ASYMPTOTIC ANALYSIS OF SINGULAR PROBLEMS IN PERFORATED CYLINDERS

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Abstract. In this paper we deal with elliptic problems having terms singular in the variable u which represents the solution. The problems are posed in cylinders Ω_n^ε of height $2n$ and perforated according to a parameter ε. We study existence, uniqueness and asymptotic behaviour of the solutions u_n^ε as the cylinders become infinite ($n \to +\infty$) and the size of the holes decreases while the number of the holes increases ($\varepsilon \to 0$).

1 Introduction

In this paper we deal with elliptic problems having terms singular in the variable u which represents the solution. These singular terms appear in situations as chemical heterogeneous catalysts, in the study of non-newtonian fluids, boundary layer phenomena for viscous fluids, as well as in the theory of heat conduction in electrically conducting materials. Let us give an example of this last situation according to [8] The region Ω_n is a cylinder (of height $2n$) of the three dimensional space occupied by an electrical conductor. Then each point in Ω_n becomes a source of heat as a current is passed through Ω_n. Let $u(x,t)$ be the temperature at the point $x \in \Omega_n$ and at the time t, let $\beta(x,t)$ the function which describes the local voltage drop in Ω_n and let $\sigma(u)$ the electrical resistivity depending on the temperature u. Then the rate of generation of heat at any point x and any time t is $\frac{\beta^2(x,t)}{\sigma(u)}$. If we assume that the specific heat c and the thermal conductivity k are constant in Ω_n then the temperature satisfies the equation

$$cu_t - k\Delta u = \beta^2(x,t)/\sigma(u).$$

In many applications it is natural to assume that σ is a positive function of u which is increasing with u and which tends to zero with u. Thus

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the differential equation is singular in the sense that the right hand side becomes unbounded at $u = 0$.

We will study the stationary equation related to the previous model in a domain Ω_n^ε perforated according to a parameter ε, allowing the thermal conductivity k to depend on the point x.

The asymptotic analysis in perforated domains has been extensively studied in connection with many applications in different fields of sciences and engineering. Let us mention in particular the theory of the so called *metamaterials*. Metamaterials gain their properties not from their composition, but from their structure. Their precise shape, geometry, size, orientation and arrangement can affect the waves of light or sound in an unconventional manner, creating material properties which are unachievable with conventional materials.

The new qualitative properties, in particular electromagnetic responses (*e.g.* anisotropy chirality and optical activity, artificial magnetism, structural colors from photonic band gaps) are determined by the geometry in the small-scale (see for instance [14], [16]).

According to the previous discussion the problems we deal with in this paper may provide a model for heat diffusion in cylindric metamaterials which are electrical conductors. More precisely in the present paper we consider, for $n \in \mathbb{R}^+$, $\Omega_n \subset \mathbb{R}^N$ defined by

$$\Omega_n = (-n, n) \times \omega$$

where ω is an open bounded subset in R^{N-1}. We consider also the perforated domains Ω_n^ε obtained by removing some closed sets T_i^ε of $\mathbb{R}^N$, $1 \leq i \leq \nu(\varepsilon)$ from Ω_n. The domain is defined by

$$\Omega_n^\varepsilon = \Omega_n \setminus \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon.$$

In the perforated domain Ω_n^ε we shall consider the following singular problem

$$\begin{cases} i)\, u_n^\varepsilon \in H_0^1(\Omega_n^\varepsilon, \partial\omega), \\ ii)\, u_n^\varepsilon \geq 0 \quad \text{a.e. in } \Omega_{\mathrm{n}}^\varepsilon, \\ iii)\, \displaystyle\int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi < +\infty, \\ iv)\, \displaystyle\int_{\Omega_n^\varepsilon} A(x) D u_n^\varepsilon D\varphi = \int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi \\ \forall \varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega),\ \varphi \geq 0 \end{cases}$$

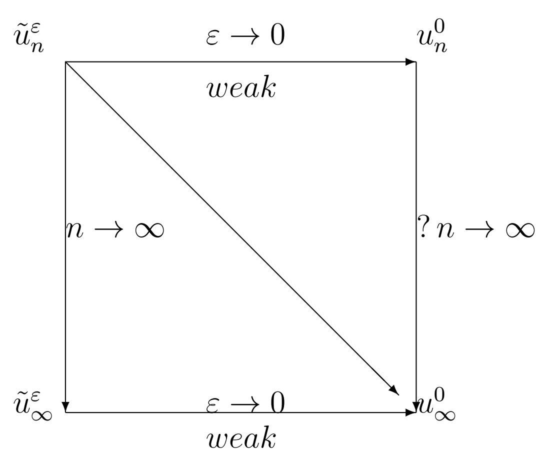
where the matrix A satisfies condition (2.6), the singular term F satisfies (2.7) and (2.8) (below) and

$$H_0^1(\Omega_n^\varepsilon, \partial\omega) = \{v \in H^1(\Omega_n^\varepsilon) : v = 0 \text{ on } (-n,n)\times\partial\omega \text{ and on } \partial\big(\cup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon\big)\}.$$

For any fixed n and ε, the existence of solutions u_n^ε is proved according to the outline of the proof of [9] (see Theorem 3.3). We note that we cannot apply directly Theorems 4.1 and 4.2 of [9], because we prescribe different boundary conditions which are the natural ones since we are also interested in the asymptotic analysis as n tends to $+\infty$. More precisely the functions in the space $H_0^1(\Omega_n^\varepsilon, \partial\omega)$ need to vanish only on the lateral boundary $(-n,n)\times\partial\omega$ while the setting for Theorems 4.1 and 4.2 of [9] is the space $H_0^1(\Omega_n^\varepsilon)$.

First we study the asymptotic behaviour of the solutions u_n^ε as the size of the holes decreases while the number of the holes increases ($\varepsilon \to 0$). Assuming that the distribution and the size of the perforation satisfy suitable conditions that can be expressed in terms of mathematical capacity of the holes, we prove that the limit u_n^0 of the solutions u_n^ε (as ε goes to zero) solves a singular problem with an extra term which takes into account the mathematical capacity of the perforation (see Theorem 4.2 and Remark 4.1).

Later on assuming that the matrix A, the datum $F(\cdot, s)$ (for any s) are 1-periodic in the x_1-direction, we investigate whether the periodicity of the data will force the solutions u_n^ε to converge towards a periodic solution u_∞^ε, as $n \to +\infty$. The answer is given in Theorem 5.1.
More precisely Theorem 5.1 gives estimates of the convergence of $\tilde{u}_n^\varepsilon$ to $\tilde{u}_\infty^\varepsilon$ (see (5.6), (5.6)), where the constants are independent of ε (and n). Here by $\tilde{u}_n^\varepsilon$ and $\tilde{u}_\infty^\varepsilon$ we denote the extensions by zero on the holes (see (4.8)). This allows us to prove that, under convenient assumptions, the sequence u_n^0 converges to a periodic solution u_∞^0 of the homogenized problem and therefore the following diagram commutes



The question whether one can prove directly the convergence of u_n^0 to u_∞^0 is still open.

2 Preliminary and notation

In the present paper we consider, for $n \in \mathbb{R}^+$, $\Omega_n \subset \mathbb{R}^N$ defined by

$$\Omega_n = (-n, n) \times \omega \tag{2.1}$$

where ω is an open bounded subset in R^{N-1}.
We consider also the perforated domains Ω_n^ε obtained by removing some closed sets T_i^ε of $\mathbb{R}^N$, $1 \le i \le \nu(\varepsilon)$ from Ω_n. The domain is defined by

$$\Omega_n^\varepsilon = \Omega_n \setminus \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon. \tag{2.2}$$

The related Sobolev spaces are

$$H_0^1(\Omega_n, \partial\omega) = \{v \in H^1(\Omega_n) : v = 0 \text{ on } (-n, n) \times \partial\omega\}, \tag{2.3}$$

(2.4)
$$H_0^1(\Omega_n^\varepsilon, \partial\omega) = \{v \in H^1(\Omega_n^\varepsilon) : v = 0 \text{ on } (-n, n) \times \partial\omega \text{ and on } \partial\big(\cup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon\big)\}.$$

In the perforated domain Ω_n^ε we shall consider the following singular problem

$$\begin{cases} i)\, u_n^\varepsilon \in H_0^1(\Omega_n^\varepsilon, \partial\omega), \\ ii)\, u_n^\varepsilon \ge 0 \quad \text{a.e. in } \Omega_{\mathrm{n}}^\varepsilon, \\ iii)\, \displaystyle\int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi < +\infty, \\ iv)\, \displaystyle\int_{\Omega_n^\varepsilon} A(x) D u_n^\varepsilon D\varphi = \int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi \\ \forall \varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega),\ \varphi \ge 0 \end{cases} \tag{2.5}$$

where the matrix A satisfies

$$\begin{cases} A(x) \in L^\infty(\Omega_n)^{N\times N}, \\ \exists \alpha > 0,\ A(x) \ge \alpha I \quad \text{a.e. } x \in \Omega_n. \end{cases} \tag{2.6}$$

The function $F : \Omega_n \times [0, +\infty[\to [0, +\infty]$ is a Carathéodory function, i.e.

$$\begin{cases} \text{for a.e. } x \in \Omega_n,\ F(x, \cdot) \text{ is continuous}, \\ \forall s \in [0, +\infty[,\ F(\cdot, s) \text{ is measurable} \end{cases} \tag{2.7}$$

and it satisfies

$$(2.8)\quad \begin{cases} i)\exists\gamma, \gamma\in(0,1], \\ ii)\exists h \ \text{and}\, \exists f \ \text{with} \ h\in L^r(\Omega_n), \ \ r\geq \frac{2^*}{2^*-1+\gamma}, \ f(x)\in L^{(2^*)'}(\Omega_n) \ \text{if} \ N>2, \\ \ \text{or} \ h,f\in L^r(\Omega_n), \ r>1, \text{if} \ N=2 \\ iii)\, h(x)\geq 0, \ f(x)\geq 0 \ \text{a.e.} \ x\in\Omega_n, \\ \text{such that} \\ iv)\, 0\leq F(x,s)\leq \frac{h(x)}{s^\gamma}+f(x) \ \text{a.e.} \ x\in\Omega_n, \forall s>0. \end{cases}$$

For any fixed n and ε, the existence of solutions u_n^ε is proved according to the outline of the proof of [9]. We show boundedness of solutions u_n^ε under the stronger assumptions, i.e. $h, f \in L^r(\Omega_n), r > \frac{N}{2}$. The uniqueness of the solution is established under the further condition

$$(2.9)\qquad F(x,s)\leq F(x,t) \ \text{a.e.} \ x\in\Omega_n, \ \forall s,\forall t, \ 0\leq t\leq s$$

(see Theorem 3.3). Moreover we are interested in the asymptotic behaviour of the solutions to problem (2.5) as the cylinder Ω_n becomes infinite ($n \to +\infty$) and the size of the holes becomes smaller and smaller while the number of the holes increases ($\varepsilon \to 0$).

Under convenient assumptions on the perforation, we show, again according to [9], that u_n^ε, up to a subsequence, converges to u_n^0 which solves in Ω_n the homogenized problem

$$(2.10)\qquad \begin{cases} i)\, u_n^0\in H_0^1(\Omega_n,\partial\omega)\cap L^2(\Omega_n,d\mu) \\ ii)\, u_n^0(x)\geq 0 \ \text{a.e.} \ x\in\Omega_n, \\ iii)\, \displaystyle\int_{\Omega_n} F(x,u_n^0)\varphi<+\infty \\ iv)\, \displaystyle\int_{\Omega_n} A(x)Du_n^0D\varphi+\int_{\Omega_n} u_n^0\varphi d\mu=\int_{\Omega_n} F(x,u_n^0)\varphi \\ \forall\varphi\in H_0^1(\Omega_n)\cap L^2(\Omega_n,d\mu), \ \varphi\geq 0 \end{cases}$$

where μ is a nonnegative measure which belongs to $H^{-1}(\Omega_n)$ (see Section 4).

We denote by $\mathcal{D}(\mathcal{O})$ the space of the functions $C^\infty(\mathcal{O})$ whose support is compact and included on $\mathcal{O}$, and by $\mathcal{D}'(\mathcal{O})$ the space of distributions on $\mathcal{O}$.

For every $s \in \mathbb{R}$ we define

$$s^+ = max\{s, 0\}, \ s^- = max\{0, -s\}.$$

For $l : \mathcal{O} \longrightarrow [0, +\infty]$ a measurable function we denote

$$\{l = 0\} = \{x \in \mathcal{O} : l(x) = 0\}, \ \{l > 0\} = \{x \in \mathcal{O} : l(x) > 0\}.$$

From now on we will denote by $|\mathcal{O}|$ the N-dimensional Lebesgue measure of the set $\mathcal{O}$.
Moreover we define the unit cell

$$Q = (0,1) \times \omega \tag{2.11}$$

and we denote by Q^ε the perforated cell

$$Q^\varepsilon = Q \setminus \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon \tag{2.12}$$

where T_i^ε are closed set of $(0,1) \times \mathbb{R}^{N-1}$, $1 \leq i \leq \nu(\varepsilon)$.
The related Sobolev spaces are
(2.13)
$$H^1_{0,per}(Q, \partial\omega) = \{v \in H^1_0(Q, \partial\omega) : v(x) = v(x+e_1) \ \forall x \in \partial Q \cap \{x_1 = 0\}\}$$
(2.14)
$$H^1_{0,per}(Q^\varepsilon, \partial\omega) = \{v \in H^1_0(Q^\varepsilon, \partial\omega) : v(x) = v(x+e_1) \ \forall x \in \partial Q^\varepsilon \cap \{x_1 = 0\}\}.$$

In the next Section 3 we will prove also existence and uniqueness results for the following problem:

$$\begin{cases} i)\, u^\varepsilon_\infty \in H^1_{0,per}(Q^\varepsilon, \partial\omega), \\ ii)\, u^\varepsilon_\infty \geq 0 \quad \text{a.e. in } Q^\varepsilon, \\ iii)\, \displaystyle\int_{Q^\varepsilon} F(x, u^\varepsilon_\infty)\varphi < +\infty, \\ iv)\, \displaystyle\int_{Q^\varepsilon} A(x) D u^\varepsilon_\infty D\varphi = \int_{Q^\varepsilon} F(x, u^\varepsilon_\infty)\varphi \\ \forall \varphi \in H^1_{0,per}(Q^\varepsilon, \partial\omega), \ \varphi \geq 0. \end{cases} \tag{2.15}$$

We recall some important features of the solutions to problems (2.5) (2.15)

Remark 2.1. Condition iii) in (2.5) implies in particular that, denoting a solution u^ε_n by u, $F(x, u(x))$ is finite almost everywhere on Ω^ε_n, i.e. that

$$|\{x \in \Omega^\varepsilon_n : u(x) = 0 \text{ and } F(x, 0) = +\infty\}| = 0. \tag{2.16}$$

Actually the following stronger results hold true

$$|\{x \in \Omega^\varepsilon_n : u(x) = 0 \text{ and } 0 < F(x, 0) \leq +\infty\}| = 0, \tag{2.17}$$

and

$$\int_{\Omega_n^\varepsilon} F(x,u)v = \int_{\{u>0\}} F(x,u)v \quad \forall v \in H_0^1(\Omega_n^\varepsilon),\ v \geq 0 \tag{2.18}$$

where u is a solution to problem (2.5). We refer to Proposition 3.3 in [9] for the proof. Claim (2.17) is also equivalent to

$$\begin{cases} \{x \in \Omega_n^\varepsilon : u(x) = 0\} \subset \{x \in \Omega_n^\varepsilon : F(x,0) = 0\} \\ \text{except for a set of zero measure,} \end{cases} \tag{2.19}$$

and also to

$$\begin{cases} \{x \in \Omega_n^\varepsilon : 0 < F(x,0) \leq +\infty\} \subset \{x \in \Omega_n^\varepsilon : u(x) > 0\} \\ \text{except for a set of zero measure.} \end{cases}$$

Remark 2.2. By using the strong maximum principle we can show that any solution u to problem (2.5) satisfies

$$\text{either } u \equiv 0 \text{ or} |\{x \in \Omega_n^\varepsilon : u(x) = 0\}| = 0. \tag{2.20}$$

We refer to Proposition 3.4 in [9] for the proof.

The previous results hold also if the operator $-div\, A(x)Du$ is replaced by $-div\, A(x)Du + a_0 u$, with $a_0 \in L^\infty(\Omega_n^\varepsilon)$, $a_0 \geq 0$.
In the framework of homogenization with many small holes (see Section 4), this will be the case if the measure μ belongs to $L^\infty(\Omega_n^\varepsilon)$. We recall that if μ belongs to $H^{-1}(\cdot)$ the strong maximum principle may fail even for linear problems (a counterexample can be constructed by a result of [5] when $N \geq 3$).

Remark 2.3. If $u \equiv 0$ is the solution to problem (2.5) then condition(2.19) implies that $F(x,0) \equiv 0$. Conversely, if $F(x,0) \not\equiv 0$, $u \equiv 0$ is not a solution to problem (2.5) and then (2.20) implies that

$$u(x) > 0 \text{ a.e. } x \in \Omega_n^\varepsilon.$$

Remark 2.4. Remarks 2.1, 2.2 and 2.3 hold true for any solution to problem (2.15).

Remark 2.5. Once we got a solution to (2.5), actually the equation iv) holds true for any $\varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$, splitting any function $\varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$ as $\varphi = \varphi^+ - \varphi^-$ and taking into account iii) of (2.5). The same remark holds for (2.15).

3 Existence results

In this Section we state our existence and uniqueness results in the notations introduced in Section 2.

We point out that Poincaré type inequalities hold in our framework. More precisely, by a direct calculation we get

Proposition 3.1. *For any function $v \in H^1_0(\Omega_n, \partial\omega)$ we have*

$$\int_{\Omega_n} v^2 \leq C_P^2 \int_{\Omega_n} |Dv|^2 \tag{3.1}$$

where $C_P = C_P(\omega)$ does not depend on n. The same inequality holds true in the space $H^1_0(\Omega^\varepsilon_n, \partial\omega)$ with a constant which does not depend on n and ε.

Proposition 3.2. *For any function $v \in H^1_0(\Omega_n, \partial\omega)$ we have*

$$(\int_{\Omega_n} v^q)^{\frac{1}{q}} \leq C(n)(\int_{\Omega_n} |Dv|^2)^{\frac{1}{2}} \tag{3.2}$$

where either $q = 2^*$ and $C(n) = C_S(\omega)$ if $N \geq 3$ or $q \geq 1$ (arbitrary) and $C(n) = C_S(\omega)\, n^{\frac{1}{q}}$ if $N = 2$. Here the constant C_S does not depend on n. The same inequality holds true in the space $H^1_0(\Omega^\varepsilon_n, \partial\omega)$ with a constant C_S which does not depend on n and ε.

Proof. Suppose $N \geq 3$. Then for any function $u \in H^1_0(\Omega_n, \partial\omega)$ there exists an extension $u^* \in H^1_0(\Omega_{n+1}, \partial\omega)$ such that

$$\|u^*\|_{H^1(\Omega_{n+1})} \leq C_E\|u\|_{H^1(\Omega_n)} \tag{3.3}$$

where the constant C_E is independent of n (see [12]).
Consider the function ρu^* where $0 \leq \rho \leq 1$ is a cut-off function such that $\rho = \rho(x_1) = 1$ on the set $|x_1| < n$, $\rho = 0$ on the set $|x_1| \geq n + 1$ and it is continuous and piecewise affine. Then we have

$$(\int_{\Omega_n} |u|^{2^*})^{\frac{2}{2^*}} \leq (\int_{\Omega_{n+1}} |\rho u^*|^{2^*})^{\frac{2}{2^*}} \leq C_{G,T} \int_{\Omega_{n+1}} |D(\rho u^*)|^2$$
$$\leq 2C_{G,T}C_E^2\|u\|^2_{H^1(\Omega_n)} \leq 2(1 + C_P^2)C_{G,T}C_E^2\|Du\|^2_{L^2(\Omega_n)}$$

where we have used Theorem 7.10 in [10] and estimates (3.3) and (3.1). If $N = 2$ the proof goes on in the same way and using Hölder inequality the extra term $n^{\frac{1}{q}}$ appears. □

We state the following theorem.

Theorem 3.3. *Assume that the matrix A and the function F satisfy* (2.6)*,* (2.7) *and* (2.8)*.Then for any fixed $n, \varepsilon \in \mathbb{R}^+$ there exists a solution u^ε_n to the problem* (2.5)*. If in addition we assume $h(x) \in L^r(\Omega^\varepsilon_n)$, and $f(x) \in L^r(\Omega^\varepsilon_n)$, $r > \frac{N}{2}$, then any solution u^ε_n belongs to $L^\infty(\Omega^\varepsilon_n)$. Moreover if we assume* (2.9) *then the solution is unique.*

We note that we cannot apply directly Theorems 4.1, 4.2 and 4.4 of [9], because we prescribe different boundary conditions. More precisely the functions in the space $H_0^1(\Omega_n^\varepsilon, \partial\omega)$ need to vanish only on the lateral boundary $(-n, n) \times \partial\omega$ while the setting for Theorems 4.1, 4.2 and 4.4 of [9] is the space $H_0^1(\Omega_n^\varepsilon)$. Hence we are going to sketch the proof, pointing out the main differences and referring to [9] for further details.

Proof. We split our proof in five steps. We point out that in the following proof the parameters ε, $n \in \mathbb{R}^+$ are fixed.

Step 1. We put $F_k(x, s) = min(F(x, s), k)$, for any $k \in \mathbb{R}^+$.
By a fixed point argument in $L^2(\Omega_n^\varepsilon)$, we can prove that there exists a solution of the problem

$$\begin{cases} v_k \in H_0^1(\Omega_n^\varepsilon, \partial\omega) \\ \displaystyle\int_{\Omega_n^\varepsilon} A(x) D v_k D\varphi = \int_{\Omega_n^\varepsilon} F_k(x, v_k^+)\varphi \quad \forall \varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega). \end{cases} \tag{3.4}$$

By choosing as test function $-v_k^-$ and using Poincaré inequality (see (3.1)) we prove that $v_k \geq 0$ a.e. in Ω_n^ε .

Step 2. By choosing v_k as test function in (3.4) and $r = \frac{2^*}{2^*-1+\gamma}$, if $N > 2$, we obtain

$$\alpha \int_{\Omega_n^\varepsilon} |Dv_k|^2 \leq \Big\{ \int_{\Omega_n^\varepsilon} h^r \Big\}^{\frac{1}{r}} \Big\{ \int_{\Omega_n^\varepsilon} |v_k|^{r'(1-\gamma)} \Big\}^{\frac{1}{r'}} + \Big\{ \int_{\Omega_n^\varepsilon} f^{(2^*)'} \Big\}^{\frac{1}{(2^*)'}} \Big\{ \int_{\Omega_n^\varepsilon} |v_k|^{2^*} \Big\}^{\frac{1}{2^*}} =$$

$$\Big\{ \int_{\Omega_n^\varepsilon} h^r \Big\}^{\frac{1}{r}} \Big\{ \int_{\Omega_n^\varepsilon} |v_k|^{2^*} \Big\}^{\frac{1-\gamma}{2^*}} + \Big\{ \int_{\Omega_n^\varepsilon} f^{(2^*)'} \Big\}^{\frac{1}{(2^*)'}} \Big\{ \int_{\Omega_n^\varepsilon} |v_k|^{2^*} \Big\}^{\frac{1}{2^*}} \leq$$

$$\leq C_S^{1-\gamma} \|h\|_{L^r(\Omega_n^\varepsilon)} \Big\{ \int_{\Omega_n^\varepsilon} |Dv_k|^2 \Big\}^{\frac{1-\gamma}{2}} + C_S \|f\|_{L^{(2^*)'}(\Omega_n^\varepsilon)} \Big\{ \int_{\Omega_n^\varepsilon} |Dv_k|^2 \Big\}^{\frac{1}{2}}$$

(see (3.2)). Then

$$\int_{\Omega_n^\varepsilon} |Dv_k|^2 \leq C_1 \Big(\|h\|^{\frac{2}{1+\gamma}}_{L^r(\Omega_n^\varepsilon)} + \|f\|^2_{L^{(2^*)'}(\Omega_n^\varepsilon)} \Big) \tag{3.5}$$

where $C_1 = C_1(\alpha, \gamma, C_S)$. Hence for every $\varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$

$$\int_{\Omega_n^\varepsilon} F_k(x, v_k)\varphi \leq C_2 \|D\varphi\|_{L^2(\Omega_n^\varepsilon)} (\|h\|^{\frac{1}{1+\gamma}}_{L^r(\Omega_n^\varepsilon)} + \|f\|_{L^{(2^*)'}(\Omega_n^\varepsilon)}) \tag{3.6}$$

where $C_2 = C_2(\alpha, \gamma, \|A\|_{L^\infty}, C_S)$ is independent of k.
Analogously, if $N = 2$ we choose h and f in $L^r(\Omega_n^\varepsilon)$ with r an arbitrary

number greater then 1 and we get

$$\int_{\Omega_n^\varepsilon} |Dv_k|^2 \le C_1\Big(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{2}{1+\gamma}} n^{\frac{2(r-1)}{r(\gamma+1)}} + \|f\|_{L^r(\Omega_n^\varepsilon)}^2 n^{\frac{2(r-1)}{r}}\Big) \tag{3.7}$$

where $C_1 = C_1(\alpha, \gamma, C_S)$. Hence for every $\varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$

$$\int_{\Omega_n^\varepsilon} F_k(x, v_k)\varphi \le C_2\|D\varphi\|_{L^2(\Omega_n^\varepsilon)}(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{1}{1+\gamma}} n^{\frac{r-1}{r(\gamma+1)}} + \|f\|_{L^r(\Omega_n^\varepsilon)} n^{\frac{r-1}{r}}) \tag{3.8}$$

where $C_2 = C_2(\alpha, \gamma, \|A\|_{L^\infty}, C_S)$ is independent of k.

Step 3. By the previous steps we get that there exists $u_n^\varepsilon \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$, $u_n^\varepsilon \ge 0$ a.e. in Ω_n^ε such that, up to a subsequence, as $k \to +\infty$:

$$v_k \rightharpoonup u_n^\varepsilon \text{ in } H_0^1(\Omega_n^\varepsilon, \partial\omega) \text{ weakly.} \tag{3.9}$$

Note that the function u_n^ε satisfies either $(N > 2)$

$$\int_{\Omega_n^\varepsilon} |Du_n^\varepsilon|^2 \le C_1\Big(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{2}{1+\gamma}} + \|f\|_{L^{(2^*)'}(\Omega_n^\varepsilon)}^2\Big) \tag{3.10}$$

or $(N = 2)$

$$\int_{\Omega_n^\varepsilon} |Du_n^\varepsilon|^2 \le C_1\Big(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{2}{1+\gamma}} n^{\frac{2(r-1)}{r(\gamma+1)}} + \|f\|_{L^r(\Omega_n^\varepsilon)}^2 n^{\frac{2(r-1)}{r}}\Big) \tag{3.11}$$

where C_1 is independent of ε and n. Then, for every $\varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$, since F is a Carathéodory function, by either(3.6) or (3.8) and Fatou's Lemma we have

$$\int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi \le C_2\|D\varphi\|_{L^2(\Omega_n^\varepsilon)}(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{1}{1+\gamma}} + \|f\|_{L^{(2^*)'}(\Omega_n^\varepsilon)})\ (N > 2) \tag{3.12}$$

$$\int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi \le C_2\|D\varphi\|_{L^2(\Omega_n^\varepsilon)}(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{1}{1+\gamma}} n^{\frac{r-1}{r(\gamma+1)}} + \|f\|_{L^r(\Omega_n^\varepsilon)} n^{\frac{r-1}{r}})\ (N = 2) \tag{3.13}$$

where C_2 is independent of ε and n. Let us define for $\delta > 0$ the following function

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \le s \le \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \le s \le 2\delta, \\ 0, & \text{if } 2\delta \le s. \end{cases} \tag{3.14}$$

and choose in (3.4) as test function $\varphi_M Z_\delta(v_k)$ where $\varphi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$, $\varphi \geq 0$ and

$$\varphi_M = min\{\varphi, M\}$$

we get

$$\int_{\{v_k \leq \delta\}} F_k(x, v_k)\varphi_M \leq \int_{\Omega_n^\varepsilon} A(x)Dv_k D(\varphi_M Z_\delta(v_k)) \leq \int_{\Omega_n^\varepsilon} A(x)Dv_k D\varphi_M Z_\delta(v_k). \tag{3.15}$$

We deduce by (3.15), arguing as in Theorem 4.2 in [9], that

$$\limsup_\delta \limsup_k \int_{\{v_k \leq \delta\}} F_k(x, v_k)\varphi_M = 0. \tag{3.16}$$

Moreover by (3.16) we can deduce as in [9]

$$\int_{\{u_n^\varepsilon = 0\}} F(x, u_n^\varepsilon)\varphi_M = 0 \tag{3.17}$$

and by (3.17) also

$$\lim_\delta \lim_k \int_{\{v_k \geq \delta\}} F_k(x, v_k)\varphi_M = \int_{\{u_n^\varepsilon > 0\}} F(x, u_n^\varepsilon)\varphi_M = \int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi_M. \tag{3.18}$$

Then we pass to the limit as $k \to +\infty$ in (3.4) and we obtain

$$\int_{\Omega_n^\varepsilon} A(x)Du_n^\varepsilon D\varphi_M = \int_{\Omega_n^\varepsilon} F(x, u_n^\varepsilon)\varphi_M. \tag{3.19}$$

Now we easily pass to the limit as $M \to +\infty$ in the left hand side of (3.19). In the right hand side we can use Lebesgue Theorem since $F(x, u_n^\varepsilon)\varphi_M \leq F(x, u_n^\varepsilon)\varphi$ for every $M > 0$ and $F(x, u_n^\varepsilon)\varphi \in L^1(\Omega_n^\varepsilon)$ by (3.12) or (3.13) . This conclude the proof of the existence of a solution.

Step 4. Let us assume now that $h(x)$ and $f(x)$ belong to $L^r(\Omega_n^\varepsilon)$, $r > \frac{N}{2}$. We use Stampacchia method ([15]), Poincaré inequality (see (3.2)) and Sobolev embedding theorem. More precisely, given a solution u to problem (2.5) we choose as test function in (2.5) $(u-m)^+$ with $m \geq 1$. Note that, with this choice, we can confine ourselves to the set $\{u \geq m\}$ where we are far from the singularity. Hence we can prove that

$$\|u\|_{L^\infty(\Omega_n^\varepsilon)} \leq C_3, \quad C_3 = C_3(|\Omega_n^\varepsilon|, r, \|h\|_{L^r}, \|f\|_{L^r}, \alpha) \tag{3.20}$$

where C_3 is an increasing function with respect to the measure of the domain.

Step 5. Finally, let us assume (2.9) and we prove the uniqueness. We denote by u and w two possible solutions to problem (2.5).

We choose in iv) of (2.5) as test function $\phi = u - w$ (see also Remark 2.5) and we obtain

$$\int_{\Omega_n^\varepsilon} A(x)DuD(u-w) = \int_{\Omega_n^\varepsilon} F(x,u)(u-w)$$
$$\int_{\Omega_n^\varepsilon} A(x)DwD(u-w) = \int_{\Omega_n^\varepsilon} F(x,w)(u-w)$$

Taking the difference between the two equations, the uniqueness follows by the fact that the singular term is non increasing in the second variable. □

Remark 3.4. We note that if $N = 2$ and we assume the more restrictive conditions $f \in L^2(\Omega_n^\varepsilon)$ and $h \in L^{\frac{2}{1+\gamma}}(\Omega_n^\varepsilon)$ instead of (2.8)-ii) we can improve estimates (3.7) and (3.8) by using Poincaré inequality (3.1) instead of (3.2). In this case we get

$$\int_{\Omega_n^\varepsilon} |Dv_k|^2 \leq C_1\Big(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{2}{1+\gamma}} + \|f\|_{L^2(\Omega_n^\varepsilon)}^2\Big),\ r = \frac{2}{1+\gamma} \tag{3.21}$$

and

$$\int_{\Omega_n^\varepsilon} F_k(x,v_k)\varphi \leq C_2\|D\varphi\|_{L^2(\Omega_n^\varepsilon)}(\|h\|_{L^r(\Omega_n^\varepsilon)}^{\frac{1}{1+\gamma}} + \|f\|_{L^2(\Omega_n^\varepsilon)}),\ r = \frac{2}{1+\gamma}. \tag{3.22}$$

We deal now with Problem (2.15).

With the notations of Section 2, following the outline of Theorems 4.1, 4.3 and 4.4 of [9], we can prove the following result. We note that, in view of the periodic conditions, the space $H^1_{0,per}(Q^\varepsilon, \partial\omega)$ defined by (2.14) can be identified with the space $H^1_0(Q^\varepsilon_\#)$ where $Q^\varepsilon_\#$ is N-dimensional "torus".

Theorem 3.5. *Assume that the matrix A and the function F satisfy (2.6), (2.7) and (2.8),where Ω_n is replaced by Q^ε. Then there exists a solution u^ε_∞ to problem (2.15). If in addition we assume $h(x) \in L^r(Q^\varepsilon)$, and $f(x) \in L^r(Q^\varepsilon)$, $r > \frac{N}{2}$, then any solution u^ε_∞ belongs to $L^\infty(Q^\varepsilon)$. Moreover if we assume (2.9) then the solution is unique.*

Remark 3.6. Actually condition (2.8) iv) can be weakened allowing more general functions $F(x,s)$ whose growth in the s variable may be different in different regions of the domain, for example functions of the type

$$0 \leq F(x,s) \leq \frac{h(x)}{s^\gamma} + \frac{g(x)}{s^\delta} + f(x) \text{ a.e. } x \in \Omega_n, \forall s > 0$$

where $1 \geq \gamma > \delta > 0$ and h, g and f are nonnegative functions such that

$$
(3.23) \qquad \begin{cases} h \in L^r(\Omega_n), \ \text{ with } r \geq \frac{2^*}{2^*-1+\gamma} \text{ if } N > 2 \\ g \in L^q(\Omega_n), \ \text{ with } q \geq \frac{2^*}{2^*-1+\delta} \text{ if } N > 2 \\ f \in L^p(\Omega_n) \ \text{ with } p = (2^*)' \text{ if } N > 2 \\ \text{or } h, g, f \in L^p(\Omega_n) \text{ with } p > 1 \text{ if } N = 2. \end{cases}
$$

A simple example is

$$
F(x,s) = \frac{h(x)}{s^\gamma}\left(2 + \sin\frac{1}{s}\right) + \frac{g(x)}{s^\delta} + f(x) \text{ a.e. } x \in \Omega_n, \ \forall s > 0,
$$

with h, g, f satisfying (3.23).

Remark 3.7. By the strong maximum principle (see Remark 2.2) either u_n^ε is identically zero or $u_n^\varepsilon > 0$ a.e. in Ω_n^ε; the same holds true for u_∞^ε in Q^ε (see Theorem 8.17 in [10]). Note that u_n^ε, as well as u_∞^ε, is identically zero if and only if $F(x,0) = 0$, by Remark 2.3.

Remark 3.8. Comparing the summability exponents for the data which appear in Theorem 3.3 and Theorem 3.5, we note that, if $N = 2$, the solutions u_n^ε and u_∞^ε to problems (2.5) and (2.15) are bounded. If instead $N > 2$ the summability exponent of the function h, $\frac{2^*}{2^*-1+\gamma}$ is always strictly less than $\frac{N}{2}$.

4 Homogenization results

We state now the homogenization results for problems studied in the previous sections.

We consider also the perforated domains Ω_n^ε obtained by removing some closed sets T_i^ε of $\mathbb{R}^N$, $1 \leq i \leq \nu(\varepsilon)$ from Ω_n. The domain is defined in (2.2)

We assume that the sequence of the domains Ω_n^ε is such that there exist a sequence of functions w^ε, a distribution $\mu \in \mathcal{D}'(\mathbb{R}^N)$ and such that

$$
(4.1) \qquad w^\varepsilon \in H^1(\Omega_n),
$$

$$
(4.2) \qquad 0 \leq w^\varepsilon \leq 1 \text{ a.e. } x \in \Omega_n,
$$

$$
(4.3) \qquad w^\varepsilon = 0 \text{ on } \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon,
$$

$$
(4.4) \qquad w^\varepsilon \rightharpoonup 1 \text{ in } H^1(\Omega_n) \text{ weakly, in } L^\infty(\Omega_n) \text{ weakly-star,}
$$

$$\mu = \mu_n \in (H_0^1(\Omega_n, \partial\omega))', \tag{4.5}$$

$$\begin{cases} \displaystyle\int_{\Omega_n} {}^tA(x)Dw^\varepsilon D(\phi v_\varepsilon) \to \langle \mu, \phi v\rangle_{(H_0^1(\Omega_n,\partial\omega))', H_0^1(\Omega_n,\partial\omega)} \\ \text{for every } v^\varepsilon \in H^1(\Omega_n),\ v_\varepsilon = 0 \text{ on } \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon \\ \text{such that } v^\varepsilon \rightharpoonup v \text{ weakly in } H^1(\Omega_n) \\ \text{for every } \phi \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n). \end{cases} \tag{4.6}$$

Remark 4.1. We note that, by choosing $v^\varepsilon = w^\varepsilon$, $v = 1$ and $\phi \geq 0$, $\phi \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n)$, we can prove, as in Proposition 1.1 of [3], that μ is a linear positive functional on $H_0^1(\Omega_n, \partial\omega)$. Let us observe that, by Riesz' Theorem any linear positive functional on $H_0^1(\Omega_n, \partial\omega)$ is a nonnegative Radon measure on Ω_n. To this aim we can revised the proof of the classical Riesz' Theorem (see Theorem 2.14 of [13]) by replacing the space $C_0(\Omega_n)$ by the space $C^*(\overline{\Omega_n})$ of the functions in $C^1(\overline{\Omega_n})$ whose support has positive distance from the lateral boundary $[-n, n] \times \partial\omega$ and by choosing in the definition of the measure relative open sets in $\overline{\Omega_n}$.

Assumptions (4.1),(4.2) (4.3), (4.4), (4.5) and (4.6) are satisfied in the model case described below. The matrix $A(x)$ is the identity (therefore the operator $-div\, A(x)D$ is $-\Delta$). In $\mathbb{R}^N, N \geq 2$ we introduce a periodically distributed N-dimensional cubic lattice of cubes of size 2ε, P_i^ε such that 0 belongs to the set of the vertices of the lattice. In any cube P_i^ε we put a ball of radius ε, B_i^ε (centered in the center of the cube) and the hole T_i^ε is the balls of radius r^ε concentric with B_i^ε with r^ε given by

$$\begin{cases} r^\varepsilon = C_0 \varepsilon^{N/(N-2)} \text{ if } N \geq 3, \\ r^\varepsilon = \exp -\dfrac{C_0}{\varepsilon^2} \text{ if } N = 2, \end{cases}$$

for some C_0. The measure μ is given by

$$\begin{cases} \mu = \dfrac{S_N(N-2)}{2^N} C_0^{N-2} \text{ if } N \geq 3, \\ \mu = \dfrac{2\pi}{4}\dfrac{1}{C_0} \text{ if } N = 2. \end{cases}$$

where S_N denotes the area of the unit sphere in $\mathbb{R}^N$. According to [3] we can construct the sequence w_ε by setting

$$\begin{cases} w_\varepsilon = 0 \text{ in } T_i^\varepsilon \\ \triangle w_\varepsilon = 0 \text{ in } B_i^\varepsilon \setminus T_i^\varepsilon \\ w_\varepsilon = 1 \text{ in } P_i^\varepsilon \setminus B_i^\varepsilon. \end{cases} \tag{4.7}$$

w_ε continuous at interface. In the quoted paper [3] the authors prove that

$$\begin{cases} -\triangle w_\varepsilon = \mu_\varepsilon - \gamma_\varepsilon \\ \mu_\varepsilon \to \mu \text{ strongly in } H^{-1}(\Omega_n) \\ \langle \gamma_\varepsilon, v_\varepsilon \rangle = 0 \ \forall v^\varepsilon \in H_0^1(\Omega_n), \ v^\varepsilon = 0 \text{ in } T_i^\varepsilon. \end{cases}$$

where the measure μ_ε denotes the mass supported by ∂B_i^ε and γ^ε the mass supported by ∂T_i^ε.
Note that our test functions do not vanish on the subsets of the boundary given by $\{n\} \times \omega$ and $\{-n\} \times \omega$ hence by applying the Green formula to the integral in (4.6), two boundary integrals appear that may diverge as ε goes to zero. In order to avoid this situation we suppose that the parameters ε and n of the domain Ω_n^ε satisfy the further conditions $n \in \mathbb{N}$ and $\varepsilon = \frac{1}{m}$, $m \in \mathbb{N}$. In fact if $\varepsilon = \frac{1}{2m}$, $m \in \mathbb{N}$, the sets $\{n\} \times \omega \cap \bar{P}_i^\varepsilon$ and $\{-n\} \times \omega \cap \bar{P}_i^\varepsilon$ are subsets of a face of the cube P_i^ε, then $w^\varepsilon = 1$ (see (4.7)) and the integral vanishes. Moreover, if $\varepsilon = \frac{1}{2m+1}$, then the sets $\{n\} \times \omega \cap \bar{P}_i^\varepsilon$ and $\{-n\} \times \omega \cap \bar{P}_i^\varepsilon$ are subsets either of a face of the cube P_i^ε or of the median section of the cube P_i^ε. In the latter case the boundary integrals vanish since the normal to the boundary is orthogonal to the radial direction of the annulus $B_i^\varepsilon \setminus T_i^\varepsilon \cap \{x_1 = n\}$. In Figure 1 we have $\varepsilon = \frac{1}{2}$ and we see the lattice of size 2ε in grey, the holes T_i^ε in red and the boundary of the domain $\partial\Omega_n^\varepsilon$ in green. In Figure 2 we have $\varepsilon = \frac{1}{3}$.

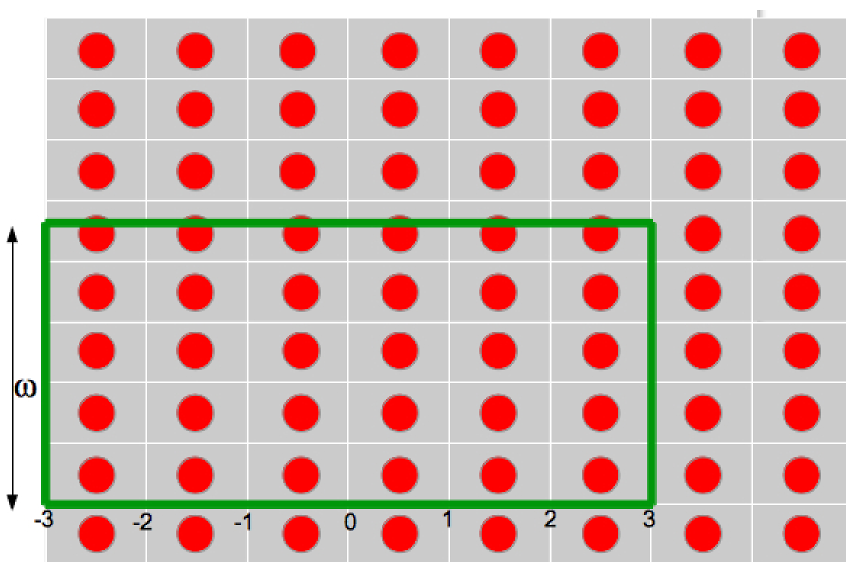


Figure 1: the lattice and the domain Ω_n^ε, $n = 3$, $\varepsilon = \frac{1}{2}$.

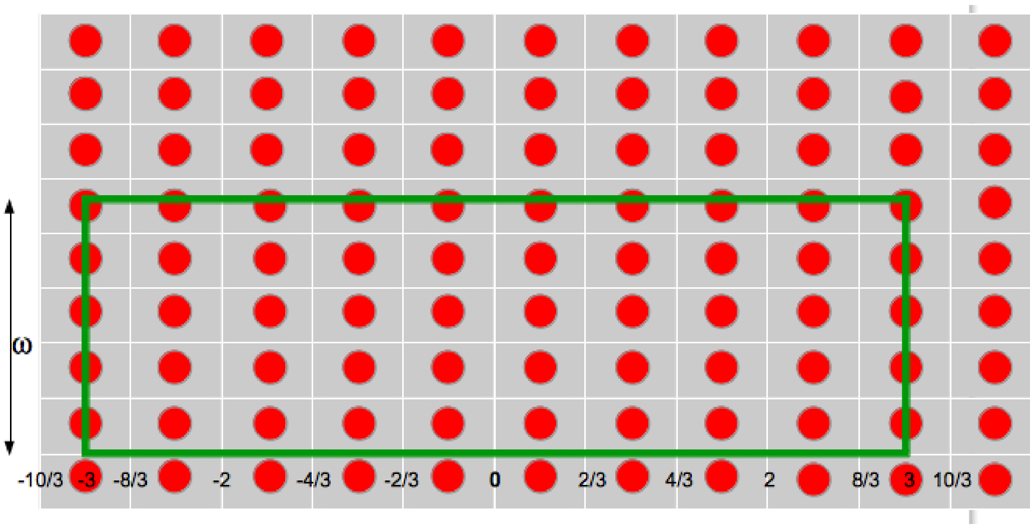


Figure 2: the lattice and the domain Ω_n^ε, $n = 3$, $\varepsilon = \frac{1}{3}$.

In the present paper, for every function z^ε in $H_0^1(\Omega_n^\varepsilon, \partial\omega)$ (see(1)) we denote by $\tilde{z}^\varepsilon$ the extension to Ω_n:

$$\tilde{z}^\varepsilon(x) = \begin{cases} z^\varepsilon(x) & \text{in } \Omega_n^\varepsilon, \\ 0 & \text{in } \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon \cap \Omega_n, \end{cases} \tag{4.8}$$

then $\tilde{z}^\varepsilon \in H_0^1(\Omega_n, \partial\omega)$, $\|\tilde{z}^\varepsilon\|_{L^2(\Omega_n)} = \|z^\varepsilon\|_{L^2(\Omega_n^\varepsilon)}$ and $\|D\tilde{z}^\varepsilon\|_{L^2(\Omega_n)} = \|Dz^\varepsilon\|_{L^2(\Omega_n^\varepsilon)}$.

Theorem 4.2. *Let the matrix A and the function F satisfy* (2.6),(2.7), (2.8). *Let the sequence of perforated sets Ω_n^ε fulfill* (4.1), (4.2), (4.3), (4.4), (4.5) *and* (4.6) *and denote by u_n^ε a solution to problem* (2.5). *Then, for any fixed n, the sequence $\tilde{u}_n^\varepsilon$ defined by* (4.8), *has a subsequence (still denoted by $\tilde{u}_n^\varepsilon$) satisfying (as $\varepsilon \to 0$)*

$$\tilde{u}_n^\varepsilon \rightharpoonup u_n^0 \ \text{ in } H_0^1(\Omega_n, \partial\omega) \text{ weakly,}$$

where u_n^0 is a solution to

$$\begin{cases} i)\, u_n^0 \in H_0^1(\Omega_n, \partial\omega) \cap L^2(\Omega_n, d\mu), \\ ii)\, u_n^0(x) \geq 0 \ a.e.\ x \in \Omega_n, \\ iii) \displaystyle\int_{\Omega_n} F(x, u_n^0)\varphi < +\infty, \\ iv) \displaystyle\int_{\Omega_n} A(x)Du_n^0 D\varphi + \int_{\Omega_n} u_n^0 \varphi d\mu = \int_{\Omega_n} F(x, u_n^0)\varphi \\ \forall \varphi \in H_0^1(\Omega_n, \partial\omega) \cap L^2(\Omega_n, d\mu),\ \varphi \geq 0. \end{cases} \tag{4.9}$$

If in addition we assume $h(x) \in L^r(\Omega_n)$, and $f(x) \in L^r(\Omega_n)$, $r > \frac{N}{2}$, then the solutions $\tilde{u}_n^\varepsilon$ and u_n^0 belong to $L^\infty(\Omega_n)$ and, up to a subsequence,

$$\tilde{u}_n^\varepsilon \rightharpoonup u_n^0 \ \text{ in } L^\infty(\Omega_n) \quad \text{weakly-star.}$$

Moreover if we assume (2.9) *then the solution of* (4.9) *is unique. As a consequence the whole sequence $\tilde{u}_n^\varepsilon$ converges to u_n^0.*

We note that we cannot apply directly Theorem 5.1 of [9], because we prescribe different boundary conditions. More precisely the functions in the space $H_0^1(\Omega_n^\varepsilon, \partial\omega)$ need to vanish only on the lateral boundary $(-n, n) \times \partial\omega$ while the setting for Theorem 5.1 of [9] is the space $H_0^1(\Omega_n^\varepsilon)$. Actually the proof is similar, modulo some slight modifications. However we prefer to present here the modified proof for sake of completeness.

Proof. We split our proof in six steps.
We point out that in this proof n is fixed. Note that the function $\tilde{u}_n^\varepsilon$ defined in (4.8) satisfies, in view of (3.10) or (3.11),

$$\|\tilde{u}_n^\varepsilon\|_{H_0^1(\Omega_n, \partial\omega)} \le C_1 \tag{4.10}$$

where C_1 does not depend on ε.
Then, up to a subsequence, we have

$$\tilde{u_n^\varepsilon} \rightharpoonup u_n^0 \quad \text{in } H_0^1(\Omega_n, \partial\omega) \text{ and } \tilde{u_n^\varepsilon} \to u_n^0 \quad \text{a.e. in } \Omega_n. \tag{4.11}$$

In particular condition ii) in (4.9) is satisfied.

Step 1

In view of assumptions (4.1), (4.2) and (4.3), one has

$$w^\varepsilon \psi \in H_0^1(\Omega_n^\varepsilon, \partial\omega) \cap L^\infty(\Omega_n^\varepsilon), \ \forall \psi \in H_0^1(\Omega_n, \partial\omega) \cap L^\infty(\Omega_n),$$

and

$$\|w^\varepsilon \psi\|_{H_0^1(\Omega_n^\varepsilon, \partial\omega)} \le C_4(\|D\psi\|_{L^2(\Omega_n)} + \|\psi\|_{L^\infty(\Omega_n)}),$$

where

$$C_4 = \max_\varepsilon\{1, \|Dw^\varepsilon\|_{L^2(\Omega_n)}\}.$$

We now fix $\psi \in H_0^1(\Omega_n, \partial\omega) \cap L^\infty(\Omega_n)$, $\psi \ge 0$, and we use $\varphi^\varepsilon = w^\varepsilon \psi \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$ as test function in (2.5). We obtain using (4.8)

$$\int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} D\psi \, w^\varepsilon + \int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} Dw^\varepsilon \, \psi = \int_{\Omega_n} \widetilde{F(x, u_n^\varepsilon)} w^\varepsilon \psi, \tag{4.12}$$

where by $\widetilde{F(\cdot, u_n^\varepsilon(\cdot))}$ denote the extension to Ω_n by zero on the holes, as in (4.8). Equation (4.12) in particular implies by (4.4) and (4.10) that

$$\int_{\Omega_n} \widetilde{F(x, u_n^\varepsilon)} w^\varepsilon \psi \le C_5 \tag{4.13}$$

where C_5 is independent of ε.

We now claim that for a subsequence, still labelled by ε,

$$\chi_{\Omega_n^\varepsilon} \to 1 \text{ a.e. in } \Omega_n; \tag{4.14}$$

indeed, from $w^\varepsilon \chi_{\Omega_n^\varepsilon} = w^\varepsilon$ a.e. in Ω_n, which results from (4.3), (4.2) and (4.4) we get

$$\begin{cases} \chi_{\Omega_n^\varepsilon} = \chi_{\Omega_n^\varepsilon} w^\varepsilon + \chi_{\Omega_n^\varepsilon}(1 - w^\varepsilon) = w^\varepsilon + \chi_{\Omega_n^\varepsilon}(1 - w^\varepsilon) \to 1 \\ \text{in } L^2(\Omega_n) \text{ strongly.} \end{cases}$$

We deduce from (4.14) and (4.11) that

$$\widetilde{F(x, u_n^\varepsilon)} \to F(x, u_n^0) \text{ a.e. } x \in \Omega. \tag{4.15}$$

Using (4.13), (4.4) and (4.15) and applying Fatou's Lemma implies that

$$\int_\Omega F(x, u_n^0)\psi < +\infty \ \ \forall \psi \in H_0^1(\Omega_n, \partial\omega) \cap L^\infty(\Omega_n), \ \psi \ge 0. \tag{4.16}$$

Step 2 Equation (4.12) can be rewritten, for any $\phi \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n), \phi \ge 0$, as

(4.17)
$$\begin{cases} \displaystyle\int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} D\phi \, w^\varepsilon + \int_{\Omega_n} {}^t\!A(x) Dw^\varepsilon D(\phi \tilde{u_n^\varepsilon}) - \int_{\Omega_n} {}^t\!A(x) Dw^\varepsilon D\phi \, \tilde{u_n^\varepsilon} = \\ \displaystyle = \int_{\Omega_n} \widetilde{F(x, u_n^\varepsilon)} w^\varepsilon \phi. \end{cases}$$

Using (4.11), (4.3), (4.4) and (4.6), we can easily pass to the limit in the left-hand side of (4.17), and we obtain

(4.18)
$$\begin{cases} \displaystyle\int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} D\phi \, w^\varepsilon + \int_{\Omega_n} {}^t\!A(x) Dw^\varepsilon D(\phi \tilde{u_n^\varepsilon}) - \int_{\Omega_n} {}^t\!A(x) Dw^\varepsilon D\phi \, \tilde{u_n^\varepsilon} \to \\ \displaystyle \to \int_{\Omega_n} A(x) Du_n^0 D\phi + \langle \mu, u_n^0 \phi \rangle_{(H_0^1(\Omega_n, \partial\omega))', H_0^1(\Omega_n, \partial\omega)} \end{cases}$$

Let us observe that $\mu \in (H_0^1(\Omega_n, \partial\omega))'$ and, by Riesz' Theorem, it is a nonnegative Radon measure on Ω_n (see Remark 4.1). Moreover any function of $H^1(\Omega_n)$ is defined μ- a.e. and it is μ- measurable for any nonnegative Borel measure μ which does not charge Borel sets of zero capacity (see [6]). Then we can write

$$\langle \mu, u_n^0 \phi \rangle_{(H_0^1(\Omega_n,\partial\omega))',H_0^1(\Omega_n,\partial\omega)} = \int_{\Omega_n} u_n^0 \phi d\mu \tag{4.19}$$

for any $\phi \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n)$.

Step 3

We split for any $\delta > 0$ the right-hand side of (4.17) as

$$\int_{\Omega_n} \widetilde{F(x,u_n^\varepsilon)} w^\varepsilon \phi = \int_{\Omega_n} \widetilde{F(x,u_n^\varepsilon)} w^\varepsilon \phi \chi_{\{0\le \tilde{u^\varepsilon} \le \delta\}} + \int_{\Omega_n} \widetilde{F(x,u_n^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u_n^\varepsilon} > \delta\}}. \tag{4.20}$$

We deal with the first term in the right-hand side of (4.20) and we use $\phi^\varepsilon = w^\varepsilon \phi Z_\delta(u_n^\varepsilon)$ as test function in (2.5), where $Z_\delta(s)$ is defined by (3.14) and where $\phi \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n)$, $\phi \ge 0$. We get, by proceeding as in the fourth step of the proof of Theorem 5.1 in [9]

$$\begin{cases} \displaystyle\int_{\Omega_n} \widetilde{F(x,u_n^\varepsilon)} w^\varepsilon \phi \chi_{\{0\le \tilde{u}_n^\varepsilon \le \delta\}} \le \\ \displaystyle\le \int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} Dw^\varepsilon\, \phi Z_\delta(\tilde{u_n^\varepsilon}) + \int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} D\phi\, w^\varepsilon Z_\delta(\tilde{u_n^\varepsilon}). \end{cases} \tag{4.21}$$

Let us define the function $Y_\delta(s)$ by

$$Y_\delta(s) = \int_0^s Z_\delta(\sigma) d\sigma, \quad \forall s \ge 0,$$

and observe that $Y_\delta(u_n^\varepsilon) \in H_0^1(\Omega_n^\varepsilon, \partial\omega)$, then

$$\begin{cases} \displaystyle\int_{\Omega_n} A(x) D\tilde{u_n^\varepsilon} Dw^\varepsilon \phi Z_\delta(\tilde{u_n^\varepsilon}) = \int_{\Omega_n} {}^tA(x) Dw^\varepsilon DY_\delta(\tilde{u_n^\varepsilon}) \phi = \\ \displaystyle\int_{\Omega_n} {}^tA(x) Dw^\varepsilon D(\phi Y_\delta(\tilde{u_n^\varepsilon})) - \int_{\Omega_n} {}^tA(x) Dw^\varepsilon D\phi\, Y_\delta(\tilde{u_n^\varepsilon}). \end{cases} \tag{4.22}$$

Using now (4.6), (4.11), the fact that

$$Y_\delta(\tilde{u_n^\varepsilon}) \rightharpoonup Y_\delta(u_n^0) \text{ in } H^1(\Omega_n) \text{ weakly},$$

and (4.4) proves that the right-hand side of (4.22) tends to

$$\langle \mu, \phi Y_\delta(u_n^0) \rangle_{(H_0^1(\Omega_n,\partial\omega))',H_0^1(\Omega_n,\partial\omega)}$$

as ε tends to zero for $\delta > 0$ fixed. Turning back to (4.21), passing to the limit in the last term of (4.21), and using (4.22) and the latest result, we have proved that for every $\delta > 0$ fixed

$$(4.23)\quad \begin{cases} \limsup_{\varepsilon} \int_{\Omega_n} \widetilde{F(x, u_n^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u}_n^\varepsilon \le \delta\}} \le \\ \le \langle \mu, \phi Y_\delta(u_n^0) \rangle_{(H_0^1(\Omega_n, \partial\omega))', H_0^1(\Omega_n, \partial\omega)} + \int_{\Omega_n} A(x) D u_n^0 D\phi \, Z_\delta(u_n^0). \end{cases}$$

We now pass to the limit in (4.23) as δ tends to zero.

For the first term of the right-hand side of (4.23), we use (4.19) and the fact that $0 \le Y_\delta(s) \le \frac{3}{2}\delta$ for every $s \ge 0$; we get

$$0 \le \langle \mu, \phi Y_\delta(u_n^0) \rangle_{(H_0^1(\Omega_n, \partial\omega))', H_0^1(\Omega_n, \partial\omega)} = \int_{\Omega_n} \phi Y_\delta(u_n^0) d\mu \le \frac{3}{2}\delta \int_{\Omega_n} \phi d\mu \to 0 \text{ as } \delta \to 0.$$

For the second term of the right-hand side of (4.23) we have

$$\int_{\Omega_n} A(x) D u_n^0 D\phi Z_\delta(u_n^0) \to \int_{\Omega_n} A(x) D u_n^0 D\phi \chi_{\{u_n^0 = 0\}} = 0 \text{ as } \delta \to 0.$$

Hence we proved that

$$(4.24)\quad \lim_{\delta} \limsup_{\varepsilon} \int_{\Omega_n} \widetilde{F(x, u_n^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u}_n^\varepsilon \le \delta\}} = 0.$$

We note that passing to the limit in the second term of the right-hand side of (4.20) is easier since we are far away from the singularity. We refer to the fifth step of the proof of Theorem 5.1 in [9] and by choosing conveniently δ we get

$$(4.25)\quad \lim_{\delta} \lim_{\varepsilon} \int_{\Omega_n} \widetilde{F(x, u_n^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u_n^\varepsilon} > \delta\}} = \int_{\Omega_n} F(x, u_n^0) \phi \chi_{\{u_n^0 > 0\}} = \int_{\Omega_n} F(x, u_n^0) \phi.$$

Step 4

We come back to (4.17). Collecting together (4.18), (4.19), (4.20), (4.24) and (4.25) we have proved that

$$(4.26)\quad \begin{cases} \forall \phi \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n), \phi \ge 0, \\ \int_{\Omega_n} A(x) D u_n^0 D\phi + \int_{\Omega_n} u_n^0 \phi d\mu = \int_{\Omega_n} F(x, u_n^0) \phi. \end{cases}$$

Let us now take $\psi \in H_0^1(\Omega_n, \partial\omega) \cap L^\infty(\Omega_n), \psi \ge 0$.

Consider a sequence ψ_m such that

$$\begin{cases} \psi_m \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n),\ \psi_m \geq 0,\ \|\psi_m\|_{L^\infty(\Omega_n)} \leq C, \\ \operatorname{supp} \psi_m \subset [-n,n] \times \omega, \\ \psi_m \to \psi\ \ H^1(\Omega_n) \text{ and quasi-everywhere in } \Omega_n. \end{cases}$$

Define

$$\hat{\psi}_m = min\{\psi_m, \psi\};$$

then

$$\begin{cases} \hat{\psi}_m \in H_0^1(\Omega_n, \partial\omega) \cap L^\infty(\Omega_n),\ \hat{\psi}_m \geq 0,\ \|\hat{\psi}_m\|_{L^\infty(\Omega_n)} \leq C, \\ \operatorname{supp} \hat{\psi}_m \subset \operatorname{supp} \psi_m \subset [-n,n] \times \omega, \\ \hat{\psi}_m \to \psi\ H^1(\Omega_n) \text{ and quasi-everywhere in } \Omega_n. \end{cases}$$

For the moment let m be fixed, let ρ_η be a sequence of mollifiers and denote by $\star$ the convolution operator. We extend the function $\hat{\psi}_m$ to $\mathbb{R}^N$ and we denote the extension with the same symbol $\hat{\psi}_m$. Due to the geometry of our domain, by using a procedure of reflexion and cut-off functions, we can assume that, for η sufficiently small, the support of $\chi_{\Omega_n} \cdot \hat{\psi}_m \star \rho_\eta$ is included in a fixed compact $K_m \subset [-n,n] \times \omega$, and $\hat{\psi}_m \star \rho_\eta \in H_0^1(\Omega_n, \partial\omega) \cap W^{1,\infty}(\Omega_n)$, $\hat{\psi}_m \star \rho_\eta \geq 0$. We can therefore use $\phi = \hat{\psi}_m \star \rho_\eta$ as test function in (4.26). We get

(4.27)

$$\int_{\Omega_n} A(x) D u_n^0 D(\hat{\psi}_m \star \rho_\eta) + \int_{\Omega_n} u_n^0 (\hat{\psi}_m \star \rho_\eta) d\mu = \int_{\Omega_n} F(x, u_n^0)(\hat{\psi}_m \star \rho_\eta).$$

Let us pass to the limit in each term of this equation (for m fixed) as η tends to zero. In the right-hand side we use the estimate (4.16), the fact that $\|\hat{\psi}_m \star \rho_\eta\|_{L^\infty(\Omega_n)} \leq \|\hat{\psi}_m\|_{L^\infty(\Omega_n)}$, and the almost convergence of $\hat{\psi}_m \star \rho_\eta$ to $\hat{\psi}_m$ together with Lebesgue's dominated convergence Theorem.

In the first term of the left-hand side we use the strong convergence of $\hat{\psi}_m \star \rho_\eta$ to $\hat{\psi}_m$ in $H^1(\Omega_n)$. As far as the second term in the left-hand side of (4.27), we note that this strong convergence implies (for a subsequence) the quasi-everywhere convergence and therefore the μ-almost everywhere convergence of $\hat{\psi}_m \star \rho_\eta$ to $\hat{\psi}_m$. We use again Lebesgue's dominated convergence Theorem, this time in $L^1(\Omega_n; d\mu)$, and the fact that (see Section 2.2 in [6])

$$0 \leq u_n^0 (\hat{\psi}_m \star \rho_\eta) \leq u_n^0 \|\hat{\psi}_m\|_{L^\infty(\Omega_n; d\mu)} = u_n^0 \|\hat{\psi}_m\|_{L^\infty(\Omega_n)}\ \ \mu\text{-a.e. } x \in \Omega_n.$$

Hence we pass to the limit in the second term of the left-hand side. We have proved that

$$\int_{\Omega_n} A(x)Du_n^0 D\hat{\psi}_m + \int_{\Omega_n} u_n^0\hat{\psi}_m d\mu = \int_{\Omega_n} F(x,u_n^0)\hat{\psi}_m. \tag{4.28}$$

We now pass to the limit in each term of (4.28) as m tends to infinity. This is easy in the right-hand side by Lebesgue's dominated convergence Theorem since $\hat{\psi}_m$ tends almost everywhere to ψ, since

$$0 \le F(x,u_n^0)\hat{\psi}_m \le F(x,u_n^0)\psi \text{ a.e. } x \in \Omega_n,$$

and since the latest function belongs to $L^1(\Omega_n)$ (see (4.16)). This is also easy in the first term of the left-hand side of (4.28) since $\hat{\psi}_m$ tends to ψ strongly in $H^1(\Omega)$. Also $\hat{\psi}_m$ converges to ψ quasi-everywhere, therefore μ-almost everywhere and we easily pass to the limit in the second term of the left-hand side of (4.28) by Lebesgue's dominated convergence Theorem since (see Section 2.2 in [6])

$$0 \le u_n^0\hat{\psi}_m \le u^0\psi \le u_n^0\|\psi\|_{L^\infty(\Omega_n;d\mu)} = u_n^0\|\psi\|_{L^\infty(\Omega_n)} \ \ \mu\text{-a.e. } x \in \Omega_n$$

and since $u_n^0 \in H^1(\Omega_n) \subset L^1(\Omega_n; d\mu)$ (see Section 2.2 in [6]).

We have proved that

$$\begin{cases} \forall\psi \in H_0^1(\Omega_n,\partial\omega)\cap L^\infty(\Omega_n), \psi \ge 0, \\ \displaystyle\int_{\Omega_n} A(x)Du_n^0 D\psi + \int_{\Omega_n} u_n^0\psi d\mu = \int_{\Omega_n} F(x,u_n^0)\psi. \end{cases} \tag{4.29}$$

Step 5

Let us finally prove that $u_n^0 \in L^2(\Omega_n; d\mu)$. We set for any $m > 0$ $T_m(v) = min\{v,m\}$. Taking $\psi = T_m(u_n^0) \in H_0^1(\Omega_n,\partial\omega)\cap L^\infty(\Omega_n)$ in (4.29) and using the coercitivity (2.6) of A, the growth condition (2.8 *iv*) of F and Fatou's Lemma we obtain

$$u_n^0 \in L^2(\Omega_n; d\mu).$$

This allows to use nonnegative test functions in $H_0^1(\Omega_n,\partial\omega)\cap L^2(\Omega_n,d\mu)$ and therefore (4.9) holds true.

Step 6 If we assume $h(x) \in L^r(\Omega_n)$, and $f(x) \in L^r(\Omega_n)$, $r > \frac{N}{2}$, by (3.20) we deduce that u_n^0 belongs to $L^\infty(\Omega_n)$ and the sequence converges, up to a subsequence, also in $L^\infty(\Omega_n)$ weakly-star.

Finally assuming (2.9) we prove the uniqueness of the solution in the same way as in Step 5 of the proof of Theorem 3.3, noting that the measure μ is nonnegative. The proof of Theorem 4.2 is now complete. □

Remark 4.3. We note that conditions (4.5) and (4.6) differ from the classical ones given in [3] and in [9] i.e.

$$\mu \in H^{-1}(\mathcal{O}), \tag{4.30}$$

$$\begin{cases} \displaystyle\int_{\Omega_n} {}^tA(x)Dw^\varepsilon D(\phi v_\varepsilon) \to \langle \mu, \phi v\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})} \\ \text{for every } v^\varepsilon \in H^1(\mathcal{O}),\ v_\varepsilon = 0 \text{ on } \bigcup_{i=1}^{\nu(\varepsilon)} T_i^\varepsilon, \\ \text{such that } v^\varepsilon \rightharpoonup v \text{ weakly in } H^1(\mathcal{O}) \\ \text{for every } \phi \in \mathcal{D}(\mathcal{O}). \end{cases} \tag{4.31}$$

Actually if we assume (4.1), (4.2), (4.3), (4.4), (4.30) and (4.31) we are able to prove that the function u_n^0 is a solution in a weaker sense because under the previous assumptions the space of the admissible test functions (for equation iv in (4.9)) is the space $H_0^1(\Omega_n) \cap L^2(\Omega_n, d\mu)$ that is a proper subspace of $H_0^1(\Omega_n, \partial\omega) \cap L^2(\Omega_n, d\mu)$. Moreover, even assuming condition (2.9), uniqueness may fail.

Next theorem concerns the periodic case. In the periodic case the natural conditions are (4.30) and (4.31) where the domain $\mathcal{O}$ is replaced by the domain Q (and the sequence $\mathcal{O}^\varepsilon$ by the sequence Q^ε). Indeed in this case the space $H^1_{0,per}(Q, \partial\omega)$ defined in (2.13) can be identified with the space $H_0^1(Q_\#)$ where $Q_\#$ is N-dimensional torus.

From now on, according to (4.8), we denote by $\tilde{u}^\varepsilon_\infty$ the extension to Q of u^ε_∞ (by zero on the holes).

Theorem 4.4. *Assume that the matrix A and the function F satisfy* (2.6)*,* (2.7)*,* (2.8)*. Assume also that the sequence of perforated sets Q^ε fulfills* (4.1)*,* (4.2)*,* (4.3)*,* (4.4)*,* (4.30) *and* (4.31) *and denote by u^ε_∞ a solution to problem* (2.15)*. Then the sequence $\tilde{u}^\varepsilon_\infty$ has a subsequence (still denoted by $\tilde{u}^\varepsilon_\infty$) satisfying (as $\varepsilon \to 0$)*

$$\tilde{u}^\varepsilon_\infty \rightharpoonup u^0_\infty \ \text{ in } H^1_{0,per}(Q, \partial\omega) \text{ weakly and a.e. in } Q$$

where u^0_∞ is a solution of

$$\begin{cases} i)\ u^0_\infty \in H^1_{0,per}(Q, \partial\omega) \cap L^2(Q, d\mu), \\ ii)\ u^0_\infty(x) \geq 0 \text{ a.e. } x \in Q, \\ iii)\ \displaystyle\int_Q F(x, u^0_\infty)\varphi < +\infty \\ iv)\ \displaystyle\int_Q A(x)Du^0_\infty D\varphi + \int_Q u^0_\infty \varphi d\mu = \int_Q F(x, u^0_\infty)\varphi \\ \forall \varphi \in H^1_{0,per}(Q, \partial\omega) \cap L^2(Q, d\mu),\ \varphi \geq 0. \end{cases} \tag{4.32}$$

If in addition we assume $h(x) \in L^r(Q)$ *and* $f(x) \in L^r(Q)$, $r > \frac{N}{2}$, *then the solutions* $\tilde{u}^\varepsilon_\infty$ *and* u^0_∞ *belong to* $L^\infty(Q)$ *and, up to a subsequence,*

$$\tilde{u}^\varepsilon_\infty \rightharpoonup u^0_\infty \text{ in } L^\infty(Q) \text{ weakly-star.}$$

Moreover if we assume (2.9) *then problem* (2.15) *as well as problem* (4.32) *admit unique solution,* u^ε_∞ *and* u^0_∞ *respectively. As a consequence the whole sequence* $\tilde{u}^\varepsilon_\infty$ *converges to* u^0_∞*.*

Proof. The proof can be carried on as in Theorem 5.1 of [9] as, in view of the periodic conditions, the space $H^1_{0,per}(Q, \partial\omega) \cap L^2(Q, d\mu)$ can be identified with the space $H^1_0(Q_\#) \cap L^2(Q_\#, d\mu)$ where $Q_\#$ is the N-dimensional torus. □

Remark 4.5. Once we got iv) in (4.9), actually the equation holds true for any $\varphi \in H^1_0(\Omega_n, \partial\omega) \cap L^2(\Omega_n, d\mu)$. The same remark holds true for Problem (4.32).

Remark 4.6. Note that (2.19) applied to the function u^0_∞ and to the domain Q gives

$$\begin{cases} \{x \in Q : u^0_\infty(x) = 0\} \subset \{x \in Q : F(x, 0) = 0\} \\ \text{except for a set of zero measure.} \end{cases} \tag{4.33}$$

5 T-periodic problems in one direction.

In this section we assume that the matrix A, the datum $F(x, s)$ (for any fixed s) and the functions h and f in (2.8) are 1- periodic in the x_1-direction in the set $\Omega_\infty = \mathbb{R} \times \omega$.

We are interested to find conditions on the domains Ω^ε_n in order to force any sequence of solutions u^ε_n to problems (2.5) to converge to a solution u^ε_∞ to problem (2.15).

Our choice of the domain Ω^ε_n should guarantee that Ω^ε_n is *union* of $2n$ copies of Q^ε.

A possible framework is the following.

Let Ω_n and Q be as in (2.1) and (2.11), i.e.

$$\Omega_n = (-n, n) \times \omega, \; Q = (0, 1) \times \omega,$$

where ω is an open bounded subset in $\mathbb{R}^{N-1}$.

We recall that

$$Q^\varepsilon = Q \setminus \bigcup_{i=1}^{\nu(\varepsilon)} T^\varepsilon_i$$

where T^ε_i are closed set of $(0, 1) \times \mathbb{R}^{N-1}$, $1 \le i \le \nu(\varepsilon)$.

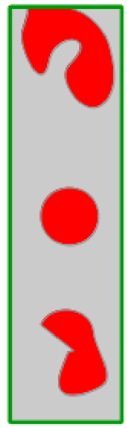


Figure 3: the domain Q^ε, the holes in red.

We define

(5.1) $$Q_k^\varepsilon = Q^\varepsilon + (k, 0) \text{ with } k \in \mathbb{Z},\ 0 \in \mathbb{R}^{N-1}$$

(5.2) $$\Omega_n^\varepsilon = \left(\cup_{k=-n}^{k=n-1} \bar{Q}_k^\varepsilon \right)^o.$$

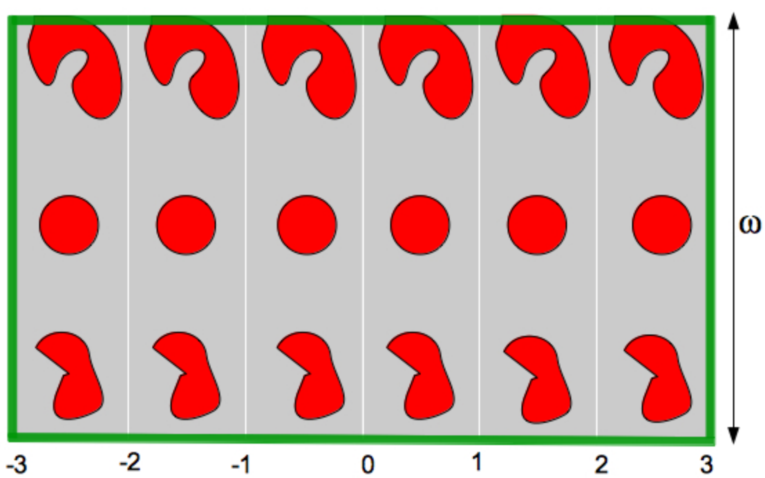


Figure 4: the domain Ω_n^ε, $n = 3$, the holes in red.

Let us suppose that

(5.3) $$A(x) \in L^\infty(\mathbb{R} \times \omega)^{N \times N} \text{1-periodic in } x_1\text{-direction.}$$

Analogously

(5.4) $$h, f \text{ and } F(\cdot, s)\ \forall s \in [0, +\infty) \text{ are 1-periodic in } x_1\text{-direction}$$

and

$$(5.5)\qquad \begin{cases} h \in L^r(K\times\omega), r=\frac{2^*}{2^*-1+\gamma},\ f\in L^{(2^*)'}(K\times\omega), \text{if } N\geq 3\\ \text{or}\quad h,f\in L^r(K\times\omega),\ r>1, \text{if } N=2\\ \forall K \text{ compact in } \mathbb{R}.\end{cases}$$

We recall that $\tilde{v}$ denote the extension by zero on the holes of a function v defined in a perforated domain (see (4.8)). If $v \in H^1_{0,per}(Q,\partial\omega)$, for any fixed $n\in\mathbb{N}$ we denote by $v_\#$ the function in $H^1_0(\Omega_n,\partial\omega)$ which is 1-periodic in the x_1-direction and which extends the function v to Ω_n. For sake of simplicity, from now on, we omit the symbol # and we simply write $\tilde{u}^\varepsilon_\infty$ instead of $(\tilde{u}^\varepsilon_\infty)_\#$. In a similar way, we denote by u^ε_∞ the 1-periodic extension from Q^ε to Ω^ε_n.

The main result of this section is the following one.

Theorem 5.1. *Let u^ε_n, $n\in\mathbb{N}$, and u^ε_∞ be the solutions of Problems* (2.5) *and* (2.15) *(respectively). Let us assume* (2.6)-(2.9) *and* (5.1)-(5.5)*, the following estimates hold true:*

$$(5.6)\qquad \begin{cases} \|\tilde{u}^\varepsilon_n-\tilde{u}^\varepsilon_\infty\|_{H^1(\Omega_{\frac{n}{2}})}\leq Ce^{-\beta n}\Big(n^{\frac{1}{r(1+\gamma)}}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)}+n^{\frac{1}{(2^*)'}}|f\|_{L^{(2^*)'}(Q)}\Big)\\ \textit{where } r=\frac{2^*}{2^*-1+\gamma},\ \textit{if } N\geq 3\end{cases}$$

$$(5.7)\qquad \begin{cases} \|\tilde{u}^\varepsilon_n-\tilde{u}^\varepsilon_\infty\|_{H^1(\Omega_{\frac{n}{2}})}\leq Ce^{-\beta n}\Big(n^{\frac{1}{1+\gamma}}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)}+n\|f\|_{L^r(Q)}\Big)\\ \textit{where } r>1,\ \textit{if } N=2\end{cases}$$

and β is a positive constant independent of n and ε.

In order to prove the previous theorem we need some preliminary results.

Lemma 5.2. *In notations and assumptions of Theorem 5.1 we have*

$$(5.8)\qquad -div\,A(x)D\tilde{u}^\varepsilon_\infty=\widetilde{F(x,u^\varepsilon_\infty)}\ \textit{in the sense of}\ \mathcal{D}'(\mathbb{R}\times\omega)$$

where $\widetilde{F(x,u^\varepsilon_\infty)}$ denote the extension by zero to $\mathbb{R}\times\omega$.
In particular

$$(5.9)\qquad \int_{\Omega^\varepsilon_n}A(x)Du^\varepsilon_\infty D\varphi=\int_{\Omega^\varepsilon_n}F(x,u^\varepsilon_\infty)\varphi\ \ \forall\varphi\in H^1_0(\Omega^\varepsilon_n).$$

Proof. The proof of (5.8) can be done as in Lemma 2.4 of [4].
By (5.8) we deduce in particular that $F(x, u^\varepsilon_\infty) \in L^1_{loc}(\mathbb{R} \times \omega)$. Let us take $\varphi \in H^1_0(\Omega^\varepsilon_n)$, $\varphi \geq 0$ and for $M > 0$ we denote by $\varphi_M = min\{M, \varphi\}$; then there exists a sequence $\varphi_{m,M} \in \mathcal{D}(\Omega^\varepsilon_n), \varphi_{m,M} \geq 0$ converging to φ_M strongly in $H^1_0(\Omega^\varepsilon_n)$ and weakly-star in $L^\infty(\Omega^\varepsilon_n)$ $(m \to +\infty)$. Consider now the sequence defined by

$$\hat{\varphi}_{m,M} = min\{\varphi_{m,M}, \varphi_M\}.$$

Note that $\hat{\varphi}_{m,M} \in H^1_0(\Omega^\varepsilon_n) \cap L^\infty(\Omega^\varepsilon_n)$, $\|\hat{\varphi}_{m,M}\|_{L^\infty(\Omega^\varepsilon_n)} \leq M$ and $supp\, \hat{\varphi}_{m,M} \subset K_m \subset \Omega^\varepsilon_n$, K_m compact set. For any fixed m we approximate $\hat{\varphi}_{m,M}$ by means of mollifiers $\hat{\varphi}_{m,\eta} = \hat{\varphi}_{m,M} \star \rho_\eta$; for η sufficiently small we can suppose that $supp\, \hat{\varphi}_{m,\eta} \subset \tilde{K}_m \subset \Omega^\varepsilon_n$, $\tilde{K}_m$ compact set.

We can take $\hat{\varphi}_{m,\eta}$ as test function in (5.8). Passing to the limit on η, we obtain

$$\int_{\Omega^\varepsilon_n} A(x) Du^\varepsilon_\infty D\hat{\varphi}_{m,M} = \int_{\Omega^\varepsilon_n} F(x, u^\varepsilon_\infty)\hat{\varphi}_{m,M} \tag{5.10}$$

by Lebesgue Theorem and the fact that $0 \leq \hat{\varphi}_{m,\eta} \leq M$. Now we pass to the limit on m. We note that the sequence $\hat{\varphi}_{m,M}$ converges strongly in $H^1(\Omega^\varepsilon_n)$ and in particular a.e. in Ω^ε_n to φ_M. By (5.10) and Fatou's lemma, we deduce that $F(x, u^\varepsilon_\infty)\varphi_M \in L^1(\Omega^\varepsilon_n)$. As

$$0 \leq F(x, u^\varepsilon_\infty)\hat{\varphi}_{m,M} \leq F(x, u^\varepsilon_\infty)\varphi_M$$

using Lebesgue's theorem we can pass to the limit in (5.10) obtaining

$$\int_{\Omega^\varepsilon_n} A(x) Du^\varepsilon_\infty D\varphi_M = \int_{\Omega^\varepsilon_n} F(x, u^\varepsilon_\infty)\varphi_M. \tag{5.11}$$

Repeating the previous argument, we pass to the limit as $M \to +\infty$ in 5.11 and we complete the proof of (5.9). ☐

Lemma 5.3. *In notations and assumptions of Theorem 5.1 we have that here exists a constant C independent of $n \in \mathbb{N}$ and ε such that*

$$\begin{cases} \displaystyle\int_{\Omega_n} |D\tilde{u}^\varepsilon_n|^2 \leq C\Big(\|h\|^{\frac{2}{1+\gamma}}_{L^r(Q)} n^{\frac{2}{r(1+\gamma)}} + \|f\|^2_{L^{(2^*)'}(Q)} n^{\frac{2}{(2^*)'}}\Big), \\ r = \frac{2^*}{2^*-1+\gamma}\ , if\ N \geq 3 \end{cases} \tag{5.12}$$

$$\begin{cases} \displaystyle\int_{\Omega_n} |D\tilde{u}^\varepsilon_n|^2 \leq C\Big(n^{\frac{2}{1+\gamma}} \|h\|^{\frac{2}{1+\gamma}}_{L^r(Q)} + n^2 \|f\|^2_{L^r(Q)}\Big), \\ r > 1,\ if\ N = 2. \end{cases} \tag{5.13}$$

Proof. Let us take as test function in (2.5) u_n^ε, we consider $N > 2$ and we choose $r = \frac{2^*}{2^*-1+\gamma}$. We obtain

$$\alpha \int_{\Omega_n^\varepsilon} |Du_n^\varepsilon|^2 \le \Big\{ \int_{\Omega_n^\varepsilon} h^r \Big\}^{\frac{1}{r}} \Big\{ \int_{\Omega_n^\varepsilon} |u_n^\varepsilon|^{r'(1-\gamma)} \Big\}^{\frac{1}{r'}} + \Big\{ \int_{\Omega_n^\varepsilon} f^{(2^*)'} \Big\}^{\frac{1}{(2^*)'}} \Big\{ \int_{\Omega_n^\varepsilon} |u_n^\varepsilon|^{2^*} \Big\}^{\frac{1}{2^*}} =$$

$$\Big\{ \int_{\Omega_n^\varepsilon} h^r \Big\}^{\frac{1}{r}} \Big\{ \int_{\Omega_n^\varepsilon} |u_n^\varepsilon|^{2^*} \Big\}^{\frac{1-\gamma}{2^*}} + \Big\{ \int_{\Omega_n^\varepsilon} f^{(2^*)'} \Big\}^{\frac{1}{(2^*)'}} \Big\{ \int_{\Omega_n^\varepsilon} |u_n^\varepsilon|^{2^*} \Big\}^{\frac{1}{2^*}} \le$$

$$\le C_S^{1-\gamma} \|h\|_{L^r(\Omega_n^\varepsilon)} \Big\{ \int_{\Omega_n^\varepsilon} |Du_n^\varepsilon|^2 \Big\}^{\frac{1-\gamma}{2}} + C_S \|f\|_{L^{(2^*)'}(\Omega_n^\varepsilon)} \Big\{ \int_{\Omega_n^\varepsilon} |Du_n^\varepsilon|^2 \Big\}^{\frac{1}{2}}$$

and then

$$\int_{\Omega_n} |D\tilde{u}_n^\varepsilon|^2 \le C\Big(\|h\|_{L^r(Q)}^{\frac{2}{1+\gamma}} n^{\frac{2}{r(1+\gamma)}} + \|f\|^2_{L^{(2^*)'}(Q)} n^{\frac{2}{(2^*)'}} \Big), \tag{5.14}$$

where we have used the fact that the functions h and f are 1-periodic in Ω_∞ in the x_1 variable and the Poincaré inequality (5.3).
Analogously, if $N = 2$, we obtain inequality (5.13), by choosing the exponent r an arbitrary number greater than 1. □

We are now in position to prove Theorem 5.1

Proof. Let us denote x as $x = (x_1, x')$ and consider $n \in \mathbb{N}$. We take as test function in (5.9) and (2.5) the function

$$(u_\infty^\varepsilon - u_n^\varepsilon)\rho$$

where $\rho = \rho(x_1) = 1$ on the set $|x_1| < l_1$ where $l_1 = [\frac{n}{2\eta}]\eta + \eta,$ where η is a small positive parameter, $\rho = 0$ on the set $|x_1| \ge l_1 + \eta$; the function ρ is nonnegative, continuous and piecewise affine.
In view of Remark 2.5 this is an admissible test function in (2.5).
Subtracting the two equations we get, for large n,

$$\begin{aligned} \int_{\Omega_n} A(x) D(\tilde{u}_\infty^\varepsilon - \tilde{u}_n^\varepsilon) D((\tilde{u}_\infty^\varepsilon - \tilde{u}_n^\varepsilon)\rho) \\ = \int_{\Omega_n^\varepsilon} A(x) D(u_\infty^\varepsilon - u_n^\varepsilon) D((u_\infty^\varepsilon - u_n^\varepsilon)\rho) \\ = \int_{\Omega_n^\varepsilon} (F(x, u_\infty^\varepsilon) - F(x, u_n^\varepsilon))(u_\infty^\varepsilon - u_n^\varepsilon)\rho \le 0 \end{aligned} \tag{5.15}$$

where the last inequality is due to the fact that the singular function $F(x, s)$ is decreasing in the s variable. By (5.15) and the coercivity

assumption we get
(5.16)

$$\alpha\int_{\Omega_{l_1}}|D(\tilde{u}^\varepsilon_\infty-\tilde{u}^\varepsilon_n)|^2\leq-\int_{\Omega^\varepsilon_{l_1+\eta}\cap supp\,\rho'}A(x)D(u^\varepsilon_\infty-u^\varepsilon_n)D\rho(u^\varepsilon_\infty-u^\varepsilon_n)$$

$$\leq C_7\frac{1}{\eta}\int_{D^\varepsilon_{l_1+\eta}}|D(u^\varepsilon_\infty-u^\varepsilon_n)||u^\varepsilon_\infty-u^\varepsilon_n|$$

where $D^\varepsilon_{l_1+\eta}=\Omega^\varepsilon_{l_1+\eta}\setminus\Omega^\varepsilon_{l_1}$.
Using Poincaré inequality we get

$$\frac{1}{\eta}\int_{D^\varepsilon_{l_1+\eta}}|D(u^\varepsilon_\infty-u^\varepsilon_n)||u^\varepsilon_\infty-u^\varepsilon_n|\leq\frac{C_P}{\eta}\int_{D^\varepsilon_{l_1+\eta}}|D(u^\varepsilon_\infty-u^\varepsilon_n)|^2.$$

Then we deduce

$$\int_{\Omega_{l_1}}|D(\tilde{u}^\varepsilon_\infty-\tilde{u}^\varepsilon_n)|^2\leq\frac{C_7C_P}{\alpha\eta+C_7C_P}\int_{\Omega^\varepsilon_{l_1+\eta}}|D(u^\varepsilon_\infty-u^\varepsilon_n)|^2. \tag{5.17}$$

We iterate now the proceedure by m steps where $m=[\frac{n}{2\eta}]-1$, we get

$$\int_{\Omega_{\frac{n}{2}}}|D(\tilde{u}^\varepsilon_\infty-\tilde{u}^\varepsilon_n)|^2\leq\big(\frac{C_7C_P}{\alpha\eta+C_7C_P}\big)^{\frac{n}{2\eta}-2}\int_{\Omega^\varepsilon_n}|D(u^\varepsilon_\infty-u^\varepsilon_n)|^2$$

$$\leq C_8e^{-\frac{\alpha_\eta n}{\eta}}\int_{\Omega^\varepsilon_n}|D(u^\varepsilon_\infty-u^\varepsilon_n)|^2$$

where $C_8=(1+\frac{\alpha}{C_7C_P})^2$ and $\alpha_\eta=\frac{1}{2}\ln(1+\frac{\alpha\eta}{C_7C_P})$.
By Lemma 5.3, if $N\geq 3$ we get

$$\int_{\Omega_n}|D\tilde{u}^\varepsilon_n|^2\leq C\Big(\|h\|^{\frac{2}{1+\gamma}}_{L^r(Q)}n^{\frac{2}{r(1+\gamma)}}+\|f\|^2_{L^{(2^*)'}(Q)}n^{\frac{2}{(2^*)'}}\Big),$$

with $r=\frac{2^*}{2^*-1+\gamma}$. In a similar way we can also prove, starting by iv) of (2.15) that

$$\int_{\Omega_n}|D\tilde{u}^\varepsilon_\infty|^2\leq C\Big(\|h\|^{\frac{2}{1+\gamma}}_{L^r(Q)}n^{\frac{2}{r(1+\gamma)}}+\|f\|^2_{L^{(2^*)'}(Q)}n^{\frac{2}{(2^*)'}}\Big).$$

Then

$$\int_{\Omega_{\frac{n}{2}}}|D(\tilde{u}^\varepsilon_\infty-\tilde{u}^\varepsilon_n)|^2\leq Ce^{-\frac{\alpha_\eta n}{\eta}}\Big(\|h\|^{\frac{2}{1+\gamma}}_{L^r(Q)}n^{\frac{2}{r(1+\gamma)}}+\|f\|^2_{L^{(2^*)'}(Q)}n^{\frac{2}{(2^*)'}}\Big)$$

As $\lim\limits_{\eta\to0^+}\dfrac{\alpha_\eta}{\eta}=\dfrac{\alpha}{2C_7C_P}$,

$$\|D(\tilde{u}^\varepsilon_n-\tilde{u}^\varepsilon_\infty)\|_{L^2(\Omega_{\frac{n}{2}})}\leq Ce^{-\beta n}\Big(\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)}n^{\frac{1}{r(1+\gamma)}}+\|f\|_{L^{(2^*)'}(Q)}n^{\frac{1}{(2^*)'}}\Big)$$

with $\beta < \frac{\alpha}{2C_7C_P}$. Therefore the proof of (5.6) is complete. If $N = 2$ we get (5.7) analogously, using estimates (5.13). □

If we assume further conditions on the distribution of the perforations, we can improve the previous result. More precisely, let $Y = (0,1)^N$ be the reference cell and T a closed set of Y with non empty interior. We set

$$T_a^\varepsilon = \varepsilon(T + a),\ Y_a^\varepsilon = \varepsilon(Y + a),\ a \in \mathbb{Z}^N, \tag{5.18}$$

$$T^\varepsilon = \bigcup_{a \in I^\varepsilon} T_a^\varepsilon \qquad Q^\varepsilon = Q \setminus T^\varepsilon \tag{5.19}$$

where $I^\varepsilon = \{a \in \mathbb{Z}^N : T_a^\varepsilon \subset (0,1) \times \mathbb{R}^{N-1}\}$ and Ω_n^ε is defined according to (5.1) and (5.2).

Then for any $v \in H_0^1(\Omega_n^\varepsilon, \partial\omega)(n \in \mathbb{N})$, the following Poincaré inequality holds true

$$\|v\|^2_{L^2(\Omega_n^\varepsilon)} \le C_P^2 \varepsilon^2 \|Dv\|^2_{L^2(\Omega_n^\varepsilon)}. \tag{5.20}$$

Using this inequality we can improve the result of Theorem 5.1. More precisely we can show the following result.

Theorem 5.4. *Let u_n^ε and u_∞^ε be the solutions of Problems* (2.5) *and* (2.15) *(respectively) and $\tilde{u}_n{}^\varepsilon$ and $\tilde{u}_\infty^\varepsilon$ their extensions by zero to Ω_n and Q (respectively), $n \in \mathbb{N}$. Let us assume* (2.6)-(2.8), (5.1)-(5.5) *and* (5.18), (5.19),

$$F(x,s) = F_1(x,s) + F_2(x,s) \tag{5.21}$$

where $F_1(x,s)$ satisfies (2.9) *and $F_2(x,s)$ is Lipschitz continuous with respect to s uniformly in x and Lipschtz constant L, then the following estimates hold true: if $N \ge 3$ and $r = \frac{2^*}{2^*-1+\gamma}$*

$$\|D(\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon)\|_{L^2(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\Big(n^{\frac{1}{r(1+\gamma)}}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n^{\frac{1}{(2^*)'}}|f\|_{L^{(2^*)'}(Q)}\Big), \tag{5.22}$$

$$\|\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon\|_{L^2(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\varepsilon\Big(n^{\frac{1}{r(1+\gamma)}}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n^{\frac{1}{(2^*)'}}|f\|_{L^{(2^*)'}(Q)}\Big), \tag{5.23}$$

or if $N = 2$ and $r > 1$

$$\|D(\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon)\|_{L^2(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\Big(n^{\frac{1}{1+\gamma}}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n\|f\|_{L^r(Q)}\Big), \tag{5.24}$$

$$\|\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon\|_{L^2(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\varepsilon\ \Big(n^{\frac{1}{1+\gamma}}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n\|f\|_{L^r(Q)}\Big). \tag{5.25}$$

Here β is a positive constant independent of n and ε and $\varepsilon \le \varepsilon_0 = \varepsilon_0(\alpha, C_P, L)$.

Proof. We just want to point out the way to handle the new term in the right hand side of (5.15), due to the presence of the non monotone function $F_2(x,s)$:

$$X = \int_{\Omega_n^\varepsilon} (F_2(x,u_\infty^\varepsilon) - F_2(x,u_n^\varepsilon))(u_\infty^\varepsilon - u_n^\varepsilon)\rho$$

We have, using (5.20),

$$|X| \le L \int_{\Omega^\varepsilon_{l_1+\eta}} (u_\infty^\varepsilon - u_n^\varepsilon)^2\rho \le C_P^2\varepsilon^2 L \int_{\Omega^\varepsilon_{l_1+\eta}} |D(u_\infty^\varepsilon - u_n^\varepsilon)|^2 =$$

$$C_P^2\varepsilon^2 L \int_{\Omega^\varepsilon_{l_1}} |D(u_\infty^\varepsilon - u_n^\varepsilon)|^2 + C_P^2\varepsilon^2 L \int_{D^\varepsilon_{l_1+\eta}} |D(u_\infty^\varepsilon - u_n^\varepsilon)|^2.$$

Taking into account this new term in (5.16), we arrive to the analogous of (5.17) i.e.

$$\int_{\Omega_{l_1}} |D(\tilde{u}_\infty^\varepsilon - \tilde{u}_n^\varepsilon)|^2 \le \frac{C_7C_P + \eta C_P^2\varepsilon^2 L}{\alpha\eta + C_7C_P} \int_{\Omega^\varepsilon_{l_1+\eta}} |D(u_\infty^\varepsilon - u_n^\varepsilon)|^2.$$

Now we can proceed exactly as in the proof of Theorem 5.1 we obtain estimates (5.22). Using inequality (5.20) we get also (5.23). If $N = 2$ we get (5.24) and (5.25) analogously, using estimates (5.13). □

Remark 5.5. A simple example of function $F(x,s)$ satisfying the assumptions of the previous Theorem is

$$F(x,s) = \frac{1}{s^\gamma}\left(2 + \sin s^2\right) + f(x) \text{ a.e. } x \in \Omega,\ \forall s > 0,$$

where $f(x) \in L^{(2^*)'}_{loc}(\mathbb{R} \times \omega)$, 1-periodic in x_1-direction.

Remark 5.6. We note that if $N = 2$ and we assume $f \in L^2(K \times \omega)$ and $h \in L^{\frac{2}{1+\gamma}}(K \times \omega)$ instead of (5.5) we can improve estimates (5.7),(5.24) and (5.25) by using Poincaré inequality (3.1) instead of (3.2). In this case we get, under the assumptions of Theorem 5.1,

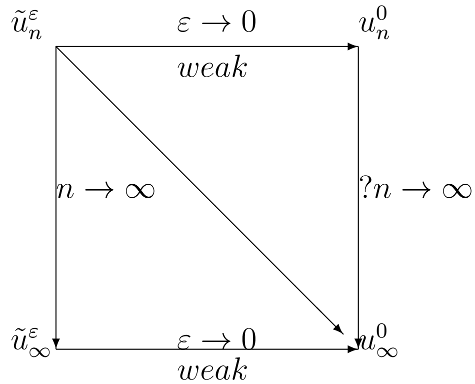
$$\|\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon\|_{H^1(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\Big(n^{\frac12}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n^{\frac12}\|f\|_{L^2(Q)}\Big),\ r = \frac{2}{1+\gamma}$$

and, under the assumptions of Theorem 5.4,

(5.26)

$$\begin{cases} \|D(\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon)\|_{L^2(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\Big(n^{\frac12}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n^{\frac12}\|f\|_{L^2(Q)}\Big), \\ \|\tilde{u}_n^\varepsilon - \tilde{u}_\infty^\varepsilon\|_{L^2(\Omega_{\frac{n}{2}})} \le Ce^{-\beta n}\varepsilon\ \Big(n^{\frac12}\|h\|^{\frac{1}{1+\gamma}}_{L^r(Q)} + n^{\frac12}\|f\|_{L^2(Q)}\Big), \\ \text{where } r = \frac{2}{1+\gamma}. \end{cases}$$

Remark 5.7. We want to stress that the constants which appear in (5.6), (5.7) or in (5.22)-(5.25) are independent of ε and n. This allows us to prove that, under the assumptions of Theorems 4.2, 4.4 and 5.1 (or Theorem 5.4) the sequence u_n^0 converges to u_∞^0 and therefore the following diagram commutes with respect to the H^1-convergence.



The question whether one can prove directly the convergence of u_n^0 to u_∞^0 is still open.

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References

[1] L. Boccardo and J. Casado-Díaz, *Some properties of solutions of some semilinear elliptic singular problems and applications to the G-convergence.* Asymptotic Analysis **86** (2014), 1–15.

[2] L. Boccardo and L. Orsina, *Semilinear elliptic equations with singular nonlinearities.* Calc. Var. Partial Differential Equations **37** (2010), 363–380.

[3] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs, I et II.* In *Nonlinear partial differential equations and their applications, Collège de France Seminar, Vol. II and Vol. III*, ed. by H. Brezis and J.-L. Lions. Research Notes in Math. 60 and 70, Pitman, London, (1982), 98-138 and 154-178.

[4] M. Chipot and Y. Xie, *Elliptic problems with periodic data: an asymptotic analysis.* J.Math. Pures Appl. **85** (2006), 345–370.

[5] G. Dal Maso and U. Mosco, *The Wiener modulus of a radial measure.* Houston J. Math **15** no. 1 (1989), 35-57.

[6] G. Dal Maso and F. Murat, *Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains.* Ann. Inst. H. Poincaré Anal. non linéaire **21** (2004), 445–486.

[7] P. Donato, S. Mardare and B. Vernescu, *From Stokes to Darcy in infinite cylinders: do limits commute?.* Differential and Integral Equations **26** 9-10 (2013), 949-974.

[8] W. Fulks and J. S. Maybee, *A singular nonlinear equation.* Osaka J. Math., **12** (1960), 1-19.
[9] D. Giachetti, P.J. Martínez-Aparicio and F. Murat, *An elliptic equation with a mild singularity at $u = 0$: existence and homogenization.* Submitted.
[10] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Classics in Mathematics, Springer Verlag, Berlin Heidelberg New York (1983).
[11] V. G. Maz'ya , *Sobolev spaces.* Springer Series in Soviet Mathematics, Springer Verlag, Berlin (1985).
[12] V. G. Maz'ya and S.V. Poborchi, *Differential functions on bad domains.* World Scientific, Singapore (1997).
[13] W. Rudin, *Functional Analysis.* Mc Graw-Hill, New York (1973).
[14] A. Sihvola, *Metamaterials in electromagnetics.* Metamaterials **1** (2007), 2-11.
[15] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus.* Ann. Inst. Fourier (Grenoble) **15** (1965), 189–258.
[16] S. Zouhdi, A. Sihvola and M. Arsalane, *Advances in Electromagnetics of Complex Media and Metamaterials.* NATO Science Series **89** (2002).

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