A fractional wavelet Galerkin method for the fractional diffusion problem

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Abstract

The aim of this paper is to solve some fractional differential problems having time fractional derivative by means of a wavelet Galerkin method that uses the fractional scaling functions introduced in [13] as approximating functions. These refinable functions, which are a generalization of the fractional B-splines [16], have many interesting approximation properties. In particular, their fractional derivatives have a closed form that involves just the fractional difference operator. This allows us to construct accurate and efficient numerical methods to solve fractional differential problems. Some numerical tests on a fractional diffusion problem will be given.

Keywords: Fractional diffusion problem, Wavelet Galerkin method, Fractional refinable function

1. Introduction

Fractional calculus arises in several fields, from physics to continuum mechanics, from signal processing to electromagnetism. In particular, fractional differential equations are recently used to model a variety of real world phenomena, such as wave propagation in porous materials, diffusive phenomena in biological tissue, viscoelasticity. For a survey on the subject see, for instance, [8, 9, 11, 15] and references therein.

The goal of this paper is to numerically solve diffusion problems having fractional time derivative using the fractional refinable functions introduced in [13] as approximating functions. These functions, which are a generalization of the fractional B-splines [16], have many interesting approximation properties. In particular, their fractional derivatives have a closed form that involves just the fractional difference operator so that it can be easily evaluated.

This allows us to construct an accurate and efficient numerical method, the *fractional wavelet Galerkin method*, which uses a wavelet Galerkin method in space and a collocation method in time.

The paper is organized as follows. In Section 2, a space-time fractional diffusion problem is presented and the definition of fractional derivative is given. The numerical method we propose to solve this fractional diffusion problem is introduced in Section 3. In Section 4, the class of fractional refinable functions we are interested in is described and their main properties are recalled. We will exploit the properties of the refinable functions in the class to efficiently evaluate the coefficients of the linear, algebraic system arising from the fractional wavelet Galerkin method. Finally, in Section 5 the numerical results obtained in some tests will be displayed.

2. A fractional diffusion problem

We consider the *space-fractional time differential diffusion problem* [3, 10]

$$\begin{cases} D_t^\gamma\, u(t,x) - \dfrac{\partial^2}{\partial x^2}\, u(t,x) = f(t,x)\,, & t \in [0,T]\,, \quad x \in [0,1]\,, \\ u(0,x) = 0\,, & \qquad\qquad\quad x \in [0,1]\,, \\ u(t,0) = u(t,1) = 0\,, & t \in [0,T]\,, \end{cases} \tag{1}$$

where $D_t^\gamma u$, $0 < \gamma < 1$, denotes the *partial fractional derivative* with respect to the time t. Here, we follow the Riemann-Liouville definition, i.e.

$$D_t^\gamma\, u(t,x) = \frac{1}{\Gamma(1-\gamma)}\,\frac{\partial}{\partial t}\int_0^t \frac{u(\tau,x)}{(t-\tau)^\gamma}\,d\tau\,, \tag{2}$$

where Γ is the Euler's gamma function

$$\Gamma(\gamma+1) = \int_0^\infty s^\gamma\, \mathrm{e}^{-s}\, ds\,. \tag{3}$$

We notice that due to the homogeneous initial condition, for the function $u(t,x)$, solution of the differential problem (1), the Riemann-Liouville definition (2) coincides with the Caputo definition [14]. One of the advantage of the Riemann-Liouville definition is in that the usual differentiation operator in the Fourier domain can be easily extended to the fractional case, i.e.

$$\mathcal{F}\big(D_t^\gamma\, v(t)\big) = (i\omega)^\gamma \mathcal{F}(v(t))\,, \tag{4}$$

where $\mathcal{F}(v)$ denotes the Fourier transform of the function v. Moreover, the Riemann-Liouville definition coincides with the Grunwald-Letnikov definition

$$D_t^\gamma\, u(t,x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{\frac{t}{h}}(-1)^k\binom{\gamma}{k}\,u(t-kh,x)\,, \tag{5}$$

where

$$\binom{\gamma}{k} := \frac{\Gamma(\gamma+1)}{k!\,\Gamma(\gamma-k+1)}\,, \qquad k \in \mathbb{Z}_0^+\,, \tag{6}$$

are the generalized binomial coefficients ($\mathbb{Z}_0^+$ denotes the set $\mathbb{Z}^+ \cup 0$). They reduce to the usual binomial coefficients when γ is an integer. We note that the Grunwald-Letnikov definition (5) is easier to use when addressing numerical solution of fractional differential problems.
In the following section we will introduce the wavelet Galerkin method to numerically solve the fractional differential problem (1).

3. The wavelet Galerkin method

Let us introduce the sequence of approximating spaces

$$V_j = \text{span}\,\{\varphi_{jk} = \varphi(2^j\ \cdot - k), k \in \mathbb{Z}\}\,, \qquad j \in \mathbb{Z}\,, \tag{7}$$

where φ is a refinable function, i.e. a function defined through a *refinement mask* $\mathbf{a} = \{a_k \in \mathbb{R}, k \in \mathbb{Z}\}$ and a *refinement equation*

$$\varphi = \sum_{k\in\mathbb{Z}} a_k\,\varphi(2\ \cdot - k)\,. \tag{8}$$

Suitable conditions on the mask coefficients $\{a_k\}$ ensure the existence of a unique function φ solution of (8), belonging to $L^2(\mathbb{R})$ and such that the space sequence $\{V_j\}$ forms a multiresolution analysis (see [12] for details).

For any j held fix, the wavelet Galerkin method looks for an approximating function

$$u_j(t,x) = \sum_{k\in\mathbb{Z}} c_k(t)\,\varphi_{jk}(x) \in V_j \tag{9}$$

that solves the variational problem

$$\begin{cases} (D_t^\gamma u_j\,, \varphi_{jk}) - \left(\frac{\partial}{\partial x^2}\,u_j, \varphi_{jk}\right) = (f, \varphi_{jk})\,, & k \in \mathbb{Z}\,, \\ u_j(0,x) = 0\,, & x \in [0,1]\,, \\ u_j(t,0) = 0\,, \quad u_j(t,1) = 0\,, & t \in [0,T]\,, \end{cases} \tag{10}$$

where $(f,g) = \int_0^1 f\,g$. Now, writing (10) in a weak form and using (9) we get the system of fractional ordinary differential equations

$$\begin{cases} M\,D_t^\gamma\,C(t) = L\,C(t) + F(t)\,, \quad t \in [0,T]\,, \\ C(0) = 0\,, \end{cases} \tag{11}$$

where $C(t) = (c_k(t))_{k\in\mathbb{Z}}$ is the unknown vector. The connecting coefficients, i.e. the entries of the mass matrix $M = (m_{ki})_{k,i\in\mathbb{Z}}$, of the stiffness matrix $L = (\ell_{ki})_{k,i\in\mathbb{Z}}$ and of the load vector $F(t) = (f_k(t))_{k\in\mathbb{Z}}$, are given by

$$m_{ki} = \int_0^1 \varphi_{jk}\,\varphi_{ji}\,, \qquad \ell_{ki} = \int_0^1 \varphi'_{jk}\,\varphi'_{ji}\,, \qquad f_k(t) = \int_0^1 f(t,\cdot)\,\varphi_{jk}\,.$$

Due to the refinability of φ, the entries of M and L can be evaluated by solving an eigenvalue-eigenvector problem [2]. The entries of F can be evaluated by quadrature formulas especially designed for wavelet methods [1, 4].

To solve the fractional differential system (11) we use a collocation method on dyadic nodes. For an integer value of T, let $t_p = p/2^s$, $p = 0,\ldots,T\,2^s$, where s is a given integer greater than 1, be a set of dyadic nodes in the dyadic interval $[0,T]$. Now, assuming

$$c_k(t) = \sum_{r\in\mathbb{Z}} \lambda_{rk}\,\chi_{sr}(t)\,, \qquad k\in\mathbb{Z}\,, \tag{12}$$

where $\chi_{sr}(t) = \chi(2^s\,t - r)$ is χ a refinable function, and collocating (11) on the nodes t_p, we get the linear system

$$\begin{cases} (M\otimes A - L\otimes B)\,\Lambda = F \\ X^T\Lambda = 0 \end{cases} \tag{13}$$

where $\Lambda = (\lambda_{rk})_{r,k,\in\mathbb{Z}}$,

$$A = (a_{pr})_{0\le p\le 2^sT, r\in\mathbb{Z}}\,, \qquad a_{pr} = D_t^\gamma\,\chi_{sr}(t_p)\,, \tag{14}$$

$$B = (b_{pr})_{0\le p\le 2^sT, r\in\mathbb{Z}}\,, \qquad b_{pr} = \chi_{sr}(t_p)\,, \tag{15}$$

$F = (f_k(t_p))_{k,\in\mathbb{Z},0\le p\le 2^sT}$ and $X = (\chi_{sr}(0))_{r\in\mathbb{Z}}$. Since the refinable functions are compactly supported or have fast dacay, in practical applications the linear system (13) has finite dimension and the unknown vector Λ can be recovered by solving (13) in the least square sense.
We notice that the entries of B and X, which involve just the values of χ_{sr} on the dyadic nodes t_p, can be evaluated by usual refinement techniques [12]. On the other hand, we must pay a special attention to the evaluation of the entries of A since they involve the values of the fractional derivative $D_t^\alpha\chi_{rs}$. In the following section we will exploit the properties of a particular class of refinable functions to accurately evaluate the matrix A.

4. A class of fractional refinable functions

In [13] a class of *fractional refinable functions* was introduced. The refinable functions belonging to this class are identified through the family of refinement

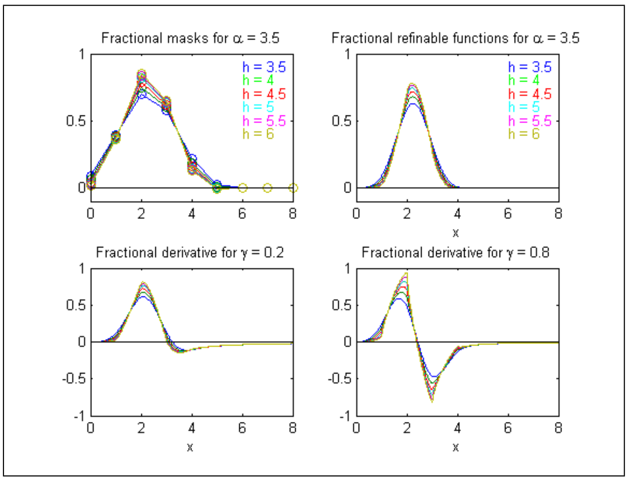

Figure 1: Top panels: the fractional mask and the fractional refinable function for $\alpha = 3.5$. Bottom panels: the fractional derivative when $\gamma = 0.2, 0.8$.

masks $\mathbf{a}^{(\alpha,h)} = \{a_k^{(\alpha,h)}, k \in \mathbb{Z}_+\}$ that depend on the real parameters $\alpha \geq 2$ and $h \geq \alpha$. The mask coefficients have an explicit espression given by

$$a_k^{(\alpha,h)} = \frac{1}{2^h}\left[\binom{\alpha+1}{k} + 4(2^{h-\alpha}-1)\binom{\alpha-1}{k-1}\right], \qquad k \in \mathbb{Z}_0^+ . \tag{16}$$

(We assume $\binom{\alpha}{k} = 0$ when $k < 0$.)
For each admissible values of α and h, the corresponding *refinable function* turns out to be the unique solution of the *refinement equation*

$$\varphi^{(\alpha,h)} = \sum_{k\in\mathbb{Z}_0^+} a_k^{(\alpha,h)}\, \varphi^{(\alpha,h)}(2\cdot -k)\,. \tag{17}$$

We note that setting $h = \alpha$ in (16) gives

$$a_k^{(\alpha)} := a_k^{(\alpha,\alpha)} = \frac{1}{2^\alpha}\binom{\alpha+1}{k}, \qquad k \in \mathbb{Z}_0^+ , \tag{18}$$

which are the mask coefficients of the *fractional B-spline* so that $\varphi^{(\alpha,\alpha)}$ coincides with the fractional B-spline B^α introduced in [16]. B^α is a piecewise polynomial of real degree α and reduces to the classical integer degree polynomial B-spline for integer values of α (see [16] for details). Thus, the class of functions $\varphi^{(\alpha,h)}$ can be seen as a generalization of the fractional B-splines and this is the reason why they are called *fractional refinable functions.*

Even if $\varphi^{(\alpha,h)}$ has compact support only when α has an integer value, for any admissible value of α and h, $\varphi^{(\alpha,h)}$ belongs to $L^2(\mathbb{R})$ and decays to the infinity rather rapidly. Moreover, $\varphi^{(\alpha,h)}$ gives rise to a *multiresolution analysis* of $L^2(\mathbb{R})$. As a consequence, the approximating spaces

$$V_j^{(\alpha,h)} = \text{span}\,\{\varphi_{jk}^{(\alpha,h)} = \varphi^{(\alpha,h)}(2^j\ \cdot - k), k \in \mathbb{Z}\}\,, \qquad j \in \mathbb{Z}\,, \tag{19}$$

are suitable to be used in the wavelet Galerkin method.
As for the order of exactness, we recall that $\varphi^{(\alpha,h)}$ has order of polynomial exactness $d = \lceil \alpha \rceil - 1$, i.e. there exist sequences $\{q_k^l\}$ such that

$$x^l = \sum_{k\in\mathbb{Z}} q_k^l\, \varphi^{(\alpha,h)}(x-k)\,, \quad l = 0, \ldots, \lceil \alpha \rceil - 2\,, \tag{20}$$

(see [13] for details and further properties).

A remarkable property of the fractional refinable functions, especially useful in the numerical solution of fractional differential problems, is the following differentiation rule [13]:

$$D_x^\gamma\, \varphi^{(\alpha,h)} = \Delta^\gamma\, \varphi^{(\alpha-\gamma,h)} = \sum_{k\in\mathbb{Z}^+} (-1)^k \binom{\alpha}{k} \varphi^{(\alpha-\gamma,h)}(\cdot - k)\,, \qquad \gamma > 0\,. \tag{21}$$

This makes fractional differentiation very easy since fractional derivatives can be evaluated just by a finite difference formula. As a consequence, the computation of the coefficients a_{pr} in (14) is very efficient, as we will show in the following section.
In Figure 1 (upper panels) the mask coefficients $a_k^{(3.5,h)}$, $k = 0, 1, \ldots, 8$, and the corresponding refinable function $\varphi^{(3.5,h)}$ are displayed for different values of h. In Figure 1 (lower panels) the fractional derivatives $D_x^\gamma\, \varphi^{(3.5,h)}$ are displayed for $\gamma = 0.2$ and $\gamma = 0.8$. The pictures show that $\varphi^{(3.5,h)}$ decays rapidly to infinity and is almost positive.

5. A case study

In this section we will apply the fractional wavelet Galerkin method using the fractional refinable functions $\varphi^{(\alpha,h)}$ to solve the fractional diffusion problem (1) when $f(t,x) = \frac{2}{\Gamma(3-\gamma)}\, t^{2-\alpha}\, \sin(2\pi x) + 4\pi^2\, t^2\, \sin(2\pi x)$. In this case the exact solution is $u(t,x) = t^2\, \sin(2\pi x)$ (cf. [3]).
To numerically solve the problem we approximate $u(t,x)$ by using the fractional refinable functions described in the previous section. More precisely, for some given set of admissible values $H = (\alpha_1, h_1, \alpha_2, h_2)$, we assume

$$u_j^{(H)}(t,x) = \sum_{k\in\mathbb{Z}} c_k^{(\alpha_1,h_1)}(t)\, \varphi_{jk}^{(\alpha_2,h_2)}(x) \in V_j^{(\alpha_2,h_2)} \tag{22}$$

as approximating function with

$$c_k^{(\alpha_1,h_1)}(t) = \sum_{r\in\mathbb{Z}} \lambda_{rk}\, \varphi_{sr}^{(\alpha_1,h_1)}(t)\,, \qquad k \in \mathbb{Z}\,. \tag{23}$$

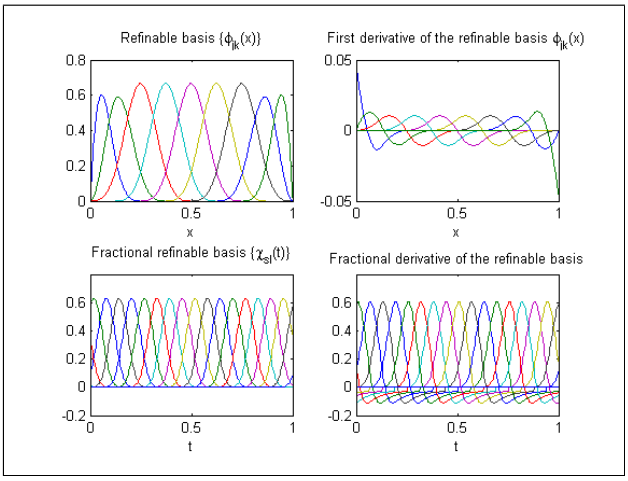

Figure 2: The refinable basis $\{\varphi_{3k}^{(3,3)}\}$ in the interval [0,1] and its first derivative (upper panels) and the refinable basis $\{\varphi_{4k}^{(3.5,3.5)}\}$ and its fractional derivative for $\gamma = 0.2$ (lower panels).

Since just the ordinary first derivative of $\varphi_{jk}^{(\alpha_2,h_2)}$ is involved in the numerical method, we can assume α_2 integer. In this case $\varphi^{(\alpha_2,h_2)}$ reduces to the compactly supported refinable function belonging to the general class introduced in [5] so that the connecting coefficients can be evaluated as in [7]. Moreover, the existence and uniqueness of the solution of the Galerkin problem (10) and the convergence of the wavelet Galekin method follows from some results in [7].
On the other hand, the fractional refinable function $\varphi^{(\alpha_1,h_1)}$ has a fast decay to the infinity so that it can be assumed compactly supported and the series in (23) are well approximated by only few terms.

In the following tests we set $\alpha_1 = 3.5$, $h_1 \geq 3.5$, $\alpha_2 = 3$, $h_2 \geq 3$. The refinable basis for $\alpha_2 = 3$ and $h_2 = 3$, i.e. the cubic B-spline basis, on the interval [0,1] at resolution level $j = 3$ is displayed in Figures 2-3 (upper panels) along its first derivative (we refer to [6] for details on the construction of refinable bases on bounded intervals). The refinable basis for $\alpha_2 = 3.5$ and $h_2 = 3.5$ on the interval [0,1] at resolution level $s = 4$ is displayed in Figures 2-3 (lower panels) along its 0.2 and 0.8 fractional derivatives.

In Figure 4, the numerical solution $u_j^{(H)}$ and the error $u - u_j^{(H)}$ obtained when using the refinable basis in Figures 2-3 are displayed.

6. Conclusions

We exploit the properties of a class of fractional refinable functions to construct a wavelet method to solve a space-fractional time partial diffusion problem. The main advantage in using fractional refinable functions is in that they satisfy an

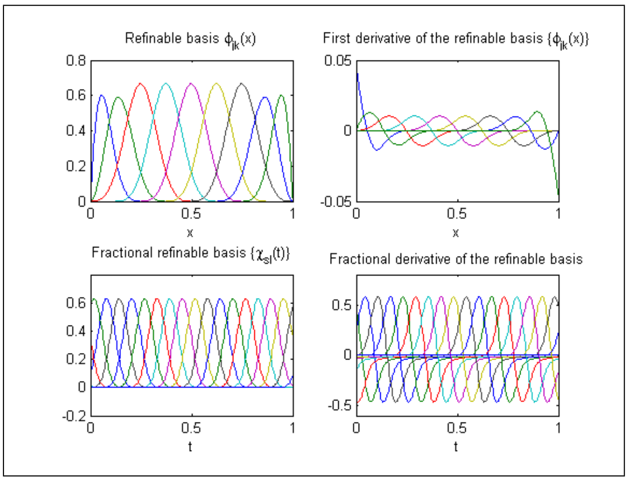

Figure 3: The refinable basis $\{\varphi_{3k}^{(3,3)}\}$ in the interval [0,1] and its first derivative (upper panels) and the refinable basis $\{\varphi_{4k}^{(3.5,3.5)}\}$ and its fractional derivative for $\gamma = 0.8$ (lower panels).

exact and very simple formula for the evaluation of the fractional derivatives so that the resulting fractional wavelet Galerkin method is accurate and efficient, as shown in the numerical tests in Section 4.
In a forthcoming paper we will address the solution of fractional diffusion problems in $\mathbb{R}^d$, $d \leq 3$, by the fractional wavelet Galerkin problem.

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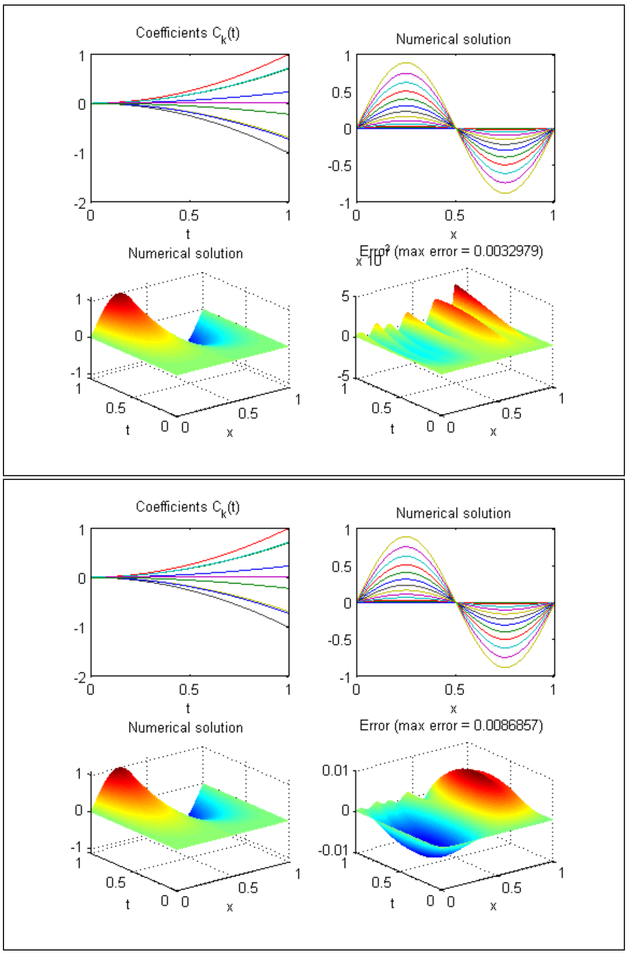

Figure 4: The numerical solution and the relative error for $\gamma = 0.2$ (upper panels) and $\gamma = 0.8$ (lower panels).

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