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# Existence of solutions to a non-variational singular elliptic system with unbounded weights

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In this paper we prove an existence result for the following singular elliptic system

$$\begin{cases} z > 0 \text{ in } \Omega\,, \; z \in W_0^{1,p}(\Omega): & -\Delta_p z = a(x) z^{q-1} u^{\theta}\,, \\ u > 0 \text{ in } \Omega\,, \; u \in W_0^{1,p}(\Omega): & -\Delta_p u = b(x) z^{q} u^{\theta-1}\,, \end{cases}$$

where $\Omega$ is a bounded open set in $\mathbb{R}^{\mathrm{N}}$ ($\mathrm{N} \geq 2$), $-\Delta_p$ is the $p$-laplacian operator, $a(x)$ and $b(x)$ are suitable Lebesgue functions and $q > 0$, $0 < \theta < 1$, $p > 1$ are positive parameters satisfying suitable assumptions.

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## 1 Introduction and main result

In this paper we are concerned with the following nonlinear singular elliptic system

$$\begin{cases} z > 0 \text{ in } \Omega\,, \; z \in W_0^{1,p}(\Omega): & -\Delta_p z = a(x) z^{q-1} u^{\theta}, \\ u > 0 \text{ in } \Omega\,, \; u \in W_0^{1,p}(\Omega): & -\Delta_p u = b(x) z^{q} u^{\theta-1}, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded open set in $\mathbb{R}^N$ ($N \geq 2$), the *weights* $a(x)$ and $b(x)$ satisfy

$$a(x) \in L^{\infty}(\Omega), \quad b(x) \in L^{m}(\Omega) \text{ for some } m > \frac{N}{p}, \quad \begin{cases} 0 \leq a(x), \quad a(x) \not\equiv 0 \\ 0 < \beta \leq b(x) \end{cases} \quad \text{a.e. in } \Omega, \tag{1.2}$$

and the parameters $q, \theta, p$ are chosen so that

$$0 < \theta < 1, \quad 0 < q < p - \theta, \tag{1.3}$$

and

$$\begin{cases} \theta + 1 < p < N, \\ p \neq 1 + \sqrt{q\theta}. \end{cases} \tag{1.4}$$

In (1.3) and (1.4) the upper bound on $q$, the lower bound on $p$ and the last condition are technical assumptions needed in the proof of our main result (see Section 3).

Being $\theta < 1$, at least the second equation in (1.1) has a lower order term that is singular with respect to the solution $u$, in the sense that it blows up when the solution is zero; additionally, when $q < 1$, also the right hand side of the first equation in (1.1) becomes singular with respect to $z$.

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For the reader's convenience, we recall that the $p$-laplacian operator, with $p > 1$, acts on a function $v \in W^{1,p}(\Omega)$ as $-\Delta_p v := -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right)$.

Elliptic boundary value problems with singular lower order terms have been widely studied in the past. For what concerns the case of a single equation, we recall the pioneering papers [7], [14] and the more recents works [3], [10], [12], [15]–[17]. When considering elliptic systems with singular lower order terms, beyond the variational cases studied in [4], [11], we recall the non-variational case with higher order terms that are linear with respect to the solutions studied in [6], as well as [13], [18], where the authors study Lane–Emdem systems with negative exponents.

Going further in details, in [4] and [11] the variational counterpart of system (1.1) has been studied for $p = 2$ and for the general case $p > 1$ respectively; precisely, in [11] the author considers the following system

$$\begin{cases} z > 0 \text{ in } \Omega\,, \ z \in W_0^{1,p}(\Omega) : & -\Delta_p z = q z^{q-1} u^{\theta}, \\ u > 0 \text{ in } \Omega\,, \ u \in W_0^{1,p}(\Omega) : & -\Delta_p u = \theta z^{q} u^{\theta-1}, \end{cases} \tag{1.5}$$

namely system (1.1) with $a(x) \equiv q$, $b(x) \equiv \theta$ and $q, \theta, p$ positive real numbers such that

$$\begin{cases} 0 < \theta < 1, \quad 0 < q < p^* - \theta, \\ 1 < p < \mathrm{N}. \end{cases} \tag{1.6}$$

We underline that, on one hand, assumption (1.6) covers a wider range for the parameters $q, \theta, p$ with respect to (1.3) and (1.4) but, on the other hand, the weights in the lower order terms of (1.1), namely $a(x)$ and $b(x)$, are more general with respect to the ones of (1.5), that are in fact chosen in such a way to give a variational structure to the system. Indeed, the particular choice $a(x) \equiv q$ and $b(x) \equiv \theta$ allows to consider (1.1) as a system of Euler-Lagrange equations associated to the following functional

$$J(v, w) = \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v_+^{\theta} w_+^{q},$$

which is well-defined on $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ but is not differentiable in $v$ (being $\theta < 1$) and also in $w$, if $q < 1$. Thus, in order to obtain the existence of a finite energy solution $(u, z)$ to (1.5), the idea is to introduce a suitable approximation $J_n$ of $J$, depending whether $q \geq 1$ or $0 < q < 1$, allowing the definition of the Gateaux derivative of $J$ along every direction of the subspace $W_0^{1,q}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$; it is then possible to find a solution to (1.5) by passing to the limit in a sequence of critical points of $J_n$.

Differently from the case studied in [11], here we no longer have a variational structure and in order to prove the existence of a solution to (1.1), we can not proceed as in the proof of [11, Theorem 1.1].

The present work means to give a new contribution in the literature on non-variational singular elliptic systems, proving that (1.1) admits a finite energy solution $(u, z) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, as enlightened in the following Theorem.

**Theorem 1.1** *Suppose that* (1.2)*,* (1.3) *and* (1.4) *hold. Then there exists* $(u,\ z) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *positive solution to* (1.1) *in the following sense:*

$$\begin{cases} \displaystyle\int_\Omega |\nabla z|^{p-2}\nabla z \nabla \psi = \int_\Omega a(x) z^{q-1} u^{\theta} \psi \quad \forall \psi \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi = \int_\Omega b(x) z^{q} u^{\theta-1} \varphi \quad \forall \varphi \in W_0^{1,p}(\Omega)\,. \end{cases} \tag{1.7}$$

*Moreover, both u and z belong to* $L^\infty(\Omega)$.

**Remark 1.2** If $q \geq 1$, applying the generalized Hölder inequality and using (1.2)–(1.4), we can easily prove that the right hand side of the first equation in (1.7) is well defined; on the other side, when $0 < q < 1$, to prove that the right hand side of either the first equation or the second one is well defined, we have to proceed by steps, as it will be clarified later on in the proof of Theorem 1.1.

The proof of Theorem 1.1 is based on an application of the Schauder's Fixed Point Theorem, that we recall here for reader's convenience.

**Theorem 1.3** (Schauder's Fixed Point Theorem) *Let X be a Banach space,* $S : X \to X$ *be a continuous map such that* $\overline{S(C)}$ *is compact for every* $C \subset X$ *bounded and* $\mathcal{K}$ *be a convex, closed and bounded subset of X that is invariant for S. Then S has at least a fixed point in* $\mathcal{K}$.


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The lack of variational structure for the system (1.1) prevents also to prove that the solution we find is different from zero as done in [11], where this follows by using the property that such a solution is obtained as limit of a sequence of non-trivial critical point of a certain approximating functionals $J_n$. The idea here is to define the invariant set $\mathcal{K}$ in such a way it is far from zero, so that the solution $(u, z)$ to (1.1) found by applying Theorem 1.3, namely found as a fixed point of some map $S$ defined on $\mathcal{K}$, will not be identically zero.

We underline that the method proposed in this paper could be a first step to address the problem of the existence of solutions to more general singular and non-variational systems of elliptic equations through fixed point arguments. In particular we here have in mind the case in which both the equations are singular with respect to their solutions and have non-constant weights in the lower order terms.

We close this Introduction with a short plan of the paper. In Section 2 we give some preliminary results needed in order to prove our main result. Section 3 is the core of the paper. At first, we treat the trivial case $a(x) \equiv A$ and $b(x) \equiv B$, with $A, B$ positive constants, showing that elementary computations lead to the existence of a solution to (1.1) by taking advantage of the existence of a solution in the case $A = q$ and $B = \theta$ (see [11]); the second part of Section 3 is entirely devoted to the proof of Theorem 1.1 under its general assumptions.

## 2 Notations and preliminaries

For the sake of simplicity we will often use the simplified notation

$$\int_\Omega f := \int_\Omega f(x)\, dx,$$

where no ambiguity on the integration variable is possible. Moreover we denote by $r^*$ the *Sobolev conjugate* of $1 \le r < N$, given by $\dfrac{Nr}{N-r}$, and by $r' = \dfrac{r}{r-1}$ the *Hölder conjugate* of $1 < r < \infty$ (if $r = 1$ we define $r' = \infty$, while if $r = \infty$ we define $r' = 1$).

If $v : \Omega \to \mathbb{R}$ is a measurable function, we define the *positive part* of the function $v$ as

$$v^+ = \max(v, 0),$$

and the truncation function $T_k$ at the level $k \in \mathbb{N}$ as

$$T_k(v) = \max\{-k, \min(k, v)\}.$$

We denote with $\varphi_1^\mu$ *the first eigenfunction of the* $p$*-laplacian operator with weight* $\mu = \mu(x)$, where $0 \le \mu(x) \in L^m(\Omega)$, with $m > \frac{N}{p}$ and $\mu(x) \not\equiv 0$. Namely $0 \le \varphi_1^\mu \in W_0^{1,p}(\Omega)$, $\varphi_1^\mu \not\equiv 0$ and it solves

$$\begin{cases} -\mathrm{div}\left(|\nabla \varphi_1^\mu|^{p-2}\nabla\varphi_1^\mu\right) = \lambda_1^\mu \mu(x)\left(\varphi_1^\mu\right)^{p-1} & \text{in} \quad \Omega, \\ \varphi_1^\mu = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\lambda_1^\mu = \inf\left\{\int_\Omega |\nabla v|^p : \int_\Omega \mu(x)|v|^p = 1\right\}.$$

It can be proven that $\varphi_1^\mu$ is strictly positive in $\Omega$ (see [8, Proposition 3.2]) and that $\varphi_1^\mu \in L^\infty(\Omega)$ (see [9, Lemma 3.5]).

Moreover we denote by $C_0^1(\Omega)$ the set of $C^1$ functions on $\Omega$ with compact support.

The following two results will be used in the proof of Theorem 1.1 and represent a generalization of [1, Theorem 2.2, Theorem 2.4].

**Theorem 2.1** *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded open set and $\gamma \in \mathbb{R}^+$. Assume that $0 \le f \in L^1(\Omega)$ is not identically zero and that there exists a locally positive distributional solution $v \in W_0^{1,p}(\Omega)$ to the following singular problem*

$$\begin{cases} -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) = \dfrac{f(x)}{v^\gamma} & \text{in} \quad \Omega, \\ v > 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

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*namely we assume that there exists* $v \in W_0^{1,p}(\Omega)$ *such that*

$$\forall\, \omega \subset\subset \Omega \;\exists\, c_\omega > 0 : \; v \geq c_\omega \text{ in } \omega, \tag{2.2}$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi = \int_\Omega \frac{f\varphi}{v^\gamma} \quad \forall \varphi \in C_0^1(\Omega). \tag{2.3}$$

*Then* $v$ *satisfies*

$$\frac{f\psi}{v^\gamma} \in L^1(\Omega) \quad \forall \psi \in W_0^{1,p}(\Omega), \tag{2.4}$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \psi = \int_\Omega \frac{f\psi}{v^\gamma} \quad \forall \psi \in W_0^{1,p}(\Omega). \tag{2.5}$$

Proof. Let $\psi \in W_0^{1,p}(\Omega)$ and $\varphi_n \in C_0^1(\Omega)$ be a sequence of smooth functions converging to $\psi$ in $W_0^{1,p}(\Omega)$. We take

$$\varphi := \left[(\varepsilon^{p'} + |\varphi_n - \varphi_k|^{p'})^{\frac{1}{p'}} - \varepsilon\right] \in C_0^1(\Omega),$$

as a test function in (2.3) and we find

$$\begin{aligned}
\int_\Omega \frac{f}{v^\gamma}\left[(\varepsilon^{p'} + |\varphi_n - \varphi_k|^{p'})^{\frac{1}{p'}} - \varepsilon\right] &= \int_\Omega \frac{|\nabla v|^{p-2}\nabla v \nabla(\varphi_n - \varphi_k)|\varphi_n - \varphi_k|^{p'-1}}{\left[(\varepsilon^{p'} + |\varphi_n - \varphi_k|^{p'})^{\frac{1}{p}}\right]} \\
&\leq \left(\int_\Omega |\nabla v|^p\right)^{\frac{1}{p'}} \left(\int_\Omega \frac{|\nabla(\varphi_n - \varphi_k)|^p |\varphi_n - \varphi_k|^{p'}}{\varepsilon^{p'} + |\varphi_n - \varphi_k|^{p'}}\right)^{\frac{1}{p}} \\
&\leq \|v\|_{W_0^{1,p}}^{\frac{p}{p'}} \|\varphi_n - \varphi_k\|_{W_0^{1,p}}.
\end{aligned}$$

Since $f, v \geq 0$, using Fatou's lemma as $\varepsilon \to 0$ we obtain

$$\int_\Omega \frac{f}{v^\gamma}|\varphi_n - \varphi_k| \leq \|v\|_{W_0^{1,p}}^{\frac{p}{p'}} \|\varphi_n - \varphi_k\|_{W_0^{1,p}},$$

deducing that $\frac{f\varphi_n}{v^\gamma}$ is a Cauchy sequence in $L^1(\Omega)$. Since $\varphi_n$ converges almost everywhere to $\psi$, this implies that $\lim_{n\to\infty} \int_\Omega \frac{f\varphi_n}{v^\gamma} = \int_\Omega \frac{f\psi}{v^\gamma}$. In particular (2.4) holds and we can pass to the limit as $n \to \infty$ in

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi_n = \int_\Omega \frac{f\varphi_n}{v^\gamma},$$

obtaining (2.5).

**Theorem 2.2** *Assume* $f_1, f_2 \in L^m(\Omega)$ *be two nonnegative functions such that the corresponding solutions* $v_1, v_2$ *of* (2.1) *are in* $W_0^{1,p}(\Omega)$. *Then* $f_1 \geq f_2$ *implies* $v_1 \geq v_2$.

*In particular, if* (2.1) *admits a finite energy solution* $v \in W_0^{1,p}(\Omega)$ *in the sense of* (2.2) *and* (2.3), *then* $v$ *is unique.*

Proof. Since $v_1, v_2$ are finite energy solutions, (2.5) leads to

$$\int_\Omega |\nabla v_2|^{p-2}\nabla v_2 \nabla(v_2 - v_1)^+ = \int_\Omega \frac{f_2(v_2 - v_1)^+}{v_2^\gamma},$$

$$-\int_\Omega |\nabla v_1|^{p-2}\nabla v_1 \nabla(v_2 - v_1)^+ = -\int_\Omega \frac{f_1(v_2 - v_1)^+}{v_1^\gamma}.$$

Summing up we obtain

$$0 \leq \int_\Omega \left(|\nabla v_2|^{p-2}\nabla v_2 - |\nabla v_1|^{p-2}\nabla v_1\right) \nabla(v_2 - v_1)^+$$


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$$= \int_\Omega \left(\frac{f_2}{v_2^\gamma} - \frac{f_1}{v_1^\gamma}\right)(v_2 - v_1)^+$$
$$\le \int_\Omega f_1 \left(\frac{1}{v_2^\gamma} - \frac{1}{v_1^\gamma}\right)(v_2 - v_1)^+ \le 0,$$

implying $\chi_{\{v_2 \ge v_1\}} = 0$ almost everywhere, as desired.

## 3 Proof of Theorem 1.1

### 3.1 The case with constant weigths

We start by considering the easiest case where $a(x)$ and $b(x)$ are constant in $\Omega$, i.e. $a(x) \equiv A$ and $b(x) \equiv B$ for some $A, B \in \mathbb{R}^+$. We show that elementary computations lead to the existence of a solution to (1.1); the main tool we are going to use is the existence of a solution to the variational counterpart of (1.1), as it was proven in [11].

Precisely, let $(w, v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ be a positive solution to

$$\begin{cases} -\Delta_p w = q\, w^{q-1} v^\theta, \\ -\Delta_p v = \theta\, w^q v^{\theta-1}, \end{cases}$$

and let us look for a solution $(z, u)$ to

$$\begin{cases} -\Delta_p z = A\, z^{q-1} u^\theta, \\ -\Delta_p u = B\, z^q u^{\theta-1}, \end{cases} \tag{3.1}$$

of the form $z = \frac{w}{\rho}$ and $u = \frac{v}{\sigma}$, being $\rho, \sigma \in \mathbb{R}^+$ constants. By substituting we get

$$-\Delta_p z = \frac{-\Delta_p w}{\rho^{p-1}} = \frac{q}{\rho^{p-1}}(\rho z)^{q-1}(\sigma u)^\theta = \frac{q\sigma^\theta}{\rho^{p-q}} z^{q-1} u^\theta,$$
$$-\Delta_p u = \frac{-\Delta_p v}{\sigma^{p-1}} = \frac{\theta}{\sigma^{p-1}}(\rho z)^q(\sigma u)^{\theta-1} = \frac{\theta\rho^q}{\sigma^{p-\theta}} z^q u^{\theta-1},$$

which translates into the following relation between the couples $(A, B)$ and $(\rho, \sigma)$

$$\begin{cases} A = A(\rho, \sigma) = \dfrac{q\sigma^\theta}{\rho^{p-q}}, \\[2mm] B = B(\rho, \sigma) = \dfrac{\theta\rho^q}{\sigma^{p-\theta}}, \end{cases}$$

Being the Jacobian of the map $\mathcal{T} : (\rho, \sigma) \to (A, B)$ different from zero for every $(\rho, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$ (as a consequence of the fact that $q \neq p - \theta$ by assumption), the map $\mathcal{T}$ is invertible. This means that, once $q$ and $\theta$ are fixed, for all $A, B \in \mathbb{R}^+$ we can find positive constants $\rho$ and $\sigma$ such that the couple $(w/\rho, v/\sigma)$ solves (3.1).

**Remark 3.1** Thanks to the previous computations, it is possible to prove the existence of a solution to (1.1) in the special case where both $a(x)$ and $b(x)$ are $L^\infty$ functions bounded from below by positive constants. Indeed, one can use a sub/supersolution argument (see, for instance, [5] and the references therein) to find a solution $(z, u)$ to (1.1) that satisfies

$$z_\alpha \le z \le z_{\alpha'} \quad \text{and} \quad u_\beta \le u \le u_{\beta'} \quad \text{a.e. in} \quad \Omega,$$

being $\alpha, \alpha', \beta, \beta' \in \mathbb{R}^+$ such that

$$\alpha \le a(x) \le \alpha', \quad \text{and} \quad \beta \le b(x) \le \beta' \quad \text{a.e. in} \quad \Omega,$$

and being $(z_\alpha, u_\beta)$ and $(z_{\alpha'}, u_{\beta'})$ solutions to (3.1) with the choices $(A, B) = (\alpha, \beta)$ and $(A, B) = \tilde{}(\alpha', \beta')$ respectively.

For shortness, we will omit here further details on such a procedure, being the case of $a(x)$ and $b(x)$ bounded from above and from below by positive constants included in the statement of Theorem 1.1, which we are going to prove in the next subsection.

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### 3.2 The case with general weights satisfying (1.2)

As already pointed out, the main tool we mean to use to prove the existence of a solution to (1.1) is Theorem 1.3. To this aim, we need to deal with three main parts: at first, after the identification of the map $S$, we define the set $\mathcal{K}$ and we prove its invariance under the map $S$; subsequently, we treat the continuity and the compactness of $S$.

Let us start defining the map $S$ as

$$\begin{aligned} S : X := L^s(\Omega) &\longrightarrow L^s(\Omega) \longrightarrow L^s(\Omega), \\ v \quad &\longrightarrow \quad z \quad\quad \longrightarrow \quad u, \end{aligned} \tag{3.2}$$

where $s > \frac{N\theta}{p}$, $z$ solves the following problem

$$\begin{cases} -\Delta_p z = a(x) z^{q-1} v^\theta & \text{in} \quad \Omega, \\ z = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

and, once $z$ is given, $u$ is defined as a solution to

$$\begin{cases} -\Delta_p u = b(x) z^q u^{\theta-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.4}$$

We show that $z$ and $u$ are $W_0^{1,p}$-functions, so that, if the map $S$ admits a fixed point, it turns to be a solution to (1.1) by definition.

Let us thus define $\mathcal{K}$ as

$$\mathcal{K} := \left\{ v \in L^s(\Omega) \text{ s.t. } v \geq \varepsilon\varphi_1^a \right\} \cap \overline{B_R\left(L^s(\Omega)\right)},$$

being as before $s > \frac{N\theta}{p}$ and $\varepsilon, R > 0$. Recall that $\varphi_1^a$ is the first eigenfunction of the $p$-laplacian operator in $\Omega$ with homogeneous Dirichlet boundary conditions and weight $a \equiv a(x)$.

The set $\mathcal{K}$ is obviously a convex, closed and bounded subset of $L^s(\Omega)$.

#### 3.2.1 The invariance of the set $\mathcal{K}$ in the case $0 < q < 1$

In this case both the equations in (1.1) are singular with respect to their solutions.

The first step is to prove the existence of a solution to (3.3) with $v \in \mathcal{K}$, so that $a(x)v^\theta \in L^r$ for some $r > \frac{N}{p}$ and $a(x)v \not\equiv 0$ (we recall that $v \geq \varepsilon\varphi_1^a$). The existence of a positive distributional solution $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the singular equation (3.3) follows immediately from the results of [10]. Precisely, it is proven that there exists a solution $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the following sense

$$\int_\Omega |\nabla z|^{p-2} \nabla z \nabla \psi = \int_\Omega a(x) v^\theta z^{q-1} \psi \quad \forall\, \psi \in C_0^1(\Omega).$$

Moreover, since $z \in W_0^{1,p}(\Omega)$ and because of Theorems 2.1 and 2.2, we deduce that the previous formulation can be extended to $W_0^{1,p}(\Omega)$ test functions and that $z$ is unique, namely $z$ is the unique weak solution to (3.3).

By similar computations, since $0 \not\equiv b(x) z^q \in L^m(\Omega)$ with $m > \frac{N}{p}$, we obtain the existence of a unique weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.4).

Clearly, in order to prove that $\mathcal{K}$ is invariant, we need to choose $R$ such that $u \in \overline{B_R(L^s(\Omega))}$, being $v \in \overline{B_R(L^s(\Omega))}$; also, we need to estimate from below $u$ with $\varepsilon\varphi_1^a$, and this goes through an estimate on $z$. Precisely, by taking $(\varepsilon\varphi_1^a - z)^+$ as a test function in the equation solved by the difference $\varepsilon\varphi_1^a - z$, we get the following inequality

$$\begin{aligned} &\int_\Omega \left( |\nabla \varepsilon\varphi_1^a|^{p-2} \nabla \varepsilon\varphi_1^a - |\nabla z|^{p-2} \nabla z \right) \nabla (\varepsilon\varphi_1^a - z)^+ \\ &\quad \leq \int_\Omega \left( a(x) v^\theta (\varepsilon\varphi_1^a)^{q-1} - a(x) v^\theta z^{q-1} \right) (\varepsilon\varphi_1^a - z)^+, \end{aligned} \tag{3.5}$$

that is satisfied if we choose $\varepsilon$ such that $\varepsilon\varphi_1^a$ is a subsolution to

$$-\Delta_p z = a(x) v^\theta z^{q-1}.$$


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Precisely we ask for

$$\lambda_1^a a(x)\varepsilon^{p-1}(\varphi_1^a)^{p-1} \le a(x)v^\theta(\varepsilon\varphi_1^a)^{q-1},$$

and this holds true if we choose $\varepsilon$ such that

$$\varepsilon \le \frac{1}{(\lambda_1^a)^{\frac{1}{p-q-\theta}}} \cdot \frac{1}{\|\varphi_1^a\|_{L^\infty(\Omega)}}, \tag{3.6}$$

where we recall that $p - q - \theta > 0$ by assumption (1.3). Since the right hand side in (3.5) is nonpositive (and, trivially, the left hand side is nonnegative), we deduce $\varepsilon\varphi_1^a \le z$ almost everywhere in $\Omega$.

We want to show that $u \ge \varepsilon\varphi_1^a$ in $\Omega$. Taking $(\varepsilon\varphi_1^a - u)^+$ as a test function in the problem solved by the difference between $\varepsilon\varphi_1^a$ and $u$, we find out that

$$\begin{aligned}
&\int_\Omega \left(|\nabla\varepsilon\varphi_1^a|^{p-2}\nabla\varepsilon\varphi_1^a - |\nabla u|^{p-2}\nabla u\right)\nabla(\varepsilon\varphi_1^a - u)^+ \\
&\quad = \int_\Omega \left(\lambda_1^a a(x)\varepsilon^{p-1}(\varphi_1^a)^{p-1} - b(x)z^q u^{\theta-1}\right)(\varepsilon\varphi_1^a - u)^+ \\
&\quad \le \int_\Omega \left(\lambda_1^a a(x)\varepsilon^{p-1}(\varphi_1^a)^{p-1} - b(x)(\varepsilon\varphi_1^a)^{q+\theta-1}\right)(\varepsilon\varphi_1^a - u)^+.
\end{aligned} \tag{3.7}$$

Since $\theta + q < p$, if we choose

$$\varepsilon \le \left(\frac{\beta}{\lambda_1^a\|a(x)\|_{L^\infty(\Omega)}}\right)^{\frac{1}{p-\theta-q}} \frac{1}{\|\varphi_1^a\|_{L^\infty(\Omega)}}, \tag{3.8}$$

being $\beta$ as in (1.2), the right hand side of (3.7) is nonpositive. Hence if $\varepsilon$ satisfies both (3.6) and (3.8), that is if

$$\varepsilon = \min\left\{\left(\frac{\beta}{\lambda_1^a\|a(x)\|_{L^\infty(\Omega)}}\right)^{\frac{1}{p-\theta-q}} \frac{1}{\|\varphi_1^a\|_{L^\infty(\Omega)}}, \frac{1}{(\lambda_1^a)^{\frac{1}{p-\theta-q}}\|\varphi_1^a\|_{L^\infty(\Omega)}}\right\}, \tag{3.9}$$

we can conclude that

$$u \ge \varepsilon\varphi_1^a \quad \text{a.e. in} \quad \Omega.$$

In order to prove the invariance of $\mathcal{K}$, the last step is to choose $R$ such that $u \in \overline{B_R(L^s(\Omega))}$. From [10, Theorem 4.1], since $q > 0$, we have, for the solution to (3.3)

$$\|z\|_{L^\infty(\Omega)} \le C_1\|a(x)\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}\|v^\theta\|_{L^r(\Omega)}^{\frac{1}{p-1}} = C_1\|a(x)\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}\|v\|_{L^s(\Omega)}^{\frac{\theta}{p-1}}, \tag{3.10}$$

where $C_1$ is a positive constant independent of $z, v, a$, $r > \frac{N}{p}$ while $s > \frac{N\theta}{p}$ as usual. Analogously, since $\theta > 0$ and $m > \frac{N}{p}$, it holds

$$\begin{aligned}
\|u\|_{L^\infty(\Omega)} &\le C_2\|b(x)z^q\|_{L^m(\Omega)}^{\frac{1}{p-1}} \\
&\le C_2\|b(x)\|_{L^m(\Omega)}^{\frac{1}{p-1}}\|z\|_{L^\infty(\Omega)}^{\frac{q}{p-1}} \\
&\le C_1^{\frac{q}{p-1}}\, C_2\|b(x)\|_{L^m(\Omega)}^{\frac{1}{p-1}}\|a(x)\|_{L^\infty(\Omega)}^{\frac{q}{(p-1)^2}}\|v\|_{L^s(\Omega)}^{\frac{\theta q}{(p-1)^2}} \\
&\le C_1^{\frac{q}{p-1}}\, C_2C_3R^{\frac{\theta q}{(p-1)^2}},
\end{aligned} \tag{3.11}$$

where $C_2 > 0$ is a constant independent of $z, u, b$ and $C_3 = \|b(x)\|_{L^m(\Omega)}^{\frac{1}{p-1}}\|a(x)\|_{L^\infty(\Omega)}^{\frac{q}{(p-1)^2}}$. If we impose $C_1^{\frac{q}{p-1}}\, C_2C_3R^{\frac{\theta q}{(p-1)^2}} = R$, using the second assumption on $p$ in (1.4) we find

$$R = \left(C_1^{\frac{q}{p-1}}\, C_2C_3\right)^{-\frac{1}{\left(\frac{\theta q}{(p-1)^2}-1\right)}}. \tag{3.12}$$

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By choosing $\varepsilon$ and $R$ as in (3.9) and (3.12) respectively, the set $\mathcal{K}$ is invariant.

**Remark 3.2** We underline that, in this case, the assumption (1.4) can be weakened requiring

$$\theta + q < p < N \quad \text{and} \quad p \neq 1 + \sqrt{q\theta}.$$

### 3.2.2 The invariance of the set $\mathcal{K}$ in the case $q \geq 1$

If $q = 1$, by the classical theory for nonlinear elliptic equations, we immediately obtain the existence of a solution $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.3).

Conversely, if $1 < q < p - \theta$, in order to prove the existence of a solution to (3.3), we proceed by a sub/supersolution argument. To this end, let us look for $\varepsilon$ such that $\varepsilon\varphi_1^a$ is a subsolution to (3.3), namely we require

$$-\Delta_p(\varepsilon\varphi_1^a) = \lambda_1^a a(x)\varepsilon^{p-1}(\varphi_1^a)^{p-1} \leq a(x)v^\theta(\varepsilon\varphi_1^a)^{q-1} \quad \text{in} \quad \Omega.$$

In particular, the previous inequality holds true if $\varepsilon$ satisfies

$$\varepsilon^{p-(\theta+q)}(\varphi_1^a)^{p-(\theta+q)} \leq \frac{1}{\lambda_1^a}$$

which, since $\theta + q < p$, leads to

$$\varepsilon \leq \frac{1}{(\lambda_1^a)^{\frac{1}{p-\theta-q}}\|\varphi_1^a\|_{L^\infty(\Omega)}}. \tag{3.13}$$

Let now $h \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the solution to the following problem

$$\begin{cases} -\Delta_p h = a(x)v^\theta & \text{in} \quad \Omega, \\ h = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and let us choose $T \in \mathbb{R}^+$ such that $Th$ is a super-solution to (3.3), that is we impose

$$-\Delta_p(Th) = T^{p-1}a(x)v^\theta \geq (Th)^{q-1}a(x)v^\theta \quad \text{in} \quad \Omega.$$

In particular, this inequality holds true if we require

$$T \geq \|h\|_{L^\infty(\Omega)}^{\frac{q-1}{p-q}}. \tag{3.14}$$

In order to prove the existence of a solution to (3.3), we need to show that there exist $\varepsilon$ and $T$, satisfying (3.13) and (3.14) respectively, such that $\varepsilon\varphi_1^a \leq Th$ in $\Omega$. To this purpose, we choose $(\varepsilon\varphi_1^a - Th)^+$ as a test function in the equation solved by the difference between $\varepsilon\varphi_1^a$ and $Th$, obtaining

$$\begin{aligned} &\int_\Omega \left(|\nabla\varepsilon\varphi_1^a|^{p-2}\nabla\varepsilon\varphi_1^a - |\nabla Th|^{p-2}\nabla Th\right)\nabla(\varepsilon\varphi_1^a - Th)^+ \\ &\quad = \int_\Omega \left(\lambda_1^a a(x)\varepsilon^{p-1}(\varphi_1^a)^{p-1} - T^{p-1}a(x)v^\theta\right)(\varepsilon\varphi_1^a - Th)^+ \\ &\quad \leq \int_\Omega \left(\lambda_1^a a(x)\varepsilon^{p-1}(\varphi_1^a)^{p-1} - T^{p-1}a(x)(\varepsilon\varphi_1^a)^\theta\right)(\varepsilon\varphi_1^a - Th)^+ \\ &\quad = \int_\Omega (\varepsilon\varphi_1^a)^\theta a(x)\left(\lambda_1^a(\varepsilon\varphi_1^a)^{p-1-\theta} - T^{p-1}\right)(\varepsilon\varphi_1^a - Th)^+. \end{aligned}$$

The right hand side of the previous equality is nonpositive if

$$\lambda_1^a(\varepsilon\varphi_1^a)^{p-1-\theta} - T^{p-1} \leq 0,$$

that is we have to ask $T^{p-1} \geq \lambda_1^a\left(\varepsilon\|\varphi_1^a\|_{L^\infty(\Omega)}\right)^{p-1-\theta}$. We underline that $p - 1 - \theta$ is positive because of assumption (1.3).


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Gathering all the above as $q > 1$, if we choose

$$T \geq \max\left(\|h\|_{L^\infty}^{\frac{q-1}{p-q}}, {\lambda_1^a}^{\frac{1}{p-1}}\left(\varepsilon\|\varphi_1^a\|_{L^\infty(\Omega)}\right)^{\frac{p-1-\theta}{p-1}}\right) \quad \text{and} \quad \varepsilon \leq \left(\frac{1}{\lambda_1^a}\right)^{\frac{1}{p-(\theta+q)}} \frac{1}{\|\varphi_1^a\|_{L^\infty(\Omega)}},$$

there exists a solution $z \in W_0^{1,p}(\Omega)$ to (3.3) such that $\varepsilon\varphi_1^a \leq z \leq Th$. In particular $z \in L^\infty(\Omega)$.

Once a solution $z$ to (3.3) is given, we look for a solution to (3.4) with $b(x)z^q$ as a datum. As before, since $0 \not\equiv b(x)z^q \in L^m(\Omega)$ with $m > \frac{N}{p}$, from [10] we can conclude that there exists a positive distributional solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.4); also, reasoning as in the case $0 < q < 1$, we deduce that $u \in W_0^{1,p}(\Omega)$ is a weak solution and that it is unique.

Moreover, if $\varepsilon$ satisfies both (3.13) and (3.8), namely if

$$\epsilon = \min\left\{\left(\frac{\beta}{\lambda_1^a\|a(x)\|_{L^\infty(\Omega)}}\right)^{\frac{1}{p-\theta-q}} \frac{1}{\|\varphi_1^a\|_{L^\infty(\Omega)}}, \frac{1}{(\lambda_1^a)^{\frac{1}{p-\theta-q}}\|\varphi_1^a\|_{L^\infty(\Omega)}}\right\} \tag{3.15}$$

proceeding as in (3.7) we obtain

$$u \geq \varepsilon\varphi_1^a \quad \text{a.e. in} \quad \Omega.$$

Once again, in order to conclude the proof of the invariance of $\mathcal{K}$, we need to choose $R$ such that $u \in \overline{B_R(L^s(\Omega))}$. In this case we have

$$\|z\|_{L^\infty(\Omega)} \leq T\|h\|_{L^\infty(\Omega)} \leq TC_1\left\|a(x)v^\theta\right\|_{L^r(\Omega)}^{\frac{1}{p-1}} = TC_1\|a(x)\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}\|v\|_{L^s(\Omega)}^{\frac{\theta}{p-1}}, \tag{3.16}$$

where $C_1$ is a positive constant independent of $z, v, a$, $r > \frac{N}{p}$, $s > \frac{N\theta}{p}$. Since $m > \frac{N}{p}$, using once again [10, Theorem 4.1] we deduce

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} &\leq C_2\|b(x)z^q\|_{L^m(\Omega)}^{\frac{1}{p-1}} = C_2\|b(x)\|_{L^m(\Omega)}^{\frac{1}{p-1}}\|z\|_{L^\infty(\Omega)}^{\frac{q}{p-1}} \\ &\leq (TC_1)^{\frac{q}{p-1}} C_2\|b(x)\|_{L^m(\Omega)}^{\frac{1}{p-1}}\|a(x)\|_{L^\infty(\Omega)}^{\frac{q}{(p-1)^2}}\|v\|_{L^s(\Omega)}^{\frac{\theta q}{(p-1)^2}} \\ &\leq (TC_1)^{\frac{q}{p-1}} C_2C_3R^{\frac{\theta q}{(p-1)^2}}\end{aligned} \tag{3.17}$$

where, as before, $C_2$ is a positive constant independent of $z, u, b$ and $C_3$ is given by $\|b(x)\|_{L^m(\Omega)}^{\frac{1}{p-1}}\|a(x)\|_{L^\infty(\Omega)}^{\frac{q}{(p-1)^2}}$. Finally, if we impose

$$(TC_1)^{\frac{q}{p-1}} C_2C_3R^{\frac{\theta q}{(p-1)^2}} = R,$$

using once again the second assumption on $p$ in (1.4) we find

$$R = \left((TC_1)^{\frac{q}{p-1}} C_2C_3\right)^{-\frac{1}{\left(\frac{\theta q}{(p-1)^2}-1\right)}}. \tag{3.18}$$

The choice of $\varepsilon$ and $R$ as in (3.15) and (3.18) respectively makes $\mathcal{K}$ an invariant set.

**Remark 3.3** We stress that the map $S$ is well defined. Indeed, if $0 < q < 1$, the solution $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (3.3) is unique so that the source of problem (3.4) is well identified. If $q = 1$, the uniqueness of $z$ follows by the classical theory for nonlinear elliptic equations. Finally, if $q > 1$, we find $z$ through a sub/supersolution argument and it can be uniquely identified as the unique solution to (3.3) found by monotone iterations starting from the given subsolution.

### 3.2.3 Continuity of $S$

Let us suppose that $v_n \in \mathcal{K}$ is such that $v_n \to v$ in $L^s(\Omega)$ and let $u_n := S(v_n)$, being $S$ defined in (3.2). We want to prove that there exists $u \in L^s(\Omega)$ such that, up to subsequences, $u_n \to u \in L^s(\Omega)$ and that $u = S(v)$.

From now on all the convergences results must be intended up to subsequences.

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Starting from $v_n$, for each $n \in \mathbb{N}$ we can construct $z_n$ and $u_n$ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, sequences of solutions to (3.3) and to (3.4) respectively.

Thanks to (3.10) and (3.16), we conclude that

$$\|z_n\|_{L^\infty(\Omega)} \leq C\|v_n\|_{L^s(\Omega)}^{\frac{\theta}{p-1}} \quad \forall\, q > 0, \tag{3.19}$$

where $C$ is a positive constant independent of $n \in \mathbb{N}$. Hence $z_n$ is bounded in $L^\infty(\Omega)$ and, in particular, there exists $z \in L^r(\Omega)$ for all $1 \leq r < \infty$ such that $z_n \to z$ almost everywhere and $z_n \to z$ in $L^r(\Omega)$ for all $1 \leq r < \infty$.

Moreover, thanks to (3.11) and (3.17), there exists a positive constant $C$ independent of $n \in \mathbb{N}$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq C\|z_n\|_{L^\infty(\Omega)}^{\frac{q}{p-1}} \quad \forall\, q > 0. \tag{3.20}$$

As before, there exists $u \in L^r(\Omega)$ for all $1 \leq r < \infty$ such that $u_n \to u$ in $L^r(\Omega)$ for all $1 \leq r < \infty$. In particular $u_n$ strongly converges to $u$ in $L^s(\Omega)$.

To conclude, we need to prove that $u = S(v)$. At first, we check that we can pass to the limit in the following equality

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\psi = \int_\Omega b(x) z_n^q u_n^{\theta-1}\psi \quad \forall\, \psi \in W_0^{1,p}(\Omega), \tag{3.21}$$

where we use test functions in $W_0^{1,p}(\Omega)$ because, from Theorem 2.1, it holds

$$b(x) z_n^q u_n^{\theta-1} \in W^{-1,p'}(\Omega) \quad \forall n \in \mathbb{N}.$$

Since $u_n \in \mathcal{K}$ for all $n \in \mathbb{N}$, we get

$$b(x) z_n^q u_n^{\theta-1} \leq b(x) z_n^q (\varepsilon\varphi_1)^{\theta-1} \quad \text{a.e. in} \quad \Omega.$$

In particular $b(x)\dfrac{z_n^q}{u_n^{1-\theta}}$ is bounded in $L^m(\Omega)$ with respect to $n \in \mathbb{N}$, being $z_n$ bounded in $L^\infty(\Omega)$. Moreover, taking $u_n$ as a test function in (3.21), we find

$$\int_\Omega |\nabla u_n|^p \leq \|b(x)\|_{L^1(\Omega)}\|z_n\|_{L^\infty(\Omega)}^q \|u_n\|_{L^\infty(\Omega)}^\theta \operatorname{meas}(\Omega) = C,$$

where the constant $C > 0$ is independent of $n \in \mathbb{N}$; in particular $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Then, if we take $u_n - u$ as a test function in (3.21) we obtain

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n - u) \\ &\quad = \int_\Omega b(x) z_n^q u_n^{\theta-1}(u_n - u) - \int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_n - u), \end{aligned} \tag{3.22}$$

and we can conclude that the right hand side of (3.22) goes to zero as $n \to \infty$. Indeed $u_n$ strongly converges to $u$ in $L^r(\Omega)$ for all $1 \leq r < \infty$, implying that it converges to $u$ in $L^{m'}(\Omega)$. Then, since $b(x) z_n^q u_n^{\theta-1}$ is bounded in $L^m(\Omega)$ with respect to $n \in \mathbb{N}$, the first term of the right hand side of (3.22) goes to zero.

Going further, because of the weak convergence of $u_n$ to $u$ in $W_0^{1,p}(\Omega)$, also the second term on the right hand side of (3.22) goes to zero and we can apply [2, Lemma 5] to deduce that

$$u_n \to u \quad \text{in} \quad W_0^{1,p}(\Omega).$$

This implies that we can pass to the limit in the left hand side of (3.21) with $W_0^{1,p}(\Omega)$ test functions.

Finally, since the sequence of $L^1(\Omega)$ functions $b(x) z_n^q u_n^{\theta-1}\psi$ is bounded in $L^1(\Omega)$ with respect to $n \in \mathbb{N}$ for each $\psi \in W_0^{1,p}(\Omega)$ and converges almost everywhere to $b(x) z^q u^{\theta-1}\psi$, by the Lebesgue Theorem we can pass to the limit also in the right hand side of (3.21), obtaining the desired result.

In order to ensure that $u = S(v)$, it remains only to prove the uniqueness of the solution to our problem. To this aim we start noticing that, if $\lim_{n\to\infty} v_n = v \in \mathcal{K}$, the function $z$, source of (3.4), is well identified (see Remark 3.3). Moreover the solution $u \in W_0^{1,p}(\Omega)$ to (3.4) with $b(x)z^q$ as a datum is unique, since we can take test functions


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in the whole $W_0^{1,p}(\Omega)$ (see Theorem 2.2). Due to the uniqueness of $u$, it follows that $\lim_{n\to\infty} S(v_n) = S(v)$ and the map $S$ is continuous.

### 3.2.4 Compactness of $S$

If $v_n \in \mathcal{K}$ is bounded in $L^s(\Omega)$ and $u_n = S(v_n)$, we can construct $z_n$ such that (3.19) and (3.20) hold. Hence, proceeding as in the first part of the proof of the continuity of $S$, we can conclude that there exists $u \in L^s(\Omega)$ such that $u_n$ strongly converges to $u$ in $L^s(\Omega)$ up to subsequences. This ensure that $\overline{S(C)}$ is compact for every $C \subset L^s(\Omega)$ bounded.

We complete the proof of Theorem 1.1 by applying Theorem 1.3. We thus obtain the existence of at least one couple $(u, z) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ that solves (1.1) in the sense of (1.7).

**Remark 3.4** Let us observe that the condition $a(x) \in L^\infty(\Omega)$, assumed in (1.2), is needed in order to find a suitable value of $\varepsilon$ such that the last term in the right hand side of (3.7) is nonpositive, both for $0 < q < 1$ and for $q \geq 1$. However we underline that, alternatively to (1.2), for any $q > 0$ we can also require

$$0 < \beta \leq a(x) \equiv b(x) \in L^m(\Omega) \quad \text{for some} \quad m > \frac{N}{p}. \tag{3.23}$$

In this case the negativity of the last term in (3.7) is preserved by choosing

$$\varepsilon \leq \left(\frac{\beta}{\lambda_1^a}\right)^{\frac{1}{p-\theta-q}} \frac{1}{\|\varphi_1^a\|_{L^\infty(\Omega)}},$$

while a straightforward computation shows that, under the assumption (3.23), the invariant set $\mathcal{K}$ has to be defined as

$$\mathcal{K} := \left\{v \in L^s(\Omega) \text{ s.t. } v \geq \varepsilon\varphi_1^a\right\} \cap \overline{B_R\left(L^s(\Omega)\right)},$$

where now $s > \frac{N\theta m}{mp-N}$.

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