

# A Fibonacci control system with application to hyper-redundant manipulators

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**Abstract** We study a model for snake-like robots based on the Fibonacci sequence. The present paper includes an investigation of the reachable workspace, a more general analysis of the control system underlying the model, its reachability and local controllability properties. In addition, we establish some fractal properties of the reachable workspace by means the theory of iterated function systems.

**Keywords** Redundant manipulators · Fibonacci sequence · Discrete control · Self-similar dynamics · Expansions in complex bases

## 1 Introduction

The aim of this paper is to give a model of a planar hyper-redundant manipulator, which is analogous in morphology to robotic snakes and tentacles, based on a discrete linear dynamical system involving the Fibonacci sequence. This approach is motivated by the ubiquitous presence of Fibonacci numbers in nature (see [3] and [27]) and, in particular, in human limbs [24].

The robot proposed in the present paper is a planar manipulator with rigid links and with an arbitrarily large number of degrees of freedom, i.e., it belongs to the class of

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so-called macroscopically serial hyper-redundant manipulators—the term was first introduced in [5]. The device is controlled by a sequence of couples of discrete actuators on the junctions, ruling both the length and orientation of every link.

Hyper-redundant architecture was intensively studied back to the late 60s, when the first prototype of hyper-redundant robot arm was built [1].

The interest of researchers in devices with redundant controls is motivated by their ability to avoid obstacles and to perform new forms of locomotion and grasping—see for instance [2,4,6].

Crucial as it is, the effective control of hyper-redundant manipulator is difficult for its redundancy; see, for example [23]. For instance, the number of points of the reachable workspace increases exponentially with the number of degrees of freedom. In this paper, we employ the self-similarity of the Fibonacci sequence to provide alternative techniques of investigation of the reachable workspace based on combinatorics and on fractal geometry.

The main purpose of the present paper is to provide a theoretical background suitable for applications to inverse kinematic problems in a fashion like [9], where the analysis of the reachable workspace is used to design an algorithm solving the inverse kinematic problem in linear time with respect to the number of actuators. Furthermore, in [17], the design of a manipulator modeling human arm and with link lengths following the Fibonacci sequence provides a method for the self-collision avoidance problem. We believe that analogous geometrical properties can be extended to manipulators which are inspired by other biological forms, through the self-similarity induced by Fibonacci numbers. We motivate the choice of discrete controls via their precision with low cost compared to their continuous counterparts.

Hyper-redundant manipulators considered here are planar manipulators. This is only a first step in exploring an approach that, to the best of our knowledge, could add novelty to the existing literature in this field; therefore, for future work, its extension to the three-dimensional case represents a natural progress of this paper.

We finally mention that the workspaces of planar manipulators in the above-cited papers (e.g., [9]) are quite different from those depicted here. This is mainly due to the fact that we represent only a subset of the workspace, corresponding to the particular subclass of full-rotation configurations whose relation with fractal geometry is the most striking. Furthermore, unlike above-mentioned works, our robotic device has a telescopic structure modeled by the possibility of ruling not only the orientation but also the length of each link: we believe this additional feature to possibly affect the shape of the workspace.

The theoretical background relies on the theory of iterated function systems—see [11] for a general introduction on the topic. The approach proposed here is inspired by the relation between robotics and theory of expansions in non-integer bases that was first introduced in [8] and later applied to planar manipulators in [19–21] and [22]. For an overview on the expansions in non-integer bases, we refer to the Rényi's seminal paper [26] and to the papers [10,25]. For the geometrical aspects of the expansions in complex base, namely the arguments that are more related to problem studied here, we refer to the papers [12–14,16]. The techniques developed in the

present paper to study the full-rotation configuration generalize previous results in [18].

### 1.1 Brief description of the main results

A discrete dynamical system models the position of the extremal junction of the manipulator. The model includes two binary control parameters on every link. The first control parameter, denoted  $u_n$ , rules the length of the  $n$ th link  $l_n := u_n f_n q^{-n}$ , where  $f_n$  is the  $n$ th Fibonacci number and  $q$  is a constant scaling ratio. The other control,  $v_n$ , rules the angle between the current link and its predecessor, denoted  $\omega_n := (\pi - \omega)v_n$ , where  $\omega$  a fixed angle in  $(0, \pi)$ . Therefore, when  $v_n = 0$ , the  $n$ th link is collinear with its predecessor, and when  $v_n = 1$ , it forms a fixed angle  $\pi - \omega \in (0, \pi)$  with the  $n - 1$ th link. In Sect. 2 we show that, under these assumptions, the position of the  $n$ th junction,  $x_n(\mathbf{u}, \mathbf{v})$  is ruled by the relation

$$x_n(\mathbf{u}, \mathbf{v}) = x_{n-1} + u_n \frac{f_n}{q^n} e^{-i\omega \sum_{h=0}^n v_h} \quad (1)$$

where  $\mathbf{u} = (u_j)$ ,  $\mathbf{v} = (v_j) \in \{0, 1\}^\infty$ . By assuming that the  $n$ th junction is positioned at time  $n$  (namely by reading the index  $n$  as a discrete time variable) above equation may be reinterpreted as a discrete control system, whose trajectories model the configurations of the manipulator. This is a stationary problem: indeed, at this stage of the investigation we are interested on the reachable workspace of the manipulator (namely a static feature of robot) rather than its kinematics. In this setting, if the number of the links is finite, say it is equal to  $N$ , then the position of the end effector of the manipulator (i.e., the position of its extremal junction) is represented by  $x_N(\mathbf{u}, \mathbf{v})$ . We call reachable workspace the set

$$W_{N,q,\omega} := \{x_N(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \{0, 1\}^N\}.$$

By allowing an infinite number of the links, we also may introduce the definition of the asymptotic reachable workspace

$$W_{\infty,q,\omega} := \left\{ \lim_{N \rightarrow \infty} x_N(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \{0, 1\}^\infty \right\}.$$

The first main results, Theorems 1 and 2, deal with some asymptotic controllability properties of the manipulator.

Indeed, the investigation begins with the study of the quantity

$$L(\mathbf{u}) := \sum_{n=0}^{\infty} \frac{u_n f_n}{q^n}$$

called the total length of the manipulator.<sup>1</sup> First of all, we notice that the condition  $q > \varphi$ , where  $\varphi = (1 + \sqrt{5})/2$  is the Golden Ratio, ensures the convergence of above series. Theorem 1 is a first investigation of the behavior of the set of possible total lengths

$$L_{\infty,q} := \{L(\mathbf{u}) \mid \mathbf{u} \in \{0, 1\}^\infty\}$$

as  $q \rightarrow \infty$ . In particular, we show that if  $q$  is less than or equal to  $1 + \sqrt{3}$  then  $L_{\infty,q}$  is an interval. This estimate is sharp, indeed we shall also prove that when  $q > 1 + \sqrt{3}$  then  $L_{\infty,q}$  is a disconnected set. In other words, Theorem 1 states that we can arbitrarily set the length of manipulator within the range  $[0, L(\mathbf{1})]$  (where we have set  $\mathbf{1} := (1, 1, \dots, 1, \dots)$ ) if and only if the scaling ratio  $q$  belongs to the range  $(\varphi, 1 + \sqrt{3}]$ . The proof of Theorem 1 is constructive and an explicit algorithm is given.

Theorem 1 turns out to be also a useful tool to prove sufficient conditions for the local asymptotic controllability of the control system underlying the model (see Theorem 2) that is the possibility of placing the end effector of the manipulator arbitrarily close to any point belonging to a sufficiently small neighborhood of the origin. More precisely, Theorem 2 states that, under some technical assumptions (namely we assume that the maximal rotation angle  $\omega$  is of the form  $2d\pi/p$  for some  $d, p \in \mathbb{N}$ ), if  $q$  belongs to a certain range, then the asymptotic reachable workspace contains a neighborhood of the origin.<sup>2</sup>

The approach in the investigation of  $L_{\infty,q}$  and  $R_{\infty,q,\omega}$ , the latter defined as

$$R_{\infty,q,\omega} := \left\{ \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k e^{i\omega k}} \mid \mathbf{u} \in \{0, 1\}^\infty \right\},$$

strongly relies on the particular choice of the lengths of the links,  $l_n(u_n) := u_n f_n q^{-n}$ , and in particular, on the fact that, fixing  $\mathbf{u} = (u_n)$  the “backward” sequence  $\tilde{L}_n(\mathbf{u}) = \sum_{j=0}^n l_j(u_{n-j})$  satisfies the recursive, contractive relation

$$\tilde{L}_{n+1}(\mathbf{u}) = \frac{u_n + \tilde{L}_n(\mathbf{u})}{q} + \frac{\tilde{L}_{n-1}(\mathbf{u})}{q^2}. \quad (2)$$

A suitable generalization of (2) is interpreted as a discrete control dynamical system, the Fibonacci control system, which is investigated by means of combinatorial arguments.

We then use a generalization of above approach to study a suitable subset of  $R_{\infty,q,\omega}$ , the set of full-rotation configurations (namely the configurations corresponding to the choice  $\mathbf{v} = \mathbf{1}$ ). This approach is motivated by the fact that the full-rotation configurations satisfy a contractive, recursive relation similar to (2).

<sup>1</sup> Notice that  $L(\mathbf{u}) = L(\mathbf{u}, \mathbf{v})$  for all  $\mathbf{v} \in \{0, 1\}^\infty$  where  $L(\mathbf{u}, \mathbf{v}) := \sum_{n=1}^{\infty} |x_n(\mathbf{u}, \mathbf{v}) - x_{n-1}(\mathbf{u}, \mathbf{v})|$ .

<sup>2</sup> Actually, we prove that such a neighborhood is indeed a polygon which is symmetric with respect to the origin.

The third main result of the present paper concerns the characterization of  $L_{\infty,q,\omega}$  and the set of full-rotation configurations in terms of the attractor of a suitable iterated function system (IFS). This approach gives access to well-established results in fractal geometry to further investigate the topological properties of the reachable workspace, and to use known efficient algorithms for the generation of self-similar sets (e.g., random iteration algorithm) to have a numerical approximation of the asymptotic reachable set.

In what follows, we show some numerical simulations approximating the asymptotic reachable set associated with full-rotation configurations. However, a deeper exploitation of these potential applications is beyond the purposes of present work.

We finally remark that for all  $N \geq 0$  we have the inclusion  $W_{N,q,\omega} \subset W_{\infty,q,\omega}$  and, consequently, the Hausdorff distance between  $W_{N,q,\omega}$  and  $W_{\infty,q,\omega}$  satisfies

$$\begin{aligned} d_H(W_{N,q,\omega}, W_{\infty,q,\omega}) &= \sup_{x_\infty \in W_{\infty,q,\omega}} \inf_{x_N \in W_{N,q,\omega}} |x_\infty - x_N| \\ &\leq \sum_{k=N+1}^{\infty} \frac{f_k}{q^k} \leq \frac{q}{q^N(q^2 - q - 1)}. \end{aligned}$$

The above relation establishes a global error estimate for the approximation of  $W_{\infty,q,\omega}$  with  $W_{N,q,\omega}$ ; hence, every above-mentioned asymptotic controllability property is inherited by a practical implementable manipulator with a finite number of links  $N$  by paying an explicitly given, exponential decaying cost in terms of precision.

## 1.2 Organization of the paper

In Sect. 2, we introduce the model and we state the main results on the density of the reachable workspace. The remaining part of the paper is devoted to the analysis of the dynamical system underlying the model. Section 3 includes the introduction of such Fibonacci control system and to its preliminary properties. In Sects. 3.1 and 3.2, we establish some properties of reachability and local controllability. Finally, in Sect. 4, we establish a relation with the theory of iterated function systems and we point out some parallelisms with classical expansions in non-integer bases.

## 2 A model for a snake-like robot

Throughout this section we introduce a model for a snake-like robot. We assume links and junctions to be thin, so to be, respectively, approximated with their middle axes and barycentres. We also assume axes and barycentres to be coplanar and, by employing the isometry between  $\mathbb{R}^2$ , we use the symbols  $x_0, x_1, \dots, x_n \in \mathbb{C}$  to denote the position of the barycentres of the junctions; therefore, the length  $l_n$  of the  $n$ th link is

$$l_n = |x_n - x_{n-1}|. \quad (3)$$

We assume  $l_n$  to be ruled by a binary control  $u_n$ , and in particular,

$$l_n := u_n \frac{f_n}{q^n}. \quad (4)$$

where  $(f_n)$  is Fibonacci sequence, namely  $f_0 = f_1 := 1$  and  $f_{n+2} = f_{n+1} + f_n$  for all  $n \geq 0$ .

Now, consider the quantity

$$L(\mathbf{u}) = \sum_{n=0}^{\infty} l_n(u_n) \quad \text{with } \mathbf{u} = (u_n) \in \{0, 1\}^{\infty}$$

representing the total length of the configuration of the snake-like robot corresponding to the control  $\mathbf{u}$ .

*Remark 1* To simplify subsequent notations, we shall fix as the base of the manipulator the point  $x_{-1} = 0$ , so that the 0th link is well defined and it may be of length either 0 or 1.

We shall also use the quantity

$$S(q, h, p) := \sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}}. \quad (5)$$

The most general form of this definition will be used only in Sect. 3.2. At this stage, it is useful to introduce for brevity, the notation

$$S(q) := S(q, 0, 1) = \sum_{n=0}^{\infty} \frac{f_n}{q^n} = \begin{cases} \frac{q^2}{q^2 - q - 1} & \text{if } q > \varphi; \\ +\infty & \text{if } q \in (0, \varphi] \end{cases} \quad (6)$$

where  $\varphi := \frac{1+\sqrt{5}}{2}$  denotes the Golden Mean.

*Remark 2* If  $q > \varphi$  then for every  $\mathbf{u} \in \{0, 1\}^{\infty}$ , one has  $L(\mathbf{u}) \in [L(\mathbf{0}), L(\mathbf{1})] = [0, S(q)]$ .

In what follows we show that if the scaling ratio  $q$  belongs to a fixed interval and if we allow the number of links to be infinite, then we may constraint the total length of the snake-like robot  $L(\mathbf{u})$  to be any value in the interval  $[0, S(q)]$ .

**Theorem 1** *If  $q \in (\varphi, 1 + \sqrt{3}]$  then for every  $\bar{L} \in [0, S(q)]$  there exists a binary control sequence  $\mathbf{u} \in \{0, 1\}^{\infty}$  such that*

$$L(\mathbf{u}) = \bar{L}.$$

**Remark 3** The proof of Theorem 1 is postponed to Sect. 3.1 below.

We now continue the building of the model. In view of (3), if  $x_0 = 0$  one has for every  $n$

$$x_n(\mathbf{u}) = \sum_{k=0}^n u_k \frac{f_k}{q^k e^{i\omega_k}}, \quad (7)$$

where  $-\omega_k \in (-\pi, \pi]$  is the argument of  $x_k - x_{k-1}$  for  $k = 1, \dots, n$  and, consequently, it represents the orientation of the  $k$ th link with respect to the global reference system given by the real and imaginary axes.

**Example 1** If the angle between two consecutive links is constantly equal to  $\pi - \omega \in [0, 2\pi)$ , then  $\omega_n = n\omega \bmod (-\pi, \pi]$ .

So far we introduced a control sequence ruling the length of each link. We now endow the model with another binary control sequence  $\mathbf{v} = (v_n)$ , ruling the angle between two consecutive links. In the model, the angle between two consecutive links is either  $\pi$  or  $\pi - \omega$  for some fixed  $\omega \in (0, \pi)$ . If  $v_n = 0$  then the angle between the  $n - 1$ th link and the  $n$ th link is  $\pi$ , while if  $v_n = 1$  then the angle between the  $n - 1$ th link and the  $n$ th link is  $\pi - \omega$  so that

$$v_n = \begin{cases} 1 & \text{rotation of the angle } \omega \text{ of the } n\text{-th link;} \\ 0 & \text{no rotation.} \end{cases} \quad (8)$$

We notice that, under these assumptions,  $\omega_n = \omega_n(\mathbf{v})$  in (7) is indeed a controlled quantity, while  $L(\mathbf{u})$  is yet independent from  $\mathbf{v}$ .

**Proposition 1** Let  $n \geq 0$  and  $u_j = 1$  and  $v_j \in \{0, 1\}$  for  $j = 1, \dots, n$ . Then

$$\omega_n = \sum_{j=0}^n v_j \omega \bmod (-\pi, \pi] \quad (9)$$

*Proof* We adopt the notation  $\text{Arg}(z) \in (-\pi, \pi]$  to represent the principal value of the argument function  $\arg(z)$ . In view of (7)

$$w_{n+1} = -\text{Arg}(x_{n+1}(\mathbf{u}) - x_n(\mathbf{u})). \quad (10)$$

On the other hand,  $x_n$  is the vertex of the angle between the  $n$ th link and the  $n + 1$ th link; therefore, we have the relations

$$\text{Arg}(x_{n+1}(\mathbf{u}) - x_n(\mathbf{u})) - \text{Arg}(x_n(\mathbf{u}) - x_{n-1}(\mathbf{u})) \bmod (-\pi, \pi] = -v_{n+1} \omega \quad (11)$$

By a comparison between (10) and (11) we get

$$w_{n+1} = w_n + v_{n+1}\omega \mod (-\pi, \pi]. \quad (12)$$

and, consequently, the claim.  $\square$

**Remark 4** We notice that if  $u_n = 0$  then any choice of  $\omega_n(\mathbf{v})$  satisfies 7. So, if the link is not extended, the rotation of the angle is meant as a rotation of the reference frame of the link.

For example, if  $v_n = v_{n+1} = u_{n-1} = u_{n+1} = 1$  and  $u_n = 0$ , one has that  $x_{n-1} = x_n$  but the angle formed by the  $n - 1$ th junction and the  $n + 1$ th junction is  $\pi - 2\omega$ .

In view of Proposition 1 and of above Remark, we set  $\omega_n(\mathbf{v}) := \sum_{j=0}^n v_j \omega$ , so that the complete control system for the joints of manipulator reads:

$$x_n(\mathbf{u}, \mathbf{v}) = \sum_{k=0}^n u_k \frac{f_k}{q^k} e^{-i\omega \sum_{j=0}^k v_j}. \quad (13)$$

The second main result describes the topology of the asymptotic reachable workspace when the rotation angle  $\omega$  is rational with respect to  $\pi$ , namely it satisfies  $\omega = 2\pi \frac{d}{p}$  for some  $d, p \in \mathbb{N}$ . One has a local controllability result when the scaling ratio  $q$  is lower than a threshold depending on  $p$ , which we denote  $q(p)$ . In particular,  $q(p)$  is defined as the greatest real solution of the equation

$$\sum_{k=0}^{\infty} \frac{f_{pk}}{q^{pk}} = 2.$$

In Sect. 3.2.2 we give a closed formula for  $q(p)$ .

**Theorem 2** *If  $\omega = 2\pi \frac{d}{p}$  for some  $d, p \in \mathbb{N}$  and if  $q \in (\varphi, q(p)]$  then the asymptotic reachable workspace*

$$W_{\infty, q, \omega} := \left\{ \lim_{n \rightarrow \infty} x_n(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in \{0, 1\}^{\infty} \right\}$$

*contains a neighborhood of the origin.*

The proof of Theorem 2 is postponed to Sect. 3.2.1.

### 3 A Fibonacci control system

Throughout this section, we introduce an auxiliary control system that we call *Fibonacci control system* and we study its asymptotic reachable set.

We shall see that the reachability properties of the Fibonacci control system are somehow inherited by manipulator (modeled in previous section as the sequence of junctions  $x_n(\mathbf{u}, \mathbf{v})$ ) and that this relation provides an indirect proof of Theorems 1 and 2.



To gradually introduce Fibonacci control system, we begin with some remarks on particular configurations of  $x(\mathbf{u}, \mathbf{v})$ .

We notice that for every  $\mathbf{u}$

$$x(\mathbf{u}, \mathbf{0}) = \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k} = L(\mathbf{u})$$

and

$$x(\mathbf{u}, \mathbf{1}) = \sum_{k=0}^{\infty} u_k \frac{f_k}{q^k e^{i\omega k}} = \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}, \quad \text{where } z = qe^{i\omega}.$$

Then both Theorems 1 and 2 are related to the study of the set

$$R_{\infty}(z) := \left\{ \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k} \mid u_k \in \{0, 1\} \right\}.$$

Indeed

$$L_{\infty}(q) = \{L(\mathbf{u}) \mid \mathbf{u} \in \{0, 1\}^{\infty}\} = R_{\infty}(q)$$

and

$$W_{\infty, q, \omega} \supseteq \{x(\mathbf{u}, \mathbf{1}) \mid \mathbf{u} \in \{0, 1\}^{\infty}\} = R_{\infty}(qe^{i\omega})$$

In particular, the relation with Theorem 2 becomes clear by noticing that if we are able to show that  $R_{\infty}(qe^{i\omega})$  is a neighborhood of the origin, then the claim of Theorem 2 follows.

*Remark 5* Notice that if  $|z| > \varphi$  then  $R(z)$  is well defined and it is a compact set. Indeed one has

$$\lim_{n \rightarrow \infty} \left| \sum_{k=n}^{\infty} u_k \frac{f_k}{z^k} \right| \leq \lim_{n \rightarrow \infty} \sum_{k=0}^n \left| \frac{f_k}{z^k} \right| \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \frac{\varphi^{k-1}}{|z|^k} = 0.$$

(for the proof of the estimate  $f_k \leq \varphi^{k-1}$  see Proposition 5 below) and, consequently, the convergence of the series  $\sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}$ . Furthermore, one has

$$\left| \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k} \right| \leq \varphi^{-1} \left( 1 + \frac{1}{1 - \varphi/|z|} \right)$$

thus  $R(z)$  is a bounded set. Finally  $R(z)$  is closed by the continuity of the map

$$\mathbf{u} \mapsto \sum_{k=0}^{\infty} u_k \frac{f_k}{z^k}$$

with respect to the topology on infinite sequences induced by the distance  $d(\mathbf{u}, \mathbf{v}) = 2^{-\min\{k | u_k \neq v_k\}}$ .

In view of above reasoning, in what follows we shall focus on the study of  $R_\infty(z)$ , by constructing the theoretical background necessary to prove Theorems 1 and 2 and by investigating further properties of  $R_\infty(z)$ .

We finally introduce the Fibonacci control system

$$\begin{cases} \bar{x}_0(\mathbf{u}) = u_0 \\ \bar{x}_1(\mathbf{u}) = u_1 + \frac{u_0}{z} \\ \bar{x}_{n+2}(\mathbf{u}) = u_{n+2} + \frac{\bar{x}_{n+1}(\mathbf{u})}{z} + \frac{\bar{x}_n(\mathbf{u})}{z^2}. \end{cases} \quad (\text{F})$$

so that  $\bar{x}_n(\mathbf{u})$  is the (discrete) trajectory corresponding to the control  $\mathbf{u} \in \{0, 1\}^\infty$ .

*Remark 6* The first terms of  $\bar{x}_n(\mathbf{u})$  are

$$u_0, u_1 + \frac{u_0}{z}, u_2 + \frac{u_1}{z} + \frac{2u_0}{z^2}, u_3 + \frac{u_2}{z} + \frac{2u_1}{z^2} + \frac{3u_0}{z^3}, \dots$$

In general, by an inductive argument it is possible to prove that for each  $n \in \mathbb{N}$

$$\bar{x}_n(\mathbf{u}) = \sum_{k=0}^n \frac{f_k}{z^k} u_{n-k}.$$

We finally point out that the above equality implies that  $R_\infty(z)$  contains  $\{\bar{x}_n(\mathbf{u}) \mid \mathbf{u} \in \{0, 1\}^\infty, n \in \mathbb{N}\}$ , i.e., the reachable set of the system (F).

### 3.1 Asymptotical reachable set in real case

Throughout this section, we consider a real number  $q > \varphi$  and we show that  $R_\infty(q) = [0, S(q)]$  if and only if  $q \leq 1 + \sqrt{3}$  (namely we prove Theorem 1.<sup>3</sup>) For brevity, we specialize the definition of  $S(q, h, p)$  given in (5) as follows:

$$S(q, h) := \sum_{k=0}^{\infty} \frac{f_{h+k}}{q^k} = \frac{q^2 f_h + q f_{h-1}}{q^2 - q - 1} \quad (14)$$

Last equality can be proved by a simple inductive argument. We also shall use the following recursive relation

$$S(q, h) = q(S(q, h-1) - f_{h-1}). \quad (15)$$

Finally note that  $S(q, 0) = S(q)$ .

<sup>3</sup> Indeed the claim immediately follows by recalling the equality  $\{L(\mathbf{u}) \mid \mathbf{u} \in \{0, 1\}^\infty\} = R_\infty(q)$ .

**Lemma 1** Let  $q > \varphi$ . For every  $h$

$$f_h \leq \frac{S(q, h+1)}{q} \quad (16)$$

if and only if  $q \leq 1 + \sqrt{3}$ .

*Proof* First of all note that for every  $h$

$$\frac{f_{h+1}}{f_h} \geq 1 = \frac{f_1}{f_0}$$

and consequently  $q \leq 1 + \sqrt{3}$  if and only if

$$q \leq \frac{1}{2} \left( \frac{f_{h+1}}{f_h} + 1 \right) + \sqrt{\frac{1}{4} \left( \frac{f_{h+1}}{f_h} + 1 \right)^2 + 2} \quad \text{for every } h.$$

This, together with  $q > \varphi > 0$  implies that  $q \leq 1 + \sqrt{3}$  is equivalent to

$$f_h \leq \frac{q f_{h+1} + f_h}{q^2 - q - 1} \left( = \frac{S(q, h+1)}{q} \right) \quad \text{for every } h.$$

□

**Theorem 3** Let  $q \leq 1 + \sqrt{3}$  and  $x \in [0, S(q, 0)]$  and consider the sequences  $(r_h)$  and  $(u_h)$  defined by

$$\begin{cases} r_0 = x; \\ u_h = \begin{cases} 1 & \text{if } r_h \in [f_h, S(q, h)] \\ 0 & \text{otherwise} \end{cases} \\ r_{h+1} = q(r_h - u_h f_h) \end{cases} \quad (17)$$

Then

$$x = \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k \quad (18)$$

and, consequently,  $R_{\infty}(q) = [0, S(q, 0)]$ . Moreover, if  $q > 1 + \sqrt{3}$  then  $R_{\infty} \subsetneq [0, S(q, 0)]$ .

*Proof* Fix  $x \in [0, S(q, 0)]$  and first of all note that

$$x = \sum_{k=0}^h \frac{f_k}{q^k} u_k + \frac{r_{h+1}}{q^{h+1}} \quad \text{for all } h \geq 0. \quad (19)$$

Indeed above equality can be shown by induction on  $h$ . For  $h = 0$  one has  $r_1 = q(x - u_0 f_0)$  and consequently  $x = f_0 u_0 + r_1/q$ . Assume now (19) as inductive hypothesis. Then

$$r_{h+2} = q^{h+2} \left( x - \sum_{k=0}^h \frac{f_k}{q^k} u_k \right) - q f_{h+1} u_{h+1}$$

and, consequently,

$$x = \sum_{k=0}^{h+1} \frac{f_k}{q^k} u_k + \frac{r_{h+2}}{q^{h+2}}.$$

Now we claim that if  $q \leq 1 + \sqrt{3}$  then

$$r_h \in [0, S(q, h)] \quad \text{for every } h. \quad (20)$$

We show the above inclusion by induction. If  $h = 0$  then the claim follows by the definition of  $r_0$  and by the fact that  $x \in [0, S(q, 0)]$ . Assume now (20) as inductive hypothesis. One has  $r_h \in [0, S(q, h)] = [0, f_h] \cup [f_h, S(q, h)]$ . If  $r_h \in [0, f_h]$  then  $r_{h+1} = q r_h \in [0, q f_h] \subseteq [0, S(q, h+1)]$ —where the last inclusion follows by Lemma 1. If otherwise  $r_h \in [f_h, S(q, h)]$  then  $r_{h+1} = q(r_h - f_h) \subseteq [0, q(S(q, h) - f_h)] = [0, S(q, h+1)]$ —see (15).

Recalling  $f_n \sim \varphi^n$  as  $n \rightarrow \infty$ , one has

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k &= \lim_{h \rightarrow \infty} \sum_{k=0}^{h-1} \frac{f_k}{q^k} u_k \stackrel{(19)}{=} x - \lim_{h \rightarrow \infty} \frac{r_h}{q^h} \stackrel{(20)}{\geq} x - \lim_{h \rightarrow \infty} \frac{S(q, h)}{q^h} \\ &\stackrel{(14)}{=} x - \lim_{h \rightarrow \infty} \frac{q^2 f_{h+1} + q f_h}{q^h (q^2 - q - 1)} = x. \end{aligned}$$

On the other hand,

$$\sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k = x - \lim_{h \rightarrow \infty} \frac{r_h}{q^h} \leq x$$

and this proves (18). It follows by the arbitrariness of  $x$  that if  $q \leq 1 + \sqrt{3}$  then  $R_{\infty} = [0, S(q, 0)]$ .

Finally assume  $q > 1 + \sqrt{3}$ . By Lemma 1 there exists  $x \in (S(q, 1)/q, f_1)$ . To find a contradiction, assume  $x \in R_{\infty}$ . Then

$$x = u_0 f_0 + \frac{1}{q} \sum_{k=0}^{\infty} \frac{f_{k+1}}{q^k} u_{k+1}$$

Note that  $u_0 \neq 1$  because  $x < f_1 = 1$ . Then  $u_0 = 0$  and

$$x = \frac{1}{q} \sum_{k=0}^{\infty} \frac{f_{k+1}}{q^k} u_{k+1} \leq \frac{1}{q} \sum_{k=0}^{\infty} \frac{f_{k+1}}{q^k} = \frac{S(q, 1)}{q}$$

but this contradicts  $x \in (S(1, q)/q, f_1)$ . Then  $x \in [0, S(q, 0)] \setminus R_{\infty}$  and this concludes the proof.  $\square$

### 3.2 Asymptotical reachable set in complex case

Throughout this section we investigate  $R_{\infty}(z)$  with  $z = qe^{i\omega}$  and  $\omega = \frac{d}{p}2\pi$ ;  $d, p \in \mathbb{N}$ . First of all we notice that  $z^p = q^p$  and consequently

$$\sum_{k=0}^{\infty} u_k \frac{f_k}{z^k} = \sum_{h=0}^{p-1} z^{-h} \sum_{k=0}^{\infty} u_{pk+h} \frac{f_{pk+h}}{q^{pk}}. \quad (21)$$

Above equality implies that if  $p \geq 2$  and if

$$R_{\infty}^h := \left\{ \sum_{k=0}^{\infty} \frac{u_{pk+h} f_{pk+h}}{q^{pk}} \mid u_{pk+h} \in \{0, 1\} \right\}$$

is an interval (and not a disconnected set) then

$$R_{\infty}(z) = \left\{ \sum_{k=0}^{\infty} \frac{f_k}{z^k} u_k \mid u_k \in \{0, 1\} \right\} = \sum_{j=0}^{p-1} z^{-h} R_{\infty}^h$$

is a polygon containing the origin in its interior—note that  $\min R_{\infty}^h = 0$ . In what follows, we show that if  $q$  is small enough, then such a local controllability condition is satisfied.

By Definition 5, so that  $R_{\infty}^h \subset [0, S(q, h, p)]$  for every  $h = 0, \dots, p-1$  and from simple inductive arguments, we have the following recursive relation

$$S(q, h, p) = f_{h-1} S(q, 1, p) + f_{h-2} S(q, 0, p) \quad (22)$$

Moreover, one has

$$S(q, p, p) = q^p (S(q, 0, p) - f_0) \quad (23)$$

$$S(q, p+1, p) = q^p(S(q, 1, p) - f_1) \quad (24)$$

and, more generally,

$$S(q, p+h, p) = q^p(S(q, h, p) - f_h). \quad (25)$$

**Example 2** Let  $q = 2$  and  $p = 4$ . In view of (22),

$$\begin{aligned} R_\infty^0 &\subseteq [0, S(2, 0, 4)] \\ R_\infty^1 &\subseteq [0, S(2, 1, 4)] \\ R_\infty^2 &\subseteq [0, S(2, 0, 4) + S(2, 1, 4)] \\ R_\infty^3 &\subseteq [0, S(2, 0, 4) + 2S(2, 1, 4)]. \end{aligned}$$

See Sect. 3.2.2 for the explicit calculation of  $S(q, h, p)$ . In Theorem 4 below, we show that above inclusions are actually equalities, so that

$$R_\infty = R_\infty^0 - \frac{i}{2}R_\infty^1 - \frac{1}{4}R_\infty^2 + \frac{i}{8}R_\infty^3$$

is a rectangle in the complex plane—see Fig. 1.

**Lemma 2** If  $q \leq q(p)$  then for every  $h \in \mathbb{N}$

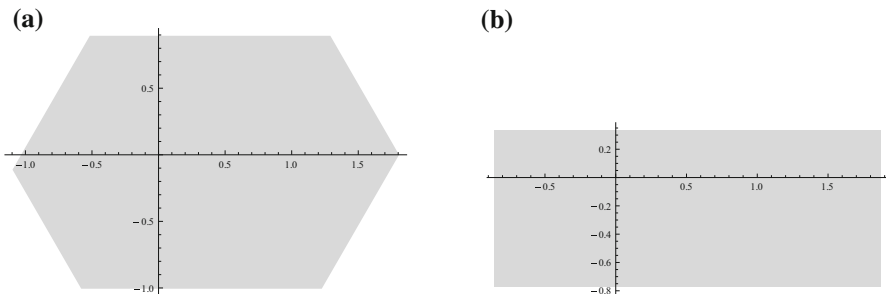
$$S(q, p, p+h) \geq q^p f_h. \quad (26)$$

*Proof* The case  $h = 0$  follows by the definition of  $q(p)$  and by (23). If  $h = 1$  then

$$S(q, p, p+1) \geq S(q, p, p) \geq q^p f_0 = q^p f_1.$$

Fix now  $h \geq 2$  and now (26) as inductive hypothesis for every integer lower than  $h$ . It follows by (22)

$$S(q, p, p+h) = f_{h-1}S(q, 1, p) + f_{h-2}S(q, 0, p) \geq 2(f_{h-1} + f_{h-2}) = 2f_h;$$



**Fig. 1** By Theorem 4,  $R_\infty(2e^{i2\pi/p})$  with  $p = 3, 4$  is a polygon. **a**  $R_\infty(2e^{i2\pi/3})$ , **b**  $R_\infty(2e^{i\pi/2})$

therefore, by (25), we finally get

$$S(q, p, p + h) = q^p(S(q, h, p) - f_h) \geq q^p f_h.$$

□

Finally let us define  $q(p)$  as the greatest solution of the equation

$$S(q, 0, p) = 2f_0 = 2$$

Note that if  $q \leq q(p)$  then  $S(q, 0, p) \geq 2$ .

**Remark 7** The value  $q(p)$  is explicitly calculated in Sect. 3.2.2 below. Among other results, we shall show

$$q(p) = \begin{cases} \left( \frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 - 8} \right)^{\frac{1}{p}} & p \text{ even;} \\ \left( \frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 + 8} \right)^{\frac{1}{p}} & p \text{ odd.} \end{cases} \quad (27)$$

We notice that above equality implies  $q(p) \sim f(p)^{1/p} \sim \varphi$  as  $p \rightarrow \infty$ .

**Example 3**  $q(1) = 1 + \sqrt{3}$ ,  $q(2) = \sqrt{\frac{1}{2}(5 + \sqrt{17})}$ ,  $q(3) = \sqrt[3]{\frac{1}{2}(7 + \sqrt{57})}$ ,  $q(4) = \sqrt[4]{6 + \sqrt{34}}$ .

**Lemma 3** Let  $p, h \in \mathbb{N}$  and let  $q \leq q(p)$ . For  $x \in [0, S(q, h, p)]$  consider the sequences  $(r_n)$  and  $(u_n)$  defined by

$$\begin{cases} r_0 = x; \\ u_n = \begin{cases} 1 & \text{if } r_n \in [f_n, S(q, np + h)] \\ 0 & \text{otherwise} \end{cases} \\ r_{n+1} = q^p(r_n - u_n f_{np+h}). \end{cases} \quad (28)$$

Then

$$x = \sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}} u_k \quad (29)$$

and, consequently,  $R_{\infty}^h = [0, S(q, 0, p)]$ . Moreover, if  $q > q(p)$  then  $R_{\infty} \subsetneq [0, S(q, 0, p)]$ .

**Proof** Fix  $h \in \mathbb{N}$  and  $x \in [0, S(q, 0, p)]$ . First of all note that

$$x = \sum_{k=0}^n \frac{f_{pk+h}}{q^{pk}} u_k + \frac{r_{n+1}}{q^{p(n+1)}} \quad \text{for all } n. \quad (30)$$

Indeed for  $h = 0$  one has  $r_1 = q^p(x - u_0 f_h)$  and consequently  $x = f_h u_0 + r_1/q^p$ . Assume now (30) as inductive hypothesis. Then

$$\begin{aligned} r_{n+2} &= q^p(r_{n+1} - u_{n+1} f_{p(n+1)+h}) \\ &= q^{p(n+2)} \left( x - \sum_{k=0}^n \frac{f_{kp+h}}{q^{pk}} u_k \right) - q^{p(n+2)} f_{p(n+1)+h} u_{n+1} \end{aligned}$$

and, consequently,

$$x = \sum_{k=0}^{n+1} \frac{f_{kp+h}}{q^{kp}} u_k + \frac{r_{n+2}}{q^{p(n+2)}}.$$

Now, we claim that for every  $n$  if  $q \leq q(p)$  then

$$r_n \in [0, S(q, pn + h, p)]. \quad (31)$$

We show the above inclusion by induction. If  $h = 0$  then the claim follows by the definition of  $r_0$  and by the fact that  $x \in [0, S(q, h, p)]$ . Assume now (31) as inductive hypothesis. One has  $r_n \in [0, S(q, pn + h, p)] = [0, f_{pn+h}) \cup [f_{pn+h}, S(q, pn + h, p)]$ . If  $r_n \in [0, f_{pn+h})$  then  $r_{n+1} = q^p r_n \in [0, q^p f_{pn+h}] \subseteq [0, S(q, (n+1)p + h, p)]$ —where the last inclusion follows by Lemma 2. If otherwise  $r_n \in [f_{pn+h}, S(q, pn + h, p)]$  then  $r_{n+1} = q^p(r_n - f_{pn+h}) \subseteq [0, q(S(q, pn + h, p) - f_{pn+h})] = [0, S(q, p(n+1) + h, p)]$ —see (25).

Recalling  $f_n \sim \varphi^n$  as  $n \rightarrow \infty$ , one has

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}} u_k &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{f_{pk+h}}{q^{pk}} u_k \stackrel{(30)}{=} x - \lim_{n \rightarrow \infty} \frac{r_n}{q^{pn}} \stackrel{(31)}{\geq} x - \lim_{n \rightarrow \infty} \frac{S(q, pn + h, p)}{q^{pn}} \\ &\stackrel{(22)}{=} x - \lim_{n \rightarrow \infty} \frac{f_{pn+h-1} S(q, 1, p) + f_{pn+h-2} S(q, 0, p)}{q^{ph}} = x. \end{aligned}$$

On the other hand,

$$\sum_{k=0}^{\infty} \frac{f_{pk+h}}{q^{pk}} u_k = x - \lim_{n \rightarrow \infty} \frac{r_n}{q^{pn}} \leq x$$

and this proves (29). It follows by the arbitrariness of  $x$  that if  $q \leq q(p)$  then  $R_{\infty}^h = [0, S(q, 0, p)]$ . Finally assume  $q > q(p)$ . By Lemma 1 there exists  $x \in (S(q, h, p)/q^p, f_h)$ . To find a contradiction, assume  $x \in R_{\infty}^h$ . Then

$$x = u_0 f_h + \frac{1}{q^p} \sum_{k=0}^{\infty} \frac{f_{p(k+1)+h}}{q^{pk}} u_{k+1}$$



Note that  $u_0 \neq 1$  because  $x < f_h$ . Then  $u_0 = 0$  and

$$x = \frac{1}{q^p} \sum_{k=0}^{\infty} \frac{f_{p(k+1)+h}}{q^{pk}} u_{k+1} \leq \frac{1}{q^p} \sum_{k=0}^{\infty} \frac{f_{p(k+1)}}{q^{pk}} = \frac{S(q, h, p)}{q^p}$$

but this contradicts  $x \in (S(q, h, p)/q^p, f_h)$ . Then  $x \in [0, S(q, 0, p)] \setminus R_{\infty}^h$  and this concludes the proof.  $\square$

**Theorem 4** *If  $\varphi < |z| \leq q(p)$  then  $R_{\infty}(z)$  is a polygon on the complex plane containing the origin.*

*Proof* It follows by Lemma 3 and by

$$R_{\infty}(z) = \left\{ \sum_{k=0}^{\infty} \frac{f_k}{z^k} u_k \mid u_k \in \{0, 1\} \right\} = \sum_{h=0}^{p-1} z^{-h} R_{\infty}^h.$$

$\square$

### 3.2.1 Proof of Theorem 2

Theorem 2 immediately follows by

$$R_{\infty}(qe^{i\omega}) = \{x(\mathbf{u}, \mathbf{1}) \mid \mathbf{u} \in \{0, 1\}^{\infty}\} \subset W_{\infty, q, \omega}$$

and by Theorem 4.

### 3.2.2 An explicit formula for $q(p)$

By a comparison between (22), (23) and (24),  $S(q, 0, p)$  and  $S(q, 1, p)$  are solutions of the following system of equations

$$\begin{cases} q^p(S(q, 0, p) - f_0) = f_{p-1}S(q, 1, p) + f_{p-2}S(q, 0, p) \\ q^p(S(q, 1, p) - f_1) = f_pS(q, 1, p) + f_{p-1}S(q, 0, p) \end{cases} \quad (32)$$

$$\begin{cases} (q^p - f_{p-2})S(q, 0, p) - f_{p-1}S(q, 1, p) = f_0q^p \\ -f_{p-1}S(q, 0, p) + (q^p - f_p)S(q, 1, p) = f_1q^p \end{cases} \quad (33)$$

whose solution is

$$S(q, 0, p) = \frac{\begin{vmatrix} f_0q^p & -f_{p-1} \\ f_1q^p & q^p - f_p \end{vmatrix}}{\begin{vmatrix} q^p - f_{p-2} & -f_{p-1} \\ -f_{p-1} & q^p - f_p \end{vmatrix}}, \quad (34)$$

$$S(q, 1, p) = \frac{\begin{vmatrix} q^p - f_{p-2} & f_0 q^p \\ -f_{p-1} & f_1 q^p \end{vmatrix}}{\begin{vmatrix} q^p - f_{p-2} & -f_{p-1} \\ -f_{p-1} & q^p - f_p \end{vmatrix}}. \quad (35)$$

We now show that the solutions in (34) and (35) are well defined.

**Proposition 2** *Let*

$$\Delta_p(q) := \begin{vmatrix} q^p - f_{p-2} & -f_{p-1} \\ -f_{p-1} & q^p - f_p \end{vmatrix} = (q^p - f_{p-2})(q^p - f_p) - f_{p-1}^2$$

*Then*

$$\Delta_p(q) = q^{2p} - (f_{p-2} + f_p)q^p + (-1)^p \quad (36)$$

*and the real roots of  $\Delta_p(q)$  are  $\pm\varphi$  and  $\pm(\varphi - 1)$  if  $p$  is even and  $-\varphi$  and  $\varphi - 1$  if  $p$  is odd.*

*In particular, if  $q > \varphi$  then  $\Delta_p \neq 0$ .*

*Proof* The equality in (36) follows by Cassini identity for  $p \geq 2$

$$f_{p-2}f_p - f_{p-1}^2 = (-1)^p.$$

Now, we notice that  $\Delta_p(q) = 0$  if and only if

$$\begin{cases} z = q^p \\ z^2 - (f_{p-2} + f_p)z + (-1)^p = 0. \end{cases}$$

We first discuss the case of an even  $p$ . When  $p$  is even then  $\Delta_p(q)$  has exactly 4 real solutions

$$\begin{aligned} q_{1,2}^{\text{even}} &= \pm \sqrt[p]{\frac{1}{2}(f_{p-2} + f_p) - \frac{1}{2}\sqrt{(f_{p-2} + f_p)^2 + 4}}, \\ q_{3,4}^{\text{even}} &= \pm \sqrt[p]{\frac{1}{2}(f_{p-2} + f_p) + \frac{1}{2}\sqrt{(f_{p-2} + f_p)^2 + 4}}. \end{aligned}$$

Now, for every  $p \in \mathbb{N}$  one has that the Golden Mean  $\varphi$  satisfies

$$\varphi^p = f_{p-1}\varphi + f_{p-2}$$

and, consequently,

$$\begin{aligned} \varphi^{2p} &= (f_{p-1}\varphi + f_{p-2})^2 \\ &= f_{p-1}^2\varphi^2 + 2f_{p-1}f_{p-2}\varphi + f_{p-2}^2 \\ &= (f_{p-1}^2 + 2f_{p-1}f_{p-2})\varphi + f_{p-1}^2 + f_{p-2}^2. \end{aligned}$$

This, together with  $\Delta(q) = \Delta(-q)$  and Cassini identity, implies

$$\Delta_p(\varphi) = \Delta_p(-\varphi) = f_{p-1}(f_{p-1} + f_{p-2} - f_p)\varphi + f_{p-1}^2 - f_p f_{p-2} + 1 = 0.$$

Moreover, since  $\varphi - 1 = 1/\varphi$  and  $\Delta(q) = \Delta(-q)$ ,

$$\Delta_p(\varphi - 1) = \Delta_p(1 - \varphi) = \Delta_p(1/\varphi) = \frac{\Delta_p(\varphi)}{\varphi^{2p}} = 0.$$

This concludes the proof for the even case.

Now, if  $p$  is odd, then  $\Delta_p(q) = 0$  has exactly two real solutions

$$q_{1,2}^{odd} = \sqrt[p]{\frac{1}{2}(f_{p-2} + f_p) - \frac{1}{2}\sqrt{(f_{p-2} + f_p)^2 - 4}}.$$

Again by Cassini identity

$$\begin{aligned}\Delta_p(\varphi) &= \varphi^{2p} - (f_{p-2} + f_p)\varphi + 1 \\ &= f_{p-1}(f_{p-1} + f_{p-2} - f_p)\varphi + f_{p-1}^2 - f_p f_{p-2} - (-1)^p = 0.\end{aligned}$$

Since  $1 - \varphi = -1/\varphi$ , we finally obtain

$$\Delta_p(1 - \varphi) = \Delta_p(-1/\varphi) = -\frac{\Delta_p(\varphi)}{\varphi^{2p}} = 0.$$

□

*Example 4* For  $p = 1$ , we already showed

$$S(q, 0, 1) = S(q) = \frac{q^2}{q^2 - q - 1} \quad S(q, 1, 1) = S(q) = \frac{q^2 + q}{q^2 - q - 1}.$$

For  $p = 2$ , namely when  $z = -q$ ,

$$S(q, 0, 2) = \frac{q^2(q^2 - 1)}{q^4 - 3q^2 + 1} \quad S(q, 1, 2) = \frac{q^4}{q^4 - 3q^2 + 1}.$$

For  $p = 3$ , namely when  $z$  is a rescaled cubic root of unity,

$$S(q, 0, 3) = \frac{q^3(q^3 - 1)}{q^6 - 4q^3 - 1} \quad S(q, 1, 3) = \frac{q^6 + q^3}{q^6 - 4q^3 - 1}.$$

For  $p = 4$

$$S(q, 0, 4) = \frac{q^4(q^4 - 2)}{q^8 - 7q^4 + 1} \quad S(q, 1, 4) = \frac{q^8 + q^4}{q^8 - 7q^4 + 1}.$$

We now give a closed formula for  $q(p)$ , see Fig. 2.

**Proposition 3** For every  $p \in \mathbb{N}$

$$q(p) = \begin{cases} \left( \left( \frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 - 8} \right)^{\frac{1}{p}} & p \text{ even;} \\ \left( \left( \frac{1}{2}(f_{p-2} + 2f_p) + \frac{1}{2}\sqrt{(f_{p-2} + 2f_p)^2 + 8} \right)^{\frac{1}{p}} & p \text{ odd.} \end{cases} \quad (37)$$

*Proof* We recall that  $q(p)$  is defined as the greatest solution of  $\sum_{k=0}^{\infty} \frac{f_{kp}}{q^{kp}} = 2$ , namely of

$$S(q, 0, p) = \frac{q^{2p} - f_{p-2}q^p}{q^{2p} - (f_{p-2} + f_p)q^p + (-1)^p} = 2.$$

Solving above equation, one gets

$$q^{2p} + (-f_{p-2} - 2f_p)q^p + 2(-1)^p = 0$$

and finally (37).  $\square$

## 4 A characterization of the reachable set via iterated function systems

Throughout this section we characterize  $R_{\infty}(q)$ , with  $q \in \mathbb{R}$ ,  $q > \varphi$ , as a projection on  $\mathbb{R}$  of the attractor of a (linear) Iterated Function System defined on  $\mathbb{R}^2$ .

### 4.1 Some basic facts about IFSs

An iterated function system (IFS) is a set of contractive functions  $G_j : X \rightarrow X$ , where  $(X, \mathbf{d})$  is a metric space. We recall that a function  $f$  is contractive if for every  $x, y \in X$

$$d(f(x), f(y)) < c \cdot d(x, y)$$

for some  $c < 1$ . In [15], Hutchinson showed that every finite IFS, namely every IFS with finitely many contractions, admits a unique non-empty compact fixed point  $Q$  with respect to the Hutchinson operator

$$\mathcal{G} : S \mapsto \bigcup_{j=1}^J G_j(S).$$

Moreover, for every non-empty compact set  $S \subseteq \mathbb{C}$

$$\lim_{k \rightarrow \infty} \mathcal{G}^k(S) = Q.$$

The attractor  $Q$  is a self-similar set and it is the only bounded set satisfying  $\mathcal{F}(Q) = Q$ .

## 4.2 The reachable set is a projection of the attractor of an IFS

Let  $q > \varphi$ ,  $\mathbf{v} \in \mathbb{R}^2$  and consider the linear map from  $\mathbb{R}^2$  onto itself

$$F_{q,\mathbf{v}}(\bar{\mathbf{x}}) = \mathbf{v} + A(q)\bar{\mathbf{x}}$$

where

$$A(q) = \begin{pmatrix} \frac{1}{q} & \frac{1}{q^2} \\ 1 & 0 \end{pmatrix}.$$

We notice that if  $\bar{x}(\mathbf{u})$  is a trajectory of the system (F) with  $z = q$ , then

$$\begin{pmatrix} \bar{x}_{n+2}(\mathbf{u}) \\ \bar{x}_{n+1}(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} u_{n+2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{q} & \frac{1}{q^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_{n+1}(\mathbf{u}) \\ \bar{x}_n(\mathbf{u}) \end{pmatrix},$$

namely

$$(\bar{x}_{n+2}(\mathbf{u}), \bar{x}_{n+1}(\mathbf{u}))^T = F_{q,(u_{n+2},0)}(\bar{x}_{n+1}(\mathbf{u}), \bar{x}_n(\mathbf{u}))^T. \quad (38)$$

We now introduce the concept at the base of the symbolic dynamics, which is a particular application from  $\mathbf{u} \in \{0, 1\}^\infty$  to itself that iterates in a natural way.

**Definition 1** The application  $\sigma : \{0, 1\}^\infty \rightarrow \{0, 1\}^\infty$  defined by

$$\sigma(\mathbf{u}) = \sigma(u_0, u_1, u_2, \dots) = (u_1, u_2, \dots) \quad (39)$$

is said unit shift.

Set  $x(\mathbf{u}) := x(\mathbf{u}, \mathbf{0}) = \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k$  (see Definition (13)) and define

$$\begin{aligned} Q_\infty &:= \{(x(\mathbf{u}), x(\sigma(\mathbf{u}))) \mid \mathbf{u} \in \{0, 1\}^\infty\} \\ &= \left\{ \left( \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k, \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_{k+1} \right) \mid \mathbf{u} \in \{0, 1\}^\infty \right\}. \end{aligned}$$

**Proposition 4** For every  $q > \varphi$

$$\bigcup_{u \in \{0,1\}} F_{q,(u,0)}(\mathcal{Q}_\infty) = \mathcal{Q}_\infty.$$

*Proof* Let  $\mathbf{u} = (u_0, u_1, \dots) \in \{0, 1\}^\infty$ . One has

$$F_{q,(u_0,0)}(x(\sigma(\mathbf{u})), x(\sigma^2(\mathbf{u}))) = (x(\mathbf{u}), x(\sigma(\mathbf{u})))$$

and this implies  $\mathcal{Q}_\infty \subseteq \bigcup_{u \in \{0,1\}} F_{q,(u,0)}(\mathcal{Q}_\infty)$ . Now let  $u \in \{0, 1\}$  and  $\mathbf{d} \in \{0, 1\}^\infty$ . Define  $\mathbf{u} = (u, \mathbf{d}) = (u, d_0, d_1, \dots)$  and note that  $\sigma(\mathbf{u}) = \mathbf{d}$ . One has

$$F_{q,(u,0)}(x(\mathbf{d}), x(\sigma(\mathbf{d}))) = (x(\mathbf{u}), x(\mathbf{d})) = (x(\mathbf{u}), x(\sigma(\mathbf{u})))$$

and this implies the inclusion  $\bigcup_{u \in \{0,1\}} F_{q,(u,0)}(\mathcal{Q}_\infty) \subseteq \mathcal{Q}_\infty$ .  $\square$

Note that in general  $F_{q,\mathbf{v}}$  is not a contractive map. However, the spectral radius of  $A(q)$ , say  $\rho(q)$ , satisfies

$$\rho(q) = \frac{\varphi}{q} < 1 \quad \text{for every } q > \varphi$$

Then

$$\lim_{k \rightarrow \infty} A^k(q) = 0.$$

In particular, there exists  $k(q)$  such that for every  $k \geq k(q)$

$$\|A^k(q)\| := \max_{\mathbf{x} \neq (0,0)} \frac{\|A^k(q)\mathbf{x}\|}{\|\mathbf{x}\|} < 1.$$

*Example 5* Let  $k = 2$ . One has

$$\|A^2(q)\|_2 = \frac{q^4 + 5q^2 + 1}{q^6},$$

see Sect. 4.3 for a detailed computation of  $\|A^k(q)\|$ . Therefore,  $\|A^2(q)\| < 1$  if and only if

$$q^6 - q^4 - 5q^2 - 1 > 0,$$

namely  $k(q) = 2$  for every  $q > \bar{q} \simeq 1.69299$  where  $\bar{q}$  is the unique positive solution of equation  $q^6 - q^4 - 5q^2 - 1 = 0$ .

Now, for every binary sequence of length  $k$ , say  $\mathbf{u}_k$ , define the vector

$$\mathbf{v}(\mathbf{u}_k) := \sum_{h=0}^{k-1} A^h(q) \begin{pmatrix} u_{k+1-h} \\ 0 \end{pmatrix}.$$

and for every  $k$ , the vector function:

$$G_{q,\mathbf{u}_k}(\mathbf{x}) = \mathbf{v}(\mathbf{u}_k) + A^k(q)\mathbf{x} = \sum_{h=0}^{k-1} A^h(q) \begin{pmatrix} u_{k+1-h} \\ 0 \end{pmatrix} + A^k(q)\mathbf{x}.$$

One has that for  $k = 1$

$$G_{q,\mathbf{u}_1} = F_{q,(u_2,0)} \quad (40)$$

and, more generally,

$$G_{q,\mathbf{u}_k} = F_{q,(u_{k+1},0)} \circ F_{q,(u_k,0)} \circ \cdots \circ F_{q,(u_2,0)}. \quad (41)$$

**Remark 8** If  $\mathbf{u}_k = (u_{n+2}, \dots, u_{n+1+k})$  then

$$(x_{n+1+k}, x_{n+k})^T = G_{q,\mathbf{u}_k}(x_{n+1}, x_n)^T.$$

**Theorem 5** For  $k \geq k(q)$  and for every  $\mathbf{u}_k \in \{0, 1\}^k$ , the map  $G_{q,\mathbf{u}_k}$  is a contraction and

$$\bigcup_{\mathbf{u}_k \in \{0,1\}^k} G_{q,\mathbf{u}_k}(\mathcal{Q}_\infty) = \mathcal{Q}_\infty. \quad (42)$$

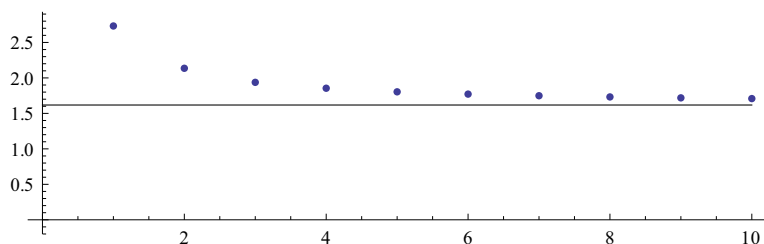
Moreover,  $\mathcal{Q}_\infty(q)$  is the attractor of a two-dimensional linear Iterated Function System (IFS)

$$\mathcal{G}_{q,k} := \{G_{q,\mathbf{u}_k} \mid \mathbf{u}_k \in \{0, 1\}^k\};$$

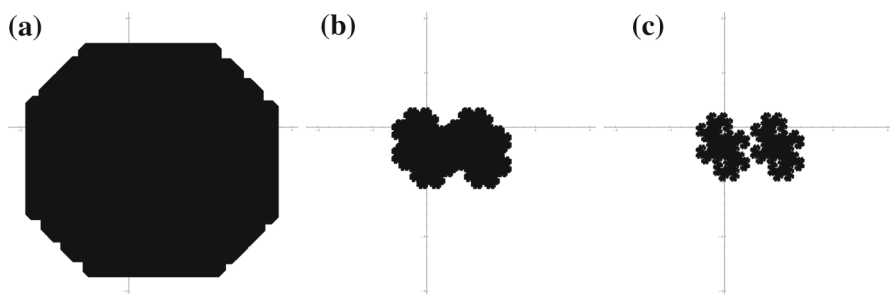
namely for every compact set  $X \subset \mathbb{R}^2$ , one has

$$\lim_{n \rightarrow \infty} \mathcal{G}_{q,k}^n(X) = \mathcal{Q}_\infty(q).$$

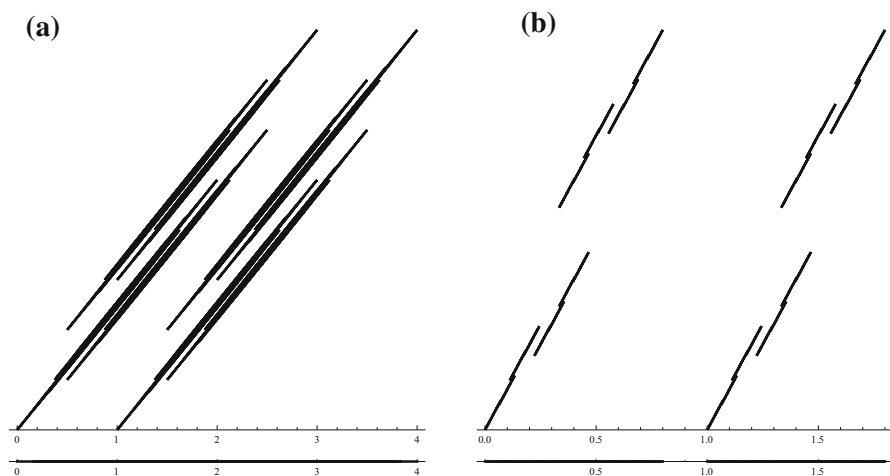
*Proof* By the definition of  $k(q)$ , for each  $\mathbf{u}_k \in \{0, 1\}^k$ ,  $G_{q,\mathbf{u}_k}$  is a contractive map. The equality (42) follows by Proposition 4 and by (41). The second part of the statement follows by the fact that in general the unique invariant compact set of an IFS is also an attractor, see for instance [11].  $\square$



**Fig. 2**  $q(p)$  for  $p = 1, \dots, 10$ . Note that  $q(p)$  tends to  $\varphi$  as  $p \rightarrow \infty$ . Indeed it suffices to recall  $f_p \sim \varphi^p$  to have  $\lim_{p \rightarrow \infty} q(p)/\varphi = 1$



**Fig. 3** Approximations of  $R_\infty(z)$  with  $z = (q(p) + h)e^{i\pi/4}$  and  $h = 0, 0.4, 0.5$ . Note that, by Theorem 4, if  $h = 0$  then  $R_\infty(z)$  is indeed an octagon. See Sect. 4 and, in particular, Remark 9 for a description of the approximation techniques. **a**  $h = 0$ , **b**  $h = 0.4$ , **c**  $h = 0.5$



**Fig. 4** An approximation of  $Q_\infty(q)$ , with  $q = 2, 3$ , and of its projection on  $x$ -axis  $R_\infty(q)$ . It is obtained by 4 iterations of the IFS  $\mathcal{G}_{q,2}$  with initial datum  $[0, S(q)] \times [0, S(q)]$ . **a**  $q = 2$ , **b**  $q = 3$

**Remark 9** [Some remarks on the approximation of  $R_\infty$  in the complex case.] Theorem 5 gives an operative way to approximate  $Q_\infty(q)$  and, consequently,  $R_\infty(q)$ , see Figs. 3, 4.



Above reasonings apply when considering as a base a complex number  $z = qe^{i\omega}$ , so that  $Q_\infty(z) \subset \mathbb{C} \times \mathbb{C}$ . Note that

$$Q_\infty(z) \subset H(z) := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid \max\{|\Re(z_h)|, |\Im(z_h)|\} \leq S(|z|), h = 1, 2\}$$

and  $\lim_{n \rightarrow \infty} \mathcal{G}_{z,k}^n(H(z)) = Q_\infty(z)$ . Then one may approximate  $Q_\infty(z)$  by iteratively applying  $\mathcal{G}_{z,k}$  to  $H(z)$ . To this end, it is possible to employ the isometry between  $\mathbb{C}$  and  $\mathbb{R}^2$  to set the problem on  $\mathbb{R}^4$ . Then the real-valued counterpart of  $H(z)$  is the hypercube

$$\tilde{H}(z) := \{\mathbf{x} \in \mathbb{R}^4 \mid |\mathbf{x}|_{\max} \leq S(|z|)\}$$

while we denote by  $\tilde{\mathcal{G}}_{z,k}$  and by  $\tilde{\mathcal{G}}_{z,\mathbf{u}}$  the real-valued counterparts of  $\mathcal{G}_{z,k}$  and of  $G_{z,\mathbf{u}}$ , respectively, so that

$$\mathcal{G}_{z,k}^n(x) = \bigcup_{\mathbf{u} \in \{0,1\}^{nk}} G_{z,\mathbf{u}}(x).$$

We then may get a bi-dimensional representation of an approximation of  $R_\infty(z)$  by projecting  $\tilde{\mathcal{G}}_{z,k}^n(\tilde{H}(z))$  on  $\mathbb{R}^2$ . However, this yields some complexity issues in numerical simulations. Indeed a brute force attack consists in applying  $\tilde{\mathcal{G}}_{z,k}^n$  to a four-dimensional grid rastering  $\tilde{H}(z)$  and then projecting the result on  $\mathbb{R}^2$ . Thus, the generation of an image with  $5 N \times N$  pixels involves the computation of  $2^{kn} N^4$  points.

To restrain the computational cost, we employed the geometric properties of  $\tilde{\mathcal{G}}_{z,\mathbf{u}}$ . Indeed for every  $\mathbf{u}$ ,  $\tilde{\mathcal{G}}_{z,\mathbf{u}}$  is an affine map, thus it preserves parallelism and convexity. In view of these properties, we considered only the 16 vertices of  $\tilde{H}(z)$ , say  $\mathbf{x}_j$ , with  $j = 1, \dots, 16$ . Our method consists in computing the  $\tilde{\mathcal{G}}_{z,\mathbf{u}}(\mathbf{x}_j)$ 's separately, in projecting the result (namely  $2^{kn}$  points) on  $\mathbb{R}^2$  and finally on computing their convex hull, employing the fact that this projection, say  $\pi$ , preserves convexity, too. In other words, we employed the identity

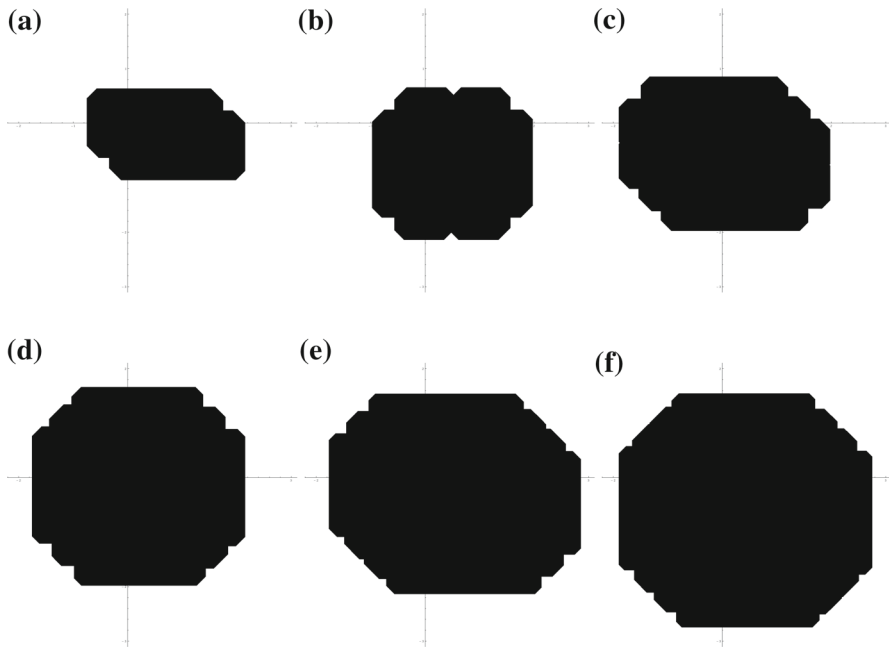
$$\pi(\tilde{\mathcal{G}}_{z,\mathbf{u}}(\tilde{H}(z))) = \pi(\tilde{\mathcal{G}}_{z,\mathbf{u}}(\text{co}(\{\mathbf{x}_j\}))) = \text{co}(\pi(\tilde{\mathcal{G}}_{z,\mathbf{u}}(\mathbf{x}_j))),$$

so that

$$\tilde{\mathcal{G}}_{z,k}^n(\tilde{H}(z)) = \bigcup_{\mathbf{u} \in \{0,1\}^{kn}} \text{co}(\pi(\tilde{\mathcal{G}}_{z,\mathbf{u}}(\mathbf{x}_j))).$$

With this method we need to compute  $2^{kn} \cdot 16$  points and we may possibly store the result on a vectorial format, instead of a raster one. See Figs. 5 and 6 for some examples.

*Remark 10* [Some remarks on the analogies with expansions in non-integer bases] We notice that the  $G_{q,\mathbf{u}_k}$ 's share the same scaling factor,  $A^k(q)$ , and they differ for



**Fig. 5** Various iterations of  $\tilde{G}_{z,k}^n(\tilde{H}(z))$  with  $z = q(8)e^{i\pi/4}$ . **a**  $n = 1$ , **b**  $n = 2$ , **c**  $n = 3$ , **d**  $n = 4$ , **e**  $n = 5$ , **f**  $n = 6$

the translation component  $\mathbf{v}(\mathbf{u}_k)$ . A similar structure also emerges for the one-step recursion case, generating power series with coefficients in  $\{0, 1\}$ . Indeed

$$\tilde{x}_n = \sum_{k=0}^n \frac{u_{n-k}}{q^k} \Leftrightarrow \begin{cases} \tilde{x}_0 = u_0 \\ \tilde{x}_{n+1} = u_{n+1} + \frac{\tilde{x}_n}{q} \end{cases} \quad (43)$$

and setting

$$\tilde{R}_\infty(q) := \left\{ \sum_{k=0}^{\infty} \frac{u_k}{q^k} \mid u_k \in \{0, 1\} \right\}$$

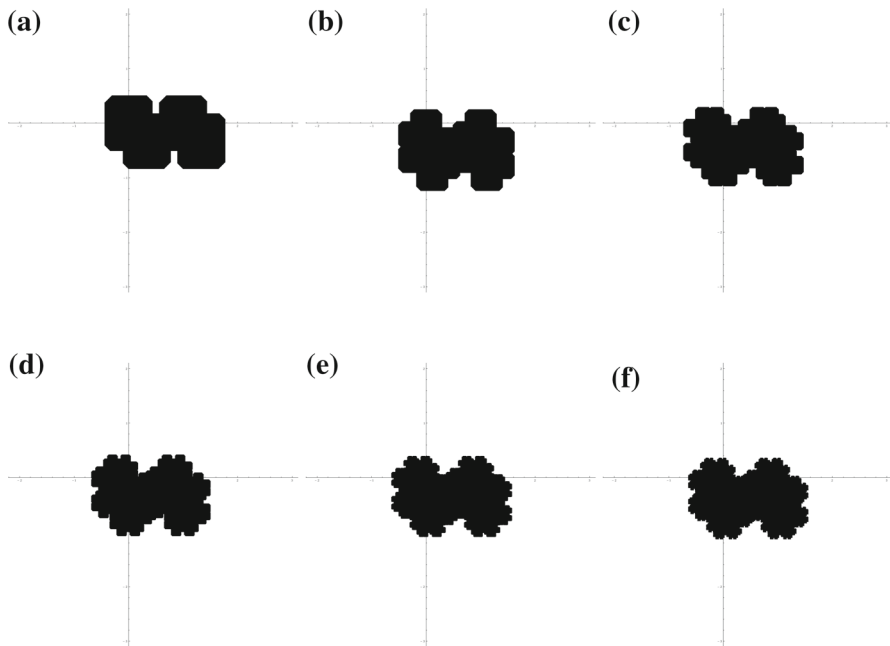
one has that

$$\tilde{R}_\infty(q) = \bigcup_{u \in \{0,1\}} \tilde{G}_{q,c}(\tilde{R}_\infty)$$

where

$$\tilde{G}_{q,u}(\tilde{x}) = u + \frac{\tilde{x}}{q}.$$

The differences and analogies between the two systems can be summarized as follows:



**Fig. 6** Various iterations of  $\tilde{G}_{z,2}^n(\tilde{H}(z))$  with  $z = (q(8) + 0.3)e^{i\pi/4}$ . Notice the similarity with the twin-dragon curve, generated by expansions in complex base with argument again  $\pi/4$ . **a**  $n = 1$ , **b**  $n = 2$ , **c**  $n = 3$ , **d**  $n = 4$ , **e**  $n = 5$ , **f**  $n = 6$

1. both systems are related to power series;
2.  $\tilde{R}_\infty(q)$  can be generated by a one-step recursive algorithm and it is the attractor of a one-dimensional IFS, the radius of convergence is 1. The buffer needed (i.e., the number of digits the IFS depends on) is constantly equal to 1;
3.  $R_\infty(q)$  can be generated by a two-step recursive algorithm and it is the attractor of a two-dimensional IFS, the radius of convergence is  $\varphi$ . The buffer needed,  $k(q)$ , depends on  $q$  and it goes to infinity as  $q$  tends to  $\varphi$  from above.

### 4.3 A sufficient contractivity condition

In what follows, we provide an upper estimate for  $k(q)$ .

**Proposition 5** (An upper estimate for the Fibonacci sequence) *For every  $n \in \mathbb{N}$*

$$f_{n+1} \leq \varphi^n.$$

*Proof* By induction on  $n$ . First, as base cases, we will consider the cases when  $n = 1$  and  $n = 2$ . Note that  $1 < \varphi < 2$ . By adding 1 to each term in the inequality, we obtain  $2 < \varphi + 1 < 3$ . The two inequalities together yield

$$1 < \varphi < 2 < \varphi + 1 < 3.$$

Using the relation  $\varphi + 1 = \varphi^2$  and the first few Fibonacci numbers, we can rewrite this as

$$f_2 < \varphi < f_3 < \varphi^2 < f_4$$

which shows that the statement is true for  $n = 1$  and  $n = 2$ . Now, as the induction hypothesis, suppose that  $f_{i+1} < \varphi_i < f_{i+2}$  for all  $i$  such that  $0 \leq i \leq k + 1$ .

$$f_{k+2} < \varphi^{k+1} < f_{k+3}$$

and

$$f_{k+1} < \varphi^k < f_{k+2}.$$

Adding each term of the two inequalities, we obtain

$$f_{k+2} + f_{k+1} < \varphi^{k+1} + \varphi^k < f_{k+3} + f_{k+2}.$$

Using the relation  $\varphi^{k+1} + \varphi^k = \varphi^{k+2}$  and the first few Fibonacci numbers, we can rewrite this inequality as

$$f_{k+3} < \varphi^{k+2} < f_{k+4}$$

which shows that the inequality holds for  $n = k + 2$ .  $\square$

**Lemma 4** (Explicit computation of  $A^k(q)$ ) *For every  $q > \varphi$  and for every  $k \in \mathbb{N}$*

$$A^k(q) = \frac{1}{q^{k+1}} \begin{pmatrix} f_{k+1}q & f_k \\ f_k q^2 & f_{k-1}q \end{pmatrix}. \quad (44)$$

*Proof* By induction on  $k$ . Base step,  $k = 1$ , is trivially satisfied. Assume now (44) as inductive hypothesis. For  $k + 1$  we have

$$A^{k+1}(q) = A^k(q)A(q) = \frac{1}{q^{k+2}} \begin{pmatrix} (f_{k+1} + f_k)q & f_{k+1} \\ (f_k + f_{k-1})q^2 & f_k \end{pmatrix} = \frac{1}{q^{k+2}} \begin{pmatrix} f_{k+2}q & f_{k+1} \\ f_{k+1}q^2 & f_k q \end{pmatrix}.$$

and this concludes the proof.  $\square$

**Proposition 6** *For every  $q > \varphi$*

$$k(q) \leq \frac{\ln \left( \frac{1}{\varphi^2 q^2} (q^4 + 3q^2 + 1) \right)}{2 (\ln q - \ln \varphi)}. \quad (45)$$

*Proof* Fix  $k$  and set

$$B(q) := \begin{pmatrix} f_{k+1}q & f_k \\ f_k q^2 & f_{k-1}q \end{pmatrix}$$

so that, by Lemma 4, one has

$$A^k(q) = \frac{1}{q^{k+1}} B(q).$$

Denote by  $\lambda_{\max}(A)$  the greatest eigenvalue of  $A$  in modulus. One has that the matrix norm consistent with Euclidean norm satisfies the following identity

$$\|A\| := \max_{\mathbf{x} \neq (0,0)} \|A\mathbf{x}\| = \sqrt{\lambda_{\max}(A^T A)}.$$

Then

$$\|A^k(q)\| = \frac{\|B(q)\|}{q^{k+1}} = \sqrt{\lambda_{\max}(B^T(q)B(q))}.$$

The product matrix  $B^T(q)B(q)$  has the form:

$$B^T(q)B(q) = \begin{pmatrix} f_{k+1}^2 q^2 + f_k^2 q^4 & f_k f_{k+1} q + f_k f_{k-1} q^3 \\ f_k f_{k+1} q + f_k f_{k-1} q^3 & f_k^2 + f_{k-1}^2 q^2 \end{pmatrix}.$$

The characteristic polynomial  $p(\lambda)$  associated to  $B^T(q)B(q)$  is hence

$$p(\lambda) = \lambda^2 - \lambda (f_{k+1}^2 q^2 + f_k^2 q^4 + f_k^2 + f_{k-1}^2 q^2) + q^4 (f_{k-1}^2 f_{k+1}^2 + f_k^4 - 2f_{k+1} f_{k-1} f_k^2).$$

The free term of characteristic polynomial is linked to algebraic identities involving the Fibonacci numbers,

$$f_{k-1}^2 f_{k+1}^2 + f_k^4 - 2f_{k+1} f_{k-1} f_k^2 = 1.$$

In fact

$$f_{k-1}^2 f_{k+1}^2 + f_k^4 - 2f_{k+1} f_{k-1} f_k^2 = (f_k^2 - f_{k-1} f_{k+1})^2$$

involving Cassini's identity

$$f_{n-1} f_{n+1} - f_n^2 = (-1)^{n+1},$$

Then, the characteristic polynomial becomes:

$$p(\lambda) = \lambda^2 - \lambda (f_{k+1}^2 q^2 + f_k^2 q^4 + f_k^2 + f_{k-1}^2 q^2) + q^4.$$

Set  $\bar{\lambda}_{\max} = f_k^2 q^4 + (f_{k+1}^2 + f_{k-1}^2)q^2 + f_k^2$  and note that

$$\lambda_{\max}(B^T(q)B(q)) = \frac{1}{2} \left( \bar{\lambda}_{\max} + \sqrt{\lambda_{\max}^2 - 4q^2} \right) \leq \bar{\lambda}_{\max}.$$

Furthermore, by Proposition 5 we have

$$\bar{\lambda}_{\max} \leq \varphi^{2k-2} q^4 + (\varphi^{2k} + \varphi^{2k-4})q^2 + \varphi^{2k-2} = \varphi^{2k-2}(q^4 + 3q^2 + 1)$$

and finally

$$\|A^k(q)\| = \frac{\lambda_{\max}}{q^{2k+2}} \leq \frac{\bar{\lambda}_{\max}}{q^{2k+2}} \leq \frac{\varphi^{2k-2}}{q^{2k+2}}(q^4 + 3q^2 + 1).$$

Consequently, if

$$\frac{\varphi^{2k-2}}{q^{2k+2}}(q^4 + 3q^2 + 1) < 1$$

then  $\|A^k(q)\| < 1$ . To solve above inequality with respect to  $k$  we apply the logarithm, requiring that the final report is less than 0:

$$2k \ln \left( \frac{\varphi}{q} \right) + \ln \left( \frac{1}{\varphi^2 q^2} (q^4 + 3q^2 + 1) \right) < 0.$$

We finally obtain that if

$$k > \frac{\ln \left( \frac{1}{\varphi^2 q^2} (q^4 + 3q^2 + 1) \right)}{2 (\ln q - \ln \varphi)}$$

then  $\|A^k(q)\| < 1$  and hence the claim.  $\square$

It is well known that  $f_k$  is the closest integer to  $\frac{\varphi^k}{\sqrt{5}}$ . Therefore, it can be found by rounding in terms of the nearest integer function:  $f_k = \left\lfloor \frac{\varphi^k}{\sqrt{5}} \right\rfloor$ ,  $k \geq 0$ . This gives a very sharp inequality. In fact, if  $k$  is an even number, then  $f_k = \left\lfloor \frac{\varphi^k}{\sqrt{5}} \right\rfloor < \frac{\varphi^k}{\sqrt{5}}$ , i.e.,  $f_{2k} = \left\lfloor \frac{\varphi^{2k}}{\sqrt{5}} \right\rfloor < \frac{\varphi^{2k}}{\sqrt{5}}$ . We notice that  $\frac{\varphi^k}{\sqrt{5}} < \varphi^{k-1}$ . By the same procedure applied previously, we get

$$\bar{\lambda}_{\max} \leq \frac{q^2}{5} (\varphi^{2k-2} + \varphi^{2k+2}) + \frac{\varphi^{2k}}{5} (q^4 + 1).$$

We have

$$\begin{aligned} \frac{\bar{\lambda}_{\max}}{q^{2k+2}} &\leq \left(\frac{\varphi}{q}\right)^{2k} \frac{1}{5q^2} \left(\frac{q^2}{\varphi^2} + q^2\varphi^2 + q^4 + 1\right) \leq 1 \\ &\Leftrightarrow 2k \ln\left(\frac{\varphi}{q}\right) + \ln\left(\frac{1}{5\varphi^2} + \frac{\varphi^2}{5} + \frac{q^2}{5} + \frac{1}{5q^2}\right) \leq 0 \end{aligned}$$

whence

$$k \geq \frac{\ln\left(\frac{1}{5\varphi^2} + \frac{\varphi^2}{5} + \frac{q^2}{5} + \frac{1}{5q^2}\right)}{2(\ln q - \ln \varphi)} \quad (q > \varphi) \quad (46)$$

for  $k$  even.

*Remark 11* Now we want to compare the values of  $k(q)$ , and suppose that  $k(q)$  of (46) is greater than (45).

$$\frac{\ln\left(\frac{1}{5\varphi^2} + \frac{\varphi^2}{5} + \frac{q^2}{5} + \frac{1}{5q^2}\right)}{2(\ln q - \ln \varphi)} > \frac{\ln\left(1 + \frac{1}{\varphi^4} + \frac{q^2}{\varphi^2} + \frac{1}{q^2\varphi^2}\right)}{2(\ln q - \ln \varphi)},$$

i.e.,

$$q^4(\varphi^4 - 5\varphi^2) + q^2(\varphi^2 + \varphi^6 - 5\varphi^4 - 5) - 5\varphi^2 + \varphi^4 > 0$$

which does not admit solution. Then

$$\frac{\ln\left(\frac{1}{5\varphi^2} + \frac{\varphi^2}{5} + \frac{q^2}{5} + \frac{1}{5q^2}\right)}{2(\ln q - \ln \varphi)} < \frac{\ln\left(1 + \frac{1}{\varphi^4} + \frac{q^2}{\varphi^2} + \frac{1}{q^2\varphi^2}\right)}{2(\ln q - \ln \varphi)}.$$

## 5 Conclusions

We studied the workspace of a hyper-redundant manipulator, modeling a snake-like robot with links decaying as a scaled Fibonacci sequence. We give a formal proof of the results, highlighted by numerical simulations based on a fractal geometry approach. The main novelty of the paper consists in the exploitation of self-similar structure (induced by the dependence on the Fibonacci sequence) for a combinatoric study of the reachable workspace. We finally notice that, by the arbitrariness of the number of links, the asymptotic properties of the model (e.g., the possibility of setting an arbitrary global length for the manipulator) extend by approximation to the case with a finite number of links with arbitrary small tolerance.

Possible developments of the present work include the search for solutions for inverse kinematic problems in a fashion like [23], and for obstacle avoidance algorithms similar to those presented in the Chirikjian and Burdick's seminal report [5].

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