Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

A QUASI-LINEAR NONLOCAL VENTTSEL' PROBLEM OF AMBROSETTI–PRODI TYPE ON FRACTAL DOMAINS

Maria Rosaria Lancia

Dipartimento di Scienze di Base e Applicate per I'Ingegneria
Sapienza Università di Roma Roma, Italy
Via A. Scarpa 16, 00161 Roma, Italy

Alejandro Vélez - Santiago*

Department of Mathematical Sciences
University of Puerto Rico at Mayagüez, Mayagüez, Puerto Rico 00681

Paola Vernole

Dipartimento di Scienze di Base e Applicate per I'Ingegneria
Sapienza Università di Roma Roma, Italy
Via A. Scarpa 16, 00161 Roma, Italy

(Communicated by the associate editor name)

Abstract. We investigate the solvability of the Ambrosetti–Prodi problem for the p-Laplace operator Δ_p with Venttsel' boundary conditions on a two-dimensional open bounded set with Koch-type boundary, and on an open bounded three-dimensional cylinder with Koch-type fractal boundary. Using a priori estimates, regularity theory and a sub-supersolution method, we obtain a necessary condition for the non-existence of solutions (in the weak sense), and the existence of at least one globally bounded weak solution. Moreover, under additional conditions, we apply the Leray-Schauder degree theory to obtain results about multiplicity of weak solutions.

1. **Introduction and the main result.** Let $\Omega_2 \subseteq \mathbb{R}^2$ be a bounded domain with a Koch-type fractal boundary $\Gamma_2 := \partial\Omega_2$, and let $\Omega_3 \subseteq \mathbb{R}^3$ be a bounded cylinder with a Koch-type fractal boundary $\partial\Omega_3 := \overline{\Omega_3} \setminus \Omega_3 = \Gamma_3 \cup (\Omega_2 \times \{0\}) \cup (\Omega_2 \times \{1\})$, where $\Gamma_3 := \Gamma_2 \times I$, for $I := [0,1]$ (see section 2 for more details on the construction of these sets). Given $p \in (1,\infty)$, denote by $\lambda_N(\cdot)$ the usual N-dimensional Lebesgue measure in Ω_N ($N \in \{2,3\}$), and by $\mu(\cdot) := \mathcal{H}^d(\cdot)$ the d-dimensional Hausdorff measure, for d the Hausdorff dimension of the fractal boundary Γ_i ($i \in \{2,3\}$). In order to pose our problem of interest, because the notion of the normal derivative (in the classical sense) may not make sense for non-Lipschitz domains, we need to define the notion of extended normal derivative (e.g. [49, Definition 4.1]).

2010 *Mathematics Subject Classification.* 35J62, 35J92, 35D30, 35B45, 35B65.
Key words and phrases. Venttsel' boundary conditions, Koch snowflake domain, Weak solutions, A priori estimates, Sub-supersolution method, Leray-Schauder degree theory.
* Corresponding author: A.Velez-Santiago.

Definition 1.1. Let μ be a Borel measure supported on $\Gamma := \partial\Omega$, and let $u \in W^{1,1}_{loc}(\Omega)$ be such that $|\nabla u|^{p-2}\nabla u \cdot \nabla v \in L^1(\Omega, dx)$ for all $v \in C^1(\overline{\Omega})$. If there exists a function $f \in L^1_{loc}(\mathbb{R}^N, dx)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx = \int_\Omega f v\, dx + \int_\Gamma v\, d\mu,$$

for all $v \in C^1(\overline{\Omega})$, then we say that μ is the p-**generalized normal derivative** of u, and we denote

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu_\mu} := \mu.$$

Note that if Ω is "sufficiently regular", for instance, a bounded Lipschitz domain, taking μ the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$, which in such case coincides with the classical surface measure on Γ, it follows that the above notion of generalized normal derivative coincide with the classical definition of the normal derivative.

We consider the solvability of the quasi-linear nonlocal elliptic problems of Ambrosetti-Prodi type, formally given by

$$\begin{cases} -\Delta_p u + \Theta_{\Omega} u \;=\; f(x,u) + \xi\phi + h & \text{in } \Omega_2, \\ \mathcal{A}_2 u + |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu_\mu} + \Theta_{\Gamma} u \;=\; 0 & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta_p u + \Theta_{\Omega} u \;=\; f(x,u) + \xi\phi + h & \text{in } \Omega_3, \\ \mathcal{A}_3 u + |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu_\mu} + \Theta_{\Gamma} u \;=\; 0 & \text{on } \Gamma_3, \\ \dfrac{\partial u}{\partial \nu_{\lambda_2}} \;=\; 0 & \text{on } \partial\Omega_3 \setminus \Gamma_3. \end{cases} \tag{1.2}$$

Given $D \subseteq \mathbb{R}^N$ arbitrary, we consider

$$D := \begin{cases} D_2, & \text{if } N = 2, \\ D_3, & \text{if } N = 3, \end{cases} \quad \text{and} \quad \mu := \begin{cases} \mu|_{\Gamma_2}, & \text{if } N = 2, \\ \mu|_{\Gamma_3}, & \text{if } N = 3, \end{cases} \tag{1.3}$$

In this paper, D will be either Ω, or Γ. Then, we assume that ϕ, $h \in L^\infty(\Omega)$ with $\phi > 0$ a.e. in Ω, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the conditions

$$\liminf_{s\to\infty} \frac{f(x,s)}{|s|^{p-2}s} > 0 \qquad \text{and} \qquad \limsup_{s\to-\infty} \frac{f(x,s)}{|s|^{p-2}s} < 0 \tag{1.4}$$

(the limit being uniform in $x \in \Omega$), and that $\xi \in \mathbb{R}$ is a parameter. Here we recall that $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator on Ω, $\Theta_{\Gamma} : \mathbb{B}^p_\alpha(\Omega) \to \mathbb{B}^p_\alpha(\Omega)^*$ and $\Theta_{\Gamma} : \mathbb{B}^p_\beta(\Gamma) \to \mathbb{B}^p_\beta(\Gamma)^*$ are given by

$$(\Theta_{\Omega} u)v := \int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))(v(x)-v(y))}{|x-y|^{\varsigma p+N}}\right) dxdy \tag{1.5}$$

and

$$(\Theta_{\Gamma} u)v := \int_\Gamma\int_\Gamma \left(\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))(v(x)-v(y))}{|x-y|^{\gamma p+d}}\right) d\mu_x d\mu_y\,, \tag{1.6}$$

respectively. Furthermore, the Venttsel' maps $\mathcal{A}_2 : D(\mathcal{E}^{(p)}) \to D(\mathcal{E}^{(p)})^*$ and $\mathcal{A}_3 : \mathcal{D}(\Gamma_3) \to \mathcal{D}(\Gamma_3)^*$ are defined by

$$(\mathcal{A}_2 u)v := \int_{\Gamma_2} d\mathcal{L}^{(p)}(u,v) := \mathcal{E}^{(p)}(u,v), \tag{1.7}$$

and

$$(\mathcal{A}_3 u)v := \int_I [(\mathcal{A}_2 u)v]\, ds + \int_{\Gamma_2}\int_I (|\partial_s u|^{p-2}\partial_s u)\partial_s v\, ds d\mu, \tag{1.8}$$

for $d\mathcal{L}^{(p)}(\cdot,\cdot)$ the p-Lagrangian form on Γ_2 (see section 3 for complete details about the p-energy form).

When we consider the corresponding eigenvalue problem associated with the Laplace operator Δ_p, such eigenvalue problem is formulated by

$$-\Delta_p v = \lambda |u|^{p-2}u \quad \text{in } \Omega,$$

subject to the general dynamical boundary conditions described in the equations (1.1) and (1.2). The above structure of the Then one has the first eigenvalue, given by the zero eigenvalue $\lambda_1 = 0$, from where we see that the condition (1.4) can be regarded as a sort of of an eigenvalue crossing of f. Henceforth, one can find constants M, $\eta_0 > 0$ such that

$$f(x,s) \ \geq\ \eta_0 |s|^{p-2}s - M, \qquad \text{if } \ s > 0, \tag{1.9}$$

and

$$f(x,s) \ \geq\ -\eta_0 |s|^{p-2}s - M, \qquad \text{if } \ s < 0. \tag{1.10}$$

Moreover, we will assume in addition that there exist constants α_0, $\gamma_0 > 0$ such that

$$f(x,s) + \alpha_0 |s|^{p-2}s \quad \text{is non-decreasing in } \ s \ \text{ on } \ [-L, L], \ \text{ for every } \ L > 0, \tag{1.11}$$

and

$$f(x,s) \ \leq\ \gamma_0(1 + |s|^{q-1}), \qquad \text{for all } \ s \in \mathbb{R}, \quad \text{and uniformly for } \ x \in \Omega, \tag{1.12}$$

where

$$1 \ < \ q \ < \begin{cases} \frac{Np}{N-p}, & \text{if } 1 < p < N, \\ \infty, & \text{if } p \geq N. \end{cases} \tag{1.13}$$

Next, we put:

$$V_p(\overline{\Omega_2}) := \left\{ u \in W^{1,p}(\Omega_2) \ : \ u|_{\Gamma_2} \in D(\mathcal{E}^{(p)}) \right\} \tag{1.14}$$

and

$$V_p(\overline{\Omega_3}) := \left\{ u \in W^{1,p}(\Omega_3) \ : \ u|_{\Gamma_3} \in \mathcal{D}(\Gamma_3) \right\}. \tag{1.15}$$

Then we define the function space:

$$\mathcal{W}_p(\overline{\Omega}) := \begin{cases} V_p(\overline{\Omega_2}), & \text{if } N = 2, \\ V_p(\overline{\Omega_3}), & \text{if } N = 3. \end{cases} \tag{1.16}$$

One has that $\mathcal{W}_p(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{V_p(\overline{\Omega})} := \left(\|u\|_{W^{1,p}(\Omega)} + \int_I [\mathcal{E}^{(p)}(u,u)]\, ds + \sigma_N \int_{\Gamma_2}\int_I |\partial_s v|^p\, ds d\mu \right)^{\frac{1}{p}}, \tag{1.17}$$

for $u \in \mathcal{W}_p(\overline{\Omega})$, where

$$\sigma_N := \begin{cases} 0, & \text{if } N = 2, \\ 1, & \text{if } N = 3. \end{cases} \tag{1.18}$$

We next introduce the notion of weak solutions for our boundary value problems (1.1) and (1.4).

Definition 1.2. (a) A function $u \in \mathcal{W}_p(\overline{\Omega})$ a called a **weak solution of either problem (1.1) ($N = 2$), or problem (1.2) ($N = 3$)**, if

$$\Lambda_p(u,\varphi) = \int_\Omega f(x,u)\varphi\,dx + \int_\Omega (\xi\phi + h)\varphi\,dx, \qquad \forall\,\varphi \in \mathcal{W}_p(\overline{\Omega}), \tag{1.19}$$

where

$$\Lambda_p(v,w) := \int_\Omega |\nabla v|^{p-2}\nabla v \nabla w\,dx + \int_I [\mathcal{E}^{(p)}(v,w)]\,ds + \\ + \sigma_{_N} \int_{\Gamma_2}\int_I (|\partial_s v|^{p-2}\partial_s v)\partial_s w\,ds d\mu + (\Theta_{_\Omega} v)w + (\Theta_{_\Gamma} v)w, \tag{1.20}$$

for $v,\, w \in \mathcal{W}_p(\overline{\Omega})$, where $\sigma_{_N}$ is defined as in (1.18).

(b) We call $u \in \mathcal{W}_p(\overline{\Omega})$ a **weak subsolution of problem (1.1) ($N = 2$), or problem (1.2) ($N = 3$)**, if

$$\Lambda_p(u,\varphi) \le \int_\Omega f(x,u)\varphi\,dx + \int_\Omega (\xi\phi + h)\varphi\,dx, \qquad \forall\,\varphi \in \mathcal{W}_p(\overline{\Omega}))^+, \tag{1.21}$$

where we recall that $\mathcal{W}_p(\overline{\Omega})^+ := \{u \in \mathcal{W}_p(\overline{\Omega}) \mid u \ge 0\}$. Moreover, $u \in \mathcal{W}_p(\overline{\Omega})$ is a **weak supersolution of problem (1.1) ($N = 2$), or problem (1.2) ($N = 3$)**, if the reverse inequality in (1.21) holds.

We now make the main assumptions that will be crucial for the main results.

Assumption 1.3. Given $N \in \{2,3\}$, $p \in (1,\infty)$, $d \in (N-p,N) \cap (0,N)$, let $\Omega \subseteq \mathbb{R}^N$ be given in accordance with (1.3) (for $D = \Omega$), assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the conditions (1.4), (1.11), and (1.12), let $\phi,\, h \in L^\infty(\Omega)$ with $\phi > 0$ a.e. in Ω, and let $\xi \in \mathbb{R}$ be a parameter.

We recall that if $p \ge N$, by the Sobolev embedding theorem, all the discussions and estimates of the paper become much easier, with many simplifications (see section 2). Therefore, when deriving a priori estimates, we will concentrate on the critical case, namely, when $p < N$.

Now we state the first main result of this paper, namely, an existence and nonexistence result.

Theorem 1.4. *Suppose that all the conditions of Assumption 1.3 are fulfilled. Then there exist parameters $\xi_0,\, \xi_0' \in \mathbb{R}$ such that*

1. *problems (1.1) and (1.2) are not solvable (over $\mathcal{W}_p(\overline{\Omega})$) for all $\xi > \xi_0$;*
2. *if $\xi \le \xi_0'$, then problems (1.1) and (1.2) are both solvable (over $\mathcal{W}_p(\overline{\Omega})$). Furthermore, the weak solutions of the equations (1.1) and (1.2) are globally bounded over $\overline{\Omega}$.*

In the preceding theorem, if $p \in (1, N]$, it is unknown (up to the present time) whether weak solutions of either problem (1.1), or problem (1.4), are globally continuous over $\overline{\Omega}$. Consequently, because of the general structure of the elliptic equations (1.1) and (1.4), we have not been able to deduce further results, such as multiplicity results, for the case $p \in (1, N]$. However, when $p > N$, then one can refine more the above result, as we see in our second main result of the paper.

Theorem 1.5. *Assume (in addition to the conditions in Theorem 1.4) that $p > N$, Then there exists a parameter $\xi_0 \in \mathbb{R}$, such that*

(1) problems (1.1) and (1.2) are not solvable (over $\mathcal{W}_p(\overline{\Omega})$) for all $\xi > \xi_0$;

(2) for each $\xi \leq \xi_0$, problems (1.1) and (1.2) admit respective minimal solution (in the weak sense), which are Hölder continuous over $\overline{\Omega}$.

Moreover, when f is locally Hölder continuous in $\mathbb{R}$, and uniformly a.e. on Ω, one gets that

(3) there exists $\xi_1 \leq \xi_0$ such that each of the problems (1.1) and (1.2) have at least two distinct solutions, whenever $\xi < \xi_1$.

(4) If in addition $f \in C(\overline{\Omega} \times \mathbb{R})$, then $\xi_1 = \xi_0$.

The literature related to Ambrosetti–Prodi-type problems is extensive, and has mostly concentrated in the Dirichlet problem. The motivation of this problem comes from the pioneering paper by Ambrosetti and Prodi [3]. The results in [3] opened the door to many generalizations, and further investigation of problems of this type, but in different frameworks (structures and boundary conditions). In particular, the Ambrosetti–Prodi problem for the p-Laplace operator and Dirichlet boundary conditions has been considered in [2, 4, 5, 25, 37, 39, 40], among many others. It is important to mention that for the Dirichlet problem, the regularity theory is not addressed,since the domain is assumed to be smooth (and thus the regularity results are standard and known). Complications related to the Dirichlet problem of Ambrosetti–Prodi-type arise when proving the non-existence of solutions, mainly because in this case, the constant functions are not in the corresponding function space $W_0^{1,p}(\Omega)$ where the Dirichlet problem is posed, and also because the first eigenvalue for the Dirichlet problem is strictly positive. To obtain non-existence results for the Dirichlet problem, some key estimates, such as the positivity of the first eigenvalue and Picone's identity, are required. The smoothness of the domain and the regularity of the solution play a crucial role in the application of such identity.

The Ambrosetti–Prodi problem for other boundary conditions is less known. On smooth domains, the Ambrosetti–Prodi problem with (local) Neumann boundary conditions has been addressed in [19, 20, 44, 46], and recently a generalization of the (local) Neumann problem of Ambrosetti–Prodi type to a large class of non-smooth domains has been considered in [47]. Furthermore, a nonlocal version of the Neumann problem was investigated in [48], where the domain was assumed to be a bounded Lipschitz domain, and in the same paper, the author introduced for the first time the Venttsel' problem (also denoted by Wentzell problem) of Ambrosetti–Prodi type (on bounded Lipschitz domains). To our knowledge, there is no literature concerning the Venttsel' problem of Ambrosetti–Prodi-type, other than the results in [48], where the Venttsel' operator is given by the p-Laplace-Beltrami operator over Γ It is important to point out that in this case, unlike the case of the Dirichlet problem, non-existence results are easier to be established, but on the other hand, other results and a priori estimates become much harder to be handled.

In the present paper, we turn our attention to the solvability of the nonlocal Ambrosetti–Prodi type problem with Venttsel' boundary conditions, on the domain Ω with fractal-like boundary Γ, given as in (1.3). Such boundary value problem has never been investigated before, and to our knowledge, there is no literature regarding the nonlocal Venttsel' problem Ambrosetti–Prodi-type on fractal domains. In fact, the only known results for (local) Ambrosetti–Prodi problems on fractal domains appears in [47] for the Neumann problem. Furthermore, when $\Omega = \Omega_3 \subseteq \mathbb{R}^3$, problem (1.2) is a mixed nonlocal Venttsel' - Neumann boundary value problem. To our knowledge, there are no results in the literature concerning Ambrosetti–Prodi problems with mixed boundary conditions. We also point out that problems (1.1) and (1.2), have a nonlocal term both in the interior of the domain and on the boundary.

We point out that in order to consider Venttsel' boundary conditions we introduce a suitable "surrogate" of the p-Laplacian on the boundary Γ of Ω. This operator has been introduced in the case $p = 2$ in [30, 29, 28], and recently generalized to the quasi-linear case in [16, 27] (for the case when $\Omega = \Omega_2 \subseteq \mathbb{R}^2$). Fractal boundaries and fractal layers are of great interest for those applications in which the surface effects are enhanced with respect to the surrounding volume (see [13, 14, 15] for details and motivations). Therefore, the interpretation of the equations (1.1) and (1.2) in a suitable sense represents substantial results for the theory of quasi-linear boundary value problems, as well as for the analysis on fractal domains.

We outline the plan of the paper. In section 2 we provide a brief construction of the domains under consideration, fix the notations, definitions, and state some intermediate well-know results that will be applied in the subsequent sections. In section 3, we define the p-energy functional related to the Ventssel'-type operator on Ω_2 and Ω_3. Section 4 concerns the regularity theory for weak solutions to problems (1.1) and (1.2). Under the general conditions outlined in Assumption 1.3, assuming that the boundary value problems (1.1) and (1.2) are solvable, we perform a priori estimates for weak solutions to both problems, as well as a priori estimates for the difference of weak solutions. As a consequence, we show that weak solutions of both problems (1.1) and (1.2) are globally bounded. For bounded Lipschitz domains and for the particular case $q = p$ (in (1.12)), these a priori estimates have been obtained in [48], and the same results for optimal growth condition (1.12) have been established for the (local) Neumann problem in [47]. Our approach will be inspired by the ones employed in [48, 47] but since we are dealing with more general conditions and assumptions, there are also substantial differences and generalizations in the present problem. Such variants will be addressed in detail. In section 5, we establish an alternative version of a sub-supersolution method for the boundary value problems (1.1) and (1.2), which will be a key tool for the establishment of the main results of the paper. To conclude, in section 6 we prove the main results of the paper, namely, Theorem 1.4 and Theorem 1.5.

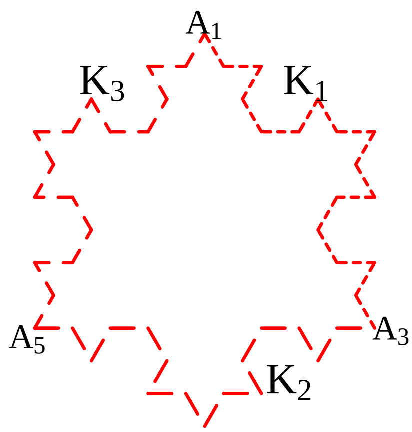

FIGURE 1. Pre-fractal Koch snowflake

2. **Preliminaries and intermediate results.** In this section we present some important (well-known) definitions, fix the notations that will be carried out in the subsequent sections, and state some known results that will be used in the later sections. All the arguments will be given under the conditions of Assumption 1.3.

2.1. **Geometry.** We denote by $|P - P_0|$ the Euclidean distance in $\mathbb{R}^n$ and by $B(P_0, r) = \{P \in \mathbb{R}^n : |P - P_0| < r\}$, $P_0 \in \mathbb{R}^n, r > 0$, the euclidean ball.
By the Koch snowflake Γ_2, we denote the union of three com-planar Koch curves K_1, K_2 and K_3. We assume that the junction points A_1, A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$.

K_1 is the uniquely determined self-similar set with respect to a family Ψ^1 of four suitable contractions $\psi_1^{(1)}, ..., \psi_4^{(1)}$, with respect to the same ratio $\frac{1}{3}$. Let $V_0^{(1)} := \{A_1, A_3\}$, $\psi_{i_1 \dots i_h} := \psi_{i_1} \circ \dots \circ \psi_{i_h}$, $V_{i_1 \dots i_h}^{(1)} := \psi_{i_1 \dots i_h}^{(1)}(V_0^{(1)})$ and

$$V_h^{(1)} := \bigcup_{i_1 \dots i_h = 1}^{4} V_{i_1 \dots i_h}^{(1)}.$$

We set $i|h = (i_1, i_2, \dots, i_h)$, $V_\star^{(1)} := \cup_{h \geq 0} V_h^{(1)}$. It holds that $K_1 = \overline{V_\star^{(1)}}$. Now let K_0 denote the unit segment whose endpoints are A_1 and A_3. We set $K_{i_1 \dots i_h} = \psi_{i_1 \dots i_h}(K_0)$ and $V(K_{i_1 \dots i_h}) = V_{i_1 \dots i_h}$. In a similar way, it is possible to approximate K_2, K_3 by the sequences $(V_h^{(2)})_{h \geq 0}$, $(V_h^{(3)})_{h \geq 0}$, and denote their limits by $V_\star^{(2)}, V_\star^{(3)}$.

In order to approximate Γ_2, we define the increasing sequence of finite sets of points $\mathcal{V}^h := \cup_{i=1}^{3} V_h^{(i)}, h \geq 1$ and $\mathcal{V}_\star := \cup_{h \geq 1} \mathcal{V}^h$. It holds that $\mathcal{V}_\star = \cup_{i=1}^{3} V_\star^{(i)}$ and $\Gamma_2 = \overline{\mathcal{V}_\star}$.
The Hausdorff dimension of the Koch snowflake is given by $d_2 = \frac{\log(4)}{\log(3)}$.
One can define, in a natural way, a finite Borel measure $\mu|_{\Gamma_2}$ supported on Γ_2 by

$$\mu|_{\Gamma_2} := \mu_1 + \mu_2 + \mu_3, \tag{2.1}$$

where μ_i denotes the normalized d_2-dimensional Hausdorff measure, restricted to K_i, $i = 1, 2, 3$.

Further, for any $n \geq 1$, we define a discrete measure on $V_n^{(i)}$ by:

$$\mu_n^i := \frac{1}{4^n} \sum_{p \in V_n^{(i)}} \delta_{\{p\}}, \tag{2.2}$$

where $\delta_{\{p\}}$ denotes the Dirac measure at the point p. We have the following result:

Proposition 2.1. *(see [31]) The sequence $(\mu_n^i)_{n\geq 1}$ is weakly convergent (i.e. in $C(K_i)'$) to the measure μ_i.*

The measure $\mu|_{\Gamma_2}$ is a d_2-measure (see [24]), that is, there exist two positive constants c_1, c_2, such that

$$c_1 r^{d_2} \leq \mu_{\Gamma_2}(B(P,r) \cap \Gamma_2) \leq c_2 r^{d_2},\ \forall P \in \Gamma_2$$

where $d_2 = \frac{\log(4)}{\log(3)}$.

In the following we denote by

$$F_{h+1} = \bigcup_{i=1}^{3} K_i^{(h+1)} \tag{2.3}$$

the closed polygonal curve approximating Γ_2 at the $(h+1)$-th step. We define $S_h = F_h \times I$, where $I = [0,1]$. By $\Omega_h \subset \mathbb{R}^2$ we denote the open bounded set having as boundary F_h. We denote by Q_h the three-dimensional cylindrical domain having S_h as "lateral surface" and the sets $\Omega_h \times \{0\}$ and $\Omega_h \times \{1\}$ as bases.

In an analogous way, we define the cylindrical-type surface $\Gamma_3 = \Gamma_2 \times I$ and we denote by Ω_2 the open bounded two-dimensional domain with boundary Γ_2. Also, by Ω_3 we denote the open cylindrical domain having Γ_3 as lateral surface, and the sets $\Omega_2 \times \{0\}$ and $\Omega_2 \times \{1\}$ as bases. We denote the points of Γ_3 and S_h by the couple $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plain containing Γ_2 and F_h respectively (for Γ_3 and S_h) and s is the coordinate of the orthogonal projection of P on the interval $(0,1)$, that is $(x_1, x_2) \in \Gamma_2$ (or $(x_1, x_2) \in F_h$ for the pre-fractal case) and $s \in I$.

We introduce on Γ_3 the measure

$$\mathrm{d}\mu|_{\Gamma_3} = \mathrm{d}\mu|_{\Gamma_2} \times \mathrm{d}s, \tag{2.4}$$

where $\mathrm{d}s$ is the one-dimensional Lebesgue measure on I. It follows that the measure $\mu|_{\Gamma_3}$ is also a d_3-measure for $d_3 := 1 + \frac{\log(4)}{\log(3)}$. Then we set

$$d := \begin{cases} d_2, & \text{if } N = 2, \\ d_3, & \text{if } N = 3, \end{cases} \tag{2.5}$$

and μ is considered as in 1.3.

2.2. **Functional spaces.** By $W^{1,p}(\Omega)$ we mean the well-known L^p-based Sobolev space. Also, given $E \subseteq \mathbb{R}^m$ a d-set in the sense of [24], and given $\beta \in (0,1)$, $d \in (0, N]$, we define the *Besov space* $\mathbb{B}_\beta^p(E, \eta)$ as the set of functions $u \in L^p(E, d\eta)$ for which the semi-norm

$$\mathcal{N}_\beta^p(u, E, \eta) := \left(\int_E \int_E \left[\frac{|u(x) - u(y)|}{|x-y|^{\beta + \frac{d}{p}}} \right]^p d\eta_x d\eta_y \right)^{\frac{1}{p}} \tag{2.6}$$

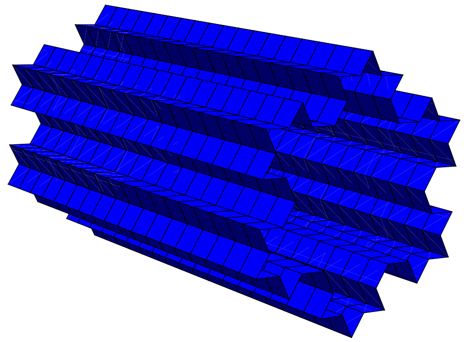

FIGURE 2. Surface S_3

is finite. Additionally, for $r,\, s \in [1,\infty)$, or $r = s = \infty$, we will often refer at times to the Banach Space defined in [50], i.e.,

$$\mathbb{X}^{r,s}(\Omega,\Gamma) := \{(f,g)\,:\, f \in L^r(\Omega),\; g \in L^s(\Gamma)\}\,,$$

endowed with norm

$$|||(f,g)|||_{r,s} := \|(f,g)\|_{\mathbb{X}^{r,s}(\Omega,\Gamma)} := \|f\|_{r,\Omega} + \|g\|_{s,\Gamma}\,, \quad \text{ if } r,\, s \in [1,\infty),$$

and

$$|||(f,g)|||_{\infty} := \|(f,g)\|_{\mathbb{X}^{\infty,\infty}(\Omega,\Gamma)} := \max\left\{\|f\|_{\infty,\Omega},\, \|g\|_{\infty,\Gamma}\right\}.$$

We stress that when $r = s$, we will write $\mathbb{X}^r(\Omega,\Gamma) := \mathbb{X}^{r,s}(\Omega,\Gamma)$ and $|||(f,g)|||_r := |||(f,g)|||_{r,s}$, respectively. For more information and properties regarding the Sobolev spaces under the definitions above, refer to [7, 9, 24, 38, 43] (among others). For completeness, we quote the following properties (e.g. [7, 18, 23, 32, 24, 51, 52, 53]).

- $W^{1,p}(\Omega) \hookrightarrow L^{\frac{pN}{N-p}}(\Omega)$, and hence there exists a constant $c_1 > 0$, such that

$$\|u\|_{\frac{pN}{N-p},\Omega} \leq c_1 \|u\|_{W^{1,p}(\Omega)}, \qquad \forall\, u \in W^{1,p}(\Omega). \tag{2.7}$$

Moreover, if $1 < r < \frac{pN}{N-p}$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.

- There exists a linear continuous operator from $W^{1,p}(\Omega)$ into $L^{\frac{pd}{N-p}}(\Gamma)$, and thus there exists a constant $c_2 > 0$, such that

$$\|u\|_{\frac{pd}{N-p},\Gamma} \leq c_2 \|u\|_{W^{1,p}(\Omega)}, \qquad \forall\, u \in W^{1,p}(\Omega). \tag{2.8}$$

Moreover, if $1 < s < \frac{pd}{N-p}$, then the operator from $W^{1,p}(\Omega)$ into $L^s(\Gamma)$ is compact.

- If $p = N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and the trace of $W^{1,p}(\Omega)$ lies in $L^r(\Gamma)$ continuously, for all $q,\, r \in [1,\infty)$, while in the case $p > N$, we have that $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for some $\delta \in (0,1)$.

- There exists a constant $c_3 > 0$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq c_3 \|\nabla u\|_{p,\Omega}, \tag{2.9}$$

for every $u \in W^{1,p}(\Omega)$ such that $\int_E u\,dx = 0$ for a measurable set $E \subseteq \Omega$ with $\lambda_N(E) > 0$.

- If $p \in (1, N)$ and $s \in (0, 1)$, then there exists a linear continuous mapping from $W^{1,p}(\Omega)$ into $\mathbb{B}^p_s(\Omega)$ and a constant $c_4 > 0$ such that

$$\|u\|_{\mathbb{B}^p_s(\Omega)} \le c_4 \|u\|_{W^{1,p}(\Omega)}, \qquad \textit{for all } u \in W^{1,p}(\Omega). \tag{2.10}$$

- If $p \in (1, N)$, then there exists a linear continuous mapping from $W^{1,p}(\Omega)$ into $\mathbb{B}^p_{1-\frac{N-d}{p}}(\Gamma, \mu)$ and a constant $c_5 > 0$ such that

$$\|u\|_{\mathbb{B}^p_{1-\frac{N-d}{p}}(\Gamma,\mu)} \le c_5 \|u\|_{W^{1,p}(\Omega)}, \qquad \textit{for all } u \in W^{1,p}(\Omega). \tag{2.11}$$

We complete this section by collecting some well-known analytical results that will be applied throughout the subsequent section.

Lemma 2.2. *(see [41]) Let $\varphi = \varphi(t)$ be a nonnegative, non-increasing function on a half line $\{t \ge k_0 \ge 0\}$, such that there exist c, $\alpha > 0$, and there exists $\delta > 1$ with*

$$\varphi(h) \le c\,(h-k)^{-\alpha} \varphi(k)^{\delta},$$

for $h > k \ge k_0$. Then

$$\varphi(k_0 + \varsigma) = 0,$$

where

$$\varsigma^{\alpha} = c\,\varphi(k_0)^{\delta-1} 2^{\alpha\delta(\delta-1)}.$$

Lemma 2.3. *(see [26]) Let u be an integrable function over a bounded open set D, such that for arbitrary $k \ge k_0 > 0$,*

$$\int_D (u-k)\chi_{\{u>k\}}\,dx \le \gamma k^{\alpha} \lambda_N(\{x \in D \mid u(x) > k\})^{1+\epsilon},$$

where γ, α, ϵ are constants such that $\epsilon > 0$ and $0 \le \alpha \le 1+\epsilon$. Then there exists a constant C, depending on γ, α, ϵ, k_0, and $\left\|u\chi_{\{u>k\}}\right\|_{1,D}$, such that $\|u\|_{\infty,D} \le C$.

Proposition 2.4. *(see [6]) Let a, $b \in \mathbb{R}^N$ and $r \in (1, \infty)$. Then there exists a constant $c_r > 0$ such that*

$$\left(|a|^{r-2}a - |b|^{r-2}b\right)(a-b) \ge c_r\left(|a| + |b|\right)^{r-2} |a-b|^2 \ge 0. \tag{2.12}$$

If in addition $r \in [2, \infty)$, then there exists a constant $c_r^* \in (0, 1]$ such that

$$\left(|a|^{r-2}a - |b|^{r-2}b\right)(a-b) \ge c_r^* |a-b|^r, \tag{2.13}$$

and also in this case there is a constant $c_r' \in (0, 1]$ such that

$$sgn(a-b)\left(|a|^{r-2}a - |b|^{r-2}b\right) \ge c_r' |a-b|^{r-1}. \tag{2.14}$$

Finally, if $r \in (1, 2]$ and $\epsilon > 0$, then for each a, $b \in \mathbb{R}^N$ with $|a-b| \ge \epsilon \min\{|a|, |b|\}$, we find a constant $c_{r,\epsilon} > 0$ such that

$$\left\langle |a|^{r-2}a - |b|^{r-2}b, a-b\right\rangle \ge c_{r,\epsilon} |a-b|^r, \tag{2.15}$$

and

$$sgn(a-b)\left(|a|^{r-2}a - |b|^{r-2}b\right) \ge c_{r,\epsilon} |a-b|^{r-1}. \tag{2.16}$$

Proposition 2.5. *(see [6]) Let ξ, ς, c, τ, $\varrho \in [0,\infty)$ and $r \in [1,\infty)$, and assume that the parameter $\xi \leq \tau\epsilon^{-\varrho}\varsigma + \epsilon^r c$ for all $\epsilon > 0$. Then one has*

$$\xi \;\leq\; (\tau+1)\left[\varsigma^{\frac{r}{r+\varrho}}\, c^{\frac{\varrho}{r+\varrho}} + \varsigma\right]. \tag{2.17}$$

3. **The p-forms on Koch-type domains.** In [11] the p-energy forms on the Koch curve have been constructed.
For $f : V_\star^{(i)} \to \mathbb{R}$, we define for $1 < p < \infty$

$$\mathcal{E}^{(p)}_{n,i}[f] = \frac{1}{p}4^{(p-1)n} \sum_{i_1,\dots,i_n=1}^{4} \sum_{\xi,\eta\in V_0^{(i)}} |f(\psi_{i_1\dots i_n}(\xi)) - f(\psi_{i_1\dots i_n}(\eta))|^p, \tag{3.1}$$

it has been shown that the sequence $\mathcal{E}^{(p)}_n[f]$ is non-decreasing, and by defining for $f : V_\star^{(i)} \to \mathbb{R}$

$$\mathcal{E}^{(p)}_i[f] = \lim_{n\to\infty} \mathcal{E}^{(p)}_{n,i}[f], \tag{3.2}$$

the set

$$\mathcal{F}^{(p)}_{\star,i} = \{f : V_\star^{(i)} \to \mathbb{R} : \mathcal{E}^{(p)}[f] < \infty\} \tag{3.3}$$

does not degenerate to a space containing only constant functions. Each $f \in \mathcal{F}^{(p)}_{\star,i}$ can be uniquely extended in $C(K_i)$. We denote this extension on K_i still by f and we define the space

$$D(\mathcal{E}^{(p)}_i) = \{f \in C(K_i) : \mathcal{E}^{(p)}[f] < \infty\}, \tag{3.4}$$

where $\mathcal{E}^{(p)}_i[f] := \mathcal{E}^{(p)}_i[f|_{V^\star}]$. Hence $D(\mathcal{E}^{(p)}_i) \subset C(K_i) \subset L^p(K_i,\mu)$. Moreover, $(\mathcal{E}^{(p)}_i, D(\mathcal{E}^{(p)}_i))$ is a non-negative energy functional in $L^p(K_i,\mu_i)$ and the following result holds.

Theorem 3.1. *(see [11]) The following properties hold.*

(i) $D(\mathcal{E}^{(p)}_i)$ *is complete in the norm* $\|f\|_{D(\mathcal{E}^{(p)}_i)} := \|f\|_{L^p(K^i,\mu_i)} + (\mathcal{E}^{(p)}_i[f])^{1/p}$.

(ii) $D(\mathcal{E}^{(p)}_i)$ *is dense in* $L^p(K_i,\mu_i)$.

(iii) $D(\mathcal{E}^{(q)}_i) \subset D(\mathcal{E}^{(p)}_i)$, *for* $1 < p \leq q < \infty$.

3.1. **p-Lagrangians on the Koch curve and on the Koch snowflake.** In this subsection, we recall the main properties of the p-Lagrangian on the Koch curve. For the concept of Lagrangians on fractals, i.e. the notion of a measure valued local energy, we refer to [22] and [42].

We also have the following:

Proposition 3.2. *(see [10]) Let A be any subset of K_i. For every $u \in D\left(\mathcal{E}^{(p)}_i\right)$, the sequence of measures given by*

$$\widetilde{\mathcal{L}}^{(p)}_{n,i}(u)(A) := \frac{4^{(p-1)n}}{p} \sum_{i_1,\dots,i_n=1}^{4} \sum_{\substack{\xi,\eta\in V_0^{(i)} \\ \psi_{i_1\dots i_n}(\xi),\psi_{i_1\dots i_n}(\eta)\in A}} |u(\psi_{i_1\dots i_n}(\xi)) - u(\psi_{i_1\dots i_n}(\eta))|^p. \tag{3.5}$$

converges to a positive semidefinite additive Borel measure

$$\widetilde{\mathcal{L}}_i^{(p)}(u)(A) := \lim_{n\to\infty} \widetilde{\mathcal{L}}_{n,i}^{(p)}(u)(A) \tag{3.6}$$

the so-called p-Lagrangian measure on K_i. Moreover it holds that for every $p>1$,

- $\tilde{\mathcal{L}}_i^{(p)}$ *is positive semidefinite and convex*
- $\tilde{\mathcal{L}}_i^{(p)}$ *is homogeneous of degree p*
- $\tilde{\mathcal{L}}_i^{(p)}$ *is such that* $\|u\| = (\int_{K_i} |u|^p d\mu + \int_{K_i} d\tilde{\mathcal{L}}_i^{(p)}(u))^{1/p}$ *is a norm in* $D\left(\mathcal{E}_i^{(p)}\right)$
- *for every* $u,v \in D\left(\mathcal{E}_i^{(p)}\right)$ *there exists in the weak $*$ topology of M (the set of Radon measures) the following limit*

$$\lim_{t\to 0} \frac{\widetilde{\mathcal{L}}_i^{(p)}(u+tv) - \widetilde{\mathcal{L}}_i^{(p)}(u)}{t} = \left\langle \partial \widetilde{\mathcal{L}}_i^{(p)}(u), v\right\rangle$$

Moreover, it holds that

$$\mathcal{E}_i^{(p)}(u,v) = \int_{K_i} d\mathcal{L}_i^{(p)}(u,v), \quad u,v \in D\left(\mathcal{E}_i^{(p)}\right). \tag{3.7}$$

where we set $\mathcal{L}_i^{(p)}(u,v) : D\left(\mathcal{E}_i^{(p)}\right) \times D\left(\mathcal{E}_i^{(p)}\right) \to M$, *with*

$$\mathcal{L}_i^{(p)}(u,v) = \left\langle \partial \widetilde{\mathcal{L}}_i^{(p)}(u), v\right\rangle. \tag{3.8}$$

We note that by straightforward calculations it holds that:

$$\left\langle \widetilde{\mathcal{L}}_{n,i}^{(p)}(u), v\right\rangle = \frac{4^{(p-1)n}}{p} \sum_{i_1,\dots,i_n=1}^{4} \sum_{\xi,\eta\in V_0^{(i)}} |u(\psi_{i_1\dots i_n}(\xi)) - u(\psi_{i_1\dots i_n}(\eta))|^{p-2}$$
$$(u(\psi_{i_1\dots i_n}(\xi)) - u(\psi_{i_1\dots i_n}(\eta)))(v(\psi_{i_1\dots i_n}(\xi)) - v(\psi_{i_1\dots i_n}(\eta))),$$

so proceeding as in [10] we have

$$\lim_{n\to\infty} \left\langle \widetilde{\mathcal{L}}_{n,i}^{(p)}(u), v\right\rangle = \left\langle \widetilde{\mathcal{L}}_i^{(p)}(u), v\right\rangle. \tag{3.9}$$

Finally by proceeding as in [21, Section 4.1-4.2] one can define on Γ_2 a p-Lagrangian $(\mathcal{L}^{(p)}, D(\mathcal{E}^{(p)}))$ and a p-energy form $(\mathcal{E}^{(p)}, D(\mathcal{E}^{(p)}))$. More precisely $\mathcal{E}^{(p)}(u,v) = \int_{\Gamma_2} d\mathcal{L}^{(p)}(u,v)$, for every $u,v \in D(\mathcal{E}^{(p)})$ where $D(\mathcal{E}^{(p)}) = \{u \in C(\Gamma_2) : u|_{K_i} \in D(\mathcal{E}_i^{(p)})\}$, *for* $i = 1,2,3$. In particular it holds (see [21, Thoerem 4.6]) that

$$\mathcal{E}^{(p)}[u] = \sum_{i=1}^{3} \mathcal{E}_i^{(p)}[u|_{K_i}] \tag{3.10}$$

We recall that from [12, Theorem 4.1], it follows that $D(\mathcal{E}^{(p)})$ can be characterized in terms of Lipschitz spaces with equivalent norms:

$$D(\mathcal{E}^{(p)}) = Lip_{d_f,d_f}(p,\infty,\Gamma_2),$$

$$D(\mathcal{E}^{(p)}) \subset \mathbb{B}_\alpha^p(\Gamma_2, \mu|_{\Gamma_2}), \text{ for every } \alpha < d. \tag{3.11}$$

We now define the energy form on Γ_3:

$$E_{\Gamma_3}[u] = \frac{1}{p}\int_I \mathcal{E}^{(p)}[u]\mathrm{d}s + \frac{1}{p}\int_{\Gamma_2}\int_I |\partial_s u|^p \mathrm{d}s\mathrm{d}\mu|_{\Gamma_3} \tag{3.12}$$

with domain $\mathcal{D}(\Gamma_3)$ defined as

$$\mathcal{D}(\Gamma_3) = \overline{C(\Gamma_3) \cap L^p([0,1]; D(\mathcal{E}^{(p)})) \cap W^{1,p}([0,1]; L^p(\Gamma_2))}^{\|\cdot\|_{\mathcal{D}(\Gamma_3)}}, \tag{3.13}$$

where $\|\cdot\|_{\mathcal{D}(\Gamma_3)}$ is the intrinsic norm

$$\|u\|_{\mathcal{D}(\Gamma_3)} = \left(E_{\Gamma_3}[u] + \|u\|^p_{p,\Gamma_3}\right)^{\frac{1}{p}}. \tag{3.14}$$

We now give an embedding result for the domain $\mathcal{D}(\Gamma_3)$. Unlike the two dimensional case where there is a characterization of the functions in $D(\mathcal{E}^{(p)})$ in terms of the so-called Lipschitz space, for $\mathcal{D}(\Gamma_3)$ we do not have such characterization, but the following result holds.

Proposition 3.3. *(see [17]) $\mathcal{D}(S) \hookrightarrow \mathbb{B}^p_{\bar\beta}(\Gamma_3, \mu|_{\Gamma_3})$ for any $0 < \bar\beta < 1$.*

From the above results it follows in particular that the spaces $V_p(\bar\Omega_2)$ and $V_p(\bar\Omega_3)$ are non trivial. Finally, we recall that for the remaining of the paper, we will consider the sets Ω, Γ, and the measure μ on Γ, in accordance with (1.3).

4. **Global regularity for weak solutions.** The following section is devoted to establish global regularity results for weak solutions of both Eq. (1.1) and (1.2), under the conditions of Assumption 1.3.

Given $u \in \mathcal{W}_p(\overline{\Omega})$ and $k \geq 0$ a fixed real number, we put

$$u_k := (u-k)^+, \qquad u_k^- := (u^- - k)^+, \qquad \text{and} \qquad \hat u_k := (|u|-k)^+ \operatorname{sgn}(u). \tag{4.1}$$

Clearly u_k, u_k^-, $\hat u_k \in \mathcal{W}_p(\overline{\Omega})$. Set

$$A_k := \{x \in \overline{\Omega} \mid |u| > k\}, \quad A_k^+ := \{x \in \overline{\Omega} \mid u > k\}, \quad A_k^- := \{x \in \overline{\Omega} \mid u^- > k\}. \tag{4.2}$$

Then, for each $D \subseteq \mathbb{R}^N$ such that $D \cap \overline{\Omega} \neq \emptyset$, taking into account (4.1) and (4.2), we write

$$D_k := D \cap A_k, \qquad D_k^{\pm} := D \cap A_k^{\pm}. \tag{4.3}$$

We stress that throughout the remaining of the paper, the measurable set D will be either the interior Ω, or the boundary $\Gamma := \partial\Omega$. Finally, we put

$$p_{_N} := \frac{Np}{N-p} \qquad \text{and} \qquad p_{_d} := \frac{pd}{N-p}. \tag{4.4}$$

Next, in view of the above notations, we now derive L^∞-type estimates for weak solutions of both Eq. (1.1) and Eq. (1.2), and a priori estimates for the difference of weak solutions of problem (1.1), and problem (1.2).

Proposition 4.1. *Let $u \in \mathcal{W}_p(\overline{\Omega})$. Then it holds that $\mathcal{E}^{(p)}(u, u_k^-) \leq 0$ and $E_S(u, u_k^-) \leq 0$.*

Proof. Since

$$\mathcal{E}^{(p)}(w,v) := \int_{\Gamma_2} d\tilde{\mathcal{L}}^{(p)}(w,v) = \left\langle \partial\mathcal{L}^{(p)}(w), v \right\rangle = \lim_{n\to\infty} \left\langle \partial\tilde{\mathcal{L}}^{(p)}_n(w), v \right\rangle, \tag{4.5}$$

for w, $v \in D(\mathcal{E}^{(p)})$, where

$$\left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(w), v\right\rangle := \frac{1}{p}\sum_{\kappa_i=1}^{4}\sum_{\xi,\varsigma\in V_0^{(i)}}\sum_{\xi\neq\varsigma}\frac{|w(\psi_{\kappa_i}(\xi))-w(\psi_{\kappa_i}(\varsigma))|^{p-2}}{|\psi_{\kappa_i}(\xi)-\psi_{\kappa_i}(\varsigma)|^{dp}} \times\left(w(\psi_{\kappa_i}(\xi))-w(\psi_{\kappa_i}(\varsigma))\right)\left(v(\psi_{\kappa_i}(\xi))-v(\psi_{\kappa_i}(\varsigma))\right)\mu_n, \tag{4.6}$$

for $\kappa_i := i_1,\dots,i_n$. Then, one sees easily that $\left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(u), u_k^-\right\rangle \leq 0$ for each $n\in\mathbb{N}$, and thus passing to thee limit, we deduce that $\mathcal{E}^{(p)}(u,u_k^-)\leq 0$. From the results for the two dimensional case and the monotonicity of the integral it follows that $E_{\Gamma_3}(u,u_k^-)\leq 0$. □

For $u,\, v\in\mathcal{W}_p(\overline{\Omega})$, we define

$$K^{(p)}(u,v) := \int_I[\mathcal{E}^{(p)}(u,v)]\,dt + \sigma_{_N}\int_{\Gamma_2}\int_I|\partial_t u|^{p-2}uv\,dtd\mu, \tag{4.7}$$

for $\sigma_{_N}$ defined as in (1.18). Then, (1.20) becomes

$$\Lambda_p(u,v) := \int_\Omega|\nabla v|^{p-2}\nabla u\nabla v\,dx + K^{(p)}(u,v) + (\Theta_{_\Omega}u)v + (\Theta_{_\Gamma}u)v$$

Then we have the following key result.

Theorem 4.2. *Suppose that ξ belongs to a bounded interval. If $u\in\mathcal{W}_p(\overline{\Omega})$ is a weak solution of either (1.1), or (1.2), then there exists a constant $C_\xi\geq 0$ large enough (and independent of u) such that $\|u\|_{\infty,\Omega}\leq C_\xi$.*

Proof. We will prove both cases at the same time. Indeed, let $u\in\mathcal{W}_p(\overline{\Omega})$ be a weak solution of either (1.1), or (1.2). We will show that the L^∞-norm of both u^- and u^+ are bounded by C_ξ.

• First we show that $\|u^-\|_{\infty,\Omega}\leq C_\xi$ for C_ξ large enough. Indeed, given $k>0$ a real number, let $u_k^-\in\mathcal{W}_p(\overline{\Omega})$ be the function defined in (4.1). It is easy to see that $(\Theta_{_\Omega}u)u_k^-\leq 0$ and $(\Theta_{_\Gamma}u)u_k^-\leq 0$ (recall that $\Theta_{_\Omega}$ and $\Theta_{_\Gamma}$ are defined as in (1.5) and (1.6), respectively). From Proposition 4.1 and the definition of $K^{(p)}(\cdot,\cdot)$ in (4.7), it follows that $K^{(p)}(u,u_k^-)\leq 0$. By testing (1.19) with u_k^- and applying (1.10), we obtain

$$\begin{aligned}\|\nabla u_k^-\|^p_{p,\Omega_k^-} &\leq -\Lambda_p(u,u_k^-) = -\int_{\Omega_k^-}f(x,u)u_k^-\,dx - \int_{\Omega_k^-}(\xi\phi+h)u_k^-\,dx\\ &\leq \eta_0\int_{\Omega_k^-}(u^--k)^p\,dx + \left(\eta_0k^{p-1}+M+|\xi|\|\phi\|_{\infty,\Omega}+\|h\|_{\infty,\Omega}\right)\int_{\Omega_k^-}(u^--k)\,dx.\end{aligned}$$

Now, by Hölder's inequality together with (2.7), one sees that

$$\int_{\Omega_k^-}(u^--k)^p\,dx \leq \lambda_N(\Omega_k^-)^{\frac{p}{N}}c_1\left(\|\nabla u_k^-\|^p_{p,\Omega_k^-} + \int_{\Omega_k^-}(u^--k)^p\,dx\right). \tag{4.8}$$

Hence, combining (4.8) with the first estimate, and rearranging, yields that

$$\left(\lambda_N(\Omega_k^-)^{-\frac{p}{N}} - c_1(1+\eta_0)\right)\int_{\Omega_k^-}(u^--k)^p\,dx$$

$$\leq c_1 \left(\eta_0 k^{p-1} + M + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right) \int_{\Omega_k^-} (u^- - k)\, dx. \tag{4.9}$$

We note that given $u \in \mathcal{W}_p(\overline{\Omega})$ a weak solution of either (1.1), or (1.2) we can test (1.19) with the function $w \equiv 1$. Then, we have $(\Theta_\Omega u)1 = (\Theta_\Gamma u)1 = 0$, and also in view of (4.5) and (4.6), we deduce that $\mathcal{E}^{(p)}(u,1) = 0$. Hence, applying (1.9) and (1.10), we are lead to the following:

$$0 = \int_\Omega (f(x,u) + \xi\phi + h)\, dx \geq \eta_0 \int_\Omega |u|^{p-1}\, dx - (|\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega} + M)\lambda_N(\Omega),$$

which implies that

$$\int_\Omega |u|^{p-1}\, dx \leq \frac{|\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega} + M}{\eta_0} \lambda_N(\Omega) := C_\Omega. \tag{4.10}$$

On the other hand, since

$$\int_\Omega |u|^{p-1}\, dx \geq \int_{\Omega_k^-} (u^-)^{p-1}\, dx \geq k^{p-1}\lambda_N(\Omega_k^-),$$

taking into account (4.10) we get

$$\lambda_N(\Omega_k^-) \leq \frac{1}{k^{p-1}} \int_\Omega |u|^{p-1}\, dx \leq \frac{C_\Omega}{k^{p-1}}. \tag{4.11}$$

Therefore, (4.11) implies that $\lim_{k\to\infty} \lambda_N(\Omega_k^-) = 0$, and thus we can find a constant $k_0 > 0$ sufficiently large (and independent of u), such that

$$\lambda_N(\Omega_k^-)^{-\frac{p}{N}} - c_1(1+\eta_0) \geq \frac{\lambda_N(\Omega_k^-)^{-\frac{p}{N}}}{2}, \qquad \text{for all } k \geq k_0. \tag{4.12}$$

By Hölder's inequality and (4.9) one has

$$\int_{\Omega_k^-} (u^- - k)\, dx \leq \lambda_N(\Omega_k^-)^{\frac{1}{p'}} \left(\frac{c_1(\eta_0 k^{p-1} + M + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega})}{\lambda_N(\Omega_k^-)^{-\frac{p}{N}} - c_1(1+\eta_0)} \int_{\Omega_k^-} (u^- - k)\, dx \right)^{1/p},$$

and thus combining this inequality with (4.12) we have

$$\begin{aligned}
\int_{\Omega_k^-} (u^- - k)\, dx &\leq \lambda_N(\Omega_k^-)^{1+\frac{p'}{N}} \left(\frac{c_1(\eta_0 k^{p-1} + M + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega})}{1 - c_1(1+\eta_0)\lambda_N(\Omega_k^-)^{-\frac{p}{N}}} \right)^{1/(p-1)} \\
&\leq \lambda_N(\Omega_k^-)^{1+\frac{p'}{N}} k \left[\frac{c_1}{2} \left(\eta_0 + \frac{M + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}}{k^{p-1}} \right) \right]^{1/(p-1)} \\
&\leq \lambda_N(\Omega_k^-)^{1+\frac{p'}{N}} k \left[\frac{c_1}{2} \left(\eta_0 + \frac{M + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}}{k_0^{p-1}} \right) \right]^{1/(p-1)}.
\end{aligned} \tag{4.13}$$

From this last calculation together with Lemma 2.3 we obtain that $\|u^-\|_{\infty,\Omega}$ is bounded by a constant depending on p, N, η_0, M, ξ, $\|\phi\|_{\infty,\Omega}$, $\|h\|_{\infty,\Omega}$, and $\|u^-\|_{1,\Omega_k^-}$. Furthermore, examining the proof of Lemma 2.12 (see [26, proof of Lemma 5.1]), it remains to find an upper bound for $\|u^-\|_{1,\Omega_k^-}$, independent of k and u. To achieve this, test (1.19) with $v=-u^-$. Then we see that in this case $(\Theta_\Omega u)v\geq 0$, $(\Theta_\Gamma u)v\geq 0$, and $K^{(p)}(u,v)\geq 0$. Hence, recalling (1.10), and applying Young's inequality, we deduce that for all $\epsilon>0$,

$$\begin{aligned}0 \,\le\, \|\nabla u^-\|^p_{p,\Omega}+\mathcal{E}^{(p)}(u^-,u^-) \,\le\, \Lambda_p(u,v) &= -\int_{\Omega_k^-}(f(x,u)+\xi\phi+h)u^-\,dx\\ &\le\, -\eta_0\int_\Omega (u^-)^p\,dx+(M+|\xi|\|\phi\|_{\infty,\Omega}+\|h\|_{\infty,\Omega})\int_{\Omega_k^-}u^-\,dx\\ &\le\, -\eta_0\int_\Omega (u^-)^p\,dx+\frac{(M+|\xi|\|\phi\|_{\infty,\Omega}+\|h\|_{\infty,\Omega})^{p'}\lambda_N(\Omega)}{\epsilon^{1/(p-1)}}+\epsilon\int_\Omega (u^-)^p\,dx.\end{aligned}$$

Letting $\epsilon=\eta_0/2>0$ and rearranging, one sees that

$$\int_\Omega (u^-)^p\,dx \,\le\, \left(\frac{2(M+|\xi|\|\phi\|_{\infty,\Omega}+\|h\|_{\infty,\Omega})}{\eta_0}\right)^{p'}\lambda_N(\Omega), \tag{4.14}$$

and thus (4.14) gives

$$\|u^-\|_{1,\Omega_k^-} \,\le\, \lambda_N(\Omega)^{\frac{1}{p'}}\left(\int_\Omega (u^-)^p\,dx\right)^{1/p} \,\le\, \left(\frac{2(M+|\xi|\|\phi\|_{\infty,\Omega}+\|h\|_{\infty,\Omega})}{\eta_0}\right)^{\frac{1}{p-1}}\lambda_N(\Omega), \tag{4.15}$$

proving the desired conclusion.

• Next, we show that $\|u^+\|_{\infty,\Omega}\leq C_\xi$. Given $k>0$ a real number, let $u_k^+\in W^{1,p}(\Omega)$ be the function defined in (4.1). Proceeding as in [50, proof of Theorem 5.7] and recalling (4.5) and (4.6), we see that $(\Theta_\Omega u)u_k^+\geq 0$, $(\Theta_\Gamma u)u_k^+\geq 0$, and $K^{(p)}(u,u_k^+)\geq 0$. Furthermore, as in the previous case, one can easily deduce that $\lim\limits_{k\to\infty}\lambda_N(\Omega_k^+)=0$. Then, testing (1.19) with u_k^+ and applying (1.12), we obtain

$$\begin{aligned}\|\nabla u_k^+\|^p_{p,\Omega_k^+} \,\le\, \Lambda_p(u,u_k^+) &= \int_{\Omega_k^+} f(x,u)u_k^+\,dx+\int_{\Omega_k^+}(\xi\phi+h)u_k^+\,dx\\ &\le\, \gamma_0\int_{\Omega_k^+}(u-k)^q\,dx+\big(\gamma_0(k^{q-1}+1)+|\xi|\|\phi\|_{\infty,\Omega}+\|h\|_{\infty,\Omega}\big)\int_{\Omega_k^+}(u-k)\,dx.\end{aligned} \tag{4.16}$$

At this point, it suffices to consider the case when $q\geq p$. For if $1<q<p$, then in view of (1.12), by virtue of Young's inequality, there exist constants $\gamma_0^*>0$ and $q^*\geq p>q$ such that

$$f(x,s) \,\le\, \gamma_0(1+|s|^{q-1}) \,\le\, \gamma_0^*(1+|s|^{q^*-1}),\quad \text{for all } s\in\mathbb{R},\ \text{ and uniformly for } x\in\Omega.$$

Then, since $\lambda_N(\Omega_k^+)\overset{k\to\infty}{\longrightarrow}0$ (proof similar as previous case), we select $k_1>0$ large enough, such that $\|\nabla u_k^+\|_{p,\Omega_k^+}\leq 1$ for all $k\geq k_1$. Thus for each $k\geq k_1$, we have

$$\|\nabla u_k^+\|^q_{p,\Omega_k^+} \,\le\, \|\nabla u_k^+\|^p_{p,\Omega_k^+} \qquad (\,q\geq p\,). \tag{4.17}$$

Applying Hölder's inequality and recalling (2.7), one sees that

$$\int_{\Omega_k^+} (u-k)^q\,dx \,\le\, \lambda_N(\Omega_k^+)^{1-\frac{q(N-p)}{Np}}\, c_1 \left(\|\nabla u_k^+\|^q_{p,\Omega_k^-} + \lambda_N(\Omega_k^+)^{(\frac{q}{p}-1))} \int_{\Omega_k^+} (u-k)^q\,dx \right). \tag{4.18}$$

In view of (4.18), (4.16) and (4.17), we have

$$\begin{aligned}\left(\lambda_N(\Omega_k^+)^{\frac{q(N-p)}{Np}-1} - c_1(\lambda_N(\Omega_k^+)^{\frac{q}{p}-1} + \gamma_0) \right) \int_{\Omega_k^+} (u-k)^q\,dx \\ \le c_1\left(\gamma_0(k^{q-1}+1) + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right) \int_{\Omega_k^+} (u-k)\,dx. \end{aligned}\tag{4.19}$$

In the same way as before, we can find a constant $k_2 > 0$ sufficiently large (and independent of u), such that

$$\lambda_N(\Omega_k^+)^{\frac{q(N-p)}{Np}-1} - c_1(\lambda_N(\Omega_k^+)^{\frac{q}{p}-1} + \gamma_0) \,\ge\, \frac{\lambda_N(\Omega_k^+)^{\frac{q(N-p)}{Np}-1}}{2}, \quad \text{for all } k \ge k_2. \tag{4.20}$$

Also, we select $k_3 > 0$ such that

$$\lambda_N(\Omega_k^+) \,\le\, \frac{1}{2}\lambda_N(\Omega) \quad \text{and} \quad \lambda_N(\Omega_k) \,\le\, \frac{1}{2}\lambda_N(\Omega), \qquad \text{for all } k \ge k_3. \tag{4.21}$$

Then, for each $k \ge k_0 := \max\{k_1, k_2, k_3\}$, we apply Hölder's inequality in (4.19) to obtain that

$$\int_{\Omega_k^+} (u-k)\,dx \,\le\, \lambda_N(\Omega_k^+)^{1-\frac{1}{q}} \left(\frac{c_1\{\gamma_0(1+k^{q-1}) + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\}}{\lambda_N(\Omega_k^+)^{\frac{q(N-p)}{Np}-1} - c_1(\lambda_N(\Omega_k^+)^{\frac{q}{p}-1} + \gamma_0)} \int_{\Omega_k^+} (u-k)\,dx \right)^{1/q}. \tag{4.22}$$

Combining (4.22) with (4.20) yields that

$$\int_{\Omega_k^+} (u-k)\,dx \,\le\, \lambda_N(\Omega_k^+)^{1+\frac{1}{q-1}\left(1-\frac{q}{p_N}\right)} k \left[\frac{c_1}{2}\left(\gamma_0 + \frac{\gamma_0 + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}}{k_0^{q-1}} \right) \right]^{1/(q-1)},$$

where we recall that the constant p_N is given by (4.4). This last calculation together with Lemma 2.3 entail that $\|u^+\|_{\infty,\Omega}$ is bounded by a constant depending on p, N, γ_0, ξ, $\|\phi\|_{\infty,\Omega}$, $\|h\|_{\infty,\Omega}$, and $\|u^+\|_{1,\Omega_k^+}$. To complete the proof, we must bound $\|u^+\|_{1,\Omega_k^+}$ by a constant independent of u. In fact, noticing that $k_0 := \max\{k_1, k_2, k_3\}$, using (4.17) and (1.12), and applying Hölder's inequality together with Young's inequality, we have the following calculation.

$$\begin{aligned}\|\nabla u_{k_0}^+\|^{p_N}_{p,\Omega_{k_0}^+} \,&\le\, \|\nabla u_{k_0}^+\|^p_{p,\Omega_{k_0}^+} \,\le\, \Lambda_p(u, u_{k_0}^+) \\ &= \int_{\Omega_{k_0}^+} f(x,u)u_{k_0}^+\,dx + \int_{\Omega_{k_0}^+} (\xi\phi + h)u_{k_0}^+\,dx \\ &\le \gamma_0 \int_{\Omega_{k_0}^+} (u)^{q-1}(u-k_0)\,dx + \left(\gamma_0 + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right) \int_{\Omega_{k_0}^+} (u-k_0)\,dx\end{aligned}$$

$$\leq \epsilon \|u^+\|^{p_N}_{p_N,\Omega^+_{k_0}} + \left(\frac{2}{\epsilon}\right)^{\frac{q}{p_N-q}} \left\{\gamma_0 \lambda_N(\Omega)^{\frac{q-1}{q}}\right\}^{\frac{p_N}{p_N-q}} +$$

$$+ \left(\frac{2}{\epsilon}\right)^{\frac{1}{p_N-1}} \left(\gamma_0 + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right)^{\frac{p_N}{p_N-1}} \lambda_N(\Omega), \quad (4.23)$$

for all $\epsilon > 0$. On the other hand, taking into account (4.21) and the selection of $k_0 > 0$, since $\lambda_N\left(\Omega \setminus \Omega^+_{k_0}\right) > \frac{1}{2}\lambda_N(\Omega) > 0$, and $u^+_{k_0} = 0$ over $\Omega \setminus \Omega^+_{k_0}$, one applies (2.7) and (2.9) to deduce that

$$\|u^+\|^{p_N}_{p_N,\Omega^+_{k_0}} \leq C_p \left(\|u^+_{k_0}\|^{p_N}_{p_N,\Omega^+_{k_0}} + k_0^{p_N} \lambda_N(\Omega)\right) \leq C'_p \left(\|\nabla u^+_{k_0}\|^{p_N}_{p,\Omega^+_{k_0}} + k_0^{p_N} \lambda_N(\Omega)\right), \quad (4.24)$$

for some constant $C_p > 0$, and for $C'_p := C_p \max\{1, c_1 c_3\}$. Selecting $\epsilon := 1/2$, combining (4.23) and (4.24), and applying Young's inequality, we get

$$\|u^+\|_{1,\Omega^+_k} \leq \|u^+\|_{1,\Omega^+_{k_0}} \leq \left\{2C'_p \left(\|\nabla u^+_{k_0}\|^{p_N}_{p,\Omega^+_{k_0}} + (k_0^{p_N} + 1)\lambda_N(\Omega)\right)\right\}^{1/p_N} \leq C^*_\xi(k_0), \quad (4.25)$$

where

$$C^*_\xi(k_0) := \left\{2C'_p \left((1 + k_0^{p_N})\lambda_N(\Omega) + 4^{\frac{q}{p_N-q}} \left\{\gamma_0 \lambda_N(\Omega)^{\frac{q-1}{q}}\right\}^{\frac{p_N}{p_N-q}} + \right.\right.$$

$$\left.\left. +4^{\frac{1}{p_N-1}} \left(\gamma_0 + |\xi| \|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right)^{\frac{p_N}{p_N-1}} \lambda_N(\Omega)\right)\right\}^{1/p_N}, \quad (4.26)$$

which is independent of u and k. Thus we have found the desired bound for $\|u^+\|_{1,\Omega^+_{k_0}}$, completing the proof of the theorem.

□

Next, we establish that weak solutions of the Ambrosetti–Prodi problem (1.1) are globally bounded. Such result is automatically true when $p > N$ (by virtue of the Sobolev inequality). Thus we will prove it for the critical case, namely, when $1 < p < N$ (the case $p \geq N$ follows in an even simpler way). Furthermore, an a priori estimate for the difference of weak solutions is established.

Theorem 4.3. *Let Assumption 1.3 hold, let f_1, $f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions satisfying the conditions (1.4), (1.11), and (1.12), let ϕ_1, ϕ_2, h_1, $h_2 \in L^\infty(\Omega)$ with ϕ_1, $\phi_2 > 0$ a.e. in Ω, and let ξ_1, $\xi_2 \in \mathbb{R}$ be parameters*

(a) *If $u \in \mathcal{W}_p(\overline{\Omega})$ is a weak solutions of either (1.1), or (1.2), then there exists a constant $c_6(\xi) > 0$ such that*

$$|||(u, u|_\Gamma)|||^{p-1}_\infty \leq c_6(\xi). \quad (4.27)$$

(b) *Assume that $p > \frac{2N}{N+2}$. If u_1, $u_2 \in \mathcal{W}_p(\overline{\Omega})$ are weak solutions of either (1.1), or (1.2), related to f_1, ϕ_1, h_1, ξ_1 and f_2, ϕ_2, h_2, ξ_2, respectively, then there* *exists a constant* $c_7(\xi_1,\xi_2)>0$ *such that*

$$|||(u_1-u_2, u_1|_\Gamma - u_2|_\Gamma)|||_\infty^{p-1} \le c_7(\xi_1,\xi_2). \tag{4.28}$$

Proof. We only prove part (b), for part (a) follows similarly (and even in a simpler way). Again we deal both cases at once. As the arguments run similarly as in [48, Theorem 3.2] and [49, Theorem 5.1], we will only sketch the main steps of the proof. Let f_1, f_2 and ξ_1,ξ_2 be as in the theorem. Recall that we are assuming all the conditions of Assumption 1.3.

• First suppose that $2N(N+2)^{-1}<p<2$. Let u_1, $u_2\in\mathcal{W}_p(\overline{\Omega})$ be weak solutions of either (1.1), or (1.2), related to f_1, ξ_1 and f_2, ξ_2, respectively. Let $w:=u_1-u_2$, and let $k\ge k_0>0$ be a real number, where $k_0>0$ denotes the constant appearing in the proof of Theorem 3.1. Now let $\hat{w}_k$ be defined by (4.1) (but now with respect to $w:=u_1-u_2$). Then $\hat{w}_k\in\mathcal{W}_p(\overline{\Omega})$, with $(\nabla\hat{w}_k)|_\Omega=(\nabla w)\chi_{\Omega_k}$. Next, for $\epsilon\in(0,1]$, let $C_{p,\epsilon}:=\min\{c_p,c_p^*,c'_p,c_{p,\epsilon}\}>0$, where c_p, c_p^*, c'_p, $c_{p,\epsilon}$ denote the constants in Proposition 2.4 (for $r=p$). Then there exists a constant $C_p>0$ such that

$$C_{p,\epsilon}\ \ge\ C_p^{-1}\epsilon^{2-p}, \qquad \text{for all } \epsilon\in(0,1]. \tag{4.29}$$

Now, we consider the following nonlinear form $\Lambda_p^{(n)}(\cdot,\cdot)$ by

$$\Lambda_p^{(n)}(u,v) := \Lambda_p(u,v) - \mathcal{E}^{(p)}(u,v) + \left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(u), v\right\rangle, \tag{4.30}$$

for u, $v\in\mathcal{W}_p(\overline{\Omega})$, where the form $\Lambda_p(\cdot,\cdot)$ is given by (1.20), and the last term is defined in (4.6). By the Dominated Convergence Theorem, one has

$$\lim_{n\to\infty}\int_I \Lambda_p^{(n)}(u,v)\,dt = \Lambda_p(u,v)$$

for all u, $v\in\mathcal{W}_p(\overline{\Omega})$. Then, we define the following sets:

$$\Omega^1(\epsilon) := \{x\in\Omega:\ |\nabla w(x)|\ge\epsilon\min\{|\nabla u_1(x)|,|\nabla u_2(x)|\}\},$$

$$\Omega^2(\epsilon) := \{(x,y)\in\Omega\times\Omega:\ |w(x)-w(y)|\ge\epsilon\min\{|u_1(x)-u_1(y)|,|u_2(x)-u_2(y)|\}\}.$$

$$\Gamma^1(\epsilon) := \{(x,y)\in\Gamma\times\Gamma:\ |w(x)-w(y)|\ge\epsilon\min\{|u_1(x)-u_1(y)|,|u_2(x)-u_2(y)|\}\}.$$

$$\Omega_k^1(\epsilon):=\Omega^1(\epsilon)\cap\Omega_k, \quad \Omega_k^2(\epsilon):=\Omega^2(\epsilon)\cap(\Omega_k\times\Omega_k), \quad \Gamma_k^1(\epsilon):=\Gamma^1(\epsilon)\cap(\Gamma_k\times\Gamma_k),$$

$$V_{0,k}^{(i)}(\epsilon):=\Gamma^1(\epsilon)\cap V_{0,k}^{(i)}.$$

$$\widetilde{\Omega}_k^1(\epsilon):=\Omega_k\setminus\Omega_k^1(\epsilon), \quad \widetilde{\Omega}_k^2(\epsilon):=(\Omega_k\times\Omega_k)\setminus\Omega_k^2(\epsilon), \quad \widetilde{\Gamma}_k^1(\epsilon):=(\Gamma_k\times\Gamma_k)\setminus\Gamma_k^1(\epsilon),$$

$$\widetilde{V}_{0,k}^{(i)}(\epsilon):=(V_{0,k}^{(i)}\times V_{0,k}^{(i)})\setminus V_{0,k}^{(i)}(\epsilon).$$

Here we recall that $V_{0,k}^{(i)}:=\{\xi\in V_0^{(i)}\ :\ w(\psi_{\kappa_i}(\xi))\ge k\}$, for $\kappa_i:=i_1,\ldots,i_n$ (e.g. (4.6)). Then, as $(\Theta_\Omega w)\hat{w}_k\ge(\Theta_\Omega\hat{w}_k)\hat{w}_k$, $(\Theta_\Gamma w)\hat{w}_k\ge(\Theta_\Gamma\hat{w}_k)\hat{w}_k$, and $\left\langle\partial\tilde{\mathcal{L}}_n^{(p)}(w),\hat{w}_k\right\rangle\ge\left\langle\partial\tilde{\mathcal{L}}_n^{(p)}(\hat{w}_k),\hat{w}_k\right\rangle$ (e.g. [50, Proof of Theorem 5.7]), Applying Proposition 2.4 for $\epsilon\in(0,1]$, we obtain that

$$
\begin{aligned}
&\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\\
&\geq\; C_{p,\epsilon}\left(\Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k)-\int_{\widetilde{\Omega}_k^1(\epsilon)}|\nabla\hat{w}_k|^p\,dx-\int_{\widetilde{\Omega}_k^2(\epsilon)}\frac{|\hat{w}_k(x)-\hat{w}_k(y)|^p}{|x-y|^{\varsigma p+N}}\,dxdy+\right.\\
&\left.-\int_{\widetilde{\Gamma}_k^1(\epsilon)}\frac{|\hat{w}_k(x)-\hat{w}_k(y)|^p}{|x-y|^{\gamma p+d}}\,d\mu_x d\mu_y-\frac{1}{p}\sum_{\kappa_i=1}^{4}\sum_{\xi,\varsigma\in\widetilde{V}_{0,k}^{(i)}(\epsilon)\ \xi\neq\varsigma}\frac{|\hat{w}_k(\psi_{\kappa_i}(\xi))-\hat{w}_k(\psi_{\kappa_i}(\varsigma))|^p}{|\psi_{\kappa_i}(\xi)-\psi_{\kappa_i}(\varsigma)|^{dp}}\right)\\
&\geq\; C_{p,\epsilon}\left[\Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k)-\epsilon^p\int_{\widetilde{\Omega}_{k,\epsilon}^1}\left(|\nabla u_1|^p+|\nabla u_2|^p\right)\,dx\right]+\\
&\quad-\epsilon^p C_{p,\epsilon}\left(\int_{\widetilde{\Omega}_k^2(\epsilon)}\frac{|u_1(x)-u_1(y)|^p}{|x-y|^{\varsigma p+N}}\,dxdy+\int_{\widetilde{\Omega}_k^2(\epsilon)}\frac{|u_2(x)-u_2(y)|^p}{|x-y|^{\varsigma p+N}}\,dxdy\right)+\\
&\quad-\epsilon^p C_{p,\epsilon}\left(\int_{\widetilde{\Gamma}_k^1(\epsilon)}\frac{|u_1(x)-u_1(y)|^p}{|x-y|^{\gamma p+d}}\,d\mu_x d\mu_y+\int_{\widetilde{\Gamma}_k^1(\epsilon)}\frac{|u_2(x)-u_2(y)|^p}{|x-y|^{\gamma p+d}}\,d\mu_x d\mu_y\right)+\\
&\qquad\qquad-\epsilon^p C_{p,\epsilon}\left(\frac{1}{p}\sum_{\kappa_i=1}^{4}\sum_{\xi,\varsigma\in\widetilde{V}_{0,k}^{(i)}(\epsilon)\ \xi\neq\varsigma}\frac{|u_1(\psi_{\kappa_i}(\xi))-u_1(\psi_{\kappa_i}(\varsigma))|^p}{|\psi_{\kappa_i}(\xi)-\psi_{\kappa_i}(\varsigma)|^{dp}}\right)+\\
&\qquad\qquad-\epsilon^p C_{p,\epsilon}\left(\frac{1}{p}\sum_{\kappa_i=1}^{4}\sum_{\xi,\varsigma\in\widetilde{V}_{0,k}^{(i)}(\epsilon)\ \xi\neq\varsigma}\frac{|u_2(\psi_{\kappa_i}(\xi))-u_2(\psi_{\kappa_i}(\varsigma))|^p}{|\psi_{\kappa_i}(\xi)-\psi_{\kappa_i}(\varsigma)|^{dp}}\right).
\end{aligned}
$$

Thus in virtue of (4.29) we see that

$$
\begin{aligned}
\Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k)\;&\leq\;\frac{1}{C_{p,\epsilon}}\left[\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\right]+\epsilon^p\left[\Lambda_p^{(n)}(u_1,u_1)+\Lambda_p^{(n)}(u_2,u_2)\right)\\
&\leq\;\frac{C_p}{\epsilon^{2-p}}\left[\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\right]+\epsilon^p\left[\Lambda_p^{(n)}(u_1,u_1)+\Lambda_p^{(n)}(u_2,u_2)\right].
\end{aligned}
$$

Applying Proposition 2.5 with

$$
r:=p,\qquad\tau:=C_p,\qquad\varrho:=2-p,\qquad\xi:=\Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k),
$$

$$
\varsigma:=\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k),\quad\text{and}\quad c:=\Lambda_p^{(n)}(u_1,u_1)+\Lambda_p^{(n)}(u_2,u_2),
$$

we get from (2.17) that

$$
\begin{aligned}
\Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k)\;\leq\;(C_p+1)&\left(\left[\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\right]^{\frac{p}{2}}\left[\Lambda_p^{(n)}(u_1,u_1)+\Lambda_p^{(n)}(u_2,u_2)\right]^{\frac{2-p}{2}}\right)+\\
&+(C_p+1)\left[\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\right].
\end{aligned}\tag{4.31}
$$

Since

$$
0\;\leq\;\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\;\leq\;4\left[\Lambda_p^{(n)}(u_1,u_1)+\Lambda_p^{(n)}(u_2,u_2)\right],
$$

one gets from (4.31) that

$$\Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k) \le C'_p\left(\left[\Lambda_p^{(n)}(u_1,\hat{w}_k)-\Lambda_p^{(n)}(u_2,\hat{w}_k)\right]^{\frac{p}{2}}\left[\Lambda_p^{(n)}(u_1,u_1)+\Lambda_p^{(n)}(u_2,u_2)\right]^{\frac{2-p}{2}}\right), \tag{4.32}$$

where $C'_p := (C_p+1)(1+2^{2-p}) > 0$. Since

$$\Lambda_p(\hat{w}_k,\hat{w}_k) = \Lambda_p^{(n)}(\hat{w}_k,\hat{w}_k) + \mathcal{E}^{(p)}(\hat{w}_k,\hat{w}_k) - \left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(\hat{w}_k),\hat{w}_k\right\rangle,$$

using this in (4.32), passing to the limit, and applying the Dominated Convergence Theorem, we arrive at

$$\Lambda_p(\hat{w}_k,\hat{w}_k) \le C\left([\Lambda_p(u_1,\hat{w}_k)-\Lambda_p(u_2,\hat{w}_k)]^{\frac{p}{2}}\,[\Lambda_p(u_1,u_1)+\Lambda_p(u_2,u_2)]^{\frac{2-p}{2}}\right). \tag{4.33}$$

In the following, the constant $C>0$ may vary from line to line. Next, recalling that $u_1,\, u_2 \in \mathcal{W}_p(\overline{\Omega})$ solve (1.19) related to $f_1,\, \xi_1$ and $f_2,\, \xi_2$, respectively, using (1.10) and (1.12) (depending on the sign of $u_1,\, u_2$), and applying Theorem 4.2, gives

$$\begin{aligned}\Lambda_p(u_1,u_1)+\Lambda_p(u_2,u_2) &= \sum_{i=1}^{2}\left(\int_\Omega f_i(x,u_i)u_i\,dx + \int_\Omega(\xi_i\phi_i+h_i)u_i\,dx\right)\\ &\le C\sum_{i=1}^{2}\left(\|u_i\|^p_{p,\Omega}+\|u_i\|^q_{q,\Omega}+\lambda_N(\Omega)+\frac{|\xi_i|\|\phi_i\|_{\infty,\Omega}+\|h_i\|_{\infty,\Omega}}{C}\|u_i\|_{1,\Omega}\right) \le M_{\xi_1,\xi_2},\end{aligned} \tag{4.34}$$

for some constant $M_{\xi_1,\xi_2}>0$ which depends on the L^∞ bounds on Theorem 4.2 (with respect to $\xi_1,\, \xi_2$). We recall (4.21) and the selection of $k_0>0$ to notice that $\lambda_N\,(\Omega\setminus\Omega_k) > \frac{1}{2}\lambda_N(\Omega) > 0$, and $\hat{w}_k = 0$ over $\Omega\setminus\Omega_k$. Thus, combining (1.9), (1.10), (1.12), (2.7), (2.9), (4.33), (4.34), and Theorem 4.2, we deduce that

$$\begin{aligned}\|\hat{w}_k\|^p_{\mathcal{W}_p(\overline{\Omega})} &\le C\,\Lambda_p(\hat{w}_k,\hat{w}_k)\\ &\le CM_{\xi_1,\xi_2}^{1-\frac{p}{2}}\left(\int_{\Omega_k}|f(x,u_1)-f(x,u_2)||\hat{w}_k|\,dx+\int_{\Omega_k}|(\xi_1\phi_1+h_1)-(\xi_2\phi_2+h_2)||\hat{w}_k|\,dx\right)^{\frac{p}{2}}\\ &\le CM^*_{\xi_1,\xi_2}|||\vec{\chi}_{k,\rho}|||^{\frac{p}{2}}_{p'_N,r_d}\|\hat{w}_k\|^{\frac{p}{2}}_{\mathcal{W}_p(\overline{\Omega})},\end{aligned} \tag{4.35}$$

for $r_d := dp(Np-N+p)^{-1} > 0$, where

$$M^*_{\xi_1,\xi_2} := M_{\xi_1,\xi_2}^{1-\frac{p}{2}}\left(\sum_{i=1}^{2}\left(|C_{\xi_i}|^{p-1}+|C_{\xi_i}|^{q-1}+|\xi_i|\|\phi_i\|_{\infty,\Omega}+\|h_i\|_{\infty,\Omega}\right)+M+\gamma_0\right)^{\frac{p}{2}}$$

and

$$\vec{\chi}_k := (\chi_{A_k},\chi_{A_k}|_\Gamma) \qquad (\,\chi_E \text{ the characteristic function of a set } E\,).$$

Now let $h>k$, and observe that $A_h\subseteq A_k$ and that $|\hat{w}_k| \ge h-k$ over A_h (recall (4.2) for the definition of the set A_k). In view of (2.7), (2.8) and (4.35), we get

$$(h-k)^{\frac{p}{2}}|||\vec{\chi}_h|||^{\frac{p}{2}}_{p_N,p_d} \le CM^*_{\xi_1,\xi_2}|||\vec{\chi}_k|||^{\frac{p}{2}}_{p'_N,r_d}. \tag{4.36}$$

Setting

$$\Phi_p(t) := |||\vec{\chi}_t|||^{\frac{p}{2}}_{p_N, p_d} \tag{4.37}$$

for each $t \in [0, \infty)$, one gets from (4.36) that

$$\Phi_p(h) \leq C M^*_{\xi_1,\xi_2}(h-k)^{-\frac{p}{2}} \left(\Phi_p(k)^{\frac{p_N}{p'_N}} + \Phi_p(k)^{\frac{p_d}{r_d}} \right). \tag{4.38}$$

As $2N(N+2)^{-1} < p < 2$, we see that

$$\frac{p_N}{p'_N} = \frac{p_d}{r_N} = \frac{Np - N + p}{N - p} := \delta > 1,$$

and thus it follows from (4.38) that

$$\Phi_p(h) \leq 2C M^*_{\xi_1,\xi_2}(h-k)^{-\frac{p}{2}} \Phi_p(k)^{\delta}. \tag{4.39}$$

we apply Lemma 2.2 to the function $\Phi_p(\cdot)$ to conclude that

$$\Phi_p(\zeta) = 0, \quad \text{ for } \zeta^{p/2} := C M^*_{\xi_1,\xi_2} \Phi_p(k_0)^{\delta - 1}. \tag{4.40}$$

Now (4.40) implies that there exists a constant $c^* > 0$ such that

$$|u - v| \leq c^* \left(M^*_{\xi_1,\xi_2}\right)^{2/p} \quad \text{ a.e. on } \overline{\Omega}, \tag{4.41}$$

which shows (4.28) for the case $2N(N+2)^{-1} < p < 2$.

• Assume now that $p \in [2, \infty)$. As mentioned before (in section 1), it suffices to consider the critical case, namely, when $p < N$ (since for $p \geq N$ the embedding results are much more sharp, and much better). The proof of this case follows as in the previous one (and even in a simpler way), we will only sketch the main steps of it. Given u_1, $u_2 \in \mathcal{W}_p(\overline{\Omega})$ weak solutions of either (1.1), or (1.2), related to f_1, ξ_1 and f_2, ξ_2, respectively, let $w := u_1 - u_2$, let $k \geq k_0$ be a real number (for $k_0 > 0$ defined as in the previous case), and let $\hat{w}_k$ be defined by (4.1). A direct calculation (similar to the previous case) shows that

$$\Lambda_p(u_1, \hat{w}_k) - \Lambda_p(u_2, \hat{w}_k) \geq \varsigma_0 \Lambda_p(\hat{w}_k, \hat{w}_k) \geq \varsigma'_0 \|\hat{w}_k\|^p_{\mathcal{W}_p(\overline{\Omega})}, \tag{4.42}$$

for some constants ς_0, $\varsigma'_0 > 0$. Proceeding exactly as in the previous case, one sees that

$$(h-k)^{p-1} |||\vec{\chi}_h|||^{p-1}_{p_N, p_d} \leq M'_{\xi_1,\xi_2} |||\vec{\chi}_k|||_{p'_N, r_d}, \tag{4.43}$$

for some constant $M'_{\xi_1,\xi_2} > 0$ (that can be computed similarly as in the previous case), for $h > k$, where we recall that $r_d := dp(Np - N + p)^{-1}$. Letting

$$\Psi_p(t) := |||\vec{\chi}_t|||^{p-1}_{p_N, p_d} \quad \text{ and } \quad \delta' := \left(\frac{1}{p-1}\right)\left(\frac{Np - N + p}{N - p}\right) > 1,$$

one can proceed in the same way as before to conclude that

$$|u-v| \le C'_{\xi_1,\xi_2} \quad \text{a.e. on } \overline{\Omega}, \tag{4.44}$$

for some constant $C'_{\xi_1,\xi_2}>0$. Therefore, (4.44) leads to (4.28) when $p\in[2,N)$, and completes the proof of the theorem. □

5. **Sub-supersolution method for nonlocal equations.** In this part we will derive an alternative sub-supersolution method for the problems (1.1) and (1.2), which will be very useful in the establishment of the main results of the paper. Although some arguments will follow similar approaches as in [33, 34], to our knowledge, there is no sub-supersolution method for nonlocal equations of type (1.1) and (1.2). Recall that all the arguments will be carried out under the assumptions (1.4), (1.11), and (1.12).

To begin our discussion, given $F\subseteq\mathcal{W}_p(\overline{\Omega})$ closed and convex (where we recall that $\mathcal{W}_p(\overline{\Omega})$ is defined by (1.16)), we consider the following variational inequality

$$\begin{cases} \Lambda_p(u,\varphi-u) \ge \displaystyle\int_\Omega f(x,u)(\varphi-u)\,dx+\int_\Omega(\xi\phi+h)(\varphi-u)\,dx, & \forall\,\varphi\in F,\\ u\in F, \end{cases} \tag{5.1}$$

where we recall that $\Lambda_p(\cdot,\cdot)$ is defined by (1.20).

Before giving the corresponding definitions for subsolutions and supersolutions of the Eq. (5.1), we will fix some additional notations that will be frequently used in this section. Indeed, for $u,\,v\in\mathcal{W}_p(\overline{\Omega})$ and $D,\,F\subseteq\mathcal{W}_p(\overline{\Omega})$, we denote $u\vee v:=\max\{u,v\}$, $u\wedge v:=\min\{u,v\}$, and

$$D\star F:=\{u\star v:\ u\in D,\ v\in F\}, \qquad u\star F:=\{u\}\star F,$$

where $\star$ denotes either $\vee$ or $\wedge$.

Definition 5.1. A function $\hat{u}\in\mathcal{W}_p(\overline{\Omega})$ is said to be a **subsolution** of (5.1), if $\hat{u}\vee F\subseteq F$, and

$$\Lambda_p(\hat{u},\varphi-\hat{u}) \ge \int_\Omega f(x,\hat{u})(\varphi-\hat{u})\,dx+\int_\Omega(\xi\phi+h)(\varphi-\hat{u})\,dx, \qquad \forall\,\varphi\in\hat{u}\wedge F. \tag{5.2}$$

Also, $\check{u}\in\mathcal{W}_p(\overline{\Omega})$ is said to be a *supersolution* of (5.1), if $\check{u}\wedge F\subseteq F$, and

$$\Lambda_p(\check{u},\varphi-\check{u}) \ge \int_\Omega f(x,\check{u})(\varphi-\check{u})\,dx+\int_\Omega(\xi\phi+h)(\varphi-\check{u})\,dx, \qquad \forall\,\varphi\in\check{u}\vee F. \tag{5.3}$$

Suppose that $\hat{u}_1,\dots,\hat{u}_k\in\mathcal{W}_p(\overline{\Omega})$ (resp. $\check{u}_1,\dots,\check{u}_m\in\mathcal{W}_p(\overline{\Omega})$) are subsolutions (resp. supersolutions) of (5.1) (for $k,\,m\in\mathbb{N}$). Then, we will set

$$\hat{u}_0:=\max\{\hat{u}_1,\dots,\hat{u}_k\}, \qquad \text{and} \qquad \check{u}_0:=\min\{\check{u}_1,\dots,\check{u}_m\}. \tag{5.4}$$

Then, we can state and prove the following important result.

Theorem 5.2. *Let $\hat{u}_1, \ldots, \hat{u}_k \in \mathcal{W}_p(\overline{\Omega})$ (resp. $\check{u}_1, \ldots, \check{u}_m \in \mathcal{V}_p(\Omega)$) be subsolutions (resp. supersolutions) of (5.1) (for $k, m \in \mathbb{N}$). If $\hat{u}_1, \ldots, \hat{u}_k$ (resp. $\check{u}_1, \ldots, \check{u}_m$) are bounded functions in Ω, with $\hat{u}_0 \leq \check{u}_0$, then there exists a solution $u \in \mathcal{W}_p(\overline{\Omega})$ of (5.1) such that $\hat{u}_0 \leq u \leq \check{u}_0$.*

Proof. Again, we restrict our proof to the critical case $1 < p < N$.

Choose $r \in (1, \infty)$ such that

$$\max\{p, q\} < r < \begin{cases} \frac{Np}{N-p}, & \text{if } 1 < p < N, \\ \infty, & \text{if } p \geq N. \end{cases} \tag{5.5}$$

Then we consider the following truncating-regularizing function: $\Psi : \Omega \times \mathbb{R} \to \mathbb{R}$, by

$$\Psi(x, s) := \begin{cases} (s - \check{u}_0(x))^{r-1}, & \text{if } s > \check{u}_0(x), \\ 0, & \text{if } \hat{u}_0(x) \leq s \leq \check{u}_0(x), \\ -(\hat{u}_0(x) - s)^{r-1}, & \text{if } s < \hat{u}_0(x), \end{cases} \tag{5.6}$$

for each $(x, s) \in \Omega \times \mathbb{R}$. Since $\hat{u}_0$, $\check{u}_0 \in L^\infty(\Omega)$, we can find constants α_1, $\alpha_1' > 0$ such that

$$|\Psi(x, s)| \leq \alpha_1 + \alpha_1' |s|^{r-1}, \qquad \text{for a.e. } (x, s) \in \Omega \times \mathbb{R}. \tag{5.7}$$

Now, define the operator $\mathcal{S}_p : \mathcal{W}_p(\overline{\Omega}) \to \mathcal{W}_p(\overline{\Omega})^*$, by:

$$(\mathcal{S}_p v) w := \int_\Omega \Psi(x, v) w \, dx, \qquad \text{for } v, w \in \mathcal{W}_p(\overline{\Omega}). \tag{5.8}$$

Since the embedding $\mathcal{W}_p(\overline{\Omega}) \hookrightarrow L^r(\Omega)$ is compact, it follows that $\mathcal{S}_p$ is well-defined, bounded, and completely continuous. Moreover, a direct calculation shows that

$$(\mathcal{S}_p v) v \geq \alpha_2' \int_\Omega |v|^r \, dx - \alpha_2, \qquad \text{for all } v \in \mathcal{W}_p(\overline{\Omega}), \tag{5.9}$$

for some constants α_2, $\alpha_2' > 0$ (cf. [33, Proof of Theorem 3.1]). Next, define the operator $\mathcal{B}_p : \mathcal{W}_p(\overline{\Omega}) \to \mathcal{W}_p(\overline{\Omega})^*$, by

$$(\mathcal{B}_p(\cdot))(\cdot) := \Lambda_p(\cdot, \cdot), \tag{5.10}$$

where we recall that $\Lambda_p(\cdot, \cdot)$ is defined by (1.20). Recalling the definition of $\mathcal{W}_p(\overline{\Omega})$ and in view of (2.10) and (2.11), it is easy to see that $\mathcal{B}_p$ is bounded, continuous, and monotone. Moreover, it is coercive in the sense that

$$(\mathcal{B}_p v) v \geq c_0 \left(\|\nabla v\|_{p,\Omega}^p + K^{(p)}(v, v) + (\Theta_\Omega v + \Theta_\Gamma v) v \right), \tag{5.11}$$

for every $v \in \mathcal{W}_p(\overline{\Omega})$, and for some constant $c_0 > 0$ (recall that $K^{(p)}(\cdot, \cdot)$, Θ_Ω, and Θ_Γ are given by (4.7), (1.5), and (1.6), respectively). Furthermore, we define another operator $\mathcal{G}_p : \mathcal{W}_p(\overline{\Omega}) \to \mathcal{W}_p(\overline{\Omega})^*$, by

$$(\mathcal{G}_p v) w := -\int_\Omega f(x, \tau_{00} v) w \, dx + \sum_{j=1}^m \int_\Omega |f(x, \tau_{0j} v) - f(x, \tau_{00} v)| \, w \, dx +$$

$$- \sum_{i=1}^{k} \int_{\Omega} |f(x, \tau_{i0}v) - f(x, \tau_{00}v)|\, w\, dx - \int_{\Omega} (\xi\phi + h)v\, dx, \tag{5.12}$$

for each v, $w \in \mathcal{W}_p(\overline{\Omega})$, where

$$(\tau_{ij}v)(x) := \begin{cases} \check{u}_j(x), & \text{if } v(x) > \check{u}_j(x), \\ v(x), & \text{if } \hat{u}_i(x) \leq v(x) \leq \check{u}_j(x), \\ \hat{u}_i(x), & \text{if } v(x) < \hat{u}_i(x), \end{cases} \tag{5.13}$$

(where k, $m \in \mathbb{N}$, $i \in \{0, \ldots, k\}$, and $j \in \{0, \ldots, m\}$). From the boundedness of $\hat{u}_1, \ldots, \hat{u}_k$ (resp. $\check{u}_1, \ldots, \check{u}_m$) over Ω, the continuity of the mapping $\tau_{ij} : \mathcal{W}_p(\overline{\Omega}) \to \mathcal{W}_p(\overline{\Omega})$, and the compactness of the embedding $\mathcal{W}_p(\overline{\Omega}) \hookrightarrow L^r(\Omega)$, one can deduce that $\mathcal{G}_p$ is well-defined, bounded, and completely continuous over $\mathcal{W}_p(\overline{\Omega})$. Then we consider the variational inequality, formally given by

$$\begin{cases} (\mathcal{S}_p v + \mathcal{B}_p v + \mathcal{G}_p v)(\varphi - v) \geq 0, & \forall \varphi \in F, \\ v \in F, \end{cases} \tag{5.14}$$

where $\mathcal{S}_p$, $\mathcal{B}_p$, and $\mathcal{G}_p$ are defined by (5.8), (5.10), and (5.12), respectively. From the properties of $\mathcal{S}_p$, $\mathcal{B}_p$, and $\mathcal{G}_p$, it follows that the operator $\mathcal{T}_p := \mathcal{S}_p + \mathcal{B}_p + \mathcal{G}_p$ is bounded, and pseudo-monotone over $\mathcal{W}_p(\overline{\Omega})$. (cf. [45, pag. 40-41]). Moreover, we claim that $\mathcal{T}_p$ is coercive, in the sense that

$$\lim_{\|v\|_{\widetilde{\mathcal{W}}_p(\overline{\Omega})} \to \infty} \frac{(\mathcal{T}_p v)(v - v_0)}{\|v\|_{\widetilde{\mathcal{W}}_p(\overline{\Omega})}} = \infty, \tag{5.15}$$

for each bounded function $v_0 \in F$, where we recall the semi-norm

$$\|w\|^p_{\widetilde{\mathcal{W}}_p(\overline{\Omega})} := \|\nabla w\|^p_{p,\Omega} + K^{(p)}(w, w), \quad \text{for } w \in \mathcal{W}_p(\overline{\Omega}).$$

In fact, we first observe that

$$(\mathcal{T}_p v)(v - v_0) \geq (\mathcal{S}_p v)v + (\mathcal{B}_p v)v - |(\mathcal{S}_p v)v_0| - |(\mathcal{B}_p v)v_0| - \int_{\Omega} (|\xi|\phi + |h|)(|v| + |v_0|)\, dx +$$
$$- \left(\int_{\Omega} |f(x, \tau_{00}v)|(|v| + |v_0|)\, dx + \sum_{j=1}^{m} \int_{\Omega} (|f(x, \tau_{0j}v)| + |f(x, \tau_{00}v)|)\, (|v| + |v_0|)\, dx \right) +$$
$$- \sum_{i=1}^{k} \int_{\Omega} (|f(x, \tau_{i0}v)| + |f(x, \tau_{00}v)|)\, (|v| + |v_0|)\, dx. \tag{5.16}$$

Define now the operator $\mathcal{B}_p^{(n)}$ by

$$(\mathcal{B}_p^{(n)}(\cdot))(\cdot) := \Lambda_p^{(n)}(\cdot, \cdot), \tag{5.17}$$

for the form $\Lambda_p^{(n)}(\cdot, \cdot)$ given by (4.30). As before, one has

$$(\mathcal{B}_p u)v = \lim_{n \to \infty} \int_I [(\mathcal{B}_p^{(n)} u)v]\, dt, \qquad \forall\, u,\, v \in \mathcal{W}_p(\overline{\Omega}).$$

Moreover, by virtue of Young's inequality and the following inequality for p-Lagrangian (see 2.9 in [8]), $\mathcal{L}^{(p)}(u, v) \leq \epsilon \mathcal{L}^{(p)}(u, u) + c_\epsilon \mathcal{L}^{(p)}(v, v)$, and thus

$$|(\mathcal{B}_p^{(n)}v)v_0| \le \epsilon\left(\|\nabla v\|^p_{p,\Omega} + \left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(v), v\right\rangle + (\Theta_\Omega v + \Theta_\Gamma v)v\right)$$
$$+ C_\epsilon\left(\|\nabla v_0\|^p_{p,\Omega} + \left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(v_0), v_0\right\rangle + (\Theta_\Omega v_0 + \Theta_\Gamma v_0)v_0\right),$$

for every $\epsilon > 0$, and for some constant $C_\epsilon > 0$. Passing to the limit in the above inequality, one deduces from the Dominated Convergence Theorem that

$$|(\mathcal{B}_p v)v_0| \le \epsilon\left(\|\nabla v\|^p_{p,\Omega} + K^{(p)}(v,v) + (\Theta_\Omega v + \Theta_\Gamma v)v\right)$$
$$+ C_\epsilon\left(\|\nabla v_0\|^p_{p,\Omega} + K^{(p)}(v_0,v_0) + (\Theta_\Omega v_0 + \Theta_\Gamma v_0)v_0\right), \tag{5.18}$$

for every $\epsilon > 0$. Furthermore, recalling the definition of $\tau_{ij}(\cdot)$, using (1.12), (5.7), and Young's inequality, we get that

$$|(\mathcal{S}_p v)v_0| \le \alpha_1\|v_0\|_{1,\Omega} + \alpha_1'\int_\Omega |v|^{r-1}|v_0|\,dx \le \epsilon\|v\|^r_{r,\Omega} + C_\epsilon'\|v_0\|^r_{r,\Omega} + C_1, \tag{5.19}$$

and

$$\int_\Omega |f(x,\tau_{ij}v)|(|v|+|v_0|)\,dx + \int_\Omega (|\xi|\phi + |h|)(|v|+|v_0|)\,dx$$
$$\le \gamma_0\int_\Omega |v|^{q-1}(|v|+|v_0|)\,dx + \left(\gamma_0 + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right)\int_\Omega(|v|+|v_0|)\,dx$$
$$= \gamma_0\|v\|^q_{q,\Omega} + \int_\Omega |v|^{q-1}|v_0|\,dx + \left(\gamma_0 + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right)\int_\Omega(|v|+|v_0|)\,dx$$
$$\le (3\gamma_0+1)\epsilon\,\|v\|^r_{r,\Omega} + C_\epsilon'\gamma_0\|v_0\|^{\frac{r}{r-q+1}}_{\frac{r}{r-q+1},\Omega} + \left(\gamma_0 + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right)\|v_0\|_{1,\Omega} +$$
$$+ C_\epsilon'\left\{2\gamma_0 + \left(|\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}\right)\right\}\lambda_N(\Omega)$$
$$\le (3\gamma_0+1)\epsilon\,\|v\|^r_{r,\Omega} + M_\epsilon(v_0,\gamma_0,\xi,\|\phi\|_{\infty,\Omega},\|h\|_{\infty,\Omega}), \tag{5.20}$$

for every $\epsilon > 0$, and for some constants C_ϵ, C_ϵ', C_1, $C_2 > 0$, and for a constant $M_\epsilon(v_0,\gamma_0,\xi,\|\phi\|_{\infty,\Omega},\|h\|_{\infty,\Omega}) > 0$. Taking into account (5.9), (5.11) and the boundedness of v_0, selecting $\epsilon > 0$ suitably, and combining (5.18), (5.19) and (5.20) into (5.16), we deduce that

$$(\mathcal{T}_p v)(v - v_0) \ge C_3\|v\|^p_{\mathcal{W}_p(\overline{\Omega})} - M^*(v_0,\gamma_0,\xi,\|\phi\|_{\infty,\Omega},\|h\|_{\infty,\Omega}), \tag{5.21}$$

for all $v \in \mathcal{W}_p(\overline{\Omega})$ and $v_0 \in F$ bounded, for some constant $C_3 > 0$, and for a constant $M^*(v_0,\gamma_0,\xi,\|\phi\|_{\infty,\Omega},\|h\|_{\infty,\Omega}) > 0$. Thus, (4.36) leads to (5.15), proving the desired claim, and as a consequence, the above properties for the operator $\mathcal{T}_p$ ensure that Eq. (5.1) has a solution $u \in F$ (cf. [35]). To complete the proof of the theorem, we show that $\hat{u}_i \le u \le \check{u}_j$ for all $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,m\}$. Indeed, to establish the first inequality, we use the fact that $\hat{u}_i$ is a subsolution of (5.1), and we put $v_i := u + (\hat{u}_i - u)^+ = \hat{u}_i \vee u \in F$. Then, testing (5.14) with the function v_i, we get

$$0 \le \Lambda_p(u,(\hat{u}_i - u)^+) + \int_\Omega \Psi(x,u)(\hat{u}_i - u)^+\,dx - \int_\Omega(\xi\phi + h)(\hat{u}_i - u)^+\,dx +$$

$$
-\int_{\Omega} f(x,\tau_{00}u)(\hat{u}_i-u)^+\,dx+\sum_{j=1}^{m}\int_{\Omega}|f(x,\tau_{0j}u)-f(x,\tau_{00}u)|\,(\hat{u}_i-u)^+\,dx+ \\
-\sum_{s=1}^{k}\int_{\Omega}|f(x,\tau_{s0}u)-f(x,\tau_{00}u)|\,(\hat{u}_i-u)^+\,dx. \tag{5.22}
$$

On the other hand, as $\hat{u}_i\in F$ is a subsolution of (5.1), testing (5.2) with the function $w_i:=\hat{u}_i-(\hat{u}_i-u)^+=\hat{u}_i\wedge u\in\hat{u}_i\wedge F$, one sees that

$$
0\,\le\,-\Lambda_p(\hat{u}_i,(\hat{u}_i-u)^+)+\int_{\Omega}(\xi\phi+h)(\hat{u}_i-u)^+\,dx+\int_{\Omega}f(x,\hat{u}_i)(\hat{u}_i-u)^+\,dx. \tag{5.23}
$$

Adding (5.22) and (5.23), we get

$$
0\,\le\,\Lambda_p(u,(\hat{u}_i-u)^+)-\Lambda_p(\hat{u}_i,(\hat{u}_i-u)^+)+\int_{\Omega}[f(x,\hat{u}_i)-f(x,\tau_{00}u)](\hat{u}_i-u)^+\,dx+ \\
+\int_{\Omega}\Psi(x,u)(\hat{u}_i-u)^+\,dx+\sum_{j=1}^{m}\int_{\Omega}|f(x,\tau_{0j}u)-f(x,\tau_{00}u)|\,(\hat{u}_i-u)^+\,dx+ \\
-\sum_{s=1}^{k}\int_{\Omega}|f(x,\tau_{s0}u)-f(x,\tau_{00}u)|\,(\hat{u}_i-u)^+\,dx. \tag{5.24}
$$

Now, for each $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,m\}$, we see that $u(x)<\hat{u}_i(x)\le\hat{u}_0(x)\le\check{u}_0(x)\le\check{u}_j(x)$ for every $x\in\{y\in\overline{\Omega}:\,u(y)<\hat{u}_i(y)\}$. Therefore we have $\tau_{00}u=\tau_{0j}u=\hat{u}_0$ and $\tau_{i0}u=\hat{u}_i$ over $\{y\in\overline{\Omega}:\,u(y)<\hat{u}_i(y)\}$. The combination of all these properties entails that

$$
\int_{\Omega}[f(x,\hat{u}_i)-f(x,\tau_{00}u)](\hat{u}_i-u)^+\,dx-\sum_{s=1}^{k}\int_{\Omega}|f(x,\tau_{s0}u)-f(x,\tau_{00}u)|\,(\hat{u}_i-u)^+\,dx \\
\le\int_{\Omega\cap\{u<\hat{u}_i\}}\left(f(x,\hat{u}_i)-f(x,\hat{u}_0)-|f(x,\hat{u}_0)-f(x,\hat{u}_i)|\right)(\hat{u}_i-u)^+\,dx\,\le\,0, \tag{5.25}
$$

and

$$
\int_{\Omega}|f(x,\tau_{0j}u)-f(x,\tau_{00}u)|\,(\hat{u}_i-u)^+\,dx \\
=\int_{\Omega\cap\{u<\hat{u}_i\}}|f(x,\hat{u}_0)-f(x,\hat{u}_0)|\,(\hat{u}_i-u)^+\,dx=0. \tag{5.26}
$$

Moreover, using (2.12), it is clear that

$$
\Lambda_p^{(n)}(u,(\hat{u}_i-u)^+)-\Lambda_p^{(n)}(\hat{u}_i,(\hat{u}_i-u)^+)\,\le\,0,
$$

where we recall the definition of $\Lambda_p^{(n)}(\cdot,\cdot)$ given in (4.30). Thus by the Dominated Convergence Theorem, we obtain that

$$
\Lambda_p(u,(\hat{u}_i-u)^+)-\Lambda_p(\hat{u}_i,(\hat{u}_i-u)^+)\,\le\,0, \tag{5.27}
$$

and from here we combine (5.24), (5.25), (5.26), and (5.27), to get that

$$
0\le\int_{\Omega}\Psi(x,u)(\hat{u}_i-u)^+\,dx\,\le\,-\int_{\Omega\cap\{u<\hat{u}_i\}}(\hat{u}_i-u)^r\,dx\,\le\,0. \tag{5.28}
$$

Inequality (5.28) shows that $u \geq \hat{u}_i$ a.e. in Ω, for each $i \in \{1, \ldots, k\}$. Now, similar arguments can be employed to deduce as well that $u \leq \check{u}_j$ for every $j \in \{1, \ldots, m\}$, and consequently $\hat{u}_0 \leq u \leq \check{u}_0$ a.e. on Ω. In addition, recalling (5.6) and (5.13), we have that $\Psi(x, u) = 0$ and $\tau_{ij} u = u$ a.e. in Ω (for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$). Therefore,

$$(\mathcal{G}_p u)v = -\int_\Omega f(x, u)v\, dx - \int_\Omega (\xi\phi + h)v\, dx, \qquad \forall\, v \in E,$$

and consequently (5.14) becomes (5.1), as desired.

□

Remark 1. Consider the problem (5.1) for $F = \mathcal{W}_p(\overline{\Omega})$. Then one can observe that in this case, the variational inequality (5.1) becomes the variational equality (1.19) given in Definition 1.2. Thus in this case, a solution of (5.1) becomes a weak solution of problem (1.1). Furthermore, one can observe that given $w \in \mathcal{W}_p(\overline{\Omega})$, one has

$$w \vee \mathcal{W}_p(\overline{\Omega}) = \{v \in \mathcal{W}_p(\overline{\Omega}) \mid v \geq w \ \text{ a.e. in } \Omega\}$$

and

$$w \wedge \mathcal{W}_p(\overline{\Omega}) = \{v \in \mathcal{W}_p(\overline{\Omega}) \mid v \leq w \ \text{ a.e. in } \Omega\}.$$

Consequently, in this case the notion of weak subsolution and supersolution in Definition 5.1 coincides with the formulation for weak subsolutions and supersolutions given in Definition 1.2(b).

6. **Proof of the main result.** In this section we focus our attention in establishing the main results of this paper, namely, Theorem 1.4 and Theorem 1.5, assuming all the conditions in Assumption 1.3. Our approach will be follow arguments similar as in [48] (with some ideas motivated by the results in [19]). Thus, the proofs of some intermediate results will be omitted, but as our problem under consideration is much more general and several results require generalizations and modifications, more details will be given when needed. In particular, we will give proofs to most of the results here, and outline the main steps, especially when the generalizations come into play.

We begin by providing a non-existence result.

Lemma 6.1. *If*

$$\xi > \frac{(M + \|h\|_{\infty,\Omega})\lambda_N(\Omega)}{\|\phi\|_{1,\Omega}} > 0,$$

then the problems (1.1) and (1.2) have no weak solution over $\mathcal{W}_p(\overline{\Omega})$.

Proof. We argue as in the proof of Theorem 4.2. In fact, if $u \in \mathcal{W}_p(\overline{\Omega})$ is a weak solution of (1.1), we test (1.19) with the function $w \equiv 1$. Because $\left\langle \partial\tilde{\mathcal{L}}_n^{(p)}(u), 1 \right\rangle = 0$, it follows that $\Lambda_p(u, 1) = 0$. By using (1.9) and (1.10), we get

$$0 = \int_\Omega f(x, u)\, dx + \int_\Omega (\xi\phi + h)\, dx$$

$$\geq \eta_0 \int_\Omega |u|^{p-1}\,dx + \xi\|\phi\|_{1,\Omega} - (M + \|h\|_{\infty,\Omega})\lambda_N(\Omega),$$

Since $\lambda_N(\Omega) > 0$ and $\|\phi\|_{1,\Omega} > 0$ (since $\phi > 0$ a.e. in Ω), the preceding inequality implies that

$$\xi \leq \frac{(M + \|h\|_{\infty,\Omega})\lambda_N(\Omega)}{\|\phi\|_{1,\Omega}},$$

which establishes the desired claim. □

Lemma 6.2. *For each $\xi \in \mathbb{R}$, the constant*

$$c_\xi := -\left(\frac{|\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega} + M}{\eta_0}\right)^{1/(p-1)} \tag{6.1}$$

is a weak subsolution of (1.1) and (1.2), and every constant $c < c_\xi$ is a strict subsolution of both (1.1) and (1.2). Moreover, if $u \in \mathcal{W}_p(\overline{\Omega})$ is a weak solution of (1.1) $(N = 2)$, or (1.2) $(N = 3)$, then $u(x) \geq c_\xi$ for every $x \in \overline{\Omega}$.

Proof. For simplicity, assume that $\Omega = \Omega_3 \in \mathbb{R}^3$ (the other case is analogous). Given c_ξ defined as in (6.1), by virtue of (1.10) we have

$$f(x, c_\xi) \geq \eta_0|c_\xi|^{p-2}c_\xi - M = |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega}. \tag{6.2}$$

Selecting $\varphi \in \mathcal{W}_p(\overline{\Omega})^+$ arbitrarily, multiplying the Eq. (6.2) by φ, and integrating over Ω, we get

$$\begin{aligned}\int_\Omega f(x, c_\xi)\varphi\,dx &\geq |\xi|\|\phi\|_{\infty,\Omega}\int_\Omega \varphi\,dx + \|h\|_{\infty,\Omega}\int_\Omega \varphi\,dx \\ &\geq |\xi|\int_\Omega \phi\varphi\,dx + \int_\Omega |h|\varphi\,dx \geq -\int_\Omega (\xi\phi + h)\varphi\,dx.\end{aligned} \tag{6.3}$$

Thus from (6.3), we get

$$0 = \Lambda_p(c_\xi, \varphi) \leq \int_\Omega f(x, c_\xi)\varphi\,dx + \int_\Omega (\xi\phi + h)\varphi\,dx,$$

proving that c_ξ is a weak subsolution of the problem (1.2). Furthermore, if follows immediately that if c is a constant less than c_ξ, then c becomes a strict subsolution of the Eq. (1.2). It remains to show that c_ξ is a lower bound for any weak solution $u \in \mathcal{W}_p(\overline{\Omega})$ of (1.2). We suppose that the contrary holds, that is, assume that there exists a constant $\epsilon_0 > 0$ such that the function $(c_\xi - \epsilon_0 - u(x_0))^+$ is strictly positive for some $x_0 \in \overline{\Omega}$. Given $\epsilon \in (0, \epsilon_0)$, we set

$$\Omega_{\epsilon,\xi} := \{x \in \Omega :\ u(x) < c_\xi - \epsilon\} \neq \emptyset \quad \text{and} \quad u_{\epsilon,\xi} := (c_\xi - \epsilon - u)^+ \in \mathcal{W}_p(\overline{\Omega})^+.$$

Notice that $(\Theta_\Omega u)u_{\epsilon,\xi} \leq 0$, $(\Theta_\Gamma u)u_{\epsilon,\xi} \leq 0$, and also in view of (4.5) and (4.6), we deduce that $K^{(p)}(u, u_{\epsilon,\xi}) \leq 0$. As $\nabla u_{\epsilon,\xi} = -\nabla u$, testing (1.19) with the function $u_{\epsilon,\xi}$ and applying (1.9), we obtain

$$\begin{aligned}\|\nabla u\|^p_{p,\Omega_{\epsilon,\xi}} &\leq -\Lambda_p(u, u_{\epsilon,\xi}) = -\int_{\Omega_{\epsilon,\xi}} f(x, u)u_{\epsilon,\xi}\,dx - \int_{\Omega_{\epsilon,\xi}} (\xi\phi + h)u_{\epsilon,\xi}\,dx \\ &\leq -\eta_0\int_{\Omega_{\epsilon,\xi}} |u|^{p-1}u_{\epsilon,\xi}\,dx + (M + |\xi|\|\phi\|_{\infty,\Omega} + \|h\|_{\infty,\Omega})\int_{\Omega_{\epsilon,\xi}} u_{\epsilon,\xi}\,dx\end{aligned}$$

$$\le -\left(\eta_0|c_\xi - \epsilon|^{p-1} - |\xi|\|\phi\|_{\infty,\Omega} - \|h\|_{\infty,\Omega} - M\right)\int_{\Omega_{\epsilon,\xi}} u_{\epsilon,\xi}\,dx.$$

Recalling the definition of c_ξ and the selection $\epsilon \in (0,\epsilon_0)$, the above estimate yields $\|\nabla u\|^p_{p,\Omega_{\epsilon,\xi}} < 0$, a clear contradiction. Consequently, $u(x) \ge c_\xi$ for all $x \in \overline{\Omega}$, completing the proof. $\square$

Now we establish the first main result of this article.

Proof. [Theorem 1.4] Assume the condition of the theorem. From Lemma 6.1 one has the statements (1).
To prove the other assertion, for simplicity (as in the previous lemma) we assume that $\Omega = \Omega_3 \in \mathbb{R}^3$. Given ξ, $\xi' \in \mathbb{R}$ with $\xi \le \xi_0'$, we notice that the zero function 0 is a weak supersolution of (1.2) if and only if $\xi\phi(x) \le -f(x,0)-h(x)$ for a.e. $x \in \Omega$. Thus, set

$$\xi_0' := \inf_{x\in\Omega}\left\{\frac{-f(x,0)-h(x)}{\max\{1,\phi(x)\}}\right\}.$$

If $\xi \le \xi_0'$, then we get that

$$-f(x,0)-h(x) \ \ge\ \xi\max\{1,\phi(x)\} \ \ge\ \xi\phi(x) \quad \text{ for a.e. } x\in\Omega,$$

and consequently 0 is a weak supersolution of (1.2) for all $\xi \le \xi_0'$. On the other hand, by Lemma 6.2, the negative constant c_ξ given by (6.1) is a weak subsolution of (1.2) for all ξ. Hence, Theorem 5.2 and Remark 1 imply that (1.2) is solvable for all $\xi \le \xi'$. Moreover, if (1.2) is solvable for some parameter ξ, and $u \in \mathcal{W}_p(\overline{\Omega})$ is the corresponding weak solution, as $\phi > 0$ a.e. in Ω, it follows that u is a weak supersolution of (1.2) corresponding to the parameter ζ, for $\zeta < \xi$. Henceforth, by Lemma 6.2, Theorem 5.2 and Remark 1, we deduce that (1.2) is solvable for all $\zeta < \xi$. The last statement of the theorem follows directly from Theorem 4.3. $\square$

Next we establish the second main result of the paper.

Proof. [Theorem 1.5] Again, without loss of generality, we assume that $\Omega = \Omega_3 \in \mathbb{R}^3$. Because $p \in (N,\infty)$, by the Sobolev embedding theorem (see section 2) we have $\mathcal{W}_p(\overline{\Omega}) \hookrightarrow C^{0,\delta}(\overline{\Omega})$ for some $\delta \in (0,1)$. If $\xi \le \xi_0$, then Theorem 1.4 asserts the existence of a weak solution of the problem (1.2). In view of the proof of Theorem 1.4, recalling Lemma 6.1, we observe that the set

$$S = \{\xi \in \mathbb{R} \colon \text{ (1.2) is solvable}\}$$

is non-empty, and bounded from above by $\xi_0 := \sup(S)$. Clearly $(-\infty,\xi_0) \subseteq S$. Now let $\{\xi_n\}$ be a non-decreasing sequence of real numbers, such that $\lim_{n\to\infty} \xi_n = \xi_0$. Denote by u_{ξ_n} the weak solutions of (1.2) with respect to the parameters ξ_n ($n \in \mathbb{N}$). By virtue of Theorem 4.2 and the Sobolev embedding, the family $\{u_{\xi_n}\}_{n\in\mathbb{N}}$ is equicontinuous and pointwise bounded. Consequently, by Ascoli's theorem, we can find a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\lim_{k\to\infty} u_{\xi_{n_k}} = u_{\xi_0}$ in $C(\overline{\Omega})$. Therefore, u_{ξ_0} is a weak solution of (1.2), which implies that $S = (-\infty,\xi_0]$. It follows that there exists a minimal weak solution of Eq. (1.2) over the set $\{w \in \mathcal{W}_p(\Omega) : w \ge c_\xi \text{ in } \overline{\Omega}\}$, which proves (1) and (2). To show (3) and (4), we put

$$\xi_1 := \sup_{s\in\mathbb{R}} \inf_{x\in\Omega} \left\{ \frac{-f(x,s)-h(x)}{\max\{1,\phi(x)\}} \right\}. \tag{6.4}$$

As $h \in L^\infty(\Omega)$, recalling (1.9) and (1.10), we see that $-f(x,s)-h(x)$ is bounded from above, and thus ξ_1 is well-defined. If $\xi < \xi_1$, then one can find $s_0 \in \mathbb{R}$ such that

$$\xi < \inf_{x\in\Omega} \left\{ \frac{-f(x,s_0)-h(x)}{\max\{1,\phi(x)\}} \right\}.$$

Thus, s_0 is a weak supersolution of (1.2), c_ξ is a weak subsolution of (1.2) (c_ξ given by (6.1)), and $c_\xi \le s_0$ by virtue of Lemma 6.2. Consequently, from Theorem 5.2 and Remark 1 we deduce that there exists a weak solution $u_\xi \in \mathcal{W}_p(\overline{\Omega})$ with $c_\xi \le u_\xi \le s_0$. Therefore $\xi_1 \le \xi_0$. Given f locally Hölder continuous over $\mathbb{R}$, and uniformly continuous a.e. on Ω, let $\xi < \xi_1$. We search a second distinct weak solution of (1.2). To achieve this, we will employ the well-known Leray-Schauder degree theory. Indeed, we can find a constant $t > s_0$ such that

$$\xi < \inf_{x\in\Omega} \left\{ \frac{-f(x,t)-h(x)}{\max\{1,\phi(x)\}} \right\},$$

and thus t is also a weak supersolution of (1.2). Moreover, by Lemma 6.2, any constant c with $c < c_\xi$ is a strict weak subsolution of (1.2). Then we define the open set

$$U := \{v \in C(\overline{\Omega}) : \, c < v(x) < t \text{ for all } x \in \overline{\Omega}\}. \tag{6.5}$$

Clearly $u_\xi \in U$, and furthermore, by (1.11), one can find a constant $\tau > 0$ such that $f(x,u) + \tau|u|^{p-2}u$ is non-decreasing in u over $[c,t]$. Since $u_\xi \in \mathcal{W}_p(\overline{\Omega}) \cap U$ is a weak solution of (1.2), it follows that u_ξ is a fixed point of the compact operator $\mathcal{K}_\xi : C(\overline{\Omega}) \to C(\overline{\Omega})$, defined by $\mathcal{K}_\xi v = w$, for w a weak solution of the boundary value problem

$$\begin{cases} -\Delta_p w + \Theta_\Omega w + \tau|w|^{p-2}w \;=\; f(x,v) + \tau|v|^{p-2}v + \xi\phi + h \;\text{ in }\; \Omega, \\ \mathcal{A}_V w + |\nabla w|^{p-2}\dfrac{\partial w}{\partial \nu} + \Theta_\Gamma w \;=\; 0 \qquad\quad \text{on } \Gamma \end{cases} \tag{6.6}$$

The existence of the solution $w \in \mathcal{W}_p(\overline{\Omega}) \cap C(\overline{\Omega})$ of problem (6.6) can be obtained in a similar way as in [27, section 4]. If $\deg(I-\mathcal{K}_\xi, U, 0)$ is not well-defined (that is, if $0 \in (I-\mathcal{K}_\xi)(\partial U)$), the conclusion (3) follows. Otherwise, let $\psi \in U$, $\alpha \in [0,1]$, and define the mapping $\mathcal{T}_\alpha : C(\overline{\Omega}) \to C(\overline{\Omega})$, by

$$\mathcal{T}_\alpha v := \alpha\mathcal{K}_\xi v + (1-\alpha)\psi.$$

Notice that

$$-\Delta_p c + \Theta_\Omega c + \tau|c|^{p-2}c \;\le\; -\Delta_p(\mathcal{K}_\xi v) + \Theta_\Omega(\mathcal{K}_\xi v) + \tau|\mathcal{K}_\xi v|^{p-2}\mathcal{K}_\xi v \;\le\; -\Delta_p t + \Theta_\Omega t + \tau|t|^{p-2}t$$

for all $v \in U$, it follows from the Weak Comparison Principle that $\mathcal{K}_\xi v \in \overline{U}$ for each $v \in U$, that is, $\mathcal{K}_\xi v(x) \in [c,t]$ for each $x \in \overline{\Omega}$. Since U is convex and $\psi \in U$, one gets that $\mathcal{T}_\alpha$ maps U into U for every $\alpha \in (0,1]$, which yields that $0 \notin (I-\mathcal{T}_\alpha)(\partial U)$ for all $\alpha \in [0,1]$. In this way, $\deg(I-\mathcal{T}_\alpha, U, 0)$ is well-defined and independent of

α. Moreover, as $\psi \in U$ and $\mathcal{T}_0 v = \psi$ for all $v \in U$, we obtain that

$$\deg(I - \mathcal{K}_\xi, U, 0) = \deg(I - \mathcal{T}_\alpha, U, 0) = \deg(I - \mathcal{T}_0, U, 0) = 1. \tag{6.7}$$

On the other hand, for a sufficiently large constant $\vartheta > c_6(\xi) > 0$ (for $c_6(\xi)$ the constant in (4.28)), set

$$D_\vartheta := \left\{ v \in C(\overline{\Omega}) : \|v\|_{C(\overline{\Omega})} < \vartheta \right\}. \tag{6.8}$$

By virtue of Theorem 4.3(a), we observe that if $\xi' \in [\xi, \xi_0 + 1]$ and $u_{\xi'}$ is a fixed point of $\mathcal{K}_{\xi'}$, then $u_{\xi'} \in D_\vartheta$. As ξ is fixed, and $\xi < \xi_1 \leq \xi_0$, we can suppose that $U \subseteq V_\varrho$. Since $\mathcal{K}_{\xi_0+1}$ does not have fixed points, this fact together with the above arguments entail that

$$\deg(I - \mathcal{K}_\xi, D_\vartheta, 0) = \deg(I - \mathcal{K}_{\xi_0+1}, D_\vartheta, 0) = 0. \tag{6.9}$$

Consequently, (6.7) and (6.8) imply that

$$\deg(I - \mathcal{K}_\xi, D_\vartheta - U, 0) = -1, \tag{6.10}$$

which says that problem (1.2) admits a weak solution $\tilde{u} \in \mathcal{W}_p(\overline{\Omega}) \setminus U$. This establishes the assertion (3). Finally, suppose that $f \in C(\overline{\Omega} \times \mathbb{R})$. For $\xi < \xi_0$, let $u_{\xi_0} \in \mathcal{W}_p(\overline{\Omega})$ be the minimal weak solution of (1.2) with respect to ξ_0. Then, u_{ξ_0} is a weak supersolution of (1.2) corresponding to ξ. Since $c_\xi \leq u_{\xi_0}$, by Theorem 5.2, there is a weak solution u_ξ of (1.2) fulfilling $c_\xi \leq u_\xi \leq u_{\xi_0}$. As $f \in C(\overline{\Omega} \times \mathbb{R})$, we can find a constant γ sufficiently small, such that

$$f(x, u_{\xi_0}) - f(x, u_{\xi_0} + \gamma) + (\xi_0 - \xi)\phi(x) \, \geq \, 0. \tag{6.11}$$

Letting $w_{\xi_0} := u_{\xi_0} + \gamma$, one gets from (6.11) that

$$\begin{aligned} -\Delta_p w_{\xi_0} + \Theta_\Omega w_{\xi_0} &= f(x, u_{\xi_0} + \gamma) + t\phi(x) + [f(x, u_{\xi_0}) - f(x, u_{\xi_0} + \gamma) + (\xi_0 - \xi)\phi(x)] \\ &\geq \, f(x, u_{\xi_0} + \gamma) + t\phi(x), \end{aligned} \tag{6.12}$$

which entails that w_{ξ_0} is a weak supersolution of (1.2) corresponding to t (for γ small enough). Recalling that any $c < c_\xi$ is a strict weak subsolution of (1.2), one concludes that $c < u_\xi < w_{\xi_0}$. From here, one can proceed exactly as in the previous case to achieve the same conclusion as in statement (3). Consequently, in this case, $\xi_1 = \xi_0$, establishing (4) and completing the proof of the theorem. □

Acknowledgements. M.R. Lancia and P. Vernole have been supported by **the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM)**.

REFERENCES

[1] R. Adams, *Sobolev Spaces.* Academic Press, New York, 1975.

[2] S.Aizicovici, N.S.Papageorgiou and V.Staicu, Sublinear and superlinear Ambrosetti–Prodi problems for the Dirichlet p-Laplacian. *Nonlinear Analysis* **95** (2014), 263–280.

[3] A.Ambrosetti and G.Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces. *Annali Mat. Pura Appl.* **93** (1972), 231–247.

[4] D.Arcoya and D.Ruiz, The Ambrosetti–Prodi problem for the p-Laplacian operator. *Comm. Partial Differential Equations* **31** (2006), 849–865.

[5] M.Arias and M.Cuesta, A one side superlinear Ambrosetti–Prodi problem for the Dirichlet p-laplacian. *J. Math. Anal. Appl.* **367** (2010), 499–507.

[6] M.Biegert, A priori estimate for the difference of solutions to quasi-linear elliptic equations. *Manuscripta Math.* **133** (2010), 273–306.

[7] M.Biegert, On trace of Sobolev functions on the boundary of extension domains. *Proc. Amer. Math. Soc.* **137** (2009), 4169–4176.

[8] M. Biroli and P.Vernole, Strongly local nonlinear Dirichlet functionals and forms, *Adv. Math. Sci. Appl.*, **15**, (2005), 655–682,

[9] V. I. Burenkov. *Sobolev Spaces on Domains.* TEUBNER-TEXTE zur Mathematik, Vol. 137, 1998.

[10] R. Capitanelli, Homogeneous p-Lagrangians and self-similarity, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* , **5** (2003), 215–235.

[11] R. Capitanelli, Nonlinear energy forms on certain fractal curves, *J. Nonlinear Convex Anal.* **3** (2002), no. 1, 67–80.

[12] R.Capitanelli and M.R.Lancia, Nonlinear energy forms and Lipschitz spaces on the Koch curve. *J. Convex Anal.* **9** (2002), 245–257.

[13] M.Cefalo, G.Dell'acqua and M.R.Lancia, Numerical approximation of transmission problems across Koch-type highly conductive layers. *AMC* **218** (2012), 5453–5473.

[14] M.Cefalo and M.R.Lancia, An optimal mesh generation algorithm for domains with Koch type boundaries. *Math. Comput. Simulation* **106** (2014), 133–162.

[15] M.Cefalo, M.R.Lancia and H.Liang, Heat flow problems across fractal mixtures: regularity results of the solutions and numerical approximation *Differential and Integral equations* **26** (2013), 1027–1054.

[16] S.Creo, M.R.Lancia, A.Vélez - Santiago and P.Vernole, Approximation of a nonlinear fractal energy functional on varying Hilbert spaces. *Comm. Pure Appl. Anal.* **17** (2018), 647–669.

[17] S.Creo and V.Regis Durante Convergence and density results for parabolic quasi-linear Venttsel' problems in fractal domains. *To appear on DCDS-S 2019*

[18] D.Danielli, N.Garofalo and D-H.Nhieu, Non-doubling Ahlfors Measures, Perimeter Measures, and the Characterization of the Trace Spaces of Sobolev Functions in Carnot-Carathéodory Spaces. *Mem. Amer. Math. Soc.* **182** (2006), no. **857**.

[19] F.O.de Paiva and M.Montenegro, An Ambrosetti-Prodi-type result for a quasilinear Neumann problem. *Proc. Edinburgh Math. Soc.* **55** (2012), 771–780.

[20] F.O.de Paiva and A.E.Presoto, A Neumann problem of Ambrosetti-Prodi type. *J. Fixed Point Theory Appl.* **18** (2016), 189–200.

[21] U.Freiberg and M. R. Lancia, Energy form on a closed fractal curve, *Z. Anal. Anwendungen.* **23** (2004), 115-135.

[22] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter Studies in Mathematics, vol. 19, Berlin Eds. Bauer Kazdan, Zehnder 1994.

[23] P. Hajłasz, P. Koskela and H. Tuominen, Sobolev embeddings, extensions and measure density condition. *J. Functional Analysis* **254** (2008), 1217–1234.

[24] A. Jonsson and H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$. Math. Rep. Vol. 2 Part I, Academic Publisher, Harwood, 1984.

[25] E.Koizumi and K.Schmitt, Ambrosetti–Prodi-type problems for quasilinear elliptic problems. *Diff. Integ. Equations* **18** (2005), 241–262.

[26] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations, in: Mathematics in Science and Engineering, 46.* Academic Press, New York-London, 1968.

[27] M. R. Lancia, A. Vélez - Santiago and P. Vernole, Quasi-linear Venttsel' problems with non-local boundary conditions on fractal domains. *Nonlinear Anal. Real Worl Appl.* **35** (2017), 265–291.

[28] M. R. Lancia and P. Vernole, Semilinear Venttsel problems in fractal domains. *Applied Mathematics* **5** (2014), 1820–1833.
[29] M. R. Lancia and P. Vernole, Venttsel problems in fractal domains. *J. Evolution Equations* **14** (2014), 681–712.
[30] M. R. Lancia V. Regis Durante and P. Vernole, Asymptotics for Venttsel' problems for operators in non divergence form in irregular domains, Discrete Contin. Dyn. Syst. Ser. S,**9**, (2016), no. 5, 1493-1520.
[31] M. R. Lancia and M. A. Vivaldi, Lipschitz spaces and Besov traces on self-similar fractals, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* **23**, (1999), 101–106.
[32] V. Lappalainen and A. Lehtonen, Embedding of Orliz-Sobolev spaces in Hölder spaces. *Annales Academiæ Scientiarum Fennicæ* **14** (1989), 41–46.
[33] V. K. Le, On a sub-supersolutions method for variational inequalities with Leray-Lions operators in variable exponent spaces. *Nonlinear Analysis* **71** (2009), 3305–3321.
[34] V. K. Le and K. Schmitt, Some general concepts of sub- and supersolutions for nonlinear elliptic problems. *Topol. Methods Nonlinear Anal.* **28** (2006), 87–103.
[35] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.
[36] J. Maly and U. Mosco, Remarks on measure-valued Lagrangians on homogeneous spaces. *Ricerche Mat.*, **48**, (1999), 217-231.
[37] J. Mawhin, *Ambrosetti–Prodi-type results in nonlinear boundary value problems*. Lecture Notes in Mathematics, Volume 1285, pp. 290–313 (Springer), 1985.
[38] V. G. Maz'ya, *Sobolev Spaces*. Springer-Verlag, Berlin, 1985.
[39] T. J. Miotto, Superlinear Ambrosetti–Prodi problem for the p-Laplacian operator. *Nonlinear Differ. Equ. Appl. NoDEA* **17** (2010), 337–353.
[40] T. J. Miotto, On the Ambrosetti-Prodi problem for a system involving p-Laplacian operator. *Math Nachr.* **289** (2016), 67–84.
[41] M. R. V. Murthy and G. Stampacchia, Boundary value problems for some degenerate-elliptic operators. *Annali Mat. Pura Appl.* **80** (1968), 1–122.
[42] U. Mosco, Lagrangian metrics on fractals. *Proc. Symp. Appl. Math,* **54**, *Amer. Math. Soc., R.Spigler and S. Venakides eds.* (1998), 301–323.
[43] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*. Masson, Paris, 1967.
[44] D. D. Repovš, Ambrosetti-Prodi problem with degenerate potential and Neumann boundary conditions. *Electronic J. Differential Equations* **41** (2018), 1–10.
[45] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Amer. Math. Soc., Providence, RI, 1997.
[46] E. Sovrano, Ambrosetti-Prodi type result to a Neumann problem via a topological approach. *Discrete Contin. Dyn. Syst. Ser. S* **11** (2009), 345–355.
[47] A. Vélez-Santiago, A quasi-linear Neumann problem of Ambrosetti–Prodi type on extension domains. *Nonlinear Analysis* **160** (2017), 191–210.
[48] A. Vélez-Santiago, Ambrosetti–Prodi-type problems for quasi-linear elliptic equations with nonlocal boundary conditions. *Calc. Var. PDEs* **54** (2015), 3439–3469.
[49] A. Vélez-Santiago, Global regularity for a class of quasi-linear local and nonlocal elliptic equations on extension domains. *J. Functional Analysis* **269** (2015), 1–46.
[50] A.Vélez-Santiago and M.Warma, A class of quasi-linear parabolic and elliptic equations with nonlocal Robin boundary conditions. *J. Math. Anal. Appl.* **372** (2010), 120–139.
[51] H. Wallin, The trace to the boundary of Sobolev spaces on a snowflake. *Manuscripta Math.* **73** (1991), 117–125.
[52] M. Warma, The fractional relative capacity and the fractional Laplacian with Neuman and Robin boundary conditions on open sets. *Potential Analysis.* **42** (2015), 499–547.
[53] W.P.Ziemer, *Weakly Differentiable Functions*. Springer, New York, 1989.

E-mail address: maria.lancia@sbai.uniroma1.it
E-mail address: alejandro.velez2@upr.edu, alejovelez32@gmail.com
E-mail address: paola.vernole@uniroma1.it