# Controllability of spin-boson systems

Ugo Boscain, Paolo Mason, Gianluca Panati, and Mario Sigalotti

**Abstract**

In this paper we study the so-called spin-boson system, namely a two-level system in interaction with a distinguished mode of a quantized bosonic field. We give a brief description of the controlled Rabi and Jaynes–Cummings models and we discuss their appearance in the mathematics and physics literature. We then study the controllability of the Rabi model when the control is an external field acting on the bosonic part. Applying geometric control techniques to the Galerkin approximation and using perturbation theory to guarantee non-resonance of the spectrum of the drift operator, we prove approximate controllability of the system, for almost every value of the interaction parameter.

# 1 Introduction and description of the models

## 1.1 The Rabi model *alias* the standard spin-boson model

In the Hilbert space $\mathcal{H} = L^2(\mathbf{R}, \mathbf{C}) \otimes \mathbf{C}^2$, we consider the Schrödinger equation

$$\mathrm{i}\,\partial_t \psi = H_{\mathrm{Rabi}} \psi, \tag{1}$$

where

$$H_{\mathrm{Rabi}} = \frac{\omega}{2}(P^2 + X^2) \otimes \mathbb{1} + \frac{\Omega}{2}\mathbb{1} \otimes \sigma_3 + g\,X \otimes \sigma_1, \tag{2}$$

for some fixed positive constants $\omega$ and $\Omega$, and $g \in \mathbf{R}$, where $X$ is the multiplication operator defined by $(X\psi)(x) = x\psi(x)$, and $P = -i\partial_x$. Here and in the following we use $\sigma_1, \sigma_2, \sigma_3$ to denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the natural isomorphism $\mathcal{H} \cong L^2(\mathbf{R}, \mathbf{C}^2)$, one rewrites (1) as the system

$$\mathrm{i}\,\partial_t \psi_+ = \frac{\omega}{2}(-\partial_x^2 + x^2)\psi_+ + \frac{\Omega}{2}\psi_+ + gx\psi_-$$

$$\mathrm{i}\,\partial_t \psi_- = \frac{\omega}{2}(-\partial_x^2 + x^2)\psi_- - \frac{\Omega}{2}\psi_- + gx\psi_+.$$

In the physics literature, it is customary to introduce the *raising* and *lowering* matrices

$$\sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\,\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\,\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Setting

$$a = \frac{1}{\sqrt{2}}(X + \mathrm{i}\,P), \qquad a^\dagger = \frac{1}{\sqrt{2}}(X - \mathrm{i}\,P),$$

and omitting tensors, $H_{\mathrm{Rabi}}$ appears in physics literature as

$$H_{\mathrm{Rabi}} = \omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\Omega}{2}\sigma_3 + \frac{g}{\sqrt{2}}(a + a^\dagger)(\sigma_- + \sigma_+). \tag{3}$$

Modeling of spin-boson interactions was initiated by Rabi in the 30's [21, 22]. To our knowledge, the Hamiltonian (3) was first derived from a more fundamental model in the milestone paper [13], where also the simpler Jaynes-Cummings Hamiltonian appears (see Section 1.2). A throughout discussion on various aspects of this model can be found in [3]. Recent results on its integrability have been obtained in [7].

The Hamiltonian (3) is ubiquitous in the literature, since it encodes one of the simplest possible couplings between a harmonic oscillator and a two-level system. The physical interpretation of the two factors appearing in the tensor product varies according to the context. We briefly outline the interpretation in cavity QED in the following digression.

### 1.1.1 Digression: the Rabi model in the context of cavity QED

While we refer to the specific literature for an exhaustive treatement of the problem [13, 11, 23], we briefly sketch the derivation of the Hamiltonian (3) in the context of *Cavity Quantum Electro Dynamics* (CQED). In a typical CQED experiment, an atom moves across a Fabry–Perot cavity interacting with the quantized electromagnetic (EM) field of the cavity. The atom is initially prepared in a special state, to guarantee both a strong interaction with the cavity field and a decay time (to the atomic ground state) longer than the experiment timescale. The first goal is obtained by choosing a hydrogenoid atom (e.g., a Rubidium atom) prepared in a *circular Rydberg state* with high orbital angular momentum, and so with a large dipole moment. The second goal requires instead that transitions to the other atomic states are not resonant with the radiation from the environment, typically in the visible or infrared spectrum. Since the atomic energies are $E_n = -\frac{R_{\mathrm{Rb}}}{n^2}$ (where $R_{\mathrm{Rb}}$ is the Rydberg constant, corrected for the Rubidium), one select $n \in \mathbf{N}$ in such a way that the transition frequency $\Omega_{\mathrm{at}} := \frac{1}{\hbar}(E_n - E_{n-1})$ is in the range of the microwaves (in the case of Rb, this is obtained by choosing $n = 51$). The cavity is designed in such a way that a normal mode of the EM field has frequency $\omega$, which is almost resonant with $\Omega_{\mathrm{at}}$, i.e., $|\Omega_{\mathrm{at}} - \omega| \ll \omega, \Omega_{\mathrm{at}}$.

Under these conditions, the only appreciable interactions are those among the two eigenstates $|n\rangle$ and $|n-1\rangle$, and the distinguished mode of the EM field with frequency $\omega$. The linear space generated by $|n\rangle$ and $|n-1\rangle$ corresponds to the factor $\mathbf{C}^2$ in $\mathcal{H}$, while the quantized EM mode yields a quantum harmonic oscillator, corresponding to the factor $L^2(\mathbf{R}, \mathbf{C})$. This explains the structure of the first two terms in (3).

A deeper understanding of (3) is obtained by deriving it, heuristically, from the Pauli–Fierz model, see [29, Chapter 13]. We consider a hydrogenoid atom, as, e.g., Rubidium, interacting with the quantized electromagnetic (EM) field in the cavity. The electron, with effective mass $m_*$ and charge $-e$, is described by a form factor $\varphi \in C_0^\infty(\mathbf{R}^3)$, which is assumed to be a radial positive function, normalized so that $\int_{\mathbf{R}^3} \varphi(x)dx = 1$. The atomic core is supposed at rest at the origin or, equivalently, moving across the cavity with constant velocity.

Within the Pauli–Fierz model, the quantized EM fields are described by operator-valued distributions acting on the Hilbert space $\mathcal{H}_{\mathrm{at}} \otimes \mathcal{H}_{\mathrm{f}}$, where $\mathcal{H}_{\mathrm{at}} = L^2(\mathbf{R}^3, dr)$ corresponds to the atom, while

$$\mathcal{H}_{\mathrm{f}} = \bigoplus_{n \in \mathbf{N}} \left( L^2(\mathbf{R}^3, dk) \otimes \mathbf{C}^2 \right)^{\otimes n}_{\mathrm{sym}}$$

is the bosonic Fock space corresponding to the field [29]. In the Coulomb gauge, the vector potential (in free space) reads

$$A_\varphi(r) = \sum_{\lambda=1}^2 \int_{\mathbf{R}^3} dk \ \sqrt{\frac{\hbar c^2}{2\omega(k)}} \ \widehat{\varphi}(k) \, e_\lambda(k) \ \left( \mathrm{e}^{\mathrm{i} k \cdot r} \otimes a(k, \lambda) + \mathrm{e}^{-\mathrm{i} k \cdot r} \otimes a(k, \lambda)^\dagger \right), \qquad (4)$$

where $\omega(k) = c|k|$, $\widehat{\varphi}$ denotes the Fourier transform of $\varphi$, the three vectors $\{k/|k|, e_1(k), e_2(k)\}$ form an orthonormal basis at every point $k \in \mathbf{R}^3$, and $a(k, \lambda)$ and $a(k, \lambda)^\dagger$ satisfy the canonical commutations relations. Analogously, the transverse electric field reads

$$E_{\varphi, \perp}(r) = \sum_{\lambda=1}^2 \int_{\mathbf{R}^3} dk \ \sqrt{\frac{\hbar \omega(k)}{2}} \ \widehat{\varphi}(k) \, e_\lambda(k) \ \mathrm{i} \left( \mathrm{e}^{\mathrm{i} k \cdot r} \otimes a(k, \lambda) - \mathrm{e}^{-\mathrm{i} k \cdot r} \otimes a(k, \lambda)^\dagger \right). \qquad (5)$$

Actually, since the field is confined in the cavity, described by a compact set $C \subset \mathbf{R}^3$, the integral appearing in (4) and (5) can be replaced by an infinite sum $\sum_{k \in \mathcal{I}}$, where $\mathcal{I}$ labels the solutions to the eigenvalue equation $\Delta A = \lambda_k A$ satisfying $\nabla \cdot A = 0$ and appropriate boundary conditions on $\partial C$, see [29, Sec. 13.6].

The *(spinless) Pauli–Fierz Hamiltonian* is the self-adjoint operator

$$H = \frac{1}{2m_*} \left( \mathrm{i}\hbar \nabla_r + \frac{e}{c} A_\varphi(r) \right)^2 + \mathbb{1} \otimes H_{\mathrm{f}} - e^2 \, V_{\varphi, \mathrm{Coul}}(r) \otimes \mathbb{1}, \qquad (6)$$

where

$$H_{\mathrm{f}} = \sum_{\lambda=1}^2 \int_{\mathbf{R}^3} dk \, \hbar \omega(k) \, a(k, \lambda)^\dagger a(k, \lambda)$$

corresponds to the field energy, while

$$V_{\varphi, \mathrm{Coul}}(r) = \int_{\mathbf{R}^3} dk \ \frac{|\widehat{\varphi}(k)|^2}{|k|^2} \mathrm{e}^{-\mathrm{i} k \cdot r} = \int_{\mathbf{R}^3} dx \int_{\mathbf{R}^3} dy \ \frac{\varphi(x - r)\varphi(y)}{4\pi|x - y|}$$

is the Coulomb interaction smeared through $\varphi$.

To derive the Rabi Hamiltonian (3) from (6), several approximations are in order. [1]

First, since the electron is bound to the atomic core, "one loses little by evaluating the vector potential at the origin instead of at $r$, the position of the electron" [29, Sec. 13.7]. Within such approximation, and after a suitable change of gauge, the Hamiltonian becomes

$$H_{\mathrm{dip}} = \left( -\frac{\hbar^2}{2m_*} \Delta_r + V_{\varphi, \mathrm{Coul}}(r) \right) \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} - e \, r \cdot E_{\varphi, \perp}(0) + \epsilon_{\mathrm{dip}}(r), \qquad (7)$$

---

[1] Obviously, in the comparison, the order of factors in the tensor product should be reversed, so that $L^2(\mathbf{R}, \mathbf{C}) \otimes \mathbf{C}^2$ is an effective reduced space for $\mathcal{H}_{\mathrm{f}} \otimes \mathcal{H}_{\mathrm{at}}$. Since both conventions are well-established, we did not dare to reverse the order of factors in any of the two models, namely in (2) and (6).

where $r \cdot E(0) = \sum_{j=1}^{3} r_j \otimes E_j(0)$. The expression for $H_{\mathrm{dip}}$ involves the dipole operator $D = -e\, r$, thus explaining the name *dipole approximation*. Here a quadratic potential appears, namely

$$\epsilon_{\mathrm{dip}}(r) = \frac{1}{2}\left(\frac{2}{3}\int_{\mathbf{R}^3} dk\, e^2 |\widehat{\varphi}(k)|^2\right) r^2,$$

which is usually neglected in this context.

As a second crucial approximation, *only a distinguished mode of the cavity field* is considered as interacting with the atom. If $k_*$ is its wavevector, the integrals (or sums) above reduce to the evaluation at $k_*$. Accordingly, and assuming that $\widehat{\varphi}(k_*) = 1$, one has

$$
\begin{aligned}
A_\varphi(0) &= \sum_{\lambda=1}^{2}\sqrt{\frac{\hbar c^2}{2\omega(k_*)}}\; e_\lambda(k_*)\; \left[\mathbb{1}\otimes\left(a(k_*,\lambda) + a(k_*,\lambda)^\dagger\right)\right]\\
E_{\varphi,\perp}(0) &= \sum_{\lambda=1}^{2}\sqrt{\frac{\hbar\omega(k_*)}{2}}\; e_\lambda(k_*)\; \left[\mathbb{1}\otimes\mathrm{i}\left(a(k_*,\lambda) - a(k_*,\lambda)^\dagger\right)\right]. \qquad (8)\\
H_{\mathrm{f}} &= \sum_{\lambda=1}^{2}\hbar\omega(k_*)\, a(k_*,\lambda)^\dagger a(k_*,\lambda).
\end{aligned}
$$

Neglecting the sum over the polarization vectors $e_\lambda$'s, we select an index $\lambda_* \in \{1,2\}$ and we set $\omega(k_*)\equiv\omega$ and $a(k_*,\lambda_*)\equiv a_{\mathrm{c}}$, where the subscript c stands for *cavity*. Then one has the correspondences

$$A_\varphi(0) \simeq a_{\mathrm{c}} + a_{\mathrm{c}}^\dagger = \sqrt{2}X_{\mathrm{c}}, \qquad \text{and} \qquad E_{\varphi,\perp}(0) \simeq \mathrm{i}(a_{\mathrm{c}} - a_{\mathrm{c}}^\dagger) = -\sqrt{2}P_{\mathrm{c}}, \qquad (9)$$

where the last equality refers to the fact that the operators $\frac{1}{\sqrt{2}}(a_{\mathrm{c}} + a_{\mathrm{c}}^\dagger)$ and $\frac{-\mathrm{i}}{\sqrt{2}}(a_{\mathrm{c}} - a_{\mathrm{c}}^\dagger)$ can be identified, up to unitary equivalence, with a canonical Schrödinger pair $(X_{\mathrm{c}}, P_{\mathrm{c}})$ acting on $L^2(\mathbf{R}, dq)$. Next, an inverse Fourier transform $\mathcal{F}^{-1} : L^2(\mathbf{R}, dq) \to L^2(\mathbf{R}, dx)$ is performed, which intertwines the pair $(X_{\mathrm{c}}, -P_{\mathrm{c}})$ with the pair $(P, X)$, where $P = -\mathrm{i}\partial_x$ and $(X\psi)(x) = x\psi(x)$. Correspondingly, $\mathcal{F}^{-1} a_{\mathrm{c}} \mathcal{F} = -\mathrm{i}a$, so that $H_{\mathrm{f}}$ is unitarily transformed into $\tilde{H}_{\mathrm{f}} = \hbar\omega_{\mathrm{c}} a^\dagger a$, which corresponds to the first addendum in (3), up to an irrelevant constant. Moreover,

$$A_\varphi(0) \simeq -\mathrm{i}(a - a^\dagger) = \sqrt{2}P, \qquad \text{and} \qquad E_{\varphi,\perp}(0) \simeq a + a^\dagger = \sqrt{2}X, \qquad (10)$$

which shows that the interaction term (the third addendum in (2) or in (3)) is proportional to the electric field, which is assumed to be uniform in view of the dipole approximation.

Third, one assumes that effectively *only two atomic states* $|n-1\rangle \equiv |g\rangle$ and $|n\rangle \equiv |e\rangle$ are appreciably coupled to the radiation field. Then, the factor $\mathcal{H}_{\mathrm{at}}$ is replaced by a 2-dimensional Hilbert space

$$\mathcal{H}_{2L} = \mathrm{Span}_{\mathbf{C}}\{|g\rangle, |e\rangle\} \simeq \mathbf{C}^2,$$

and the atomic Hamiltonian (the first term in (7)) by the 2-level Hamiltonian $\frac{\hbar\Omega_{\mathrm{at}}}{2}\sigma_3$, where $\hbar\Omega_{\mathrm{at}} = E_n - E_{n-1}$.

Finally, if the two selected atomic states are *circular Rydberg states* one has that the dipole operator is off-diagonal in the basis $\{|g\rangle, |e\rangle\}$, namely

$$\langle g|\, D\, |g\rangle = 0, \qquad \langle e|\, D\, |e\rangle = 0.$$

Moreover, there exist $d > 0$ and a versor $e_{\mathrm{at}} \in \mathbf{C}^3$ such that $d\, e_{\mathrm{at}} = \langle e|\, D\, |g\rangle$. We identify a state $\psi_{\mathrm{at}} \in \mathcal{H}_{2L}$ with the pair $(\psi_g, \psi_e) \in \mathbf{C}^2$ via $\psi_{\mathrm{at}} = \psi_g\, |g\rangle + \psi_e\, |e\rangle$, so that

$$D\psi_{\mathrm{at}} = d\, (\psi_g\, e_{\mathrm{at}}^*\, |e\rangle + \psi_e\, e_{\mathrm{at}}\, |g\rangle)\,.$$

By using the latter fact, one obtains

$$D \cdot E_{\varphi,\perp}(0) = \sqrt{\frac{\hbar\omega(k_*)}{2}} d\, e_{\lambda_*}(k_*) \cdot \left[ e_{\mathrm{at}}^* \left( \sigma_+ \otimes \mathrm{i}(a_{\mathrm{c}} - a_{\mathrm{c}}^\dagger) \right) + e_{\mathrm{at}} \left( \sigma_- \otimes \mathrm{i}(a_{\mathrm{c}} - a_{\mathrm{c}}^\dagger) \right) \right], \qquad (11)$$

where, with a little abuse of notation, we wrote $\sigma_- = |g\rangle\,\langle e|$ and $\sigma_+ = |e\rangle\,\langle g|$. When adapting the previous formula to concrete physical situations, by specializing to the actual values of $e_{\mathrm{at}}$ and $e_\lambda(k_*)$, one obtains some terms analogous to the ones appearing in (3), possibly with different weights. In particular, if $e_{\mathrm{at}}^* = e_{\mathrm{at}}$, one obtains a coupling term proportional to $\sigma_1 \otimes P_{\mathrm{c}}$, which, after the Fourier transform mentioned above, becomes $\sigma_1 \otimes X$ and so agrees with the third addendum in (2) or in (3).

This concludes the heuristic "derivation" of the Rabi Hamiltonian, starting from a fundamental model as the one given by (6). On the other hand, a rigorous derivation of the Hamiltonian (3) from a mathematical model of non-relativistic QED, as, e.g., the *Pauli–Fierz model*, in a suitable scaling limit, is to our knowledge absent from the literature.

## 1.2 The Jaynes–Cummings model

The Rabi model presented in the previous section has been considered difficult to treat in view of the widespread opinion that it is not explicitly solvable (cf. [7]).

For this reason Jaynes and Cummings [13] proposed an approximated Hamiltonian, which is obtained from $H_{\mathrm{Rabi}}$ by neglecting the so-called counter-rotating terms. More precisely, the interaction term in $H_{\mathrm{Rabi}}$, namely

$$\frac{g}{\sqrt{2}} (a^\dagger + a)\sigma_1 = \frac{g}{\sqrt{2}} (a^\dagger \sigma_- + a\sigma_+ + a^\dagger \sigma_+ + a\sigma_-),$$

is replaced by a new interaction term where the last two terms in the right-hand side are neglected under the assumption that

$$|\Omega - \omega| \ll \Omega, \qquad g \ll \Omega, \omega. \qquad (12)$$

According to this procedure, one obtains

$$H_{\mathrm{JC}} = \omega \left( a^\dagger a + \frac{1}{2} \right) + \frac{\Omega}{2}\sigma_3 + \frac{g}{\sqrt{2}} (a^\dagger \sigma_- + a\sigma_+) \qquad (13)$$

$$= \frac{\omega}{2}(P^2 + X^2) \otimes \mathbb{1} + \mathbb{1} \otimes \frac{\Omega}{2}\sigma_3 + \frac{g}{2}(X \otimes \sigma_1 - P \otimes \sigma_2).$$

This model is considerably simpler than the Rabi one, since it admits the conserved quantity

$$C = a^\dagger a \otimes \mathbb{1} + \mathbb{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which represents the total number of excitations. Indeed, the interaction term in $H_{\mathrm{JC}}$ contains two parts: the term $a^\dagger \sigma_-$ creates an excitation of the bosonic mode and destroys one of the two-level system, while $a\sigma_+$ acts in the opposite way.

The heuristic justification of the approximation leading to (13) is based on separation of time scales. Indeed, when considering the dynamics in the interaction picture, the terms $a^\dagger \sigma_-$ and $a\sigma_+$, which conserve the total number of excitations of the system, evolve periodically with frequency $|\Omega - \omega|$, while the remaining terms $a^\dagger \sigma_+$ and $a\sigma_-$ oscillate with frequency $\Omega + \omega$. If the *almost-resonance condition* (12) is satisfied, the latter oscillations average to zero on the longer time scale $|\Omega - \omega|^{-1}$.
On the other hand, a rigorous mathematical justification for this approximation seems absent from the literature. However, as already noticed by Rouchon in [27], it seems possible to obtain it by adapting the methods developed in [20, 19, 30]. A similar task, by analytical methods, has been accomplished by Ervedoza and Puel on a related model [12].

Notice that in applications to circuits QED the assumption $g \ll \Omega, \omega$ is in general not satisfied, since it is possible to achieve spin-filed interactions which are not much smaller than the field energy [28]. This is a motivation for the direct study of the Rabi model, which is pursued here.

## 1.3 Modeling an external control field

In a wide variety of experimental situations one can act on the system by an external field. The goal of the controller could be to lead the system from a given initial state to a prescribed final one. For spin-boson models, this amounts to study the control problem

$$\mathrm{i}\,\partial_t \psi(t) = H_0 \psi(t) + H_{\mathrm{c}}(u(t))\psi(t), \tag{14}$$

where $H_0$ represents either the Rabi or the Jaynes–Cummings Hamiltonian and $H_{\mathrm{c}}$ is a self-adjoint operator describing the coupling between the system and the controlled external field. The operator $H_{\mathrm{c}}$ depends on the control function $u$ which, in general, takes values in $\mathbf{C}^m$.

In most cases the external field can act on the bosonic mode only, while the spin mode is not directly accessible. This leads to a control Hamiltonian of the form

$$H_{\mathrm{c}}(u(t)) = h_{\mathrm{c}}(u(t)) \otimes \mathbb{1}. \tag{15}$$

where $h_{\mathrm{c}}(u(t))$ is a self-adjoint operator in $L^2(\mathbf{R})$. One of the simplest form for the operator $h_{\mathrm{c}}(u(t))$ is the following

$$h_{\mathrm{c}}(u(t)) = u(t)X. \tag{16}$$

As argued in Section 1.1.1, in the context of cavity QED and within the dipole approximation, the operator $X$ corresponds to the electric field at the origin, see (10). Therefore, in this context, the control (16) corresponds to an external EM field which rescales the value of $E_{\varphi,\perp}(0)$ by a time-dependent factor $u(t)$. Notice that the linearity in $X$ is a consequence of the *dipole approximation*.

Simmetrically, one might also consider a control consisting of an external field which modulates the value of $A_\varphi(0)$, as e.g., an externally-controlled magnetic field. In view of (10), this amounts to consider a control term $\widetilde{h}_{\mathrm{c}}(u(t)) = u(t)P$, a problem which will be investigated elsewhere.

## 1.4   Other related controlled models

Controllability of finite-dimensional approximations of spin-boson systems have first been obtained in the physics literature, via constructive methods, in [17] and [16]. The results have been extended to the full infinite-dimensional model in [31] via analysis of non-resonances and in [1] by adiabatic arguments (see also [6]).

A mathematical description of the controllability properties of a related model is given in [12]. The main technical tool is an explicit estimate of the approximation error with respect to the Lamb–Dicke limit.

For the counterpart of the Eberly–Law model for more than one trapped ion an approximate controllability result is obtained in [4] (see also [24]), based on the analysis on a sequence of nested finite-dimensional systems. In the recent papers [15, 18] the (approximate) controllability of the system is established by considering different families of controlled dynamics.

## 2   Approximate controllability of the controlled Rabi model

We consider here the approximate controllability problem for a system of the form (14), where $H_0$ is the Rabi Hamiltonian and $H_c$ takes the form (15)-(16).

The problem under consideration is then

$$
\begin{cases}
i\partial_t \psi = H_{\text{Rabi}}\psi + u(X \otimes \mathbb{1})\psi, \\
u \in [0, \delta], \\
|\Omega - \omega| \ll \Omega, \\
\delta, \Omega > 0.
\end{cases}
\tag{17}
$$

The main result of the paper is the following. The precise definition of approximate controllability will be given in the next section. It basically means that, for every choice of the initial and final state, there exists an admissible control law $u$, depending on the time, which steers the initial state arbitrarily close to the final one.

**Theorem 1.** *System* (17) *is approximately controllable for almost every* $g \in \mathbf{R}$.

The proof of Theorem 1 goes by studying the applicability of a general approximate controllability result in dependence on the parameter $g$. In order to do so, we have to use perturbation theory in the parameter $g$ up to order 4.

## 3   An approximate controllability result

We are going to recall a general controllability result for bilinear quantum systems in an abstract setting.

In a separable Hilbert space $H$, endowed with the Hermitian product $\langle \cdot, \cdot \rangle$, we consider the following control system

$$
\frac{d}{dt}\psi = (A + u(t)B)\psi, \quad u(t) \in U,
\tag{18}
$$

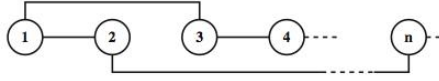where $(A, B, U)$ satisfies the following assumption.

Figure 1: Each vertex of the graph represents an eigenstate of $A$ (when the spectrum is not simple, several nodes may be attached to the same eigenvalue). An edge links two vertices if and only if $B$ connects the corresponding eigenstates. In this example, $\langle \phi_1, B\phi_2 \rangle$ and $\langle \phi_1, B\phi_3 \rangle$ are not zero, while $\langle \phi_1, B\phi_4 \rangle = \langle \phi_2, B\phi_3 \rangle = \langle \phi_2, B\phi_4 \rangle = 0$.

**Assumption 1.** *$U$ is a subset of $\mathbf{R}$ and $(A, B)$ is a pair of (possibly unbounded) linear operators in $H$ such that*

1. *$A$ is skew-adjoint on its domain $D(A)$;*

2. *there exists a Hilbert basis $(\phi_k)_{k \in \mathbf{N}}$ of $H$ consisting of eigenvectors of $A$: for every $k$, $A\phi_k = \mathrm{i}\lambda_k \phi_k$ with $\lambda_k$ in $\mathbf{R}$;*

3. *for every $j$ in $\mathbf{N}$, $\phi_j$ is in the domain $D(B)$ of $B$;*

4. *$A + uB$ is essentially skew-adjoint for every $u \in U$;*

5. *$\langle B\phi_j, \phi_k \rangle = 0$ for every $j, k$ in $\mathbf{N}$ such that $\lambda_j = \lambda_k$ and $j \neq k$.*

If $(A, B, U)$ satisfies Assumption 1, then $A + uB$ generates a unitary group $t \mapsto e^{t(A+uB)}$. By concatenation, one can define the solution of (18) for every piecewise constant function $u$ taking values in $U$, for every initial condition $\psi_0$ given at time $t_0$. We denote this solution by $t \mapsto \Upsilon^u_{t,t_0} \psi_0$.

For every $j, k \in \mathbf{N}$, we denote $b_{jk} = \langle \phi_j, B\phi_k \rangle$. A pair $(j, k)$ in $\mathbf{N}^2$ is a *non-resonant transition* of $(A, B)$ if $b_{jk} \neq 0$ and, for every $l, m$, $|\lambda_j - \lambda_k| = |\lambda_l - \lambda_m|$ implies $\{j, k\} = \{l, m\}$ or $\{l, m\} \cap \{j, k\} = \emptyset$.

A subset $S$ of $\mathbf{N}^2$ is a *chain of connectedness* of $(A, B)$ if for every $j, k$ in $\mathbf{N}$, there exists a finite sequence $p_1 = j, p_2, \ldots, p_r = k$ for which $(p_l, p_{l+1}) \in S$ for every $l$ and $\langle \phi_{p_{l+1}}, B\phi_{p_l} \rangle \neq 0$ for every $l = 1, \ldots, r - 1$. A chain of connectedness $S$ of $(A, B)$ is *non-resonant* if every $(j, k)$ in $S$ is a non-resonant  transition of $(A, B)$.

**Definition 2.** Let $(A, B, U)$ satisfy Assumption 1. We say that (18) is *approximately controllable* if for every $\varepsilon > 0$, for every $\psi_0, \psi_1 \in H$, there exists a piecewise constant function $u_\varepsilon : [0, T_\varepsilon] \to U$ such that $\|\Upsilon^u_{T_\varepsilon, 0} \psi_0 - \psi_1\| < \varepsilon$.

**Theorem 3** ([5])**.** *Assume that $[0, \delta] \subset U$ for some $\delta > 0$ and let $(A, B, U)$ satisfy Assumption 1 and admit a non-resonant chain of connectedness. Then system (18) is approximately controllable.*

# 4    Proof of Theorem 1

In this section, we consider the approximate controllability problem for (17) and we prove Theorem 1. The proof of the theorem is based on a suitable application of Theorem 3.

The strategy of the proof is the following. We first show in Subsection 4.1.1 that, for almost every $g$ in $\mathbf{R}$, some relevant pairs of eigenvalues of $H_{\text{Rabi}}$ satisfy the non-resonance condition, see (22). This goal is achieved by exploiting the analyticity of the eigenvalues and by using perturbation theory. Then, in Subsection 4.1.2, we prove that these pairs of eigenvalues correspond to non-resonant transitions, according to the definition above. The degenerate case $\omega = \Omega$ is treated separately in Section 4.2

Preliminarily, we introduce some additional notations. Denote by $H_{\text{Rabi},0}$ the Hamiltonian $H_{\text{Rabi}}$ where we set $g = 0$. The general form of $H_{\text{Rabi}}$ is then

$$H_{\text{Rabi}} = H_{\text{Rabi},0} + gV, \tag{19}$$

where $V$ is the operator $X \otimes \sigma_1$.

Let $(\varphi_j)_{j \in \mathbf{N}}$ be the standard Hilbert basis of $L^2(\mathbf{R}, \mathbf{C})$ given by real eigenfunctions of $-\partial_x^2 + x^2$, so that $(-\partial_x^2 + x^2)\varphi_j = (2j+1)\varphi_j$ and $\int_{\mathbf{R}} x\varphi_j(x)\varphi_{j+1}(x)dx = \sqrt{(j+1)/2}$ for $j \geq 0$.

Let $\nu_1 = (1,0)^T$, $\nu_{-1} = (0,1)^T$ be the canonical orthonormal basis of $\mathbf{C}^2$. Based on $(\varphi_j)_{j \in \mathbf{N}}$, we obtain a Hilbert basis of factorized eigenstates $\Phi_{j,s} = \varphi_j \otimes \nu_s$, $j \in \mathbf{N}$, $s \in \{-1, 1\}$, of $H_{\text{Rabi},0}$ whose corresponding eigenvalues are

$$E_{j,s} = \omega \left( j + \frac{1}{2} \right) + \frac{s}{2}\Omega. \tag{20}$$

If $\Omega$ is not an integer multiple of $\omega$ then each eigenvalue $E_{j,s}$ is simple. (See Figure 2.) In the following, for ease of notations, we write in bold the elements of $\mathbf{N} \times \{-1, 1\}$, and for every $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$ we define $n(\mathbf{j}), s(\mathbf{j})$ in such a way that $\mathbf{j} = (n(\mathbf{j}), s(\mathbf{j}))$.

For $g \in \mathbf{R}$, denote by $E_{\mathbf{j}}^g$, $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$, the eigenvalues of $H_{\text{Rabi}}$ repeated according to their multiplicities, and by $\Phi_{\mathbf{j}}^g$, $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$, an orthonormal basis of corresponding eigenstates. By a consequence of perturbation theory, detailed in the Appendix, we can assume that, up to the choice of a suitable labelling, each map $\mathbf{R} \ni g \mapsto (E_{\mathbf{j}}^g, \Phi_{\mathbf{j}}^g) \in \mathbf{R} \times L^2(\mathbf{R}, \mathbf{C})$ is analytic, with $E_{\mathbf{j}}^0 = E_{\mathbf{j}}$, where $E_{\mathbf{j}}$ is defined in (20). When $E_{\mathbf{j}}$ is simple we can assume, in addition, that $\Phi_{\mathbf{j}}^0 = \Phi_{\mathbf{j}}$.

Under the assumption $|\Omega - \omega| \ll \Omega$, the only case in which $H_{\text{Rabi},0}$ has multiple eigenvalues is when $\omega = \Omega$. If this is the case all eigenvalues different from the lowest one are double. Equation (32) allows to identify the splitting of the two-dimensional eigenspaces induced by the perturbation $g \mapsto H_{\text{Rabi},0} + gV$, as detailed in Section 4.2.

In order to study the first and higher-order derivatives of $g \mapsto E_{j,s}^g$ at $g = 0$, it is useful to introduce the quantities

$$V_{\mathbf{i},\mathbf{j}} = \langle \Phi_{\mathbf{i}}, (x \otimes \sigma_1)\Phi_{\mathbf{j}} \rangle = \left( \delta_{n(\mathbf{i}),n(\mathbf{j})-1}\sqrt{\frac{n(\mathbf{j})}{2}} + \delta_{n(\mathbf{i}),n(\mathbf{j})+1}\sqrt{\frac{n(\mathbf{j})+1}{2}} \right) (1 - \delta_{s(\mathbf{i}),s(\mathbf{j})}). \tag{21}$$

9

## 4.1   Case $\omega \neq \Omega$

### 4.1.1   Step I: Relevant eigenvalue pairs are non-resonant.

Let us first prove that for almost every $g \in \mathbf{R}$ and every $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{N} \times \{-1, 1\}$, with $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$, one has $E_{\mathbf{i}}^g - E_{\mathbf{j}}^g \neq E_{\mathbf{k}}^g - E_{\mathbf{l}}^g$. In order to do so, we observe that it is enough to show that for fixed $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{N} \times \{-1, 1\}$ as before, the set

$$S_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} = \{g \mid E_{\mathbf{i}}^g - E_{\mathbf{j}}^g \neq E_{\mathbf{k}}^g - E_{\mathbf{l}}^g\} \tag{22}$$

is of full measure. By the analytic dependence on $g$ of the eigenvalues of $H_{\mathrm{Rabi}}$, this is equivalent to say that $g \mapsto E_{\mathbf{i}}^g - E_{\mathbf{j}}^g$ and $g \mapsto E_{\mathbf{k}}^g - E_{\mathbf{l}}^g$ have different Taylor expansions at $g = 0$.

Let us consider the Taylor expansion

$$E_{\mathbf{j}}^g = E_{\mathbf{j}} + \sum_{m=1}^{\infty} g^m E_{\mathbf{j}}^{(m)}.$$

The computation of the coefficients $E_{\mathbf{j}}^{(m)}$ carried on below is based on the Rayleigh–Schrödinger series (see, for instance, [26, Chapter XII]).

First of all we observe that $E_{\mathbf{i}} - E_{\mathbf{j}} = E_{\mathbf{k}} - E_{\mathbf{l}}$ is equivalent to $n(\mathbf{i}) - n(\mathbf{j}) = n(\mathbf{k}) - n(\mathbf{l})$ and $s(\mathbf{i}) - s(\mathbf{j}) = s(\mathbf{k}) - s(\mathbf{l})$, in the case in which $\Omega$ is not an integer multiple of $\omega/2$. If $\Omega = (2m + 1)\omega/2$ for some non-negative integer $m$, then $E_{\mathbf{i}} - E_{\mathbf{j}} = E_{\mathbf{k}} - E_{\mathbf{l}}$ implies

$$n(\mathbf{i}) + n(\mathbf{l}) - n(\mathbf{j}) - n(\mathbf{k}) = \frac{2m + 1}{4} \big(s(\mathbf{j}) + s(\mathbf{k}) - s(\mathbf{i}) - s(\mathbf{l})\big),$$

and thus, if the left-hand side is an integer number different from zero, it must be $|s(\mathbf{j}) + s(\mathbf{k}) - s(\mathbf{i}) - s(\mathbf{l})| = 4$, that is $s(\mathbf{j}) = s(\mathbf{k}) = -s(\mathbf{i}) = -s(\mathbf{l})$.

The term $E_{\mathbf{j}}^{(1)}$ coincides with $V_{\mathbf{j},\mathbf{j}} = \langle \Phi_{\mathbf{j}}, (X \otimes \sigma_1) \Phi_{\mathbf{j}} \rangle$, thus we deduce from (21) that $E_{\mathbf{j}}^{(1)} = 0$ for every $\mathbf{j}$.

Following [26] we have that

$$E_{\mathbf{j}}^{(2)} = -\sum_{\mathbf{m} \neq \mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-1} V_{\mathbf{j},\mathbf{m}} V_{\mathbf{m},\mathbf{j}}.$$

Thus

$$
\begin{aligned}
E_{\mathbf{j}}^{(2)} &= -\sum_{\mathbf{m} \neq \mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-1} (1 - \delta_{s(\mathbf{j}),s(\mathbf{m})})^2 \left( \delta_{n(\mathbf{j}),n(\mathbf{m})-1} \sqrt{\frac{n(\mathbf{j}) + 1}{2}} + \delta_{n(\mathbf{j}),n(\mathbf{m})+1} \sqrt{\frac{n(\mathbf{j})}{2}} \right)^2 \\
&= -(E_{n(\mathbf{j})+1,-s(\mathbf{j})} - E_{\mathbf{j}})^{-1} \frac{n(\mathbf{j}) + 1}{2} - (E_{n(\mathbf{j})-1,-s(\mathbf{j})} - E_{\mathbf{j}})^{-1} \frac{n(\mathbf{j})}{2} \\
&= -(\omega - s(\mathbf{j})\Omega)^{-1} \frac{n(\mathbf{j}) + 1}{2} + (\omega + s(\mathbf{j})\Omega)^{-1} \frac{n(\mathbf{j})}{2} \\
&= \frac{\omega + s(\mathbf{j})\Omega(2n(\mathbf{j}) + 1)}{2(\Omega^2 - \omega^2)}.
\end{aligned}
$$

Notice that the computation above is correct also for $n(\mathbf{j}) = 0$, even if in this case $E_{n(\mathbf{j})-1,-s(\mathbf{j})}$ is not a well defined eigenvalue of $H_{\mathrm{Rabi},0}$. Indeed, in this case the term

$$(E_{n(\mathbf{j})-1,-s(\mathbf{j})} - E_\mathbf{j})^{-1}\frac{n(\mathbf{j})}{2}$$

counts as zero.

Let us identify the values $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ such that $E_\mathbf{i}^{(2)} - E_\mathbf{j}^{(2)} = E_\mathbf{k}^{(2)} - E_\mathbf{l}^{(2)}$ under the assumption that $E_\mathbf{i} - E_\mathbf{j} = E_\mathbf{k} - E_\mathbf{l}$. Recall that we also assume that $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$. From the above expression of $E_\mathbf{j}^{(2)}$ we have

$$s(\mathbf{i})\,(2n(\mathbf{i}) + 1) - s(\mathbf{j})\,(2n(\mathbf{j}) + 1) = s(\mathbf{k})\,(2n(\mathbf{k}) + 1) - s(\mathbf{l})\,(2n(\mathbf{l}) + 1)\,. \tag{23}$$

If $\Omega = (2m + 1)\omega/2$ for some non-negative integer $m$ and $s(\mathbf{j}) = s(\mathbf{k}) = -s(\mathbf{i}) = -s(\mathbf{l})$ then (23) gives $n(\mathbf{i})+n(\mathbf{j})+n(\mathbf{k})+n(\mathbf{l})+2 = 0$, which is impossible being the addends non-negative.

The remaining case is when $n(\mathbf{i}) - n(\mathbf{j}) = n(\mathbf{k}) - n(\mathbf{l})$ and $s(\mathbf{i}) - s(\mathbf{j}) = s(\mathbf{k}) - s(\mathbf{l})$, in which case

$$s(\mathbf{i})n(\mathbf{i}) - s(\mathbf{j})n(\mathbf{j}) = s(\mathbf{k})n(\mathbf{k}) - s(\mathbf{l})n(\mathbf{l}). \tag{24}$$

Then, either $s(\mathbf{i}) = s(\mathbf{j})$, which implies $s(\mathbf{k}) = s(\mathbf{l})$ and then, by (24), $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l})$, or $s(\mathbf{i}) = -s(\mathbf{j})$, which implies $s(\mathbf{k}) = -s(\mathbf{l})$ and then, by (24), $n(\mathbf{i}) + n(\mathbf{j}) = n(\mathbf{k}) + n(\mathbf{l})$. In the latter case it must be $\mathbf{i} = \mathbf{k}$ and $\mathbf{j} = \mathbf{l}$, which is excluded by assumption. Therefore the nontrivial quadruples $\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}$ satisfying both the equalities $E_\mathbf{i} - E_\mathbf{j} = E_\mathbf{k} - E_\mathbf{l}$ and $E_\mathbf{i}^{(2)} - E_\mathbf{j}^{(2)} = E_\mathbf{k}^{(2)} - E_\mathbf{l}^{(2)}$ are those for which $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l})$ and $n(\mathbf{i}) - n(\mathbf{j}) = n(\mathbf{k}) - n(\mathbf{l})$.

Let us now evaluate the terms $E_\mathbf{j}^{(3)}$ as in [26]. We have

$$E_\mathbf{j}^{(3)} = \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{p}\neq\mathbf{j}} (E_\mathbf{m} - E_\mathbf{j})^{-1}(E_\mathbf{p} - E_\mathbf{j})^{-1}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{p}}V_{\mathbf{p},\mathbf{j}} - \sum_{\mathbf{m}\neq\mathbf{j}}(E_\mathbf{m} - E_\mathbf{j})^{-2}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{j}}.$$

Since $V_{\mathbf{a},\mathbf{b}} \neq 0$ only if $s(\mathbf{a}) = -s(\mathbf{b})$ it turns out that $V_{\mathbf{j},\mathbf{m}}$ and $V_{\mathbf{m},\mathbf{p}}$ are different from 0 only if $s(\mathbf{j}) = s(\mathbf{p}) = -s(\mathbf{m})$, but then $V_{\mathbf{p},\mathbf{j}} = 0$. Thus, recalling that $V_{\mathbf{j},\mathbf{j}} = 0$, we have $E_\mathbf{j}^{(3)} = 0$ for every $\mathbf{j} \in \mathbf{N} \times \{-1, 1\}$.

We are going to complete the proof that the set $S_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}$ defined as in (22) has full measure by showing that if $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{N}\times\{-1, 1\}$ are such that $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$, $\mathbf{i} \neq \mathbf{j}$, $n(\mathbf{i})-n(\mathbf{j}) = n(\mathbf{k})-n(\mathbf{l})$ (which follows from $E_\mathbf{i} - E_\mathbf{j} = E_\mathbf{k} - E_\mathbf{l}$) and $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l})$ (which follows from $E_\mathbf{i}^{(2)} - E_\mathbf{j}^{(2)} = E_\mathbf{k}^{(2)} - E_\mathbf{l}^{(2)}$), then $E_\mathbf{i}^{(4)} - E_\mathbf{j}^{(4)} \neq E_\mathbf{k}^{(4)} - E_\mathbf{l}^{(4)}$.

The general formula for $E_\mathbf{j}^{(4)}$ (see [26]) is

$$E_\mathbf{j}^{(4)} = -\sum_{\mathbf{m}\neq\mathbf{j},\mathbf{p}\neq\mathbf{j},\mathbf{q}\neq\mathbf{j}} (E_\mathbf{m} - E_\mathbf{j})^{-1}(E_\mathbf{p} - E_\mathbf{j})^{-1}(E_\mathbf{q} - E_\mathbf{j})^{-1}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{p}}V_{\mathbf{p},\mathbf{q}}V_{\mathbf{q},\mathbf{j}} \tag{25}$$

$$+ \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{p}\neq\mathbf{j}} V_{\mathbf{j},\mathbf{j}}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{p}}V_{\mathbf{p},\mathbf{j}}[(E_\mathbf{m} - E_\mathbf{j})^{-1}(E_\mathbf{p} - E_\mathbf{j})^{-2} + (E_\mathbf{m} - E_\mathbf{j})^{-2}(E_\mathbf{p} - E_\mathbf{j})^{-1}]$$

$$+ \sum_{\mathbf{m}\neq\mathbf{j},\mathbf{p}\neq\mathbf{j}} (E_\mathbf{m} - E_\mathbf{j})^{-2}(E_\mathbf{p} - E_\mathbf{j})^{-1}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{p}}V_{\mathbf{p},\mathbf{j}} - \sum_{\mathbf{m}\neq\mathbf{j}}(E_\mathbf{m} - E_\mathbf{j})^{-3}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{j}}^2.$$

11

Since $V_{\mathbf{j},\mathbf{j}} = 0$, only the first and third term of the right-hand side must be evaluated.

Let us compute the first term in (25). In order to avoid null terms we must assume $s(\mathbf{j}) = -s(\mathbf{m}) = s(\mathbf{p}) = -s(\mathbf{q})$ and thus $n(\mathbf{j}) \neq n(\mathbf{p})$. Therefore the only nonzero terms in the sum are given by $n(\mathbf{j}) = n(\mathbf{m}) + 1 = n(\mathbf{p}) + 2 = n(\mathbf{q}) + 1$ (if $n(\mathbf{j}) > 1$) and $n(\mathbf{j}) = n(\mathbf{m}) - 1 = n(\mathbf{p}) - 2 = n(\mathbf{q}) - 1$. We have

$$\sum_{\mathbf{k}\neq\mathbf{j},\mathbf{l}\neq\mathbf{j},\mathbf{i}\neq\mathbf{j}} (E_{\mathbf{k}} - E_{\mathbf{j}})^{-1}(E_{\mathbf{l}} - E_{\mathbf{j}})^{-1}(E_{\mathbf{i}} - E_{\mathbf{j}})^{-1}V_{\mathbf{j},\mathbf{k}}V_{\mathbf{k},\mathbf{l}}V_{\mathbf{l},\mathbf{i}}V_{\mathbf{i},\mathbf{j}} =$$

$$(-\omega - s(\mathbf{j})\Omega)^{-1}(-2\omega)^{-1}(-\omega - s(\mathbf{j})\Omega)^{-1}\left(\frac{n(\mathbf{j})}{2}\right)\left(\frac{n(\mathbf{j}) - 1}{2}\right)$$

$$+ (\omega - s(\mathbf{j})\Omega)^{-1}(2\omega)^{-1}(\omega - s(\mathbf{j})\Omega)^{-1}\left(\frac{n(\mathbf{j}) + 1}{2}\right)\left(\frac{n(\mathbf{j}) + 2}{2}\right).$$

Notice that the formula is correct also in the case where $n(\mathbf{j}) = 0$ or $n(\mathbf{j}) = 1$.

Let us now compute the third term in (25). As before, to avoid null terms we assume $s(\mathbf{j}) = -s(\mathbf{m}) = -s(\mathbf{p})$. The nonzero terms in the sum are given by $n(\mathbf{m}) = n(\mathbf{j}) \pm 1$ and $n(\mathbf{p}) = n(\mathbf{j}) \pm 1$ thus we have to sum four terms. We have

$$\sum_{\mathbf{m}\neq\mathbf{j},\mathbf{p}\neq\mathbf{j}} (E_{\mathbf{m}} - E_{\mathbf{j}})^{-2}(E_{\mathbf{p}} - E_{\mathbf{j}})^{-1}V_{\mathbf{j},\mathbf{m}}V_{\mathbf{m},\mathbf{j}}V_{\mathbf{j},\mathbf{p}}V_{\mathbf{p},\mathbf{j}} = (-\omega - s(\mathbf{j})\Omega)^{-2}(-\omega - s(\mathbf{j})\Omega)^{-1}\left(\frac{n(\mathbf{j})}{2}\right)^2$$

$$+ (-\omega - s(\mathbf{j})\Omega)^{-2}(\omega - s(\mathbf{j})\Omega)^{-1}\frac{n(\mathbf{j})}{2}\frac{n(\mathbf{j}) + 1}{2}$$

$$+ (\omega - s(\mathbf{j})\Omega)^{-2}(-\omega - s(\mathbf{j})\Omega)^{-1}\frac{n(\mathbf{j})}{2}\frac{n(\mathbf{j}) + 1}{2}$$

$$+ (\omega - s(\mathbf{j})\Omega)^{-2}(\omega - s(\mathbf{j})\Omega)^{-1}\left(\frac{n(\mathbf{j}) + 1}{2}\right)^2.$$

By summing up all the terms one sees that, for fixed $s = s(\mathbf{j})$, the term $E_{\mathbf{j}}^{(4)}$ depends quadratically on $n(\mathbf{j})$, i.e.

$$E_{\mathbf{j}}^{(4)} = C_0(s(\mathbf{j})) + C_1(s(\mathbf{j}))n(\mathbf{j}) + C_2(s(\mathbf{j}))n(\mathbf{j})^2,$$

where the coefficient $C_2(s(\mathbf{j}))$ is given by

$$C_2(s(\mathbf{j})) = s(\mathbf{j})\frac{\Omega(\omega^2 + 3\Omega^2)}{2(\omega^2 - \Omega^2)^3} \neq 0.$$

So, if $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ are such that $s(\mathbf{i}) = s(\mathbf{j}) = s(\mathbf{k}) = s(\mathbf{l}) = s$ and $n(\mathbf{i}) - n(\mathbf{j}) = n(\mathbf{k}) - n(\mathbf{l})$, we have

$$E_{\mathbf{i}}^{(4)} - E_{\mathbf{j}}^{(4)} = E_{\mathbf{k}}^{(4)} - E_{\mathbf{l}}^{(4)} \iff C_1(s)(n(\mathbf{i}) - n(\mathbf{j})) + C_2(s)(n(\mathbf{i})^2 - n(\mathbf{j})^2) =$$
$$C_1(s)(n(\mathbf{k}) - n(\mathbf{l})) + C_2(s)(n(\mathbf{k})^2 - n(\mathbf{l})^2)$$
$$\iff C_2(s)(n(\mathbf{i})^2 - n(\mathbf{j})^2) = C_2(s)(n(\mathbf{k})^2 - n(\mathbf{l})^2)$$
$$\iff C_2(s)(n(\mathbf{i}) + n(\mathbf{j})) = C_2(s)(n(\mathbf{k}) + n(\mathbf{l}))$$
$$\iff n(\mathbf{i}) = n(\mathbf{k}) \text{ and } n(\mathbf{j}) = n(\mathbf{l}).$$

This concludes the proof that for almost every $g \in \mathbf{R}$ and every $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbf{N} \times \{-1, 1\}$, with $(\mathbf{i}, \mathbf{j}) \neq (\mathbf{k}, \mathbf{l})$ and $\mathbf{i} \neq \mathbf{j}$, one has $E_{\mathbf{i}}^g - E_{\mathbf{j}}^g \neq E_{\mathbf{k}}^g - E_{\mathbf{l}}^g$.
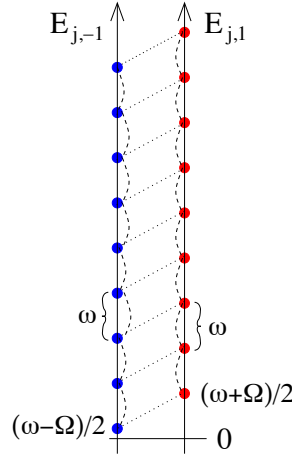
Figure 2: The dashed lines connect eigenvalues of $H_{\text{Rabi}}$ that are "coupled" by the controlled Hamiltonian when $g = 0$, while the dotted lines connect eigenvalues that are coupled by the controlled Hamiltonian for almost all $g \neq 0$.

### 4.1.2 Step 2: Coupling of the relevant energy levels.

The proof of Theorem 1 is then concluded, thanks to Theorem 3, if we show that the controlled Hamiltonian $x \otimes \mathbb{1}$ couples, directly or indirectly, all the energy levels for almost all $g \in \mathbf{R}$.

More precisely, we show below that $\langle \Phi_{\mathbf{j}}^g, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^g \rangle \neq 0$ for almost every $g \in \mathbf{R}$ for all $\mathbf{j}, \mathbf{k}$ such that $s(\mathbf{j}) = s(\mathbf{k})$ and $|n(\mathbf{j}) - n(\mathbf{k})| = 1$ or $s(\mathbf{j}) = -s(\mathbf{k})$ and $n(\mathbf{j}) = n(\mathbf{k})$. See Figure 2. As before, it is enough to show that the corresponding Taylor series in $g$ is nonzero.

Set $\Phi_{\mathbf{j}}^g = \Phi_{\mathbf{j}} + \sum_{m=1}^{\infty} g^m \Phi_{\mathbf{j}}^{(m)}$. We have

$$\langle \Phi_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}} \rangle = \left( \delta_{n(\mathbf{j}),n(\mathbf{k})-1} \sqrt{\frac{n(\mathbf{j})+1}{2}} + \delta_{n(\mathbf{j}),n(\mathbf{k})+1} \sqrt{\frac{n(\mathbf{j})}{2}} \right) \delta_{s(\mathbf{j}),s(\mathbf{k})}.$$

This is enough to say that $\langle \Phi_{\mathbf{j}}^g, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^g \rangle \neq 0$ for almost every $g$ for all $\mathbf{j}, \mathbf{k}$ such that $s(\mathbf{j}) = s(\mathbf{k})$ and $|n(\mathbf{j}) - n(\mathbf{k})| = 1$.

The term $\Phi_{\mathbf{j}}^{(1)}$ can be characterized through the relation

$$(H_{\text{Rabi},0} + gx \otimes \sigma_1)(\Phi_{\mathbf{j}} + g\Phi_{\mathbf{j}}^{(1)} + o(g)) = (E_{\mathbf{j}} + gE_{\mathbf{j}}^{(1)} + o(g))(\Phi_{\mathbf{j}} + g\Phi_{\mathbf{j}}^{(1)} + o(g)). \quad (26)$$

Regrouping the first-order terms in (26) we get

$$H_{\text{Rabi},0}\Phi_{\mathbf{j}}^{(1)} + (x \otimes \sigma_1)\Phi_{\mathbf{j}} - E_{\mathbf{j}}\Phi_{\mathbf{j}}^{(1)} - E_{\mathbf{j}}^{(1)}\Phi_{\mathbf{j}} = 0. \quad (27)$$

Denote by $\Pi$ the orthogonal projection on the orthogonal complement to $\Phi_{\mathbf{j}}$. Applying $\Pi$ to (27), we get

$$(H_{\text{Rabi},0} - E_{\mathbf{j}}\mathbb{1})\Phi_{\mathbf{j}}^{(1)} + \Pi(x \otimes \sigma_1)\Phi_{\mathbf{j}} = 0.$$

Notice that the orthogonal complement to $\Phi_{\mathbf{j}}$ is an invariant space for the operator $H_{\text{Rabi},0} - E_{\mathbf{j}}\mathbb{1}$, which is invertible when restricted to it. We write $(H_{\text{Rabi},0} - E_{\mathbf{j}}\mathbb{1})^{-1}$ to denote its inverse

(whose values are in the orthogonal complement to $\Phi_{\mathbf{j}}$). Thus,

$$\Phi_{\mathbf{j}}^{(1)} = -(H_{\text{Rabi},0} - E_{\mathbf{j}}\mathbb{1})^{-1}\Pi(x \otimes \sigma_1)\Phi_{\mathbf{j}} = \sum_{\mathbf{l} \neq \mathbf{j}}(E_{\mathbf{j}} - E_{\mathbf{l}})^{-1}\langle\Phi_{\mathbf{l}}, (x \otimes \sigma_1)\Phi_{\mathbf{j}}\rangle\Phi_{\mathbf{l}}.$$

The linear term in the Taylor expansion of $\langle\Phi_{\mathbf{j}}^g, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^g\rangle$ with respect to $g$ is given by

$$\langle\Phi_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^{(1)}\rangle + \langle\Phi_{\mathbf{j}}^{(1)}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}\rangle = \sum_{\mathbf{l} \neq \mathbf{k}}(E_{\mathbf{k}} - E_{\mathbf{l}})^{-1}\langle\Phi_{\mathbf{l}}, (x \otimes \sigma_1)\Phi_{\mathbf{k}}\rangle\langle\Phi_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi_{\mathbf{l}}\rangle +$$

$$\sum_{\mathbf{l} \neq \mathbf{j}}(E_{\mathbf{j}} - E_{\mathbf{l}})^{-1}\langle\Phi_{\mathbf{l}}, (x \otimes \sigma_1)\Phi_{\mathbf{j}}\rangle\langle\Phi_{\mathbf{l}}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}\rangle.$$

Taking into account only the nonzero terms in the first sum gives $s(\mathbf{j}) = s(\mathbf{l}) = -s(\mathbf{k})$, $|n(\mathbf{j}) - n(\mathbf{l})| = 1$ and $|n(\mathbf{k}) - n(\mathbf{l})| = 1$, so for fixed $\mathbf{j}, \mathbf{k}$ we only have (at most) two terms. We assume $n(\mathbf{j}) = n(\mathbf{k})$, thus we have

$$\langle\Phi_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^{(1)}\rangle = \left[(-\omega + s(\mathbf{k})\Omega)^{-1}\frac{n(\mathbf{k}) + 1}{2} + (\omega + s(\mathbf{k})\Omega)^{-1}\frac{n(\mathbf{k})}{2}\right]$$

$$= \frac{\omega + s(\mathbf{k})\Omega(1 + 2n(\mathbf{k}))}{2(\Omega^2 - \omega^2)}.$$

Similarly,

$$\langle\Phi_{\mathbf{j}}^{(1)}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}\rangle = \frac{\omega - s(\mathbf{k})\Omega(1 + 2n(\mathbf{k}))}{2(\Omega^2 - \omega^2)}.$$

Hence,

$$\langle\Phi_{\mathbf{j}}^{(1)}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}\rangle + \langle\Phi_{\mathbf{j}}, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^{(1)}\rangle = \frac{\omega}{\Omega^2 - \omega^2},$$

which is different from zero for every $\mathbf{k}$. Thus the Taylor series of $\langle\Phi_{\mathbf{j}}^g, (x \otimes \mathbb{1})\Phi_{\mathbf{k}}^g\rangle$ is nonzero, which concludes the proof of the theorem in the case $\omega \neq \Omega$.

## 4.2 Case $\omega = \Omega$

Let us assume in this section that $\omega = \Omega$. Hence

$$E_{j,1} = E_{j+1,-1} = \omega(j + 1)$$

for every $j \in \mathbf{N}$ and $E_{0,-1} = 0$.

Recall that, as $g$ varies in $\mathbf{R}$, a complete set of eigenpairs for $H_{\text{Rabi}}$ can be given in the form $(E_{\mathbf{j}}^g, \Phi_{\mathbf{j}}^g)_{\mathbf{j} \in \mathbf{N} \times \{-1,1\}}$ with each $E_{\mathbf{j}}^g$ and $\Phi_{\mathbf{j}}^g$ depending analytically on $g$ and $E_{\mathbf{j}}^0 = E_{\mathbf{j}}$. Notice that each $\Phi_{\mathbf{j}}^g$ can be taken in $L^2(\mathbf{R}, \mathbf{R}) \otimes \mathbf{C}^2$ and that we can take $\Phi_{0,-1}^0 = \Phi_{0,-1} = \varphi_0 \otimes \nu_{-1}$.

The eigenfunctions $\Phi_{j,1}^0, \Phi_{j+1,-1}^0$ are characterized by the relation

$$\langle\Phi_{j,1}^0, (X \otimes \sigma_1)\Phi_{j+1,-1}^0\rangle = 0.$$

(This follows by standard perturbation relations recalled in the Appendix, see equation (32).)

Since, moreover, $\Phi^0_{j,1}$ and $\Phi^0_{j+1,-1}$ are orthogonal and of norm one, one gets, up to signs,

$$\{\Phi^0_{j,1}, \Phi^0_{j+1,-1}\} = \left\{\frac{\Phi_{j,1} + \Phi_{j+1,-1}}{\sqrt{2}}, \frac{\Phi_{j,1} - \Phi_{j+1,-1}}{\sqrt{2}}\right\}.$$

Let us fix, by convention,

$$\Phi^0_{j,1} = \frac{\Phi_{j,1} + \Phi_{j+1,-1}}{\sqrt{2}}, \qquad \Phi^0_{j+1,-1} = \frac{\Phi_{j,1} - \Phi_{j+1,-1}}{\sqrt{2}}.$$

With this choice (32) yields

$$\frac{d}{dg}E^g_{j,1}\big|_{g=0} = \sqrt{\frac{j+1}{2}}, \qquad \frac{d}{dg}E^g_{j+1,-1}\big|_{g=0} = -\sqrt{\frac{j+1}{2}}.$$

Assume now that

$$E^0_{\mathbf{j}_1} - E^0_{\mathbf{j}_2} = E^0_{\mathbf{j}_3} - E^0_{\mathbf{j}_4}, \tag{28}$$

$$\frac{d}{dg}\big|_{g=0}(E^g_{\mathbf{j}_1} - E^g_{\mathbf{j}_2}) = \frac{d}{dg}\big|_{g=0}(E^g_{\mathbf{j}_3} - E^g_{\mathbf{j}_4}), \tag{29}$$

and let us prove that $\mathbf{j}_1 = \mathbf{j}_3$ and $\mathbf{j}_2 = \mathbf{j}_4$. Write $E^0_{\mathbf{j}_l} = \omega k_l$ for $l = 1, \ldots, 4$ for some $k_1, \ldots, k_4 \in \mathbf{N}$. Then (28) implies that $k_1 = k_2 + p$ and $k_3 = k_4 + p$ for some $p \in \mathbf{Z}$.

Assume that $k_1, \ldots, k_4 \neq 0$ (the other cases being similar). Equality (29) then yields

$$s_1\sqrt{k_2 + p} - s_2\sqrt{k_2} = s_3\sqrt{k_4 + p} - s_4\sqrt{k_4} \tag{30}$$

for some $s_1, \ldots, s_4 \in \{-1, 1\}$ and we are left to prove that $s_1 = s_3$, $s_2 = s_4$ and $k_2 = k_4$. Without loss of generality we can assume that $k_2 + p = \max\{k_2, k_2 + p, k_4, k_4 + p\}$ and that $s_1 = 1$. Then $p$ and the left-hand side in (30) are nonnegative, which implies that $s_3 = 1$. We can then rewrite (30) as

$$\sqrt{k_2 + p} - s_2\sqrt{k_2} = \sqrt{k_4 + p} - s_4\sqrt{k_4} \tag{31}$$

If $s_2 = s_4$ then, by monotonicity of $k \mapsto \sqrt{k + p} \pm \sqrt{k}$ we deduce that $k_2 = k_4$ and we are done. Finally, if $s_2 = -s_4$ then either $s_2 = -1$, and then $k_4 > k_2$, or $s_2 = 1$ and then $\sqrt{k_2 + p} = \sqrt{k_4 + p} + \sqrt{k_4} + \sqrt{k_2} > \sqrt{p} + \sqrt{k_2} \geq \sqrt{k_2 + p}$, leading in both cases to a contradiction.

In order to apply Theorem 3 we are left to prove that the controlled Hamiltonian $X \otimes \mathbb{1}$ couples, directly or indirectly, all the elements of the basis $(\Phi^0_{\mathbf{j}})_{\mathbf{j} \in \mathbf{N} \times \{-1,1\}}$.

First notice that

$$\langle \Phi^0_{j,1}, (X \otimes \mathbb{1})\Phi^0_{j+1,1}\rangle = \frac{1}{2}\left(\sqrt{\frac{j+1}{2}} + \sqrt{\frac{j+2}{2}}\right) \neq 0, \qquad j \in \mathbf{N},$$

meaning that $X \otimes \mathbb{1}$ couples all basis elements of the type $\Phi^0_{j,1}$.

Similarly, one has that $\langle \Phi^0_{j,-1}, (X \otimes \mathbb{1})\Phi^0_{j+1,-1}\rangle \neq 0$ for every $j \in \mathbf{N}$. Finally, one easily checks that $\langle \Phi^0_{j,1}, (X \otimes \mathbb{1})\Phi^0_{j,-1}\rangle \neq 0$ for every $j \in \mathbf{N}$, completing the proof.

# Appendix: global real analyticity of eigenpairs

In this section we detail the proof of the analyticity of the eigenpairs $(E_{\mathbf{j}}^g, \Phi_{\mathbf{j}}^g)$ as functions of $g \in \mathbf{R}$. This result is somehow *folklore* and follows from well-known theorems in linear perturbation theory ([14, 26]). Since the latter are spread over the literature, we prefer to sketch here the main ideas of the argument, which is also useful in other contexts, as e.g. for the degenerate Gell-Man and Low theorem ([8, 9, 10]).

Recall that, given two self-adjoint operators $H_0, W$ on a separable Hilbert space, $W$ is said to be *Kato-small with respect to $H_0$* if $D(H_0) \subset D(W)$ and for every $a > 0$ there exists $b > 0$ such that $\|W\psi\| \le a\|H_0\psi\| + b\|\psi\|$ for every $\psi \in D(H_0)$. According to this definition, it is easy to check that $V$, defined as in (19), is Kato-small with respect to $H_{\mathrm{Rabi},0}$.

**Proposition 4.** *Let $H_0, W$ be self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Assume that $W$ is Kato-small with respect to $H_0$ and that $H_0$ has compact resolvent. Then $H_g = H_0 + gW$ is self-adjoint with compact resolvent for every $g \in \mathbf{R}$, hence it admits a complete orthonormal system of eigenfunctions $\{\psi_j(g)\}_{j\in\mathbf{N}}$ and the corresponding eigenvalues $\{E_j(g)\}_{j\in\mathbf{N}}$ have finite multiplicity and do not accumulate at any finite point. Moreover, up to a suitable choice of labelling, for each $j \in \mathbf{N}$ the function $g \mapsto (E_j(g), \psi_j(g))$ is analytic from $\mathbf{R}$ to $\mathbf{R} \times \mathcal{H}$.*

*Proof.* Self-adjointness of $H_g$ follows from [25, Theorem X.12]. Let us first prove that $H_g$ has compact resolvent for every $g \in \mathbf{R}$ (we provide a direct proof; see also [14, Theorem VII.2.4]). Since [(2)]

$$(H_g - \mathrm{i}\mathbb{1})^{-1} = (H_0 - \mathrm{i}\mathbb{1})^{-1}\left(\mathbb{1} - g\,W(H_g - \mathrm{i}\mathbb{1})^{-1}\right)$$

and $(H_0 - \mathrm{i}\mathbb{1})^{-1}$ is compact, we are left to prove that $W(H_g - \mathrm{i}\mathbb{1})^{-1}$ is bounded. Notice that for every $a > 0$ there exists $b > 0$ such that $\|W\psi\| \le a\|(H_g - \mathrm{i}\mathbb{1})\psi\| + b\|\psi\|$ for every $\psi \in D(H_0) = D(H_g - \mathrm{i}\mathbb{1})$. Setting $\psi = (H_g - \mathrm{i}\mathbb{1})^{-1}\phi$ for $\phi \in \mathcal{H}$, we get

$$\|W(H_g - \mathrm{i}\mathbb{1})^{-1}\phi\| \le a\|\phi\| + b\|(H_g - \mathrm{i}\mathbb{1})^{-1}\phi\| \le (a+b)\|\phi\|,$$

where the last inequality follows from $\|(H_g - \mathrm{i}\mathbb{1})\eta\|\|\eta\| \ge |\langle(H_g - \mathrm{i}\mathbb{1})\eta, \eta\rangle| \ge \|\eta\|^2$, for $\eta \in D(H_g - \mathrm{i}\mathbb{1})$.

If $E_j(g)$ is simple then, by Rellich's theorem [26, Theorem XII.8], up to a reordering of the spectrum of $H_{g'}$ for $g'$ in a neighborhood of $g$, the eigenpair parameterization $g' \mapsto (E_j(g'), \psi_j(g'))$ is analytic near $g$. The previous result can be generalized to the case of eigenvalues of multiplicity $m \ge 2$. Indeed, if $E_{j_1}(g) = \cdots = E_{j_m}(g)$ and $j_1, \ldots, j_m$ are distinct, we can assume, up to relabelling, that $g' \mapsto (E_{j_1}(g'), \psi_{j_1}(g')), \ldots, g' \mapsto (E_{j_m}(g'), \psi_{j_m}(g'))$ are analytic near $g$ ([26, Theorem XII.13]). Moreover,

$$\langle\psi_{j_l}(g), W\psi_{j_k}(g)\rangle = \delta_{lk}\frac{d}{dg}E_{j_l}(g), \quad l, k = 1, \ldots, m. \tag{32}$$

(See, for instance, [2].)

In order to describe the spectrum by a countable family of functions which are globally analytic on $\mathbf{R}$, we should ensure that all such locally analytic functions $E_j(\cdot)$ can be extended indefinitely.

---

[(2)] Notice that $(H_g - \mathrm{i}\mathbb{1})^{-1}$ maps $\mathcal{H}$ into $D(H_g) \subset D(W)$, so that $W(H_g - \mathrm{i}\mathbb{1})^{-1}$ is well-defined on the whole Hilbert space $\mathcal{H}$.

Fix $j$ and let $I$ be the maximal interval containing 0 such that, up to relabelling, $E_j$ is analytic on $I$. Let us prove that, for every $\ell \in (0, +\infty)$, $[-\ell, \ell] \subset I$.

Set $a = (2\ell)^{-1}$. Take $b > 0$ such that $\|W\psi\| \le a\|H_0\psi\| + b\|\psi\|$ for every $\psi \in D(H_0)$. Hence, for $\psi \in D(H_0)$ and $g \in [-\ell, \ell]$,

$$\|W\psi\| \le a\|(H_0 + gW)\psi\| + a|g|\|W\psi\| + b\|\psi\|$$
$$\le a\|H_g\psi\| + \frac{\|W\psi\|}{2} + b\|\psi\|$$

and therefore

$$\|W\psi\| \le 2a\|H_g\psi\| + 2b\|\psi\|, \quad g \in [-\ell, \ell].$$

It follows from (32) that, for every $g \in I \cap [-\ell, \ell]$,

$$\left| \frac{d}{dg} E_j(g) \right| = |\langle \psi_j(g), W\psi_j(g) \rangle| \le \|W\psi_j(g)\| \le 2a\|H_g\psi_j(g)\| + 2b\|\psi_j(g)\|.$$

Hence,

$$\left| \frac{d}{dg} |E_j(g)| \right| \le 2a|E_j(g)| + 2b, \qquad \text{for almost every } g \in I \cap [-\ell, \ell].$$

A standard application of Gronwall inequality yields that $E_j(\cdot)$ is Lipschitz on $I \cap [-\ell, \ell]$. In particular the limits of $E_j(\cdot)$ at the boundary of $I \cap [-\ell, \ell]$ are well defined and they are eigenvalues of the corresponding perturbed operators (see e.g. [26, Theorem XII.7]). Therefore $E_j(\cdot)$ can be extended indefinitely on $[-\ell, \ell]$ and thus also on the whole real line. $\qquad\square$

# Acknowledgements

# References

[1] R. Adami and U. Boscain. Controllability of the Schrödinger equation via intersection of eigenvalues. In *Proceedings of the 44th IEEE Conference on Decision and Control, December 12-15*, pages 1080–1085, 2005.

[2] J. H. Albert. Genericity of simple eigenvalues for elliptic PDE's. *Proc. Amer. Math. Soc.*, 48:413–418, 1975.

[3] L. Allen and J. Eberly. *Optical Resonance and Two-level Atoms*. Dover Books on Physics Series. Dover, 1975.

[4] A. M. Bloch, R. W. Brockett, and C. Rangan. Finite Controllability of Infinite-Dimensional Quantum Systems. *IEEE Transactions on Automatic Control*, 55(8):1797–1805, Aug. 2010.

[5] U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti. A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule. *Commun. Math. Phys.*, 311(2):423–455, 2012.

[6] U. V. Boscain, F. Chittaro, P. Mason, and M. Sigalotti. Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues. *IEEE Trans. Automat. Control*, 57(8):1970–1983, 2012.

[7] D. Braak. Integrability of the Rabi model. *Physical Review Letters*, 107(10):100401, 2011.

[8] C. Brouder, G. Panati, and G. Stoltz. Gell-Mann and Low formula for degenerate unperturbed states. *Ann. Henri Poincaré*, 10(7):1285–1309, 2009.

[9] C. Brouder, G. Panati, and G. Stoltz. Many-body green function of degenerate systems. *Phys. Rev. Lett.*, 103(23):230401, 2009.

[10] C. Brouder, G. Stoltz, and G. Panati. Adiabatic approximation, Gell-Mann and Low theorem, and degeneracies: A pedagogical example. *Phys. Rev. A*, 78(4):042102, Oct. 2008.

[11] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons & Atoms: Introduction to Quantum Electrodynamics*. John Wiley & Sons, 1989.

[12] S. Ervedoza and J.-P. Puel. Approximate controllability for a system of Schrödinger equations modeling a single trapped ion. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 26(6):2111–2136, Nov. 2009.

[13] E. Jaynes and F. Cummings. Comparison of quantum and semiclassical radiation theories with application to the beam maser. *Proc. IEEE*, pages 89–109, 1963.

[14] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[15] M. Keyl, R. Zeier, and T. Schulte-Herbrueggen. Controlling Several Atoms in a Cavity. *Arxiv*, Jan. 2014.

[16] B. Kneer and C. Law. Preparation of arbitrary entangled quantum states of a trapped ion. *Physical Review A*, 57(3):2096–2104, Mar. 1998.

[17] C. Law and J. Eberly. Arbitrary control of a quantum electromagnetic field. *Physical Review Letters*, 76(7):1055–1058, Feb. 1996.

[18] E. Paduro and M. Sigalotti. Approximate controllability of the two trapped ions system. Preprint, 2014.

[19] G. Panati, H. Spohn, and S. Teufel. Space-Adiabatic Perturbation Theory. *Adv. Theor. Math. Phys.*, 7(1):49, Jan. 2002.

[20] G. Panati, H. Spohn, and S. Teufel. Space-adiabatic perturbation theory in quantum dynamics. *Phys. Rev. Lett.*, 88(25 Pt 1):250405, 2002.

[21] I. I. Rabi. On the process of space quantization. *Phys. Rev.*, 49:324, 1936.

[22] I. I. Rabi. Space quantization in a gyrating magnetic field. *Phys. Rev.*, 51:652, 1937.

[23] J. M. Raimond, M. Brune, and S. Haroche. Colloquium: Manipulating quantum entanglement with atoms and photons in a cavity. *Rev. Mod. Phys.*, 73:053805, 2001.

[24] C. Rangan, A. Bloch, C. Monroe, and P. Bucksbaum. Control of trapped-ion quantum states with optical pulses. *Physical Review Letters*, 92(11):10, Mar. 2004.

[25] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[26] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[27] P. Rouchon. Quantum systems and control. *ARIMA Rev. Afr. Rech. Inform. Math. Appl.*, 9:325–357, 2008.

[28] D. Schuster, A. Houch, J. Schreirer, A. Wallraff, J. Gambetta, A. Blais, L. Frunzio, J. Majer, B. Johnson, and M. Devoret. Resolving photon-number states in a superconducting circuit. *Nature*, 45:515–518, 2007.

[29] H. Spohn. *Dynamics of Charged Particles and their Radiation Field.* Cambridge University Press, 2004.

[30] S. Teufel. *Adiabatic Perturbation Theory in Quantum Dynamics.* Springer, 2003.

[31] H. Yuan and S. Lloyd. Controllability of the coupled spin-$\frac{1}{2}$ harmonic oscillator system. *Physical Review A*, 75:052331, May 2007.