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On the inequality $F(x, D^2u) \geq f(u) + g(u)|Du|^q$

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Abstract We consider fully nonlinear degenerate elliptic equations with zero and first order terms. We provide a priori upper bounds and characterize the existence of entire subsolutions under growth conditions on the lower order coefficients which extend the classical Keller–Osserman condition for semilinear equations.

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1 Introduction

In 1957 Keller [26] and Osserman [33] simultaneously and independently proved that, for a given positive, continuous and nondecreasing function f, the semilinear differential inequality

$$\Delta u \geq f(u) \tag{1.1}$$

possesses an entire solution $u : \mathbb{R}^n \to \mathbb{R}$ if and only if

$$\int_0^{+\infty} \frac{dt}{\left(\int_0^t f(s)\,ds\right)^{1/2}} = +\infty. \tag{1.2}$$

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Moreover, if the Keller–Osserman condition 1.2 fails and u satisfies 1.1 in a proper open subset $\Omega \subset \mathbb{R}^n$, then u is bounded from above by a universal function of the distance from the boundary $\partial\Omega$, determined only by f. These results turned out to be rich of consequences and applications, and numerous generalizations have been established in the subsequent literature. Some of the most recent results will be recalled below.

Carrying on the study started in [14], the aim of the present paper is to establish analogous results for the more general fully nonlinear differential inequality

$$F(x, D^2u) \geq f(u) + g(u)\,|Du|^q, \quad x \in \mathbb{R}^n. \tag{1.3}$$

Here, f and g are assumed to be continuous and monotone increasing, with f positive, q belongs to $(0, 2]$ and F is a second order degenerate elliptic operator, that is a continuous function $F : \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ satisfying $F(x, O) = 0$ and the (normalized) ellipticity condition

$$0 \leq F(x, X + Y) - F(x, X) \leq \operatorname{tr}(Y), \quad \forall\, x \in \mathbb{R}^n, \quad X, Y \in \mathcal{S}_n, \quad Y \geq O,$$

$\mathcal{S}_n$ being the space of symmetric $n \times n$ real matrices equipped with the usual partial ordering.

The model cases for F that we have in mind are the degenerate maximal Pucci operator $\mathcal{M}_{0,1}^+$ defined by

$$\mathcal{M}_{0,1}^+(X) = \sum_{\mu_i > 0} \mu_i(X) \tag{1.4}$$

and

$$\mathcal{P}_k^+(X) = \mu_{n-k+1}(X) + \cdots + \mu_n(X) \tag{1.5}$$

where $\mu_1(X) \leq \mu_2(X) \leq \cdots \leq \mu_n(X)$ are the ordered eigenvalues of the matrix X.

The Pucci extremal operators, in the uniformly elliptic case, have been extensively studied in a monograph by Caffarelli and Cabré, see [11]. Let us recall here that the operator 1.4 is maximal in the class of degenerate elliptic operators vanishing at $X = O$. In particular, for any $1 \leq k \leq n$ and for all $X \in \mathcal{S}_n$, one has

$$\mathcal{P}_k^+(X) \leq \mathcal{M}_{0,1}^+(X).$$

Hence, if u satisfies inequality

$$\mathcal{P}_k^+(D^2u) \geq f(u) + g(u)\,|Du|^q, \tag{1.6}$$

or inequality 1.3, then u is also a solution of

$$\mathcal{M}_{0,1}^+(D^2u) \geq f(u) + g(u)\,|Du|^q. \tag{1.7}$$

As for the operators $\mathcal{P}_k^+$, we refer to the recent works of Harvey and Lawson [24] and Caffarelli et al. [12], see also Amendola et al. [3] and Galise [20] and the references

therein for further results. We just point out here that such degenerate operators arise in several frameworks, e.g. the level set approach to geometric evolution problems, see Ambrosio and Soner [2] and Giga [22], or the PDE approach to the convex envelope problem, see Oberman and Silvestre [32].

Since the considered operators are in non-divergence form, a natural approach to the analysis of the partial differential inequality 1.3 is that of viscosity solutions. So, by solution of 1.3 or 1.7 or 1.6 we always mean an upper semicontinuous subsolution in the viscosity sense.

Let us present now our results in a rather informal way. A key point is that while we assume f to be positive we will not make a sign assumption on the function g, so that the first order terms can be of "absorbing" or "reaction" type.

Let us discuss first the case where $\lim_{t\to+\infty} g(t) > 0$. In this case, the necessary and sufficient "sublinearity" condition 1.2 which rules the semilinear case 1.1 should be of course generalized in order to take in proper account the first order terms. We prove indeed that inequality 1.7 possesses an entire viscosity solution (see Theorem 3.3 below) if and only if

$$q \leq 1 \quad \text{and} \quad \int_0^{+\infty} \frac{dt}{\left(\int_0^t f(s)\,ds\right)^{1/2} + \left(\int_0^t g^+(s)\,ds\right)^{1/(2-q)}} = +\infty. \quad (1.8)$$

The same condition is also proved, in Theorem 3.5, to be necessary and sufficient for the existence of an entire viscosity solution of 1.6, provided that $g(t) \geq 0$ for all $t \in \mathbb{R}$. In particular, we see that if $q > 1$ then no entire subsolutions can exist, independently of how slow the growth of f and g is, whereas growth restrictions both on f and g are needed in the case $q \leq 1$.

Moreover, in Theorem 3.6 we show that if condition 1.8 is violated and u satisfies inequality 1.3 in a proper open subset $\Omega \subset \mathbb{R}^n$, then u satisfies the universal pointwise upper bound

$$u(x) \leq \mathcal{R}^{-1}(d(x))$$

where $d(x) = \text{dist}(x, \partial\Omega)$ and $\mathcal{R}$ is given (assuming for simplicity $g \geq 0$) by the formula

$$\mathcal{R}(a) = \begin{cases} 2\left(\dfrac{n}{2-q}\right)^{1/(2-q)} \displaystyle\int_a^{+\infty} \dfrac{dt}{\left(\int_a^t f(s)\,ds\right)^{1/2} + \left(\int_a^t g(s)\,ds\right)^{1/(2-q)}} & \text{if } q < 2 \\[2em] \sqrt{\dfrac{n}{2}} \displaystyle\int_a^{+\infty} \dfrac{dt}{\left(\int_a^t e^{\frac{2}{n}\int_s^t g(\tau)\,d\tau} f(s)\,ds\right)^{1/2}} & \text{if } q = 2 \end{cases}.$$

Assume now that $\lim_{t\to+\infty} g(t) \leq 0$. In this case, the zero and the first order terms in inequality 1.7 are competing with each other due to their opposite signs. Our analysis shows that in this case entire viscosity subsolutions exist if and only if a relaxed version of condition 1.2 involving f, g and q holds true, namely

$$\int_0^{+\infty} \frac{dt}{\left(\int_0^t e^{-2\int_s^t \left(-\frac{g(\tau)}{f(\tau)}\right)^{2/q} f(\tau)\, d\tau} f(s)\, ds\right)^{1/2}} = +\infty. \tag{1.9}$$

In particular, if $\lim_{t\to+\infty} g(t) < 0$, then 1.9 is proved to be equivalent to

$$\int_0^{+\infty} \left[\frac{1}{\left(\int_0^t f(s)\, ds\right)^{1/2}} + \frac{1}{f(t)^{1/q}} \right] dt = +\infty. \tag{1.10}$$

Note that the above condition becomes, for $q = 2$, the "subquadratic" growth condition

$$\int_0^{+\infty} \frac{dt}{f(t)^{1/2}} = +\infty.$$

Moreover, also in this case as in the previously discussed one, if condition 1.9 fails and u satisfies 1.3 in any open subset $\Omega \subset \mathbb{R}^n$, then u is universally estimated from above by an explicit function of the distance from the boundary determined by f, g and q, see Theorem 3.6.

As in the original papers [26,33] our strategy for proving the above results, see Theorems 3.3 and 3.5, is based on comparison with radially symmetric solutions of 1.6 and 1.7, after a detailed analysis of the existence of entire maximal solutions of the associated ordinary differential equation. As a matter of fact, entire solutions of 1.7 exist if and only if entire radially symmetric solutions exist. Remarkably, this fact is proved by a comparison argument which works also in the currently considered degenerate cases, without any a priori growth assumptions at infinity on u.

Let us further observe that the restriction $q \leq 2$ in our results is a natural growth condition consistent with the classical Bernstein–Nagumo condition (see [7,31]). Some partial results can be obtained also for $q > 2$, but we defer them to a future work. Let us just mention here that necessary conditions for the existence and peculiar properties of subsolutions of elliptic equations having positive superquadratic hamiltonians can be found in [13,28].

Let us point out finally that, as a general fact, when the Keller–Osserman type conditions do not hold true and the zero order term $f(u)$ is an odd function satisfying the sign condition $f(u)\, u \geq 0$, one can obtain universal local bounds from above and from below. This property has been largely used in the literature to obtain existence results for entire solutions as well as existence of "large" solutions in bounded domains, that is solutions blowing up on the boundary.

We did not investigate in this direction in the present paper and we just recall in this respect the result of Brezis [10] about existence and uniqueness of entire solutions of semilinear equations with $f(u) = |u|^{p-1}u$, $p > 1$. For subsequent extensions to more general divergence form principal parts and zero order terms we refer to Boccardo et al. [8,9], Leoni [29], Leoni and Pellacci [30] and D'Ambrosio and Mitidieri [16].

In the fully nonlinear framework, analogous results have been more recently obtained by Esteban et al. [18], Diaz [17] and Galise and Vitolo [21], Amendola et al. [4], and for Hessian equations, involving the k-th elementary symmetric function of the eigenvalues $\mu_1(D^2u), \ldots, \mu_n(D^2u)$ by Bao and Ji [5], Bao et al. [6], Jin et al. [25]. For application to removable singularities results see also Labutin [27].

As far as equations with gradient terms are concerned, the analogous "absorbing" property of superlinear first order terms in semilinear elliptic equations was singled out first by Lasry and Lions [28] and then extensively studied, see e.g. Alarcón et al. [1] and Felmer et al. [19] for fully nonlinear uniformly elliptic equations with purely first order terms of the form $h(|Du|)$. Moreover, some results obtained by Porretta [34] on the existence of entire solutions of the equation

$$-\Delta u + f(u) + g(u)|Du|^2 = h(x)$$

are closely related to the present paper. Let us observe that they are obtained by performing the change of unknown $v = \int_u^{+\infty} e^{-\int_0^t g(s)\,ds}\,dt$ which can be used only when the power q of the gradient term is 2.

2 On the associated ODE

In this section we perform a fairly complete qualitative analysis of the Cauchy problem

$$\begin{cases} \varphi'' + \frac{c-1}{r}\,\varphi' = f(\varphi) + g(\varphi)\,|\varphi'|^q \\ \varphi(0) = a \\ \varphi'(0) = 0 \end{cases} \tag{2.1}$$

where f, g are continuous nondecreasing functions and c, q are positive real numbers.

As indicated in the Introduction this analysis is one of the basic tools in our approach to the study of entire solutions of the elliptic PDE

$$F(x, D^2u) = f(u) + g(u)\,|Du|^q\,.$$

By solution of (2.1) in $[0, R)$ with $0 < R \leq +\infty$, we mean here and in the sequel a function $\varphi \in C^2((0, R)) \cap C([0, R))$ satisfying, moreover,

$$0 = \varphi'(0) = \lim_{r \to 0} \varphi'(r), \quad \exists \lim_{r \to 0^+} \varphi''(r) \in \mathbb{R}\,.$$

Therefore the ordinary differential equation in (2.1) has to be satisfied for $r = 0$, too. Let us observe that the existence of local solutions of (2.1) follows from the standard theory of ordinary differential equations with continuous data.

Lemma 2.1 *Let $c > 0$, $q > 0$, and f and g be continuous. Let φ be a solution of* 2.1 *in $[0, R)$; then*:

(i) *if f is positive, then φ is strictly increasing*;

(ii) *if f is positive and f, g are nondecreasing, and $c \geq 1$, then φ is convex and*

$$\varphi'(r) \leq \left(\frac{f(\varphi(r))}{g^-(\varphi(r))}\right)^{1/q} \quad \text{for all} \quad r \in [0, R), \tag{2.2}$$

(iii) *if f is positive, f, g are nondecreasing, g is non negative and $c \geq 1$, then*

$$\varphi''(r) \geq \frac{\varphi'(r)}{r} \quad \text{for all} \quad r \in [0, R). \tag{2.3}$$

Proof From 2.1 it follows that

$$c\varphi''(0) = \lim_{r \to 0^+} \left(\varphi''(r) + (c-1)\frac{\varphi'(r)}{r}\right) = f(a) > 0,$$

so that φ' is increasing, hence positive, in some interval $(0, r_0)$. Actually, one has $\varphi'(r) > 0$ in the whole interval $(0, R)$, since, if not, there should be a point $r^* \in (0, R)$ satisfying $\varphi'(r^*) = 0$ and $\varphi''(r^*) \leq 0$ on the one hand, and $\varphi''(r^*) = f(\varphi(r^*)) > 0$ on the other hand. Hence (i) is proved.

Next, let us prove (ii). Since $\varphi''(0) > 0$, there exists some $r_1 > 0$ such that $\varphi'(r) > 0$ and $\varphi''(r) > 0$ for $r \in (0, r_1]$. By contradiction, let us assume that there exists $\tau > r_1$ such that $\varphi''(\tau) < 0$. Then, the function φ' has a strict local maximum point $r_0 \in (0, \tau)$ and the set $\mathcal{R} = \{r \in (0, r_0) : \varphi'(r) = \varphi'(\tau)\}$ is non empty. Let $\sigma = \min \mathcal{R}$, so that $\varphi''(\sigma) \geq 0$. Therefore, we have found $\sigma < \tau$ where $\varphi'(\sigma) = \varphi'(\tau)$ and such that

$$\varphi''(\sigma) - \varphi''(\tau) > 0. \tag{2.4}$$

On the other hand, Eq. 2.1 tested at σ and τ yields

$$\begin{aligned}
\varphi''(\sigma) - \varphi''(\tau) = {} & (c-1)\left(\frac{\varphi'(\tau)}{\tau} - \frac{\varphi'(\sigma)}{\sigma}\right) \\
& + g(\varphi(\sigma))\varphi'(\sigma)^q - g(\varphi(\tau))\varphi'(\tau)^q \\
& + f(\varphi(\sigma)) - f(\varphi(\tau)),
\end{aligned}$$

In view of the monotonicity of f, g and φ and the assumption $c \geq 1$, the fact that $\sigma < \tau$ and that $\varphi'(\sigma) = \varphi'(\tau)$, this yields $\varphi''(\sigma) - \varphi''(\tau) \leq 0$, which contradicts 2.4. Therefore, φ is convex and increasing in $[0, R)$ and from Eq. 2.1 we immediately obtain

$$f(\varphi) + g(\varphi)(\varphi')^q \geq 0 \quad \text{in } [0, R),$$

which yields 2.2. This proves (ii).

Finally, let us prove (iii). Multiplying Eq. 2.1 by r^{c-1} and integrating between 0 and r yields

$$r^{c-1}\varphi'(r) \leq [f(\varphi(r)) + g(\varphi(r))\varphi'(r)^q] \int_0^r s^{c-1}ds = \left[\varphi''(r) + \frac{(c-1)}{r}\varphi'(r)\right]\frac{r^c}{c},$$

since φ, φ', f and g are nondecreasing. Hence 2.3 is proved. □

For the next results we focus on problem 2.1 under the following conditions

$$q \in (0, 2], c \geq 1, f, g \text{ continuous and nondecreasing, } f \text{ positive} \qquad (2.5)$$

and we obtain sharp estimates from above and from below on the function φ'.

Lemma 2.2 *Assume* (2.5). *If φ is a solution of* 2.1 *in* $[0, R)$, *then, for every* $r \in [0, R)$,

$$
\varphi'(r) \leq
\begin{cases}
2^{2/(2-q)} \left[\left(\int_a^{\varphi(r)} f(t)\, dt \right)^{1/2} + \left(\int_a^{\varphi(r)} g^+(t)\, dt \right)^{1/(2-q)} \right] & \text{if } q < 2 \\[4mm]
\left(2 \int_a^{\varphi(r)} e^{2 \int_t^{\varphi(r)} g^+(s)\, ds} f(t)\, dt \right)^{1/2} & \text{if } q = 2
\end{cases}
$$
$$(2.6)$$

and also, if $g(a) \geq 0$,

$$
\varphi'(r) \geq
\begin{cases}
\dfrac{1}{2} \left(\dfrac{2-q}{c} \right)^{1/(2-q)} \left[\left(\int_a^{\varphi(r)} f(t)\, dt \right)^{1/2} + \left(\int_a^{\varphi(r)} g^+(t)\, dt \right)^{1/(2-q)} \right] & \text{if } q < 2 \\[4mm]
\left(\dfrac{2}{c} \int_a^{\varphi(r)} e^{\frac{2}{c} \int_t^{\varphi(r)} g^+(s)\, ds} f(t)\, dt \right)^{1/2} & \text{if } q = 2
\end{cases}
$$
$$(2.7)$$

Furthermore,

$$\varphi'(r) \geq \left(\frac{q}{c^{2/q}} \int_a^{\varphi(r)} e^{-2 \int_t^{\varphi(r)} \left(\frac{g^-(s)}{f(s)} \right)^{2/q} f(s)\, ds} f(t)\, dt \right)^{1/2} \qquad (2.8)$$

and also, if $g(t) \leq 0$ for all $t \in \mathbb{R}$,

$$\varphi'(r) \leq \left(2 \int_a^{\varphi(r)} e^{-2 \int_t^{\varphi(r)} \left(\frac{g^-(s)}{f(s)} \right)^{2/q} f(s)\, ds} f(t)\, dt \right)^{1/2} \qquad (2.9)$$

Finally, if φ is a maximal solution of 2.1 *on* $[0, R)$, *with $R \leq +\infty$, then*

$$\lim_{r \to R^-} \varphi(r) = +\infty. \qquad (2.10)$$

Proof By Lemma 2.1, φ is convex and increasing in $[0, R)$, so that, from the Eq. 2.1 we immediately deduce

$$\varphi'' + g^-(\varphi)(\varphi')^q \leq f(\varphi) + g^+(\varphi)(\varphi')^q. \qquad (2.11)$$

In particular, one has

$$\varphi'' \leq f(\varphi) + g^+(\varphi)(\varphi')^q. \qquad (2.12)$$

Consider first the case $q < 2$. Multiplying 2.12 by φ' and integrating in $(0, r)$, jointly with the increasing monotonicity of φ', then yields

$$\frac{(\varphi'(r))^2}{2} \leq \int_a^{\varphi(r)} f(t)\, dt + (\varphi'(r))^q \int_a^{\varphi(r)} g^+(t)\, dt.$$

By Young inequality with exponent $2/q > 1$, we then obtain

$$\frac{(\varphi'(r))^2}{2} \leq \int_a^{\varphi(r)} f(t)\, dt + \frac{(\varphi'(r))^2}{4} + 4^{q/(2-q)} \left(\int_a^{\varphi(r)} g^+(t)\, dt \right)^{2/(2-q)},$$

which immediately gives 2.6 in the case $q < 2$.

If $q = 2$, we multiply 2.12 by $2 e^{-2 \int_a^{\varphi(r)} g^+(t)\, dt} \varphi'(r)$ and deduce

$$\left(e^{-2 \int_a^{\varphi(r)} g^+(t)\, dt} (\varphi'(r))^2 \right)' \leq 2\, e^{-2 \int_a^{\varphi(r)} g^+(t)\, dt} \varphi'(r) f(\varphi(r)).$$

Integrating in $(0, r)$ then yields 2.6 in the case $q = 2$.

On the other hand, the Eq. 2.1 may be written as

$$\frac{(r^{c-1} \varphi'(r))'}{r^{c-1}} = f(\varphi(r)) + g(\varphi(r)) \varphi'(r)^q,$$

hence

$$(r^{c-1} \varphi'(r))' \leq [f(\varphi(r)) + g^+(\varphi(r)) \varphi'(r)^q] r^{c-1},$$

and integration in $(0, r)$, by the monotonicity of φ, φ', f and g^+, yields

$$\frac{\varphi'(r)}{r} \leq \frac{1}{c} [f(\varphi(r)) + g^+(\varphi(r))(\varphi'(r))^q].$$

The above inequality inserted in 2.1 then implies

$$\varphi''(r) + g^-(\varphi(r))(\varphi'(r))^q \geq \frac{1}{c} [f(\varphi(r)) + g^+(\varphi(r))(\varphi'(r))^q]. \qquad (2.13)$$

Assume now that $g(a) \geq 0$; in this case $g(\varphi(r)) \geq 0$ for every $r \in [0, R)$ and the above inequality becomes

$$\varphi''(r) \geq \frac{1}{c} [f(\varphi(r)) + g^+(\varphi(r))(\varphi'(r))^q]. \qquad (2.14)$$

If $q = 2$, inequality 2.14 can be easily integrated as before yielding inequality 2.7 for $q = 2$.

If $q < 2$, 2.14 can be split into the two

$$\varphi''(r) \geq \frac{f(\varphi(r))}{c}, \quad \varphi''(r) \geq \frac{g^+(\varphi(r))(\varphi'(r))^q}{c}.$$

From the former it easily follows that

$$\varphi'(r) \geq \left(\frac{2}{c} \int_a^{\varphi(r)} f(t)\, dt \right)^{1/2},$$

whereas the integration of the latter yields

$$\varphi'(r) \geq \left(\frac{2-q}{c} \int_a^{\varphi(r)} g^+(t)\, dt \right)^{1/(2-q)}.$$

Adding term by term we obtain 2.7 also for $q < 2$.

Furthermore, we notice that 2.13 also gives

$$\varphi''(r) + g^-(\varphi(r))(\varphi'(r))^q \geq \frac{f(\varphi(r))}{c}. \tag{2.15}$$

If $q < 2$, by Young inequality with exponent $2/q > 1$ we obtain

$$\varphi''(r) + \frac{q}{2} c^{2/q-1} \frac{g^-(\varphi)^{2/q}}{f(\varphi)^{2/q-1}} (\varphi')^2 + \left(1 - \frac{q}{2} \right) \frac{f(\varphi)}{c} \geq \frac{f(\varphi)}{c}$$

and then, since $c \geq 1$,

$$\varphi''(r) + \frac{g^-(\varphi)^{2/q}}{f(\varphi)^{2/q-1}} (\varphi')^2 \geq \frac{\varphi''(r)}{c^{2/q-1}} + \frac{q}{2} \frac{g^-(\varphi)^{2/q}}{f(\varphi)^{2/q-1}} (\varphi')^2 \geq \frac{q}{2} \frac{f(\varphi)}{c^{2/q}},$$

which holds true, by 2.15, also for $q = 2$.

By multiplying both sides by $2\varphi' e^{2 \int_a^\varphi \left(\frac{g^-(t)}{f(t)} \right)^{2/q} f(t)\, dt}$ and by integrating in $(0, r)$ we obtain

$$e^{2 \int_a^{\varphi(r)} \left(\frac{g^-(t)}{f(t)} \right)^{2/q} f(t)\, dt} (\varphi')^2 \geq \frac{q}{c^{2/q}} \int_a^{\varphi(r)} e^{2 \int_a^t \left(\frac{g^-(s)}{f(s)} \right)^{2/q} f(s)\, ds} f(t)\, dt,$$

that is 2.8.

On the other hand, if g is nonpositive, inequality 2.11 reads

$$\varphi'' + \frac{g^-(\varphi)}{(\varphi')^{2-q}} (\varphi')^2 \leq f(\varphi),$$

which, on the account of 2.2, implies

$$\varphi'' + \frac{g^-(\varphi)^{2/q}}{f(\varphi)^{2/q-1}}(\varphi')^2 \le f(\varphi).$$

Multiplication by $2\varphi' e^{2\int_a^\varphi \left(\frac{g^-(t)}{f(t)}\right)^{2/q} f(t)\,dt}$ and integration in $(0, r)$ yields, as before,

$$e^{2\int_a^{\varphi(r)}\left(\frac{g^-(t)}{f(t)}\right)^{2/q} f(t)\,dt}(\varphi')^2 \le 2\int_a^{\varphi(r)} e^{2\int_a^t \left(\frac{g^-(s)}{f(s)}\right)^{2/q} f(s)\,ds} f(t)\,dt,$$

and 2.9 is proved.

Finally, let us prove 2.10. Since φ is convex and increasing in $[0, R)$, the limits $\lim_{r\to R^-}\varphi(r)$ and $\lim_{r\to R^-}\varphi'(r)$ exist. Now, if $R = +\infty$, then 2.10 follows by convexity and strictly increasing monotonicity. On the other hand, from estimate 2.6 it follows that if φ is bounded in $[0, R)$, then φ' is bounded as well, and this contradicts the maximality of R, if $R < +\infty$. Hence, 2.10 holds true in any case. $\qquad\square$

For the sequel, it is convenient to rewrite the estimates of Lemma 2.2 in the following equivalent formulations.

Lemma 2.3 *Assume* 2.5. *If φ is a solution of* 2.1 *for some $a \in \mathbb{R}$ and $R > 0$, then for every $r \in [0, R)$ one has*

$$r \ge \begin{cases} \dfrac{1}{2^{2/(2-q)}} \displaystyle\int_a^{\varphi(r)} \dfrac{dt}{\left[\left(\int_a^t f(s)\,ds\right)^{1/2} + \left(\int_a^t g^+(s)\,ds\right)^{1/(2-q)}\right]} & \text{if } q < 2 \\[4ex] \displaystyle\int_a^{\varphi(r)} \dfrac{dt}{\left(2\int_a^t e^{2\int_s^t g^+(r)\,dr} f(s)\,ds\right)^{1/2}} & \text{if } q = 2 \end{cases}$$

$$(2.16)$$

as well as, if $g(a) \ge 0$,

$$r \le \begin{cases} 2\left(\dfrac{c}{2-q}\right)^{1/(2-q)} \displaystyle\int_a^{\varphi(r)} \dfrac{dt}{\left(\int_a^t f(s)\,ds\right)^{1/2} + \left(\int_a^t g^+(s)\,ds\right)^{1/(2-q)}} & \text{if } q < 2 \\[4ex] \sqrt{\dfrac{c}{2}} \displaystyle\int_a^{\varphi(r)} \dfrac{dt}{\left(\int_a^t e^{\frac{2}{c}\int_s^t g^+(\tau)\,d\tau} f(s)\,ds\right)^{1/2}} & \text{if } q = 2 \end{cases}.$$

$$(2.17)$$

Furthermore, for any $q \in (0, 2]$ and $r \in (0, R)$, one has

$$r \le \frac{c^{1/q}}{\sqrt{q}} \int_a^{\varphi(r)} \frac{dt}{\left(\int_a^t e^{-2\int_s^t \left(\frac{g^-(\tau)}{f(\tau)}\right)^{2/q} f(\tau)\,d\tau} f(s)\,ds\right)^{1/2}}, \qquad (2.18)$$

and, if $g(t) \leq 0$ for all $t \in \mathbb{R}$,

$$r \geq \frac{1}{\sqrt{2}} \int_a^{\varphi(r)} \frac{dt}{\left(\int_a^t e^{-2 \int_s^t \left(\frac{g^-(\tau)}{f(\tau)} \right)^{2/q} f(\tau)\, d\tau} f(s)\, ds \right)^{1/2}}. \tag{2.19}$$

Proof We rewrite estimate 2.6 in the form

$$1 \geq \begin{cases} \dfrac{\varphi'(r)}{2^{2/(2-q)} \left[\left(\int_a^{\varphi(r)} f(t)\, dt \right)^{1/2} + \left(\int_a^{\varphi(r)} g^+(t)\, dt \right)^{1/(2-q)} \right]} & \text{if } q < 2 \\[2em] \dfrac{\varphi'(r)}{\left(2 \int_a^{\varphi(r)} e^{2 \int_t^{\varphi(r)} g^+(s)\, ds} f(t)\, dt \right)^{1/2}} & \text{if } q = 2 \end{cases}$$

Hence, integration in $[0, r]$ yields 2.16. The remaining statements 2.17, 2.18 and 2.19 are deduced in a completely analogous way from 2.7, 2.8 and 2.9 respectively. □

The next result Theorem 2.5 identifies, in the general framework of assumption 2.5, some integrability conditions on the data which are necessary and sufficient for the existence of maximal solutions of the Cauchy problem 2.1 defined on the whole interval $[0, +\infty)$.

Its proof makes use in particular of the following result from elementary calculus.

Lemma 2.4 *Let f and h be continuous functions in $\mathbb{R}$, with f positive and nondecreasing, and h nonnegative and nonincreasing. For every $a \in \mathbb{R}$ let us set*

$$\mathcal{R}(a) = \int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{-2 \int_s^t h(r)\, dr} f(s)\, ds \right)^{1/2}}.$$

Then, either $\mathcal{R}(a) \equiv +\infty$, or $\mathcal{R}(a) < +\infty$ for every $a \in \mathbb{R}$. In the latter case, $\mathcal{R} : \mathbb{R} \to (0, +\infty)$ is monotone nonincreasing, and strictly decreasing if either f or h is strictly monotone, and it satisfies

$$\lim_{a \to +\infty} \mathcal{R}(a) = 0. \tag{2.20}$$

The proof of this Lemma will be detailed below for the convenience of the reader.

Theorem 2.5 *Under Assumption 2.5, let φ be a maximal solution of the initial value problem 2.1. Then,*

(i) *if $\lim_{t \to +\infty} g(t) > 0$, then φ is globally defined in $[0, +\infty)$ if and only if*

$$q \leq 1 \quad \text{and} \quad \int_0^{+\infty} \frac{dt}{(t\, f(t))^{1/2} + (t\, g^+(t))^{1/(2-q)}} = +\infty \tag{2.21}$$

(ii) *if* $\lim\limits_{t\to+\infty} g(t) \le 0$, *then φ is globally defined in $[0, +\infty)$ if and only if*

$$\int_0^{+\infty} \frac{dt}{\left(\int_0^t e^{-2\int_s^t \left(\frac{g^-(\tau)}{f(\tau)}\right)^{2/q} f(\tau)\, d\tau} f(s)\, ds\right)^{1/2}} \, dt = +\infty \qquad (2.22)$$

(iii) *if* $\lim\limits_{t\to+\infty} g(t) < 0$, *then* 2.22 *is equivalent to*

$$\int_0^{+\infty} \left[\frac{1}{(t\, f(t))^{1/2}} + \frac{1}{f(t)^{1/q}}\right] dt = +\infty \qquad (2.23)$$

Proof (i) Assume $\lim_{t\to+\infty} g(t) > 0$. Let us first observe that condition 2.21 is equivalent to

$$q \le 1 \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{dt}{(t\, f(t))^{1/2} + (t\, g^+(t))^{1/(2-q)}} = +\infty \quad \text{for all } t_0 \ge 0.$$

By observing that, for all $a \in \mathbb{R}$ and $t \ge 2|a|$ one has

$$\frac{t}{2} f\left(\frac{t}{2}\right) \le \int_a^t f(s)\, ds \le f(t)\, (t - a) \le 2\, t\, f(t),$$

as well as

$$\frac{t}{2} g^+\left(\frac{t}{2}\right) \le \int_a^t g^+(s)\, ds \le g^+(t)\, (t - a) \le 2\, t\, g^+(t),$$

the above condition is then equivalent to

$$q \le 1 \quad \text{and} \quad \int_{2|a|}^{+\infty} \frac{dt}{\left(\int_a^t f(s)\, ds\right)^{1/2} + \left(\int_a^t g^+(s)\, ds\right)^{1/(2-q)}} = +\infty \quad \text{for all } a \in \mathbb{R},$$

and then also to

$$q \le 1 \quad \text{and} \quad \int_a^{+\infty} \frac{dt}{\left(\int_a^t f(s)\, ds\right)^{1/2} + \left(\int_a^t g^+(s)\, ds\right)^{1/(2-q)}} = +\infty \quad \text{for all } a \in \mathbb{R}.$$

$$(2.24)$$

Now, let 2.21, and therefore 2.24, be satisfied. If $\varphi \in C^2([0, R))$ is a maximal solution of 2.1 for some $a \in \mathbb{R}$, then, by letting $r \to R^-$ in 2.16 with $q \le 1$, and by using 2.10 and 2.24 we immediately infer $R = +\infty$.

Conversely, assume that for any $a \in \mathbb{R}$ a maximal solution of the Cauchy problem 2.1 belongs to $C^2([0, +\infty))$, and let us select a maximal solution φ satisfying the

initial condition $\varphi(0) = a > 0$ such that $g(a) > 0$. By inequality 2.17 it follows that, if $q > 1$, then

$$
r \leq
\begin{cases}
2\left(\dfrac{c}{2-q}\right)^{1/(2-q)} \displaystyle\int_a^{+\infty} \dfrac{dt}{(f(a)(t-a))^{1/2} + \left(g^+(a)(t-a)\right)^{1/(2-q)}} & \text{if } 1 < q < 2 \\[2em]
\sqrt{\dfrac{g^+(a)}{f(a)}} \displaystyle\int_a^{+\infty} \dfrac{dt}{\sqrt{e^{\frac{2}{c}g^+(a)(t-a)} - 1}} & \text{if } q = 2
\end{cases}
$$

which is a contradiction to the unboundedness of r. Hence, one has $q \leq 1$, and letting $r \to +\infty$ in 2.17, we obtain, by 2.10, that 2.24 is satisfied for sufficiently large a, and, therefore, 2.21 holds true.

   (ii) Assume that $\lim_{t \to +\infty} g(t) \leq 0$, so that $g(t) \leq 0$ for all $t \in \mathbb{R}$, and let $\varphi \in C^2([0, R))$ be any maximal solution of the Cauchy problem 2.1. By inequalities 2.18 and 2.19, and by 2.10, we immediately infer that $R = +\infty$ if and only if, for all $a \in \mathbb{R}$,

$$
\int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{-2\int_s^t \left(\frac{g^-(r)}{f(r)}\right)^{2/q} f(r)\,dr} f(s)\,ds\right)^{1/2}} = +\infty.
$$

By applying Lemma 2.4 with $h(r) = (g^-(r)/f(r))^{2/q} f(r)$, we deduce that the above condition is equivalent to 2.22.

   (iii) It remains to prove that, if $\lim_{t \to +\infty} g(t) = g_\infty < 0$, then 2.22 is equivalent to 2.23. First of all, let us rewrite condition 2.22 as

$$
\int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds\right)^{1/2}}\,dt = +\infty, \quad \text{for all } a \in \mathbb{R}, \tag{2.25}
$$

with h defined as above. Secondly, we remark that condition 2.23 on the growth of f as $t \to +\infty$ may be written in the equivalent form

$$
\int_a^{+\infty} \left[\frac{1}{\left(\int_a^t f(s)\,ds\right)^{1/2}} + \frac{1}{f(t)^{1/q}}\right] dt = +\infty \text{ for all } a \in \mathbb{R}. \tag{2.26}
$$

Now, we observe that, for $t \geq a$, on the one hand one has

$$
\int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds \leq \int_a^t f(s)\,ds
$$

and, on the other hand, by the monotonicity of f and h,

$$\int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds \le \frac{f(t)}{h(t)} \int_a^t e^{-2\int_s^t h(r)\,dr} h(s)\,ds \le \frac{f(t)}{h(t)} = \left(\frac{f(t)}{g^-(t)}\right)^{2/q}.$$

Therefore, for $t \ge a$, one has

$$\int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds\right)^{1/2}} \ge \frac{1}{2}\int_a^{+\infty}\left[\frac{1}{\left(\int_a^t f(s)\,ds\right)^{1/2}} + \left(\frac{g^-(t)}{f(t)}\right)^{1/q}\right] dt.$$

$$(2.27)$$

Since $g^-(t) \ge g_\infty^- > 0$, it is then clear that if 2.26 is satisfied then 2.25 holds true, that is to say 2.23 implies 2.22.

Conversely, by integration by parts we also have

$$\int_a^t f(s)\,ds - \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds = 2\int_a^t h(s)\left(\int_a^s e^{-2\int_\sigma^s h(r)\,dr} f(\sigma)\,d\sigma\right) ds,$$

and, by the nonincreasing monotonicity of the function $h(s)/g^-(s) = \left(\frac{g^-(s)}{f(s)}\right)^{2/q-1}$, it follows that

$$\int_a^t f(s)\,ds - \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds \le 2\int_a^t g^-(s)\left(\int_a^s e^{-2\int_\sigma^s h(r)\,dr} h(\sigma)\frac{f(\sigma)}{g^-(\sigma)}\,d\sigma\right) ds.$$

Hölder inequality with exponent $2/q$ then yields

$$\int_a^t f(s)\,ds - \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds$$
$$\le \int_a^t g^-(s)\left(\int_a^s 2e^{-2\int_\sigma^s h(r)\,dr} h(\sigma)\left(\frac{f(\sigma)}{g^-(\sigma)}\right)^{2/q} d\sigma\right)^{q/2}\left(\int_a^s 2e^{-2\int_\sigma^s h(r)\,dr} h(\sigma)\,d\sigma\right)^{1-q/2} ds$$
$$\le 2^{q/2}\int_a^t g^-(s)\left(\int_a^s e^{-2\int_\sigma^s h(r)\,dr} f(\sigma)\,d\sigma\right)^{q/2} ds$$

since

$$\int_a^s 2e^{-2\int_\sigma^s h(r)\,dr} h(\sigma)\,d\sigma = 1 - e^{-2\int_a^s h(r)\,dr} \le 1.$$

Then, by monotonicity,

$$\int_a^t f(s)\,ds - \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds \le 2^{q/2}\left(\int_a^t g^-(s)\,ds\right)\left(\int_a^t e^{-2\int_\sigma^t h(r)\,dr} f(\sigma)\,d\sigma\right)^{q/2}.$$

By applying further Young inequality with exponent $2/q$ we then obtain

$$\int_a^t f(s)\,ds - \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds$$

$$\leq q \int_a^t e^{-2\int_\sigma^t h(r)\,dr} f(\sigma)\,d\sigma \frac{\left(\int_a^t g^-(s)\,ds\right)^{2/q}}{\left(\int_a^t f(s)\,ds\right)^{2/q-1}} + \left(1 - \frac{q}{2}\right)\int_a^t f(s)\,ds.$$

Hence,

$$\frac{q}{2}\int_a^t f(s)\,ds \leq \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds\left[1 + q\frac{\left(\int_a^t g^-(s)\,ds\right)^{2/q}}{\left(\int_a^t f(s)\,ds\right)^{2/q-1}}\right],$$

which immediately implies

$$\int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds\right)^{1/2}}\,dt$$

$$\leq \frac{2}{\sqrt{q}}\int_a^{+\infty}\left[\frac{1}{\left(\int_a^t f(s)\,ds\right)^{1/2}} + \left(\frac{\int_a^t g^-(s)\,ds}{\int_a^t f(s)\,ds}\right)^{1/q}\right]dt. \qquad (2.28)$$

Since for t sufficiently large we have $\int_a^t f(s)\,ds \geq t/2\, f(t/2)$ and, moreover, $\int_a^t g^-(s)\,ds \leq g^-(a)\,t$, it then easily follows from inequality 2.28 that if 2.25 is satisfied, then 2.26 must be true. □

Proof of Lemma 2.4 For any fixed $a \in \mathbb{R}$ let us define the real function

$$\psi_a(t) = \int_a^t e^{-2\int_s^t h(r)\,dr} f(s)\,ds,$$

which is the solution of the linear first order initial value problem

$$\begin{cases} \psi_a'(t) + 2\,h(t)\,\psi_a(t) = f(t) \\ \psi_a(a) = 0 \end{cases}.$$

ψ_a is a $C^1(\mathbb{R})$ increasing function, since, by the monotonicity of the functions f and h, for $t \geq a$, one has

$$2\,h(t)\,\psi_a(t) \leq 2\,f(t)\int_a^t e^{-2\int_s^t h(r)\,dr} h(s)\,ds < f(t).$$

Therefore, the condition

$$\mathcal{R}(a) = \int_a^{+\infty} \frac{dt}{\sqrt{\psi_a(t)}} < +\infty$$

only depends on the growth of $\psi_a(t)$ for $t \to +\infty$. We observe that, for any $a, \, a' \in \mathbb{R}$, one has

$$\psi_{a'}(t) = \psi_a(t) - e^{-2\int_{a'}^t h(r)\,dr} \psi_a(a'), \tag{2.29}$$

Now, assume there exists $a \in \mathbb{R}$ such that $\mathcal{R}(a) < +\infty$. Then, it follows that $\lim_{t\to+\infty} \psi_a(t) = +\infty$, and, by the above identity,

$$\lim_{t\to+\infty} \frac{\psi_{a'}(t)}{\psi_a(t)} = 1.$$

Hence, $\mathcal{R}(a') < +\infty$ for all $a' \in \mathbb{R}$, and the first assertion in the statement follows.

Next, let us assume $\mathcal{R}(a) < +\infty$ for all $a \in \mathbb{R}$. For $a_1 < a_2$, by the changes of variables $\tau = t - a_1 + a_2, \sigma = s - a_1 + a_2$ and $\rho = r - a_1 + a_2$, we easily obtain

$$\mathcal{R}(a_1) = \int_{a_2}^{+\infty} \frac{d\tau}{\left(\int_{a_2}^\tau e^{-2\int_\sigma^\tau h(\rho+a_1-a_2)\,d\rho} f(\sigma+a_1-a_2)\,d\sigma\right)^{1/2}} \geq \mathcal{R}(a_2),$$

being, by monotonicity, $f(\sigma + a_1 - a_2) \leq f(\sigma)$ and $h(\rho + a_1 - a_2) \geq h(\rho)$. (We also notice that the above inequality is strict if either f or h is strictly monotone)

Moreover, for any natural number n, we have

$$\mathcal{R}(n) = \int_0^\infty \xi_n(\tau)\,d\tau,$$

with

$$\xi_n(\tau) = \frac{1}{\left(\int_0^\tau e^{-2\int_\sigma^\tau h(\rho+n)\,d\rho} f(\sigma+n)\,d\sigma\right)^{1/2}}.$$

$\{\xi_n\}$ is a nonincreasing sequence of integrable functions and, by the monotone convergence theorem, it follows that

$$\lim_{a\to\infty} \mathcal{R}(a) = \int_0^\infty \frac{d\tau}{\left(f_\infty \int_0^\tau e^{-2h_\infty(\tau-\sigma)}\,d\sigma\right)^{1/2}},$$

with $f_\infty = \lim_{t\to+\infty} f(t)$ and $h_\infty = \lim_{t\to+\infty} h(t)$. If $f_\infty < +\infty$, then

$$\lim_{a\to\infty} \mathcal{R}(a) = +\infty,$$

which is a contradiction to the nonincreasing monotonicity of $\mathcal{R}$. Therefore, if $\mathcal{R}(a) < \infty$, then we necessarily have that $f_\infty = +\infty$ and 2.20 is satisfied. $\qquad\square$

Remark 2.6 As a byproduct of Theorem 2.5 we in particular obtain that either all maximal solutions of Cauchy problem 2.1 blow up for finite R, or all maximal solutions are globally defined on $[0, +\infty)$, independently on the initial datum a, but only according to the growth as $t \to +\infty$ of f and g measured through conditions 2.21, 2.22 and 2.23 respectively. This is consistent with classical results (see [23]), since under our assumption $q \leq 2$ the Nagumo's growth condition holds.

Remark 2.7 As a consequence of the nondecreasing monotonicity of the functions $t \, f(t)$ and $t \, g^+(t)$, condition 2.21 can be easily proved to be equivalent to the two conditions

$$\int_0^{+\infty} \frac{dt}{(t f(t))^{1/2}} = +\infty \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{dt}{(t \, g^+(t))^{1/(2-q)}} = +\infty,$$

for every $t_0 > 0$ such that $g^+(t_0) > 0$. If $0 < \lim_{t \to +\infty} g(t) < +\infty$, the second integral is infinite because $q \leq 1$ and therefore in this case 2.21 becomes the usual Keller–Osserman condition

$$\int_0^{+\infty} \frac{dt}{(t \, f(t))^{1/2}} = +\infty. \tag{2.30}$$

If $\lim_{t \to +\infty} g(t) = +\infty$, then 2.21 restricts the growth at infinity both for f and g; for instance, for power like blowing up functions $g(t) \simeq t^\alpha$ for $t \to +\infty$, then 2.21 requires $\alpha \leq 1 - q$, and for $q = 1$ at most logarithmic growth $g(t) \simeq (\ln t)^\alpha$ with $\alpha \leq 1$ is allowed.

Remark 2.8 On the other hand, condition 2.23 amounts to requiring that

$$\int_0^{+\infty} \frac{dt}{(t f(t))^{1/2}} = +\infty \quad \text{or} \quad \int_0^{+\infty} \frac{dt}{(f(t))^{1/q}} = +\infty,$$

and, for $q = 2$, it is equivalent to

$$\int_0^{+\infty} \frac{dt}{(f(t))^{1/2}} = +\infty.$$

For functions f having a power growth at infinity, say $f(t) \simeq t^\alpha$ for $t \to +\infty$, 2.23 means that $0 \leq \alpha \leq \max\{1, q\}$, but it includes also functions of the form $f(t) \simeq t \, (\ln t)^\alpha$ with arbitrary $\alpha \geq 0$ if $q > 1$, and $0 \leq \alpha \leq 2$ if $q \leq 1$.

Remark 2.9 Let us consider the case $\lim_{t \to +\infty} g(t) = 0$. By inequality 2.27, it follows that

$$\int_0^{+\infty} \left[\frac{1}{(t \, f(t))^{1/2}} + \left(\frac{g^-(t)}{f(t)} \right)^{1/q} \right] dt = +\infty \tag{2.31}$$

is a sufficient condition in order to have that all maximal solutions are globally defined in $[0, +\infty)$. On the other hand, inequality 2.28 shows that if there exists a global

maximal solution $\varphi \in C^2([0, +\infty))$, then

$$\int_0^{+\infty} \left[\frac{1}{(t \, f(t))^{1/2}} + \left(\frac{\int_0^t g^-(s) \, ds}{\int_0^t f(s) \, ds} \right)^{1/q} \right] dt = +\infty. \tag{2.32}$$

Conditions 2.31 and 2.32 in general are not equivalent (also for $q < 2$). An easy example occurs for $q = 2$, $f(t) \simeq t \, (\ln t)^\alpha$ and $g(t) \simeq -1/t$ for $t \to +\infty$. In this case, 2.31 is satisfied if and only if $\alpha \leq 2$ whereas 2.32 holds true up to $\alpha \leq 3$. We observe that, in this case, 2.22 actually requires $\alpha \leq 2$.

On the other hand, if $q < 2$ and $c_1/t^\beta \leq g^-(t) \leq c_2/t^\beta$ for positive constants c_1, c_2 and β and for t sufficiently large, then, regardless the behaviour of f, 2.31 and 2.32 can be easily proved to be equivalent. In such a case, they both are more explicit formulations of 2.22.

Remark 2.10 Let us explicitly remark that for $g \equiv 0$, condition 2.22 reduces to the classical Keller–Osserman condition 2.30. An analogous condition is also recovered when $g > 0$ in the limit $q \to 0$. Indeed, for $q \to 0$ condition 2.21 becomes 2.30 applied to the positive nondecreasing nonlinearity $f(t) + g(t)$.

3 Viscosity subsolutions of fully nonlinear degenerate elliptic equations

In this section we apply the previous results on the Cauchy problem 2.1 to derive a priori upper estimates and necessary and sufficient conditions for the existence of entire viscosity solutions of inequalities of the form

$$F(x, D^2 u) \geq f(u) + g(u) \, |Du|^q, \tag{3.1}$$

where $F : \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is a continuous functions which we assume always to satisfy $F(x, O) = 0$ and the (normalized) degenerate ellipticity condition

$$0 \leq F(x, X + Y) - F(x, X) \leq \operatorname{tr}(Y), \quad \forall x \in \mathbb{R}^n, \; X, \, Y \in \mathcal{S}^n, \; Y \geq O,$$

$\mathcal{S}^n$ being the space of symmetric $n \times n$ real matrices equipped with the usual ordering.

Special attention will be devoted to study in particular subsolutions of the equation

$$\mathcal{M}_{0,1}^+(D^2 u) = f(u) + g(u) \, |Du|^q \tag{3.2}$$

and of the equation

$$\mathcal{P}_k^+(D^2 u) = f(u) + g(u) \, |Du|^q. \tag{3.3}$$

Let us immediately observe that, by the maximality of operator $\mathcal{M}_{0,1}^+$ in the class of second order degenerate elliptic operators, if u is a viscosity solution of 3.1 then u is a subsolution of 3.2.

As a first result, we show that radially symmetric solutions of the above equations can be obtained from solutions $\varphi \in C^2([0, R))$ of problem 2.1.

Lemma 3.1 (i) *Let f and g be continuous nondecreasing functions, with f positive. For any $q > 0$, if $\varphi \in C^2([0, R))$ is a solution of the Cauchy problem* 2.1 *with $c = n$, then $\Phi(x) = \varphi(|x|) \in C^2(B_R)$ is a classical solution of Eq.* 3.2 *in the ball B_R.*

(ii) *Let f and g be continuous nondecreasing functions, with f positive and g non-negative. For any $q > 0$, if $\varphi \in C^2([0, R))$ is a solution of the Cauchy problem* 2.1 *with $c = k$, then $\Phi(x) = \varphi(|x|) \in C^2(B_R)$ is a classical solution of Eq.* 3.3 *in the ball B_R.*

Proof A direct computation shows that, if $\Phi(x) = \varphi(|x|)$, then

$$
D^2\Phi(x) = \begin{cases} \varphi''(0)\, I_n & \text{if } x = 0 \\ \frac{\varphi'(|x|)}{|x|} I_n + \left(\varphi''(|x|) - \frac{\varphi'(|x|)}{|x|}\right) \frac{x}{|x|} \otimes \frac{x}{|x|} & \text{if } x \neq 0 \end{cases}
$$

Since $\varphi \in C^2([0, R))$, $\varphi'(0) = 0$ and $\varphi''(0) = \lim_{r \to 0} \varphi'(r)/r$, then Φ belongs to $C^2(B_R)$. Moreover, Φ is convex since φ is convex and increasing, and the very definition of operator $\mathcal{M}_{0,1}^+$ yields

$$
\mathcal{M}_{0,1}^+(D^2\Phi(x)) = \varphi''(|x|) + (n-1)\frac{\varphi'(|x|)}{|x|}.
$$

Therefore, if φ solves 2.1 with $c = n$, then Φ solves 3.2.

Analogously, we observe that, if $g \geq 0$ and φ solves 2.1 with $c = k$, then $\varphi''(|x|) \geq \frac{\varphi'(|x|)}{|x|}$ by Lemma 2.1 (iii). Therefore

$$
\mathcal{P}_k^+(D^2\Phi(x)) = \varphi''(|x|) + (k-1)\frac{\varphi'(|x|)}{|x|},
$$

and Φ is a solution of 3.3. $\qquad\square$

In the next result we recall a version of comparison principle that will be needed in the sequel. It is an immediate consequence of the definition of sub/supersolution when one of the functions to be compared is smooth. For the general regularizing argument needed to compare merely viscosity sub and super solutions we refer to [15].

Proposition 3.2 *Let f, g be continuous functions, with f strictly increasing and g nondecreasing, and let further $u \in USC(B_R)$ and $\Phi \in C^2(B_R)$ satisfy*

$$
F(x, D^2u) - f(u) - g(u) |Du|^q \geq 0 \geq F(x, D^2\Phi) - f(\Phi) - g(\Phi) |D\Phi|^q \quad \text{in } B_R
$$

and

$$
\limsup_{|x| \to R^-} (u(x) - \Phi(x)) \leq 0.
$$

Then $u(x) \leq \Phi(x)$ for all $x \in B_R$.

Proof By contradiction, suppose $u - \Phi$ has a positive maximum at some interior point $x_0 \in B_R$. By using $\Phi(x) + u(x_0) - \Phi(x_0)$ as test function at x_0 in the definition of viscosity subsolution for u it follows that

$$F(x_0, D^2\Phi(x_0)) \geq f(u(x_0)) + g(u(x_0)) |D\Phi(x_0)|^q,$$

which, by the strict monotonicity of f and the monotonicity of g, contradicts the fact that Φ is a supersolution. □

Our first main result provides necessary and sufficient conditions for the existence of entire subsolutions of Eq. 3.2.

Theorem 3.3 *Let f, g be continuous nondecreasing functions, with f positive and strictly increasing.*

(i) *If $\lim_{t \to +\infty} g(t) > 0$, then there exists $u \in USC(\mathbb{R}^n)$ entire viscosity subsolution of 3.2 if and only if condition 2.23 is satisfied.*
(ii) *If $\lim_{t \to +\infty} g(t) = 0$, then there exists $u \in USC(\mathbb{R}^n)$ entire viscosity subsolution of 3.2 if and only if condition 2.22 is satisfied.*
(iii) *If $\lim_{t \to +\infty} g(t) < 0$, then there exists $u \in USC(\mathbb{R}^n)$ entire viscosity subsolution of 3.2 if and only if condition 2.21 is satisfied.*

Proof (i) If condition 2.21 is satisfied, then any maximal solution of the Cauchy problem 2.1 with $c = n$, which is globally defined in $[0, +\infty)$ by Theorem 2.5 (i), gives, by Lemma 3.1, a smooth entire (sub)solution of Eq. 3.2.

Conversely, assume by contradiction that there exists $u \in USC(\mathbb{R}^n)$ entire subsolution of 3.2 and 2.21 does not hold true. Let us consider a maximal solution $\varphi(r)$ of the Cauchy problem 2.1 with $c = n$ and $a < u(0)$. Then, again by Theorem 2.5 (i), $\varphi(r)$ blows up for $r = R(a) \in (0, +\infty)$. On the other hand, by Lemma 3.1 and Proposition 3.2, the functions u and $\Phi(x) = \varphi(|x|)$ can be compared in $B_{R(a)}$ and we obtain the contradiction

$$u(0) \leq \Phi(0) = a < u(0).$$

In the same way, statements (ii) and (iii) follow by Theorem 2.5 (ii) and (iii) respectively. □

Remark 3.4 By the maximality of operator $\mathcal{M}_{0,1}^+$, conditions 2.21, 2.22 and 2.23 are necessary conditions for the existence of entire viscosity solutions of inequality 3.1 respectively in the cases $\lim_{t \to +\infty} g(t) > 0$, $\lim_{t \to +\infty} g(t) = 0$ and $\lim_{t \to +\infty} g(t) < 0$.

The same proof of Theorem 3.3 can be applied to subsolutions of Eq. 3.3. But, in this case, the extra assumption $g(t) \geq 0$ is needed in order to have correspondence between solutions of 2.1 and radially symmetric solutions of 3.3.

Theorem 3.5 *Let f, g be continuous nonnegative nondecreasing functions, with f positive and strictly increasing. There exists an entire viscosity subsolution $u \in USC(\mathbb{R}^n)$ of Eq. 3.3 if and only if condition 2.21 is satisfied.*

The comparison argument used in the proof of Theorem 3.3 actually can be applied in order to estimate from above viscosity solutions of inequality 3.1 in any open subset $\Omega \subset \mathbb{R}^n$. If $\partial\Omega \neq \emptyset$, for $x \in \mathbb{R}^n$, we set

$$d(x) = \text{dist}(x, \partial\Omega)$$

to denote the distance function from the boundary $\partial\Omega$. In order to state our universal upper bounds we need to invert strictly decreasing functions $\mathcal{R} : \mathbb{R} \to (0, +\infty)$ such that $\lim_{a\to+\infty} \mathcal{R}(a) = 0$. For $b > 0$, we will denote by $\mathcal{R}^{-1}(b)$ the unique real number a such that $\mathcal{R}(a) = b$ if it exists, and $\mathcal{R}^{-1}(b) = -\infty$ otherwise.

Theorem 3.6 *Let f, g be continuous nondecreasing functions, with f positive, strictly increasing and such that*

$$\lim_{t\to+\infty} f(t) = +\infty \quad \text{if} \ \lim_{t\to+\infty} g(t) < +\infty. \tag{3.4}$$

Let further $\Omega \subset \mathbb{R}^n$ be an open domain with non empty boundary, and, for $q \in (0, 2]$, let $u \in USC(\Omega)$ be a viscosity solution of 3.1 in Ω.

(i) *Assume that $\lim_{t\to+\infty} g(t) > 0$ and condition 2.23 is not satisfied. Then, pointwisely in Ω, one has*

$$u(x) \leq \max\{t_0, \ \mathcal{R}^{-1}(d(x))\}, \tag{3.5}$$

where $t_0 = \inf\{t \in \mathbb{R} : g(t) \geq 0\}$ and $\mathcal{R} : \mathbb{R} \to (0, +\infty)$ is defined as

$$\mathcal{R}(a) = \begin{cases} 2\left(\dfrac{n}{2-q}\right)^{1/(2-q)} \displaystyle\int_a^{+\infty} \dfrac{dt}{\left(\int_a^t f(s)\,ds\right)^{1/2} + \left(\int_a^t g^+(s)\,ds\right)^{1/(2-q)}} & \text{if } q < 2 \\[4ex] \sqrt{\dfrac{n}{2}} \displaystyle\int_a^{+\infty} \dfrac{dt}{\left(\int_a^t e^{\frac{2}{n}\int_s^t g^+(\tau)\,d\tau} f(s)\,ds\right)^{1/2}} & \text{if } q = 2 \end{cases} . \tag{3.6}$$

(ii) *Assume $\lim_{t\to+\infty} g(t) = 0$ and condition 2.22 is not satisfied. Then, pointwisely in Ω, one has*

$$u(x) \leq \mathcal{R}^{-1}(d(x)),$$

where $\mathcal{R} : \mathbb{R} \to (0, +\infty)$ is defined as

$$\mathcal{R}(a) = \dfrac{n^{1/q}}{\sqrt{q}} \int_a^{+\infty} \dfrac{dt}{\left(\int_a^t e^{-2\int_s^t \left(\frac{g^-(r)}{f(r)}\right)^{2/q} f(r)\,dr} f(s)\,ds\right)^{1/2}} . \tag{3.7}$$

(iii) *Assume $\lim_{t\to+\infty} g(t) < 0$ and condition 2.23 is not satisfied. Then, pointwisely in Ω, one has*

$$u(x) \leq \mathcal{R}^{-1}(d(x)),$$

where $\mathcal{R} : \mathbb{R} \to (0, +\infty)$ is defined as

$$\mathcal{R}(a) = 2\frac{n^{1/q}}{q} \int_a^{+\infty} \left[\frac{1}{\left(\int_a^t f(s)\,ds\right)^{1/2}} + \left(\frac{\int_a^t g^-(s)\,ds}{\int_a^t f(s)\,ds}\right)^{1/q} \right] dt. \quad (3.8)$$

Proof (i) As already observed, u is also a subsolution in Ω of Eq. 3.2. Let $x_0 \in \Omega$ be fixed, and let us set $d_0 = d(x_0)$. In particular, $u \in USC(\overline{B}_R(x_0))$ is a subsolution of 3.2 in $B_R(x_0)$ for every $0 < R < d_0$. On the other hand, we know from Theorem 2.5 (i) that, for any $a \in \mathbb{R}$, a maximal solution φ_a of 2.1 is defined on a maximal bounded interval $[0, R(a))$, with $0 < R(a) < +\infty$. Moreover, by 2.10 and by estimate 2.17 of Lemma 2.3, it follows that, if $a \geq t_0$, then $R(a) \leq \mathcal{R}(a)$, with $\mathcal{R}(a)$ defined by 3.6. Also, Lemma 3.1 implies that the function $\Phi_a(x) = \varphi_a(|x - x_0|)$ solves Eq. 3.2 in $B_{R(a)}(x_0)$. Therefore, by Proposition 3.2, it follows that

$$u(x_0) \leq \Phi_a(x_0) = a, \quad (3.9)$$

provided that $a \geq t_0$ is such that $\mathcal{R}(a) < d_0$. Now, by arguing as in the proof of Lemma 2.4, the function $\mathcal{R}$, which is well defined by definition if $q > 1$ and by assumption if $q \leq 1$, is easily seen to be strictly decreasing. If $\mathcal{R}(t_0) \leq d_0$, then 3.5 follows by 3.9. If $\mathcal{R}(t_0) > d_0$, in order to conclude it is enough to show that $\lim_{a \to +\infty} \mathcal{R}(a) = 0$.

If $q = 2$, we observe that, for a sufficiently large such that $g^+(a) > 0$, one has

$$\mathcal{R}(a) \leq \sqrt{\frac{2\,g^+(a)}{n\,f(a)}} \int_a^{+\infty} \frac{dt}{\left(e^{\frac{2}{n}g^+(a)\,(t-a)} - 1\right)^{1/2}} = \pi\sqrt{\frac{n}{2\,g^+(a)\,f(a)}},$$

and the conclusion follows by assumption 3.4.

If $1 < q < 2$, we note that, again for a such that $g^+(a) > 0$ and for every $\delta > 0$, one has

$$\mathcal{R}(a) \leq 2\left(\frac{n}{2-q}\right)^{1/(2-q)} \left[\int_a^{a+\delta} \frac{dt}{\sqrt{f(a)\,(t-a)}} + \int_{a+\delta}^{+\infty} \frac{dt}{\left(g^+(a)\,(t-a)\right)^{1/(2-q)}} \right]$$

$$= 2\left(\frac{n}{2-q}\right)^{1/(2-q)} \left[2\sqrt{\frac{\delta}{f(a)}} + \frac{2-q}{q-1}\left(\frac{\delta^{1-q}}{g^+(a)}\right)^{1/(2-q)} \right].$$

By letting $a \to +\infty$ and by using assumption 3.4 we obtain that

$$\lim_{a \to +\infty} \mathcal{R}(a) \leq \frac{2}{q-1}\left(\frac{n}{\lim\limits_{a \to +\infty} g(a)}\right)^{1/(2-q)} ((2-q)\,\delta)^{(1-q)/(2-q)},$$

or

$$\lim_{a \to +\infty} \mathcal{R}(a) \leq 4 \left(\frac{n}{2-q} \right)^{1/(2-q)} \sqrt{\frac{\delta}{\lim\limits_{a \to +\infty} f(a)}},$$

and the conclusion follows by letting $\delta \to +\infty$ or $\delta \to 0$ respectively.

Finally, if $0 < q \leq 1$, then, since 2.21 is not satisfied, by Remark 2.8 we have that either

$$\int_a^{+\infty} \frac{dt}{\left(\int_a^t f(s) \, ds \right)^{1/2}} < +\infty \quad \text{for every } a \in \mathbb{R},$$

or

$$\int_a^{+\infty} \frac{dt}{(t \, g^+(t))^{1/(2-q)}} < +\infty \quad \text{for every } a > 0 \text{ such that } g^+(a) > 0.$$

In the former case, we simply observe that

$$\mathcal{R}(a) \leq 2 \left(\frac{n}{2-q} \right)^{1/(2-q)} \int_a^{+\infty} \frac{dt}{\left(\int_a^t f(s) \, ds \right)^{1/2}}$$

and that the right hand side goes to 0 as $a \to +\infty$ by applying Lemma 2.4 with $h \equiv 0$. In the latter case, we notice that necessarily one has

$$0 = \lim_{t \to +\infty} \begin{cases} \dfrac{t^{1-q}}{g(t)} & \text{if } q < 1 \\[2ex] \dfrac{\ln t}{g(t)} & \text{if } q = 1 \end{cases}.$$

Then, by a similar argument as above, for every $\delta > 0$ sufficiently small, we can estimate, up to the constant factor $2 \left(\frac{n}{2-q} \right)^{1/(2-q)}$,

$$\mathcal{R}(a) \leq \left[\int_a^{a+\delta} \frac{dt}{\sqrt{f(a)(t-a)}} + \int_{a+\delta}^{2a} \frac{dt}{(g(a)(t-a))^{1/(2-q)}} + \int_{2a}^{+\infty} \frac{dt}{\left(\frac{t}{2} g \left(\frac{t}{2} \right) \right)^{1/(2-q)}} \right].$$

By letting first $a \to +\infty$ and then $\delta \to 0$ we get that $\lim_{a \to +\infty} \mathcal{R}(a) = 0$, and then 3.5 is proved also in the present case, i.e. for $q \leq 1$.

(ii) We repeat the same comparison argument as above, and conclude by using estimate 2.18 of Lemma 2.3 and by applying Lemma 2.4.

(iii) We argue again as in (i) and (ii), and we further apply inequality 2.28. The function $\mathcal{R}$ is easily proved also in this case to be strictly decreasing and to tend to 0 as $a \to +\infty$ by analogous arguments as in (i). □

Remark 3.7 The hypothesis [3.4](#) is needed in the case $\lim_{t \to +\infty} g(t) > 0$ and $q > 1$, whereas it is a consequence of the failure of conditions [2.21](#), [2.22](#) and [2.21](#) in the other cases. It guarantees that the amplitude $R(a)$ of the maximal interval of existence of a maximal solution of the Cauchy problem [2.1](#) satisfies

$$\lim_{a \to +\infty} R(a) = 0.$$

To show its necessity, we can use the same counterexample exhibited in [34]. Indeed, let us assume $q = 2$ and $g \equiv 1$, and for any $a \in \mathbb{R}$ let $\varphi \in C^2([0, R(a)))$ be a maximal solution of

$$\begin{cases} \varphi'' + \dfrac{n-1}{r} \varphi' = f(\varphi) + (\varphi')^2 \\ \varphi(0) = a, \ \varphi'(0) = 0 \end{cases}.$$

Then, the radial function $\Phi(x) = \varphi(|x|)$ satisfies also

$$\begin{cases} -\Delta\Phi + f(\Phi) + |D\Phi|^2 = 0 & \text{in } B_{R(a)} \\ \Phi = +\infty & \text{on } \partial B_{R(a)} \end{cases},$$

so that, the function $\Psi(x) = e^{-\Phi(x)}$ solves

$$\begin{cases} -\Delta\Psi = f\left(\ln \dfrac{1}{\Psi}\right)\Psi & \text{in } B_{R(a)} \\ \Psi > 0 \ \text{in } B_{R(a)}, \ \Psi = 0 & \text{on } \partial B_{R(a)} \end{cases}.$$

By the properties of the first eigenvalue of the laplacian associated with homogeneous Dirichlet boundary conditions, it then follows that

$$\lim_{t \to +\infty} f(t) \geq \lambda_1(B_{R(a)}),$$

where $\lambda_1\left(B_{R(a)}\right)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in $B_{R(a)}$. Therefore, if we denote by λ_1 the first eigenvalue of $-\Delta$ in B_1 and if $\lim_{t \to +\infty} f(t) < +\infty$, then we deduce

$$R(a) \geq \sqrt{\frac{\lambda_1}{\lim_{t \to +\infty} f(t)}} \quad \forall a \in \mathbb{R}.$$

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