A QUASI-LINEAR NONLOCAL VENTTSEL’ PROBLEM OF AMBROSETTI–PRODI TYPE ON FRactal DOMAINS

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(Communicated by the associate editor name)

Abstract. We investigate the solvability of the Ambrosetti–Prodi problem for the \( p \)-Laplace operator \( \Delta_p \) with Venttsel’ boundary conditions on a two-dimensional open bounded set with Koch-type boundary, and on an open bounded three-dimensional cylinder with Koch-type fractal boundary. Using a priori estimates, regularity theory and a sub-supersolution method, we obtain a necessary condition for the non-existence of solutions (in the weak sense), and the existence of at least one globally bounded weak solution. Moreover, under additional conditions, we apply the Leray-Schauder degree theory to obtain results about multiplicity of weak solutions.

1. Introduction and the main result. Let \( \Omega_2 \subseteq \mathbb{R}^2 \) be a bounded domain with a Koch-type fractal boundary \( \Gamma_2 := \partial \Omega_2 \), and let \( \Omega_3 \subseteq \mathbb{R}^3 \) be a bounded cylinder with a Koch-type fractal boundary \( \partial \Omega_3 := \overline{\Omega_3} \setminus \Omega_3 = \Gamma_3 \cup (\Omega_2 \times \{0\}) \cup (\Omega_2 \times \{1\}) \), where \( \Gamma_3 := \Gamma_2 \times I \), for \( I := [0, 1] \) (see section 2 for more details on the construction of these sets). Given \( p \in (1, \infty) \), denote by \( \lambda_N(\cdot) \) the usual \( N \)-dimensional Lebesgue measure in \( \Omega_N \) (\( N \in \{2, 3\} \)), and by \( \mu(\cdot) := H^d(\cdot) \) the \( d \)-dimensional Hausdorff measure, for \( d \) the Hausdorff dimension of the fractal boundary \( \Gamma_i \) (\( i \in \{2, 3\} \)). In order to pose our problem of interest, because the notion of the normal derivative (in the classical sense) may not make sense for non-Lipschitz domains, we need to define the notion of extended normal derivative (e.g. [49, Definition 4.1]).
Given $D$ arbitrary, we consider

$$D := \begin{cases} D_2, & \text{if } N = 2, \\ D_3, & \text{if } N = 3 \end{cases}$$

and

$$\mu := \begin{cases} \mu_{|D_2}, & \text{if } N = 2, \\ \mu_{|D_3}, & \text{if } N = 3 \end{cases}$$

In this paper, $D$ will be either $\Omega$, or $\Gamma$. Then, we assume that $\phi, h \in L^\infty(\Omega)$ with $\phi > 0$ a.e. in $\Omega$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$\liminf_{s \to -\infty} \frac{f(x, s)}{|s|^{p-2}s} > 0 \quad \text{and} \quad \limsup_{s \to -\infty} \frac{f(x, s)}{|s|^{p-2}s} < 0$$

(1.6)
respectively. Furthermore, the Venttsel’ maps $A_2: D(E^{(p)}) \to D(E^{(p)})^*$ and $A_3: D(\Gamma_3) \to D(\Gamma_3)^*$ are defined by

\[
(A_2 u)v := \int_{\Gamma_2} d\mathcal{L}^{(p)}(u, v) := E^{(p)}(u, v),
\]

and

\[
(A_3 u)v := \int_I [(A_2 u)v] ds + \int_{\Gamma_2} \int_I |\partial_s u|^{p-2} \partial_s u \partial_s v d\sigma d\mu,
\]

for $d\mathcal{L}^{(p)}(\cdot, \cdot)$ the $p$-Lagrangian form on $\Gamma_2$ (see section 3 for complete details about the $p$-energy form).

When we consider the corresponding eigenvalue problem associated with the Laplace operator $\Delta_p$, such eigenvalue problem is formulated by

\[-\Delta_p v = \lambda |u|^{p-2} u \text{ in } \Omega,
\]

subject to the general dynamical boundary conditions described in the equations (1.1) and (1.2). The above structure of the Then one has the first eigenvalue, given by the zero eigenvalue $\lambda_1 = 0$, from where we see that the condition (1.4) can be regarded as a sort of an eigenvalue crossing of $f$. Henceforth, one can find constants $M, \eta_0 > 0$ such that

\[f(x, s) \geq \eta_0 |s|^{p-2} s - M, \quad \text{if } s > 0,
\]

and

\[f(x, s) \geq -\eta_0 |s|^{p-2} s - M, \quad \text{if } s < 0.
\]

Moreover, we will assume in addition that there exist constants $\alpha_0, \gamma_0 > 0$ such that

\[f(x, s) + \alpha_0 |s|^{p-2} s \text{ is non-decreasing in } s \text{ on } [-L, L], \text{ for every } L > 0,
\]

and

\[f(x, s) \leq \gamma_0 (1 + |s|^{q-1}), \quad \text{for all } s \in \mathbb{R}, \text{ and uniformly for } x \in \Omega,
\]

where

\[1 < q < \begin{cases} \frac{Np}{N-p}, & \text{if } 1 < p < N, \\ \infty, & \text{if } p \geq N. \end{cases}
\]

Next, we put:

\[V_p(\Omega) := \begin{cases} V_p(\Omega_2), & \text{if } N = 2, \\ V_p(\Omega_3), & \text{if } N = 3. \end{cases}
\]

Then we define the function space:

\[W_p(\Omega) := \begin{cases} V_p(\Omega_2), & \text{if } N = 2, \\ V_p(\Omega_3), & \text{if } N = 3. \end{cases}
\]

One has that $W_p(\Omega)$ is a Banach space with respect to the norm

\[\|u\|_{W_p(\Omega)} := \left( \|u\|_{W^{1,p}(\Omega)}^2 + \int_I |E^{(p)}(u, u)| ds + \sigma_N \int_{\Gamma_2} \int_I |\partial_s v|^{p} d\sigma d\mu \right)^{\frac{1}{2}},
\]

for $u \in W_p(\Omega)$, where

\[\sigma_N := \begin{cases} 0, & \text{if } N = 2, \\ 1, & \text{if } N = 3. \end{cases}
\]
We next introduce the notion of weak solutions for our boundary value problems (1.1) and (1.4).

Definition 1.2. (a) A function \( u \in W^p(\Omega) \) a called a weak solution of either problem (1.1) \((N = 2)\), or problem (1.2) \((N = 3)\), if
\[
\Lambda_p(u, \varphi) = \int_\Omega f(x, u) \varphi \, dx + \int_\Omega (\xi \phi + h) \varphi \, dx, \quad \forall \varphi \in W^p(\Omega),
\]
(1.19)
where
\[
\Lambda_p(v, w) := \int_\Omega |\nabla v|^{p-2} \nabla v \nabla w \, dx + \int_I [\mathcal{E}(v) \, (v, w)] \, ds + \sigma_N \int_{\Gamma} |\partial_s v|^{p-2} \partial_s v \partial_s w \, ds d\mu + (\Theta_{\Omega} v) w + (\Theta_{\Gamma} v) w,
\]
(1.20)
for \( v, w \in W^p(\Omega) \), where \( \sigma_N \) is defined as in (1.18).

(b) We call \( u \in W^p(\Omega) \) a weak subsolution of problem (1.1) \((N = 2)\), or problem (1.2) \((N = 3)\), if
\[
\Lambda_p(u, \varphi) \leq \int_\Omega f(x, u) \varphi \, dx + \int_\Omega (\xi \phi + h) \varphi \, dx, \quad \forall \varphi \in W^p(\Omega)^+,
\]
(1.21)
where we recall that \( W^p(\Omega)^+ := \{ u \in W^p(\Omega) \mid u \geq 0 \} \). Moreover, \( u \in W^p(\Omega) \) is a weak supersolution of problem (1.1) \((N = 2)\), or problem (1.2) \((N = 3)\), if the reverse inequality in (1.21) holds.

We now make the main assumptions that will be crucial for the main results.

Assumption 1.3. Given \( N \in \{2, 3\} \), \( p \in (1, \infty) \), \( d \in (N - p, N) \cap (0, N) \), let \( \Omega \subseteq \mathbb{R}^N \) be given in accordance with (1.3) (for \( D = \Omega \)), assume that \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying the conditions (1.4), (1.11), and (1.12), let \( \phi, h \in L^\infty(\Omega) \) with \( \phi > 0 \) a.e. in \( \Omega \), and let \( \xi \in \mathbb{R} \) be a parameter.

We recall that if \( p \geq N \), by the Sobolev embedding theorem, all the discussions and estimates of the paper become much easier, with many simplifications (see section 2). Therefore, when deriving a priori estimates, we will concentrate on the critical case, namely, when \( p < N \).

Now we state the first main result of this paper, namely, an existence and non-existence result.

Theorem 1.4. Suppose that all the conditions of Assumption 1.3 are fulfilled. Then there exist parameters \( \xi_0, \xi'_0 \in \mathbb{R} \) such that

1. problems (1.1) and (1.2) are not solvable (over \( W^p(\Omega) \)) for all \( \xi > \xi_0 \);
2. if \( \xi \leq \xi_0 \), then problems (1.1) and (1.2) are both solvable (over \( W^p(\Omega) \)). Furthermore, the weak solutions of the equations (1.1) and (1.2) are globally bounded over \( \Omega \).
In the preceding theorem, if \( p \in (1, N] \), it is unknown (up to the present time) whether weak solutions of either problem (1.1), or problem (1.4), are globally continuous over \( \Omega \). Consequently, because of the general structure of the elliptic equations (1.1) and (1.4), we have not been able to deduce further results, such as multiplicity results, for the case \( p \in (1, N] \). However, when \( p > N \), then one can refine more the above result, as we see in our second main result of the paper.

**Theorem 1.5.** Assume (in addition to the conditions in Theorem 1.4) that \( p > N \), then there exists a parameter \( \xi_0 \in \mathbb{R} \), such that

1. problems (1.1) and (1.2) are not solvable (over \( W_{p,0}(\Omega) \)) for all \( \xi > \xi_0 \);
2. for each \( \xi \leq \xi_0 \), problems (1.1) and (1.2) admit respective minimal solution (in the weak sense), which are H"older continuous over \( \Omega \).

Moreover, when \( f \) is locally H"older continuous in \( \mathbb{R} \), and uniformly a.e. on \( \Omega \), one gets that

3. there exists \( \xi_1 \leq \xi_0 \) such that each of the problems (1.1) and (1.2) have at least two distinct solutions, whenever \( \xi < \xi_1 \).
4. If in addition \( f \in C(\Omega \times \mathbb{R}) \), then \( \xi_1 = \xi_0 \).

The literature related to Ambrosetti–Prodi-type problems is extensive, and has mostly concentrated in the Dirichlet problem. The motivation of this problem comes from the pioneering paper by Ambrosetti and Prodi [3]. The results in [3] opened the door to many generalizations, and further investigation of problems of this type, but in different frameworks (structures and boundary conditions). In particular, the Ambrosetti–Prodi problem for the \( p \)-Laplace operator and Dirichlet boundary conditions has been considered in [2, 4, 5, 25, 37, 39, 40], among many others. It is important to mention that for the Dirichlet problem, the regularity theory is not addressed, since the domain is assumed to be smooth (and thus the regularity results are standard and known). Complications related to the Dirichlet problem of Ambrosetti–Prodi-type arise when proving the non-existence of solutions, mainly because in this case, the constant functions are not in the corresponding function space \( W_{0}^{1,p}(\Omega) \) where the Dirichlet problem is posed, and also because the first eigenvalue for the Dirichlet problem is strictly positive. To obtain non-existence results for the Dirichlet problem, some key estimates, such as the positivity of the first eigenvalue and Picone’s identity, are required. The smoothness of the domain and the regularity of the solution play a crucial role in the application of such identity.

The Ambrosetti–Prodi problem for other boundary conditions is less known. On smooth domains, the Ambrosetti–Prodi problem with (local) Neumann boundary conditions has been addressed in [19, 20, 44, 46], and recently a generalization of the (local) Neumann problem of Ambrosetti–Prodi type to a large class of non-smooth domains has been considered in [47]. Furthermore, a nonlocal version of the Neumann problem was investigated in [48], where the domain was assumed to be a bounded Lipschitz domain, and in the same paper, the author introduced for the first time the Venttsel’ problem (also denoted by Wentzell problem) of Ambrosetti–Prodi type (on bounded Lipschitz domains). To our knowledge, there is no literature
concerning the Venttsel’ problem of Ambrosetti–Prodi-type, other than the results in [48], where the Venttsel’ operator is given by the $p$-Laplace-Beltrami operator over $\Gamma$. It is important to point out that in this case, unlike the case of the Dirichlet problem, non-existence results are easier to be established, but on the other hand, other results and a priori estimates become much harder to be handled.

In the present paper, we turn our attention to the solvability of the nonlocal Ambrosetti–Prodi type problem with Venttsel’ boundary conditions, on the domain $\Omega$ with fractal-like boundary $\Gamma$, given as in (1.3). Such boundary value problem has never been investigated before, and to our knowledge, there is no literature regarding the nonlocal Venttsel’ problem Ambrosetti–Prodi-type on fractal domains. In fact, the only known results for (local) Ambrosetti–Prodi problems on fractal domains appears in [47] for the Neumann problem. Furthermore, when $\Omega = \Omega_3 \subseteq \mathbb{R}^3$, problem (1.2) is a mixed nonlocal Venttsel’ - Neumann boundary value problem. To our knowledge, there are no results in the literature concerning Ambrosetti–Prodi problems with mixed boundary conditions. We also point out that problems (1.1) and (1.2), have a nonlocal term both in the interior of the domain and on the boundary.

We point out that in order to consider Venttsel’ boundary conditions we introduce a suitable “surrogate” of the $p$-Laplacian on the boundary $\Gamma$ of $\Omega$. This operator has been introduced in the case $p = 2$ in [30, 29, 28], and recently generalized to the quasi-linear case in [16, 27] (for the case when $\Omega = \Omega_2 \subseteq \mathbb{R}^2$). Fractal boundaries and fractal layers are of great interest for those applications in which the surface effects are enhanced with respect to the surrounding volume (see [13, 14, 15] for details and motivations). Therefore, the interpretation of the equations (1.1) and (1.2) in a suitable sense represents substantial results for the theory of quasi-linear boundary value problems, as well as for the analysis on fractal domains.

We outline the plan of the paper. In section 2 we provide a brief construction of the domains under consideration, fix the notations, definitions, and state some intermediate well-know results that will be applied in the subsequent sections. In section 3, we define the $p$-energy functional related to the Venttsel’-type operator on $\Omega_2$ and $\Omega_3$. Section 4 concerns the regularity theory for weak solutions to problems (1.1) and (1.2). Under the general conditions outlined in Assumption 1.3, assuming that the boundary value problems (1.1) and (1.2) are solvable, we perform a priori estimates for weak solutions to both problems, as well as a priori estimates for the difference of weak solutions. As a consequence, we show that weak solutions of both problems (1.1) and (1.2) are globally bounded. For bounded Lipschitz domains and for the particular case $q = p$ (in (1.12)), these a priori estimates have been obtained in [48], and the same results for optimal growth condition (1.12) have been established for the (local) Neumann problem in [47]. Our approach will be inspired by the ones employed in [48, 47] but since we are dealing with more general conditions and assumptions, there are also substantial differences and generalizations in the present problem. Such variants will be addressed in detail. In section 5, we establish an alternative version of a sub-supersolution method for the boundary value problems (1.1) and (1.2), which will be a key tool for the establishment of the main results of the paper. To conclude, in section 6 we prove the main results of the paper, namely, Theorem 1.4 and Theorem 1.5.
2. Preliminaries and intermediate results. In this section we present some important (well-known) definitions, fix the notations that will be carried out in the subsequent sections, and state some known results that will be used in the later sections. All the arguments will be given under the conditions of Assumption 1.3.

2.1. Geometry. We denote by $|P - P_0|$ the Euclidean distance in $\mathbb{R}^n$ and by $B(P_0, r) = \{ P \in \mathbb{R}^n : |P - P_0| < r \}$, $P_0 \in \mathbb{R}^n, r > 0$, the euclidean ball.

By the Koch snowflake $\Gamma_2$, we denote the union of three co-planar Koch curves $K_1, K_2$ and $K_3$. We assume that the junction points $A_1, A_3$ and $A_5$ are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$.

$K_1$ is the uniquely determined self-similar set with respect to a family $\Psi^1$ of four suitable contractions $\psi^{(1)}_1, \ldots, \psi^{(1)}_4$, with respect to the same ratio $\frac{1}{3}$. Let $V^{(1)}_0 := \{A_1, A_3\}$, $\psi_{i_1 \ldots i_h} := \psi_{i_1} \circ \ldots \circ \psi_{i_h}$, $V^{(1)}_{i_1 \ldots i_h} := \psi_{i_1 \ldots i_h}(V^{(1)}_0)$ and

$$V^{(1)}_h := \bigcup_{i_1 \ldots i_h = 1}^4 V^{(1)}_{i_1 \ldots i_h}.$$ 

We set $i|h = (i_1, i_2, \ldots, i_h)$, $V^{(1)}_i := \bigcup_{h \geq 0} V^{(1)}_h$. It holds that $K_1 = \bigcup_{h \geq 0} V^{(1)}_h$. Now let $K_0$ denote the unit segment whose endpoints are $A_1$ and $A_3$. We set $K_{i_1 \ldots i_h} = \psi_{i_1 \ldots i_h}(K_0)$ and $V(K_{i_1 \ldots i_h}) = V_{i_1 \ldots i_h}$. In a similar way, it is possible to approximate $K_2, K_3$ by the sequences $(V^{(2)}_h)_{h \geq 0}$, $(V^{(3)}_h)_{h \geq 0}$, and denote their limits by $V^{(2)}$, $V^{(3)}$.

In order to approximate $\Gamma_2$, we define the increasing sequence of finite sets of points $V^h := \bigcup_{i=1}^3 V^{(i)}_h, h \geq 1$ and $V_* := \bigcup_{h \geq 1} V^h$. It holds that $V_* = \bigcup_{i=1}^3 V^{(i)}_*$ and $\Gamma_2 = \overline{V_*}$.

The Hausdorff dimension of the Koch snowflake is given by $d_2 = \frac{\log(4)}{\log(3)}$.

One can define, in a natural way, a finite Borel measure $\mu|_{\Gamma_2}$ supported on $\Gamma_2$ by

$$\mu|_{\Gamma_2} := \mu_1 + \mu_2 + \mu_3,$$

where $\mu_i$ denotes the normalized $d_2$-dimensional Hausdorff measure, restricted to $K_i$, $i = 1, 2, 3$. 

![Figure 1. Pre-fractal Koch snowflake](image-url)
Further, for any \( n \geq 1 \), we define a discrete measure on \( V^{(i)}_n \) by:

\[
\mu_n^i := \frac{1}{4^n} \sum_{p \in V^{(i)}_n} \delta_{\{p\}},
\]

where \( \delta_{\{p\}} \) denotes the Dirac measure at the point \( p \). We have the following result:

**Proposition 2.1.** (see [31]) The sequence \( (\mu_n^i)_{n \geq 1} \) is weakly convergent (i.e. in \( C(K_i') \)) to the measure \( \mu_i \).

The measure \( \mu|_{\Gamma_2} \) is a \( d_2 \)-measure (see [24]), that is, there exist two positive constants \( c_1, c_2 \), such that

\[
c_1 r^{d_2} \leq \mu|_{\Gamma_2}(B(P, r) \cap \Gamma_2) \leq c_2 r^{d_2}, \forall P \in \Gamma_2
\]

where \( d_2 = \frac{\log(4)}{\log(3)} \).

In the following we denote by

\[
F_{h+1} = \bigcup_{i=1}^3 K_i^{(h+1)}
\]

the closed polygonal curve approximating \( \Gamma_2 \) at the \((h+1)\)-th step. We define \( S_h = F_h \times I \), where \( I = [0, 1] \). By \( \Omega_h \subset \mathbb{R}^2 \) we denote the open bounded set having as boundary \( F_h \). We denote by \( Q_h \) the three-dimensional cylindrical domain having \( S_h \) as “lateral surface” and the sets \( \Omega_h \times \{0\} \) and \( \Omega_h \times \{1\} \) as bases.

In an analogous way, we define the cylindrical-type surface \( \Gamma_3 = \Gamma_2 \times I \) and we denote by \( \Omega_2 \) the open bounded two-dimensional domain with boundary \( \Gamma_2 \). Also, by \( \Omega_3 \) we denote the open cylindrical domain having \( \Gamma_3 \) as lateral surface, and the sets \( \Omega_2 \times \{0\} \) and \( \Omega_2 \times \{1\} \) as bases. We denote the points of \( \Gamma_3 \) and \( S_h \) by the couple \( P = (x, y) \), where \( x = (x_1, x_2) \) are the coordinates of the orthogonal projection of \( P \) on the plain containing \( \Gamma_2 \) and \( F_h \) respectively (for \( \Gamma_3 \) and \( S_h \)) and \( s \) is the coordinate of the orthogonal projection of \( P \) on the interval \((0, 1)\), that is \((x_1, x_2) \in \Gamma_2 \) (or \((x_1, x_2) \in F_h \) for the pre-fractal case) and \( s \in I \).

We introduce on \( \Gamma_3 \) the measure

\[
d|_{\Gamma_3} = d\mu|_{\Gamma_2} \times ds,
\]

where \( ds \) is the one-dimensional Lebesgue measure on \( I \). It follows that the measure \( \mu|_{\Gamma_3} \) is also a \( d_3 \)-measure for \( d_3 := 1 + \frac{\log(4)}{\log(3)} \). Then we set

\[
d := \begin{cases} 
  d_2, & \text{if } N = 2, \\
  d_3, & \text{if } N = 3,
\end{cases}
\]

and \( \mu \) is considered as in 1.3.

**2.2. Functional spaces.** By \( W^{1,p}(\Omega) \) we mean the well-known \( L^p \)-based Sobolev space. Also, given \( E \subset \mathbb{R}^m \) a \( d \)-set in the sense of [24], and given \( \beta \in (0, 1) \), \( d \in (0, N] \), we define the Besov space \( \mathbb{B}^\beta_p(E, \eta) \) as the set of functions \( u \in L^p(E, d\eta) \) for which the semi-norm

\[
\mathcal{N}_{\beta}^p(u, E, \eta) := \left( \int_E \int_E \left[ \frac{|u(x) - u(y)|}{|x - y|^{\beta + \frac{d}{2}}} \right]^p \, d\eta_x d\eta_y \right)^{\frac{1}{p}}
\]

(2.6)
is finite. Additionally, for \( r, s \in [1, \infty) \), or \( r = s = \infty \), we will often refer at times to the Banach Space defined in [50], i.e.,

\[
X^{r,s}(\Omega, \Gamma) := \{(f, g) : f \in L^r(\Omega), \ g \in L^s(\Gamma)\},
\]

endowed with norm

\[
\| (f, g) \|_{r,s} := \| f \|_{r,\Omega} + \| g \|_{s,\Gamma}, \quad \text{if} \ r, s \in [1, \infty),
\]

and

\[
\| (f, g) \|_{\infty} := \max \{ \| f \|_{\infty,\Omega}, \| g \|_{\infty,\Gamma} \}.
\]

We stress that when \( r = s \), we will write \( X^r(\Omega, \Gamma) := X^{r,r}(\Omega, \Gamma) \) and \( \| (f, g) \|_{r} := \| (f, g) \|_{r,r} \), respectively. For more information and properties regarding the Sobolev spaces under the definitions above, refer to [7, 9, 24, 38, 43] (among others). For completeness, we quote the following properties (e.g. [7, 18, 23, 32, 24, 51, 52, 53]).

\begin{itemize}
  \item \( W^{1,p}(\Omega) \hookrightarrow L^{\frac{pN}{N-p}}(\Omega) \), and hence there exists a constant \( c_1 > 0 \), such that

  \[
  \| u \|_{\frac{pN}{N-p},\Omega} \leq c_1 \| u \|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{2.7}
  \]

Moreover, if \( 1 < r < \frac{pN}{N-p} \), then the embedding \( W^{1,p}(\Omega) \hookrightarrow L^r(\Omega) \) is compact.

  \item There exists a linear continuous operator from \( W^{1,p}(\Omega) \) into \( L^{\frac{pd}{N-p}}(\Gamma) \), and thus there exists a constant \( c_2 > 0 \), such that

  \[
  \| u \|_{\frac{pd}{N-p},\Gamma} \leq c_2 \| u \|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{2.8}
  \]

Moreover, if \( 1 < s < \frac{pd}{N-p} \), then the operator from \( W^{1,p}(\Omega) \) into \( L^s(\Gamma) \) is compact.

  \item If \( p = N \), then \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) and the trace of \( W^{1,p}(\Omega) \) lies in \( L^r(\Gamma) \) continuously, for all \( q, r \in [1, \infty) \), while in the case \( p > N \), we have that \( W^{1,p}(\Omega) \hookrightarrow C^{0,\delta}(\Omega) \) for some \( \delta \in (0,1) \).

  \item There exists a constant \( c_3 > 0 \) such that

  \[
  \| u \|_{W^{1,p}(\Omega)} \leq c_3 \| \nabla u \|_{p,\Omega}, \tag{2.9}
  \]
for every \( u \in W^{1,p}(\Omega) \) such that \( \int_{\Omega} u \, dx = 0 \) for a measurable set \( E \subseteq \Omega \) with \( \lambda_N(E) > 0 \).

- If \( p \in (1, N) \) and \( s \in (0, 1) \), then there exists a linear continuous mapping from \( W^{1,p}(\Omega) \) into \( B^s_p(\Omega) \) and a constant \( c_4 > 0 \) such that
  \[
  \|u\|_{B^s_p(\Omega)} \leq c_4 \|u\|_{W^{1,p}(\Omega)}, \quad \text{for all } u \in W^{1,p}(\Omega). \tag{2.10}
  \]

- If \( p \in (1, N) \), then there exists a linear continuous mapping from \( W^{1,p}(\Omega) \) into \( B^{p/(N-p)}_1(\Gamma, \mu) \) and a constant \( c_5 > 0 \) such that
  \[
  \|u\|_{B^{p/(N-p)}_1(\Gamma, \mu)} \leq c_5 \|u\|_{W^{1,p}(\Omega)}, \quad \text{for all } u \in W^{1,p}(\Omega). \tag{2.11}
  \]

We complete this section by collecting some well-known analytical results that will be applied throughout the subsequent section.

**Lemma 2.2.** (see [41]) Let \( \varphi = \varphi(t) \) be a nonnegative, non-increasing function on a half line \( \{t \geq k_0 \geq 0\} \), such that there exist \( c, \alpha > 0 \), and there exists \( \delta > 1 \) with

\[
\varphi(h) \leq c (h - k)^{-\alpha} \varphi(k)^\delta,
\]

for \( h > k \geq k_0 \). Then

\[
\varphi(k_0 + \varsigma) = 0,
\]

where

\[
\varsigma^\alpha = c \varphi(k_0)^{\delta - 1} 2^{\alpha(\delta - 1)}.
\]

**Lemma 2.3.** (see [26]) Let \( u \) be an integrable function over a bounded open set \( D \), such that for arbitrary \( k \geq k_0 > 0 \),

\[
\int_D (u - k) \chi_{\{u > k\}} \, dx \leq \gamma k^\alpha \lambda_N(\{x \in D \mid u(x) > k\})^{1 + \epsilon},
\]

where \( \gamma, \alpha, \epsilon \) are constants such that \( \epsilon > 0 \) and \( 0 \leq \alpha \leq 1 + \epsilon \). Then there exists a constant \( C \), depending on \( \gamma, \alpha, \epsilon, k_0 \), and \( \|u\chi_{\{u > k\}}\|_{1,D} \), such that \( \|u\|_{\infty,D} \leq C \).

**Proposition 2.4.** (see [6]) Let \( a, b \in \mathbb{R}^N \) and \( r \in (1, \infty) \). Then there exists a constant \( c_r > 0 \) such that

\[
(|a|^{r-2}a - |b|^{r-2}b) \cdot (a - b) \geq c_r (|a| + |b|)^{r-2} |a - b|^2 \geq 0. \tag{2.12}
\]

If in addition \( r \in [2, \infty) \), then there exists a constant \( c_r^* \in (0, 1] \) such that

\[
(|a|^{r-2}a - |b|^{r-2}b) \cdot (a - b) \geq c_r^* |a - b|^r, \tag{2.13}
\]

and also in this case there is a constant \( c_r^* \in (0, 1] \) such that

\[
\text{sgn}(a - b) \left(|a|^{r-2}a - |b|^{r-2}b\right) \geq c_r^* |a - b|^{r-1}. \tag{2.14}
\]

Finally, if \( r \in (1, 2] \) and \( \epsilon > 0 \), then for each \( a, b \in \mathbb{R}^N \) with \( |a - b| \geq \epsilon \min\{|a|, |b|\} \), we find a constant \( c_{r, \epsilon} > 0 \) such that

\[
\langle |a|^{r-2}a - |b|^{r-2}b, a - b \rangle \geq c_{r, \epsilon} |a - b|^r, \tag{2.15}
\]

and

\[
\text{sgn}(a - b) (|a|^{r-2}a - |b|^{r-2}b) \geq c_{r, \epsilon} |a - b|^{r-1}. \tag{2.16}
\]
Proposition 2.5. (see [6]) Let $\xi, \zeta, c, \tau, \varphi \in [0, \infty)$ and $r \in [1, \infty)$, and assume that the parameter $\xi \leq \tau c^{-\zeta} + \varphi$ for all $\varepsilon > 0$. Then one has
\[
\xi \leq (\tau + 1) \left[ \frac{r}{r+\varepsilon} \right] c^{-\frac{r}{r+\varepsilon}} + \zeta. \tag{2.17}
\]

3. The $p$-forms on Koch-type domains. In [11] the $p$-energy forms on the Koch curve have been constructed. For $f : V^{(i)}_* \to \mathbb{R}$, we define for $1 < p < \infty$
\[
E^{(p)}_{n,i}[f] = \frac{1}{p} 4^{(p-1)n} \sum_{i_1, \ldots, i_n=1}^{4} \|f|_{V^{(i)}_0} \|_p \sum_{\xi, \eta \in V^{(i)}_0} |f(\psi_{i_1 \ldots i_n}(\xi)) - f(\psi_{i_1 \ldots i_n}(\eta))|^p, \tag{3.1}
\]

it has been shown that the sequence $E^{(p)}_{n,i}[f]$ is non-decreasing, and by defining for $f : V^{(i)}_* \to \mathbb{R}$
\[
E^{(p)}_i[f] = \lim_{n \to \infty} E^{(p)}_{n,i}[f], \tag{3.2}
\]

the set
\[
E^{(p)}_* = \{ f : V^{(i)}_* \to \mathbb{R} : E^{(p)}_i[f] < \infty \} \tag{3.3}
\]
does not degenerate to a space containing only constant functions. Each $f \in E^{(p)}_*$ can be uniquely extended in $C(K_i)$. We denote this extension on $K_i$ still by $f$ and we define the space
\[
D(E^{(p)}_i) = \{ f \in C(K_i) : E^{(p)}_i[f] < \infty \}, \tag{3.4}
\]

where $E^{(p)}_i[f] := E^{(p)}_i[f]_{V_i}$. Hence $D(E^{(p)}_i) \subset C(K_i) \subset L^p(K_i, \mu_i)$. Moreover, $(E^{(p)}_i, D(E^{(p)}_i))$ is a non-negative energy functional in $L^p(K_i, \mu_i)$ and the following result holds.

Theorem 3.1. (see [11]) The following properties hold.

(i) $D(E^{(p)}_i)$ is complete in the norm \( \|f\|_{D(E^{(p)}_i)} := \|f\|_{L^p(K_i, \mu_i)} + (E^{(p)}_i[f])^{1/p} \).

(ii) $D(E^{(p)}_i)$ is dense in $L^p(K_i, \mu_i)$.

(iii) $D(E^{(p)}_i) \subset D(E^{(p)}_i)$, for $1 < p \leq q < \infty$.

3.1. $p$-Lagrangians on the Koch curve and on the Koch snowflake. In this subsection, we recall the main properties of the $p$-Lagrangian on the Koch curve. For the concept of Lagrangians on fractals, i.e. the notion of a measure valued local energy, we refer to [22] and [42].

We also have the following:

Proposition 3.2. (see [10]) Let $A$ be any subset of $K_i$. For every $u \in D(E^{(p)}_i)$, the sequence of measures given by
\[
\tilde{E}^{(p)}_{n,i}(u)(A) := \frac{4^{(p-1)n}}{p} \sum_{i_1, \ldots, i_n=1}^{4} \sum_{\xi, \eta \in V^{(i)}_0} |u(\psi_{i_1 \ldots i_n}(\xi)) - u(\psi_{i_1 \ldots i_n}(\eta))|^p, \tag{3.5}
\]
converges to a positive semidefinite additive Borel measure
\[ \tilde{E}_i^{(p)}(u)(A) := \lim_{n \to \infty} E_{n,i}^{(p)}(u)(A) \]
the so-called \( p \)-Lagrangian measure on \( K_1 \). Moreover it holds that for every \( p > 1 \),
- \( \tilde{E}_i^{(p)} \) is positive semidefinite and convex
- \( \tilde{E}_i^{(p)} \) is homogeneous of degree \( p \)
- \( \tilde{E}_i^{(p)} \) is such that \( \| u \| = (\int_{K_1} |u|^p d\mu + \int_{K_1} d\tilde{E}_i^{(p)}(u))^{1/p} \) is a norm in \( D(\tilde{E}_i^{(p)}) \)
- for every \( u, v \in D(\tilde{E}_i^{(p)}) \) there exists in the weak * topology of \( M \) (the set of Radon measures) the following limit
\[ \lim_{t \to 0} \frac{\tilde{E}_i^{(p)}(u + tv) - \tilde{E}_i^{(p)}(u)}{t} = \left\langle \partial \tilde{E}_i^{(p)}(u), v \right\rangle \]
Moreover, it holds that
\[ \tilde{E}_i^{(p)}(u, v) = \int_{K_1} d\tilde{E}_i^{(p)}(u, v), \quad u, v \in D(\tilde{E}_i^{(p)}) \]
where we set \( \mathcal{L}_i^{(p)}(u, v) : D(\tilde{E}_i^{(p)}) \times D(\tilde{E}_i^{(p)}) \to M \), with
\[ \mathcal{L}_i^{(p)}(u, v) = \left\langle \partial \tilde{E}_i^{(p)}(u), v \right\rangle . \]

We note that by straightforward calculations it holds that:
\[ \left\langle \tilde{E}_{n,i}^{(p)}(u), v \right\rangle = \frac{4^{(p-1)n}}{p} \sum_{i_1, \ldots, i_n} \sum_{\xi, \eta \in \nu_n \iota} |u(\psi_{i_1 \ldots i_n}(\xi)) - u(\psi_{i_1 \ldots i_n}(\eta))|^p - 2 \]
\[ (u(\psi_{i_1 \ldots i_n}(\xi)) - u(\psi_{i_1 \ldots i_n}(\eta)))(v(\psi_{i_1 \ldots i_n}(\xi)) - v(\psi_{i_1 \ldots i_n}(\eta))), \]
so proceeding as in \([10]\) we have
\[ \lim_{n \to \infty} \left\langle \tilde{E}_{n,i}^{(p)}(u), v \right\rangle = \left\langle \tilde{E}_i^{(p)}(u), v \right\rangle . \] (3.9)

Finally by proceeding as in \([21, Section 4.1-4.2]\) one can define on \( \Gamma_2 \) a \( p \)-Lagrangian \( (\mathcal{L}^{(p)}, D(\mathcal{L}^{(p)})) \) and a \( p \)-energy form \( (\mathcal{E}^{(p)}, D(\mathcal{E}^{(p)})) \). More precisely \( \mathcal{E}^{(p)}(u, v) = \int_{\Gamma_2} d\mathcal{L}^{(p)}(u, v) \), for every \( u, v \in D(\mathcal{E}^{(p)}) \) where \( D(\mathcal{E}^{(p)}) = \{ u \in C(\Gamma_2) : u|_{K_i} \in D(\tilde{E}_i^{(p)}) \} \), for \( i = 1, 2, 3 \). In particular it holds (see \([21, Theorem 4.6]\)) that
\[ \mathcal{E}^{(p)}[u] = \sum_{i=1}^{3} \mathcal{E}_i^{(p)}[u|_{K_i}] \] (3.10)

We recall that from \([12, Theorem 4.1]\), it follows that \( D(\mathcal{E}^{(p)}) \) can be characterized in terms of Lipschitz spaces with equivalent norms:
\[ D(\mathcal{E}^{(p)}) = Lip_{d_\Gamma, \iota}(p, \infty, \Gamma_2), \]
\[ D(\mathcal{E}^{(p)}) \subset B^p_\alpha(\Gamma_2, \mu|_{\Gamma_2}), \quad \text{for every } \alpha < d. \] (3.11)

We now define the energy form on \( \Gamma_3 \):
\[ E_{\gamma_3}[u] = \frac{1}{p} \int_I \mathcal{E}^{(p)}[u] ds + \frac{1}{p} \int_{\Gamma_2} \int_I |\partial_s u|^p ds d\mu|_{\gamma_3} \] (3.12)
with domain \( \mathcal{D}(\Gamma_3) \) defined as
\[ D(\Gamma_3) = \mathcal{C}(\Gamma_3) \cap L^p([0,1];D(\mathcal{E}^{(p)})) \cap W^{1,p}([0,1];L^p(\Gamma_2)) \|^1_{D(\Gamma_3)}, \]  
where \( \| \cdot \|_{D(\Gamma_3)} \) is the intrinsic norm
\[ \|u\|_{D(\Gamma_3)} = \left( E_{\Gamma_3}[u] + \|u\|^p_{L^p(\Gamma)} \right)^{\frac{1}{p}}. \]  

We now give an embedding result for the domain \( D(\Gamma_3) \). Unlike the two-dimensional case where there is a characterization of the functions in \( D(E_p) \) in terms of the so-called Lipschitz space, for \( D(\Gamma_3) \) we do not have such characterization, but the following result holds.

**Proposition 3.3.** (see [17]) \( D(S) \hookrightarrow \mathbb{B}_\beta^p(\Gamma_3, \mu|_{\Gamma_3}) \) for any \( 0 < \beta < 1 \).

From the above results it follows in particular that the spaces \( V_p(\bar{\Omega}_2) \) and \( V_p(\bar{\Omega}_3) \) are non trivial. Finally, we recall that for the remaining of the paper, we will consider the sets \( \Omega, \Gamma \), and the measure \( \mu \) on \( \Gamma \), in accordance with (1.3).

4. **Global regularity for weak solutions.** The following section is devoted to establish global regularity results for weak solutions of both Eq. (1.1) and (1.2), under the conditions of Assumption 1.3.

Given \( u \in W_p(\bar{\Omega}) \) and \( k \geq 0 \) a fixed real number, we put
\[ u_k := (u-k)^+, \quad u_k^- := (u^- - k)^+, \quad \text{and} \quad \hat{u}_k := (|u| - k)^+ \text{sgn}(u). \]  

Clearly \( u_k, u_k^-, \hat{u}_k \in W_p(\bar{\Omega}) \). Set
\[ A_k := \{ x \in \bar{\Omega} \mid |u| > k \}, \quad A_k^+ := \{ x \in \bar{\Omega} \mid u > k \}, \quad A_k^- := \{ x \in \bar{\Omega} \mid u^- > k \}. \]  

Then, for each \( D \subseteq \mathbb{R}^N \) such that \( D \cap \bar{\Omega} \neq \emptyset \), taking into account (4.1) and (4.2), we write
\[ D_k := D \cap A_k, \quad D_k^+ := D \cap A_k^+, \quad D_k^- := D \cap A_k^- . \]  

We stress that throughout the remaining of the paper, the measurable set \( D \) will be either the interior \( \Omega \), or the boundary \( \Gamma := \partial \Omega \). Finally, we put
\[ p_N := \frac{Np}{N-p} \quad \text{and} \quad p_d := \frac{pd}{N-p}. \]  

Next, in view of the above notations, we now derive \( L^\infty \)-type estimates for weak solutions of both Eq. (1.1) and Eq. (1.2), and a priori estimates for the difference of weak solutions of problem (1.1), and problem (1.2).

**Proposition 4.1.** Let \( u \in W_p(\bar{\Omega}) \). Then it holds that \( \mathcal{E}^{(p)}(u, u_k^-) \leq 0 \) and \( E_S(u, u_k^-) \leq 0 \).

**Proof.** Since
\[ \mathcal{E}^{(p)}(w, v) := \int_{\Gamma_2} d\mathcal{E}^{(p)}(w, v) = \left< \partial \mathcal{L}^{(p)}(w), v \right> = \lim_{n \to \infty} \left< \partial \mathcal{L}^{(p)}_n(w), v \right> , \]  
for \( w, v \in D(\mathcal{E}^{(p)}) \), where
\[ \langle \partial E_n^{(p)}(u), v \rangle := \frac{1}{p} \sum_{k=1}^{4} \sum_{i=1}^{n} |w(\psi_{k, i}(\xi)) - w(\psi_{k, i}(\xi))|^{p-2} \times (w(\psi_{k, i}(\xi)) - w(\psi_{k, i}(\xi))) v(\psi_{k, i}(\xi)) - v(\psi_{k, i}(\xi)) \mu_n, \]  

for \( \kappa_i := i_1, \ldots, i_n \). Then, one sees easily that \( \langle \partial E_n^{(p)}(u), u_k^- \rangle \leq 0 \) for each \( n \in \mathbb{N} \), and thus passing to the limit, we deduce that \( E_n^{(p)}(u, u_k^-) \leq 0 \). From the results for the two dimensional case and the monotonicity of the integral it follows that \( E_{c_1}(u, u_k^-) \leq 0 \).

For \( u, v \in W_p(\Omega) \), we define

\[ K^{(p)}(u, v) := \int_I [E^{(p)}(u, v)] dt + \sigma_N \int_{\Gamma_2} |\partial u|^{p-2} uv dt d\mu, \]

for \( \sigma_N \) defined as in (1.18). Then, (1.20) becomes

\[ \Lambda_p(u, v) := \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx + K^{(p)}(u, v) + (\Theta_\eta u)v + (\Theta_\tau u)v \]

Then we have the following key result.

**Theorem 4.2.** Suppose that \( \xi \) belongs to a bounded interval. If \( u \in W_p(\Omega) \) is a weak solution of either (1.1), or (1.2), then there exists a constant \( C_\xi \geq 0 \) large enough (and independent of \( u \)) such that \( \|u\|_{\infty, \Omega} \leq C_\xi \).

**Proof.** We will prove both cases at the same time. Indeed, let \( u \in W_p(\Omega) \) be a weak solution of either (1.1), or (1.2). We will show that the \( L^\infty \)-norm of both \( u^- \) and \( u^+ \) are bounded by \( C_\xi \).

- First we show that \( \|u^-\|_{\infty, \Omega} \leq C_\xi \) for \( C_\xi \) large enough. Indeed, given \( k > 0 \) a real number, let \( u_k^- \in W_p(\Omega) \) be the function defined in (4.1). It is easy to see that \( (\Theta_\eta u) u_k^- \leq 0 \) and \( (\Theta_\tau u) u_k^- \leq 0 \) (recall that \( \Theta_\eta \) and \( \Theta_\tau \) are defined as in (1.5) and (1.6), respectively). From Proposition 4.1 and the definition of \( K^{(p)}(\cdot, \cdot) \) in (4.7), it follows that \( K^{(p)}(u, u_k^-) \leq 0 \). By testing (1.19) with \( u_k^- \) and applying (1.10), we obtain

\[ \|\nabla u_k^-\|_{p, \Omega_k}^p \leq -\Lambda_p(u, u_k^-) = - \int_{\Omega_k^-} f(x, u)u_k^- dx - \int_{\Omega_k^-} (\xi \phi + h)u_k^- dx \]

\[ \leq \eta_0 \int_{\Omega_k^-} (u^- - k)^p dx + (\eta_0 k^{p-1} + M + \|\phi\|_{\infty, \Omega} + \|h\|_{\infty, \Omega}) \int_{\Omega_k^-} (u^- - k) dx. \]

Now, by Hölder’s inequality together with (2.7), one sees that

\[ \int_{\Omega_k^-} (u^- - k)^p dx \leq \lambda_N(\Omega_k^-) \frac{p}{p-1} \|\nabla u_k^-\|_{p, \Omega_k}^p + \int_{\Omega_k^-} (u^- - k)^p dx. \]  

(4.8)

Hence, combining (4.8) with the first estimate, and rearranging, yields that

\[ \left( \lambda_N(\Omega_k^-)^{\frac{p}{p-1}} - c_1(1 + \eta_0) \right) \int_{\Omega_k^-} (u^- - k)^p dx \]
\[ \leq c_1 \left( \eta_0 k^{p-1} + M + |\xi||\phi|_{\infty,\Omega} + \|h\|_{\infty,\Omega} \right) \int_{\Omega_k^-} (u^- - k) \, dx. \]  

(4.9)

We note that given \( u \in \mathcal{W}_p(\Omega) \) a weak solution of either (1.1), or (1.2) we can test (1.19) with the function \( w \equiv 1 \). Then, we have \( (\Theta_\alpha u)1 = (\Theta_r u)1 = 0 \), and also in view of (4.5) and (4.6), we deduce that \( \mathcal{E}(p)(u,1) = 0 \). Hence, applying (1.9) and (1.10), we are lead to the following:

\[ 0 = \int_{\Omega} (f(x,u) + \xi \phi + h) \, dx \geq \eta_0 \int_{\Omega} |u|^{p-1} \, dx - ((|\xi||\phi|_{\infty,\Omega} + \|h\|_{\infty,\Omega} + M)\lambda_N(\Omega), \]

which implies that

\[ \int_{\Omega} |u|^{p-1} \, dx \leq \frac{|\xi||\phi|_{\infty,\Omega} + \|h\|_{\infty,\Omega} + M}{\eta_0} \lambda_N(\Omega) := C_\Omega. \]  

(4.10)

On the other hand, since

\[ \int_{\Omega} |u|^{p-1} \, dx \geq \int_{\Omega_k^-} (u^-)^{p-1} \, dx \geq k^{p-1}\lambda_N(\Omega_k^-), \]

taking into account (4.10) we get

\[ \lambda_N(\Omega_k^-) \leq \frac{1}{k^{p-1}} \int_{\Omega} |u|^{p-1} \, dx \leq \frac{C_\Omega}{k^{p-1}}. \]  

(4.11)

Therefore, (4.11) implies that \( \lim_{k \to \infty} \lambda_N(\Omega_k^-) = 0 \), and thus we can find a constant \( k_0 > 0 \) sufficiently large (and independent of \( u \)), such that

\[ \lambda_N(\Omega_k^-)^{-\frac{\theta}{\nu}} - c_1(1 + \eta_0) \geq \frac{\lambda_N(\Omega_k^-)^{-\frac{\theta}{\nu}}}{2}, \]

for all \( k \geq k_0 \).  

(4.12)

By H"older’s inequality and (4.9) one has

\[ \int_{\Omega_k^-} (u^- - k) \, dx \leq \lambda_N(\Omega_k^-)^{\frac{1}{p}} \left( \frac{c_1 (\eta_0 k^{p-1} + M + |\xi||\phi|_{\infty,\Omega} + \|h\|_{\infty,\Omega})}{\lambda_N(\Omega_k^-)^{\frac{1}{p}} - c_1(1 + \eta_0)} \right)^{1/p} \int_{\Omega_k^-} (u^- - k) \, dx, \]

and thus combining this inequality with (4.12) we have

\[ \int_{\Omega_k^-} (u^- - k) \, dx \leq \lambda_N(\Omega_k^-)^{1 + \frac{\theta}{p}} k \left[ \frac{c_1}{2} \left( \eta_0 + \frac{M + |\xi||\phi|_{\infty,\Omega} + \|h\|_{\infty,\Omega}}{k^{p-1}} \right) \right]^{1/(p-1)}, \]

(4.13)

From this last calculation together with Lemma 2.3 we obtain that \( \|u^-\|_{\infty,\Omega_k^-} \) is bounded by a constant depending on \( p, N, \eta_0, M, \xi, ||\phi||_{\infty,\Omega}, ||h||_{\infty,\Omega} \), and \( \|u^-\|_{1,\Omega_k^-} \).

Furthermore, examining the proof of Lemma 2.12 (see [26, proof of Lemma 5.1]), it remains to find an upper bound for \( \|u^-\|_{1,\Omega_k^-} \), independent of \( k \) and \( u \). To
achieve this, test (1.19) with $v = -u^-$. Then we see that in this case $(\Theta_\gamma u)v \geq 0$, $(\Theta_{\eta} u)v \geq 0$, and $K^{(p)}(u,v) \geq 0$. Hence, recalling (1.10), and applying Young's inequality, we deduce that for all $\epsilon > 0$,

$$0 \leq \|\nabla u^-\|_{p,\Omega}^p + \mathcal{E}^{(p)}(u^-, u^-) \leq \Lambda_p(u, v) = -\int_{\Omega^-} (f(x,u) + \xi \phi + h) u^- dx$$

$$\leq -\eta_0 \int_{\Omega^-} (u^-)^p dx + (M + \|\xi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}) \int_{\Omega^-} u^- dx$$

$$\leq -\eta_0 \int_{\Omega^-} (u^-)^p dx + \frac{(M + \|\xi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega})^p}{\epsilon^{\gamma/(p-1)}} \lambda_N(\Omega) + \epsilon \int_{\Omega^-} (u^-)^p dx.$$

Letting $\epsilon = \eta_0/2 > 0$ and rearranging, one sees that

$$\int_{\Omega^-} (u^-)^p dx \leq \left(\frac{2(M + \|\xi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega})}{\eta_0}\right)^{\frac{1}{p}} \lambda_N(\Omega),$$

and thus (4.14) gives

$$\|u^-\|_{1,\Omega^-} \leq \lambda_N(\Omega)^{\frac{1}{p}} \left(\int_{\Omega^-} (u^-)^p dx\right)^{\frac{1}{p}} \leq \left(\frac{2(M + \|\xi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega})}{\eta_0}\right)^{\frac{1}{p}} \lambda_N(\Omega),$$

proving the desired conclusion.

Next, we show that $\|u^+\|_{\infty,\Omega} \leq C_\xi$. Given $k > 0$ a real number, let $u_k^+ \in W^{1,p}(\Omega)$ be the function defined in (4.1). Proceeding as in [50, proof of Theorem 5.7] and recalling (4.5) and (4.6), we see that $(\Theta_\xi u)u_k^+ \geq 0$, $(\Theta_{\eta} u)u_k^+ \geq 0$, and $K^{(p)}(u, u_k^+) \geq 0$. Furthermore, as in the previous case, one can easily deduce that $\lim_{k \to \infty} \lambda_N(\Omega_k^+) = 0$. Then, testing (1.19) with $u_k^+$ and applying (1.12), we obtain

$$\|\nabla u_k^+\|_{p,\Omega_k^+}^p \leq \Lambda_p(u, u_k^+) = \int_{\Omega_k^+} f(x,u) u_k^+ dx + \int_{\Omega_k^+} (\xi \phi + h) u_k^+ dx$$

$$\leq \gamma_0 \int_{\Omega_k^+} (u - k)^q dx + \gamma_0 (k^{q-1} + 1) \|\xi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}) \int_{\Omega_k^+} (u - k) dx.$$  

At this point, it suffices to consider the case when $q \geq p$. For if $1 < q < p$, then in view of (1.12), by virtue of Young's inequality, there exist constants $\gamma_0^* > 0$ and $q^* > p > q$ such that

$$f(x,s) \leq \gamma_0 (1 + |s|^{q-1}) \leq \gamma_0^* (1 + |s|^{q^*-1}), \quad \text{for all} \quad s \in \mathbb{R}, \quad \text{and uniformly for} \quad x \in \Omega.$$ 

Then, since $\lambda_N(\Omega_k^+) \xrightarrow{k \to \infty} 0$ (proof similar as previous case), we select $k_1 > 0$ large enough, such that $\|\nabla u_k^+\|_{p,\Omega_k^+} \leq 1$ for all $k \geq k_1$. Thus for each $k \geq k_1$, we have

$$\|\nabla u_k^+\|_{q,\Omega_k^+}^q \leq \|\nabla u_k^+\|_{p,\Omega_k^+}^p \quad (q \geq p).$$  

(4.17)
Applying Hölder’s inequality and recalling (2.7), one sees that
\begin{align}
\int_{\Omega_k^+} (u-k)^q \, dx & \leq \lambda_N(\Omega_k^+) \left( \frac{q(N-p)}{Np} \right) - c_1 \left( \lambda_N(\Omega_k^+) \right)^{\frac{q}{q-1}} + \lambda_N(\Omega_k^+) \left( \frac{q}{q-1} \right) \int_{\Omega_k^+} (u-k)^q \, dx.
\end{align}

In view of (4.18), (4.16) and (4.17), we have
\begin{align}
\left( \lambda_N(\Omega_k^+) \right)^{\frac{q(N-p)}{Np} - 1} - c_1 \left( \lambda_N(\Omega_k^+) \right)^{\frac{q}{q-1}} + \gamma_0 \right) \int_{\Omega_k^+} (u-k)^q \, dx
& \leq c_1 \left( \gamma_0(kq-1) + \|\phi\|_{\infty,\omega} + \|h\|_{\infty,\omega} \right) \int_{\Omega_k^+} (u-k) \, dx.
\end{align}

In the same way as before, we can find a constant $k_2 > 0$ sufficiently large (and independent of $u$), such that
\begin{align}
\lambda_N(\Omega_k^+) \left( \frac{q(N-p)}{Np} - 1 \right) - c_1 \left( \lambda_N(\Omega_k^+) \right)^{\frac{q}{q-1}} + \gamma_0 \right) \geq \frac{\lambda_N(\Omega_k^+) \left( \frac{q(N-p)}{Np} - 1 \right)}{2}, \quad \text{for all } k \geq k_2.
\end{align}

Also, we select $k_3 > 0$ such that
\begin{align}
\lambda_N(\Omega_k^+) \leq \frac{1}{2} \lambda_N(\Omega) \quad \text{and} \quad \lambda_N(\Omega_k) \leq \frac{1}{2} \lambda_N(\Omega), \quad \text{for all } k \geq k_3.
\end{align}

Then, for each $k \geq k_0 := \max\{k_1, k_2, k_3\}$, we apply Hölder’s inequality in (4.19) to obtain that
\begin{align}
\int_{\Omega_k^+} (u-k) \, dx & \leq \lambda_N(\Omega_k^+) \left( \frac{q(N-p)}{Np} - 1 \right) - c_1 \left( \lambda_N(\Omega_k^+) \right)^{\frac{q}{q-1}} + \gamma_0 \right) \int_{\Omega_k^+} (u-k) \, dx \right)^{1/q}.
\end{align}

Combining (4.22) with (4.20) yields that
\begin{align}
\int_{\Omega_k^+} (u-k) \, dx & \leq \lambda_N(\Omega_k^+) \left( \frac{q(N-p)}{Np} - 1 \right) - c_1 \left( \lambda_N(\Omega_k^+) \right)^{\frac{q}{q-1}} + \gamma_0 \right) \int_{\Omega_k^+} (u-k) \, dx \right)^{1/(q-1)},
\end{align}

where we recall that the constant $p_N$ is given by (4.4). This last calculation together with Lemma 2.3 entail that $\|u^+\|_{\infty,\omega}$ is bounded by a constant depending on $p, N, \gamma_0, \xi, \|\phi\|_{\infty,\omega}, \|h\|_{\infty,\omega}$, and $\|u^+\|_{1,\omega}$. To complete the proof, we must bound $\|u^+\|_{1,\omega}^+$ by a constant independent of $u$. In fact, noticing that $k_0 := \max\{k_1, k_2, k_3\}$, using (4.17) and (1.12), and applying Hölder’s inequality together with Young’s inequality, we have the following calculation:
\begin{align}
\|\nabla u_{k_0}^+\|_{p,\Omega_{k_0}^+}^p & \leq \|\nabla u_{k_0}^+\|_{p,\Omega_{k_0}^+}^p \leq \Lambda_p(u, u_{k_0}^+)
= \int_{\Omega_{k_0}^+} f(x,u) u_{k_0}^+ \, dx + \int_{\Omega_{k_0}^+} (\xi \phi + h) u_{k_0}^+ \, dx
\leq \gamma_0 \int_{\Omega_{k_0}^+} (u-k_0)^{q-1} \, dx + \left( \gamma_0 + \|\phi\|_{\infty,\omega} + \|h\|_{\infty,\omega} \right) \int_{\Omega_{k_0}^+} (u-k_0) \, dx.
\end{align}
Let Assumption 1.3 hold, let Theorem 4.3. The priori estimate for the difference of weak solutions is established.

Thus we will prove it for the critical case, namely, when $p > N$ follow a simpler way. Furthermore, an a priori estimate for the difference of weak solutions is established.

**Theorem 4.3.** Let Assumption 1.3 hold, let $f_1, f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the conditions (1.4), (1.11), and (1.12), let $\phi_1, \phi_2, h_1, h_2 \in L^\infty(\Omega)$ with $\phi_1, \phi_2 > 0$ a.e. in $\Omega$, and let $\xi_1, \xi_2 \in \mathbb{R}$ be parameters

(a) If $u \in \mathcal{W}_p(\Omega)$ is a weak solution of either (1.1), or (1.2), then there exists a constant $c_6(\xi) > 0$ such that

$$\|u\|_{\mathcal{W}_p(\Omega)}^{p-1} \leq c_6(\xi).$$

(b) Assume that $p > \frac{2N}{N+2}$. If $u_1, u_2 \in \mathcal{W}_p(\Omega)$ are weak solutions of either (1.1), or (1.2), related to $f_1, \phi_1, h_1, \xi_1$ and $f_2, \phi_2, h_2, \xi_2$, respectively, then there
exists a constant $c_7(\xi_1, \xi_2) > 0$ such that

$$\|(u_1 - u_2, u_1 - u_2)\|_{p-1} \leq c_7(\xi_1, \xi_2).$$  

(4.28)

Proof. We only prove part (b), for part (a) follows similarly (and even in a simpler way). Again we deal both cases at once. As the arguments run similarly as in [48, Theorem 3.2] and [49, Theorem 5.1], we will only sketch the main steps of the proof. Let $f_1, f_2$ and $\xi_1, \xi_2$ be as in the theorem. Recall that we are assuming all the conditions of Assumption 1.3.

• First suppose that $2(N + 2)^{-1} < p < 2$. Let $u_1, u_2 \in W_p(\Omega)$ be weak solutions of either (1.1), or (1.2), related to $f_1, \xi_1$ and $f_2, \xi_2$, respectively. Let $w := u_1 - u_2$, and let $k \geq k_0 > 0$ be a real number, where $k_0 > 0$ denotes the constant appearing in the proof of Theorem 3.1. Now let $\hat{w}_k$ be defined by (4.1) (but now with respect to $w := u_1 - u_2$). Then $\hat{w}_k \in W_p(\Omega)$, with $|\nabla \hat{w}_k|_{L_q} = (\nabla w)\chi_{\Omega_k}$. Next, for $\epsilon \in (0, 1]$, let $C_{p, \epsilon} := \min\{c_p, c_p', c_{p, \epsilon}, c_{p, \epsilon}'\} > 0$, where $c_p, c_p', c_p, c_{p, \epsilon}$ denote the constants in Proposition 2.4 (for $r = p$). Then there exists a constant $C_p > 0$ such that

$$C_{p, \epsilon} \geq C_p^{-1}\epsilon^{2-p}, \quad \text{for all } \epsilon \in (0, 1].$$  

(4.29)

Now, we consider the following nonlinear form $\Lambda_{p}^{(n)}(\cdot, \cdot)$ by

$$\Lambda_{p}^{(n)}(u, v) := \Lambda_p(u, v) - \mathcal{E}(p)(u, v) + \left< \partial\mathcal{L}_{n}(p)(u), v \right>,$$  

(4.30)

for $u, v \in W_p(\Omega)$, where the form $\Lambda_p(\cdot, \cdot)$ is given by (1.20), and the last term is defined in (4.6). By the Dominated Convergence Theorem, one has

$$\lim_{n \to \infty} \int_{I} \Lambda_{p}^{(n)}(u, v) \, dt = \Lambda_{p}(u, v)$$

for all $u, v \in W_p(\Omega)$. Then, we define the following sets:

$$\Omega^1(\epsilon) := \{ x \in \Omega : |\nabla w(x)| \geq \epsilon \min\{|\nabla u_1(x)|, |\nabla u_2(x)|\}\} ,$$
$$\Omega^2(\epsilon) := \{ (x, y) \in \Omega \times \Omega : |w(x) - w(y)| \geq \epsilon \min\{|u_1(x) - u_1(y)|, |u_2(x) - u_2(y)|\}\} ,$$
$$\Gamma^1(\epsilon) := \{ (x, y) \in \Gamma \times \Gamma : |w(x) - w(y)| \geq \epsilon \min\{|u_1(x) - u_1(y)|, |u_2(x) - u_2(y)|\}\} ,$$
$$\Omega^1_{k}(\epsilon) := \Omega^1(\epsilon) \cap \Omega_k, \quad \Omega^2_{k}(\epsilon) := \Omega^2(\epsilon) \cap (\Omega_k \times \Omega_k), \quad \Gamma^1_{k}(\epsilon) := \Gamma^1(\epsilon) \cap (\Gamma_k \times \Gamma_k),$$
$$V_{0, k}(\epsilon) := \Gamma^1(\epsilon) \cap V_{0, k}(i) ,$$
$$\tilde{\Omega}^1_{k}(\epsilon) := \Omega_k \setminus \Omega^1_{k}(\epsilon), \quad \tilde{\Omega}^2_{k}(\epsilon) := (\Omega_k \times \Omega_k) \setminus \Omega^2_{k}(\epsilon), \quad \tilde{\Gamma}^1_{k}(\epsilon) := (\Gamma_k \times \Gamma_k) \setminus \Gamma^1_{k}(\epsilon),$$
$$\tilde{V}_{0, k}(\epsilon) := (V_{0, k}^{(i)} \setminus V_{0, k}^{(i)}(i)) \setminus V_{0, k}(i).$$

Here we recall that $V_{0, k}^{(i)} := \{ \xi \in V_{0, k}^{(i)} : \psi_n(\xi) \geq k \}$, for $\kappa := i_1, \ldots, i_n$ (e.g. (4.6)). Then, as $(\Theta_{\epsilon} w)\hat{w}_k \geq (\Theta_{\epsilon} w)\hat{w}_k$, $(\Theta_{\epsilon} w)\hat{w}_k \geq (\Theta_{\epsilon} w)\hat{w}_k$ and $(\partial\mathcal{L}_{n}(p)(w), \hat{w}_k) \geq (\partial\mathcal{L}_{n}(p)(\hat{w}_k), \hat{w}_k)$ (e.g. [50, Proof of Theorem 5.7]), Applying Proposition 2.4 for $\epsilon \in (0, 1]$, we obtain that
\[ \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \]
\[ \geq C_{p,\epsilon} \left( \Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k) - \int_{\Omega_1^\varepsilon} |\nabla \hat{w}_k|^p dx - \int_{\Omega_2^\varepsilon} |\hat{w}_k(x) - \hat{w}_k(y)|^p |x - y|^{-pN} dxdy + \int_{T_1^\varepsilon} \frac{|\hat{w}_k(x) - \hat{w}_k(y)|^p}{|x - y|^{p+d}} d\mu_x d\mu_y - \frac{1}{p} \sum_{\kappa = 1}^4 \sum_{\xi, \zeta \in \tilde{\Gamma}_{0,\kappa}(\varepsilon)} |\hat{w}_k(\psi_{\kappa,\xi}(\xi)) - \hat{w}_k(\psi_{\kappa,\zeta}(\zeta))|^p |\psi_{\kappa,\xi}(\xi) - \psi_{\kappa,\zeta}(\zeta)|^{dp} \right) \]
\[ \geq C_{p,\epsilon} \left[ \Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k) - e^p \int_{\Omega_1^\varepsilon} (|\nabla u_1|^p + |\nabla u_2|^p) dx \right] + \]
\[ - e^p C_{p,\epsilon} \left( \int_{\Omega_2^\varepsilon} \frac{|u_1(x) - u_1(y)|^p}{|x - y|^{p+d}} d\mu_x d\mu_y + \int_{T_2^\varepsilon} \frac{|u_2(x) - u_2(y)|^p}{|x - y|^{p+d}} d\mu_x d\mu_y \right) + \]
\[ - e^p C_{p,\epsilon} \left( \int_{T_1^\varepsilon} \frac{|u_1(x) - u_1(y)|^p}{|x - y|^{p+d}} d\mu_x d\mu_y + \int_{T_1^\varepsilon} \frac{|u_2(x) - u_2(y)|^p}{|x - y|^{p+d}} d\mu_x d\mu_y \right) + \]
\[ - e^p C_{p,\epsilon} \left( \frac{1}{p} \sum_{\kappa = 1}^4 \sum_{\xi, \zeta \in \tilde{\Gamma}_{0,\kappa}(\varepsilon)} \frac{|u_1(\psi_{\kappa,\xi}(\xi)) - u_1(\psi_{\kappa,\zeta}(\zeta))|^p}{|\psi_{\kappa,\xi}(\xi) - \psi_{\kappa,\zeta}(\zeta)|^{dp}} \right) + \]
\[ - e^p C_{p,\epsilon} \left( \frac{1}{p} \sum_{\kappa = 1}^4 \sum_{\xi, \zeta \in \tilde{\Gamma}_{0,\kappa}(\varepsilon)} \frac{|u_2(\psi_{\kappa,\xi}(\xi)) - u_2(\psi_{\kappa,\zeta}(\zeta))|^p}{|\psi_{\kappa,\xi}(\xi) - \psi_{\kappa,\zeta}(\zeta)|^{dp}} \right). \]

Thus in virtue of (4.29) we see that
\[ \Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k) \leq \frac{1}{C_{p,\epsilon}} \left[ \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \right] + e^p \left[ \Lambda_p^{(n)}(u_1, u_1) + \Lambda_p^{(n)}(u_2, u_2) \right] \]
\[ \leq \frac{C_p}{e^{2-p}} \left[ \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \right] + e^p \left[ \Lambda_p^{(n)}(u_1, u_1) + \Lambda_p^{(n)}(u_2, u_2) \right]. \]

Applying Proposition 2.5 with
\[ r := p, \quad \tau := C_p, \quad \rho := 2 - p, \quad \xi := \Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k), \]
\[ \varsigma := \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k), \quad \text{and} \quad c := \Lambda_p^{(n)}(u_1, u_1) + \Lambda_p^{(n)}(u_2, u_2), \]
we get from (2.17) that
\[ \Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k) \leq (C_p + 1) \left( \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \right)^{\frac{2}{2+p}} \left[ \Lambda_p^{(n)}(u_1, u_1) + \Lambda_p^{(n)}(u_2, u_2) \right]^{\frac{2}{2+p}} \]
\[ + (C_p + 1) \left[ \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \right]. \quad (4.31) \]

Since
\[ 0 \leq \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \leq 4 \left[ \Lambda_p^{(n)}(u_1, u_1) + \Lambda_p^{(n)}(u_2, u_2) \right], \]
one gets from (4.31) that
\[
\Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k) \leq C_p \left( \left[ \Lambda_p^{(n)}(u_1, \hat{w}_k) - \Lambda_p^{(n)}(u_2, \hat{w}_k) \right] \frac{p}{q} \left[ \Lambda_p^{(n)}(u_1, u_1) + \Lambda_p^{(n)}(u_2, u_2) \right] \right)^{\frac{q}{2-p}},
\]
where \( C_p := (C_p + 1)(1 + 2^{\frac{q}{p}}) > 0 \). Since
\[
\Lambda_p(\hat{w}_k, \hat{w}_k) = \Lambda_p^{(n)}(\hat{w}_k, \hat{w}_k) + \mathcal{E}(p)(\hat{w}_k, \hat{w}_k) - \left\langle \partial \mathcal{L}(p)(\hat{w}_k), \hat{w}_k \right\rangle,
\]
using this in (4.32), passing to the limit, and applying the Dominated Convergence Theorem, we arrive at
\[
\Lambda_p(\hat{w}_k, \hat{w}_k) \leq C \left( [\Lambda_p(u_1, \hat{w}_k) - \Lambda_p(u_2, \hat{w}_k)] \frac{p}{q} [\Lambda_p(u_1, u_1) + \Lambda_p(u_2, u_2)] \right)^{\frac{q}{2-p}}.
\]
In the following, the constant \( C > 0 \) may vary from line to line. Next, recalling that \( u_1, u_2 \in \mathcal{W}_p(\bar{\Omega}) \) solve (1.19) related to \( f_1, \xi_1 \) and \( f_2, \xi_2 \), respectively, using (1.10) and (1.12) (depending on the sign of \( u_1, u_2 \)), and applying Theorem 4.2, gives
\[
\Lambda_p(u_1, u_1) + \Lambda_p(u_2, u_2) = \sum_{i=1}^{2} \left\{ \int_{\Omega} f_i(x, u_i) u_i \, dx + \int_{\Omega} (\xi_i \phi_i + h_i) u_i \, dx \right\}
\]
\[
\leq C \sum_{i=1}^{2} \left( \|u_i\|_{p, \Omega}^p + \|u_i\|_{q, \Omega}^q + \lambda_N(\Omega) + \frac{\|\xi_i\|_{\infty, \Omega} + \|h_i\|_{\infty, \Omega} \|u_i\|_{1, \Omega}}{C} \right) \leq M_{\xi_1, \xi_2},
\]
for some constant \( M_{\xi_1, \xi_2} > 0 \) which depends on the \( L^\infty \) bounds on Theorem 4.2 (with respect to \( \xi_1, \xi_2 \)). We recall (4.21) and the selection of \( k_0 > 0 \) to notice that \( \lambda_N(\Omega \setminus \Omega_k) > \frac{1}{2} \lambda_N(\Omega) > 0 \), and \( \hat{w}_k = 0 \) over \( \Omega \setminus \Omega_k \). Thus, combining (1.9), (1.10), (1.12), (2.7), (2.9), (4.33), (4.34), and Theorem 4.2, we deduce that
\[
\|\hat{w}_k\|_{\mathcal{W}_p(\bar{\Omega})} \leq C \Lambda_p(\hat{w}_k, \hat{w}_k)
\]
\[
\leq CM_{\xi_1, \xi_2}^\ast \left( \int_{\Omega_k} |f(x, u_1) - f(x, u_2)|\hat{w}_k \, dx + \int_{\Omega_k} |(\xi_1 \phi_1 + h_1) - (\xi_2 \phi_2 + h_2)|\hat{w}_k \, dx \right)^{\frac{q}{2}}
\]
\[
\leq CM_{\xi_1, \xi_2}^\ast \|\chi_k\|_{p, \Omega}^p \|\hat{w}_k\|_{\mathcal{W}_p(\bar{\Omega})}^q, \tag{4.35}
\]
for \( r_d := dp(Np - N + p)^{-1} > 0 \), where
\[
M_{\xi_1, \xi_2}^\ast := \frac{M_{\xi_1, \xi_2}^\ast}{2} \left( \sum_{i=1}^{2} (|C_{\xi_i}|^{p-1} + |C_{\xi_i}|^{q-1} + |\xi_i|\|\phi_i\|_{\infty, \Omega} + \|h_i\|_{\infty, \Omega}) + M + \gamma_0 \right)^{\frac{q}{2}}
\]
and
\[
\chi_k := \left( \chi_{A_k}, \chi_{A_k} \right|_{\Omega_k} \quad (\chi_E \text{ the characteristic function of a set } E).
\]
Now let \( h > k \), and observe that \( A_h \subseteq A_k \) and that \( |\hat{w}_k| \geq h - k \) over \( A_h \) (recall (4.2) for the definition of the set \( A_h \)). In view of (2.7), (2.8) and (4.35), we get
\[
(h - k)^{\frac{q}{2}} \|\chi_k\|_{p, \Omega}^p \|\hat{w}_k\|_{\mathcal{W}_p(\bar{\Omega})}^q \leq CM_{\xi_1, \xi_2}^\ast \|\chi_k\|_{p, \Omega}^p. \tag{4.36}
\]
Setting
\[ \Phi_p(t) := \|\vec{\chi}_t\|_{p_N, r_d}^p \] (4.37)
for each \( t \in [0, \infty) \), one gets from (4.36) that
\[ \Phi_p(h) \leq CM^*_\xi_1,\xi_2 (h - k)^{\frac{p}{p_N}} \left( \Phi_p(k) + \Phi_p(k) \frac{p}{p_N + r_d} \right). \] (4.38)
As \( 2N(N+2)^{-1} < p < 2 \), we see that
\[ \frac{p_N}{p_N'} = \frac{p_d}{r_N} = \frac{Np - N + p}{N - p} := \delta > 1, \]
and thus it follows from (4.38) that
\[ \Phi_p(h) \leq 2CM^*_\xi_1,\xi_2 (h - k)^{\frac{p}{p_N}} \Phi_p(k)^\delta. \] (4.39)
we apply Lemma 2.2 to the function \( \Phi_p(\cdot) \) to conclude that
\[ \Phi_p(\zeta) = 0, \quad \text{for } \zeta^{p/2} := CM^*_\xi_1,\xi_2 \Phi_p(k_0)^{-1}. \] (4.40)
Now (4.40) implies that there exists a constant \( c^* > 0 \) such that
\[ |u - v| \leq c^* \left( M^*_\xi_1,\xi_2 \right)^{2/p} \quad \text{a.e. on } \overline{\Omega}, \] (4.41)
which shows (4.28) for the case \( 2N(N+2)^{-1} < p < 2 \).

\textbf{•} Assume now that \( p \in [2, \infty) \). As mentioned before (in section 1), it suffices to consider the critical case, namely, when \( p < N \) (since for \( p \geq N \) the embedding results are much more sharp, and much better). The proof of this case follows as in the previous one (and even in a simpler way), we will only sketch the main steps of it. Given \( u_1, u_2 \in W_p(\Omega) \) weak solutions of either (1.1), or (1.2), related to \( f_1, \xi_1 \) and \( f_2, \xi_2 \), respectively, let \( w := u_1 - u_2 \), let \( k \geq k_0 \) be a real number (for \( k_0 > 0 \) defined as in the previous case), and let \( \hat{w}_k \) be defined by (4.1). A direct calculation (similar to the previous case) shows that
\[ \Lambda_p(u_1, \hat{w}_k) - \Lambda_p(u_2, \hat{w}_k) \geq \varsigma_0 \Lambda_p(\hat{w}_k, \hat{w}_k) \geq \varsigma'_0 \|\hat{w}_k\|_{W_p(\Omega)}^p, \] (4.42)
for some constants \( \varsigma_0, \varsigma'_0 > 0 \). Proceeding exactly as in the previous case, one sees that
\[ (h - k)^{p-1} \|\vec{\chi}_h\|_{p_N, r_d}^{p-1} \leq M^*_\xi_1,\xi_2 \|\vec{\chi}_k\|_{p_N, r_d}^{p-1}, \] (4.43)
for some constant \( M^*_\xi_1,\xi_2 > 0 \) (that can be computed similarly as in the previous case), for \( h > k \), where we recall that \( r_d := dp(Np - N + p)^{-1} \). Letting
\[ \Psi_p(t) := \|\vec{\chi}_t\|_{p_N, r_d}^{p-1} \quad \text{and} \quad \delta' := \left( \frac{1}{p - 1} \right) \left( \frac{Np - N + p}{N - p} \right) > 1, \]
one can proceed in the same way as before to conclude that
\[ |u - v| \leq C'_{\xi_1,\xi_2} \text{ a.e. on } \Omega, \]  
(4.44)
for some constant \( C'_{\xi_1,\xi_2} > 0 \). Therefore, (4.44) leads to (4.28) when \( p \in [2,N) \), and completes the proof of the theorem.

5. Sub-supersolution method for nonlocal equations. In this part we will derive an alternative sub-supersolution method for the problems (1.1) and (1.2), which will be very useful in the establishment of the main results of the paper. Although some arguments will follow similar approaches as in [33, 34], to our knowledge, there is no sub-supersolution method for nonlocal equations of type (1.1) and (1.2). Recall that all the arguments will be carried out under the assumptions (1.4), (1.11), and (1.12).

To begin our discussion, given \( F \subseteq W^p_p(\Omega) \) closed and convex (where we recall that \( W^p_p(\Omega) \) is defined by (1.16)), we consider the following variational inequality
\[
\begin{cases}
\Lambda_p(u,\varphi - u) \geq \int_{\Omega} f(x,u)(\varphi - u) \, dx + \int_{\Omega} (\xi \phi + h)(\varphi - u) \, dx, & \forall \varphi \in F, \\
u \in F,
\end{cases}
\]  
(5.1)
where we recall that \( \Lambda_p(\cdot, \cdot) \) is defined by (1.20).

Before giving the corresponding definitions for subsolutions and supersolutions of the Eq. (5.1), we will fix some additional notations that will be frequently used in this section. Indeed, for \( u, v \in W^p_p(\Omega) \) and \( D, F \subseteq W^p_p(\Omega) \), we denote \( u \lor v := \max\{u,v\} \), \( u \land v := \min\{u,v\} \), and
\[
D \ast F := \{u \ast v : u \in D, v \in F\}, \quad u \ast F := \{u\} \ast F,
\]
where \( \ast \) denotes either \( \lor \) or \( \land \).

**Definition 5.1.** A function \( \hat{u} \in W^p_p(\Omega) \) is said to be a **subsolution** of (5.1), if \( \hat{u} \lor F \subseteq F \), and
\[
\Lambda_p(\hat{u},\varphi - \hat{u}) \geq \int_{\Omega} f(x,\hat{u})(\varphi - \hat{u}) \, dx + \int_{\Omega} (\xi \phi + h)(\varphi - \hat{u}) \, dx, \quad \forall \varphi \in \hat{u} \land F. \]  
(5.2)
Also, \( \bar{u} \in W^p_p(\Omega) \) is said to be a **supersolution** of (5.1), if \( \bar{u} \land F \subseteq F \), and
\[
\Lambda_p(\bar{u},\varphi - \bar{u}) \geq \int_{\Omega} f(x,\bar{u})(\varphi - \bar{u}) \, dx + \int_{\Omega} (\xi \phi + h)(\varphi - \bar{u}) \, dx, \quad \forall \varphi \in \bar{u} \lor F. \]  
(5.3)
Suppose that \( \hat{u}_1, \ldots, \hat{u}_k \in W^p_p(\Omega) \) (resp. \( \bar{u}_1, \ldots, \bar{u}_m \in W^p_p(\Omega) \)) are subsolutions (resp. supersolutions) of (5.1) (for \( k, m \in \mathbb{N} \)). Then, we will set
\[
\hat{u}_0 := \max\{\hat{u}_1, \ldots, \hat{u}_k\}, \quad \text{and} \quad \bar{u}_0 := \min\{\bar{u}_1, \ldots, \bar{u}_m\}. \]  
(5.4)
Then, we can state and prove the following important result.
Theorem 5.2. Let $u_1, \ldots, u_k \in W^1_p(\Omega)$ (resp. $\hat{u}_1, \ldots, \hat{u}_m \in V^p(\Omega)$) be subsolutions (resp. supersolutions) of $(5.1)$ (for $k, m \in \mathbb{N}$). If $u_1, \ldots, u_k$ (resp. $\hat{u}_1, \ldots, \hat{u}_m$) are bounded functions in $\Omega$, with $u_0 \leq u_0$, then there exists a solution $u \in W^1_p(\Omega)$ of $(5.1)$ such that $u_0 \leq u \leq \hat{u}_0$.

Proof. Again, we restrict our proof to the critical case $1 < p < N$.

Choose $r \in (1, \infty)$ such that
\[
\max \{p, q\} < r < \begin{cases} \frac{Np}{N-p}, & \text{if } 1 < p < N, \\ \infty, & \text{if } p \geq N. \end{cases} \tag{5.5}
\]

Then we consider the following truncating-regularizing function: $\Psi : \Omega \times \mathbb{R} \to \mathbb{R}$, by
\[
\Psi(x, s) := \begin{cases} (s - u_0(x))^{r-1}, & \text{if } s > u_0(x), \\ 0, & \text{if } u_0(x) \leq s \leq \hat{u}_0(x), \\ -(\hat{u}_0(x) - s)^{r-1}, & \text{if } s < u_0(x), \end{cases} \tag{5.6}
\]

for each $(x, s) \in \Omega \times \mathbb{R}$. Since $u_0, \hat{u}_0 \in L^\infty(\Omega)$, we can find constants $\alpha_1, \alpha'_1 > 0$ such that
\[
|\Psi(x, s)| \leq \alpha_1 + \alpha'_1 |s|^{r-1}, \quad \text{for a.e. } (x, s) \in \Omega \times \mathbb{R}. \tag{5.7}
\]

Now, define the operator $S_p : W^1_p(\Omega) \to W^1_p(\Omega)^*$, by:
\[
(S_p v)(x) := \int_\Omega \Psi(x, v) w \, dx, \quad \text{for } v, w \in W^1_p(\Omega). \tag{5.8}
\]

Since the embedding $W^1_p(\Omega) \hookrightarrow L^r(\Omega)$ is compact, it follows that $S_p$ is well-defined, bounded, and completely continuous. Moreover, a direct calculation shows that
\[
(S_p v)(x) \geq \alpha'_2 \int_\Omega |v|^r \, dx - \alpha_2, \quad \text{for all } v \in W^1_p(\Omega), \tag{5.9}
\]

for some constants $\alpha_2, \alpha'_2 > 0$ (cf. [33, Proof of Theorem 3.1]). Next, define the operator $B_p : W^1_p(\Omega) \to W^1_p(\Omega)^*$, by
\[
(B_p(\cdot))(\cdot) := \Lambda_p(\cdot, \cdot), \tag{5.10}
\]

where we recall that $\Lambda_p(\cdot, \cdot)$ is defined by (1.20). Recalling the definition of $W^1_p(\Omega)$ and in view of (2.10) and (2.11), it is easy to see that $B_p$ is bounded, continuous, and monotone. Moreover, it is coercive in the sense that
\[
(B_p v)(x) \geq c_0 \left( \|v\|_{L^p_\Omega}^p + K^{(p)}(v, v) + (\Theta_{\text{int}} v + \Theta_{\text{ext}} v) v \right), \tag{5.11}
\]

for every $v \in W^1_p(\Omega)$, and for some constant $c_0 > 0$ (recall that $K^{(p)}(\cdot, \cdot)$, $\Theta_{\text{int}}$, and $\Theta_{\text{ext}}$ are given by (4.7), (1.5), and (1.6), respectively). Furthermore, we define another operator $G_p : W^1_p(\Omega) \to W^1_p(\Omega)^*$, by
\[
(G_p v)(x) := - \int_\Omega f(x, v_0) w \, dx + \sum_{j=1}^m \int_\Omega \int f(x, r_0 v) - f(x, \hat{r}_0 v) \, w \, dx +
\]
for each \( v, w \in \mathcal{W}_p(\Omega) \), where

\[
(\tau_{ij}v)(x) := \begin{cases} 
\hat{u}_j(x), & \text{if } v(x) > \hat{u}_j(x), \\
v(x), & \text{if } \hat{u}_i(x) \leq v(x) \leq \hat{u}_j(x), \\
\hat{u}_i(x), & \text{if } v(x) < \hat{u}_i(x), 
\end{cases}
\]

(5.13)

(\text{where } k, m \in \mathbb{N}, i \in \{0, \ldots, k\}, \text{and } j \in \{0, \ldots, m\}). \text{ From the boundedness of } \hat{u}_1, \ldots, \hat{u}_k \text{ (resp. } \tilde{u}_1, \ldots, \tilde{u}_m) \text{ over } \Omega, \text{ the continuity of the mapping } \tau_{ij} : \mathcal{W}_p(\Omega) \rightarrow \mathcal{W}_p(\Omega), \text{ and the compactness of the embedding } \mathcal{W}_p(\Omega) \hookrightarrow L^r(\Omega), \text{ one can deduce that } G_p \text{ is well-defined, bounded, and completely continuous over } \mathcal{W}_p(\Omega). \text{ Then we consider the variational inequality, formally given by}

\[
\begin{align*}
(\mathcal{S}_p v + B_p v + G_p v)(\varphi - v) & \geq 0, \quad \forall \varphi \in F, \\
v & \in F,
\end{align*}
\]

(5.14)

where \( \mathcal{S}_p, B_p, \) and \( G_p \) are defined by (5.8), (5.10), and (5.12), respectively. From the properties of \( \mathcal{S}_p, B_p, \) and \( G_p \), it follows that the operator \( T_p := \mathcal{S}_p + B_p + G_p \) is bounded, and pseudo-monotone over \( \mathcal{W}_p(\Omega) \). (cf. [45, pag. 40-41]). Moreover, we claim that \( T_p \) is coercive, in the sense that

\[
\lim_{\|v\|_{\mathcal{W}_p(\Omega)} \to \infty} \frac{(T_p v)(v - v_0)}{\|v\|^p_{\mathcal{W}_p(\Omega)}} = \infty,
\]

(5.15)

for each bounded function \( v_0 \in F \), where we recall the semi-norm

\[
\|w\|^p_{\mathcal{W}_p(\Omega)} := \|\nabla w\|^p_{p,\Omega} + K^{(p)}(w, w), \quad \text{for } w \in \mathcal{W}_p(\Omega).
\]

In fact, we first observe that

\[
(T_p v)(v - v_0) \geq (\mathcal{S}_p v)v + (B_p v)v - |(\mathcal{S}_p v)v_0| - |(B_p v)v_0| - \int_{\Omega}(|\xi| + |h|)(|v| + |v_0|) \, dx + \\
- \left( \int_{\Omega} |f(x, \tau_{00} v)|(|v| + |v_0|) \, dx + \sum_{j=1}^{m} \int_{\Omega} |f(x, \tau_{0j} v)|(|v| + |v_0|) \, dx \right) + \\
- \sum_{i=1}^{k} \int_{\Omega} |f(x, \tau_{i0} v)|(|v| + |v_0|) \, dx.
\]

(5.16)

Define now the operator \( B_p^{(n)} \) by

\[
(B_p^{(n)}(\cdot))(\cdot) := \Lambda_p^{(n)}(\cdot, \cdot),
\]

(5.17)

for the form \( \Lambda_p^{(n)}(\cdot, \cdot) \) given by (4.30). As before, one has

\[
(B_p v) = \lim_{n \to \infty} \int f(B_p^{(n)} u) \, dt, \quad \forall u, v \in \mathcal{W}_p(\Omega).
\]

Moreover, by virtue of Young’s inequality and the following inequality for \( p \)-Lagrangian (see 2.9 in [8]), \( \mathcal{L}^{(p)}(u, v) \leq \epsilon \mathcal{L}^{(p)}(u, u) + c_r \mathcal{L}^{(p)}(v, v) \), and thus

\[
- \sum_{i=1}^{k} \int_{\Omega} |f(x, \tau_{i0} v)|(|v| + |v_0|) \, dx.
\]
\[ |(B(v) v)| \leq \epsilon \left( \| \nabla v \|_{p, \Omega}^p + \left( \partial L^{(p)}(v) \right) + (\Theta, v + \Theta, v) \right) \\
+ C \epsilon \left( \| \nabla v_0 \|_{p, \Omega}^p + \left( \partial L^{(p)}(v_0), v_0 \right) + (\Theta, v_0 + \Theta, v_0) v_0 \right), \]  
for every \( \epsilon > 0 \), and for some constant \( C \epsilon > 0 \). Passing to the limit in the above inequality, one deduces from the Dominated Convergence Theorem that

\[ |(B v) v| \leq \epsilon \left( \| \nabla v \|_{p, \Omega}^p + K(v, v) + (\Theta, v + \Theta, v) v \right) \\
+ C \epsilon \left( \| \nabla v_0 \|_{p, \Omega}^p + K(v_0, v_0) + (\Theta, v_0 + \Theta, v_0) v_0 \right), \]  
for every \( \epsilon > 0 \). Furthermore, recalling the definition of \( \tau_{ij}(\cdot) \), using (1.12), (5.7), and Young’s inequality, we get that

\[ |(S v) v_0| \leq \alpha \| v_0 \|_{1, \Omega} + \alpha' \int [v]^{-1} |v| dx \leq \epsilon \| v \|_{r, \Omega} + C' \| v_0 \|_{r, \Omega} + C_1, \]  
and

\[ \int_\Omega |f(x, \tau_{ij} v)|(|v| + |v_0|) dx + \int_\Omega (|v| + |v_0|) dx \leq \gamma_0 \int_\Omega (|v| + |v_0|) dx + (\gamma_0 + \| \| \| \omega, \| + \| \| \omega, \| ) \int_\Omega (|v| + |v_0|) dx \]

\[ = \gamma_0 \| v \|_{r, \Omega} + \int_\Omega (|v| + |v_0|) dx + (\gamma_0 + \| \| \| \omega, \| + \| \| \omega, \| ) \int_\Omega (|v| + |v_0|) dx \]

\[ \leq (3 \gamma_0 + 1) \epsilon \| v \|_{r, \Omega} + C' \gamma_0 \| v_0 \|_{r, \Omega} + (\gamma_0 + \| \| \| \omega, \| + \| \| \omega, \| ) \| v_0 \|_{1, \Omega} + \]

\[ + C' \left\{ (\gamma_0 + \| \| \| \omega, \| + \| \| \omega, \| ) \right\} \lambda (\Omega) \]

\[ \leq (3 \gamma_0 + 1) \epsilon \| v \|_{r, \Omega} + M(v_0, \gamma_0, \| \| \omega, \| + \| \| \omega, \| ), \]  
for every \( \epsilon > 0 \), and for some constants \( C_1, C_1', C_1, C_2 > 0 \), and for a constant \( M(v_0, \gamma_0, \| \| \omega, \| + \| \| \omega, \| ) > 0 \). Taking into account (5.9), (5.11) and the boundedness of \( v_0 \), selecting \( \epsilon > 0 \) suitably, and combining (5.18), (5.19) and (5.20) into (5.16), we deduce that

\[ (T(v) v - v_0) \geq C_3 \| v \|_{p, \Omega} - M^*(v_0, \| \| \omega, \| + \| \| \omega, \| ) > 0, \]  
for all \( v \in W_p(\Omega) \) and \( v_0 \in F \) bounded, for some constant \( C_3 \), and for a constant \( M^*(v_0, \| \| \omega, \| + \| \| \omega, \| ) > 0 \). Thus, (4.36) leads to (5.15), proving the desired claim, and as a consequence, the above properties for the operator \( T(v) \) ensure that Eq. (5.1) has a solution \( v \in F \) (cf. [35]). To complete the proof of the theorem, we show that \( \hat{u}_i \leq u \leq \hat{u}_j \) for all \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, m\} \). Indeed, to establish the first inequality, we use the fact that \( \hat{u}_i \) is a subsolution of (5.1), and we put \( v := u + (\hat{u}_i - u)^+ \). Then, testing (5.14) with the function \( v \), we get

\[ 0 \leq \Lambda_p(u, (\hat{u}_i - u)^+) + \int_\Omega \Psi(x, u)(\hat{u}_i - u)^+ dx - \int_\Omega (\xi \phi + h)(\hat{u}_i - u)^+ dx + \]
- \int_{\Omega} f(x, \tau_{00}u)(\hat{u}_i - u)^+ \, dx + \sum_{j=1}^{m} \int_{\Omega} |f(x, \tau_{0j}u) - f(x, \tau_{00}u)| (\hat{u}_i - u)^+ \, dx + \\
- \sum_{s=1}^{k} \int_{\Omega} |f(x, \tau_{s0}u) - f(x, \tau_{00}u)| (\hat{u}_i - u)^+ \, dx. \tag{5.22}

On the other hand, as \( \hat{u}_i \in F \) is a subsolution of (5.1), testing (5.2) with the function \( u_i := \hat{u}_i - (\hat{u}_i - u)^+ = \hat{u}_i \wedge u \in \hat{u}_i \wedge F \), one sees that

\[ 0 \leq -\Lambda_p(\hat{u}_i, (\hat{u}_i - u)^+) + \int_{\Omega} (\xi \phi + h)(\hat{u}_i - u)^+ \, dx + \int_{\Omega} f(x, \hat{u}_i)(\hat{u}_i - u)^+ \, dx. \tag{5.23} \]

Adding (5.22) and (5.23), we get

\[ 0 \leq -\Lambda_p(u, (\hat{u}_i - u)^+) - \Lambda_p(\hat{u}_i, (\hat{u}_i - u)^+) + \int_{\Omega} [f(x, \hat{u}_i) - f(x, \tau_{00}u)](\hat{u}_i - u)^+ \, dx + \\
+ \int_{\Omega} \Psi(x, u)(\hat{u}_i - u)^+ \, dx + \sum_{j=1}^{m} \int_{\Omega} |f(x, \tau_{0j}u) - f(x, \tau_{00}u)| (\hat{u}_i - u)^+ \, dx + \\
- \sum_{s=1}^{k} \int_{\Omega} |f(x, \tau_{s0}u) - f(x, \tau_{00}u)| (\hat{u}_i - u)^+ \, dx. \tag{5.24} \]

Now, for each \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, m\} \), we see that \( u(x) < \hat{u}_i(x) \leq u_0(x) \leq \hat{u}_0(x) \leq \hat{u}_j(x) \) for every \( x \in \{ y \in \Omega : u(y) < \hat{u}_i(y) \} \). Therefore we have \( \tau_{00}u = \tau_{0j}u = \hat{u}_0 \) and \( \tau_{s0}u = \hat{u}_i \) over \( \{ y \in \Omega : u(y) < \hat{u}_i(y) \} \). The combination of all these properties entails that

\[ \int_{\Omega} [f(x, \hat{u}_i) - f(x, \tau_{00}u)](\hat{u}_i - u)^+ \, dx - \sum_{s=1}^{k} \int_{\Omega} |f(x, \tau_{s0}u) - f(x, \tau_{00}u)| (\hat{u}_i - u)^+ \, dx \leq 0, \tag{5.25} \]

and

\[ \int_{\Omega} |f(x, \tau_{0j}u) - f(x, \tau_{00}u)| (\hat{u}_i - u)^+ \, dx = \int_{\Omega \setminus \{ u < \hat{u}_i \}} |f(x, \hat{u}_0) - f(x, \hat{u}_i)| (\hat{u}_i - u)^+ \, dx = 0. \tag{5.26} \]

Moreover, using (2.12), it is clear that

\[ \Lambda_p^{(n)}(u, (\hat{u}_i - u)^+) - \Lambda_p^{(n)}(\hat{u}_i, (\hat{u}_i - u)^+) \leq 0, \]

where we recall the definition of \( \Lambda_p^{(n)}(\cdot, \cdot) \) given in (4.30). Thus by the Dominated Convergence Theorem, we obtain that

\[ \Lambda_p(u, (\hat{u}_i - u)^+) - \Lambda_p(\hat{u}_i, (\hat{u}_i - u)^+) \leq 0, \tag{5.27} \]

and from here we combine (5.24), (5.25), (5.26), and (5.27), to get that

\[ 0 \leq \int_{\Omega} \Psi(x, u)(\hat{u}_i - u)^+ \, dx \leq -\int_{\Omega \setminus \{ u < \hat{u}_i \}} (\hat{u}_i - u)^r \, dx \leq 0. \tag{5.28} \]
Inequality (5.28) shows that $u \geq \hat{u}_i$ a.e. in $\Omega$, for each $i \in \{1, \ldots, k\}$. Now, similar arguments can be employed to deduce as well that $u \leq \tilde{u}_j$ for every $j \in \{1, \ldots, m\}$, and consequently $u_0 \leq u \leq u_0$ a.e. on $\Omega$. In addition, recalling (5.6) and (5.13), we have that $\Psi(x, u) = 0$ and $\tau_{ij} u = u$ a.e. in $\Omega$ (for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$). Therefore,

$$(G_p u)v = -\int_{\Omega} f(x, u)v \, dx - \int_{\Omega} (\xi \phi + h)v \, dx, \quad \forall v \in E,$$

and consequently (5.14) becomes (5.1), as desired.

**Remark 1.** Consider the problem (5.1) for $F = W_p(\Omega)$. Then one can observe that in this case, the variational inequality (5.1) becomes the variational equality (1.19) given in Definition 1.2. Thus in this case, a solution of (5.1) becomes a weak solution of problem (1.1). Furthermore, one can observe that given $w \in W_p(\Omega)$, one has

$$w \vee W_p(\Omega) = \{ v \in W_p(\Omega) \mid v \geq w \text{ a.e. in } \Omega \}$$

and

$$w \wedge W_p(\Omega) = \{ v \in W_p(\Omega) \mid v \leq w \text{ a.e. in } \Omega \}.$$

Consequently, in this case the notion of weak subsolution and supersolution in Definition 5.1 coincides with the formulation for weak subsolutions and supersolutions given in Definition 1.2(b).

6. **Proof of the main result.** In this section we focus our attention in establishing the main results of this paper, namely, Theorem 1.4 and Theorem 1.5, assuming all the conditions in Assumption 1.3. Our approach will be follow arguments similar as in [48] (with some ideas motivated by the results in [19]). Thus, the proofs of some intermediate results will be omitted, but as our problem under consideration is much more general and several results require generalizations and modifications, more details will be given when needed. In particular, we will give proofs to most of the results here, and outline the main steps, especially when the generalizations come into play.

We begin by providing a non-existence result.

**Lemma 6.1.** If

$$\xi > \frac{(M + \|h\|_{\infty, \Omega})\lambda_N(\Omega)}{\|\phi\|_{1, \Omega}} > 0,$$

then the problems (1.1) and (1.2) have no weak solution over $W_p(\Omega)$.

**Proof.** We argue as in the proof of Theorem 4.2. In fact, if $u \in W_p(\Omega)$ is a weak solution of (1.1), we test (1.19) with the function $w \equiv 1$. Because $\left< \partial \tilde{L}^{(p)}(u), 1 \right> = 0$, it follows that $\Lambda_p(u, 1) = 0$. By using (1.9) and (1.10), we get

$$0 = \int_{\Omega} f(x, u) \, dx + \int_{\Omega} (\xi \phi + h) \, dx$$
over $\Omega$, we get

$$\xi \leq \frac{(M + \|h\|_{\infty,\Omega})\lambda_N(\Omega)}{\|\phi\|_{1,\Omega}},$$

which establishes the desired claim.

**Lemma 6.2.** For each $\xi \in \mathbb{R}$, the constant

$$c_\xi := -\left(\frac{\|\xi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega} + M}{\eta_0}\right)^{1/(p-1)}$$

is a weak subsolution of (1.1) and (1.2), and every constant $c < c_\xi$ is a strict subsolution of both (1.1) and (1.2). Moreover, if $u \in W_p(\Omega)$ is a weak solution of (1.1) ($N = 2$), or (1.2) ($N = 3$), then $u(x) \geq c_\xi$ for every $x \in \Omega$.

**Proof.** For simplicity, assume that $\Omega = \Omega_3 \in \mathbb{R}^3$ (the other case is analogous). Given $c_\xi$ defined as in (6.1), by virtue of (1.10) we have

$$f(x, c_\xi) \geq \eta_0|c_\xi|^{p-2}c_\xi - M = |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}.$$  \hfill (6.2)

Selecting $\varphi \in W_p(\Omega)^+$ arbitrarily, multiplying the Eq. (6.2) by $\varphi$, and integrating over $\Omega$, we get

$$\int_{\Omega} f(x, c_\xi) \varphi \, dx \geq |\xi|\|\phi\|_{\infty,\Omega} \int_{\Omega} \varphi \, dx + \|h\|_{\infty,\Omega} \int_{\Omega} \varphi \, dx \geq |\xi| \int_{\Omega} \phi \, dx + \int_{\Omega} |h| \varphi \, dx \geq - \int_{\Omega} (\xi \phi + h) \varphi \, dx.$$  \hfill (6.3)

Thus from (6.3), we get

$$0 \leq \Lambda_p(c_\xi, \varphi) \leq \int_{\Omega} f(x, c_\xi) \varphi \, dx + \int_{\Omega} (\xi \phi + h) \varphi \, dx,$$

proving that $c_\xi$ is a weak subsolution of the problem (1.2). Furthermore, if follows immediately that if $c$ is a constant less than $c_\xi$, then $c$ becomes a strict subsolution of the Eq. (1.2). It remains to show that $c_\xi$ is a lower bound for any weak solution $u \in W_p(\Omega)$ of (1.2). We suppose that the contrary holds, that is, assume that there exists a constant $c_0 > 0$ such that the function $(c_\xi - c_0 - u(x_0))^+$ is strictly positive for some $x_0 \in \Omega$. Given $\epsilon \in (0, c_0)$, we set

$$\Omega_{c, \epsilon} := \{ x \in \Omega : u(x) < c_\xi - \epsilon \} \neq \emptyset \quad \text{and} \quad u_{c, \epsilon} := (c_\xi - \epsilon - u)^+ \in W_p(\Omega)^+.$$  

Notice that $(\Theta_\epsilon u) u_{c, \epsilon} \leq 0, (\Theta_\epsilon u) u_{c, \epsilon} \leq 0$, and also in view of (4.5) and (4.6), we deduce that $K^{(p)}(u, u_{c, \epsilon}) \leq 0$. As $\nabla u_{c, \epsilon} = -\nabla u$, testing (1.19) with the function $u_{c, \epsilon}$ and applying (1.9), we obtain

$$\|\nabla u\|_{p, \alpha, \epsilon, \xi}^p \leq -\Lambda_p(u, u_{c, \epsilon}) = - \int_{\Omega_{c, \epsilon}} f(x, u) u_{c, \epsilon} \, dx - \int_{\Omega_{c, \epsilon}} (\xi \phi + h) u_{c, \epsilon} \, dx \leq -\eta_0 \int_{\Omega_{c, \epsilon}} |u|^{p-1} u_{c, \epsilon} \, dx + (M + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}) \int_{\Omega_{c, \epsilon}} u_{c, \epsilon} \, dx.$$
Thus, set
\[ \{ \Omega \} \]

is a minimal weak solution of Eq. (1.2) over the set
\[ \{ \Omega \} \]

is a weak solution of (1.2), which implies that \( \xi \) is non-empty, and bounded from above by 1.4, recalling Lemma 6.1, we observe that the set
\[ \{ \Omega \} \]

completing the proof.

Now we establish the first main result of this article.

**Proof.** [Theorem 1.4] Assume the condition of the theorem. From Lemma 6.1 one has the statements (1).

To prove the other assertion, for simplicity (as in the previous lemma) we assume that \( \Omega = \Omega_3 \subseteq \mathbb{R}^3 \). Given \( \xi, \xi' \in \mathbb{R} \) with \( \xi \leq \xi_0 \), we notice that the zero function 0 is a weak supersolution of (1.2) if and only if \( \xi \phi(x) \leq -f(x,0) - h(x) \) for a.e. \( x \in \Omega \).

Thus, set
\[ \xi_0' := \inf_{x \in \Omega} \left\{ \frac{-f(x,0) - h(x)}{\max\{1, \phi(x)\}} \right\} . \]

If \( \xi \leq \xi_0' \), then we get that
\[ -f(x,0) - h(x) \geq \xi \max\{1, \phi(x)\} \geq \xi \phi(x) \quad \text{for a.e. } x \in \Omega, \]

and consequently 0 is a weak supersolution of (1.2) for all \( \xi \leq \xi_0' \). On the other hand, by Lemma 6.2, the negative constant \( c_\xi \) given by (6.1) is a weak subsolution of (1.2) for all \( \xi \). Hence, Theorem 5.2 and Remark 1 imply that (1.2) is solvable for all \( \xi \leq \xi_0' \). Moreover, if (1.2) is solvable for some parameter \( \xi \), and \( u \in W_p(\Omega) \) is the corresponding weak solution, as \( \phi > 0 \) a.e. in \( \Omega \), it follows that \( u \) is a weak supersolution of (1.2) corresponding to the parameter \( \zeta \), for \( \zeta < \xi \). Henceforth, by Lemma 6.2, Theorem 5.2 and Remark 1, we deduce that (1.2) is solvable for all \( \zeta < \xi \). The last statement of the theorem follows directly from Theorem 4.3. &emsp; \( \Box \)

Next we establish the second main result of the paper.

**Proof.** [Theorem 1.5] Again, without loss of generality, we assume that \( \Omega = \Omega_3 \subseteq \mathbb{R}^3 \). Because \( p \in (N_0, \infty) \), by the Sobolev embedding theorem (see section 2) we have \( W_p(\Omega) \hookrightarrow C^{0,\delta}(\Omega) \) for some \( \delta \in (0,1) \). If \( \xi \leq \xi_0 \), then Theorem 1.4 asserts the existence of a weak solution of the problem (1.2). In view of the proof of Theorem 1.4, recalling Lemma 6.1, we observe that the set
\[ S = \{ \xi \in \mathbb{R} : \text{(1.2) is solvable} \} \]

is non-empty, and bounded from above by \( \xi_0 := \sup(S) \). Clearly \( (-\infty, \xi_0) \subseteq S \).

Now let \( \{ \xi_n \} \) be a non-decreasing sequence of real numbers, such that \( \lim_{n \to \infty} \xi_n = \xi_0 \). Denote by \( u_{\xi_n} \) the weak solutions of (1.2) with respect to the parameters \( \xi_n \) (\( n \in \mathbb{N} \)).

By virtue of Theorem 4.2 and the Sobolev embedding, the family \( \{ u_{\xi_n} \}_{n \in \mathbb{N}} \) is equicontinuous and pointwise bounded. Consequently, by Ascoli’s theorem, we can find a subsequence \( \{ \xi_{n_k} \} \) of \( \{ \xi_n \} \) such that \( \lim_{k \to \infty} u_{\xi_{n_k}} = u_{\xi_0} \) in \( C(\Omega) \). Therefore, \( u_{\xi_0} \) is a weak solution of (1.2), which implies that \( S = (-\infty, \xi_0) \). It follows that there exists a minimal weak solution of Eq. (1.2) over the set \( \{ w \in W_p(\Omega) : w \geq c_\xi \text{ in } \Omega \} \),
which proves (1) and (2). To show (3) and (4), we put
\[ \xi_1 := \sup_{s \in \mathbb{R}} \inf_{x \in \Omega} \left\{ \frac{-f(x,s) - h(x)}{\max\{1, \phi(x)\}} \right\}. \tag{6.4} \]
As \( h \in L^\infty(\Omega) \), recalling (1.9) and (1.10), we see that \(-f(x,s) - h(x)\) is bounded from above, and thus \(\xi_1\) is well-defined. If \(\xi < \xi_1\), then one can find \(s_0 \in \mathbb{R}\) such that
\[ \xi < \inf_{x \in \Omega} \left\{ \frac{-f(x,s_0) - h(x)}{\max\{1, \phi(x)\}} \right\}. \]
Thus, \(s_0\) is a weak supersolution of (1.2), \(c_\xi\) is a weak subsolution of (1.2) (\(c_\xi\) given by (6.1)), and \(c_\xi \leq s_0\) by virtue of Lemma 6.2. Consequently, from Theorem 5.2 and Remark 1 we deduce that there exists a weak solution \(u_\xi \in W_p(\Omega)\) with \(c_\xi \leq u_\xi \leq s_0\). Therefore \(\xi_1 \leq \xi_0\). Given \(f\) locally Hölder continuous over \(\mathbb{R}\), and uniformly continuous a.e. on \(\Omega\), let \(\xi < \xi_1\). We search a second distinct weak solution of (1.2). To achieve this, we will employ the well-known Leray-Schauder degree theory. Indeed, we can find a constant \(t > s_0\) such that
\[ \xi < \inf_{x \in \Omega} \left\{ \frac{-f(x,t) - h(x)}{\max\{1, \phi(x)\}} \right\}, \]
and thus \(t\) is also a weak supersolution of (1.2). Moreover, by Lemma 6.2, any constant \(c\) with \(c < c_\xi\) is a strict weak subsolution of (1.2). Then we define the open set
\[ U := \{ v \in C(\overline{\Omega}) : c < v(x) < t \text{ for all } x \in \Omega \}. \tag{6.5} \]
Clearly \(u_\xi \in U\), and furthermore, by (1.11), one can find a constant \(\tau > 0\) such that \(f(x,u) + \tau|u|^{p-2}u\) is non-decreasing in \(u\) over \([c,t]\). Since \(u_\xi \in W_p(\Omega) \cap U\) is a weak solution of (1.2), it follows that \(u_\xi\) is a fixed point of the compact operator \(K_\xi : C(\Omega) \to C(\Omega)\), defined by \(K_\xi v = w\), for \(w\) a weak solution of the boundary value problem
\[
\begin{cases}
-\Delta_p w + \Theta_\alpha w + \tau|w|^{p-2}w = f(x,v) + \tau|v|^{p-2}v + \xi v + h & \text{in } \Omega, \\
A_v w + |\nabla w|^{p-2} \frac{\partial w}{\partial \nu} + \Theta_\alpha v = 0 & \text{on } \Gamma
\end{cases}
\tag{6.6}
\]
The existence of the solution \(w \in W_p(\Omega) \cap C(\Omega)\) of problem (6.6) can be obtained in a similar way as in [27, section 4]. If \(\deg(I - K_\xi, U, 0)\) is not well-defined (that is, if \(0 \in (I - K_\xi)(\partial U)\)), the conclusion (3) follows. Otherwise, let \(\psi \in U\), \(\alpha \in [0,1]\), and define the mapping \(T_\alpha : C(\Omega) \to C(\Omega)\), by
\[ T_\alpha v := \alpha K_\xi v + (1 - \alpha)\psi. \]
Notice that
\[ -\Delta_p c + \Theta_\alpha c + \tau|c|^{p-2}c \leq -\Delta_p (K_\xi v) + \Theta_\alpha (K_\xi v) + \tau|K_\xi v|^{p-2}K_\xi v \leq -\Delta_p t + \Theta_\alpha t + \tau|t|^{p-2}t \]
for all \(v \in U\), it follows from the Weak Comparison Principle that \(K_\xi v \in \overline{U}\) for each \(v \in U\), that is, \(K_\xi v(x) \in [c,t]\) for each \(x \in \Omega\). Since \(U\) is convex and \(\psi \in U\), one gets that \(T_\alpha\) maps \(U\) into \(U\) for every \(\alpha \in (0,1]\), which yields that \(0 \notin (I - T_\alpha)(\partial U)\) for all \(\alpha \in [0,1]\). In this way, \(\deg(I - T_\alpha, U, 0)\) is well-defined and independent of
Moreover, as $\psi \in U$ and $T_0\psi = \psi$ for all $v \in U$, we obtain that
\begin{equation}
\deg(I - K_\xi, U, 0) = \deg(I - T_\alpha, U, 0) = \deg(I - T_0, U, 0) = 1.
\end{equation}

On the other hand, for a sufficiently large constant $\vartheta > c_6(\xi) > 0$ (for $c_6(\xi)$ the constant in (4.28)), set
\begin{equation}
D_\vartheta := \left\{ v \in C(\Omega) : \|v\|_{C(\Omega)} < \vartheta \right\}.
\end{equation}

By virtue of Theorem 4.3(a), we observe that if $\xi' \in [\xi, \xi_0 + 1]$ and $u_{\xi'}$ is a fixed point of $K_{\xi'}$, then $u_{\xi'} \in D_\vartheta$. As $\xi$ is fixed, and $\xi < \xi_1 \leq \xi_0$, we can suppose that $U \subseteq V_\vartheta$. Since $K_{\xi_0 + 1}$ does not have fixed points, this fact together with the above arguments entail that
\begin{equation}
\deg(I - K_\xi, D_\vartheta, 0) = \deg(I - K_{\xi_0 + 1}, D_\vartheta, 0) = 0.
\end{equation}

Consequently, (6.7) and (6.8) imply that
\begin{equation}
\deg(I - K_\xi, D_\vartheta - U, 0) = -1,
\end{equation}

which says that problem (1.2) admits a weak solution $\tilde{u} \in W_0^p(\Omega) \setminus U$. This establishes the assertion (3). Finally, suppose that $f \in C(\Omega \times \mathbb{R})$. For $\xi < \xi_0$, let $u_{\xi_0} \in W_0^p(\Omega)$ be the minimal weak solution of (1.2) with respect to $\xi_0$. Then, $u_{\xi_0}$ is a weak supersolution of (1.2) corresponding to $\xi$. Since $c_\xi \leq u_{\xi_0}$, by Theorem 5.2, there is a weak solution $u_\xi$ of (1.2) fulfilling $c_\xi \leq u_\xi \leq u_{\xi_0}$. As $f \in C(\Omega \times \mathbb{R})$, we can find a constant $\gamma$ sufficiently small, such that
\begin{equation}
f(x, u_{\xi_0}) - f(x, u_{\xi_0} + \gamma) + (\xi_0 - \xi)\phi(x) \geq 0.
\end{equation}

Letting $w_{\xi_0} := u_{\xi_0} + \gamma$, one gets from (6.11) that
\begin{equation}
-\Delta_p w_{\xi_0} + \Theta_\alpha w_{\xi_0} = f(x, u_{\xi_0} + \gamma) + t\phi(x) + [f(x, u_{\xi_0}) - f(x, u_{\xi_0} + \gamma) + (\xi_0 - \xi)\phi(x)] \\
\geq f(x, u_{\xi_0} + \gamma) + t\phi(x),
\end{equation}

which entails that $w_{\xi_0}$ is a weak supersolution of (1.2) corresponding to $t$ (for $\gamma$ small enough). Recalling that any $c < c_\xi$ is a strict weak subsolution of (1.2), one concludes that $c < u_\xi < w_{\xi_0}$. From here, one can proceed exactly as in the previous case to achieve the same conclusion as in statement (3). Consequently, in this case, $\xi_1 = \xi_0$, establishing (4) and completing the proof of the theorem.

\textbf{Acknowledgements.} M.R. Lancia and P. Vernole have been supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
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