heat equation*

$$\frac{\partial}{\partial t}u_\gamma(x,t) \;=\; \frac{\partial^\gamma}{\partial|x|^\gamma}u_\gamma(x,t), \qquad x\in\mathbb{R}, t>0, \tag{5.37}$$

has solution whose Fourier transform reads

$$\widehat{u_\gamma}(\xi,t) \;=\; e^{-t|\xi|^\gamma}. \tag{5.38}$$

For $0<\gamma<2$, (5.38) corresponds to the characteristic function of the symmetric stable processes (this is a classical result due to M. Riesz himself).

5.3 From pseudo random walks to fractional pseudoprocesses

We consider in this section continuous-time pseudo random walks with steps which are pseudo random variables, that is measurable functions endowed with signed measures, and with total mass equal to one (see Lachal (2013)). In order to obtain in the limit pseudoprocesses whose signed law satisfies higher-order fractional equations we must construct sums of the form

$$\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)} \epsilon_j \, U_j^{2k+1}(1) \, Q_j^{\gamma,\beta(2k+1)}, \qquad \beta \in (0,1), k \in \mathbb{N}, \gamma > 0, \tag{5.39}$$

where

$$\epsilon_j = \begin{cases} 1, & \text{with probability } p, \\ -1, & \text{with probability } q, \end{cases} \qquad p+q=1. \tag{5.40}$$

The r.v.'s $Q_j^{\gamma,\beta(2k+1)}$ have probability distributions, for $\beta \in (0,1)$, $k \in \mathbb{N}$,

$$\Pr\left\{ Q_j^{\gamma,\beta(2k+1)} > w \right\} = \begin{cases} \left(\frac{\gamma}{w}\right)^{\beta(2k+1)}, & \text{for } w > \gamma \\ 1, & \text{for } w < \gamma. \end{cases} \tag{5.41}$$

The Poisson process $N(t)$, $t > 0$, appearing in (5.39) is homogeneous with rate

$$\lambda = \frac{1}{\Gamma(1-\beta)}, \qquad \beta \in (0,1). \tag{5.42}$$

The pseudo random variables (see Lachal (2013)) $U_j^{2k+1}(1)$ have law with Fourier transform

$$\int_{-\infty}^{\infty} dx \, e^{i\xi x} u_{2k+1}(x,1) = e^{-i\xi^{2k+1}} \tag{5.43}$$

and the function $u_{2k+1}(x,t)$, $x \in \mathbb{R}$, $t > 0$, is the fundamental solution to the odd-order heat-type equation, for $k \in \mathbb{N}$,

$$\begin{cases} \frac{\partial}{\partial t} u_{2k+1}(x,t) = (-1)^k \frac{\partial^{2k+1}}{\partial x^{2k+1}} u_{2k+1}(x,t), & x \in \mathbb{R}, t > 0, \\ u_{2k+1}(x,0) = \delta(x). \end{cases} \tag{5.44}$$

There is a vast literature devoted to odd-order heat-type equations of the form (5.44), to the behaviour of their solutions, and to the related pseudoprocesses (Beghin et al. (2007), Lachal (2003), Orsingher (1991), Orsingher and D'Ovidio (2012)).

The r.v.'s and pseudo r.v.'s appearing in (5.39) are independent and also independent from each other. We say that two pseudo r.v.'s (or pseudoprocesses) with signed density u_m^1, u_m^2, are independent if the Fourier transform $\mathcal{F}$ of the convolution $u_m^1 * u_m^2$ factorizes, that is

$$\mathcal{F}\left[u_m^1 * u_m^2\right](\xi) = \mathcal{F}\left[u_m^1\right](\xi) \, \mathcal{F}\left[u_m^2\right](\xi). \tag{5.45}$$

We are now able to state the first theorem of this section.

Theorem 5.3.1. *The following limit in distribution holds true*

$$\lim_{\gamma\to 0}\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)} \epsilon_j\, U_j^{2j+1}(1)\, Q_j^{\gamma,\beta(2k+1)} \overset{law}{=} U_1^{2k+1}\left(H_1^\beta(pt)\right) - U_2^{2k+1}\left(H_2^\beta(qt)\right), \tag{5.46}$$

where H_1^β and H_2^β are independent positively-skewed stable processes of order $0<\beta<1$ while U_1^{2k+1} and U_2^{2k+1} are independent pseudoprocesses of order $2k+1$. All the random variables $N(t)$, $t>0$, ϵ_j, $Q_j^{\gamma,\beta(2k+1)}$ are independent and also independent from the pseudo random variables $U_j^{2k+1}(1)$. The Fourier transform of the limiting pseudoprocess reads

$$\mathbb{E}e^{i\xi U_1^{2k+1}\left(H_1^\beta(pt)\right)-U_2^{2k+1}\left(H_2^\beta(qt)\right)} = e^{-t|\xi|^{\beta(2k+1)}\left(\cos\frac{\beta\pi}{2}-i\ sign(\xi)\,(p-q)\,\sin\frac{\beta\pi}{2}\right)}. \tag{5.47}$$

Proof. In view of the independence of the r.v's and pseudo random variables appearing in (5.46) we have that

$$\begin{aligned}
&\mathbb{E}e^{i\xi\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)}\epsilon_j U_j^{2k+1}(1)\, Q_j^{\gamma,\beta(2k+1)}}\\
&=\mathbb{E}\left[\mathbb{E}\left(e^{i\xi\epsilon U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)^{N\left(t\gamma^{-\beta(2k+1)}\right)}\right]\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{\beta(2k+1)}}\left(1-\mathbb{E}e^{i\xi\epsilon U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)\right\}\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{\beta(2k+1)}}\left(1-p\mathbb{E}e^{i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}-q\mathbb{E}e^{-i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)\right\}\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{\beta(2k+1)}}\left(p+q-p\mathbb{E}e^{i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}-q\mathbb{E}e^{-i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)\right\}\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{\beta(2k+1)}}\left(p\left(1-\mathbb{E}e^{i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)+q\left(1-\mathbb{E}e^{-i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)\right)\right\}.
\end{aligned} \tag{5.48}$$

We observe that

$$\begin{aligned}
&p\left(1-\mathbb{E}e^{i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)+q\left(1-\mathbb{E}e^{-i\xi U^{2k+1}(1)Q^{\gamma,\beta(2k+1)}}\right)\\
&=p\int_\gamma^\infty dw\left(1-\frac{\gamma^{\beta(2k+1)}\beta(2k+1)}{w^{\beta(2k+1)+1}}e^{i\xi^{2k+1}w^{2k+1}}\right)\\
&\quad+q\int_\gamma^\infty dw\left(1-\frac{\gamma^{\beta(2k+1)}\beta(2k+1)}{w^{\beta(2k+1)+1}}e^{-i\xi^{2k+1}w^{2k+1}}\right)
\end{aligned} \tag{5.49}$$

and therefore

$$\mathbb{E}e^{i\xi\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)}\epsilon_j U_j^{2k+1}(1)\, Q_j^{\gamma,\beta(2k+1)}} =$$

$$= \exp\left\{-\frac{\lambda t}{\gamma^{\beta(2k+1)}}\left[p\int_\gamma^\infty dw\left(1-\frac{\gamma^{\beta(2k+1)}\beta(2k+1)}{w^{\beta(2k+1)+1}}e^{i\xi^{2k+1}w^{2k+1}}\right)\right.\right.$$
$$\left.\left.+q\int_\gamma^\infty dw\left(1-\frac{\gamma^{\beta(2k+1)}\beta(2k+1)}{w^{\beta(2k+1)+1}}e^{-i\xi^{2k+1}w^{2k+1}}\right)\right]\right\}$$
$$= \exp\left\{\frac{-\lambda t}{\gamma^{\beta(2k+1)}}\left[p\left(1-e^{i(\xi\gamma)^{2k+1}}\right)-p\,i\xi^{2k+1}(2k+1)\int_\gamma^\infty\frac{dw\,\gamma^{\beta(2k+1)}e^{i(\xi w)^{2k+1}}}{w^{\beta(2k+1)-2k}}\right.\right.$$
$$\left.\left.+q\left(1-e^{-i(\xi\gamma)^{2k+1}}\right)+q\,i\xi^{2k+1}(2k+1)\int_\gamma^\infty\frac{dw\,\gamma^{\beta(2k+1)}\,e^{-i(\xi w)^{2k+1}}}{w^{\beta(2k+1)-2k}}\right]\right\}. \quad (5.50)$$

By taking the limit we get that

$$\lim_{\gamma\to 0}\mathbb{E}e^{i\xi\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)}\epsilon_j U_j^{2k+1}(1)\,Q_j^{\gamma,\beta(2k+1)}} =$$
$$= \exp\left[-\lambda t(2k+1)\left(-pi\xi^{2k+1}\int_0^\infty\frac{dw\,e^{i(\xi w)^{2k+1}}}{w^{\beta(2k+1)-2k}}+qi\xi^{2k+1}\int_0^\infty\frac{dw\,e^{-i(\xi w)^{2k+1}}}{w^{\beta(2k+1)-2k}}\right)\right]$$
$$= e^{-\lambda t\Gamma(1-\beta)\left[p\left(-i\xi^{2k+1}\right)^\beta+q(i\xi^{2k+1})^\beta\right]}. \quad (5.51)$$

By setting $\lambda = \frac{1}{\Gamma(1-\beta)}$ we obtain

$$\lim_{\gamma\to 0}\mathbb{E}e^{i\xi\sum_{j=0}^{N\left(t\gamma^{-\beta(2k+1)}\right)}\epsilon_j U_j^{2k+1}(1)\,Q_j^{\gamma,\beta(2k+1)}} = e^{-t\left(p|\xi|^{\beta(2k+1)}e^{-\frac{i\pi\beta}{2}\,\text{sign}(\xi)}+q|\xi|^{\beta(2k+1)}e^{\frac{i\pi\beta}{2}\,\text{sign}(\xi)}\right)}$$
$$= e^{-t|\xi|^{\beta(2k+1)}\left(\cos\frac{\pi\beta}{2}-i\,\text{sign}(\xi)\,(p-q)\sin\frac{\pi\beta}{2}\right)} \quad (5.52)$$

Now we consider the Fourier transform of the law of the pseudoprocess

$$V^{(2k+1)\beta}(t) = U_1^{2k+1}\left(H_1^\beta(pt)\right) - U_2^{2k+1}\left(H_2^\beta(qt)\right) \quad (5.53)$$

which reads

$$\mathbb{E}e^{i\xi V^{(2k+1)\beta}(t)} = \mathbb{E}e^{i\xi U_1^{2k+1}\left(H_1^\beta(pt)\right)}\mathbb{E}e^{-i\xi U_2^{2k+1}\left(H_2^\beta(qt)\right)}$$
$$= \left(\int_0^\infty ds\,e^{i\xi^{2k+1}s}\,h_\beta^1(s,pt)\right)\left(\int_0^\infty dz\,e^{-i\xi^{2k+1}z}\,h_\beta^2(z,qt)\right)$$
$$= e^{-t\,p\left(-i\xi^{2k+1}\right)^\beta}e^{-t\,q\left(i\xi^{2k+1}\right)^\beta}$$
$$= e^{-t\left(p|\xi|^{\beta(2k+1)}e^{-\frac{i\beta\pi}{2}\,\text{sign}(\xi)}+q|\xi|^{\beta(2k+1)}e^{\frac{i\beta\pi}{2}\,\text{sign}(\xi)}\right)}$$
$$= e^{-t|\xi|^{\beta(2k+1)}\left(\cos\frac{\pi\beta}{2}-i\,\text{sign}(\xi)\,(p-q)\sin\frac{\pi\beta}{2}\right)}, \quad (5.54)$$

and coincides with (5.52). □

Remark 5.3.2. *If $\beta = \frac{1}{2k+1}$ the Fourier transform (5.54) becomes*

$$\mathbb{E}e^{i\xi U_1^{2k+1}\left(H_1^\beta(pt)\right)}\mathbb{E}e^{-i\xi U_2^{2k+1}\left(H_2^\beta(qt)\right)} = e^{-t|\xi|\cos\frac{\pi}{2(2k+1)}+it\xi\sin\frac{\pi}{2(2k+1)}} \quad (5.55)$$

which corresponds to the characteristic function of a Cauchy r.v. with position parameter equal to $t(p-q)\sin\frac{\pi}{2(2k+1)}$ *and scale parameter* $t\cos\frac{\beta}{2(2k+1)}$*. This slightly generalizes result 1.4 of Orsingher and D'Ovidio (2012).*

For even-order pseudoprocesses we have the following limit in distribution.

Theorem 5.3.3. *If* $U^{2k}(t)$*,* $t>0$*, is an even-order pseudoprocess and* $N(t)$*,* $t>0$*, is a homogeneous Poisson process, independent from* $U^{2k}(t)$*,* $t>0$*, we have that*

$$\lim_{\gamma\to 0}\sum_{j=0}^{N\left(t\gamma^{-2k\beta}\right)} U_j^{2k}(1)Q_j^{\gamma,2k\beta} \overset{law}{=} U^{2k}\left(H^\beta(t)\right), \qquad t>0, \tag{5.56}$$

where H^β *is a stable subordinator of order* $\beta\in(0,1)$ *and* $Q_j^{\gamma,2k\beta}$ *are i.i.d. random variables with distribution*

$$\Pr\left\{Q_j^{\gamma,2k\beta}>w\right\} = \begin{cases} 1, & w<\gamma,\\ \left(\frac{\gamma}{w}\right)^{2k\beta}, & w>\gamma. \end{cases} \tag{5.57}$$

The pseudoprocess $U^{2k}(t)$ *is governed by the equation*

$$\frac{\partial}{\partial t}u_{2k}(x,t) = (-1)^{k+1}\frac{\partial^{2k}}{\partial x^{2k}}u_{2k}(x,t), \qquad x\in\mathbb{R}. \tag{5.58}$$

Proof. We start by evaluating the Fourier transform

$$\begin{aligned}
&\mathbb{E}e^{i\xi\sum_{j=0}^{N\left(t\gamma^{-2k\beta}\right)} U_j^{2k}(1)Q_j^{\gamma,2k\beta}}\\
&=\mathbb{E}\left[\mathbb{E}\left(e^{i\xi U^{2k}(1)Q^{\gamma,2k\beta}}\right)^{N\left(t\gamma^{-2k\beta}\right)}\right]\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{2k\beta}}\left(1-\mathbb{E}e^{i\xi U^{2k}(1)Q^{\gamma,2k\beta}}\right)\right\}\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{2k\beta}}\int_\gamma^\infty dy\left(1-e^{-|\xi|^{2k}y^{2k}}\right)\right\}\frac{2k\beta\gamma^{2k\beta}}{y^{2k\beta+1}}\\
&=\exp\left\{-\frac{\lambda t}{\gamma^{2k\beta}}\left[\left(1-e^{-|\xi|^{2k}\gamma^{2k}}\right)+\int_\gamma^\infty dy\, e^{-|\xi|^{2k}y^{2k}}y^{2k-1-2k\beta}\,2k\gamma^{2k\beta}\right]\right\}
\end{aligned} \tag{5.59}$$

By taking the limit we have that

$$\begin{aligned}
\lim_{\gamma\to 0} e^{i\xi\sum_{j=0}^{N\left(t\gamma^{-2k\beta}\right)} U_j^{2k}(1)Q_j^{\gamma,2k\beta}} &= e^{-\lambda t|\xi|^{2k}2k\int_0^\infty e^{-|\xi|^{2k}y^{2k}}y^{2k(1-\beta)-1}dy}\\
&= e^{-\lambda t|\xi|^{2k\beta}\int_0^\infty e^{-w}w^{-\beta}dw}\\
&= e^{-\lambda t|\xi|^{2k\beta}\Gamma(1-\beta)}
\end{aligned} \tag{5.60}$$

which coincides with

$$\mathbb{E}e^{i\xi U^{2k}\left(H^\beta(t)\right)} = \int_0^\infty e^{-s\xi^{2k}}\Pr\left\{H^\beta(t)\in ds\right\} = e^{-t|\xi|^{2k\beta}} \tag{5.61}$$

since $\lambda=\frac{1}{\Gamma(1-\beta)}$. □

Remark 5.3.4. *For $\beta = \frac{1}{k}$ the composition $U^{2k}\left(H^{\beta}(t)\right)$ has Gaussian distribution. For $\beta = \frac{1}{2k}$ we have instead the Cauchy distribution and for $\beta = \frac{1}{4k}$ we extract the inverse Gaussian corresponding to the distribution of the first passage time of a Brownian motion. The case $\beta = \frac{1}{6k}$ yields the stable law with distribution*

$$f_{\frac{1}{3}}(x) = \frac{t}{x\sqrt[3]{3x}}\, Ai\left(\frac{t}{\sqrt[3]{3x}}\right) \tag{5.62}$$

where Ai denotes the Airy function (see Orsingher and D'Ovidio (2012)).

In order to arrive at asymmetric higher-order fractional pseudoprocesses we construct pseudo random walks by adapting the Feller approach (used for asymmetric stable laws) to our context. This means that we combine independent pseudo random walks with suitable trigonometric weights as in (5.4).

Theorem 5.3.5. *Let $X_j^{\gamma,(2k+1)\beta}$ and $Y_j^{\gamma,(2k+1)\beta}$ be i.i.d. r.v.'s with distribution*

$$\Pr\left\{X^{\gamma,(2k+1)\beta} > w\right\} = \begin{cases} \left(\frac{\gamma}{w}\right)^{(2k+1)\beta}, & w > \gamma \\ 1, & w < \gamma, \end{cases} \tag{5.63}$$

and let $U^{2k+1}(t)$, $t > 0$, be a pseudoprocess of odd-order $2k+1$, $k \in \mathbb{N}$. For $0 < \beta < 1$ and $-\beta < \theta < \beta$ we have that

$$\lim_{\gamma\to 0}\left[\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}} \sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)} X_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1)\right.$$
$$\left. - \left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}} \sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)} Y_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1)\right] \overset{law}{=} Z^{\beta(2k+1),\theta} \tag{5.64}$$

where

$$\mathbb{E}e^{i\xi Z^{\beta(2k+1),\theta}} = e^{-t|\xi|^{(2k+1)\beta} e^{\frac{i\pi\theta}{2}}} \tag{5.65}$$

Proof. The Fourier transform of (5.64) is written as

$$\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}} \sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)} X_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1)}$$
$$\times\, \mathbb{E}e^{-i\xi\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}} \sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)} Y_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1)} \tag{5.66}$$

where the first member is given by

$$\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}} \sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)} X_j^{\gamma,(2k+1)\beta} U_j^{2k+1}(1)} =$$

$$= \exp\left\{-\frac{\lambda t}{\gamma^{(2k+1)\beta}}\left(1-\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}}U^{2k+1}(1)X^{(2k+1)\beta}}\right)\right\}$$

$$= \exp\left\{-\frac{\lambda t}{\gamma^{(2k+1)\beta}}\int_\gamma^\infty\left(1-e^{i\xi^{2k+1}\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{\beta}}y^{2k+1}}\right)\frac{\gamma^{(2k+1)\beta}}{y^{(2k+1)\beta+1}}(2k+1)\beta\right\}$$

$$\stackrel{\gamma\to 0}{\longrightarrow} \exp\left\{-\lambda t\, i^{-\beta}\,\xi^{(2k+1)\beta}\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)\int_0^\infty e^{-w}w^{-\beta}dw\right\}$$

$$= e^{-\lambda t|\xi|^{(2k+1)\beta}e^{-\frac{i\pi\beta\,\mathrm{sign}(\xi)}{2}}\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)\Gamma(1-\beta)}. \tag{5.67}$$

The second member of (5.64) becomes, by performing a similar calculation,

$$\mathbb{E}e^{-i\xi\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}}\sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)}Y_j^{\gamma,(2k+1)\beta}U_j^{2k+1}(1)}$$
$$\stackrel{\gamma\to 0}{\longrightarrow} e^{-\lambda t|\xi|^{(2k+1)\beta}e^{\frac{i\pi\beta\,\mathrm{sign}(\xi)}{2}}\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)\Gamma(1-\beta)}. \tag{5.68}$$

and thus for $\lambda = \frac{1}{\Gamma(1-\beta)}$ we obtain that

$$\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}}\sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)}X_j^{\gamma,(2k+1)\beta}U_j^{2k+1}(1)}$$
$$\times\mathbb{E}e^{-i\xi\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{(2k+1)\beta}}\sum_{j=0}^{N\left(t\gamma^{-(2k+1)\beta}\right)}Y_j^{\gamma,(2k+1)\beta}U_j^{2k+1}(1)}$$
$$\stackrel{\gamma\to 0}{\longrightarrow} e^{-t|\xi|^{(2k+1)\beta}e^{-\frac{i\pi\beta\,\mathrm{sign}(\xi)}{2}}\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)}e^{-t|\xi|^{(2k+1)\beta}e^{\frac{i\pi\beta\,\mathrm{sign}(\xi)}{2}}\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)}$$
$$= e^{-t|\xi|^{(2k+1)\beta}e^{\frac{i\pi\theta}{2}\,\mathrm{sign}(\xi)}} \tag{5.69}$$

□

By considering symmetric pseudo random walks with the Feller construction we arrive in the next theorem at symmetric pseudoprocesses with time scale equal to $\frac{\cos\frac{\pi\beta}{2}}{\sin\frac{\pi\beta}{2}}$, $0<\beta<1$, $-\beta<\theta<\beta$.

Theorem 5.3.6. *Let $X_j^{\gamma,2\beta k}$ and $Y_j^{\gamma,2\beta k}$ be i.i.d. r.v.'s with distribution*

$$\Pr\left\{X^{\gamma,2\beta k}>w\right\} = \begin{cases}\left(\frac{\gamma}{w}\right)^{2\beta k}, & w>\gamma\\ 1, & w<\gamma,\end{cases} \tag{5.70}$$

and let $U^{2k}(t)$, $t>0$, be a pseudoprocess of order $2k$, $k\in\mathbb{N}$. If $N(t)$ is a homogeneous Poisson process, with parameter $\lambda=\frac{1}{\Gamma(1-\beta)}$, independent from $X_j^{\gamma,2\beta k}$ and $Y_j^{\gamma,2\beta k}$ we have that

$$\lim_{\gamma\to 0}\left[\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}X_j^{\gamma,2\beta k}\,U_j^{2k}(1)\right.$$

$$+\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}Y_j^{\gamma,2\beta k}\,U_j^{2k}(1)\Bigg] \overset{law}{=} Z^{2k\beta,\theta}, \qquad t>0, \tag{5.71}$$

for $0<\beta<1$ *and* $-\beta<\theta<\beta$ *and*

$$\mathbb{E}e^{i\xi Z^{2k\beta,\theta}} = e^{-t|\xi|^{2k\beta}\frac{\cos\frac{\pi}{2}\theta}{\cos\frac{\pi}{2}\beta}} \tag{5.72}$$

Proof. The Fourier transform of (5.71) is written as

$$\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}X_j\,U_j^{2k}(1)}\,\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}Y_j\,U_j^{2k}(1)} \tag{5.73}$$

where the first member is given by

$$\begin{aligned}
&\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}X_j^{\gamma,2\beta k}\,U_j^{2k}(1)} = \\
&= \exp\left\{-\frac{\lambda t}{\gamma^{2k\beta}}\left[1-\mathbb{E}e^{i\xi\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}U^{2k}(1)X^{\gamma,2\beta k}}\right]\right\} \\
&= \exp\left\{-\frac{\lambda t}{\gamma^{2k\beta}}\left[\int_\gamma^\infty e^{-\left|\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{2k\beta}}\right|^{2k}y^{2k}}(2k\beta)\frac{\gamma^{2k\beta}}{y^{2k\beta+1}}dy\right]\right\} \\
&\overset{\gamma\to 0}{\longrightarrow} \exp\left\{-\lambda t|\xi|^{2k}\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{\beta}}2k\int_0^\infty e^{-|\xi|^{2k}\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{\beta}}y^{2k}}y^{2k-1-2k\beta}dy\right\} \\
&= e^{-\lambda t|\xi|^{2k\beta}\Gamma(1-\beta)\left[\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right]}
\end{aligned} \tag{5.74}$$

and by similar calculations the second member becomes

$$\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}Y_j^{\gamma,2\beta k}\,U_j^{2k}(1)} \overset{\gamma\to 0}{\longrightarrow} e^{-\lambda t|\xi|^{2k\beta}\Gamma(1-\beta)\left[\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right]}. \tag{5.75}$$

Thus we have that

$$\begin{aligned}
&\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}X_j^{\gamma,2\beta k}\,U_j^{2k}(1)}\,\mathbb{E}e^{i\xi\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right)^{\frac{1}{2\beta k}}\sum_{j=0}^{N\left(t\gamma^{-2\beta k}\right)}Y_j^{\gamma,2\beta k}\,U_j^{2k}(1)} \\
&\overset{\gamma\to 0}{\longrightarrow} e^{-\lambda t|\xi|^{2k\beta}\Gamma(1-\beta)\left[\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}+\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\right]} \\
&= e^{-t|\xi|^{2\beta k}\frac{\cos\frac{\pi}{2}\theta}{\cos\frac{\pi}{2}\beta}}
\end{aligned} \tag{5.76}$$

□

5.4 Governing equations

In the previous section we obtained fractional pseudoprocesses as limit of suitable pseudo random walks. In this section we will show that the limiting fractional pseudoprocesses obtained before have signed density satisfying space-fractional heat-type equations of higher-order with Riesz or Feller fractional derivatives. The order of fractionality of the governing equations is a positive real number and this is the major difference with respect to the pseudoprocesses considered so far in the literature.

We start by examining space fractional higher-order equations of order $2k\beta$, $\beta \in (0,1)$, $k \in \mathbb{N}$, which interpolate equations of the form (5.1).

Theorem 5.4.1. *The solution to the initial-value problem*

$$\begin{cases} \frac{\partial}{\partial t} v_{2k}^{\beta}(x,t) = \frac{\partial^{2k\beta}}{\partial |x|^{2k\beta}} v_{2k}^{\beta}(x,t), & x \in \mathbb{R}, t > 0, k \in \mathbb{N}, \beta \in (0,1) \\ v_{2k}^{\beta}(x,0) = \delta(x) \end{cases} \tag{5.77}$$

can be written as

$$\begin{aligned} v_{2k}^{\beta}(x,t) &= \frac{1}{\pi x} \mathbb{E}\left[\sin\left(x G^{2k}\left(\frac{1}{H^{\beta}(t)} \right) \right) \right] \\ &= \frac{1}{\pi x} \mathbb{E}\left[\sin\left(x G^{2k\beta}\left(\frac{1}{t} \right) \right) \right] \end{aligned} \tag{5.78}$$

and coincides with the law of the pseudoprocess

$$V^{2k\beta}(t) = U^{2k}\left(H^{\beta}(t) \right), \qquad t > 0, \tag{5.79}$$

where U^{2k} is related to equation (5.1) for $m = 2k$ and H^{β} is a stable subordinator independent from U^{2k}. $G^{\gamma}(t)$ is a gamma r.v. with density

$$g^{\gamma}(x,t) = \gamma \frac{x^{\gamma-1}}{t} e^{-\frac{x^{\gamma}}{t}}, \qquad x > 0, t > 0, \gamma > 0. \tag{5.80}$$

Proof. The Fourier transform of (5.77) leads to the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} \widehat{v}_{2k}^{\beta}(\xi,t) = -|\xi|^{2k\beta} \widehat{v}_{2k}^{\beta}(\xi,t) \\ \widehat{v}_{2k}^{\beta}(\xi,0) = 1, \end{cases} \tag{5.81}$$

whose unique solution reads

$$\begin{aligned} \mathbb{E} e^{i\xi V^{2k\beta}(t)} &= \int_{\mathbb{R}} dx\, e^{i\xi x} \int_0^{\infty} ds\, u_{2k}(x,s)\, h_{\beta}(s,t) \\ &= \int_0^{\infty} ds\, e^{-s\xi^{2k}} h_{\beta}(s,t) = e^{-t|\xi|^{2k\beta}}. \end{aligned} \tag{5.82}$$

In (5.82) u_{2k} is the density of U^{2k} and $h_\beta(x,t)$ is the probability density of the subordinator H^β. Now we show that the Fourier transform of (5.78) coincides with (5.82). We have that

$$\begin{aligned}\widehat{v}^{\beta}_{2k}(\xi,t) &= \int_{\mathbb{R}} dx e^{i\xi x}\,\frac{1}{\pi x}\mathbb{E}\left[\sin\left(xG^{2k}\left(\frac{1}{H^\beta(t)}\right)\right)\right] \\ &= \int_{\mathbb{R}} dx\, e^{i\xi x}\left[\int_0^\infty\int_0^\infty \frac{\sin xy}{\pi x}\Pr\left\{G^{2k}\left(\frac{1}{s}\right)\in dy\right\}\Pr\left\{H^\beta(t)\in ds\right\}\right] \\ &= \int_0^\infty\int_0^\infty \Pr\left\{G^{2k}\left(\frac{1}{s}\right)\in dy\right\}\Pr\left\{H^\beta(t)\in ds\right\}\left[\int_{\mathbb{R}} dx\, e^{i\xi x}\frac{\sin xy}{\pi x}\right].\end{aligned} \tag{5.83}$$

By considering that the Heaviside function

$$\mathcal{H}_\alpha(z) = \begin{cases}1, & z>\alpha,\\ 0, & z<\alpha\end{cases} \tag{5.84}$$

can be represented as

$$\mathcal{H}_\alpha(z) = \frac{1}{2\pi}\int_{\mathbb{R}} dw\, e^{iwz}\frac{e^{-i\alpha w}}{iw} = -\frac{1}{2\pi}\int_{\mathbb{R}} dw\, e^{-iwz}\frac{e^{i\alpha w}}{iw}, \tag{5.85}$$

we obtain that formula (5.83) becomes

$$\begin{aligned}&\widehat{v}^{\beta}_{2k}(\xi,t) = \\ &= \int_0^\infty\int_0^\infty \Pr\left\{G^{2k}\left(\frac{1}{s}\right)\in dy\right\}\Pr\left\{H^\beta(t)\in ds\right\}\left[\mathcal{H}_{-y}(\xi)-\mathcal{H}_{y}(\xi)\right] \\ &= \int_0^\infty\int_0^\infty \Pr\left\{G^{2k}\left(\frac{1}{s}\right)\in dy\right\}\Pr\left\{H^\beta(t)\in ds\right\}\left[\mathbb{I}_{[-\xi,+\infty]}(y)-\mathbb{I}_{[-\infty,\xi]}(y)\right] \\ &= \int_0^\infty\int_0^\infty dy\, ds\left(2ksy^{2k-1}e^{-sy^{2k}}\right)\mathbb{I}_{[0,\infty]}(y)\left[\mathbb{I}_{[-\xi,+\infty]}(y)-\mathbb{I}_{[-\infty,\xi]}(y)\right]h_\beta(s,t).\end{aligned} \tag{5.86}$$

For $\xi>0$ (5.86) becomes

$$\begin{aligned}\widehat{v}^{\beta}_{2k}(\xi,t) &= \int_0^\infty ds\left[1-\int_0^\xi dy\, 2ksy^{2k-1}e^{-y^{2k}s}\right]h_\beta(s,t) \\ &= \int_0^\infty ds\, e^{-\xi^{2k}s}h_\beta(s,t) = e^{-t|\xi|^{2k\beta}},\end{aligned} \tag{5.87}$$

and for $\xi<0$ (5.86) is

$$\begin{aligned}\widehat{v}^{\beta}_{2k}(\xi,t) &= \int_0^\infty ds\left[\int_{-\xi}^\infty 2ksy^{2k-1}e^{-y^{2k}s}\right]h_\beta(s,t) \\ &= \int_0^\infty ds\, e^{-|\xi|^{2k}s}\,h_\beta(s,t) = e^{-t|\xi|^{2k\beta}}.\end{aligned} \tag{5.88}$$

Since

$$\begin{aligned}\Pr\left\{G^{2k}\left(\frac{1}{H^\beta(t)}\right)\in dy\right\}/dy &= 2ky^{2k-1}\int_0^\infty se^{-sy^{2k}}h_\beta(s,t)\,ds\\ &= -\frac{\partial}{\partial y}\int_0^\infty e^{-sy^{2k}}h_\beta(s,t)\,ds\\ &= -\frac{\partial}{\partial y}e^{-y^{2k\beta}t}\\ &= \Pr\left\{G^{2k\beta}\left(\frac{1}{t}\right)\in dy\right\}/dy \end{aligned} \tag{5.89}$$

the second form of the solution (5.78) follows immediately. □

For $k \geq 1$, $\beta \in \left(0, \frac{1}{k}\right]$ the solutions (5.78) are densities of symmetric random variables, while for $\beta > \frac{1}{k}$ the functions (5.78) are sign-varying. Clearly for $\beta = 1$ we obtain the solution of even-order heat-type equations discussed in Orsingher and D'Ovidio (2012). As far as space-fractional higher-order heat-type equations we have the result of the next theorem where the governing fractional operator $\mathfrak{R}$ is obtained as a suitable combination of Weyl derivatives. The operator $\mathfrak{R}$ governing the fractional pseudoprocesses appearing in Theorem 5.3.1 is explicitely written for $\{p,q \in [0,1] : p+q=1\}$, $\{\beta \in (0,1),\, k \in \mathbb{N} : m-1 < \beta(2k+1) < m,\, m \in \mathbb{N}\}$ as

$$\begin{aligned}&\mathfrak{R}\, v^\beta_{2k+1}(x,t) =\\ &= -\frac{1}{\cos\frac{\pi\beta}{2}}\left[p\, e^{i\pi\beta k}\frac{{}^{+}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} + q\, e^{-i\pi\beta k}\frac{{}^{-}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}}\right] v^\beta_{2k+1}(x,t)\\ &= -\frac{1}{\cos\frac{\pi\beta}{2}\Gamma(m-(2k+1)\beta)}\frac{\partial^m}{\partial x^m}\left[e^{i\pi\beta k}p\int_{-\infty}^x \frac{v^\beta_{2k+1}(y,t)}{(x-y)^{(2k+1)\beta-m+1}}dy\right.\\ &\quad\left. +q\, e^{-i\pi\beta k}(-1)^m\int_x^\infty \frac{v^\beta_{2k+1}(y,t)}{(y-x)^{(2k+1)\beta-m+1}}dy\right], \end{aligned} \tag{5.90}$$

where the left and right Weyl fractional derivatives appear.

Theorem 5.4.2. *The solution to the problem*

$$\begin{cases}\frac{\partial}{\partial t}v^\beta_{2k+1}(x,t) = \mathfrak{R}\, v^\beta_{2k+1}(x,t), & x\in\mathbb{R}, t>0, \beta\in(0,1), k\in\mathbb{N},\\ v^\beta_{2k+1}(x,0) = \delta(x), \end{cases} \tag{5.91}$$

is given by the signed law of the pseudoprocess

$$\bar{V}^{\beta(2k+1)}(t) = U_1^{2k+1}\left(H_1^\beta\left(\frac{pt}{\cos\frac{\beta\pi}{2}}\right)\right) - U_2^{2k+1}\left(H_2^\beta\left(\frac{qt}{\cos\frac{\beta\pi}{2}}\right)\right), \tag{5.92}$$

where U_1^{2k+1}, U_2^{2k+1} are independent odd-order pseudoprocesses and H_1^β, H_2^β, are independent stable subordinators.

Proof. The Fourier transform of (5.90) is written as

$$\begin{aligned}
&\mathcal{F}\left[\Re\, v^{\beta}_{2k+1}(x,t)\right](\xi) = \\
&= \mathcal{F}\left[-\frac{1}{\cos\frac{\pi\beta}{2}}\left[p\, e^{i\pi\beta k}\frac{{}^{+}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}} + q\, e^{-i\pi\beta k}\frac{{}^{-}\partial^{\beta(2k+1)}}{\partial x^{\beta(2k+1)}}\right] v^{\beta}_{2k+1}(x,t)\right](\xi) \\
&= -\frac{1}{\cos\frac{\beta\pi}{2}}\left[p\, e^{i\pi\beta k}\,(-i\xi)^{\beta(2k+1)} + q\, e^{-i\pi\beta k}\,(i\xi)^{\beta(2k+1)}\right]\widehat{v}^{\beta}_{2k+1}(\xi,t) \\
&= -\frac{1}{\cos\frac{\beta\pi}{2}}|\xi|^{\beta(2k+1)}\left[pe^{-\frac{i\pi\beta}{2}\,\mathrm{sign}(\xi)} + qe^{\frac{i\pi\beta}{2}\,\mathrm{sign}(\xi)}\right]\widehat{v}^{\beta}_{2k+1}(\xi,t) \\
&= -\,|\xi|^{\beta(2k+1)}\left(1 - i\,\mathrm{sign}(\xi)\,(p-q)\tan\frac{\pi\beta}{2}\right)\widehat{v}^{\beta}_{2k+1}(\xi,t) \qquad (5.93)
\end{aligned}$$

and therefore we have that

$$\widehat{v}^{\beta}_{2k+1}(\xi,t) = e^{-t|\xi|^{\beta(2k+1)}\left(1-i(p-q)\,\mathrm{sign}(\xi)\tan\frac{\beta\pi}{2}\right)} \qquad (5.94)$$

In view of (5.54) we get

$$\begin{aligned}
\mathbb{E}e^{i\xi\bar{V}^{(2k+1)\beta}(t)} &= \mathbb{E}e^{i\xi V^{(2k+1)\beta}\left(\frac{t}{\cos\frac{\beta\pi}{2}}\right)} \\
&= e^{-t|\xi|^{\beta(2k+1)}\left(1-i\,\mathrm{sign}(\xi)\,(p-q)\tan\frac{\pi\beta}{2}\right)} \qquad (5.95)
\end{aligned}$$

and this confirms that the solution to (5.91) is given by the law of the pseudoprocess (5.92). □

Remark 5.4.3. *Since* $\frac{e^{\pm i\pi k\beta}}{\cos\frac{\beta\pi}{2}} = \frac{1}{\cos\beta(2k+1)\frac{\pi}{2}}$ *(because* $e^{i\pi k\beta} = (e^{i\pi})^{k\beta} = (e^{-i\pi})^{k\beta} = e^{-i\pi k\beta}$*) the operator* (5.90) *takes the form of the Riesz fractional derivative of order* $\beta(2k+1)$ *when* $p = q = \frac{1}{2}$.

We now pass to the derivation of the governing equation of the fractional pseudoprocesses studied in Theorem 5.3.5. We first recall the definition of the Feller space-fractional derivative which is

$${}^{F}D^{\beta,\theta}u(x) = -\left[\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin(\pi\beta)}\frac{{}^{+}\partial^{\beta}}{\partial x^{\beta}} + \frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin(\pi\beta)}\frac{{}^{-}\partial^{\beta}}{\partial x^{\beta}}\right]u(x). \qquad (5.96)$$

We recall that

$$\mathcal{F}\left[{}^{F}D^{\beta,\theta}u(x)\right](\xi) = -|\xi|^{\beta}e^{\frac{i\pi\theta}{2}\,\mathrm{sign}(\xi)}\widehat{u}(\xi), \qquad (5.97)$$

as can be shown by means of the following calculation

$$\int_{\mathbb{R}} dx\, e^{i\xi x}\,{}^{F}D^{\beta,\theta}u(x) = -\left[\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin(\pi\beta)}(-i\xi)^{\beta} + \frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin(\pi\beta)}(i\xi)^{\beta}\right]\widehat{u}(\xi)$$

$$\begin{aligned}
&= -\frac{|\xi|^\beta}{2i\sin\pi\beta}\Big[\left(e^{\frac{i\pi}{2}\beta}e^{-\frac{i\pi}{2}\theta}-e^{-\frac{i\pi}{2}\beta}e^{\frac{i\pi}{2}\theta}\right)e^{-\frac{i\pi}{2}\beta\,\mathrm{sign}(\xi)}+\\
&\quad+\left(e^{\frac{i\pi}{2}\beta}e^{\frac{i\pi}{2}\theta}-e^{-\frac{i\pi}{2}\beta}e^{-\frac{i\pi}{2}\theta}\right)e^{\frac{i\pi}{2}\beta\,\mathrm{sign}(\xi)}\Big]\widehat{u}(\xi)\\
&= \begin{cases} -\xi^\beta e^{\frac{i\pi\theta}{2}}\widehat{u}(\xi), & \xi>0,\\ -(-\xi)^\beta e^{-\frac{i\pi\theta}{2}}\widehat{u}(\xi), & \xi<0\end{cases}\\
&= -|\xi|^\beta e^{\frac{i\pi\theta}{2}\,\mathrm{sign}(\xi)}\widehat{u}(\xi)
\end{aligned} \tag{5.98}$$

where we used the results of Theorem 5.2.1. The explicit form of the Fourier transform of the solution to

$$\frac{\partial}{\partial t}u(x,t) = {}^FD^{\beta,\theta}u(x,t), \qquad u(x,0)=\delta(x), \qquad x\in\mathbb{R}, t>0, \tag{5.99}$$

is written as

$$\widehat{u}(\xi,t) = e^{-|\xi|^\beta t e^{\frac{i\pi\theta}{2}\,\mathrm{sign}(\xi)}} \tag{5.100}$$

and for $\beta\in(0,2]$, $4m-1<\theta<4m+1$, $m\in\mathbb{N}$, represents the characteristic function of a stable r.v.. The last condition on θ is due to the fact that

$$|\widehat{u}(\xi,t)|\le 1 \text{ if and only if } \cos\frac{\theta\pi}{2}\in(0,1]. \tag{5.101}$$

The condition (5.101) must be assumed also for $\beta>2$ where (5.100) however fails to be the characteristic function of a genuine r.v.. For $\theta=\beta<1$ (5.100) becomes totally negatively skewed. By interchanging $\sin(\beta-\theta)\frac{\pi}{2}$ with $\sin(\beta+\theta)\frac{\pi}{2}$ we obtain instead

$$\widehat{u}(\xi,t) = e^{-|\xi|^\beta t e^{-\frac{i\pi}{2}\theta\,\mathrm{sign}(\xi)}} \tag{5.102}$$

which is totally positively skewed for $\theta=\beta<1$.

We are now ready to prove the following Theorem.

Theorem 5.4.4. *Let $Z^{\beta(2k+1),\theta}(t)$, $t>0$, be the limiting fractional pseudoprocess studied in Theorem 5.3.5. The signed density of $Z^{\beta(2k+1),\theta}(t)$ is the solution to*

$$\begin{cases}\frac{\partial}{\partial t}z^{\beta(2k+1),\theta}(x,t) = {}^FD^{\beta(2k+1),\theta}z^{\beta(2k+1),\theta}(x,t)\\ z^{\beta(2k+1),\theta}(x,0)=\delta(x)\end{cases} \tag{5.103}$$

and coincide with the signed distribution of the composition for $\beta\in(0,1)$, $-\beta<\theta<\beta$,

$$\mathfrak{Z}^{\beta(2k+1),\theta}(t) = U_1^{2k+1}\left(H_1^\beta\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}t\right)\right)-U_2^{2k+1}\left(H_2^\beta\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}t\right)\right), \tag{5.104}$$

where H_j^β, $j = 1, 2$ *are independent stable r.v.'s and the independent pseudoprocesses* U_j^{2k+1}, $j = 1, 2$, *are related to the odd-order heat-type equation*

$$\frac{\partial}{\partial t} u_{2k+1}(x,t) = (-1)^k \frac{\partial^{2k+1}}{\partial x^{2k+1}} u_{2k+1}(x,t). \tag{5.105}$$

The positivity of the time scales in (5.104) *implies that* $-\beta < \theta < \beta$.

Proof. By profiting from the result (5.97) we note that the Fourier transform of (5.103) is written as

$$\begin{cases} \frac{\partial}{\partial t} \widehat{\mathfrak{z}}^{\beta(2k+1),\theta}(\xi,t) = -|\xi|^{\beta(2k+1)} e^{\frac{i\pi}{2}\theta \,\mathrm{sign}(\xi)} \, \widehat{\mathfrak{z}}^{\beta(2k+1),\theta}(\xi,t) \\ \widehat{\mathfrak{z}}^{\beta(2k+1),\theta}(\xi,0) = 1. \end{cases} \tag{5.106}$$

which is satisfied by the Fourier transform

$$\widehat{\mathfrak{z}}^{\beta(2k+1),\theta}(\xi,t) = e^{-t|\xi|^{\beta(2k+1)} e^{\frac{i\pi}{2}\theta\,\mathrm{sign}(\xi)}}. \tag{5.107}$$

We now prove that the Fourier transform of (5.104) coincides with (5.107). In view of the independence of the r.v.'s and pseudo r.v.'s involved we write that

$$\begin{aligned}
&\mathbb{E}e^{i\xi\mathfrak{Z}^{\beta(2k+1),\theta}(t)} = \mathbb{E}e^{i\xi\left[U_1^{2k+1}\left(H_1^\beta\left(\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}t\right)\right) - U_2^{2k+1}\left(H_2^\beta\left(\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}t\right)\right)\right]} \\
&= \left[\int_{\mathbb{R}} dx\, e^{i\xi x} \int_0^\infty ds\, u^1_{2k+1}(x,s) h^1_\beta\left(s, \frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}t\right)\right] \\
&\quad \times \left[\int_{\mathbb{R}} dx\, e^{-i\xi x} \int_0^\infty ds\, u^2_{2k+1}(x,s) h^2_\beta\left(s, \frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}t\right)\right] \\
&= \left[\int_0^\infty e^{-i\xi^{2k+1}s} h^1_\beta\left(s, \frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}t\right) ds\right] \left[\int_0^\infty e^{i\xi^{2k+1}s} h^2_\beta\left(s, \frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}t\right) ds\right] \\
&= e^{-t\frac{\sin\frac{\pi}{2}(\beta+\theta)}{\sin\pi\beta}\left(i\xi^{2k+1}\right)^\beta} e^{-t\frac{\sin\frac{\pi}{2}(\beta-\theta)}{\sin\pi\beta}\left(-i\xi^{2k+1}\right)^\beta} \\
&= e^{-\frac{t|\xi|^{\beta(2k+1)}}{\sin\pi\beta}\left[\sin\frac{\pi}{2}(\beta+\theta)e^{\frac{i\pi}{2}\beta\,\mathrm{sign}(\xi)} + \sin\frac{\pi}{2}(\beta-\theta)e^{-\frac{i\pi}{2}\beta\,\mathrm{sign}(\xi)}\right]} \\
&= e^{-\frac{t|\xi|^{\beta(2k+1)}}{2i\sin\pi\beta}\left[\left(e^{\frac{i\pi}{2}\beta}e^{\frac{i\pi}{2}\theta} - e^{-\frac{i\pi}{2}\beta}e^{-\frac{i\pi}{2}\theta}\right)e^{\frac{i\pi}{2}\beta\,\mathrm{sign}(\xi)} + \left(e^{\frac{i\pi}{2}\beta}e^{-\frac{i\pi}{2}\theta} - e^{-\frac{i\pi}{2}\beta}e^{\frac{i\pi}{2}\theta}\right)e^{-\frac{i\pi}{2}\beta\,\mathrm{sign}(\xi)}\right]} \\
&= e^{-t|\xi|^{\beta(2k+1)} e^{\frac{i\pi\theta}{2}\,\mathrm{sign}(\xi)}}
\end{aligned} \tag{5.108}$$

which coincides with (5.107). □

5.5 Some remarks

We give various forms for the density $v^\gamma(x,t)$ of symmetric pseudoprocesses of arbitrary order $\gamma > 0$. For integer values of $\gamma = 2n$ or $\gamma = 2n+1$ the analysis of the structure of these densities is presented in Orsingher and D'Ovidio (2012). We give here an analytical representation of $v^\gamma(x,t)$ for non-integer values of γ, which is an alternative to (5.78), as a power series and in integral form (involving the Mittag-Leffler functions). Furthermore, in Figure 5.1 we give some curves for special values of γ. We also give the distribution of the sojourn time of compositions of pseudoprocesses with stable subordinators (totally positively skewed stable r.v.'s).

Proposition 5.5.1. *For $\gamma > 1$ the inverse of the Fourier transform*

$$\widehat{v}^\gamma(\xi,t) \;=\; e^{-t|\xi|^\gamma} \tag{5.109}$$

can also be written as

$$\begin{aligned}
v^\gamma(x,t) &= \frac{1}{\pi}\int_0^\infty \cos(\xi x)\, e^{-t\xi^\gamma}\, d\xi \\
&= \frac{1}{\pi\gamma}\sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{(2k)!}\frac{\Gamma\left(\frac{2k+1}{\gamma}\right)}{t^{\frac{2k+1}{\gamma}}} \\
&= \frac{1}{\pi\gamma}\sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{t^{\frac{2k+1}{\gamma}}}\frac{B\left(\frac{2k+1}{\gamma},\,(2k+1)\left(1-\frac{1}{\gamma}\right)\right)}{\Gamma\left((2k+1)\left(1-\frac{1}{\gamma}\right)\right)} \\
&= \frac{1}{\pi\gamma}\sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{t^{\frac{2k+1}{\gamma}}}\frac{1}{\Gamma\left((2k+1)\left(1-\frac{1}{\gamma}\right)\right)}\int_0^1 dy\, y^{\frac{2k+1}{\gamma}-1}(1-y)^{(2k+1)\left(1-\frac{1}{\gamma}\right)-1} \\
&= \frac{1}{\pi\gamma}\int_0^1 dy\sum_{k=0}^\infty \frac{(-1)^k\left(xy^{\frac{1}{\gamma}}(1-y)^{1-\frac{1}{\gamma}}\right)^{2k}}{t^{\frac{2k+1}{\gamma}}\Gamma\left((2k+1)\left(1-\frac{1}{\gamma}\right)\right)}\, y^{\frac{1}{\gamma}-1}(1-y)^{-\frac{1}{\gamma}} \\
&= \frac{t^{-\frac{1}{\gamma}}}{\pi\gamma}\int_0^1 dy\, E_{2\left(1-\frac{1}{\gamma}\right),1-\frac{1}{\gamma}}\left(-\left(xy^{\frac{1}{\gamma}}(1-y)^{1-\frac{1}{\gamma}}\right)^2 t^{-\frac{2}{\gamma}}\right)\, y^{\frac{1}{\gamma}-1}(1-y)^{-\frac{1}{\gamma}} \\
&\overset{w=y/(1-y)}{=} \frac{t^{-\frac{1}{\gamma}}}{\pi\gamma}\int_0^\infty dw\, E_{2\left(1-\frac{1}{\gamma}\right),1-\frac{1}{\gamma}}\left(-x^2\left(\frac{w^{\frac{1}{\gamma}}}{1+w}\right)^2 t^{-\frac{2}{\gamma}}\right)\frac{w^{\frac{1}{\gamma}}}{1+w}\frac{1}{w}
\end{aligned} \tag{5.110}$$

and for $\gamma < 2$ coincides with the characteristic function of symmetric stable processes.

Formula (5.110) is an alternative to the probabilistic representation (5.78) for $\gamma = 2k\beta$. For $1 < \gamma < 2$ it represents the density of a symmetric stable r.v..

Remark 5.5.2. *We note that*

$$v^\gamma(0,t) \;=\; \frac{t^{-\frac{1}{\gamma}}}{\pi}\Gamma\left(1+\frac{1}{\gamma}\right) \tag{5.111}$$

as can be inferred from (5.110). In the neighbourhood of $x = 0$ the density $v^\gamma(x,t)$ can be written as

$$v^\gamma(x,t) \approx \frac{1}{\pi\gamma}\left(\frac{1}{t^{\frac{1}{\gamma}}}\Gamma\left(\frac{1}{\gamma}\right) - \frac{x^2}{2}\frac{\Gamma\left(\frac{3}{\gamma}\right)}{t^{\frac{3}{\gamma}}}\right)$$
$$= v^\gamma(0,t)\left(1 - x^2\frac{C_\gamma}{2t^{\frac{2}{\gamma}}}\right) \tag{5.112}$$

where

$$C_\gamma = \frac{\Gamma\left(\frac{1}{\gamma}+\frac{1}{3}\right)\Gamma\left(\frac{1}{\gamma}+\frac{2}{3}\right)3^{\frac{3}{\gamma}-\frac{1}{2}}}{2\pi}. \tag{5.113}$$

In the above calculation the triplication formula of the Gamma function (see Lebedev (1965) page 14) has been applied

$$\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right) = \frac{2\pi}{3^{3z-\frac{1}{2}}}\Gamma(3z). \tag{5.114}$$

Remark 5.5.3. *For even-order pseudoprocesses $U^{2k}(t)$, $t>0$, the distribution of the sojourn time*

$$\Gamma_t\left(U^{2k}\right) = \int_0^t \mathbb{I}_{[0,\infty)}\left(U^{2k}(s)\right) ds \tag{5.115}$$

follows the arcsine law for all $n \geq 1$ (see Krylov (1960)). Therefore the distribution of the sojourn time of $U^{2k}\left(H^\beta(t)\right)$, $t>0$, $\beta \in (0,1)$, reads

$$\Pr\left\{\Gamma_t\left(U^{2k}\left(H^\beta\right)\right) \in dx\right\} = \int_0^\infty \Pr\left\{\Gamma_s\left(U^{2k}\right) \in dx\right\}\Pr\left\{H^\beta(t) \in ds\right\}$$
$$= \frac{dx}{\pi}\int_x^\infty \frac{1}{\sqrt{x(s-x)}}\Pr\left\{H^\beta(t) \in ds\right\}. \tag{5.116}$$

In the odd-order case the distribution of the sojourn time

$$\Gamma_t\left(U^{2k+1}\right) = \int_0^t \mathbb{I}_{[0,\infty)}\left(U^{2k+1}(s)\right) ds \tag{5.117}$$

is written as (see Lachal (2003))

$$\Pr\left\{\Gamma_t\left(U^{2k+1}\right) \in dx\right\} = dx\,\frac{\sin\frac{\pi}{2k+1}}{\pi}x^{-\frac{1}{2k+1}}(t-x)^{-\frac{2k}{2k+1}}\mathbb{I}_{(0,t)}(x) \tag{5.118}$$

and thus we get

$$\Pr\left\{\Gamma_t\left(U^{2k+1}\left(H^\beta\right)\right) \in dx\right\} = \frac{dx\,\sin\frac{\pi}{2k+1}}{\pi}\int_x^\infty \frac{1}{\sqrt[2k+1]{x(s-x)^{2k}}}\Pr\left\{H^\beta(t) \in ds\right\}. \tag{5.119}$$

For $\beta = \frac{1}{2}$ the integral (5.116) can be evaluated explicitly

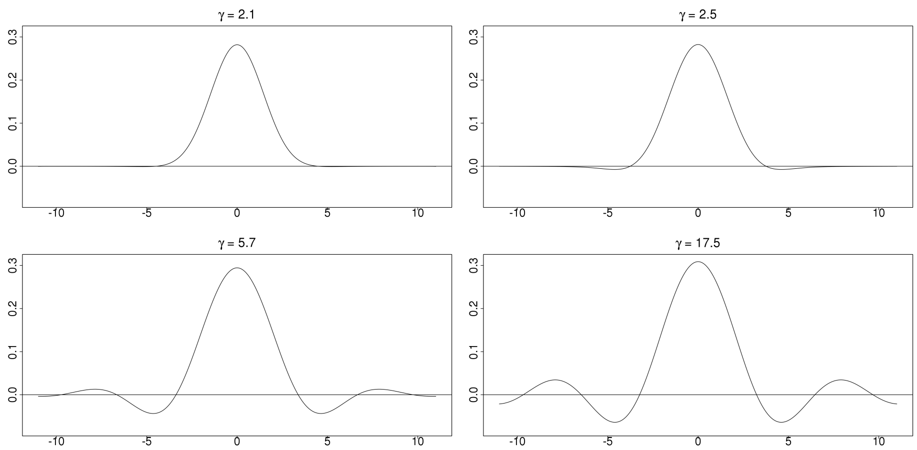
$$\begin{aligned}
\Pr\left\{\Gamma_t\left(U^{2k}\left(H^{\frac{1}{2}}\right)\right) \in dx\right\} &= \frac{dx}{\pi}\int_x^\infty \frac{1}{\sqrt{x(s-x)}}\frac{te^{-\frac{t^2}{2s}}}{\sqrt{2\pi s^3}}ds \\
&= \frac{dx\,t}{\pi\sqrt{2\pi x}}\int_0^{\frac{1}{x}} \frac{e^{-\frac{t^2}{2}y}}{\sqrt{1-xy}}dy \\
&= \frac{dx\,t}{\pi\sqrt{2\pi x^3}}\int_0^1 \frac{e^{-\frac{t^2}{2x}w}}{\sqrt{1-w}}dw \\
&= \frac{dx\,t}{\pi\sqrt{2\pi x^3}}\sum_{k=0}^\infty \left(-\frac{t^2}{2x}\right)^k \frac{1}{k!}\int_0^1 w^k\left(1-w\right)^{-\frac{1}{2}}\,dw \\
&= \frac{dx\,t}{\pi\sqrt{2x^3}}E_{1,\frac{3}{2}}\left(-\frac{t^2}{2x}\right), \qquad x>0, t>0, \qquad (5.120)
\end{aligned}$$

where

$$E_{\nu,\mu}(x) = \sum_{j=0}^\infty \frac{x^j}{\Gamma(j\nu+\mu)}, \qquad \nu,\mu>0, \qquad (5.121)$$

is the Mittag-Leffler function.

Figure 5.1: The density $v^\gamma(x,t)$ for $\gamma > 2$ displays an oscillating behaviour similar to that of the fundamental solution of even-order heat equations.



Ackwnoledgements

The authors have benefited from fruitful discussions on the topics of this paper with Dr. Mirko D'Ovidio.

Chapter 6

Time-changed pseudoprocesses on a circle

Article: Orsingher and Toaldo (2013). Even-order pseudoprocesses on a circle and related Poisson kernels.

Summary

Pseudoprocesses, constructed by means of the solutions of higher-order heat-type equations have been developed by several authors and many related functionals have been analyzed by means of the Feynman-Kac functional or by means of the Spitzer identity. We here examine even-order pseudoprocesses wrapped up on circles and derive their explicit signed density measures. We observe that circular even-order pseudoprocesses differ substantially from pseudoprocesses on the line because - for $t > \bar{t} > 0$, where $\bar{t}$ is a suitable n-dependent time value - they become real random variables. By composing the circular pseudoprocesses with positively-skewed stable processes we arrive at genuine circular processes whose distribution, in the form of Poisson kernels, is obtained. The distribution of circular even-order pseudoprocesses is similar to the Von Mises (or Fisher) circular normal and therefore to the wrapped up law of Brownian motion. Time-fractional and space-fractional equations related to processes and pseudoprocesses on the unit radius circumference are introduced and analyzed.

6.1 Introduction and preliminaries

Pseudoprocesses are connected with the fundamental solution of heat-type equations of the form

$$\frac{\partial}{\partial t}u_n(x,t) = c_n\frac{\partial^n}{\partial x^n}u_n(x,t), \qquad x\in\mathbb{R},\, t>0,\, n\in\mathbb{N}, \tag{6.1}$$

where

$$c_n = \begin{cases}(-1)^{\frac{n}{2}+1}, & \text{for even values of } n\\ \pm 1, & \text{for odd values of } n,\end{cases} \tag{6.2}$$

subject to the initial condition

$$u(x,0) = \delta(x). \tag{6.3}$$

For $n > 2$ the fundamental solutions to (6.1) are sign-varying. By means of a Wiener-type approach some authors (see for example Albeverio et al. (2011), Daletsky (1969), Daletsky and Fomin (1965), Krylov (1960), Ladohin (1962)) have constructed pseudoprocesses which we denote by $X(t)$, $t>0$ or $X_n(t)$, if we specify the order of the governing equation. In these papers the set of real functions $x: t\in[0,\infty)\to x(t)$ (sample paths) and the cylinders

$$C = \{x(t) : a_j \le x(t_j) \le b_j,\, j=1,\cdots,n\} \tag{6.4}$$

have been considered. By using the solutions u_n to (6.1) the measure of cylinders is given as

$$\mu_n(C) = \int_{a_1}^{b_1} dx_1\cdots\int_{a_n}^{b_n} dx_n \prod_{j=1}^{n} u_n(x_j-x_{j-1}, t_j-t_{j-1}). \tag{6.5}$$

In (6.5) we denote by u_n

$$u_n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi x} e^{c_n(-i\xi)^n t}. \tag{6.6}$$

For $n=2k$ and $c_{2k} = (-1)^{k+1}$ the integral (6.6) always converge as it does for the odd-order case. The measure (6.5) is extended to the field generated by cylinders (6.4) for fixed $t_1<\cdots<t_j<\cdots<t_n$. The signed measure obtained in this way is Markovian in the sense that

$$\mu_{x_0}\left\{X(t+T)\in\mathcal{B}\middle|\mathcal{F}_T\right\} = \mu_{X(T)}\left\{X(t)\in\mathcal{B}\right\}, \tag{6.7}$$

where $\mathcal{F}_T$ is the field generated as

$$\mathcal{F}_T = \sigma\left\{X(t_1)\in\mathcal{B}_1,\cdots,X(t_n)\in\mathcal{B}_n\right\}, \tag{6.8}$$

where $0 \leq t_1 \leq \cdots \leq t_n = T$. More information on properties of pseudoprocesses can be found in Cammarota and Lachal (2012), Lachal (2003) and Nishioka (2001). For pseudoprocesses with drift the reader can consult Lachal (2008).

In this paper we consider pseudoprocesses on the ring $\mathcal{R}$ of radius one, denoted by $\Theta(t)$, $t > 0$, whose signed density measures are governed by

$$\begin{cases} \frac{\partial}{\partial t} v_n(\theta, t) = c_n \frac{\partial^n}{\partial \theta^n} v_n(\theta, t), & \theta \in [0, 2\pi), t > 0, n \geq 2, \\ v_n(\theta, 0) = \delta(\theta). \end{cases} \tag{6.9}$$

The signed measures of pseudoprocesses on the line $X(t)$, $t > 0$, and those on the unit-radius ring, $\Theta(t)$, $t > 0$, can be related by

$$\{\Theta(t) \in d\theta\} = \bigcup_{m=-\infty}^{\infty} \{X(t) \in d(\theta + 2m\pi)\}, \qquad 0 \leq \theta < 2\pi. \tag{6.10}$$

This means that the pseudoprocess Θ has sample paths which are obtained from those of X by wrapping them up around the circumference $\mathcal{R}$. Counterclockwise moving sample paths of Θ correspond to increasing sample paths of X.

For $n = 2$ we have in particular the circular Brownian motion studied by Hartman and Watson (1974), Roberts and Ursell (1960), Stephens (1963). The pseudoprocesses running on $\mathcal{R}$ are called circular pseudoprocesses and are denoted either by $\Theta(t)$, $t > 0$, or $\Theta_n(t)$ if we want to clarify the order of the equation governing their distribution. We concentrate our attention on the even-order case because the odd-order wrapped-up pseudoprocesses pose qualitatively different problems of convergence of their Fourier expansion. In view of (6.10) we can write

$$v_{2n}(\theta, t) = \sum_{m=-\infty}^{\infty} u_{2n}(\theta + 2m\pi, t), \qquad 0 \leq \theta < 2\pi. \tag{6.11}$$

Equation (6.11) shows that the solution to (6.9) can be obtained by wrapping up the solution to (6.1) which reads

$$u_{2n}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{-i\xi x} e^{-\xi^{2n} t}, \qquad x \in \mathbb{R}, t > 0. \tag{6.12}$$

The function (6.12) has been investigated in special cases by Hochberg (1978), Krylov (1960), Nishioka (2001) and more in general by Lachal (2003, 2008). The sign-varying structure of (6.12) has been discovered in special cases by Bernstein (1919), Lévy (1923), Pòlya (1923), as early as at the beginning of the Twentieth century and has been more recently studied also by Li and Wong (1993).

The Fourier series of (6.11) has the remarkably simple form

$$v_{2n}(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-k^{2n} t} \cos k\theta, \qquad \theta \in [0, 2\pi). \tag{6.13}$$

For $n = 1$ we obtain the Fourier series of the law of the circular Brownian motion (see Hartman and Watson (1974)). The function

$$v_2(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t} \cos k\theta \tag{6.14}$$

is similar to the Von Mises circular normal

$$v(\theta, k) = \frac{e^{k \cos \theta}}{2\pi I_0(k)} = \frac{1}{2\pi} \left(1 + 2 \sum_{m=1}^{\infty} \frac{I_m(k)}{I_0(k)} \cos m\theta \right), \qquad \theta \in [0, 2\pi), \tag{6.15}$$

where

$$I_m(x) = \sum_{j=0}^{\infty} \left(\frac{x}{2} \right)^{2j+m} \frac{1}{k! \, \Gamma(m + j + 1)} \tag{6.16}$$

is the m-th order Bessel function. The relationship between (6.14) and (6.15) is investigated in the paper by Hartman and Watson (1974). The Von Mises circular normal represents the hitting distribution of the circumference $\mathcal{R}$ of a Brownian motion with drift starting from the center of $\mathcal{R}$. The planar Brownian motion $(R(t), \Psi(t))$, $t > 0$, with drift $\boldsymbol{k} = (k_1, k_2)$, $\|\boldsymbol{k}\| = k$, has transition function

$$\Pr\{R(t) \in d\rho, \Psi(t) \in d\varphi\} = \frac{\rho}{2\pi t} e^{-\frac{\rho^2}{2t}} e^{-\frac{k^2 t}{2}} e^{\rho k \cos \varphi} d\rho d\varphi \tag{6.17}$$

and marginal

$$\Pr\{R(t) \in d\rho\} = \frac{\rho}{t} e^{-\frac{\rho^2}{2t}} e^{-\frac{k^2 t}{2}} I_0(\rho k) \, d\rho. \tag{6.18}$$

Therefore

$$\Pr\left\{ \Psi(t) \in d\varphi \middle| R(t) \in d\rho \right\} = \frac{e^{\rho k \cos \varphi}}{2\pi I_0(\rho k)} d\varphi \tag{6.19}$$

and for $\rho = 1$ coincides with (6.15).

The analysis of the pictures of $v_{2n}(\theta, t)$ for different values of t and different values of the order $2n$, $n \in \mathbb{N}$, shows that the distributions (6.13) after a certain time become non-negative. This means that pseudoprocesses on the circle $\mathcal{R}$ behave differently from their counterparts on the line and rapidly become genuine random variables. Furthermore we remark that in small initial intervals of time the circular pseudoprocesses have signed-valued distributions with a number of minima which rapidly unify into a single minimum (located at $\theta = \pi$) which for increasing t upcrosses the zero level. This is due to the fact that in a small initial interval of time the effect produced by the central bell of the distribution has not yet spread on the whole ring $\mathcal{R}$.

The value of the absolute minimum of $v_{2n}(\theta, t)$ for $t > \bar{t}$ has the form

$$v_{2n}(\pi, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k e^{-k^{2n} t}, \qquad t > \bar{t}. \tag{6.20}$$

The graph of functions $v_{2n}(\theta, t)$ slightly differ from that of the density of circular Brownian motion as shown in figures 6.1 and 6.2. The term $k = 1$ in (6.13) is the leading term of the series and the form of the distribution $v_{2n}(\theta, t)$ is very close to that of $\frac{1}{2\pi} + \frac{1}{\pi} e^{-t} \cos\theta$.

Figure 6.1: The distributions of the fourth-order circular pseudoprocess for different values of t

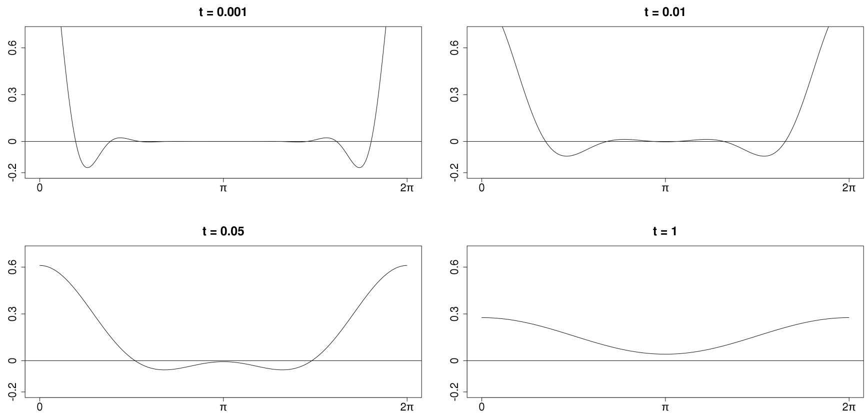
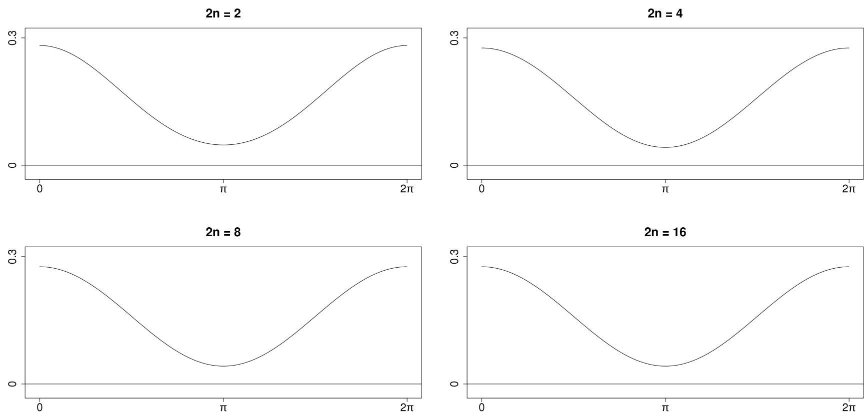


Figure 6.2: The distributions (for $t = 1$) of the circular pseudoprocesses of various order $2n$



The odd-order case is much more complicated because the solutions to equation (6.1) are asymmetric (with asymmetry decreasing for increasing values of the order n). Some properties of solutions to odd-order heat-type equations can be found in Lachal (2003, 2008). In the present paper the wrapped up solution to (6.1) gives the fundamental solution of (6.9) as

$$v_{2n+1}(\theta, t) = \sum_{k=-\infty}^{\infty} u_{2n+1}(\theta + 2k\pi, t), \qquad \theta \in [0, 2\pi)\,, \tag{6.21}$$

whose Fourier series reads

$$v_{2n+1}(\theta,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos(k^{2n+1}t + k\theta). \tag{6.22}$$

We note that for $n = 1$ the series (6.22) becomes a discrete version of the solution to (6.1) which reads

$$u_3(x,t) = \frac{1}{\sqrt[3]{3t}}\,\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right) \tag{6.23}$$

where

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\alpha x + \frac{\alpha^3}{3}\right)\,d\alpha \tag{6.24}$$

is the Airy function. The probabilistic representations of solutions of higher-order heat-type equations show that for increasing values of n the solutions $u_{2n+1}(x,t)$ and $u_{2n}(x,t)$ slightly differ. Therefore the corresponding circular version $v_{2n+1}(\theta,t)$ must have Fourier transform which converge since $v_{2n}(\theta,t)$ do.

We consider also the wrapped up stable processes $\mathfrak{S}^{2\beta}(t)$, $t>0$, and the related governing space-fractional equation.In particular we show that the law of $\mathfrak{S}^{2\beta}(2^{-\beta}t)$, $t>0$, is the fundamental solution of the space-fractional equations

$$\begin{cases} \frac{\partial}{\partial t}v_2^{\beta}(\theta,t) = -\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^{\beta} v_2^{\beta}(\theta,t), & \theta\in[0,2\pi)\,,\, t>0, \beta\in(0,1]\,, \\ v_2^{\beta}(\theta,0) = \delta(\theta), \end{cases} \tag{6.25}$$

and has Fourier expansion

$$v_2^{\beta}(\theta,t) = \frac{1}{2\pi}\left[1 + 2\sum_{m=1}^{\infty} e^{-\left(\frac{m^2}{2}\right)^{\beta}t}\cos m\theta\right]. \tag{6.26}$$

The fractional operator appearing in (6.25) is the one-dimensional fractional Laplacian which can be defined by means of the Bochner representation (see, for example, Balakrishnan (1960), Bochner (1949))

$$-\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^{\beta} = \frac{\sin\pi\beta}{\pi}\int_0^{\infty}\left(\lambda + \left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)\right)^{-1}\lambda^{\beta}\,d\lambda, \qquad \beta\in(0,1). \tag{6.27}$$

We show that formula (6.26) coincides with the distribution of the subordinated Brownian motion on the circle, $\mathfrak{B}\left(H^{\beta}(t)\right)$, $t>0$, where $H^{\beta}(t)$, $t>0$, is a stable subordinator of order $\beta\in(0,1]$ (see, for example, Baeumer and Meerschaert (2001)). Furthermore we notice that

$$\mathfrak{B}\left(2\,H^{\beta}(t)\right) \overset{\text{law}}{=} \mathfrak{S}^{2\beta}(t), \qquad t>0, \tag{6.28}$$

$\mathfrak{S}^{2\beta}(t)$, $t>0$, is a symmetric process on the ring $\mathcal{R}$ with distribution which can be obtained by its symmetric stable counterpart on the line as

$$\begin{aligned} p_{\mathfrak{S}^{2\beta}}(\theta,t) &= \sum_{m=\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{-i\xi(\theta+2m\pi)} e^{-t|\xi|^{2\beta}} \\ &= \frac{1}{2\pi}\left[1+2\sum_{k=1}^{\infty} e^{-k^{2\beta}t}\cos k\theta\right]. \end{aligned} \tag{6.29}$$

For $\beta=\frac{1}{2}$ we extract from (6.26) the Poisson kernel

$$v_2^{\frac{1}{2}}(\theta,t) = \frac{1}{2\pi}\frac{1-e^{-t\sqrt{2}}}{1+e^{-t\sqrt{2}}-2e^{-\frac{t}{\sqrt{2}}}\cos\theta}. \tag{6.30}$$

The composition of the circular pseudoprocesses $\Theta_n(t)$, $t>0$, with positively-skewed stable processes of order $\frac{1}{n}$, say $H^{\frac{1}{n}}(t)$, $t>0$, leads also to the Poisson kernel. In particular, we show that

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)\in d\theta\right\} = \frac{d\theta}{2\pi}\frac{1-e^{-2t}}{1+e^{-2t}-2e^{-t}\cos\theta}, \qquad \theta\in[0,2\pi). \tag{6.31}$$

In the odd-order case the result is different, depends on n and has the following form for $\theta\in[0,2\pi)$

$$\begin{aligned} &\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)\in d\theta\right\} \\ &= \frac{d\theta}{2\pi}\frac{1-e^{-2t\cos\frac{\pi}{2(2n+1)}}}{1+e^{-2t\cos\frac{\pi}{2(2n+1)}}-2e^{-t\cos\frac{\pi}{2(2n+1)}}\cos\left(\theta+t\sin\frac{\pi}{2(2n+1)}\right)}. \end{aligned} \tag{6.32}$$

The composition of pseudoprocesses with stable processes therefore produces genuine r.v.'s on the ring $\mathcal{R}$ as it happens on the line (see Orsingher and D'Ovidio (2012)). We note that the distribution of the composition in the even order case is independent from n (formula (6.31)), while in the odd-order case the Poisson kernel obtained depends on n and has a rather complicated structure. For $n\to\infty$ the kernel (6.32) converges pointwise to (6.31) since the asymmetry of the fundamental solutions of (6.1) (as well as that of their wrapped up counterparts) decreases. The result (6.31) offers an interesting interpretation. The Poisson kernel (6.31) can be viewed as the probability that a planar Brownian motion starting from the point with polar coordinates $(e^{-t},0)$ hits the circumference $\mathcal{R}$ in the point $(1,\Theta)$ (see Fig. 6.4a). Therefore this distribution coincides with the law of an even-order pseudoprocess running on the circumference and stopped at time $H^{\frac{1}{2n}}(t)$, $t>0$. This result is independent from n and therefore is valid also for Brownian motion. A similar interpretation holds also for circular odd-order pseudoprocesses taken at the time $H^{\frac{1}{2n+1}}(t)$, $t>0$, but starting from the point with polar coordinates (e^{-a_nt},b_nt), where $a_n=\cos\pi/(2(2n+1))$ and $b_n=\sin\pi/(2(2n+1))$.

6.2 Pseudoprocesses on a ring

In this section we consider pseudoprocesses $\Theta(t)$, $t > 0$, on the unit-radius circumference $\mathcal{R}$, whose density function $v_n(\theta, t)$, $\theta \in [0, 2\pi)$, $t > 0$, is governed by the higher order heat-type equation

$$\begin{cases} \frac{\partial}{\partial t} v_n(\theta,t) = c_n \frac{\partial^n}{\partial \theta^n} v_n(\theta,t), & \theta \in [0,2\pi), \, t > 0, \, n \geq 2, \\ v_n(\theta,0) = \delta(\theta). \end{cases} \tag{6.33}$$

The pseudoprocesses Θ_n have sample paths obtained by wrapping up the trajectories of pseudoprocesses on the line $\mathbb{R}$. Increasing sample paths on $\mathcal{R}$ correspond to counterclockwise moving motions on the ring $\mathcal{R}$. The structure of sample paths of pseudoprocesses has not been investigated in detail although some results by Lachal (Theorem 5.2, Lachal (2008)) show that there is a sort of "slight" discontinuity in their behaviour (this is confirmed by Hochberg (1978)) and the fact that the reflection principle fails (Beghin et al. (2001), Lachal (2003)).

It must be considered that the wrapping up of the sample paths and of the corresponding density measures produces in the long run genuine random variables (with non-negative measure densities in the case n is even). Our first result concerns the distribution of $\Theta_n(t)$, $t > 0$.

Theorem 6.2.1. *The solutions to the even-order heat-type equations* (6.33) *reads*

$$v_{2n}(\theta,t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-k^{2n}t} \cos k\theta, \qquad \text{for } c_{2n} = (-1)^{n+1}, \, n \geq 1, \tag{6.34}$$

Proof. We can obtain the result (6.34) in two different ways. We start by considering the even-order case where the wrapping up of the solutions to (6.1) which leads to

$$v_{2n}(\theta,t) = \sum_{m=-\infty}^{\infty} u_{2n}(\theta + 2m\pi, t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi \, e^{-i(\theta+2m\pi)\xi} e^{-\xi^{2n}t}. \tag{6.35}$$

The Fourier series expansion of the symmetric function $v_{2n}(\theta, t)$ has coefficients

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_0^{2\pi} d\theta \, \cos k\theta \left[\sum_{m=-\infty}^{\infty} u_{2n}(\theta + 2m\pi, t) \right] \\ &= 2 \sum_{m=-\infty}^{\infty} \int_m^{m+1} dy \, u_{2n}(2\pi y, t) \, \cos 2\pi k y \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} dz \, \cos kz \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{-i\xi z} e^{-\xi^{2n}t} \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{-\xi^{2n}t} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dz \left(e^{iz(k-\xi)} + e^{-iz(k+\xi)} \right) \right] \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-\xi^{2n}t}\left[\delta(\xi-k)+\delta(\xi+k)\right] \;=\; \frac{e^{-k^{2n}t}}{\pi}. \tag{6.36}$$

An alternative derivation of $v_{2n}(\theta,t)$ is based on the method of separation of variables. Thus under the assumption that $v_{2n}(\theta,t)=T(t)\psi(\theta)$ we get

$$\frac{T^{(1)}(t)}{T(t)} \;=\; \frac{\psi^{(2n)}(\theta)}{\psi(\theta)}(-1)^{n+1} \;=\; -\beta^{2n}. \tag{6.37}$$

In order to have periodic solutions we must take integer values of β and thus the general solution to (6.33) becomes

$$v_{2n}(\theta,t) \;=\; \sum_{k=-\infty}^{\infty} A_k\, e^{-k^{2n}t}\cos k\theta \;=\; A_0 + 2\sum_{k=1}^{\infty} A_k\, e^{-k^{2n}t}\cos k\theta. \tag{6.38}$$

The initial condition

$$v_{2n}(\theta,0) \;=\; \delta(\theta) \;=\; \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos k\theta \tag{6.39}$$

implies that $A_k=\frac{1}{2\pi}$, which confirms the result. □

Proposition 6.2.2. *We are able to give a third derivation of (6.34) by resorting to the probabilistic representation of fundamental solutions to even-order heat-type equations of Orsingher and D'Ovidio (2012) which reads*

$$u_{2n}(x,t) \;=\; \frac{1}{\pi x}\mathbb{E}\left\{\sin\left(xG^{2n}\left(\frac{1}{t}\right)\right)\right\} \tag{6.40}$$

for $c_n=(-1)^{n+1}$, $n\geq 1$. In (6.40) $G^{\gamma}(t^{-1})$ is a generalized gamma r.v. with density

$$g^{\gamma}(x,t) \;=\; \gamma\frac{x^{\gamma-1}}{t}e^{-\frac{x^{\gamma}}{t}}, \qquad x>0,\, t>0,\, \gamma>0. \tag{6.41}$$

Proof. We start the proof by wrapping-up the representation (6.40) as follows

$$v_{2n}(\theta,t) \;=\; \sum_{m=-\infty}^{\infty}\frac{1}{\pi(\theta+2m\pi)}\mathbb{E}\left\{\sin\left((\theta+2m\pi)G^{2n}\left(\frac{1}{t}\right)\right)\right\}. \tag{6.42}$$

Now we evaluate the Fourier coefficients of (6.42) as

$$\begin{aligned}
a_k &= \frac{1}{\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{2\pi}\frac{d\theta}{\theta+2m\pi}\mathbb{E}\left\{\sin(\theta+2m\pi)G^{2n}\left(\frac{1}{t}\right)\right\}\cos k\theta\\
&= \frac{1}{\pi^2}\mathbb{E}\left\{\int_{-\infty}^{\infty}dz\frac{\cos kz}{z}\sin\left(z\,G^{2n}\left(\frac{1}{t}\right)\right)\right\}\\
&= \frac{1}{\pi^2}\mathbb{E}\left\{\int_0^{\infty}dz\left[\frac{\sin\left(kz+z\,G^{2n}\left(\frac{1}{t}\right)\right)}{z}+\frac{\sin\left(z\,G^{2n}\left(\frac{1}{t}\right)-kz\right)}{z}\right]\right\}
\end{aligned}$$

$$= \frac{1}{\pi}\mathbb{E}\left\{\mathbb{I}_{\left[-G^{2n}\left(\frac{1}{t}\right)<k<G^{2n}\left(\frac{1}{t}\right)\right]}\right\} = \frac{1}{\pi}\Pr\left\{G^{2n}\left(\frac{1}{t}\right) > k\right\} = \frac{1}{\pi}e^{-k^{2n}t}, \tag{6.43}$$

where we used the fact that

$$\int_0^\infty dx\,\frac{\sin\alpha x}{x} = \begin{cases}\frac{\pi}{2}, & \text{if } \alpha > 0,\\ -\frac{\pi}{2}, & \text{if } \alpha < 0.\end{cases} \tag{6.44}$$

□

The calculation (6.43) shows that the density of even-order circular pseudoprocesses can be viewed as the superposition of sinusoidal waves whose amplitude corresponds to the tails of a Weibull distribution.

For the odd-order pseudoprocess we proceed formally as in the even-order case and the Fourier cofficients of

$$v_{2n+1}(\theta,t) = \sum_{m=-\infty}^{\infty} u_{2n+1}(\theta+2m\pi,t) \tag{6.45}$$

become

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{(-1)^n(-i\xi)^{2n+1}t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dz\,\left(e^{i(k-\xi)z}+e^{-i(k+\xi)z}\right)\right]\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi^{2n+1}t}\left[\delta(\xi-k)+\delta(\xi+k)\right]\\ &= \frac{1}{2\pi}\left[e^{-ik^{2n+1}}+e^{ik^{2n+1}}\right] = \frac{1}{\pi}\cos k^{2n+1}t. \end{aligned} \tag{6.46}$$

In a similar way we have that

$$\begin{aligned} b_k &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{(-1)^n(-i\xi)^{2n+1}t}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\,\left(e^{i(k-\xi)z}-e^{-i(k+\xi)z}\right)\right]\\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi\, e^{-i\xi^{2n+1}t}\left[\delta(\xi-k)-\delta(\xi+k)\right] = -\frac{1}{\pi}\sin k^{2n+1}t, \end{aligned} \tag{6.47}$$

and thus the expression of the distribution of the odd-order pseudoprocess on the circle $v_{2n+1}(\theta,t)$ becomes

$$v_{2n+1}(\theta,t) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\cos\left(k^{2n+1}t+k\theta\right). \tag{6.48}$$

For $n = 1$ the series (6.48) is similar to the integral representation of the Airy function

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\alpha x+\frac{\alpha^3}{3}\right)\,d\alpha. \tag{6.49}$$

We are not able to give a rigorous proof of the convergence of the series (6.48) but we are able to give an alternative derivation as follows. In particular we can obtain the expansion (6.48) for circular odd-order pseudoprocesses by resorting again to the probabilistic representation of the law of pseudoprocesses of Orsingher and D'Ovidio (2012) which reads

$$u_{2n+1}(x,t) = \frac{1}{\pi x}\mathbb{E}\left\{e^{-b_n x\, G^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n x\, G^{2n+1}\left(\frac{1}{t}\right)\right)\right\}, \tag{6.50}$$

where $c_{2n+1} = (-1)^n$, $n \geq 1$, $G^{\gamma}\left(t^{-1}\right)$ is a generalized gamma r.v. with density (6.41) and

$$a_n = \cos\frac{\pi}{2(2n+1)}, \qquad b_n = \sin\frac{\pi}{2(2n+1)}. \tag{6.51}$$

By wrapping-up (6.50) we obtain

$$v_{2n+1}(\theta,t) = \\ = \sum_{m=-\infty}^{\infty}\frac{1}{\pi(\theta+2m\pi)}\mathbb{E}\left\{e^{-b_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)\right)\right\}. \tag{6.52}$$

We prove that the Fourier series expansion of (6.52) coincides with (6.48).We need both the sine and cosine coefficients of the Fourier expansion because the signed laws are asymmetric. The Fourier coefficients become

$$\begin{cases} a_k = \frac{1}{\pi}\cos k^{2n+1}t, \\ b_k = -\frac{1}{\pi}\sin k^{2n+1}t. \end{cases} \tag{6.53}$$

We give with some details the evaluation of (6.53)

$$\begin{aligned} a_k =& \frac{1}{\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{2\pi} d\theta\frac{\cos k\theta}{\theta+2m\pi}\mathbb{E}\left\{e^{-b_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n(\theta+2m\pi)G^{2n+1}\left(\frac{1}{t}\right)\right)\right\} \\ =& \frac{1}{\pi^2}\mathbb{E}\left\{\int_{-\infty}^{\infty}dz\frac{\cos kz}{z}e^{-b_n zG^{2n+1}\left(\frac{1}{t}\right)}\sin\left(a_n zG^{2n+1}\left(\frac{1}{t}\right)\right)\right\} \\ =& \frac{1}{2\pi^2}\mathbb{E}\left\{\int_{-\infty}^{\infty}dz\frac{\sin\left(z\left(a_nG^{2n+1}\left(\frac{1}{t}\right)+k\right)\right)+\sin\left(z\left(a_nG^{2n+1}\left(\frac{1}{t}\right)-k\right)\right)}{z}e^{-b_n zG^{2n+1}\left(\frac{1}{t}\right)}\right\} \\ =& \frac{1}{2i2\pi^2}\mathbb{E}\left\{\int_{-\infty}^{\infty}\frac{dz}{z}\left[e^{iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)+k\right)-b_n zG^{2n+1}\left(\frac{1}{t}\right)}-e^{-iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)+k\right)-b_n zG^{2n+1}\left(\frac{1}{t}\right)}\right.\right. \\ &\left.\left.+e^{iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)-k\right)-b_n zG^{2n+1}\left(\frac{1}{t}\right)}-e^{-iz\left(a_nG^{2n+1}\left(\frac{1}{t}\right)-k\right)-b_n zG^{2n+1}\left(\frac{1}{t}\right)}\right]\right\}. \end{aligned} \tag{6.54}$$

By considering the following integral representation of the Heaviside function

$$\mathcal{H}_y(x) = -\frac{1}{2\pi}\int_{\mathbb{R}}dw\, e^{-iwx}\frac{e^{iyw}}{iw} = \int_{\mathbb{R}}dw\, e^{iwx}\frac{e^{-iyw}}{iw} \tag{6.55}$$

the coefficients a_k in (6.54) become

$$\begin{aligned} a_k &= \frac{(2n+1)t}{2\pi} \int_0^\infty dw\, w^{2n} e^{-tw^{2n+1}} \left[\mathcal{H}_k(w(a_n - ib_n)) - \mathcal{H}_k(-w(a_n+ib_n))\right. \\ &\quad \left. + \mathcal{H}_k(w(a_n+ib_n)) - \mathcal{H}_k(-w(a_n-ib_n))\right] \\ &= \frac{i(2n+1)t}{2\pi} \int_0^\infty dw\, w^{2n} e^{-iw^{2n+1}t} \mathcal{H}_k(w) + \frac{i(2n+1)t}{2\pi} \int_0^\infty dw\, w^{2n} e^{iw^{2n+1}t} \mathcal{H}_k(-w) \\ &\quad - \left[\frac{i(2n+1)t}{2\pi} \int_0^\infty dw\, w^{2n} e^{iw^{2n+1}t} \mathcal{H}_k(w) + \frac{i(2n+1)t}{2\pi} \int_0^\infty dw\, w^{2n} e^{-iw^{2n+1}t} \mathcal{H}_k(-w) \right] \\ &= \frac{i(2n+1)t}{2\pi} \left(\int_{-\infty}^\infty dw\, w^{2n} e^{-iw^{2n+1}t} \mathcal{H}_k(w) - \int_{-\infty}^\infty dw\, w^{2n} e^{iw^{2n+1}t} \mathcal{H}_k(w) \right) \\ &= \frac{1}{2\pi} \left(e^{ik^{2n+1}t} + e^{-ik^{2n+1}t} \right) = \frac{1}{\pi} \cos k^{2n+1} t. \end{aligned} \tag{6.56}$$

In order to justify the last step we can either take the Laplace transform with respect to t (see for example Orsingher and D'Ovidio (2012)) or we can apply the following trick

$$a_k = \lim_{\zeta \to 0} \frac{i(2n+1)t}{2\pi} \left(\int_k^\infty dw\, e^{-\zeta w^{2n+1}} w^{2n} e^{-iw^{2n+1}t} - \int_k^\infty dw\, e^{-\zeta w^{2n+1}} w^{2n} e^{iw^{2n+1}t} \right). \tag{6.57}$$

The coefficients b_k (6.53) can be obtained by performing similar calculation.

6.2.1 Circular Brownian motion

The circular Brownian motion $\mathfrak{B}(t)$, $t > 0$, has been analyzed by Roberts and Ursell (1960), Stephens (1963) and also by Hartman and Watson (1974). In a certain sense it can be viewed as a special case of symmetric pseudoprocesses on the ring $\mathcal{R}$. The distribution of $\mathfrak{B}(t)$, $t > 0$, has Fourier representation

$$p_{\mathfrak{B}}(\theta, t) = \frac{1}{2\pi} \left(1 + 2 \sum_{k=1}^\infty e^{-\frac{k^2 t}{2}} \cos k\theta \right), \qquad \theta \in [0, 2\pi), \tag{6.58}$$

and can be also regarded as the wrapped up distribution of the standard Brownian motion

$$p_{\mathfrak{B}}(\theta, t) = \frac{1}{\sqrt{2\pi t}} \sum_{m=-\infty}^\infty e^{-\frac{(\theta + 2m\pi)^2}{2t}}. \tag{6.59}$$

Formula (6.58) corresponds to $n = 1$ of (6.34) for the even-order case with a suitable adjustement of the time scale. The law (6.58) can be obtained directly by solving the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} p_{\mathfrak{B}}(\theta, t) = \frac{1}{2} \frac{\partial^2}{\partial \theta^2} p_{\mathfrak{B}}(\theta, t), & \theta \in [0, 2\pi),\ t > 0, \\ p_{\mathfrak{B}}(\theta, 0) = \delta(\theta). \end{cases} \tag{6.60}$$

or as the limit of a circular random walk as in Stephens (1963). The distribution of the circular Brownian motion is depicted in Figure 6.2 and looks like the Von Mises circular normal (this is the inspiring idea of the paper by Hartman and Watson (1974) in which the connection between the two distributions is investigated). For $t \to \infty$ the distribution of $\mathfrak{B}(t)$, $t > 0$, tends to the uniform law.

We note that

$$\Pr\left\{-\frac{\pi}{2} < \mathfrak{B}(t) < \frac{\pi}{2}\right\} = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{e^{-\frac{(2k+1)^2 t}{2}}}{2k+1} \tag{6.61}$$

and therefore

$$\Pr\left\{-\frac{\pi}{2} < \mathfrak{B}(t) < \frac{\pi}{2}\right\} \leq \frac{1}{2} + \frac{2}{\pi}e^{-\frac{t}{2}}, \qquad \text{valid for } t > -2\log\frac{\pi}{4} \approx 0.209. \tag{6.62}$$

The relationship between circular Brownian motion $\mathfrak{B}(t)$, $t > 0$, and Brownian motion on the line $B(t)$, $t > 0$,

$$\{\mathfrak{B}(t) \in d\theta\} = \bigcup_{m=-\infty}^{\infty} \{B(t) \in d(\theta + 2m\pi)\}, \qquad \theta \in [0, 2\pi), \tag{6.63}$$

permits us to derive the distribution of

$$\max_{0 \leq s \leq t} |\mathfrak{B}(t)|, \qquad t > 0, \tag{6.64}$$

that is the distribution of the maximal distance reached by the circular Brownian motion from the starting point. Of course the sample paths overcoming the angular distance π at least once are assigned π as maximal distance which therefore has a positive probability (converging to 1 as time tends to infinity).

Proposition 6.2.3. *For the maximal distance (6.64) we have that*

$$\begin{aligned}
\Pr\left\{\max_{0\leq s\leq t}|\mathfrak{B}(s)| < \theta\right\} &= \int_{-\theta}^{\theta} \Pr\left\{-\theta < \min_{0\leq s\leq t} B(s) < \max_{0\leq s\leq t} B(s) < \theta\right\} \\
&= \int_{-\theta}^{\theta} dy \left(\sum_{m=-\infty}^{\infty} \frac{e^{-\frac{(y-4m\theta)^2}{2t}}}{\sqrt{2\pi t}} - \sum_{m=-\infty}^{\infty} \frac{e^{-\frac{(-y+2\theta(2m-1))^2}{2t}}}{\sqrt{2\pi t}}\right) \\
&= \sum_{r=-\infty}^{\infty} (-1)^r \int_{-\frac{(1+2r)\theta}{\sqrt{t}}}^{\frac{(1-2r)\theta}{\sqrt{t}}} dw\, \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}}.
\end{aligned} \tag{6.65}$$

The related first passage time of circular Brownian motion has density which has the following form

$$\Pr\{\mathcal{T}_\theta \in dt\} = -\frac{d}{dt}\Pr\left\{\max_{0\leq s\leq t}|\mathfrak{B}(s)| < \theta\right\} dt$$

$$= \sum_{r=-\infty}^{\infty} \left[\frac{(-1)^r \, e^{-\frac{(1-2r)^2\theta^2}{2t}}}{2\sqrt{2\pi t^3}} \theta(1-2r) + \frac{(-1)^r \, e^{-\frac{(1+2r)^2\theta^2}{2t}}}{2\sqrt{2\pi t^3}} \theta(1+2r) \right]$$

$$= \left(\frac{\theta \, e^{-\frac{\theta^2}{2t}}}{\sqrt{2\pi t^3}} \right) \sum_{r=-\infty}^{\infty} (-1)^r e^{-\frac{2r^2\theta^2}{r}} \left(\cosh \frac{2r\theta^2}{t} - 2r \sinh \frac{2r\theta^2}{t} \right) \quad (6.66)$$

Curiously enough the factor $\theta e^{-\frac{\theta^2}{2t}} / \sqrt{2\pi t^3}$ coincides with the first passage time through θ of a Brownian motion on the line.

6.3 Fractional equations on the ring $\mathcal{R}$ and the related processes

In this section we consider various types of processes on the unit radius circumference $\mathcal{R}$.

6.3.1 Higher-order time-fractional equations

We start by analyzing the processes related to the solutions of time-fractional higher-order heat-type equations. We consider the time-changed pseudoprocesses $\Theta_{2n}\left(L^\nu(t)\right)$, $t > 0$, where

$$L^\nu(t) \;=\; \inf\left\{ s > 0 : H^\nu(s) \geq t \right\} \quad (6.67)$$

and where $H^\nu(t)$, $t > 0$, is a positively skewed stable process of order $\nu \in (0, 1]$. We notice that the Laplace transform of the distribution $l_\nu(x, t)$ of (6.67) reads (see for example Orsingher and Toaldo (2012))

$$\int_0^\infty dx \, e^{-\gamma x} l_\nu(x, t) \;=\; E_{\nu,1}\left(-\gamma t^\nu\right) \quad (6.68)$$

where

$$E_{\nu,1}(x) \;=\; \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\nu j + 1)}, \qquad x \in \mathbb{R}, \nu > 0, \quad (6.69)$$

is the Mittag-Leffler function. For pseudoprocesses related to time-fractional equations we have the next theorem.

Theorem 6.3.1. *The solution to the problem, for $\nu \in (0, 1]$, $n \in \mathbb{N}$,*

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} v_{2n}^\nu(\theta, t) \;=\; -\left(-\frac{\partial^2}{\partial \theta^2}\right)^n v_{2n}^\nu(\theta, t), \qquad \theta \in [0, 2\pi)\,, \, t > 0, \\ v_{2n}^\nu(\theta, 0) \;=\; \delta(\theta), \end{cases} \quad (6.70)$$

is the univariate (signed) distribution of $\Theta_{2n}(L^\nu(t))$, $t>0$, *which reads*

$$v_{2n}^\nu(\theta,t) = \frac{1}{2\pi}\left(1+2\sum_{k=1}^\infty E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta\right). \tag{6.71}$$

The time-fractional derivative in (6.70) *must be understood in the Caputo sense, that is*

$$\frac{\partial^\nu}{\partial t^\nu}v_{2n}^\nu(\theta,t) = \frac{1}{\Gamma(1-\nu)}\int_0^t \frac{\frac{\partial}{\partial s}v_{2n}^\nu(\theta,s)}{(t-s)^\nu}\,ds, \qquad 0<\nu<1. \tag{6.72}$$

Proof. The law of $\Theta_{2n}(L^\nu(t))$, $t>0$, is given by

$$\begin{aligned} v_{2n}^\nu(\theta,t) &= \frac{1}{2\pi}\int_0^\infty ds\left(1+2\sum_{k=1}^\infty e^{-k^{2n}s}\cos k\theta\right)l_\nu(s,t) \\ &= \frac{1}{2\pi}\left(1+2\sum_{k=1}^\infty E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta\right). \end{aligned} \tag{6.73}$$

Since, $\forall k\geq 1$, we have that

$$\begin{aligned} \frac{\partial^\nu}{\partial t^\nu}E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta &= -\,k^{2n}E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta \\ &= (-1)^{n+1}\frac{\partial^{2n}}{\partial\theta^{2n}}E_{\nu,1}\left(-k^{2n}t^\nu\right)\cos k\theta, \end{aligned} \tag{6.74}$$

and therefore we conclude that (6.73) satisfies the fractional equation (6.70). □

Remark 6.3.2. *For* $n=1$, *formula* (6.73) *becomes the distribution of subordinated Brownian motion* $\mathfrak{B}(L^\nu(t))$, $t>0$. *For* $\nu=1$ *we retrieve from* (6.71) *the solutions* (6.34) *of the even-order heat-type equations on* $\mathcal{R}$.

6.3.2 Space-fractional equations and wrapped up stable processes

The following Theorem represents the counterpart on $\mathcal{R}$ of the Riesz statement on the relationship between space-fractional equations and symmetric stable laws (for the non-symmetric case see the paper by Feller (1952)).

Theorem 6.3.3. *The law of the process* $\mathfrak{B}\left(H^\beta(t)\right)$, $t>0$, *is given by*

$$p_{\mathfrak{B}}^\beta(\theta,t) = \frac{1}{2\pi}\left[1+2\sum_{k=1}^\infty e^{-\left(\frac{k^2}{2}\right)^\beta t}\cos k\theta\right] \tag{6.75}$$

and solves the space-fractional equation, for $\beta\in(0,1]$,

$$\begin{cases} \frac{\partial}{\partial t}p_{\mathfrak{B}}^\beta(\theta,t) = -\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta p_{\mathfrak{B}}^\beta(\theta,t), & \theta\in[0,2\pi),\ t>0 \\ p_{\mathfrak{B}}^\beta(\theta,0) = \delta(\theta). \end{cases} \tag{6.76}$$

The fractional one-dimensional Laplacian in (6.76) is defined in (6.27) and $H^\beta(t)$, $t>0$, is a stable subordinator of order $\beta \in (0,1]$.

Proof. The law of $\mathfrak{B}\left(H^\beta(t)\right)$, $t>0$, is given by

$$p^\beta_{\mathfrak{B}}(\theta,t) = \int_0^\infty ds\, p_{\mathfrak{B}}(\theta,s)\, h_\beta(s,t) = \frac{1}{2\pi}\left[1+2\sum_{k=1}^\infty e^{-\left(\frac{k^2}{2}\right)^\beta t}\cos k\theta\right], \tag{6.77}$$

where $p_{\mathfrak{B}}$ is the law of circular Brownian motion and h_β is the density of a positively skewed random process of order $\beta \in (0,1]$. In order to check that (6.75) solves (6.76) it is convenient to write the fractional derivative appearing in (6.76) as

$$\begin{aligned}\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta &= -\frac{\sin\pi\beta}{\pi}\int_0^\infty\left(\lambda+\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)\right)^{-1}\lambda^\beta\, d\lambda \\ &= -\frac{1}{\Gamma(\beta)\Gamma(1-\beta)}\int_0^\infty \lambda^\beta\int_0^\infty e^{-u\lambda-u\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)}\,du\,d\lambda \\ &= \frac{1}{\Gamma(-\beta)}\int_0^\infty u^{-\beta-1}e^{-u\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)}\,du. \end{aligned}\tag{6.78}$$

From (6.78) we have therefore that

$$\begin{aligned}\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta\cos k\theta &= \frac{1}{\Gamma(-\beta)}\int_0^\infty du\,u^{-\beta-1}e^{-u\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)}\cos k\theta \\ &= \frac{1}{\Gamma(-\beta)}\int_0^\infty du\,u^{-\beta-1}\sum_{j=0}^\infty\frac{(-u)^j}{j!}\left(-\frac{1}{2}\right)^j\frac{\partial^{2j}}{\partial\theta^{2j}}\cos k\theta \\ &= \frac{1}{\Gamma(-\beta)}\int_0^\infty du\,u^{-\beta-1}\sum_{j=0}^\infty\frac{(-u)^j}{j!}\frac{k^{2j}}{2^j}\cos k\theta \\ &= \frac{1}{\Gamma(-\beta)}\int_0^\infty du\,u^{-\beta-1}e^{-u\frac{k^2}{2}}\cos k\theta \\ &= \left(\frac{k^2}{2}\right)^\beta\cos k\theta, \end{aligned}\tag{6.79}$$

and this shows that (6.75) satisfies (6.76). □

Remark 6.3.4. *Another way to prove that*

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^\beta\cos k\theta = \left(\frac{k^2}{2}\right)^\beta\cos k\theta \tag{6.80}$$

can be traced in the paragraph 4.6, page 428 of Balakrishnan (1960), which confirms our result.

Theorem 6.3.5. *For the wrapped up version, say $\mathfrak{S}^{2\beta}(t)$, $t>0$, of the symmetric stable processes $S^{2\beta}(t)$, $t>0$, with characteristic function $\mathbb{E}e^{i\xi S^{2\beta}(t)} = e^{-t|\xi|^{2\beta}}$, we have the following equality in distribution*

$$\mathfrak{S}^{2\beta}(t) \overset{law}{=} \mathfrak{B}\left(2H^\beta(t)\right) \overset{law}{=} \mathfrak{B}\left(H^\beta\left(2^\beta t\right)\right), \qquad t>0. \tag{6.81}$$

Proof. The density of $\mathfrak{S}^{2\beta}(t)$, $t > 0$, must be written as

$$p_{\mathfrak{S}^{2\beta}}(\theta, t) = \frac{\Pr\left\{\mathfrak{S}^{2\beta}(t) \in d\theta\right\}}{d\theta} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi \, e^{-i\xi(\theta+2m\pi)} e^{-t|\xi|^{2\beta}}. \tag{6.82}$$

The Fourier expansion of (6.82) becomes

$$p_{\mathfrak{S}^{2\beta}}(\theta, t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta \tag{6.83}$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_0^{2\pi} d\theta \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi \, e^{-i\xi(\theta+2m\pi)} e^{-t|\xi|^{2\beta}} \cos k\theta \\ &= \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi \, e^{-t|\xi|^{2\beta}} \int_m^{m+1} d\theta \, e^{-i\xi 2\pi\theta} \cos 2k\pi\theta \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\xi \, e^{-t|\xi|^{2\beta}} \int_{-\infty}^{\infty} dy \, e^{-i\xi y} \left(e^{iyk} + e^{-iyk}\right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, e^{-t|\xi|^{2\beta}} \left[\delta(\xi - k) + \delta(\xi + k)\right] = \frac{1}{\pi} e^{-tk^{2\beta}}. \end{aligned} \tag{6.84}$$

This permits us to conclude that

$$p_{\mathfrak{S}^{2\beta}}(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-k^{2\beta} t} \cos k\theta. \tag{6.85}$$

□

While the integral in (6.82) (representing the Fourier inverse of symmetric stable laws) cannot carried out, its circular analogue can be explicitely worked out and leads to the Fourier expansion (6.85).

Corollary 6.3.6. *In view of the results of Theorems 6.3.1 and 6.3.3 we have that the solution to the space-time fractional equation, for* $\beta \in (0, 1]$,

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} p_{\mathfrak{B}}^{\nu,\beta}(\theta, t) = -\left(-\frac{1}{2}\frac{\partial^2}{\partial \theta^2}\right)^\beta p_{\mathfrak{B}}^{\nu,\beta}(\theta, t), & \theta \in [0, 2\pi), \, t > 0 \\ p_{\mathfrak{B}}^{\nu,\beta}(\theta, 0) = \delta(\theta). \end{cases} \tag{6.86}$$

can be written as

$$p_{\mathfrak{B}}^{\nu,\beta}(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} E_{\nu,1}\left(-\left(\frac{k^2}{2}\right)^\beta t^\nu\right) \cos k\theta, \tag{6.87}$$

and coincides with the law of the process

$$\mathcal{F}^{\nu,\beta}(t) = \mathfrak{B}\left(H^\beta\left(L^\nu(t)\right)\right), \qquad t > 0. \tag{6.88}$$

In (6.88) H^β *is a stable subordinator of order* $\beta \in (0, 1]$ *and* L^ν *is the inverse of* H^ν *as defined in (6.67).*

Proof. Here we only derive the distribution of $\mathcal{F}^{\nu,\beta}(t)$, $t > 0$. We have that

$$\begin{aligned}
\Pr\left\{\mathcal{F}^{\nu,\beta}(t) \in d\theta\right\} &= d\theta \int_0^\infty \Pr\left\{\mathfrak{B}(s) \in d\theta\right\} \int_0^\infty \Pr\left\{H^\beta(w) \in ds\right\} \Pr\left\{L^\nu(t) \in dw\right\} \\
&= \frac{d\theta}{2\pi} + \frac{d\theta}{\pi} \int_0^\infty \sum_{k=1}^\infty e^{-\frac{k^2}{2}s} \cos k\theta \int_0^\infty \Pr\left\{H^\beta(w) \in ds\right\} \Pr\left\{L^\nu(t) \in dw\right\} \\
&= \frac{d\theta}{2\pi} + \frac{d\theta}{\pi} \int_0^\infty \sum_{k=1}^\infty e^{-\left(\frac{k^2}{2}\right)^\beta w} \cos k\theta \Pr\left\{L^\nu(t) \in dw\right\} \\
&= \frac{d\theta}{2\pi} \left[1 + 2\sum_{k=1}^\infty \cos k\theta \, E_{\nu,1}\left(-\left(\frac{k^2}{2}\right)^\beta t^\nu\right)\right], \qquad (6.89)
\end{aligned}$$

where in the last step we applied (6.68). □

6.4 From pseudoprocesses to Poisson kernels

In this section we show that the composition of pseudoprocesses of order n running on the circumference $\mathcal{R}$ with positively skewed stable processes of order $\frac{1}{n}$ leads to the Poisson kernel. This is the circular counterpart of the composition of pseudoprocesses with stable subordinators which leads to Cauchy processes. In both cases pseudoprocesses stopped at $H^{\frac{1}{n}}(t)$, $t > 0$, yield genuine random variables.

We distinguish the case where n is even from the case of odd-order pseudoprocesses. We have the first result in Theorem 6.4.1.

Theorem 6.4.1. *The composition $\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)$, $t > 0$, of the pseudoprocess Θ_{2n} with the stable process $H^{\frac{1}{2n}}(t)$, $t > 0$, has density*

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right) \in d\theta\right\} = \frac{d\theta}{2\pi} \frac{1 - e^{-2t}}{1 + e^{-2t} - 2e^{-t}\cos\theta}, \qquad n \in \mathbb{N}, \qquad (6.90)$$

and distribution function

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right) < \theta\right\} = \begin{cases} \frac{1}{\pi} \arctan\left(\frac{1+e^{-t}}{1-e^{-t}} \tan\frac{\theta}{2}\right), & \theta \in [0, \pi], \\ 1 + \frac{1}{\pi} \arctan\left(\frac{1+e^{-t}}{1-e^{-t}} \tan\frac{\theta}{2}\right), & \theta \in (\pi, 2\pi), \end{cases} \qquad (6.91)$$

which are independent from n.

Proof. We have that

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right) \in d\theta\right\} = d\theta \int_0^\infty ds \left[\frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^\infty e^{-k^{2n}s} \cos k\theta\right] h_{\frac{1}{2n}}(s,t)$$

$$= d\theta \left[\frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \cos k\theta e^{-kt} \right]$$
$$= \frac{d\theta}{2\pi} \left[1 + \frac{e^{-t+i\theta}}{1+e^{-t+i\theta}} + \frac{e^{-t-i\theta}}{1-e^{-t-i\theta}} \right]$$
$$= \frac{d\theta}{2\pi} \frac{1-e^{-2t}}{1+e^{-2t}-2e^{-t}\cos\theta}. \tag{6.92}$$

The result (6.91) is derived by applying formula 2.552(3) page 172 of Gradshteyn and Ryzhik (2007)

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \arctan \frac{\sqrt{a^2-b^2}\tan\frac{x}{2}}{a+b}, \qquad a^2 > b^2. \tag{6.93}$$

□

Remark 6.4.2. *The Poisson kernel (6.90) can be interpreted as the distribution of the process $\boldsymbol{B}(\mathfrak{T}_{\mathcal{R}})$ where $\boldsymbol{B}$ is a planar Brownian motion and $\mathfrak{T}_{\mathcal{R}} = \inf\{t > 0 : \boldsymbol{B}(t) \in \mathcal{R}\}$. In the case of Theorem 6.4.1 the planar Brownian motion starts from the point $(e^{-t}, 0)$. Therefore we have that*

$$\boldsymbol{B}(\mathfrak{T}_{\mathcal{R}}) \overset{law}{=} \Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right), \qquad t > 0. \tag{6.94}$$

This means that a pseudoprocess running on the ring $\mathcal{R}$ and stopped at a stable time $H^{\frac{1}{2n}}(t)$, $t > 0$, has the same distribution of a planar Brownian motion starting from $(e^{-t}, 0)$ at the first exit time from the unit-radius circle. The result (6.94) holds for all $n \in \mathbb{N}$ and represents the circular counterpart of the composition of pseudoprocesses on the line with stable subordinators $H^{\frac{1}{n}}(t)$, $t > 0$, which possesses a Cauchy distributed law. As $t \to \infty$ the distribution (6.90) converges to the uniform law.

Remark 6.4.3. *In view of (6.91) we note that for $\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right)$, $t > 0$, the probability of staying in the right-hand side of $\mathcal{R}$ has the remarkably simple form*

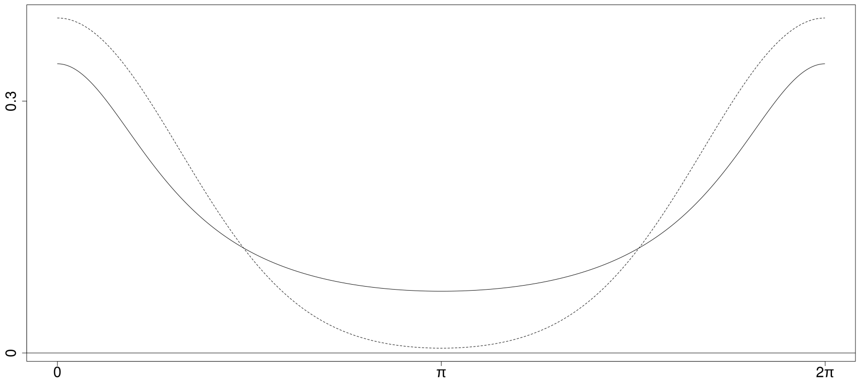
$$\Pr\left\{-\frac{\pi}{2} < \Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right) < \frac{\pi}{2}\right\} = \frac{1}{2} + \frac{2}{\pi}\arctan e^{-t}, \qquad \forall t > 0. \tag{6.95}$$

We now pass to the Poisson kernel associated to odd-order pseudoprocesses. The asymmetry implies that the density of the composition $\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$, $t > 0$, is bit more complicated than (6.90).

Theorem 6.4.4. *The composition $\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$, $t > 0$, has density*

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) \in d\theta\right\} = \frac{d\theta}{2\pi} \frac{1-e^{-2a_n t}}{1+e^{-2a_n t}-2e^{-a_n t}\cos(\theta + b_n t)}, \tag{6.96}$$

Figure 6.3: In the picture the density of the circular Brownian motion (dotted line) and the kernel (6.90) are represented.



and distribution function

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) < \theta\right\} =$$
$$= \begin{cases} \frac{1}{\pi}\left[\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2} - \arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}\right], & 0<\frac{\theta+b_nt}{2}<\pi, \\ 1+\frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2} - \frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}, & \pi<\theta<2\pi-\frac{b_nt}{2}. \end{cases} \tag{6.97}$$

where

$$a_n = \cos\frac{\pi}{2(2n+1)}, \qquad b_n = \sin\frac{\pi}{2(2n+1)}. \tag{6.98}$$

Proof. Let $h_{\frac{1}{2n+1}}(s,t)$, $s,t>0$, be the density of a positively skewed stable process of order $\frac{1}{2n+1}$. Then, in view of (6.34), we have that

$$\begin{aligned} \Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)\in d\theta\right\} &= d\theta\int_0^\infty ds\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty\cos\left(k\theta+k^{2n+1}s\right)\right)h_{\frac{1}{2n+1}}(s,t) \\ &= d\theta\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^\infty e^{-a_ntk}\cos\left(k(\theta+b_nt)\right)\right] \\ &= \frac{d\theta}{2\pi}\left[1+\frac{e^{i\theta}e^{-t(a_n-ib_n)}}{1-e^{i\theta}e^{-t(a_n-ib_n)}}+\frac{e^{-i\theta}e^{-t(a_n+ib_n)}}{1-e^{-i\theta}e^{-t(a_n+ib_n)}}\right] \\ &= \frac{d\theta}{2\pi}\frac{1-e^{-2a_nt}}{1+e^{-2a_nt}-2e^{-a_nt}\cos(\theta+b_nt)}. \end{aligned} \tag{6.99}$$

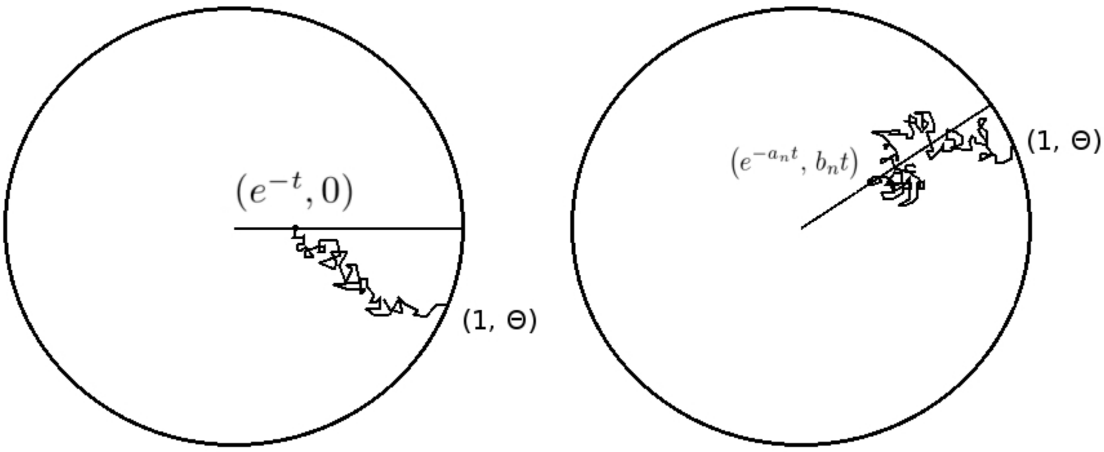
The same result can be obtained by considering that $X_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$ has the following Cauchy distribution (see Orsingher and D'Ovidio (2012))

$$\Pr\left\{X_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)\in dx\right\} = \frac{t\cos\frac{\pi}{2(2n+1)}}{\pi\left[\left(x+t\sin\frac{\pi}{2(2n+1)}\right)^2+t^2\cos^2\frac{\pi}{2(2n+1)}\right]}dx. \tag{6.100}$$

By wrapping up (6.100) we arrive at (6.96) in an alternative way. In view of (6.93) we can write

$$\begin{aligned}
&\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\theta\right\} = \\
&=\frac{1}{2\pi}\int_0^\theta dy\,\frac{1-e^{-2a_nt}}{1+e^{-2a_nt}-2e^{-a_nt}\cos\left(y+b_nt\right)} \\
&=\begin{cases}\frac{1}{\pi}\left[\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2}-\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}\right], & 0<\frac{\theta+b_nt}{2}<\pi,\\ 1+\frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{\theta+b_nt}{2}-\frac{1}{\pi}\arctan\frac{1+e^{-a_nt}}{1-e^{-a_nt}}\tan\frac{b_nt}{2}, & \pi<\theta<2\pi-\frac{b_nt}{2}.\end{cases}
\end{aligned} \tag{6.101}$$

□



(a) (b)

Figure 6.4: The distribution of the hitting point of a planar Brownian motion is obtained as subordinated circular pseudoprocess in the even case (Fig. 6.4a) and odd case (Fig. 6.4b).

Remark 6.4.5. *From* (6.101) *we arrive at the following fine expression*

$$\begin{aligned}
&\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)<\theta\right\} \\
&=\frac{1}{\pi}\arctan\left[\frac{\left(1-e^{-2a_nt}\right)\tan\frac{\theta}{2}\left(1+\tan^2\frac{b_nt}{2}\right)}{\left(1-e^{-a_nt}\right)^2+4\tan\frac{\theta}{2}\tan\frac{b_nt}{2}+\left(1+e^{-a_nt}\right)^2\tan^2\frac{b_nt}{2}}\right], \qquad \theta\in[0,2\pi),
\end{aligned} \tag{6.102}$$

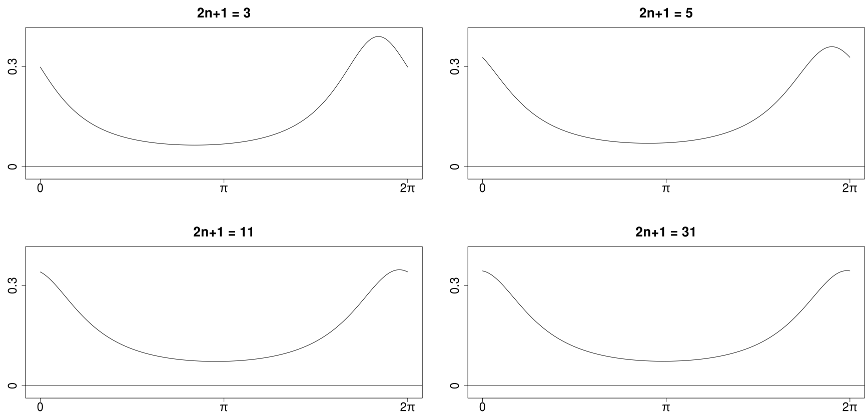


Figure 6.5: Distributions related to odd-order Poisson kernels (for $t = 1$)

from which we are able to explicitely write for $\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$ *the probability of staying in the interval* $(0, \pi)$ *as*

$$\Pr\left\{0 < \Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) < \pi\right\} = \frac{1}{\pi}\arctan\frac{\sinh a_n t}{\sin b_n t}, \qquad \forall t > 0, \tag{6.103}$$

while for $\left(0, \frac{\pi}{2}\right)$ *we obtain*

$$\begin{aligned}
&\Pr\left\{0 < \Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) < \frac{\pi}{2}\right\} \\
&= \frac{1}{\pi}\arctan\frac{\left(1 - e^{-2a_n t}\right)\left(1 + \tan^2\frac{b_n t}{2}\right)}{\left(1 - e^{-a_n t}\right)^2 + 4\tan\frac{b_n t}{2} + \left(1 + e^{-a_n t}\right)^2\tan^2\frac{b_n t}{2}} \\
&= \frac{1}{\pi}\arctan\frac{\sinh a_n t}{2\sinh^2\frac{a_n t}{2}\cos^2\frac{b_n t}{2} + e^{a_n t}\sin b_n t + 2\cosh^2\frac{a_n t}{2}\sin^2\frac{b_n t}{2}} \\
&= \frac{1}{\pi}\arctan\frac{\sinh a_n t}{\cosh a_n t - \cos b_n t + e^{a_n t}\sin b_n t}.
\end{aligned} \tag{6.104}$$

By means of the same manipulations leading to (6.104) *we arrive at the alternative form of the distribution function for* $\theta \in [0, \pi]$,

$$\Pr\left\{0 < \Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) < \theta\right\} = \frac{1}{\pi}\arctan\frac{\sinh a_n t\,\tan\frac{\theta}{2}}{\cosh a_n t - \cos b_n t + e^{a_n t}\sin b_n t\,\tan\frac{\theta}{2}}. \tag{6.105}$$

Remark 6.4.6. *In the third-order case we can arrive at the Poisson kernel* (6.96) *for* $n = 1$ *by considering that (see Orsingher and D'Ovidio (2012))*

$$\begin{aligned}
\frac{\Pr\left\{X_3\left(H^{\frac{1}{3}}(t)\right) \in dx\right\}}{dx} &= \int_0^\infty \frac{ds}{\sqrt[3]{3s}}\,Ai\left(\frac{x}{\sqrt[3]{3s}}\right)\,\frac{t}{s\sqrt[3]{3s}}\,Ai\left(\frac{t}{\sqrt[3]{3s}}\right) \\
&= \frac{\sqrt{3}t}{2\pi\left[\left(x + \frac{t}{2}\right)^2 + \frac{3t^2}{4}\right]}.
\end{aligned} \tag{6.106}$$

The wrapped up counterpart of (6.106) *becomes for* $\theta \in [0, 2\pi)$

$$\Pr\left\{\Theta_3\left(H^{\frac{1}{3}}(t)\right) \in d\theta\right\} = \sum_{m=-\infty}^{\infty} \Pr\left\{X_3\left(H^{\frac{1}{3}}(t)\right) \in d(\theta + 2m\pi)\right\} =$$

$$= d\theta \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{ds}{\sqrt[3]{3s}} Ai\left(\frac{\theta + 2m\pi}{\sqrt[3]{3s}}\right) \frac{t}{s\sqrt[3]{3s}} Ai\left(\frac{t}{\sqrt[3]{3s}}\right)$$

$$= d\theta \frac{\sqrt{3}t}{2\pi} \sum_{m=-\infty}^{\infty} \frac{t}{\left(\theta + 2m\pi + \frac{t}{2}\right)^2 + 3\frac{t^2}{4}} = \frac{d\theta\sqrt{3}t}{2} \sum_{m=-\infty}^{\infty} \int_0^\infty ds \frac{e^{-\frac{\left(\theta + 2m\pi + \frac{t}{2}\right)^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{(\sqrt{3}t/2)^2}{2s}}}{\sqrt{2\pi s^3}}$$

$$= \frac{d\theta\sqrt{3}t}{2^2\pi} \int_0^\infty ds \left[1 + 2\sum_{k=1}^{\infty} \cos\left[k\left(\theta + \frac{t}{2}\right)\right] e^{-\frac{k^2 s}{2}}\right] \frac{e^{-\frac{(\sqrt{3}t/2)^2}{2s}}}{\sqrt{2\pi s^3}}$$

$$= \frac{d\theta}{2\pi}\left[1 + 2\sum_{k=1}^{\infty} e^{-\frac{\sqrt{3}}{2}tk} \cos\left[k\left(\theta + \frac{t}{2}\right)\right]\right] = \frac{d\theta}{2\pi} \frac{1 - e^{-\sqrt{3}t}}{1 + e^{-\sqrt{3}t} - 2e^{-\sqrt{3}t}\cos\left(\theta + \frac{t}{2}\right)}, \tag{6.107}$$

which coincides with (6.96) *since* $a_1 = \frac{\sqrt{3}}{2}$ *and* $b_1 = \frac{1}{2}$.

Chapter 7

Higher-order Laplace equations

Article: Orsingher and Toaldo (2012). Shooting randomly against a line in euclidean and non-euclidean spaces.

Summary

In this paper we study a class of distributions related to the r.v. $C_n(t) = t \tan^{\frac{1}{n}} \Theta$, for different distributions of Θ. The problem is related to the hitting point of a randomly oriented ray and generalize the Cauchy distribution in different directions. We show that the distribution of $C_n(t)$ solves the Laplace equation of order $2n$, possesses even moments of order $2k < 2n - 1$, and has bimodal structure when Θ is uniform. We study also a number of distributional properties of functionals of $C_n(t)$, including those related to the arcsine law. Finally we study the same problem in the Poincaré half-plane and this leads to the hyperbolic distribution $\Pr\{\eta \in dw\} = \frac{dw}{\pi \cosh w}$ of which the main properties are explored. In particular we study the distribution of hyperbolic functions of η, the law of sums of i.i.d. r.v.'s η_j and the distribution of the area of random hyperbolic right triangles.

7.1 Introduction

In this paper we consider the random variables of the form

$$C_n(t) = \begin{cases} t \tan^{\frac{1}{n}} \Theta, & \Theta \in \left(0, \frac{\pi}{2}\right), \quad n \in \mathbb{N}, t > 0, \\ -t \tan^{\frac{1}{n}} |\Theta|, & \Theta \in \left(-\frac{\pi}{2}, 0\right), \end{cases} \tag{7.1}$$

under different assumptions on the distribution of Θ.

First of all we consider the case where the random angle Θ has distribution

$$q_n(\theta) = \frac{\sin\frac{\pi}{2n}}{\pi}\cot^{\frac{n-1}{n}}|\theta|, \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), n \in \mathbb{N},$$

and we show that in this case $C_n(t)$ has probability density

$$p_n(x,t) = \left(\frac{n\sin\frac{\pi}{2n}}{\pi}\right)\frac{t^{2n-1}}{t^{2n}+x^{2n}}, \qquad x \in \mathbb{R}, t > 0. \tag{7.2}$$

We regard (7.2) as a generalization of the classical symmetric Cauchy law under many viewpoints. First of all because, for $n = 1$, the angle has uniform distribution and the law of $C_1(t)$ becomes

$$p_1(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2}, \qquad x \in \mathbb{R}, t > 0.$$

Furthermore in this case $C_1(t) = t\tan\Theta$ represents the segment intersected by a ray shooted from the point O against the parallel t units away.

In the case $n > 1$ we mantain the same interpretation but here the angle has a law which becomes increasingly concentrated around $\theta = 0$ as n increases.

For $|x| < t$ the cumulative distribution of (7.2) has the form

$$\Pr\{C_n(t) < x\} = \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2nk+1}\left(\frac{x}{t}\right)^{2nk+1}$$

where

$$\mathcal{O}_\alpha(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2\alpha k+1}z^{2\alpha k+1}, \qquad z^2 < 1, \alpha > 0,$$

represents a generalization of the $\arctan z$ function and reduces to it for $\alpha = 1$.

The density (7.2) is a solution to the $2n$-th order Laplace equation

$$\left(\frac{\partial^{2n}}{\partial t^{2n}} + \frac{\partial^{2n}}{\partial x^{2n}}\right)p_n(x,t) = 0.$$

However (7.2) differs from the classical Cauchy because even moments of order $2k < 2n-1$ exist and is non longer infinitely divisible as the characteristic function shows.

Some other properties of the Cauchy are lost but by considering some other related distributions we are able to give a picture of generalized higher-order Cauchy distributions with interesting interlaced distributional properties. In the case Θ is uniform in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the probability density of

$$\widehat{C}_n(t) = \begin{cases} t\tan^{\frac{1}{n}}\Theta, & \Theta \in \left(0, \frac{\pi}{2}\right), \qquad n \in \mathbb{N}, t > 0, \\ -t\tan^{\frac{1}{n}}|\Theta|, & \Theta \in \left(-\frac{\pi}{2}, 0\right), \end{cases}$$

reads

$$\widehat{p}_n(x,t) \;=\; \frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+x^{2n}}, \qquad x\in\mathbb{R}, t>0. \tag{7.3}$$

The distribution has a bimodal structure and thus substantially differs from (7.2). The maxima of (7.3) are located at $x=\pm t\left(\frac{n-1}{n+1}\right)^{\frac{1}{2n}}$. For the r.v. $\widehat{C}_n(t)$ the following remarkable property holds

$$\frac{1}{\widehat{C}_n\left(\frac{1}{t}\right)} \overset{\text{i.d.}}{=} \widehat{C}_n(t).$$

In our view it is relevant that the probability law (7.3) shares with the classical Cauchy also the property that

$$\widehat{Z}_n(t) \;=\; \frac{t}{1+\left(\frac{\widehat{C}_n(t)}{t}\right)^{2n}}$$

possesses arcsine distribution, that is

$$\Pr\left\{\widehat{Z}_n(t)\in dw\right\} \;=\; \frac{dw}{\pi\sqrt{w(t-w)}}, \qquad 0<w<t.$$

Curiously enough the ratio of independent r.v.'s $\widehat{W}_n(t) = \frac{\widehat{C}^1_n(t)}{\widehat{C}^2_n(t)}$ has a distribution which generalizes that of the ratio of independent Cauchy r.v.'s; that is

$$\Pr\left\{\frac{\widehat{C}^1_n(t)}{\widehat{C}^2_n(t)}\in dw\right\} \;=\; \frac{dw}{\pi^2}\frac{nt^n|w|^{n-1}}{(t^{2n}-w^{2n})}\log\left(\frac{t}{w}\right)^{2n}, \qquad w\in\mathbb{R}, t>0.$$

However the distribution (7.3) does not satisfy an higher-order Laplace equation as (7.2) does.

The third r.v. considered below is

$$\widetilde{C}_n(t) \;=\; t\tan\Theta \tag{7.4}$$

where Θ has distribution $q_n(\theta)$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. The distribution of (7.4) is unimodal and has analytical form

$$\widetilde{p}_n(x,t) \;=\; \frac{1}{\pi}\sin\frac{\pi}{2n}\frac{t}{t^2+x^2}\left(\frac{t}{|x|}\right)^{\frac{n-1}{n}}, \qquad x\in\mathbb{R}, t>0. \tag{7.5}$$

We note that for (7.5) the r.v.

$$\widetilde{Z}_n(t) \;=\; \frac{t}{1+\left(\frac{\widetilde{C}_n(t)}{t}\right)^{2}}$$

has Beta distribution with parameters $\left(\frac{1}{2n}, 1-\frac{1}{2n}\right)$.

In the last section of the paper we consider the problem of shooting against a geodesic line in the Poincaré half-plane $\mathbb{H}_2^+$. We shoot from a point O of the x-axis, representing the infinite in $\mathbb{H}_2^+$ against a half-circumference of radius t and center O (see figure 7.6a below). The hyperbolic distance η between the points P and Q, is given by

$$\eta = \begin{cases} -\log\tan\frac{\theta}{2}, & \theta \in \left(0, \frac{\pi}{2}\right), \\ \log\tan\frac{\theta}{2}, & \theta \in \left(\frac{\pi}{2}, \pi\right), \end{cases} \tag{7.6}$$

because the metric in $\mathbb{H}_2^+$ is

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

Considering Θ uniformly distributed in $(0,\pi)$, the random variable (7.6) has probability density

$$\Pr\{\eta \in dw\} = \frac{4}{\pi}\frac{e^{-w}dw}{1+e^{-2w}} = \frac{2}{\pi}\frac{1}{\cosh w}dw, \qquad w > 0.$$

The symmetric r.v.

$$\widehat{\eta} = -\log\tan\frac{\Theta}{2}$$

has density

$$\Pr\{\widehat{\eta} \in dw\} = \frac{1}{\pi}\frac{1}{\cosh w}dw, \qquad w \in \mathbb{R}, \tag{7.7}$$

and characteristic function

$$\mathbb{E}e^{i\beta\widehat{\eta}} = \frac{1}{\cosh\frac{\beta\pi}{2}}, \qquad \beta \in \mathbb{R}.$$

The hyperbolic r.v. $\widehat{\eta}$ has the unusual property that its density and characteristic function have the same analytic form. The even-order moments

$$\mathbb{E}\widehat{\eta}^{2n} = \left(\frac{\pi}{2}\right)^{2n}|E_{2n}|$$

show an interesting relationship with the Euler numbers E_{2n}. We produce a direct derivation of the distribution

$$\Pr\{\widehat{\eta}_1+\widehat{\eta}_2 \in dw\} = \frac{2}{\pi}\frac{w}{\sinh w}, \qquad w \in \mathbb{R},$$

by means of the Cauchy residue theorem and we give also the explicit distribution of sums $\widehat{\boldsymbol{\eta}}_n = \sum_{j=1}^n \widehat{\eta}_j$ for any $n \in \mathbb{N}$. We obtain the distribution of all hyperbolic functions of $\widehat{\eta}$ and of other related functionals. For example, we prove that the law of $\sinh\widehat{\eta}$ coincides with the standard Cauchy and that

$$Y = \frac{1}{\cosh^2\widehat{\eta}} = \frac{1}{1+\sinh^2\widehat{\eta}}$$

has arcsine distribution. In the last section of the paper we also derive the distribution of the area K of the hyperbolic right triangle (see fig. 7.6a one side of which has length η defined in (7.6). We show that the distribution of K is

$$\Pr\{K \in dw\} = \frac{2}{\pi}\frac{dw}{1+\sin w}, \qquad w \in \left(0, \frac{\pi}{2}\right), \tag{7.8}$$

with mean

$$\mathbb{E}K = \frac{2}{\pi}\log 2.$$

7.2 The higher order Cauchy random variables

We consider the angular distribution

$$q_n(\theta) = \frac{\sin\frac{\pi}{2n}}{\pi}\cot^{\frac{n-1}{n}}|\theta| \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{7.9}$$

which for $n=1$ coincides with the uniform law in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The distribution (7.9) is concentrated around $\theta = 0$ (see figure 7.1) and its spread around the mean decreases as n increases. One expects that the shots must be concentrated around the target and (7.9) satisfies this requirement. In order to check that (7.9) integrates to unity we perform the following calculation

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q_n(\theta)d\theta &= \frac{2\sin\frac{\pi}{2n}}{\pi}\int_0^{\frac{\pi}{2}}\cot^{\frac{n-1}{n}}\theta\, d\theta \\ &\overset{\sin\theta=\sqrt{y}}{=} \frac{1}{\pi}\sin\frac{\pi}{2n}\int_0^1 y^{\frac{1}{2n}-1}(1-y)^{-\frac{1}{2n}}\, dy \\ &= \frac{1}{\pi}\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{2n}\right)\Gamma\left(1-\frac{1}{2n}\right) = 1,\end{aligned}$$

because $\Gamma\left(\frac{1}{2n}\right)\Gamma\left(1-\frac{1}{2n}\right) = \frac{\pi}{\sin\frac{\pi}{2n}}$ for the reflection formula of the Gamma integral.

We note that the related random variable $\cos\Theta$ with Θ distributed as (7.9) has even-order moments equal to

$$\begin{aligned}\mathbb{E}\cos^m\Theta &= 2\int_0^{\frac{\pi}{2}}\cos^m\theta q_n(\theta)\, d\theta \\ &= \frac{1}{\pi}\sin\frac{\pi}{2n}\frac{\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{m}{2}+1-\frac{1}{2n}\right)}{\Gamma\left(\frac{m}{2}+1\right)}.\end{aligned}$$

The special case $m = 2$ yields $\mathbb{E}\cos^2\Theta = 1 - \frac{1}{2n}$.

We now pass to the derivation of the distribution of

$$C_n(t) = \begin{cases} t\tan^{\frac{1}{n}}\Theta, & \Theta \in \left(0, \frac{\pi}{2}\right), \qquad n \in \mathbb{N}, t > 0, \\ -t\tan^{\frac{1}{n}}|\Theta|, & \Theta \in \left(-\frac{\pi}{2}, 0\right). \end{cases}$$

Theorem 7.2.1. *The explicit law of $C_n(t)$, where Θ possesses distribution (7.9), reads*

$$\Pr\{C_n(t) \in dx\} = \frac{n\sin\frac{\pi}{2n}}{\pi}\frac{t^{2n-1}}{t^{2n}+x^{2n}}\,dx \qquad x \in \mathbb{R}, t > 0, \tag{7.10}$$

and for $|x| < t$

$$\Pr\{C_n(t) < x\} = \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2nk+1}\left(\frac{x}{t}\right)^{2nk+1}. \tag{7.11}$$

Proof. For $x > 0$, we have that

$$\begin{aligned} \Pr\{C_n(t) < x\} &= \Pr\left\{t\tan^{\frac{1}{n}}\Theta < x\right\} \\ &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{\pi}\int_0^{\arctan\left(\frac{x}{t}\right)^n}\cot^{\frac{n-1}{n}}\theta\,d\theta. \end{aligned} \tag{7.12}$$

By taking the derivative of (7.12) with respect to x we readily have the density (7.10). In the same spirit of the previous calculation we obtain the result for $x < 0$. By means of the substitution $\tan\theta = y$ we reduce (7.12) to the form

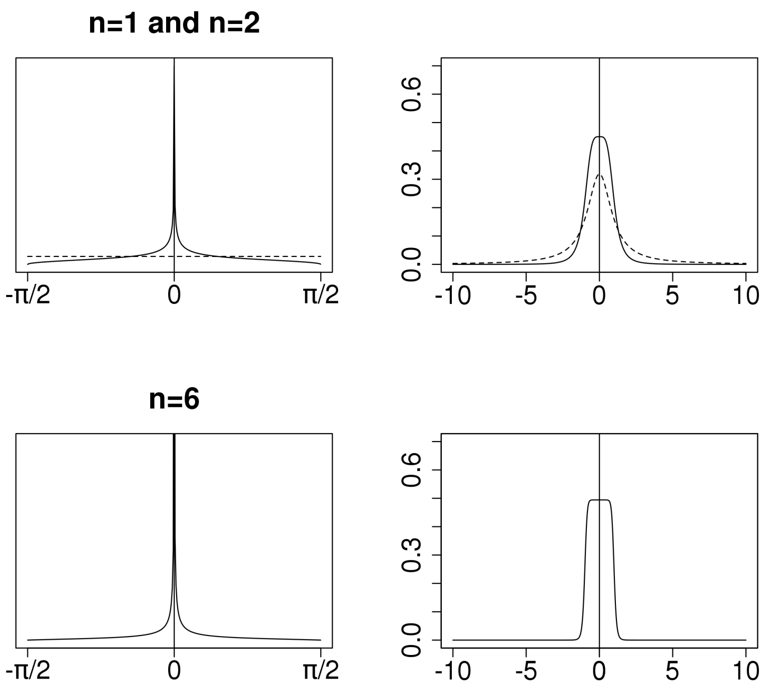
$$\begin{aligned} \Pr\{C_n(t) < x\} &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{\pi}\int_0^{\left(\frac{x}{t}\right)^n}\frac{1}{y^{\frac{n-1}{n}}}\frac{dy}{1+y^2} \\ &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{\pi}\sum_{k=0}^{\infty}(-1)^k\int_0^{\left(\frac{x}{t}\right)^n} y^{2k-1+\frac{1}{n}}\,dy \\ &= \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{x}{t}\right)^{2nk+1}}{2nk+1}, \qquad |x| < t, \end{aligned} \tag{7.13}$$

which coincides with (7.11). The intermediate step shows why the cumulative function can be written as in (7.11) for $|x| < t$. □

Remark 7.2.2. The density (7.10) has the alternative form

$$p_n(x,t) = \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\int_0^{\infty} e^{-zt}E_{2n,1}\left(-x^{2n}z^{2n}\right)\,dz, \tag{7.14}$$

Figure 7.1: The probabilty function (7.9) of the r.v. Θ (left column) and the related distribution of $C_n(t)$ (right column). The dotted lines represent the uniform law and the Cauchy density.



where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad z \in \mathbb{R}, \alpha>0, \beta>0,$$

is the Mittag-Leffler function, see for example Haubold, Mathai and Saxena (2011). The representation (7.14) permits us to show that it satisfies the Laplace equation of order $2n$. Since

$$\frac{\partial^{2n}}{\partial x^{2n}} E_{2n,1}\left(-z^{2n}x^{2n}\right) = -z^{2n} E_{2n,1}\left(-z^{2n}x^{2n}\right),$$

we have that

$$\begin{aligned} &\frac{\partial^{2n}}{\partial x^{2n}} \int_0^\infty e^{-zt} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz \\ &= -\int_0^\infty e^{-zt} z^{2n} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz \\ &= -\frac{\partial^{2n}}{\partial t^{2n}} \int_0^\infty e^{-zt} E_{2n,1}\left(-x^{2n}z^{2n}\right) dz. \end{aligned}$$

The probability density (7.10) is an unimodal function which for $n \to \infty$ converges to the uniform law in $(-t,t)$. For increasing values of n it takes the form of a rectangular wave as figure 7.1 shows.

Remark 7.2.3. The distribution function of $C_n(t)$, $t>0$, can be represented in terms of hypergeometric functions for all $w \in \mathbb{R}$ without the restriction $\left(\frac{w}{t}\right)^2 < 1$. For $w>0$ we have that

$$\begin{aligned} \Pr\{C_n(t)<w\} &= \frac{1}{2} + \frac{n \sin\frac{\pi}{2n}}{\pi} \int_0^w \frac{t^{2n-1}}{t^{2n}+x^{2n}} dx \\ &\overset{x=ty}{=} \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi} \int_0^{\frac{w}{t}} dy \int_0^\infty du\, e^{-u\left(1+y^{2n}\right)} \\ &\overset{y=x^{\frac{1}{2n}}}{=} \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{2\pi} \int_0^\infty du\, e^{-u} \int_0^{\left(\frac{w}{t}\right)^{2n}} dx\, e^{-ux} x^{\frac{1}{2n}-1} \\ &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{2\pi} \int_0^\infty e^{-u} u^{-\frac{1}{2n}} \gamma\left(\frac{1}{2n}, u\left(\frac{w}{t}\right)^{2n}\right) du \\ &= \frac{1}{2} + \frac{\sin\frac{\pi}{2n}}{2\pi} \frac{\frac{w}{t}\Gamma(1)}{\frac{1}{2n}\left(\left(\frac{w}{t}\right)^{2n}+1\right)} F\left(1,1;\frac{1}{2n}+1;\frac{\left(\frac{w}{t}\right)^{2n}}{\left(\frac{w}{t}\right)^{2n}+1}\right) \\ &= \frac{1}{2} + \frac{n\sin\frac{\pi}{2n}}{\pi} \frac{w\,t^{2n-1}}{w^{2n}+t^{2n}} F\left(1,1;\frac{1}{2n}+1;\frac{w^{2n}}{w^{2n}+t^{2n}}\right). \end{aligned} \tag{7.15}$$

In the above steps we denoted by

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt$$

the incomplete Gamma function. By

$$F\left(a,b;c;z\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{\Gamma\left(a+k\right)\Gamma\left(b+k\right)}{\Gamma\left(c+k\right)} \frac{1}{B\left(a,b\right)}\frac{z^k}{k!},$$

we denote the hypergeometric function. In the last step we used formula 6.455, page 657, of Gradshteyn and Ryzhik (2007), that is

$$\int_0^{\infty} x^{\mu-1}e^{-\beta x}\,\gamma\left(\nu,\alpha x\right)\,dx = \frac{\alpha^{\nu}\Gamma\left(\mu+\nu\right)}{\nu\left(\alpha+\beta\right)^{\mu+\nu}}\,F\left(1,\mu+\nu;\nu+1;\frac{\alpha}{\alpha+\beta}\right),$$

valid for $\Re\left(\alpha+\beta\right)>0$, $\Re\left(\beta\right)>0$, $\Re\left(\mu+\nu\right)>0$. With little changes we can see that (7.15) holds also for $w<0$. By means of formula (see Gradshteyn and Ryzhik (2007), 9.131, page 1008),

$$F(a,b;c;z) = (1-z)^{c-a-b}\,F\left(c-a,c-b;c;z\right)$$

the cumulative function (7.15) can also be written as

$$\Pr\left\{C_n(t)<w\right\} = \frac{1}{2}+\frac{n\sin\frac{\pi}{2n}}{\pi}\,\frac{w}{\left(w^{2n}+t^{2n}\right)^{\frac{1}{2n}}}\,F\left(\frac{1}{2n},\frac{1}{2n};\frac{1}{2n}+1;\frac{w^{2n}}{t^{2n}+z^{2n}}\right). \tag{7.16}$$

We note that, for $n=1$, the function (7.16) coincides with the expansion of the arctangent function,

$$\Pr\left\{C_1(t)<w\right\} = \frac{1}{2}+\frac{1}{\pi}\frac{w}{\sqrt{w^2+t^2}}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{w^2}{w^2+t^2}\right) \overset{\text{see Gradshteyn and Ryzhik (2007), 1.641, pag. 60}}{=} \frac{1}{2}+\frac{1}{\pi}\arctan\frac{w}{t}.$$

By applying the following formula

$$F\left(a,b;c;z\right) = \left(1-z\right)^{-b}\,F\left(b,c-a;c;\frac{z}{z-1}\right), \qquad \left|\frac{z}{z-1}\right|<1,$$

we can rewrite the distribution function (7.15), for $\frac{w^2}{t^2}<1$, as

$$\begin{aligned}\Pr\left\{C_n(t)<w\right\} &= \frac{1}{2}+\frac{n\sin\frac{\pi}{2n}}{\pi}\,\frac{w}{t}\,F\left(1,\frac{1}{2n},\frac{1}{2n}+1,-\frac{w^{2n}}{t^{2n}}\right)\\ &= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)}+\frac{w}{t}\sum_{k=0}^{\infty}(-1)^k\frac{(1)_k\left(\frac{1}{2n}\right)_k}{\left(\frac{1}{2n}+1\right)_k}\frac{1}{k!}\frac{w^{2nk}}{t^{2nk}}\right)\\ &= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)}+\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{1}{2n}\right)_k}{\left(\frac{2n+1}{2n}\right)_k}\frac{w^{2nk+1}}{t^{2nk+1}}\right)\end{aligned}$$

$$= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)}+\sum_{k=0}^{\infty}\frac{(-1)^k}{2nk+1}\frac{w^{2nk+1}}{t^{2nk+1}}\right). \tag{7.17}$$

In (7.17) we retrive the result (7.13) which was obtained without resorting to the hypergeometric functions.

Other useful representations of the cumulative function of $C_n(t)$ can be given in integral form as

$$\begin{aligned}
\Pr\{C_n(t)<w\} &= \frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\left(\frac{\pi}{2n\sin\left(\frac{\pi}{2n}\right)}+\sum_{k=0}^{\infty}\frac{(-1)^k}{2nk+1}\frac{w^{2nk+1}}{t^{2nk+1}}\right)\\
&= \frac{1}{2}+\frac{n\sin\frac{\pi}{2n}}{\pi}\frac{w}{t}\sum_{k=0}^{\infty}(-1)^k\left(\frac{w}{t}\right)^{2nk}\int_0^{\infty}du\, e^{-u(2nk+1)}\\
&= \frac{1}{2}+\frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\int_0^{\infty}du\, e^{-u}\frac{w}{t}\sum_{k=0}^{\infty}\left(-\frac{e^{-2nu}w^{2n}}{t^{2n}}\right)^k\\
&= \frac{1}{2}+\frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\int_0^{\infty}du\, e^{-u}\frac{t^{2n-1}}{t^{2n}+w^{2n}e^{-2nu}}.
\end{aligned} \tag{7.18}$$

Formula (7.18) can be also rewritten as

$$\begin{aligned}
&\Pr\{C_n(t)<w\} = \\
&= \frac{1}{2}+\frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\int_0^{\infty}du\, e^{-u}\int_0^{\infty}dz\, e^{-zt}E_{2n,1}\left(-\left(we^{-u}\right)^{2n}z^{2n}\right)\\
&= \frac{1}{2}+\frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\sum_{k=0}^{\infty}\frac{(-1)^k w^{2nk}}{\Gamma(2nk+1)}\int_0^{\infty}dz\, e^{-zt}z^{2nk}\int_0^{\infty}du\, e^{-u(1+2nk)}\\
&= \frac{1}{2}+\frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}w\sum_{k=0}^{\infty}\frac{(-1)^k w^{2nk}}{\Gamma(2nk+1)(1+2nk)}\int_0^{\infty}dz\, e^{-zt}z^{1+2nk-1}\\
&= \frac{1}{2}+\frac{n\sin\left(\frac{\pi}{2n}\right)}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{w}{t}\right)^{2nk+1}}{2nk+1},
\end{aligned}$$

which coincides with (7.12).

Remark 7.2.4. In force of formula 3.738 pag. 430 of Gradshteyn and Ryzhik (2007), we can give a representation of the characteristic function of (7.10) as follows

$$\begin{aligned}
\int_{-\infty}^{\infty}e^{i\beta x}p_n(x,t)dx &= \frac{2n\sin\frac{\pi}{2n}}{\pi}t^{2n-1}\int_0^{\infty}\frac{\cos\beta x}{x^{2n}+t^{2n}}dx\\
= \sin\frac{\pi}{2n}\sum_{k=1}^{n}e^{-|\beta|t\sin\frac{(2k-1)\pi}{2n}}&\sin\left(\frac{(2k-1)\pi}{2n}+|\beta|\,t\cos\frac{(2k-1)\pi}{2n}\right),
\end{aligned} \tag{7.19}$$

which coincides, for $n=1$, with the characteristic function of the Cauchy distribution.

Remark 7.2.5. Other generalizations of the Cauchy are obtained by considering two different types of r.v.'s. The first one is

$$\widehat{C}_n(t) = \begin{cases} t\tan^{\frac{1}{n}}\Theta, & \Theta\in\left(0,\frac{\pi}{2}\right) \\ -t\tan^{\frac{1}{n}}|\Theta| & \Theta\in\left(-\frac{\pi}{2},0\right), \end{cases} \tag{7.20}$$

where Θ has uniform law. The distribution function of (7.20) is

$$\Pr\left\{\widehat{C}_n(t) < x\right\} = \begin{cases} \frac{1}{2}+\frac{1}{\pi}\int_0^{\arctan\left(\frac{x}{t}\right)^n} d\theta, & x>0 \\ \frac{1}{\pi}\int_{\arctan\left(-\frac{x}{t}\right)^n}^{\frac{\pi}{2}} d\theta, & x<0, \end{cases}$$

and thus the density reads

$$\widehat{p}_n(x,t) = \frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+x^{2n}}, \qquad x\in\mathbb{R}, t>0, n\in\mathbb{N}, \tag{7.21}$$

and possesses the following representation

$$\widehat{p}_n(x,t) = \frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1}\int_0^\infty e^{-zt}E_{2n,1}\left(-x^{2n}z^{2n}\right)dz.$$

In Figure 7.2a we give a picture of density (7.21) for different values of n. It is interesting to note that the distribution is bimodal with two symmetric maxima at

$$x = \pm t\left(\frac{n-1}{n+1}\right)^{\frac{1}{2n}}, \qquad n>1.$$

Furthermore, the characteristic function of the distribution (7.21), in force of formula 3.738 of Gradshteyn and Ryzhik (2007) pag 430, reads

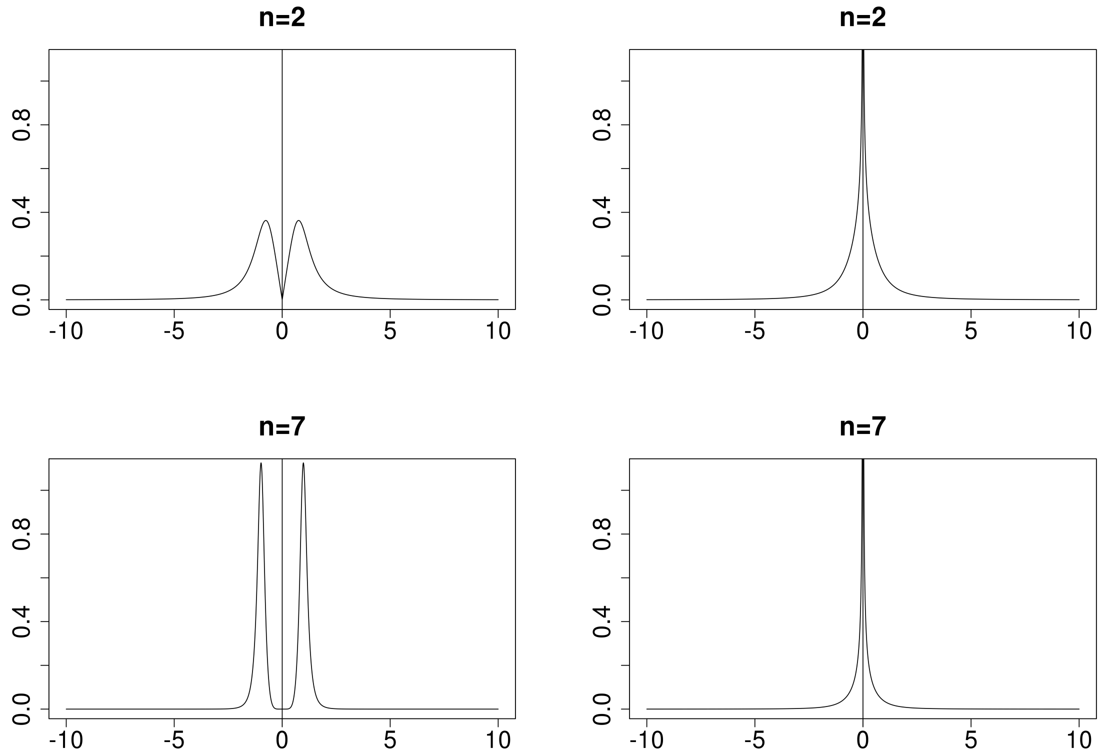
$$\int_{-\infty}^{\infty} e^{i\beta x}\widehat{p}_n(x,t)\,dx = \sum_{k=1}^n e^{-|\beta|t\sin\frac{(2k-1)\pi}{2n}}\sin\left(\frac{(2k-1)\pi}{2}+|\beta|\,t\cos\frac{(2k-1)\pi}{2n}\right).$$

For the r.v.

$$\widetilde{C}_n(t) = t\tan\Theta, \tag{7.22}$$

with Θ endowed with the distribution $q_n(\theta)$ given in (7.9), we have that

$$\begin{aligned} \widetilde{p}_n(x,t) &= \frac{d}{dx}\Pr\{t\tan\Theta<x\} \\ &= \frac{d}{dx}\left[\frac{1}{2}+\frac{\sin\frac{\pi}{2n}}{\pi}\int_{-\frac{\pi}{2}}^{\arctan\frac{x}{t}}\cot^{\frac{n-1}{n}}|\theta|\,d\theta\right] \\ &= \frac{\sin\frac{\pi}{2n}}{\pi}\frac{t}{x^2+t^2}\left(\frac{t}{|x|}\right)^{\frac{n-1}{n}}. \end{aligned} \tag{7.23}$$



(a) (b)

Figure 7.2: The probability density function of $\widehat{C}_n(t)$, (A), and $\widetilde{C}_n(t)$, (B), for different values of n.

Remark 7.2.6. Since the following identity holds

$$\frac{n}{\pi}\left(\frac{|x|}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+x^{2n}} = n|x|^{n-1}\int_0^\infty \frac{e^{-\frac{x^{2n}}{2s}}}{\sqrt{2\pi s}}t^n\frac{e^{-\frac{t^{2n}}{2s}}}{\sqrt{2\pi s^3}}\,ds,$$

for the hyperCauchy (7.21) a subordination similar to that of the classical Cauchy law can be established and reads

$$\Pr\left\{\widehat{C}_n(t)\in dx\right\} = \int_0^\infty \Pr\left\{\widehat{B}(s)\in dx\right\}\Pr\left\{T_{t^n}\in ds\right\},$$

where

$$\widehat{B}(s) = \begin{cases} |B(s)|^{\frac{1}{n}}, & B(s)>0,\\ -|B(s)|^{\frac{1}{n}}, & B(s)<0.\end{cases}$$

With $B(s)$ we denote a standard Brownian motion and T_{t^n} is defined as

$$T_{t^n} = \inf\left\{s>0\,:\,B(s) = t^n\right\}$$

Now we pass to the derivation of the moments of (7.9).

Theorem 7.2.7. *For $2n > 2k+1$, $k > 0$, we have that*

$$\begin{aligned}\mathbb{E}C_n^{2k}(t) &= \frac{\sin\frac{\pi}{2n}}{\sin\left(\frac{2k+1}{2n}\pi\right)}\,t^{2k} \\ &= \frac{t^{2k}}{\cos\frac{k\pi}{n}+\cot\frac{\pi}{2n}\sin\frac{k\pi}{n}}.\end{aligned} \tag{7.24}$$

Proof.

$$\begin{aligned}\mathbb{E}C_n^{2k}(t) &= \frac{n}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_{-\infty}^{\infty}\frac{x^{2k}}{x^{2n}+t^{2n}}\,dx \\ &= 2\frac{n}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_0^{\infty}x^{2k}\int_0^{\infty}e^{-w\left(x^{2n}+t^{2n}\right)}\,dw\,dx \\ &\overset{x=y^{\frac{1}{2n}}}{=} \frac{1}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_0^{\infty}e^{-wt^{2n}}\,dw\int_0^{\infty}e^{-wy}\,y^{\frac{2k+1}{2n}-1}\,dy \\ &= \frac{\Gamma\left(\frac{2k+1}{2n}\right)}{\pi}\sin\frac{\pi}{2n}t^{2n-1}\int_0^{\infty}e^{-wt^{2n}}\,w^{-\frac{2k+1}{2n}}\,dw \\ &= \frac{\Gamma\left(\frac{2k+1}{2n}\right)}{\pi}\sin\frac{\pi}{2n}t^{2k}\int_0^{\infty}e^{-y}\,y^{-\frac{2k+1}{2n}+1-1}\,dy \\ &= \frac{\Gamma\left(\frac{2k+1}{2n}\right)\Gamma\left(1-\frac{2k+1}{2n}\right)}{\pi}\sin\frac{\pi}{2n}t^{2k} \\ &= \frac{\sin\frac{\pi}{2n}}{\sin\left(\frac{2k+1}{2n}\pi\right)}\,t^{2k}.\end{aligned}$$

□

Remark 7.2.8. For $k=1$, formula (7.24) gives the variance of the hyperCauchy

$$\mathbb{E}C_n^2(t) = \mathrm{Var}\,C_n(t) = \frac{\sin\frac{\pi}{2n}}{\sin\frac{3\pi}{2n}}t^2 = \frac{t^2}{1+2\cos\frac{\pi}{n}}.$$

The last expression shows that the variance is a decreasing function of n.

Furthermore we have the following interesting relationships:

$$\begin{aligned}\mathbb{E}C_n^{2(n-1)}(t) &= t^{2n-2}, \\ \mathbb{E}C_n^{2(n-2)}(t) &= \frac{\sin\frac{\pi}{2n}}{\sin\frac{3\pi}{2n}}t^{2(n-2)} = t^{2n-4}\mathrm{Var}\,C_n(t) = \frac{t^{2n-2}}{1+2\cos\frac{\pi}{n}}, \\ \mathbb{E}C_n^4(t) &= \frac{t^4\mathrm{Var}\,C_n(t)}{2t^2\cos\frac{\pi}{n}-\mathrm{Var}\,C_n(t)}.\end{aligned}$$

For the distribution (7.23) it is possible to evaluate only the moment $\mathbb{E}\left|\widetilde{C}_n(t)\right|$ by performing the following calculation

$$\mathbb{E}\left|\widetilde{C}_n(t)\right| = \frac{\sin\frac{\pi}{2n}}{\pi}\int_{-\infty}^{\infty}|x|\,\frac{t}{x^2+t^2}\left(\frac{t}{|x|}\right)^{\frac{n-1}{n}}\,dx$$

$$
\begin{aligned}
&= 2\frac{\sin\frac{\pi}{2n}}{\pi}\int_0^\infty \frac{t^{2-\frac{1}{n}}x^{\frac{1}{n}}}{x^2+t^2}\,dx \\
&\overset{x=ty}{=} \frac{2\sin\frac{\pi}{2n}}{\pi}\,t\int_0^\infty \frac{y^{\frac{1}{n}}}{1+y^2}\,dy \\
&\overset{y=x^{\frac{1}{2}}}{=} \frac{\sin\frac{\pi}{2n}}{\pi}\,t\int_0^\infty e^{-u}\int_0^\infty x^{\frac{1}{2}+\frac{1}{2n}-1}e^{-ux}du\,dx \\
&= \frac{\sin\frac{\pi}{2n}}{\pi}\,\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)t\int_0^\infty e^{-u}\,u^{1-\frac{1}{2}-\frac{1}{2n}-1}\,du \\
&= \frac{\sin\frac{\pi}{2n}}{\pi}\,\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)\Gamma\left(1-\frac{1}{2}-\frac{1}{2n}\right)t \\
&= \frac{\sin\frac{\pi}{2n}}{\sin\left(\left(\frac{1}{2}+\frac{1}{2n}\right)\pi\right)}\,t \;=\; t\tan\frac{\pi}{2n}.
\end{aligned}
$$

7.3 Distributional properties of the hyperCauchy

In this section we consider a number of r.v.'s related to the hyperCauchy previously introduced. We start by examining the properties of the reciprocal of the hyperCauchy.

7.3.1 Distribution of the reciprocal

It is well known that the symmetrical Cauchy r.v. $C_1(t)$, $t > 0$, has the property that

$$\frac{1}{C_1\left(\frac{1}{t}\right)} \overset{\text{law}}{=} C_1(t).$$

For the hyperCauchy r.v.'s $C_n(t)$, $\widehat{C}_n(t)$ and $\widetilde{C}_n(t)$ we have the following theorem

Theorem 7.3.1. *We have that*

i)

$$
\begin{aligned}
\Pr\left\{\frac{1}{C_n\left(\frac{1}{t}\right)}\in dw\right\} &= \frac{n\sin\frac{\pi}{2n}}{\pi}\frac{t^{2n-1}}{t^{2n}+w^{2n}}\left(\frac{w}{t}\right)^{2n-2}dw \\
&= \left(\frac{w}{t}\right)^{2n-2}\Pr\left\{C_n(t)\in dw\right\}, \qquad w\in\mathbb{R}, t>0, \quad (7.25)
\end{aligned}
$$

ii)

$$\Pr\left\{\frac{1}{\widehat{C}_n\left(\frac{1}{t}\right)}\in dw\right\} = \Pr\left\{\widehat{C}_n(t)\in dw\right\}, \qquad w\in\mathbb{R}, t>0,$$

iii)

$$\Pr\left\{\frac{1}{\widetilde{C}_n(t)} \in dw\right\} = \frac{\sin\frac{\pi}{2n}}{\pi}\frac{t}{t^2+w^2}\left(\frac{|w|}{t}\right)^{\frac{n-1}{n}} dw$$
$$= \left(\frac{t}{|x|}\right)^{\frac{2n-2}{n}} \Pr\left\{\widetilde{C}_n(t) \in dw\right\}, \qquad w \in \mathbb{R}, t > 0.$$

Proof. The density of

$$V_n(t) = \frac{1}{C_n\left(\frac{1}{t}\right)}$$

reads

$$v_n(w,t) = \frac{d}{dw}\frac{n\sin\frac{\pi}{2n}}{\pi}\int_{\frac{1}{w}}^{\infty}\frac{\left(\frac{1}{t}\right)^{2n-1}}{\frac{1}{t^{2n}}+x^{2n}}dx$$
$$= \frac{n\sin\frac{\pi}{2n}}{\pi}\frac{t^{2n-1}}{t^{2n}+w^{2n}}\left(\frac{w}{t}\right)^{2n-2}, \qquad w \in \mathbb{R}, t > 0, \tag{7.26}$$

and for $n = 1$ we retrive the previous result of the classical Cauchy r.v.. The density (7.25) has a bimodal structure (with maxima at $x = \pm(n-1)^{\frac{1}{2n}}t)$ as illustrated in figure 7.3a.

Instead, the r.v. $\widehat{C}_n(t)$ preserves the fine property of the classical Cauchy distribution because

$$\Pr\left\{\frac{1}{\widehat{C}_n\left(\frac{1}{t}\right)} < w\right\} = \frac{n}{\pi}\int_{\frac{1}{w}}^{\infty}(t|x|)^{n-1}\frac{\left(\frac{1}{t}\right)^{2n-1}}{\frac{1}{t^{2n}}+x^{2n}}dx,$$

and so, by taking the derivative with respect to w we get

$$\Pr\left\{\frac{1}{\widehat{C}_n\left(\frac{1}{t}\right)} \in dw\right\} = \frac{n}{\pi}\left(\frac{|w|}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+w^{2n}}dw,$$

which coincides with the law of $\widehat{C}_n(t)$. For the law of the r.v. $\widetilde{C}_n(t)$ we get that

$$\Pr\left\{\frac{1}{\widetilde{C}_n\left(\frac{1}{t}\right)} < w\right\} = \frac{\sin\frac{\pi}{2n}}{\pi}\int_{\frac{1}{w}}^{\infty}\frac{\frac{1}{t}}{\frac{1}{t^2}+x^2}\left(\frac{\frac{1}{t}}{|x|}\right)^{\frac{n-1}{n}}dx,$$

and thus

$$\Pr\left\{\frac{1}{\widetilde{C}_n\left(\frac{1}{t}\right)} \in dw\right\} = \frac{\sin\frac{\pi}{2n}}{\pi}\frac{t}{t^2+w^2}\left(\frac{|w|}{t}\right)^{\frac{n-1}{n}}dw. \tag{7.27}$$

□

Distributions (7.26) and (7.27) are presented respectively in fig 7.3a and 7.3b, for different values of n and the dotted line represents the classical Cauchy density.

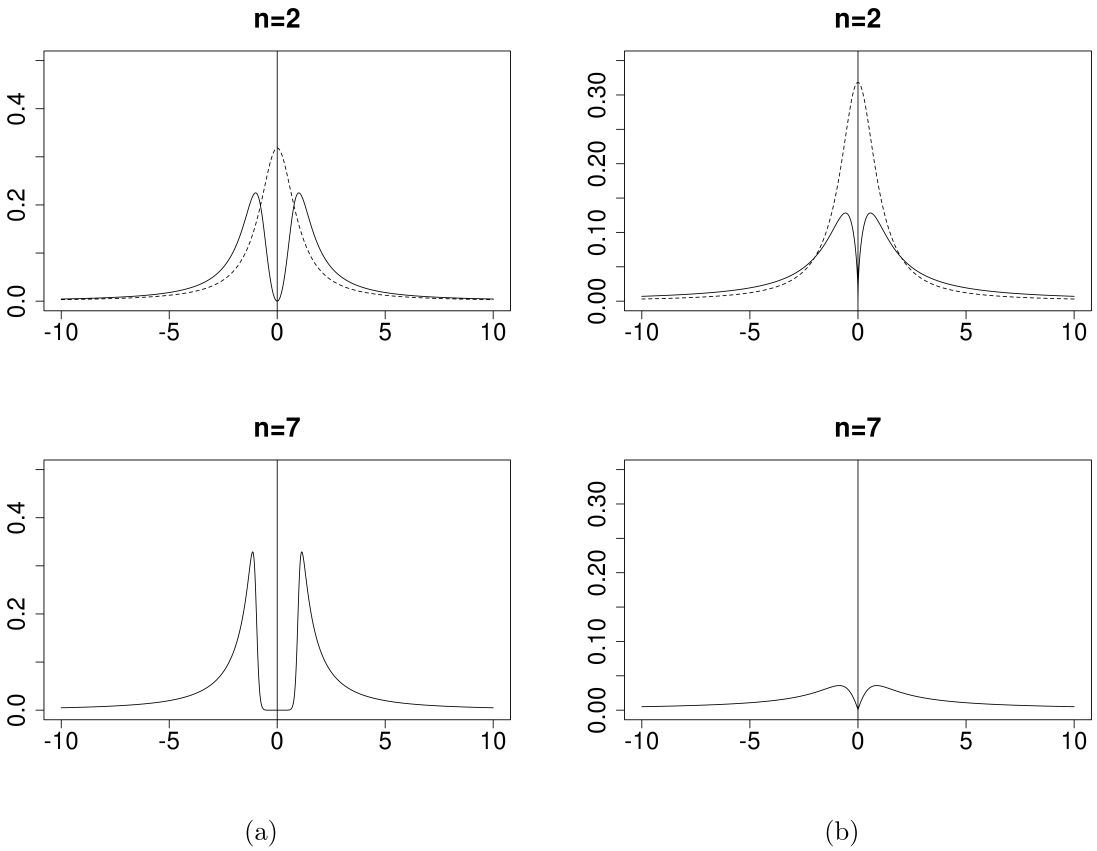


Figure 7.3: The probability density function (7.26), (A), and (7.27), (B), for two different values of n.

7.3.2 Distributions of the ratio

For the ratios of the three types of hyperCauchy distributions dealt with so far we have the following theorem.

Theorem 7.3.2. *In the following table we have the ratios of the r.v.'s and the corresponding densities*

r.v.	*density for* $w \in \mathbb{R}, t > 0$
$W_n(t) = t\frac{C_n^1(t)}{C_n^2(t)}$	$\mathfrak{w}_n(w,t) = \frac{n}{2\pi}\tan\frac{\pi}{2n} t \frac{t^{2n-2}-w^{2n-2}}{t^{2n}-w^{2n}}$
$\widehat{W}_n(t) = t\frac{\widehat{C}_n^1(t)}{\widehat{C}_n^2(t)}$	$\widehat{\mathfrak{w}}_n(w,t) = \frac{nt^n\|w\|^{n-1}}{\pi^2(t^{2n}-w^{2n})}\log\left(\frac{t}{w}\right)^{2n}$
$\widetilde{W}_n(t) = \frac{\widetilde{C}_n^1(t)}{\widetilde{C}_n^2(t)}$	$\widetilde{\mathfrak{w}}_n(w) = \frac{1}{2\pi}\tan\frac{\pi}{2n}\frac{\|w\|^{\frac{1}{n}-1}}{1-w^2}\left(1-w^{2-\frac{2}{n}}\right)$

Proof. We give a hint of the derivation of the densities above. For $w > 0$,

$$\Pr\left\{t\frac{C_n^1(t)}{C_n^2(t)} < w\right\} = \frac{1}{2} + 2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}\int_0^\infty dx\int_0^{\frac{wx}{t}} dy \frac{t^{2n-1}}{t^{2n}+x^{2n}}\frac{t^{2n-1}}{t^{2n}+y^{2n}}.$$

The density is therefore

$$\begin{aligned}\mathfrak{w}_n(w,t) &= 2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}\int_0^\infty dx\frac{x}{t}t^{4n-2}\frac{1}{t^{2n}+x^{2n}}\frac{2}{t^{2n}+\left(\frac{wx}{t}\right)^{2n}} \\ &= \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}-w^{2n}}\frac{t^{4n-2}}{t}\left[\int_0^\infty\frac{x\,dx}{t^{2n}+x^{2n}} - \frac{w^{2n}}{t^{2n}}\int_0^\infty\frac{x\,dx}{t^{2n}+\left(\frac{w^{2n}x^{2n}}{t^{2n}}\right)}\right],\end{aligned} \tag{7.28}$$

and with the change of variable $\frac{wx}{t} = y$ in the second integral of (7.28) we obtain

$$\begin{aligned}\mathfrak{w}_n(w,t) &= \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}-w^{2n}}\frac{t^{4n-2}}{t}\left(1-\frac{w^{2n}}{t^{2n}}\frac{t^2}{w^2}\right)\int_0^\infty dx\,\frac{x}{t^{2n}+x^{2n}} \\ &\overset{x=ty}{=} \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}-w^{2n}}t^{4n-3}\left(1-\frac{w^{2n-2}}{t^{2n-2}}\right)\frac{t^2}{t^{2n}}\int_0^\infty dy\,\frac{y}{1+y^{2n}} \\ &= \frac{2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}}{t^{2n}-w^{2n}}\left(t^{2n-2}-w^{2n-2}\right)\frac{t^{2n-1}}{t^{2n}}t^2\int_0^\infty dy\,\frac{y}{1+y^{2n}} \\ &= 2n^2\sin^2\left(\frac{\pi}{2n}\right)\frac{1}{\pi^2}\frac{t^{2n-2}-w^{2n-2}}{t^{2n}-w^{2n}}\frac{t}{2n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \\ &= \frac{n}{2\pi}\tan\frac{\pi}{2n}t\frac{t^{2n-2}-w^{2n-2}}{t^{2n}-w^{2n}}.\end{aligned} \tag{7.29}$$

For the r.v. $\widehat{W}_n(t)$ the density reads

$$\begin{aligned}
\widehat{\mathfrak{w}}_n(w,t) &= \frac{2n^2}{\pi^2}\int_0^\infty \frac{x}{t}\frac{t^{2n-1}}{t^{2n}+x^{2n}}\left(\frac{x}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+\left(\frac{wx}{t}\right)^{2n}}\left(\frac{wx}{t^2}\right)^{n-1}dx \\
&= \frac{2n^2t^{3n}w^{n-1}}{\pi^2}\int_0^\infty \frac{x^{2n-1}}{t^{2n}+x^{2n}}\frac{dx}{t^{4n}+(wx)^{2n}} \\
&\overset{\left(\frac{x}{t}\right)^{2n}=y}{=} \frac{2n^2t^{3n}w^{n-1}}{\pi^2}\frac{1}{t^{2n}}\frac{1}{2n}\int_0^\infty \frac{1}{1+y}\frac{dy}{t^{2n}+w^{2n}y}dy \\
&= \frac{nt^nw^{n-1}}{\pi^2\left(t^{2n}-w^{2n}\right)}\int_0^\infty\left(\frac{1}{1+y}-\frac{w^{2n}}{t^{2n}+w^{2n}y}\right)dy \\
&= \frac{nt^nw^{n-1}}{\pi^2\left(t^{2n}-w^{2n}\right)}\log\left(\frac{t}{w}\right)^{2n}. \qquad (7.30)
\end{aligned}$$

For the r.v. $\widetilde{W}_n(t)$ the density, not depending on t, has a structure different from the previous ones and is obtained by means of the following calculation

$$\begin{aligned}
\widetilde{\mathfrak{w}}_n(w) &= 2\left(\frac{\sin\frac{\pi}{2n}}{\pi}\right)^2\int_0^\infty \frac{t}{t^2+x^2}\left(\frac{t}{x}\right)^{\frac{n-1}{n}}\frac{t}{t^2+w^2x^2}\left(\frac{t}{wx}\right)^{\frac{n-1}{n}}x\,dx \\
&= 2\left(\frac{\sin\frac{\pi}{2n}}{\pi}\right)^2\frac{t^{2+2\left(\frac{n-1}{n}\right)}}{w^{\frac{n-1}{n}}}\int_0^\infty \frac{x}{x^2+t^2}\frac{x^{-2\left(\frac{n-1}{n}\right)}}{t^2+w^2x^2}dx \\
&= 2\frac{\left(\sin\frac{\pi}{2n}\right)^2}{\pi^2}\frac{t^{2\left(\frac{n-1}{n}\right)}}{w^{\frac{n-1}{n}}\left(1-w^2\right)}\left(\int_0^\infty\frac{x^{\frac{2}{n}-1}}{x^2+t^2}dx-w^2\int_0^\infty\frac{x^{\frac{2}{n}-1}}{t^2+w^2x^2}dx\right) \\
&= 2\frac{\left(\sin\frac{\pi}{2n}\right)^2}{\pi^2}\frac{t^{2\left(\frac{n-1}{n}\right)}}{w^{\frac{n-1}{n}}\left(1-w^2\right)}\left(\frac{\pi}{2t^{2-\frac{2}{n}}\sin\frac{\pi}{n}}-\frac{w^{2-\frac{2}{n}}\pi}{2t^{2-\frac{2}{n}}\sin\frac{\pi}{n}}\right) \\
&= \frac{1}{2\pi}\tan\frac{\pi}{2n}\frac{w^{\frac{1}{n}-1}}{\left(1-w^2\right)}\left(1-w^{2-\frac{2}{n}}\right). \qquad (7.31)
\end{aligned}$$

Similar calculation performed for $w<0$ yield the previous distributions for $w\in\mathbb{R}$. $\square$

Remark 7.3.3. We note that by setting $n=2$ in the law $\mathfrak{w}_n(w,t)$ we retrive the standard Cauchy density. Indeed

$$\begin{aligned}
\mathfrak{w}_2(w,t) &= \frac{1}{\pi}\tan\frac{\pi}{4}t\frac{t^2-w^2}{t^4-w^4} \\
&= \frac{1}{\pi}\frac{t}{t^2+w^2}.
\end{aligned}$$

This means that if $C_2^1(t)$ and $C_2^2(t)$ are two independent random variables with law

$$p_2(w,t) = \frac{1}{\sqrt{2}\pi}\frac{t^3}{t^4+w^4},$$

the distribution of

$$W_2(t) = \frac{C_2^1(t)}{C_2^2(t)}$$

is a standard Cauchy.

Furthermore we have that the distribution (7.30) coincides with formula (4.6) of D'Ovidio and Orsingher (2010) for $n = 1$. We can check that the r.v.

$$\left(\frac{1}{t} \frac{C_1^1(t)}{C_1^2(t)} \right)^{\frac{1}{n}},$$

where C_1^1, C_1^2 are two independent Cauchy r.v.'s, possesses distribution (7.30). In other words we have the following equality in distribution

$$t\frac{\widehat{C}_n^1(t)}{\widehat{C}_n^2(t)} \overset{\text{i.d.}}{=} \left(\frac{1}{t} \frac{C_1^1(t)}{C_1^2(t)} \right)^{\frac{1}{n}}.$$

Remark 7.3.4. In order to check that the density (7.31) integrates to unity we perform the following calculation

$$\begin{aligned}
&\int_{-\infty}^{\infty} \widetilde{\mathfrak{w}}_n(w)\, dw = \\
&= \frac{\tan\frac{\pi}{2n}}{2\pi} \int_{-\infty}^{\infty} \frac{|w|^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right) dw}{(1-w^2)} = \frac{\tan\frac{\pi}{2n}}{\pi} \int_0^{\infty} \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right) dw}{(1-w^2)} \\
&= \frac{1}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw + \frac{1}{\pi} \tan\frac{\pi}{2n} \int_1^{\infty} \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw.
\end{aligned} \tag{7.32}$$

With the change of variable $y = \frac{1}{w}$ in the second integral of (7.32), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \widetilde{\mathfrak{w}}_n(w)\, dw &= \frac{1}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw + \\
&\quad + \frac{1}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{\left(1 - y^{2-\frac{2}{n}}\right) y^{\frac{1}{n}-1}}{1-y^2} dy \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \int_0^1 \frac{w^{\frac{1}{n}-1}\left(1 - w^{2-\frac{2}{n}}\right)}{1-w^2} dw \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \sum_{k=0}^{\infty} \int_0^1 \left(w^{\frac{1}{n}-1} - w^{-\frac{1}{n}+1} \right) w^{2k} \\
&= \frac{2}{\pi} \tan\frac{\pi}{2n} \sum_{k=0}^{\infty} \left[\frac{w^{2k+\frac{1}{n}}}{2k+\frac{1}{n}} - \frac{w^{2k-\frac{1}{n}+2}}{2k-\frac{1}{n}+2} \right]_0^1
\end{aligned}$$

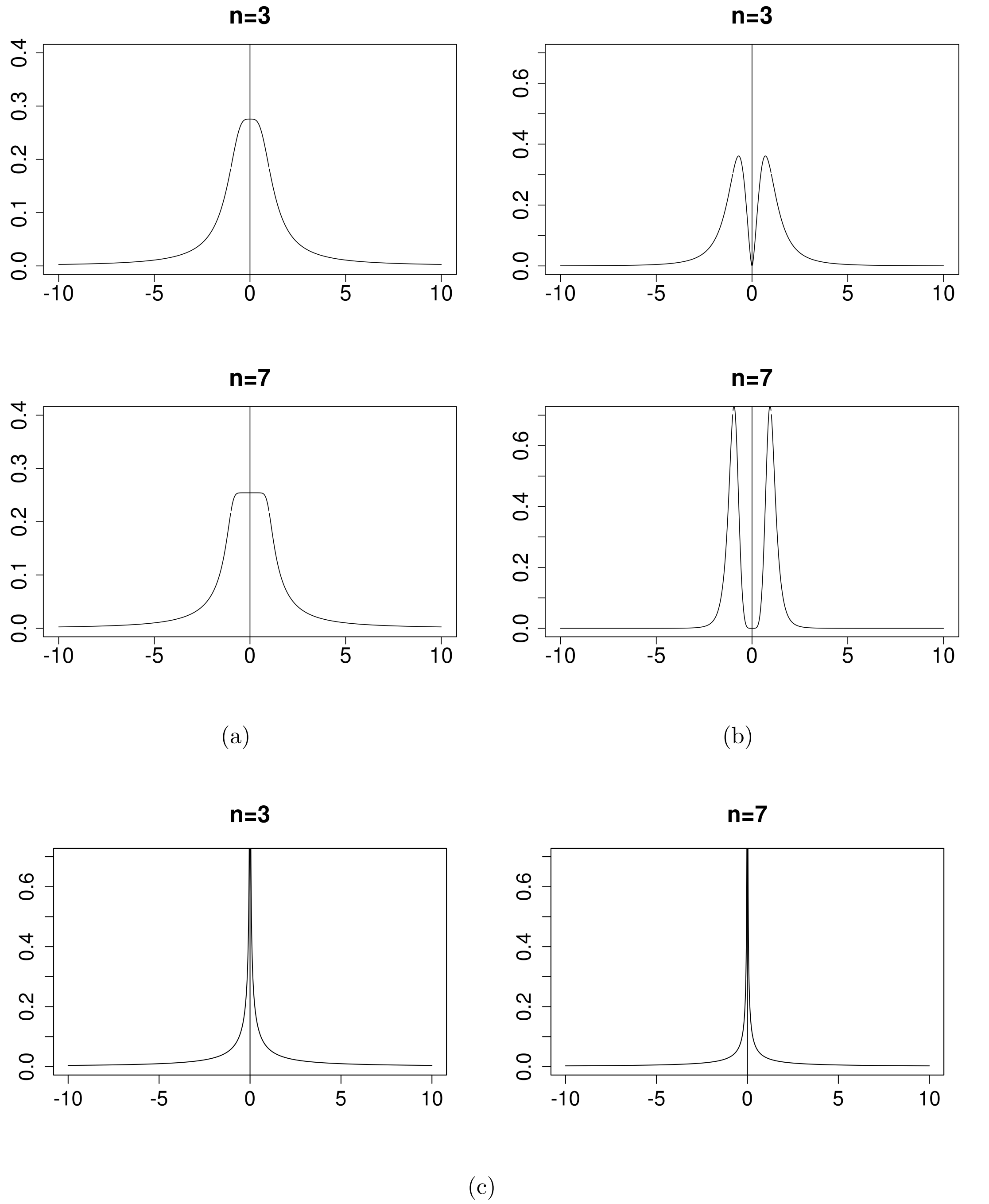


Figure 7.4: The probability density function $\mathfrak{w}_n(w,t)$, (A), $\widehat{\mathfrak{w}}_n(w,t)$, (B), and $\widetilde{\mathfrak{w}}_n(w)$, (C), for different values of n.

$$
\begin{aligned}
&= \frac{2}{\pi}\tan\frac{\pi}{2n}\sum_{k=0}^{\infty}\left(\frac{1}{2k+\frac{1}{n}} - \frac{1}{2(k+1)-\frac{1}{n}}\right) \\
&= \frac{2}{\pi}\tan\frac{\pi}{2n}\left(n + \sum_{k=1}^{\infty}\left(\frac{1}{2k+\frac{1}{n}} - \frac{1}{2k-\frac{1}{n}}\right)\right) \\
&= \frac{2}{\pi}\tan\frac{\pi}{2n}\left(n - \sum_{k=1}^{\infty}\frac{\frac{2}{n}}{(2k)^2-\frac{1}{n^2}}\right). \qquad (7.33)
\end{aligned}
$$

Considering the relationship (see Smirnov (1964) pag 410)

$$
z\cot z = 1 - \sum_{k=1}^{\infty}\frac{2z^2}{k^2\pi^2 - z^2} \qquad (7.34)
$$

and setting $z = \frac{\pi}{2n}$, we get

$$
\begin{aligned}
\frac{\pi}{2n}\cot\frac{\pi}{2n} &= 1 - \sum_{k=1}^{\infty}\frac{2\left(\frac{\pi}{2n}\right)^2}{k^2\pi^2 - \left(\frac{\pi}{2n}\right)^2} \\
&= 1 - \sum_{k=1}^{\infty}\frac{\frac{2}{n^2}}{(2k)^2 - \frac{1}{n^2}},
\end{aligned}
$$

and thus

$$
\frac{\pi}{2}\cot\frac{\pi}{2n} = n - \frac{2}{n}\sum_{k=1}^{\infty}\frac{1}{(2k)^2-\frac{1}{n^2}}. \qquad (7.35)
$$

Considering (7.35) we can rewrite (7.33) as follows

$$
\begin{aligned}
\int_{-\infty}^{\infty}\widetilde{\mathfrak{w}}_n(w)\,dw &= \frac{2}{\pi}\tan\frac{\pi}{2n}\left(n - \sum_{k=1}^{\infty}\frac{\frac{2}{n}}{(2k)^2-\frac{1}{n^2}}\right) \\
&= \frac{2}{\pi}\tan\frac{\pi}{2n}\left(\frac{\pi}{2}\cot\frac{\pi}{2n}\right) = 1.
\end{aligned}
$$

The previous calculation yields an interesting integral represention of the cotangent function. Indeed, in light of (7.33) and (7.34) we can write

$$
\begin{aligned}
\cot z &= \frac{1}{z} - \sum_{k=0}^{\infty}\frac{2z}{(2k)^2 - z^2} \\
&= \int_0^1 \frac{w^{z-1}}{1-w^2}\left(1 - w^{2(1-z)}\right)\,dw \\
&= \frac{1}{2}\int_0^{\infty}\frac{w^{z-1}}{1-w^2}\left(1 - w^{2(1-z)}\right)\,dw.
\end{aligned}
$$

For a representation of (7.29), (7.30) and (7.31), see Fig. 7.4a, 7.4b and 7.4c.

7.3.3 The higher-order arcsine law

It is well-known that for the classical Cauchy r.v., $C_1(t)$, holds the following relationship (see Chaumont and Yor (2003) pag. 104)

$$Z_1(t) = \frac{t}{1+(C_1(t))^2} \overset{\text{i.d.}}{=} \frac{1}{\pi}\frac{1}{\sqrt{w\,(t-w)}}, \qquad 0<w<t.$$

which is known as the arcsine law. For the hyperCauchy we get similar relationships.

Theorem 7.3.5. *We have the following distributions.*

r.v.	*probability density for* $0<w<t$
$Z_n(t) = \frac{t}{1+\left(\frac{\lvert C_n(t)\rvert}{t}\right)^{2n}}$	$\frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}\,w^{-\frac{1}{2n}}$
$\widehat{Z}_n(t) = \frac{t}{1+\left(\frac{\lvert \widehat{C}_n(t)\rvert}{t}\right)^{2n}}$	$\frac{1}{\pi\sqrt{(t-w)w}}$
$\widetilde{Z}_n(t) = \frac{t}{1+\left(\frac{\lvert \widetilde{C}_n(t)\rvert}{t}\right)^{2}}$	$\frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}\,w^{-\frac{1}{2n}}$

Proof. We get for $0<w<t$,

$$\Pr\left\{\widehat{Z}_n(t)<w\right\} = 2\frac{n}{\pi}\int_{t\left(\frac{t-w}{w}\right)^{\frac{1}{2n}}}^{\infty}\left(\frac{|x|}{t}\right)^{n-1}\frac{t^{2n-1}}{t^{2n}+x^{2n}}\,dx$$

and thus

$$\widehat{z}_n(w,t)\,dw = \Pr\left\{\widehat{Z}_n(t)\in dw\right\} = \frac{1}{\pi}\frac{dw}{\sqrt{w(t-w)}}.$$

For the r.v. $C_n(t)$ the distribution becomes

$$\Pr\left\{Z_n(t)<w\right\} = 2\frac{n\sin\frac{\pi}{2n}}{\pi}\int_{t\left(\frac{t-w}{w}\right)^{\frac{1}{2n}}}^{\infty}\frac{t^{2n-1}}{t^{2n}+x^{2n}}dx$$

and

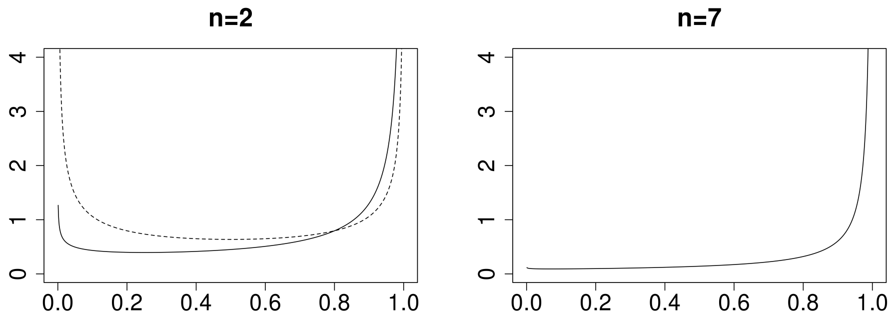
$$z_n(w,t)\,dw = \Pr\left\{Z_n(t)\in dw\right\} = \frac{\sin\frac{\pi}{2n}}{\pi}w^{-\frac{1}{2n}}\,(t-w)^{\frac{1}{2n}-1}\,dw.$$

Similar calculations for $\widetilde{Z}_n(t)$ yield

$$\Pr\left\{\widetilde{Z}_n(t)\in dw\right\} = \frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}\,w^{-\frac{1}{2n}}dw.$$

□

Figure 7.5



The density

$$z_n(w,t) \;=\; \widetilde{z}_n(w,t) \;=\; \frac{\sin\frac{\pi}{2n}}{\pi}\,(t-w)^{\frac{1}{2n}-1}\; w^{-\frac{1}{2n}}$$

is a Beta with parameters $\left(\frac{1}{2n}-1,-\frac{1}{2n}\right)$ and for increasing values of n its asymmetry increases, as shown in Fig. 7.5 for $t = 1$ (the dotted line represents the classical arcsine law).

7.4 The Hyperbolic case

Let us consider the Poincaré half-plane $\mathbb{H}_2^+ = \{x, y : x \in \mathbb{R}, y > 0\}$ (see for example Gruet (1996), Lao and Orsingher (2007)) endowed with the metric

$$ds^2 \;=\; \frac{dx^2+dy^2}{y^2}.$$

We assume that a particle is shooted from the point $O(0,0)$, see figure 7.6a, on the x-axis (representing the infinite of $\mathbb{H}_2^+$), and moves along the geodesic line joining O with an arbitrary point P on the half-circle centered at O, denoted by C_O, and with arbitrary radius t. The hyperbolic distance η between P and Q (Q is the intersection of the vertical geodesic line through O and the half-circle C_O), does not depend on t, because the half-circumferences centered at O form a system of horocycles, and will be denoted by η. Thus the hyperbolic distance η is obtained by evaluating the line integral

$$\begin{aligned} \eta \;&=\; \int_\Theta^{\frac{\pi}{2}} \frac{\sqrt{(x'(s))^2+(y'(s))^2}}{y(s)}\,ds, \qquad \Theta \in \left(0,\frac{\pi}{2}\right) \\ &=\; \int_\Theta^{\frac{\pi}{2}} \frac{ds}{\sin s} \;=\; -\log\tan\frac{\Theta}{2}, \end{aligned} \tag{7.36}$$

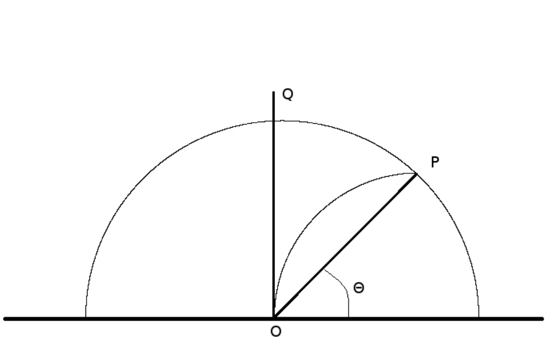
Table 7.1: In the following table we sum up our results on the hyperCauchy functionals

Variable	Law	Transformation	Law of the transformation
$C_n(t)$	$\frac{n\sin\frac{\pi}{2n}}{\pi}\frac{t^{2n-1}}{t^{2n}+x^{2n}}$	$\frac{1}{C_n\left(\frac{1}{t}\right)}$	$\frac{n\sin\frac{\pi}{2n}}{\pi}\frac{t^{2n-1}}{t^{2n}+w^{2n}}\left(\frac{w}{t}\right)^{2n-2}$
		$t\frac{C_n^1(t)}{C_n^2(t)}$	$\frac{n}{2\pi}\tan\frac{\pi}{2n}\frac{t\left(t^{2n-2}-w^{2n-2}\right)}{t^{2n}-w^{2n}}$
		$\frac{t}{1+\left(\frac{\lvert C_n(t)\rvert}{t}\right)^{2n}}$	$\frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}w^{-\frac{1}{2n}}$, for $0<w<t$
$\widehat{C}_n(t)$	$\frac{n}{\pi}\frac{t^{2n-1}}{t^{2n}+x^{2n}}\left(\frac{\lvert x\rvert}{t}\right)^{n-1}$	$\frac{1}{\widehat{C}_n\left(\frac{1}{t}\right)}$	$\frac{n}{\pi}\frac{t^{2n-1}}{t^{2n}+w^{2n}}\left(\frac{\lvert w\rvert}{t}\right)^{n-1}$
		$t\frac{\widehat{C}_n^1(t)}{\widehat{C}_n^2(t)}$	$\frac{nt^n w^{n-1}}{\pi^2(t^{2n}-w^{2n})}\log\left(\frac{t}{w}\right)^{2n}$
		$\frac{t}{1+\left(\frac{\lvert\widehat{C}_n(t)\rvert}{t}\right)^{2n}}$	$\frac{1}{\pi}w^{-\frac{1}{2}}(t-w)^{-\frac{1}{2}}$, for $0<w<t$
$\widetilde{C}_n(t)$	$\frac{\sin\frac{\pi}{2n}}{\pi}\frac{t}{t^2+x^2}\left(\frac{t}{\lvert x\rvert}\right)^{\frac{n-1}{n}}$	$\frac{1}{\widetilde{C}_n\left(\frac{1}{t}\right)}$	$\frac{\sin\frac{\pi}{2n}}{\pi}\frac{t}{t^2+w^2}\left(\frac{\lvert w\rvert}{t}\right)^{\frac{n-1}{n}}$
		$\frac{\widetilde{C}_n^1(t)}{\widetilde{C}_n^2(t)}$	$\frac{1}{2\pi}\tan\frac{\pi}{2n}\frac{\lvert w\rvert^{\frac{1}{n}-1}}{(1-w^2)}\left(1-w^{2-\frac{2}{n}}\right)$
		$\frac{t}{1+\left(\frac{\lvert\widetilde{C}_n(t)\rvert}{t}\right)^{2}}$	$\frac{\sin\frac{\pi}{2n}}{\pi}(t-w)^{\frac{1}{2n}-1}w^{-\frac{1}{2n}}$, for $0<w<t$

where Θ is the random angle formed by OP and the x-line. Formula (7.36) can be rewritten as

$$e^{-\eta} = \tan\frac{\Theta}{2}$$

which is the celebrated Lobachevsky law for the angle of parallelism.



(a) (b)

Figure 7.6: The probability density function of $\widehat{C}_n(t)$, (A), and $\widetilde{C}_n(t)$, (B), for different values of n.

If Θ is uniformly distributed in $(0,\pi)$, the non-negative random variable η (representing the hyperbolic distance of P from Q)

$$\eta = \begin{cases} -\log\tan\frac{\Theta}{2}, & \Theta \in \left(0, \frac{\pi}{2}\right), \\ \log\tan\frac{\Theta}{2}, & \Theta \in \left(\frac{\pi}{2}, \pi\right), \end{cases}$$

has distribution function

$$\begin{aligned} \Pr\{\eta < w\} &= 2\Pr\left\{0 < -\log\tan\frac{\theta}{2} < w\right\} = 2\Pr\left\{0 > \log\tan\frac{\theta}{2} > -w\right\} \\ &= 2\Pr\left\{1 > \tan\frac{\theta}{2} > e^{-w}\right\} = 2\Pr\left\{\frac{\pi}{2} > \theta > 2\arctan e^{-w}\right\} \\ &= 2\int_{2\arctan e^{-w}}^{\frac{\pi}{2}} \frac{d\theta}{\pi} = 1 - \frac{4}{\pi}\arctan e^{-w}, \qquad w > 0. \end{aligned} \tag{7.37}$$

The density related to (7.37) reads

$$\Pr\{\eta \in dw\} = \frac{4}{\pi}\frac{e^{-w}}{1+e^{-2w}}\,dw = \frac{2}{\pi}\frac{dw}{\cosh w}, \qquad w > 0.$$

If we consider the symmetric r.v. (see fig 7.7)

$$\widehat{\eta} = -\log\tan\frac{\Theta}{2}, \qquad \Theta \in (0,\pi), \tag{7.38}$$

we obtain that

$$\Pr\{\widehat{\eta} \in dw\} = \frac{1}{\pi}\frac{dw}{\cosh w}, \qquad w \in \mathbb{R}, \tag{7.39}$$

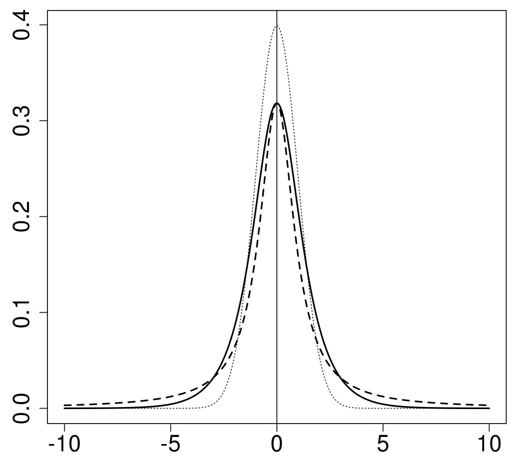
with distribution function

$$\Pr\{\widehat{\eta} < w\} = 1 - \frac{2}{\pi} \arctan e^{-w}.$$

The distribution (7.39) appears in Feller (1966) pag. 503 and emerges in the analysis of the successive overshoots by a Cauchy process in Pitman and Yor (1986).

The r.v.'s η and $\widehat{\eta}$ can be also viewed on the Poincaré disc, where the shooting point O is on the circumference and η represents the distance between Q and P (see figure 7.6b).

Figure 7.7: The density of the hyperbolic r.v. (black line) is compared with the standard normal (which has high concentration of the probability around zero) and the Cauchy law.



We give a derivation of the characteristic function of (7.39) different from the series expansion of Feller (1966). Our approach is based on the residue theorem.

Theorem 7.4.1. *The characteristic function of* (7.39) *is written as*

$$\mathbb{E}e^{i\beta\widehat{\eta}} = \frac{1}{\cosh\frac{\beta\pi}{2}}. \tag{7.40}$$

Proof. The integral (7.40) can be evaluated by means of the residue theorem applied to the function

$$f(z) = \frac{e^{i\beta\pi z}}{\cosh \pi z}, \qquad z \in \mathbb{C},$$

By considering the contour of Fig. 7.8a we have that

$$\int_{-r}^{r} \frac{e^{i\beta\pi x}dx}{\cosh \pi x} + \int_0^i \frac{e^{i\beta\pi(r+iy)}dy}{\cosh(r+iy)} + \int_r^{-r} \frac{e^{i\beta\pi(x+i)}dx}{\cosh \pi(x+i)} + \int_i^0 \frac{e^{i\beta\pi(-r+iy)}dy}{\cosh \pi(-r+iy)} =$$

$$= 2\pi i \,\mathrm{Res} f(z)|_{z=\frac{i}{2}} \,,$$

where $\mathrm{Res} f(z)|_{z=\frac{i}{2}}$ is the residue of the pole at $z = \frac{i}{2}$, the contour of integration is represented in Fig. 7.8a. By taking the limit for $r \to \infty$ the second and the fourth integral disappear and thus

$$\int_{-\infty}^{\infty} \frac{e^{i\beta\pi x} dx}{\cosh \pi x} + \int_{-\infty}^{\infty} \frac{e^{i\beta\pi x - \beta\pi} dx}{\cosh \pi x} = 2e^{-\frac{\beta\pi}{2}}$$
$$\left(1 + e^{-\beta\pi}\right) \int_{-\infty}^{\infty} \frac{e^{i\beta\pi x}}{\cosh \pi x} = 2e^{-\frac{\beta\pi}{2}}.$$

In conclusion we have that

$$\int_{-\infty}^{\infty} \frac{e^{i\beta\pi x}}{\cosh \pi x} dx = \frac{2e^{-\frac{\beta\pi}{2}}}{1 + e^{-\beta\pi}} = \frac{1}{\cosh \frac{\beta\pi}{2}},$$

which is the desired result. □

From (7.40) we obtain that

$$\mathrm{Var}\, \widehat{\eta} = \left(\frac{\pi}{2}\right)^2.$$

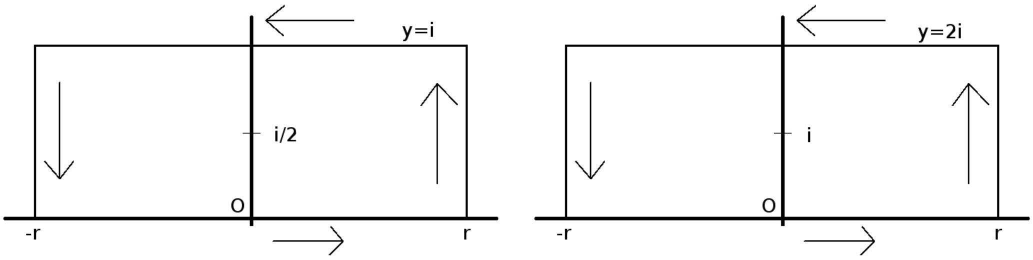
The even-order moments of $\widehat{\eta}$ can be expressed in terms of the Euler numbers E_{2n}

$$\mathbb{E}\widehat{\eta}^{2n} = \left(\frac{\pi}{2}\right)^{2n} |E_{2n}| \,,$$

in view of formula 3.523 pag 376 of Gradshteyn and Ryzhik (2007). The Euler numbers have generating function

$$\frac{1}{\cosh t} = \sum_{k=0}^{\infty} E_n \frac{t^n}{n!}, \qquad |t| < \frac{\pi}{2}. \tag{7.41}$$

Formula (7.41) gives, for $|t| < \frac{\pi}{2}$, a possible representation of the density (7.39).



(a) (b)

Figure 7.8: The contours of integration for Theorems 7.4.1 and 7.4.2.

7.4.1 Distributional Properties of the hyperbolic distribution

Theorem 7.4.2. *Let η_1 and η_2 be two independent copies of (7.38). Thus the distribution of*

$$\widehat{\boldsymbol{\eta}}_{\mathbf{2}} = \widehat{\eta}_1 + \widehat{\eta}_2, \tag{7.42}$$

is given by

$$\Pr\{\widehat{\boldsymbol{\eta}}_{\mathbf{2}} \in dx\} = \frac{2x}{\pi^2 \sinh x} dx. \tag{7.43}$$

Proof. In view of (7.40) we have

$$\Pr\{\widehat{\boldsymbol{\eta}}_{\mathbf{2}} \in dx\} = \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-izx}}{\cosh^2 \frac{z\pi}{2}} dz. \tag{7.44}$$

The inverse Fourier transform appearing in the right-hand side of (7.44) can be evaluated by means of the residue theorem, applied to the function

$$f(z) = \frac{e^{-ixz}}{\cosh^2 \frac{z\pi}{2}}, \qquad z \in \mathbb{C}, \tag{7.45}$$

along the contour of the form in Fig. 7.8b. In the same spirit of Theorem 7.4.1 we get

$$\frac{1}{2\pi}\left[\int_{-r}^{r} \frac{e^{-ixw}}{\cosh^2 \frac{w\pi}{2}} dw + \int_{r}^{-r} \frac{e^{-ix(w+2i)}}{\cosh^2 \frac{\pi}{2}(w+2i)} dw + \int_{0}^{2i} \frac{e^{-ix(r+iy)}}{\cosh \frac{\pi(r+iy)}{2}} dy + \right.$$
$$\left. + \int_{2i}^{0} \frac{e^{-ix(-r+iy)}}{\cosh \frac{\pi(-r+iy)}{2}} dy \right] = i \operatorname{Res} f(z)|_{z=i}$$

and taking the limit for $r \to \infty$ we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-ixw}}{\cosh^2 \frac{w\pi}{2}} dw = \frac{i \operatorname{Res} f(z)|_{z=i}}{1 - e^{2x}} = -\frac{i}{2\sinh x} e^{-x} \operatorname{Res} f(z)|_{z=i}. \tag{7.46}$$

The residue in $z = i$ is given by

$$\begin{aligned}
\operatorname{Res} f(z)|_{z=i} &= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{e^{-ixz}}{\cosh^2 \frac{z\pi}{2}}\right] \\
&= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{2e^{-ixz}}{1+\cosh \pi z}\right] \\
&= \lim_{z\to i} e^{-ixz}\left[\frac{-2(z-i)^2 ix + 4(z-i)}{1+\cosh \pi z} - \frac{2\pi (z-i)^2 \sinh(\pi z)}{(1+\cosh \pi z)^2}\right] \\
&= \frac{2^2 xi}{\pi^2} e^x + \lim_{z\to i} e^{-ixz}\left[\frac{4(z-i)}{1+\cosh \pi z} - \frac{2\pi (z-i)^2 \sinh \pi z}{(1+\cosh z\pi)^2}\right]
\end{aligned}$$

$$\stackrel{\text{Taylor}}{=} \frac{2^2 xi}{\pi^2} e^x + \lim_{z \to i} e^{-ixz} \left[\frac{4(z-i)}{-\frac{\pi^2}{2}(z-i)^2} + \frac{2\pi^2 (z-i)^3}{\left(-\frac{\pi^2}{2}(z-i)^2\right)^2} \right]$$
$$= \frac{2^2 xi}{\pi^2} e^x, \tag{7.47}$$

where, in the last step, we used the following Taylor's series expansions in a neighborhood of the point $z = i$

$$1 + \cosh \pi z = -\frac{(z-i)^2}{2}\pi^2 + o\left((z-i)^2\right)$$
$$\sinh \pi z = -(z-i)\pi + o(z-i).$$

In conclusion, considering (7.46) and (7.47), we obtain

$$\Pr\{\widehat{\boldsymbol{\eta}}_{\mathbf{2}} \in dx\} = \frac{2x}{\pi^2 \sinh x} dx. \tag{7.48}$$

□

Remark 7.4.3. In order to check that (7.48) integrates to unity we refer to formula 3.521 pag. 375 of Gradshteyn and Ryzhik (2007) obtaining

$$\int_{-\infty}^{\infty} \frac{2x}{\pi^2 \sinh x} dx = 1.$$

For a picture of distribution (7.43) see Fig. 7.9.

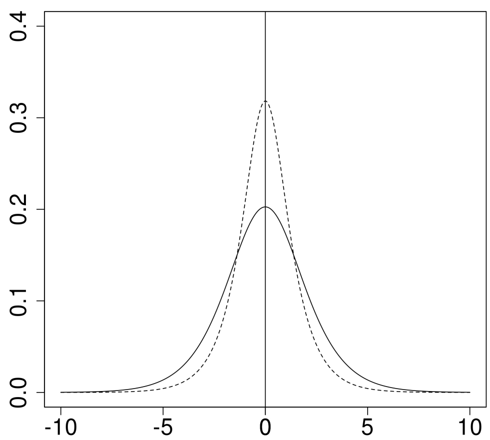


Figure 7.9: The dotted line represents the hyperbolic distribution (7.39) and the bold one represents the density (7.48) of the sum $\eta_1 + \eta_2$.

In general, for

$$\widehat{\boldsymbol{\eta}}_{\boldsymbol{n}} = \widehat{\eta}_1 + \widehat{\eta}_2 + \cdots + \widehat{\eta}_n, \qquad n \in \mathbb{N},$$

we have, in force of formula 3.985 pag 512 of Gradshteyn and Ryzhik (2007),

$$\Pr\{\widehat{\boldsymbol{\eta}}_{\boldsymbol{n}} \in dw\} = \begin{cases} \frac{4^k w}{2(2k-1)!\pi^2 \sinh w} \prod_{r=1}^{k-1} \left(\frac{w^2}{\pi^2} + r^2\right) dw, & n = 2k, 2 \le k \in \mathbb{N} \\ \\ \frac{2^{2k}}{(2k)!\pi \cosh w} \prod_{r=1}^{k} \left[\frac{w^2}{\pi^2} + \left(\frac{2r-1}{2}\right)^2\right] dw, & n = 2k+1, k \in \mathbb{N}. \end{cases} \tag{7.49}$$

The proof of (7.49) is based on the evaluation of the integral

$$\int_\Gamma f(z)\, dz, \qquad z \in \mathbb{C},$$

where

$$f(z) = \frac{e^{-ixz}}{\cosh^n \frac{\pi z}{2}}, \qquad z \in \mathbb{C},$$

and the contour Γ is that of figure 7.8b. The proof follows the same line of Theorem 7.4.2 and we arrive at

$$\int_{-\infty}^{\infty} \frac{e^{-iwx}}{\cosh \frac{w\pi}{2}} dw - \frac{e^{2x}}{(-1)^n} \int_{-\infty}^{\infty} \frac{e^{-iwx}}{\cosh^n \frac{w\pi}{2}} dw = 2\pi i \, \mathrm{Res} f(z)|_{z=i}$$

where $\mathrm{Res} f(z)|_{z=i}$ is the residue of $f(z)$ at $z = i$. The inverse Fourier transform is therefore

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iwx}}{\cosh^n \frac{w\pi}{2}} dw = \frac{i}{1 + (-1)^{n+1} e^{2x}} \mathrm{Res} f(z)|_{z=i} .$$

The evaluation of $\mathrm{Res} f(z)|_{z=i}$ leads to (7.49). For $n = 2$ we clearly retrive the result of Theorem 7.4.2.

A particle performing a random walk on the geodesic line QP of figure 7.6a, after n steps occupies the position $\widetilde{\eta}_n$ with distribution (7.49) and characteristic function

$$\mathbb{E} e^{i\beta\widehat{\eta}} = \frac{1}{\cosh^n \frac{\beta\pi}{2}}.$$

We present now some transformation of the hyperbolic distribution of $\widehat{\eta}$. We start by showing that $\sinh \widehat{\eta}$ has Cauchy distribution. We have for the r.v.

$$O(\eta) = \sinh \eta$$

that

$$\Pr\{\sinh \eta < y\} = \Pr\{\eta < \arg\sinh y\} = \Pr\left\{\eta < \log\left(y + \sqrt{y^2+1}\right)\right\}$$
$$= \int_{-\infty}^{\log\left(y+\sqrt{1+y^2}\right)} \frac{dx}{\pi \cosh x}.$$

and thus

$$\frac{\Pr\{O(\eta)\in dy\}}{dy} = \frac{1}{\pi}\frac{1}{y+\sqrt{1+y^2}}\left(1+\frac{y}{\sqrt{1+y^2}}\right)\frac{2}{e^{\log\left(y+\sqrt{1+y^2}\right)}+e^{-\log\left(y+\sqrt{1+y^2}\right)}}$$
$$= \frac{1}{\pi}\frac{y+\sqrt{1+y^2}}{y^2+1+y\sqrt{1+y^2}}\frac{1}{\sqrt{1+y^2}}$$
$$= \frac{1}{\pi\left(1+y^2\right)}.$$

Furthermore, considering the r.v. $\cosh\eta$ we get, for $w>1$

$$\Pr\{1<\cosh\eta<w\} = \frac{2}{\pi}\int_0^{\arg\cosh w}\frac{dx}{\cosh x}$$
$$= \frac{2}{\pi}\int_0^{\log\left(w+\sqrt{w^2-1}\right)}\frac{dx}{\cosh x},$$

and thus the density reads

$$\Pr\{\cosh\eta\in dw\} = \frac{2}{\pi}\frac{dw}{w\sqrt{w^2-1}}, \qquad w>1. \tag{7.50}$$

The distribution (7.50) integrates to unity since

$$\frac{2}{\pi}\int_1^\infty\frac{dw}{w\sqrt{w^2-1}} \overset{\frac{1}{w^2}=y}{=} \frac{1}{\pi}\int_0^1\frac{dy}{\sqrt{y(1-y)}} = 1.$$

The last step suggests a relationship between the r.v. $\cosh\eta$ and the arcsine law. The r.v.

$$Y = \frac{1}{\cosh^2\eta}$$

possesses arcsine distribution, as the following detailed calculation shows

$$\Pr\{Y<w\} = \Pr\left\{\eta>\arg\cosh\frac{1}{\sqrt{w}}\right\}$$
$$= \frac{2}{\pi}\int_{\log\left(\frac{1}{\sqrt{w}}+\frac{1}{\sqrt{w}}\sqrt{1-w}\right)}^\infty\frac{dx}{\cosh x}$$
$$= \frac{2}{\pi}\int_{-\frac{1}{2}\log w+\log\left(1+\sqrt{1-w}\right)}^\infty\frac{dx}{\cosh x},$$

and thus

$$\frac{\Pr\{Y\in dw\}}{dw} = \frac{1}{\pi}\left[\frac{1}{w}+\frac{1}{\sqrt{1-w}\left[1+\sqrt{1-w}\right]}\right]\frac{2}{\frac{1}{\sqrt{w}}\left(1+\sqrt{1-w}\right)+\sqrt{w}\frac{1}{1+\sqrt{1-w}}}$$
$$= \frac{2}{\pi}\frac{\sqrt{1-w}(1+\sqrt{1-w}+w)}{w\sqrt{1-w}(1+\sqrt{1-w})}\frac{\sqrt{w}(1+\sqrt{1-w})}{(1+\sqrt{1-w})^2+w}$$

$$= \frac{1}{\pi}\frac{1+\sqrt{1-w}}{\sqrt{w}\sqrt{1-w}}\frac{1}{\left(\sqrt{1-w}+1\right)} = \frac{1}{\pi}\frac{1}{\sqrt{w}\sqrt{1-w}}, \qquad 0<w<1. \tag{7.51}$$

Remark 7.4.4. Result (7.51) can be also obtained observing that

$$Y = \frac{1}{\cosh^2\eta} = \frac{1}{1+\sinh^2\eta} = \frac{1}{1+O\left(\eta\right)^2}, \tag{7.52}$$

and we have shown that O possesses Cauchy distribution. The transformation (7.52) is the classical way to obtain the arcsine law from the Cauchy distribution.

Remark 7.4.5. Let us recall the hyperbolic version of the Pythagorean theorem which reads

$$\cosh a \cosh b = \cosh c,$$

where c is the hypotenuse of the right triangle with sides a and b. Considering a and b distributed as (7.39) their hyperbolic cosine has law (7.50). The random length of the hypotenuse is therefore written as

$$\begin{aligned}\Pr\left\{\cosh\eta_1\cosh\eta_2\in dw\right\} &= dw\left(\frac{2}{\pi}\right)^2\frac{1}{w}\int_1^w\frac{dx}{\sqrt{x^2-1}\sqrt{w^2-x^2}}dx\\ &\overset{x=\cosh y}{=} dw\int_0^{\log\left(w+\sqrt{w^2-1}\right)}\frac{1}{\sqrt{w^2-\cosh^2 y}}dy.\end{aligned} \tag{7.53}$$

Remark 7.4.6. Considering the r.v.

$$\widetilde{\eta} = -\log\tan^\alpha\frac{\Theta}{2}, \qquad \alpha>0,$$

with Θ uniformly distributed in $(0,\pi)$ we get

$$\Pr\left\{\widetilde{\eta}\in dw\right\} = \frac{2}{\alpha\pi}\frac{e^{-\frac{w}{\alpha}}dw}{1+e^{-\frac{2w}{\alpha}}} = \frac{1}{\pi\alpha}\frac{dw}{\cosh\frac{w}{\alpha}}, \qquad w\in\mathbb{R}. \tag{7.54}$$

The density (7.54) is a generalization with paramater α of (7.39).

7.4.2 The area of hyperbolic random triangles

It is well known that the area A of an hyperbolic triangle is given by

$$A = \pi-(\alpha+\beta+\gamma)$$

where α, β and γ are the angles pertaining to vertices not lying on the (x-axis). A triangle which has three vertices on the x-axis has area $A=\pi$.

Let us consider the triangle with vertices O, P, and Q in Fig. 7.6a or 7.6b, thus the area K is given by $K=\frac{\pi}{2}-\alpha$ where α is the angle of the vertex $\widehat{OPQ}$, formally we have $K\in\left(0,\frac{\pi}{2}\right)$.

Table 7.2: For the hyperbolic r.v. $\widehat{\eta}$ we have the following table of distributional relationships for the related hyperbolic function.

Variable	$\sinh\widehat{\eta}$	$\cosh\widehat{\eta}$	$\tanh\widehat{\eta}$	$\tanh^2\widehat{\eta}$
Density	$\frac{1}{\pi(1+z^2)}$ $z\in\mathbb{R}$	$\frac{2}{\pi z\sqrt{z^2-1}}$ $z>1$	$\frac{1}{\pi\sqrt{1-z^2}}$ $-1<z<1$	$\frac{1}{\pi\sqrt{z(1-z)}}$ $0<z<1$
Reciprocal	$\frac{1}{\sinh\widehat{\eta}}$	$\frac{1}{\cosh\widehat{\eta}}$	$\coth\widehat{\eta}$	$\coth^2\widehat{\eta}$
Density	$\frac{1}{\pi(1+z^2)}$ $z\in\mathbb{R}$	$\frac{2}{\pi\sqrt{1-z^2}}$ $0<z<1$	$\frac{1}{\pi\lvert z\rvert\sqrt{z^2-1}}$ $z\in\mathbb{R}\setminus[-1,1]$	$\frac{1}{\pi z\sqrt{z-1}}$ $z>1$

Theorem 7.4.7. *For the random area K of the hyperbolic triangle OPQ where PQ has length η with distribution (7.39), we have that*

$$\Pr\{K\in dw\} = \frac{2}{\pi}\frac{dw}{1+\sin w}, \qquad w\in\left(0,\frac{\pi}{2}\right). \tag{7.55}$$

Proof. In view of formula

$$\tan\frac{A}{2} = \tanh\frac{a}{2}\tanh\frac{b}{2}$$

where a, b are the sides of an hyperbolic right triangle of area A, we have

$$\tan\frac{K}{2} = \tanh\frac{\eta}{2}.$$

For $w>0$

$$\begin{aligned}\Pr\{K<w\} &= \Pr\left\{\eta<2\,\mathrm{argtanh}\tan\frac{w}{2}\right\}\\ &= \frac{2}{\pi}\int_0^{\log\frac{1+\sin w}{\cos w}}\frac{1}{\cosh x}\,dx\\ &= 1-\frac{4}{\pi}\arctan\frac{\cos w}{1+\sin w},\end{aligned}$$

and thus

$$\Pr\{K\in dw\} = \frac{2}{\pi}\frac{dw}{1+\sin w}.$$

□

In view of formula 3.791 pag. 448 of Gradshteyn and Ryzhik (2007) we have

$$\mathbb{E}K = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{x}{1+\sin x}\,dx = \frac{2}{\pi}\log 2.$$

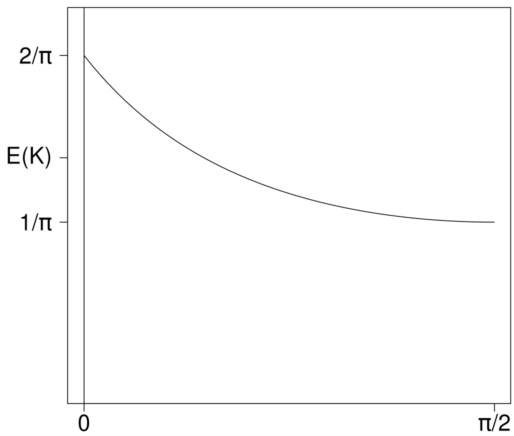


Figure 7.10: The distribution (7.55) of the random area K.

Bibliography

S. Albeverio, N. Smorodina and M. Faddeev. The probabilistic representation of the exponent of a class of pseudo-differential operators. *J. Global and Stochastic Analysis*, 1(2): 123 – 148, 2011.

H. Allouba and E. Nane. Interacting time-fractional and Δ^ν PDE's systems via Brownian-time and Inverse-stable-Lévy-time Brownian sheets. *Stochastics and Dynamics*, 13(1): 1250012, 2013.

H. Allouba and W. Zheng. Brownian-time processes: The PDE connection and the half-derivative generator. *Ann. Probab.*, 29:1780 – 1795, 2001.

B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis*, 4(4): 481 – 500, 2001.

A.V. Balakrishnan. Fractional powers of closed operators and semigroups generated by them. *Pacific J. Math.*, 10:419 – 437, 1960.

L. Beghin. Pseudoprocesses governed by higher-order fractional differential equations. *Electronic Journ. Prob.*, 13(16): 467 – 485, 2008.

L. Beghin. Fractional Gamma process and fractional Gamma-subordinated processes. *arXiv:1305.1753.*

L. Beghin, Yu. Kozachenko, E. Orsingher and L. Sakhno. On the solutions of linear odd-order heat-type equations with random initial conditions. *J. Stat. Phys.*, 127(4): 721 – 739, 2007.

L. Beghin, E. Orsingher and T. Ragozina. Joint distributions of the maximum and the process for higher-order diffusions. *Stochastic Process. Appl.*, 94(1): 71 – 93, 2001.

S. Benachour, B. Roynette and P. Vallois. Explicit solutions of some fourth order partial differential equations via iterated Brownian motion. *Seminar on Stochastic Analysis, Random Fields and Applications, Ascona, 1996. Progr. Probab.*, 45: 39 – 61, 1999.

D. Benson, R. Schumer, M. Meerschaert and S. Wheatcraft. Fractional dispersion, Lévy motions, and the MADE tracer tests. *Transport in porous media*, 42: 211 – 240, 2001.

F. Bernstein. Über das Fourierintegral $\int_0^\infty e^{-x^4} \cos tx\, dx$. *Math. Ann.*, 79: 265 – 258, 1919.

S. Bernstein. Sur les fonctions absolument monotones (french). *Acta Math*, 52; 1 – 66, 1929.

J. Bertoin. Lévy processes. *Cambridge University Press*, Cambridge, 1996.

J. Bertoin. Subordinators: examples and appications. *Lectures on probability theory and statistics (Saint-Flour, 1997)*, 1 – 91. *Lectures Notes in Math.*, 1717, Springer, Berlin, 1999.

N. Bleistein and R.A. Handelsman. Asymptotic Expansions of Integrals. *Dover Publications*, 1986.

S. Bochner. Diffusion equation and stochastic processes. *Proc. Nat. Acad. Sciences, U.S.A.*, 35:368 – 370, 1949.

S. Bochner. Harmonic analysis and the theory of probability. *University of California Pres*, Berkeley, 1955.

A. Brix. Generalized gamma measures and shot-noise Cox processes. *Adv. in Appl. Probab.*, 31(4): 929 – 953, 1999.

K. Burdzy. Some path properties of iterated Brownian motion. *Sem. Stoch. Processes*, 1992, 67-87 , Birkhaüser, Boston (ed's: E. Cinlar, K. L. Chung and M. Sharpe), 1993.

K. Burdzy. Variation of iterated Brownian motion. *Workshop and Conf. on Measure-Valued Processes, Stoch. Partial. Diff. Eq.'s and Interacting Particle Systems.*, CRM Proceedings and Lecture Notes, 1993.

K. Burdzy and D. Khoshnevisian. Brownian motion in a Brownian crack. *Ann. Applied Probab.*, 8(3): 708 – 748, 1998.

W.R. Burwell. Asymptotic expansions of generalized hypergeometric functions. *Proc. Lond. math. Soc.*, 22: 57 – 72, 1923.

D.O. Cahoy. and F. Polito. On a fractional binomial process. *Journal of Statistical Physics*, 146(3): 646-662, 2012.

V. Cammarota and A. Lachal. Joint distribution of the process and its sojourn time in a half-line $[a,+\infty)$ for pseudo-processes driven by a high-order heat-type equation. *Stochastic Process. Appl.*, 122(1): 217 – 249, 2012.

V. Cammarota and E. Orsingher. Hitting spheres on hyperbolic spaces. To appear in *Theory of Probability and its Applications*, 57(3), 2012. arXiv:1104.1043.

L. Chaumont and M. Yor. Exercises in probability. *Cambridge University Press*, 2003.

A. Chudnovsky and B. Kunin. A probabilistic model of brittle crack formation. *J. Appl. Phys.*, 62:4124 – 4129, 1987.

M. D'Ovidio. From Sturm-Liouville problems to fractional and anomalous diffusions. *Stochastic Processes and their Applications,* 122: 3513 – 3544, 2012.

M. D'Ovidio and E. Orsingher. Composition of processes and related partial differential equations. *Journal of Theoretical Probability*, 24: 342 – 375, 2010.

M. D'Ovidio, E. Orsingher and B. Toaldo. Time changed processes governed by space-time fractional telegraph equations. *Submitted*, arXiv:1206.2511v1, 2012.

M. D'Ovidio, E. Orsingher and B. Toaldo. Fractional telegraph-type equations and hyperbolic Brownian motion. To appear in *Statistics & Probability Letters*, 2014.

Yu.L. Daletsky Integration in function spaces. *Progress in Mathematics*, R.V. Gamkrelidze, ed., 4, 87 – 132, 1969

Yu.L. Daletsky and S.V. Fomin. Generalized measures in function spaces. *Theory Prob. Appl.*, 10(2): 304 – 316, 1965.

L. Debbi. Explicit solutions of some fractional partial differential equations via stable subordinators. *Journal of Applied Mathematics and Stochastic Analysis*, Vol. 5, 2006.

L. Debbi. On some properties of a higher order fractional differential operator which is not in general selfadjoint. *Applied Mathematical Sciences*, 1(27): 1325 – 1339, 2007.

A. Debiard, B. Gaveau and E. Mazet. Théorèmes de comparison en géométrie rimannienne. *Publ. Kyoto Univ.*, 12: 391 – 425, 1976.

R.D. DeBlassie. Iterated Brownian motion in an open set. *Ann. Appl. Probab.*, 14: 1529 – 1558, 2004.

K.J. Engel and R. Nagel. One-Parameter Semigroups for Linear Evolution Equations. *Springer, Graduate Texts in Mathematics*, 2000.

W. Feller. On a generalization of Marcel Riesz' potentials and the semigroups generated by them. *Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.]*, Tome Supplementaire, 1952.

W. Feller. An introduction to probability theory and its applications, vol. II. *Wiley*, New York, 1966.

S.K. Foong and S. Kanno. Properties of the telegrapher's random process with or without a trap. *Stoch. Proc. Appl.*, 53(1):147 – 173, 1994.

Y. Fujita. Integrodifferential equation which interpolates the heat equation and the wave equation.I-II. *Osaka J. Math*, I: 27(2):797 – 804, 1990. II: 27(4):309 – 321, 1990.

C.W. Gardiner. Handbook of Stochastics Methods. *Springer, Berlin*, 1985.

M.E. Gertsenshtein and V.B. Vasiliev. Waveguides with random inhomogeneities and Brownian motion in the Lobachevsky plane. *Theory Probab. Appl.*, 4:424–432, 1959.

R.K. Getoor. Infinitely divisible probabilities on the hyperbolic plane. *Pacific J. Math.*, 11:128 – 1308, 1961.

I.S. Gradshteyn and I.M. Ryzhik. Tables of integrals, series and products (seventh edition). *Academic Press*, 2007.

J.C. Gruet. Semi-groupe du mouvement brownien hyperbolique. (French) [Semigroup of hyperbolic Brownian motion]. *Stochastics Stochastics Rep.*, 56(1-2):53–61, 1996.

P. Hartman and G.S. Watson. "Normal" Distribution Functions on Spheres and the Modified Bessel Functions. *Ann. Probab.*, 2(4): 593 – 607, 1974.

H.J. Haubold, A.M. Mathai and R.K. Saxena. Mittag-Leffler Functions and Their Applications. *J. Appl. Math*, 51 pages, 2011.

K.J. Hochberg. A signed measure on path space related to Wiener measure. *Ann. Probab.*, 6(3): 433 – 458 (1978).

K. Itô. On stochastic processes. I. (Infinitely divisible laws of probability). *Japan J. Math.*, 18: 261 – 301, 1942.

N. Jacob. Pseudo Differential Operators & Markov Processes: Fourier analysis and semigroups. Volume 1. *Imperial College Press, Fourier analysis and semigroups*, 2001.

F.I. Karpelevich, V.N. Tutubalin and M.G. Shur. Limit theorems for compositions of distributions in the Lobačevskiĭ plane and space. *Theory Probab. Appl.*, 4: 432 – 436, 1959.

A.A. Kilbas, H.M. Srivastava and J.J. Trujillo. Theory and Applications of Fractional Differential Equations. *North-Holland Mathematics Studies, 204. Elsevier Science B.V.,* 2006.

T.J. Kozubowski and K. Podgórski. Distributional properties of the negative binomial Lévy process. *Probab. Math. Statist.*, 29(1): 43 –71, 2009.

V.Yu. Krylov. Some properties of the distribution corresponding to the equation $\frac{\partial u}{\partial t} = (-1)^{p+1} \frac{\partial^{2p} u}{\partial x^{2p}}$. *Soviet Math. Dokl.*, 1: 260 – 263, 1960.

A. Kumar, E. Nane and P. Vellaisamy. Time-changed Poisson processes. *Statistics & Probability Letters*, 81(12): 1899 – 1910, 2011.

A. Lachal. Distributions of sojourn time, maximum and minimum for pseudo-processes governed by higher-order heat-type equations. *Electronic Journ. Prob.*, 8:20, 1-53, 2003.

A. Lachal. First hitting time and place for pseudo-processes driven by the equation $\frac{\partial}{\partial t} = \pm \frac{\partial^N}{\partial x^N}$ subject to a linear drift. *Stoch. Proc. Appl.*, 118(1): 1 – 27, 2008.

A. Lachal. From pseudo-random walk to pseudo-Brownian motion: first exit time from a one-sided or a two-sided interval. *Submitted*, arxiv:1301.6579.

V.I. Ladohin. On non-positive distributions. (Russian) *Kazan. Gos. Univ. Učen. Zap.*, 122(4): 53 – 64, 1962.

L. Lao and E. Orsingher. Hyperbolic and fractional hyperbolic Brownian motion. *Stochastics*, 79(6): 505-522, 2007.

N. Laskin. Fractional Poisson process. *Commun. Nonlinear Sci. Numer. Simul.*, 8(3–4): 201–213, 2003.

N. N. Lebedev. Special Function and their applications. *Prentice-Hall, inc..*, 1965, Englewood Cliffs, New Jersey, United States of America.

P. Lévy. Sur une application de la derivée d'ordre non entier au calcul des probabilités. *C.R. Acad. Sci.*, 179:1118–1120, 1923.

N. Leonenko, M.M. Meerschaert and A. Sikorskii. Correlation structure of fractional Pearson diffusions. *Computers & Mathematics with Applications*, In press, 2013.

X. Li and R. Wong. Asymptotic behaviour of the fundamental solution to $\partial u/\partial t = -(-\triangle)^m u$. *Proceedings: Mathematical and Physical Sciences*, 441(1912): 423 – 432 (1993).

E.H. Lieb. The stability of matter: from atoms to stars. *Bull. Amer. Math. Soc.*, 22: 1 – 49, 1990.

H. Matsumoto and M. Yor. Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.*, 2: 312–347, 2005.

S. Mazzucchi. Probabilistic representations for the solutions of higher order differential equations. *Departmental Technical Report - Università degli Sudi di Trento. Dipartimento di Matematica*, 2013.

M.M. Meerschaert, E. Nane and P. Vellaisamy. Fractional Cauchy problems on bounded domains. *The Annals of Probability*, 37(3):979 – 1007, 2009.

M.M. Meerschaert and H.P. Scheffer. Limit theorems for continuous time random walks with infinite mean waiting times. *J. Appl. Probab.*, 41: 623 – 638, 2004.

M.M. Meerschaert, E. Nane and P. Vellaisamy. The fractional poisson process and the inverse stable subordinator. *Electron. J. Probab.*, 16(59): 1600 – 1620, 2011.

M.M. Meerschaert and A. Sikorskii. Stochastic Models for Fractional Calculus. *Walter de Gruyter*, Vol. 43 of De Gruyter Studies in Mathematics, Berlin/Boston, 2012.

M.M. Meerschaert and P. Straka. Inverse stable subordinators. *Mathematical Modeling of Natural Phenomena*, 8(2): 1 – 16, 2013.

M. Miyamoto. An extension of certain quasi-measure. *Proc. Japan Acad.*, 42: 70 – 74, 1966.

S. Momani. Analytic and approximate solutions of the space- and time-fractional telegraph equations. *Appl. Math. Comput.*, 170(2): 1126–1134, 2005.

E. Montroll and G.H. Weiss. Random walk on lattices. II. *J. Math. Phys.*, 6: 167 – 181, 1965.

R.R. Nigmatullin. The realization of the generalized transfer equation in a medium with fractal geometry. *Phys. Stat. sol*, 133b: 425 – 430, 1986.

K. Nishioka. Boundary conditions for one-dimensional biharmonic pseudo process. *Electronic Journ. Prob.*, 6(13): 1 – 27, 2001.

E. Orsingher. Processes governed by signed measures connected with third-order "heat-type" equations. *Litovsk. Mat. Sb.*, 31(2): 323 – 336, 1991; translation in *Lithuanian Math. J.*, 31(2): 220 – 231, 1992.

E. Orsingher and L. Beghin. Time-fractional telegraph equations and telegraph processes with Brownian time. *Probab. Theory Related Fields*, 128(1):141 – 160, 2004.

E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.*, 37:206 – 249, 2009.

E. Orsingher and M. D'Ovidio. Vibrations and Fractional Vibrations of Rods, Plates and Fresnel Pseudo-Processes. *J. Stat. Phys.*, 145:143 – 174, 2011.

E. Orsingher and M. D'Ovidio. Probabilistic representation of fundamental solutions to $\frac{\partial u}{\partial t} = \kappa_m \frac{\partial^m u}{\partial x^m}$. *Electronic Communications in Probability*, 17: 1-12, 2012.

E. Orsingher and M. D'Ovidio. Higher-order Laplace equations and hyper-Cauchy distributions. To appear in *Journal of Theoretical Probability*, online since 28 February 2013.

E. Orsingher and A. De Gregorio. Random flights in higher spaces. *J. Theoret. Probab.*, 20(4): 769 – 806, 2007.

E. Orsingher and F. Polito. Compositions, Random Sums and Continued Random Fractions of Poisson and Fractional Poisson Processes. *Journal of Statistical Physics*,148(2): 233–249, 2012.

E. Orsingher and F. Polito. The space-fractional Poisson process. *Statistics & Probability Letters*, 82: 852 – 858, 2012.

E. Orsingher and F. Polito. On the integral of fractional Poisson processes. *Statistics & Probability Letters*, 83(4): 1006–1017, 2013.

E. Orsingher and B. Toaldo. Space-time fractional equations and the related stable processes at random-time. *Under review*, arXiv:1207.3284, 2012.

E. Orsingher and B. Toaldo. Even-order pseudoprocesses on a circle and related Poisson kernels. *Under review*, arXiv:1212.5786, 2013.

E. Orsingher and B. Toaldo. Pseudoprocesses related to space-fractional higher-order heat-type equations. In revision in *Stochastic Analysis and Applications*, arXiv:1305.6409, 2013.

E. Orsingher and B. Toaldo. Counting processes with Bernstein intertimes and random jumps. *Under review.*

E. Orsingher and B. Toaldo. Shooting randomly against a line in Euclidean and non-Euclidean spaces. *Stochastics*, 86(1): 16 – 45, 2014.

E. Orsingher and X. Zhao. The space-fractional telegraph equation and the related fractional telegraph process. *Chin. Ann. Math.*, 24B(1):1 – 12, 2003.

R.S. Phillips. On the generation of semigroups of linear operators. Pacific J. Math.. *Pacific J. Math.*, 2:343 – 369, 1952.

J. Pitman and M. Yor. Level Crossing of a Cauchy Process. *The Annals of Probability*, 14(3):780 – 792, 1986.

G. Pòlya. On the zeros of an integral function represented by Fourier's integral. *Messenger Math.*, 52: 185 – 188, 1923.

P.H. Roberts and H.D.Ursell. Random walk on a sphere and on a Riemannian manifold. *Trans. Roy. Soc. London. Ser.*, 252(A): 317 – 356, 1960.

J. Rosiński. Tempering stable processes. *Stochastics Processes and their Applications*, 117: 677 – 707, 2007.

A.I. Saichev and G.M. Zaslavsky. Fractional kinetic equations: Solutions and applications. *Chaos*, 7: 753 – 764, 1997.

S.G. Samko, A.A. Kilbas, O.I. Marichev: Fractional Integrals and Derivatives: Theory and Applications. *Gordon and Breach*, 1006 pages, 1993.

G. Samorodnitsky. Long range dependence. *Foundation and Trends in Stochastic Systems*, 1(2): 163 – 257, 2006.

G. Samorodnitsky and M. Taqqu. Stable non-Gaussian Random Processes. *Chapman and Hall*, New York, 1944.

K. Sato. Lévy processes and infinitely divisible distributions. *Cambridge University Press*, 1999

R.K. Saxena, A.M. Mathai and H.J. Haubold. Reaction-Diffusion Systems and Nonlinear Waves. *Astrophys. Space Sci.*, 305:297 – 303, 2006.

R.K. Saxena, A.M. Mathai and H.J. Haubold. Solutions of fractional reaction-diffusion equations in terms of Mittag-Leffler functions. *arXiv:0708.2265v1*, 22 pages, 2007.

R.L. Schilling, R. Song and Z. Vondraček. Bernstein functions: theory and applications. *Walter de Gruyter GmbH & Company KG*, Vol 37 of De Gruyter Studies in Mathematics Series, 2010.

V.I. Smirnov. A course of Higher Mathematics (vol. II). *Pergamon Press*, 1964.

R. Song and Z. Vondraček. Potential theory of subordinate Brownian motion. In: *Potential Analysis of Stable Processes and its Extensions, P. Graczyk, A. Stos, editors, Lecture Notes in Mathematics 1980*: 87–176, 2009.

W. Stadje and S. Zacks. Telegraph processes with random velocities. *J. Appl. Prob.*, 41:665 – 678, 2004.

M.A. Stephens. Random walk on a circle. *Biometrika*, 50: 385 – 390, 1963.

B. Toaldo. Convolution-type derivatives, hittin-times of subordinators and time-changed C_0-semigroups. *Under review*, arXiv:1308.1327, 2013.

L. Turban. Iterated Random walk. *Europhys. Lett.*, 65(5):627 – 632, 2004.

M. Veillette and M.S. Taqqu. Using differential equations to obtain joint moments of first-passage times of increasing Lévy processes. *Stat. Probab. Letters*, 80: 697 – 705, 2010.

G. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D*, 76: 110 – 122, 1994.

V.M. Zolotarev. One-dimensional stable distributions, volume 65 of Translations of Mathematical Monographs. *American Mathematical Society, 1986.* ISBN 0-8218-4519-5. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.