Leader-Following Coordination of Nonlinear Agents under Time-varying Communication Topologies

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Abstract

The paper deals with the problem of synchronizing the outputs of a set of nonlinear agents that exchange information through a time-varying communication network in a leader-follower configuration. The dynamics of the individual followers may differ from each other and from that of the leader. The information exchange between the leader and (a limited fraction of) the followers as well as between neighboring followers only consists of the relative values of the output variables that are to be synchronized. The theory of output regulation for nonlinear systems is used to design decentralized controllers embedding an internal model of the leader dynamics. Then, under mild connectivity hypotheses, it is shown how synchronization between the local control loops can be achieved.

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I. INTRODUCTION

The problem of achieving consensus (among states and/or outputs) in a (homogeneous or heterogenous) network of systems has attracted a major attention in the past fifteen years. This area of research, which begins with a series of seminal contributions such as those of [12], [21], [18], [19], [1], [15], [4], is now pretty well established. The literature is vast and is hard to summarize in the introduction of a research paper. In what follows, we quote some recent contributions that are closer to and/or have influenced our own approach (for a more extensive survey of earlier work, see, e.g., the dissertation [30]). The present paper considers the synchronization problem for a *heterogeneous* network, i.e., a network in which the individual systems (agents) possess *different* internal dynamics. The synchronization problem for a heterogeneous network of *linear* systems connected through a time-invariant graph has been previously addressed in the papers [24], [26], [14], [16], [13], [32]. In particular, [32] for linear systems and [31] for nonlinear systems have shown that, if the outputs of the agents of a heterogenous network achieve consensus on a nontrivial trajectory, the trajectory in question is necessarily the output of some autonomous (linear or, respectively, nonlinear) system. This property is the equivalent of, in the context of the consensus problem, the celebrated internal model principle of control theory. Motivated by this observation, [13] and [32] have proposed a two-layer control structure for achieving consensus in heterogenous networks of linear systems connected through a time-invariant graph. In their approach, a network of identical local reference generators is synchronized and the theory of output regulation is used to guarantee that the outputs of the (non-identical) agents follow the (synchronized) outputs of each local generator. This approach has been recently extended in [11] to a network of *nonlinear* systems connected through a time-invariant graph. Consensus problems for a heterogeneous network of nonlinear systems have also been successfully addressed in the very recent paper [29]. The approach of [13] has also been extended in [25] to the case of a switched topology.

The consensus problem in the case of systems connected through time-varying communication graph has been successfully addressed in the milestone paper [19], which fully solves the problem in the case of a network of integrator systems under very mild connectivity conditions. This approach, though, has not been extended yet to the case of higher-dimensional linear agents, exchanging relative (full-state and/or partial state) information, let alone the case of higher-dimensional nonlinear agents.

A somewhat special synchronization scenario is the one in which states (or outputs) of (a large set of) agents (the followers) are required to asymptotically track the state (or the output) of a single autonomous system (the leader). The pattern of communication still consists of the exchange of relative information, as in the case of standard consensus problems, with the only difference that the leader receives no information from the followers. This difference is reflected in the fact that the entries of one row of the so-called adjacency matrix of the graph (the row whose index corresponds to the leader) are all zero. This setup, in which followers exchange information with neighbors but only a limited fraction of them receives (relative) feedback injection from the leader is also known as pinning control [4], [33]. The problem of consensus in such special communication setup has been successfully addressed in [20], [22] for linear systems connected through time-varying communication graphs, and in [33] for nonlinear systems. These papers address the case in which the leader and the followers have the same dynamics (namely, the case of a *homogeneous* network) and exchange (relative) *full-state* information.

The purpose of this paper is to extend the results of [20], [22] and [33] to the problem of leader-follower coordination for a *heterogeneous* network of *nonlinear* system exchanging only relative *output* information. Since in this case the followers have different dynamics, it is necessary that each one of them is locally controlled, so as to obtain a local loop that embeds a copy of the dynamics of the leader, according to the internal model principle(see [31]). However, as in the pinning control problem, only a limited fraction of the local loops may have access to (relative) information from the leader. In our setting, the communication graph that characterizes the information exchange is *time-varying*. In this case, since the dynamics of the agents are not necessarily first-order and only output (as opposite to full-state) information exchange is considered, the fundamental results of [19] cannot be directly used as such and a suitable extension is necessary, that is provided in the paper.

The paper is organized as follows. In section II we characterize the class of systems that are being studied. In section III we provide some background material about the theory of asymptotic tracking, via internal-model-based control, for nonlinear systems and the implementation of such theory in the present context. In section IV we address the problem of synchronization of the individual local loops under the given leader-follower information pattern. In section V we provide some simple example and we conclude in section VI with a summary of the results and a comparison with the existing literature.

Notations. In the paper, $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of real and nonnegative real numbers, respectively, whereas $\mathbb{N}$ denotes the set of nonnegative integers. With $\mathbb{R}^n$ we indicate the n-dimensional Euclidean space. For $\mathcal{C}$ a compact subset of $\mathbb{R}^n$, $\|x\|_{\mathcal{C}} = \min_{y\in\mathcal{C}} |x - y|$ denotes the distance from x to $\mathcal{C}$. Given a dynamical system

$$\dot{x} = f(x,t) \qquad x \in \mathbb{R}^n \tag{1}$$

in which $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a fixed vector field, and a continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the *upper directional derivative of* $V(x)$ *along* $f(x,t)$, at (x,t), is defined as

$$D^+_{f(x,t)}V(x) = \limsup_{h\to 0^+} \frac{1}{h}[V(x + hf(x,t)) - V(x)]\,.$$

Sometimes, whenever the expression of the vector field $f(x,t)$ along which the directional derivative is to be taken is excessively long, we use instead – as a subscript – the corresponding number of the equation in which $f(x,t)$ is defined, i.e., we use the shortened notation

$$D^+_{(1)}V(x) := D^+_{f(x,t)}V(x)\,.$$

In some parts of the paper we will use tools developed in the context of the theory of hybrid dynamical systems. In those parts, this work uses the framework and results of [8], from which also the notation is taken.

II. PRELIMINARIES

A. Problem formulation

As indicated in the Introduction, we consider in this paper the problem of controlling a set of N, not necessarily identical, nonlinear agents, the *followers*, in such a way that their outputs asymptotically track the output of a fixed nonlinear *leader*. It is assumed that all followers have relative degree r between control input and controlled output and are modeled in normal form as

$$\begin{aligned}\dot{z}_k &= f_k(z_k,\xi_k)\\ \dot{\xi}_{k1} &= \xi_{k2}\\ &\cdots\\ \dot{\xi}_{k,r-1} &= \xi_{kr}\\ \dot{\xi}_{kr} &= q_k(z_k,\xi_k)+b_k(z_k,\xi_k)u_k\\ y_k &= \xi_{k1}\end{aligned} \tag{2}$$

in which $z_k \in \mathbb{R}^{n_k}$, $\xi_k \in \mathbb{R}^r$ denotes the vector $\xi_k = \text{col}(\xi_{k1},\, \xi_{k2},\, \ldots\, ,\, \xi_{kr})$ and $u_k \in \mathbb{R}$. In this equation, u_k and y_k denote the control input and, respectively, the controlled output of the individual kth agent. The assumption that all agents have the same relative degree is not restrictive, since it is always possible to achieve such property by adding a suitable number of integrators to the input channel of each agent.

The outputs y_k of all such agents are requested to asymptotically track the output y_0 of a (single) leader

$$\begin{aligned}\dot{w} &= s(w)\\ y_0 &= h_0(w)\end{aligned} \tag{3}$$

in which $w \in W$, with W a compact set. The set is assumed to be invariant for the dynamics of (3).

The (decentralized) control structure consists of a set of k local feedback controllers of the form

$$\begin{aligned} \dot{\varsigma}_k &= \varphi_k(\varsigma_k, \nu_k) \\ u_k &= \varrho_k(\varsigma_k, \nu_k) \end{aligned} \tag{4}$$

exchanging information through a time-varying communication graph. Specifically, the input ν_k of each of such controllers, which represents exchange of relative information between the leader and the individual agents, is assumed to have the form

$$\nu_k(t) = a_{k0}(t)(\vartheta_0(t) - \vartheta_k(t)) + \sum_{\substack{j=1 \\ j\neq k}}^{N} a_{kj}(t)\,(\vartheta_j(t) - \vartheta_k(t)), \tag{5}$$

for $k = 1, \ldots, N$. In (5), ϑ_0 and the ϑ_j's, $j = 1, \ldots, N$ represent information taken at the leader and, respectively, at each agent, whereas the $a_{kj}\,(t)$ are non-negative-valued functions, modeling the weight of the communication link between the kth and ith agents. In the simple case in which r $= 1$,

$$\vartheta_i = y_i \qquad \forall\, i = 0, 1, \ldots, N\,.$$

The expression (5) reflects the fact that each local feedback controller communicates only with a limited number of neighbouring agents.

The problem addressed in the paper consists in the design of a set of local feedback controllers (4) in such a way that, for each k, the difference between the output y_k of the kth follower and the output y_0 of the leader (henceforth referred to as *tracking error*) asymptotically decays to 0 as time tends to ∞.

B. Basic assumptions

All functions/maps considered in the models (2)–(4) are assumed to be smooth. It is also assumed that, for some fixed pair of real numbers $0 < \underline{b} \leq \overline{b}$ the so-called high-frequency gain coefficient $b_k(z_k, \xi_k)$ of the kth agent satisfies

$$0 < \underline{b} \leq b_k(z_k, \xi_k) \leq \bar{b}\ . \tag{6}$$

The basic assumption on each of the agents (2) is that of being *strongly minimum phase*, formally specified in the following. [1] Using, as customary, $L_s\lambda(w)$ to denote the directional derivative of a function $\lambda(w)$ along a vector field $s(w)$, define

$$\xi_{\rm ss}(w) = \begin{pmatrix} h_0(w) \\ L_s h_0(w) \\ \cdots \\ L_s^{r-1} h_0(w) \end{pmatrix}$$

and observe that, if perfect tracking is achieved,

$$\xi_k(t) = \xi_{\rm ss}(w(t))\ .$$

Assumption 1: There exists a smooth map $\pi_k : W \to \mathbb{R}^{n_k}$ satisfying

$$L_s\pi_k(w) = f_k(\pi_k(w), \xi_{\rm ss}(w)) \qquad \forall w \in W\,,$$

and the system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_k &= f_k(z_k, \xi_{\rm ss}(w) + u), \end{aligned} \tag{7}$$

regarded as a system with state (w, z_k) and input u, is input-to-state stable (ISS) to the invariant set [2]

$$\mathcal{A}_k = \{(w, z_k) \in W \times \mathbb{R}^{n_k} : z_k = \pi_k(w)\}\ .$$

with a linear gain function and with an exponential decay rate. In particular, there exists a locally Lipschitz function $V_k : W \times \mathbb{R}^{n_k} \to \mathbb{R}$ such that the following holds:

[1] This assumption is customary in the literature on output regulation for nonlinear systems [17].

[2] See [27] for an introduction to the concept of input-to-state stability.

- there exist positive $\underline{a}_k$ and $\bar{a}_k$ such that

$$\underline{a}_k \|(w, z_k)\|_{\mathcal{A}_k} \leq V_k(w, z_k) \leq \bar{a}_k \|(w, z_k)\|_{\mathcal{A}_k}$$

  for all $(w, z_k) \in W \times \mathbb{R}^{n_k}$;

- there exist positive c_k and d_k such that

$$D^+_{(7)} V_k(w, z_k) \leq -c_k V_k(w, z_k) + d_k |u|$$

  for all $(w, z_k, u) \in W \times \mathbb{R}^{n_k} \times \mathbb{R}$

Remark. The existence of an ISS Lyapunov function with the properties detailed in the previous assumption implies that for all $(w(0), z(0)) \in W \times \mathbb{R}^{n_k}$ and all bounded $u(t)$, the resulting trajectory $(w(t), z(t))$ of (7) satisfies

$$\begin{aligned} &\|(w(t), z_k(t))\|_{\mathcal{A}_k} \leq \\ &\quad \max\{\lambda_k e^{-c_k t} \|(w(0), z_k(0))\|_{\mathcal{A}_k} \,,\; g_k^\circ \|u(\cdot)\|_\infty\} \end{aligned} \tag{8}$$

with $\lambda_k = 2\bar{a}_k / \underline{a}_k$ and $g_k^\circ = 2 d_k \int_0^\infty e^{-c_k(t-s)} ds / \underline{a}_k$, for all $t \geq 0$. ◁

Remark. The assumption that the gain function of (7) is linear could be weakened, requiring only linearity in a neighborhood of the origin, in which case a bound similar to the bound (8) would hold, so long as it can be guaranteed that, for some compact set U, the input function $u(\cdot)$ of (7) satisfies $|u(t)| \leq U$ for all $t \geq 0$, with g°_k a parameter depending on the set U. Weakening the assumption in this way would yield weaker convergence results. ◁

III. Standard results on asymptotic tracking

If each follower had access to a measurement of the output $y_0(t)$ of the leader, the design problem could be trivially reduced to the (independent) design of a set of N regulators, which could be accomplished by means of (relatively) standard methods developed to solve the problem of asymptotic tracking for nonlinear systems (see, for instance, [17] and references therein). Of course, this is not the case in the present setting, where the information pattern is reduced and only information exchange between neighbors is allowed, as expressed in (5). However, as shown later in the paper, results from the theory of asymptotic tracking for nonlinear systems can still be conveniently used. For this reason, we review in this section a number of such results, and put them in a form that is suited to our subsequent developments.

A. Reduction to relative degree 1

It is well known that if $r > 1$ the output of system (2) can be redefined so as to lower the relative degree to 1 while keeping the property of being strongly minimum phase. The reduction in relative degree is achieved by picking as new output the function (see [10])

$$\vartheta_k = \xi_{kr} + \sum_{j=1}^{r-1} c_j \xi_{kj} \tag{9}$$

in which the c_j 's are such that the polynomial $p(\lambda) = \lambda^{r-1} + c_{r-1}\lambda^{r-2} + \cdots + c_2\lambda + c_0$ is Hurwitz. The dynamics of (2) with ξ_{kr} replaced by ϑ_k can be seen as a system in normal form having relative degree 1 between input u_k and output ϑ_k, i.e.,

$$\begin{aligned}
\dot{z}_k &= f_k(z_k, \ell(\xi_{k1}, \ldots, \xi_{k,r-1}, \vartheta_k)) \\
\dot{\xi}_{k1} &= \xi_{k2} \\
& \cdots \\
\dot{\xi}_{k,r-1} &= -\textstyle\sum_{j=1}^{r-1} c_j \xi_{kj} + \vartheta_k \\
\dot{\vartheta}_k &= q_k(z_k, \ell(\xi_{k1}, \ldots, \xi_{k,r-1}, \vartheta_k)) \\
&\quad + \textstyle\sum_{j=1}^{r-2} c_j \xi_{k,j+1} + c_{r-1}[-\sum_{j=1}^{r-1} c_j \xi_{kj} + \vartheta_k] \\
&\quad + b_k(z_k, \ell(\xi_{k1}, \ldots, \xi_{k,r-1}, \vartheta_k)) u_k
\end{aligned}$$

in which

$$\ell(\xi_{k1}, \ldots, \xi_{k,r-1}, \vartheta_k) = \mathrm{col}(\xi_{k1}, \ldots, \xi_{k,r-1}, -\sum_{j=1}^{r-1} c_j \xi_{kj} + \vartheta_k)\,.$$

Having set

$$\vartheta_{\rm ss}(w) = L_s^{r-1} h_0(w) + \sum_{j=1}^{r-1} c_j L_s^{j-1} h_0(w)\,,$$

it is readily seen, as a standard consequence of the property that the cascade of two ISS systems is an ISS system (see [10]), that the system

$$
\begin{array}{rcl}
\dot{w} & = & s(w) \\
\dot{z}_k & = & f_k(z_k, \ell(\xi_{k1}, \ldots, \xi_{k,r-1}, \vartheta_{\rm ss}(w) + u)) \\
\dot{\xi}_{k1} & = & \xi_{k2} \\
& & \cdots \\
\dot{\xi}_{k,r-1} & = & -\sum_{j=1}^{r-1} c_j \xi_{kj} + \vartheta_{\rm ss}(w) + u
\end{array}
\tag{10}
$$

is ISS to the invariant set

$$
\begin{aligned}
\mathcal{A}'_k = \{ & (w, z_k, \xi_{i1}, \ldots, \xi_{i,r-1}) : w \in W, \\
& z_k = \pi_k(w), \xi_{i1} = h_0(w), \ldots, \xi_{i,r-1} = L_s^{r-2} h_0(w)\} \,.
\end{aligned}
$$

Hence, if (2) is strongly minimum phase (in the sense of Assumption 1), so is the system obtained after the replacement of the original output y_k with the new output ϑ_k.

In view of this calculation, from now on we restrict our analysis to the case in which all agents have relative degree 1. In this respect, it should also be observed that $\vartheta_k(t)$ is a linear combination of higher derivatives of y_k, i.e.

$$
\vartheta_k(t) = y_k^{r-1}(t) + \sum_{j=1}^{r-1} c_j y_k^{j-1}(t) \,.
$$

For the purpose of establishing the desired tracking results, classical results (see [6]) can be used to prove that partial state information such as ϑ_k can be replaced (with appropriate precautions) by a rough approximation provided by a high-gain observer driven by the actual output y_k. We will return to this issue at the end of the paper.

B. The standard internal model for each agent

In this subsection we briefly summarize some results of [17] concerning the design of a regulator for each individual agent, that will be used later in solution of the synchronization problem. Motivated by the discussion in the previous subsection, we consider the case of agents having relative degree 1 and, to simplify matters, we also assume that the "high-frequency gain" coefficient is *independent* of the state variables. In other words, we consider the case of agents modeled by equations of the form

$$\begin{aligned} \dot{z}_k &= f_k(z_k, y_k) \\ \dot{y}_k &= q_k(z_k, y_k) + b_k u_k \end{aligned} \tag{11}$$

in which b_k is a (possibly unknown) positive number.

Define $\psi_k : W \to \mathbb{R}$ via

$$L_s h_0(w) = q_k(\pi_k(w), h_0(w)) + b_k \psi_k(w)\,.$$

Based on the results of [17] it is known that there exists an integer m_k, a Hurwitz matrix $F \in \mathbb{R}^{m_k \times m_k}$, a vector $G \in \mathbb{R}^{m_k \times 1}$ such that the pair F, G is controllable, a function $\gamma_k : \mathbb{R}^{m_k} \to \mathbb{R}$ and a map $\sigma_k : W \to \mathbb{R}^{m_k}$, satisfying

$$\begin{aligned} L_s \sigma_k(w) &= F\sigma_k(w) + G\gamma_k(\sigma_k(w)) \\ \psi_k(w) &= \gamma_k(\sigma_k(w)) \end{aligned} \qquad \forall w \in W\,. \tag{12}$$

Consequently, it is possible to design, for the kth agent, an internal model of the form[3]

$$\begin{aligned} \dot{\eta}_k &= F\eta_k + G\gamma_k(\eta_k) + Gv_k \\ u_k &= \gamma_k(\eta_k) + v_k\,, \end{aligned} \tag{13}$$

in which $\eta_k \in \mathbb{R}^{m_k}$, F, G and $\gamma_k(\cdot)$ are such that (12) holds for some $\sigma_k(\cdot)$ and $v_k \in \mathbb{R}$ is a *residual* control input that will be determined later to secure the desired convergence properties. Note that the function $\gamma_k(\cdot)$ is only known (from [17]) to be continuous. However, in what follows, for convenience it will be *assumed* that the function in question is globally Lipschitz.

Define the *tracking error* of the kth agent as

$$e_k = y_k - y_0\,.$$

[3] It follows from the results of [17] that, since the number of agents is finite, it is possible to pick a single pair F, G for all agents, as the notation suggests.

The composition of (11) and (13), viewed as a system with input v_k and output e_k, has relative degree 1 and can be put in normal form by changing η_k into

$$\zeta_k = \eta_k - \frac{1}{b_k} G e_k \, .$$

The normal form in question is

$$\begin{aligned} \dot{z}_k &= f_k(z_k, h_0(w) + e_k) \\ \dot{\zeta}_k &= F\zeta_k + G\gamma_k(\sigma_k(w)) - \frac{1}{b_k}[Gq_k(z_k, h_0(w) + e_k) \\ &\quad - Gq_k(\pi_k(w), h_0(w)) - FGe_k] \\ \dot{e}_k &= q_k(z_k, h_0(w) + e_k) - q_k(\pi_k(w), h_0(w)) \\ &\quad + b_k[\gamma_k(\zeta_k + \frac{1}{b_k} G e_k) - \gamma_k(\sigma_k(w)] + b_k v_k \, . \end{aligned} \tag{14}$$

Having assumed that the agent is strongly minimum-phase and taking advantage of the fact that F is a Hurwitz matrix, it is possible to verify (using again the property that the cascade of two ISS systems is an ISS system) that the system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z}_k &= f_k(z_k, h_0(w) + e_k) \\ \dot{\zeta}_k &= F\zeta_k + G\gamma_k(\sigma_k(w)) - \frac{1}{b_k}[Gq_k(z_k, h_0(w) + e_k) \\ &\quad - Gq_k(\pi_k(w), h_0(w)) - FGe_k], \end{aligned} \tag{15}$$

viewed as a system with input e_k, is input-to-state stable to the invariant set

$$\mathcal{A}_k^{\rm a} = \{(w, z_k, \zeta_k) : w \in W, z_k = \pi_k(w), \zeta_k = \sigma_k(w)\} \, .$$

If, in addition, it is assumed that the function

$$q_k(z_k, h_0(w) + e_k) - q_k(\pi_k(w), h_0(w)) \, ,$$

which vanishes if $\|(w, z_k)\|_{\mathcal{A}_k} = 0$ and $e_k = 0$, satisfies a bound of the form

$$\begin{aligned} |q_k(z_k, h_0(w) + e_k) - q_k(\pi_k(w), h_0(w))| \leq \\ c_z \|(w, z_k)\|_{\mathcal{A}_k} + c_e |e_k| \end{aligned} \tag{16}$$

for some pair (c_z, c_e) of positive numbers independent of w, it can be concluded that (15) is ISS to the invariant set $\mathcal{A}_k^{\rm a}$ with a linear gain function and exponential decay rate. In particular, there exists a locally Lipschitz function $V^{\rm a}_k : W \times \mathbb{R}^{n_k} \times \mathbb{R}^{m_k} \to \mathbb{R}$ such that the following holds:

- there exist positive $\underline{a}_k^{\rm a}$ and $\bar{a}_k^{\rm a}$ such that

$$\underline{a}_k^{\rm a}\|(w, z_k, \zeta_k)\|_{\mathcal{A}_k^{\rm a}} \le V_k^{\rm a}(w, z_z, \zeta_k) \le \bar{a}_k^{\rm a}\|(w, z_z, \zeta_k)\|_{\mathcal{A}_k^{\rm a}}$$

  for all $(w, z_z, \zeta_k) \in W \times \mathbb{R}^{n_k} \times \mathbb{R}^{m_k}$;
- there exists positive $c_k^{\rm a}$ and $d_k^{\rm a}$ such that

$$D^+_{(15)}V_k^{\rm a}(w, z_z, \zeta_k) \le -c_k^{\rm a}V_k^{\rm a}(w, z_z, \zeta_k) + d_k^{\rm a}|e_k|$$

  for all $(w, z_z, \zeta_k) \in W \times \mathbb{R}^{n_k} \times \mathbb{R}^{m_k}$.

Finally, note that the coupling term in the last equation of (14), namely

$$\begin{aligned} q_k(z_k, h_0(w) + e_k) - q_k(\pi_k(w), h_0(w)) \\ +b_k[\gamma_k(\zeta_k + \frac{1}{b_k}Ge_k) - \gamma_k(\sigma_k(w)], \end{aligned}$$

vanishes if $\|(w, z_k, \zeta_k)\|_{\mathcal{A}_k^{\rm a}} = 0$ and $e_k = 0$.

If, as observed at the beginning of the section, each agent had access to a measurement of the output y_0 of the leader, each agent would have access to a measurement of the tracking error e_k. If this were the case, the design problem could be solved by simply picking $v_k = -\kappa e_k$, with κ a positive design parameter.

In fact, system (14) controlled by $v_k = -\kappa e_k$ might be viewed as the feedback interconnection of the following two subsystems: (i) a system with input e_k and state w, z_k, ζ_k, modeled by (15), which is ISS to the invariant set $\mathcal{A}_k^{\rm a}$ with a linear gain function and exponential decay rate; and (ii) a system with input $\|w, z_k, \zeta_k\|_{\mathcal{A}_k^{\rm a}}$ and state e_k, modelled by the last equation of (14) with $v_k = -\kappa e_k$, which can be rendered ISS to the invariant set $e_k = 0$ if κ is large enough. Thus, by the small-gain theorem for ISS systems, if the gain parameter κ were large enough, the variable $e_k(t)$ would be guaranteed to converge to 0 (while all other variables remain bounded).

This mode of control *may not be feasible, though,* because the kth agent may not have access to the kth tracking error. Thus, the structured exchange of information must be taken into account, as it will be done in the next section. We conclude the current section by rewriting in more compact form some of the relations developed so far.

C. The overall control structure

System (14) can be seen as a SISO system with input v_k and output e_k, having relative degree 1 and modeled by equations of the form (we omit indication of the dynamics of w)

$$\begin{aligned} \dot{z}_k^{\rm a} &= f_k^{\rm a}(w, z_k^{\rm a}, e_k) \\ \dot{e}_k &= q_k^{\rm a}(w, z_k^{\rm a}, e_k) + b_k v_k \,. \end{aligned}$$

in which $z_k^{\rm a} = (z_k, \zeta_k) \in \mathbb{R}^{n_k+m_k}$ and $f_k^{\rm a}(\cdot)$ and $q_k^{\rm a}(\cdot)$ are suitably defined.

Stacking all such systems together, we obtain a system with N inputs and N outputs modeled by equations of the form

$$\begin{aligned} \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + Bv \end{aligned} \qquad (17)$$

in which

$$\begin{aligned} z &= \mathrm{col}(z_1^{\rm a}, z_2^{\rm a}, \ldots, z_N^{\rm a}) \\ e &= \mathrm{col}(e_1, e_2, \ldots, e_N) \\ v &= \mathrm{col}(v_1, v_2, \ldots, v_N) \\ f(w, z, e) &= \mathrm{col}(f_1^{\rm a}(w, z_1^{\rm a}, e_1), \ldots, f_N^{\rm a}(w, z_N^{\rm a}, e_N)) \\ q(w, z, e) &= \mathrm{col}(q_1^{\rm a}(w, z_1^{\rm a}, e_1), \ldots, q_N^{\rm a}(w, z_N^{\rm a}, e_N)) \\ B &= \mathrm{diag}(b_1, b_2, \ldots, b_N) \end{aligned}$$

Note that, in view of the whole construction, if for all $k = 1, \ldots, N$ Assumption 1 and the bound (16) hold and the function $\gamma_k(\cdot)$ is globally Lipschitz, then the following holds:

(i) the system

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w,z,e)\,, \end{aligned} \tag{18}$$

viewed as a system with input e, is input-to-state stable, with a linear gain function and exponential decay rate, to a compact invariant set $\mathcal{A}^*$. In particular, there exists a locally Lipschitz function $V_z : W \times \mathbb{R}^n \to \mathbb{R}$, $n = \sum_{k=1}^{N}(n_k + m_k)$, such that:

- there exist positive $\underline{a}_z$ and $\bar{a}_z$ satisfying

  $$\underline{a}_z \|(w,z)\|_{\mathcal{A}^\star} \le V_z(w,z) \le \bar{a}_z \|(w,z)\|_{\mathcal{A}^\star}$$

  for all $(w,z) \in W \times \mathbb{R}^n$;
- there exists positive c and d such that

  $$D^+_{(18)} V_z(w,z) \le -cV_z(w,z) + d|e|\,,$$

  for all $(w,z) \in W \times \mathbb{R}^n$.

(ii) there exists a pair (K_1, K_2) of positive numbers such that

$$|q(w,z,e)| \le K_1|e| + K_2\|(w,z)\|_{\mathcal{A}^*}$$

for all w, z, e.

IV. Asymptotic coordination

In this section we present the main results of the paper. These results assume a property of connectivity, for a *time-varying* graph, which is based on the notion introduced by Moreau in [18] in his work on consensus in a network of integrator systems. A weaker version of such property is used in subsection IV-C, together with an additional assumption, to obtain the desired convergence result.

A. The communication protocol

We assume the reader is familiar with the major results about consensus of networked systems exchanging information over communication graphs and, therefore, we refrain from repeating well established definitions c oncerning g raphs. A s a nticipated i n S ection I I, t he e xchange of information between leader and followers has the expression (5), which in the present context takes the form (all agents have relative degree 1 and, therefore, $\vartheta_k = y_k$)

$$\nu_k(t) = a_{k0}(t)(y_0(t) - y_k(t)) + \sum_{\substack{j=1\\ j\neq k}}^{N} a_{kj}(t)\,(y_j(t) - y_k(t)) \tag{19}$$

for $k = 1, \ldots, N$, where $a_{kj}(t)$ is the element on the kth row and jth column of the so-called *adjacency matrix* $A(t)$ of the underlying communication *digraph*. All $a_{kj}(t)$'s are *piecewise-continuous* and *bounded* functions of time. Moreover, $a_{kj}(t) \geq 0$ and $a_{kk}(t) = 0$, for all $t \in \mathbb{R}$. Note that, in this specific case of a leader-follower configuration, $a_{0j}(t) \equiv 0$ for all $j = 1, \ldots, N$.

The signal (19), using the definition of tracking errors, can be expressed as

$$\nu_k(t) = \sum_{\substack{j=1\\ j\neq k}}^{N} a_{kj}(t)e_j - [a_{k0}(t) + \sum_{\substack{j=1\\ j\neq k}}^{N} a_{kj}(t)]e_k\,,$$

and, in compact form, as

$$\nu(t) = M(t)e(t) \tag{20}$$

in which

$$\nu = \mathrm{col}(\nu_1, \nu_2, \ldots, \nu_N)$$

and $M(t) \in \mathbb{R}^{N\times N}$ is a matrix defined as

$$\begin{aligned} m_{kj}(t) &= a_{kj}(t) \qquad\qquad \text{for } k \neq j \\ m_{kk}(t) &= -\sum_{\substack{j=0\\ j\neq k}}^{N} a_{kj}(t)\,. \end{aligned} \tag{21}$$

Remark. Note that the off-diagonal elements of $M(t)$ are non-negative and, for each $k = 1, \ldots, N$, the sum of all elements of the kth row is equal to $-a_{k0}(t)$. As a matter of fact, the negative of $M(t)$ coincides with the lower-right $N \times N$ block of the so-called *Laplacian matrix* $L(t)$ of the graph induced by the matrix $A(t)$. $\triangleleft$

The purpose of this paper is to show that the target of asymptotic tracking can be achieved by means of a control law of the form

$$v(t) = \kappa \nu(t) = \kappa M(t) e(t) \,,$$

in which $\kappa > 0$ is a gain parameter [4]. This choice, in view of (20), yields an overall controlled network which, augmented with the dynamics of the leader, reads as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, e) \\ \dot{e} &= q(w, z, e) + \kappa B M(t) e \,. \end{aligned} \tag{22}$$

Of course, the possibility of achieving this goal depends on the connectivity properties of the communication graph, which are reflected in properties of the matrix $M(t)$ which, in turn, influences the asymptotic properties of the time-varying linear system

$$\dot{e} = B M(t) e \,. \tag{23}$$

B. A digression on a Theorem of Moreau

In order to analyze the asymptotic properties of system (23), it is convenient to recall a fundamental result of Moreau (see [18]), who established appropriate connectivity assumptions under which the state $x \in \mathbb{R}^{N+1}$ of a network of $N+1$ first-order agents

$$\dot{x}_k = u_k \qquad k = 0, \ldots, N \tag{24}$$

[4]As observed earlier, if the entire vector e were available for measurement, a control law of the form $v(t) = \kappa e(t)$ would suffice to solve the problem.

controlled by

$$u_k = \sum_{\substack{j=0 \\ j\neq k}}^{N} a_{kj}(t)(x_j - x_k) \tag{25}$$

asymptotically converges to the equilibrium subspace $\mathcal{A} = \{x \in \mathbb{R}^{N+1} : x_0 = x_1 = \ldots = x_N\}$.

In the present context of a leader-follower configuration, $u_0 = 0$ and hence

$$\dot{x}_0 = 0\,.$$

Thus, without loss of generality, one can assume $x_0(t) = 0$ for all $t \in \mathbb{R}$ and describe the network in equivalent form in terms of the relative differences $e_k = x_k - x_0$ as

$$\dot{e}_k = \sum_{j=1}^{N} m_{kj}(t)e_j \qquad k = 1, \ldots, N, \tag{26}$$

in which the $m_{kj}(t)$ are the coefficients defined in (21).

The connectivity property determined in [18] under which the convergence of (24)–(25) to the equilibrium subspace takes place, can be described (in the present context of a leader-follower configuration) as follows.

Definition. The digraph associated with the adjacency matrix $A(t)$ is *uniformly connected* if there is a threshold value θ and an interval length $T > 0$ such that, for all $t \in \mathbb{R}$, in the θ-digraph [5] associated with the adjacency matrix

$$\int_t^{t+T} A(s)ds$$

all nodes may be reached from node 0. $\triangleleft$

Theorem 1 of [18] states that, if the digraph associated with the adjacency matrix $A(t)$ is uniformly connected, the equilibrium $e = 0$ of (26) is exponentially stable. Such a result can be

[5]The θ-digraph associated to an adjacency matrix $A_0(t)$ is a digraph with an arc from j to k ($k \neq j$) if and only if the element (k, j) of $A_0(t)$ is strictly larger than θ for all $t \in \mathbb{R}$.

also used to determine the asymptotic properties of system (23). In fact, it suffices to observe that the kth row of system (23) reads as

$$\dot{e}_k = b_k \sum_{j=1}^{N} m_{kj}(t) e_j \, ,$$

and hence system (23) can be interpreted as a system of the form (26) corresponding to an adjacency matrix $\tilde{A}(t)$ in which

$$\tilde{a}_{kj}(t) = b_k a_{kj}(t) \qquad k = 1, \dots, N, \quad j = 0, 1, \dots, N \, . \tag{27}$$

Since b_k is bounded as in (6), it is readily seen that, if the digraph associated with the adjacency matrix $A(t)$ is uniformly connected, so too is the digraph associated with the adjacency matrix $\tilde{A}(t)$. Thus, as an immediate corollary of Theorem 1 of [18], it can be concluded that if the digraph associated with the adjacency matrix $A(t)$ is uniformly connected, the equilibrium $e = 0$ of (23) is exponentially stable.

Theorem 1 of [18] is proven by showing the existence of a (time-independent) positive definite function of e that asymptotically decreases along trajectories. The function in question, in the present context of a leader-follower configuration and hence of the system described as in (26), is the function

$$V(e) = \max\{e_1, \dots, e_N, 0\} - \min\{e_1, \dots, e_N, 0\} \, . \tag{28}$$

This function is continuous but not continuously differentiable. However, it can be seen that this function can be bounded as

$$\underline{a}_e |e| \leq V(e) \leq \overline{a}_e |e| \qquad \forall e \in \mathbb{R}^N \, , \tag{29}$$

from which it is also seen that $V(e)$ is globally Lipschitz.

The proof of Theorem 1 of [18] shows that, if the digraph associated with the adjacency matrix $A(t)$ is uniformly connected, along any trajectory $e(t)$ of (26):

(i) the function $V(e(t))$ is non-increasing,

(ii) for some class $\mathcal{K}_\infty$ function $\gamma(\cdot)$

$$V(e(t_0+NT)) - V(e(t_0)) \leq -\gamma(|e(t_0)|) \tag{30}$$

for any t_0 (where the number T is the parameter appearing in the definition of uniform connectivity).

C. Convergence results

Motivated by the result of [18] summarized above and by the forthcoming Proposition 1, we consider in what follows the case in which the adjacency matrix $A(t)$, which characterizes the communication between agents, is such that the property indicated in the following assumption holds.

Assumption 2: There exists a globally Lipschitz function $V_e : \mathbb{R}^N \to \mathbb{R}$, bounded as in

$$\underline{a}_e|e| \leq V_e(e) \leq \bar{a}_e|e| \qquad \forall\, e \in \mathbb{R}^N$$

for some positive $\underline{a}_e$, $\bar{a}_e$, such that

$$D^+_{BM(t)e} V_e(e) \leq 0 \qquad \forall\, (e,t) \in \mathbb{R}^N \times \mathbb{R}_{\geq 0}\,. \tag{31}$$

Moreover, there exists a time T_0, a number $a > 0$ and a countable sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$ of the form $I_k = [t_{k,1}, t_{k,2}]$, with $t_{k,1} \leq t_{k,2} \leq t_{k+1,1}$ and $t_{k+1,1} - t_{k,2} \leq T_0$, such that

$$D^+_{BM(t)e} V_e(e) \leq -aV_e(e) \quad \forall\, (e,t) \in \mathbb{R}^N \times I_k\,. \tag{32}$$

As a matter of fact, using the results of [18], it is possible to check the following result.

Proposition 1: Suppose that the digraph associated with the adjacency matrix $A(t)$ is uniformly connected. Then, Assumption 2 holds.

Proof: As observed above, the digraph associated with the adjacency matrix $\tilde{A}(t)$ defined in (27) is uniformly connected. Therefore, along any trajectory $e(t)$ of (23), the function $V(e)$

defined in (28) is non-increasing and property (30) holds. The fact that $V(e)$ is non-increasing implies (31). From the inequality (30), it is easy to deduce the existence of a closed interval $I_{t_0} \subset [t_0, t_0 + NT]$ such that

$$D^+_{BM(t)e} V(e) \le -\frac{1}{2NT}\gamma(|e(t_0)|) \qquad \forall t \in I_{t_0}.$$

This inequality, in turn, using the estimate (29) for $V(e)$ and the property that $V(e(t))$ is non-increasing, can be further elaborated to yield

$$D^+_{BM(t)e} V(e) \le -\frac{1}{2NT}\gamma\left(\frac{V(e(t_0))}{\overline{a}_e}\right) \le -\frac{1}{2NT}\gamma\left(\frac{V(e(t))}{\overline{a}_e}\right).$$

Finally, is it observed that the estimates provided in [18] show that the function $\gamma(\cdot)$ on the left-hand side of (30) can be bounded as $a_0|e| \le \gamma(|e|)$ for some $a_0 > 0$. As a consequence, it is seen that

$$D^+_{BM(t)e} V(e) \le -\frac{a_0}{2NT\overline{a}_e} V(e) \quad \forall\, t \in I_{t_0}\,,$$

from which it is concluded that property (32) also holds. ■

It is seen from Proposition 1 that Assumption 2 is actually weaker than the assumption of uniform connectivity. As such, however, Assumption 2 may not be strong enough to guarantee exponential stability of (23), for the simple reason that no lower bound is prescribed on the measure of the intervals I_k. In view of this, it is convenient to strengthen Assumption 2 by requiring, for instance, that the I_k's (which, we recall, are intervals of the form $[t_{k,1}, t_{k,2}]$) satisfy, for some $n_0 \in \mathbb{N}$, $n_0 \ge 1$, and some $\tau > 0$,

$$\sum_{k=j}^{i-1} (t_{k,2} - t_{k,1}) \ge (i - j - n_0)\tau\,. \tag{33}$$

This inequality essentially expresses the property that, in the average, the intervals I_k have a guaranteed length, so as to secure – in view of (32) – that the solutions of (23) asymptotically decay to zero. The time τ, in particular, can be seen as an average length of the intervals I_k, whereas n_0 represents the maximal number of consecutive intervals I_k of zero length. As a whole, the condition can be regarded as an average dwell-time condition (see [9]). If this condition holds for *some* τ and n_0, then the solutions of (23) exponentially decay to zero. Moreover, as it will be shown in a moment, if τ and κ are sufficiently large, then it is also true that the solutions of the full system (22) are such that $e(t)$ exponentially decays to zero.

Proposition 2: Consider system (22) under Assumptions 1 and 2. There exist $\tau^\star \geq 0$ and $\kappa^\star > 0$ such that, if (33) holds for $\tau = \tau^*$ and some $n_0 \geq 1$, then the set $\mathcal{A}^\star \times \{0\}$ is globally asymptotically stable for system (22) for all $\kappa \geq \kappa^\star$.

Proof: With $V_z(w, z)$ and $V_e(e)$ the Lyapunov functions introduced respectively at the end of Section III-C and in Assumption 2, let $V_{\text{cl}} : W \times \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ be the candidate Lyapunov function for the closed-loop system defined as $V_{\text{cl}}(w, z, e) = V_z(w, z) + \beta V_e(e)$ with $\beta > 0$ yet to be chosen. By taking the upper directional derivative of $V_{\text{cl}}(\cdot)$ along (22), the last equation of which reads as

$$\dot{e} = q(w, z, e) + \kappa BM(t)e\,, \tag{34}$$

we obtain

$$D^+_{(22)}V_{\text{cl}}(w, z, e) = D^+_{(18)}V_z(w, z) + \beta D^+_{(34)}V_e(e)\,.$$

We develop separately the two terms. Regarding the derivative of $V_z(\cdot)$ we have

$$\begin{aligned} D^+_{(18)}V_z(w, z) &= -cV_z(w, z) + d|e| \\ &\leq -cV_z(w, z) + \frac{d}{\underline{a}_e}V_e(e)\,. \end{aligned}$$

Regarding the derivative of $V_e(\cdot)$, we have

$$\begin{aligned} &D^+_{(34)}V_e(e) = \limsup_{h\to 0^+} \frac{1}{h}[V_e(e + h\dot{e}) - V_e(e)] \\ &= \limsup_{h\to 0^+} \frac{1}{h}[V_e(e + h\dot{e}) - V_e(e + h\kappa BM(t)e) \\ &\quad + \limsup_{h\to 0^+} \frac{1}{h}[V_e(e + h\kappa BM(t)e) - V_e(e)] \\ &\leq \kappa D^+_{BM(t)e}V_e(e) + \limsup_{h\to 0^+} \frac{1}{h}L|hq(\cdot)| \\ &\leq \kappa D^+_{BM(t)e}V_e(e) + LK_1|e| + LK_2\|(w, z)\|_{\mathcal{A}^\star} \\ &\leq \kappa D^+_{BM(t)e}V_e(e) + L\frac{K_1}{\underline{a}_e}V_e(e) + L\frac{K_2}{\underline{a}_z}V_z(w, z) \end{aligned}$$

having denoted by L the Lipschitz constant of $V_e(\cdot)$ and having used the properties in the items (i) and (ii) at the end of Section III-C. This, bearing in mind (31) and (32), yields

$$D^+_{(34)} V_e(e) \leq -(\kappa a - L\frac{K_1}{\underline{a}_e}) V_e(e) + L\frac{K_2}{\underline{a}_z} V_z(w,z)$$

for all $t \in \{I_k\}$ and for all $(w,z,e) \in W \times \mathbb{R}^n \times \mathbb{R}^N$, and

$$D^+_{(34)} V_e(e) \leq L\frac{K_1}{\underline{a}_e} V_e(e) + L\frac{K_2}{\underline{a}_z} V_z(w,z)$$

for all $t \notin \{I_k\}$ and for all $(w,z,e) \in W \times \mathbb{R}^n \times \mathbb{R}^N$. Thus, choosing β so that

$$c - \beta L\frac{K_2}{\underline{a}_z} \geq \frac{c}{2}$$

and $\kappa^\star$ so that

$$\beta\kappa^\star a - \beta\frac{LK_1}{\underline{a}_e} - \frac{d}{\underline{a}_e} \geq \frac{c}{2}\beta$$

we have that, for all $t \in \{I_k\}$, for all $(w,z,e) \in W \times \mathbb{R}^n \times \mathbb{R}^N$, and for all $\kappa \geq \kappa^\star$,

$$\begin{aligned}
D^+_{(22)} V_{\rm cl}(w,z,e) &\leq -cV_z(w,z) + \frac{d}{\underline{a}_e} V_e(e) \\
&\quad -\beta(\kappa a - \frac{LK_1}{\underline{a}_e}) V_e(e) + \beta\frac{LK_2}{\underline{a}_z} V_z(w,z) \\
&\leq -(c - \beta\frac{LK_2}{\underline{a}_z}) V_z(w,z) - (\beta\kappa a - \beta\frac{LK_1}{\underline{a}_e} - \frac{d}{\underline{a}_e}) V_e(e) \\
&\leq -\frac{c}{2}(V_z(w,z) + \beta V_e(e)) = -\frac{c}{2} V_{\rm cl}(w,z,e) \\
&:= -\alpha_{\rm c} V_{\rm cl}(w,z,e)\,.
\end{aligned}$$

Thus, on the time interval $[t_{k,1}, t_{k,2}]$ the Lyapunov function decreases and the following estimate holds

$$V_{\rm cl}(t_{k,2}) \leq e^{-\alpha_{\rm c}(t_{k,2}-t_{k,1})} V_{\rm cl}(t_{k,1})\,.$$

Similarly, for all $t \notin \{I_k\}$, for all $(w,z,e) \in W \times \mathbb{R}^n \times \mathbb{R}^N$, and for all $\kappa \geq 0$

$$\begin{aligned}
D^+_{(22)} V_{\rm cl}(w,z,e) &\leq -cV_z(w,z) + \frac{d}{\underline{a}_e} V_e(e) \\
&\quad + \beta L \frac{K_1}{\underline{a}_e} V_e(e) + \beta L \frac{K_2}{\underline{a}_z} V_z(w,z) \\
&\leq (\beta L \frac{K_2}{\underline{a}_z} - c) V_z(w,z) + (\frac{d}{\underline{a}_e} + \beta L \frac{K_1}{\underline{a}_e}) V_e(e) \\
&\leq \alpha_{\rm d} V_{\rm cl}(w,z,e)
\end{aligned}$$

with $\alpha_{\rm d} := \max\{(\beta L K_2/\underline{a}_z - c)\,,\ (d/\underline{a}_e + \beta L K_1/\underline{a}_e)/\beta\}$. In the time intervals $(t_{k,2}, t_{k+1,1})$, in which the graph might not be connected, the growth of the Lyapunov function can be estimated as

$$V_{\rm cl}(t_{k+1,1}) \leq e^{\alpha_{\rm d}(t_{k+1,1} - t_{k,2})} V_{\rm cl}(t_{k,2}) \leq e^{\alpha_{\rm d} T_0} V_{\rm cl}(t_{k,2})\,.$$

From the inequalities thus established, it is seen that, if there exists a time τ_0 such that $t_{k,2} - t_{k,1} \geq \tau_0$ for all $k \in \mathbb{N}$ (in which case (33) is trivially satisfied with $n_0 = 1$ and $\tau = \tau_0$) and the times τ_0 and T_0 are such that

$$\alpha_{\rm c} \tau_0 \geq \alpha_{\rm d} T_0\,,$$

then the function $V_{\rm cl}(t)$ satisfies

$$V(t_{k+1,1}) \leq \bar{\alpha} V(t_{k,1}) \qquad \forall k \in \mathbb{N}$$

for some $\bar{\alpha} < 1$, in which case the result of the proposition follows.

In general, if a lower bound τ_0 for $t_{k,2} - t_{k,1}$ cannot be guaranteed, but the weaker property (33) holds, one can still establish the desired asymptotic properties by looking at the closed-loop system as an hybrid system flowing during the time intervals I_k in which the topology is connected and instantaneously jumping in the intervals $t \in [t_{k,2}, t_{k+1,1}]$. During flows the closed-loop Lyapunov function satisfies $D^+_{\mathrm{col}(\dot{w},\dot{z},\dot{e})} V_{\rm cl}(\cdot) \leq -\alpha_{\rm c} V_{\rm cl}(\cdot)$, whereas during jumps the Lyapunov function satisfies $V_{\rm cl}(\cdot)^+ \leq e^{\alpha_{\rm d} T_0} V_{\rm cl}(\cdot)$. The fact that the intervals I_k satisfy an average dwell-time condition expressed above allows one to say (see [3]) that flow and jump times of the hybrid system can be thought of as governed by a clock variable $\upsilon_{\rm c}$ flowing according to $\dot{\upsilon}_{\rm c} \in [0,\, 1/\tau]$, when $\upsilon_{\rm c} \in [0,\, n_0]$, and jumping according to $\upsilon_{\rm c}^+ = \upsilon_{\rm c} - 1$, when $\upsilon_{\rm c} \in [1,\, n_0]$. We thus endow the closed-loop system with the clock variable and study the resulting hybrid system flowing according to

$$\begin{array}{rcl} \dot{\upsilon}_{\rm c} & \in & [0, 1/\tau] \\ w & = & s(w) \\ \dot{z} & = & f(w,z,e) \\ \dot{e} & = & q(w,z,e) + \kappa B M(t) e \end{array} \tag{35}$$

when $(\upsilon_{\rm c}, w, z, e) \in [0, n_0] \times W \times \mathbb{R}^n \times \mathbb{R}$ and jumping according to

$$\begin{array}{rcl} \upsilon_{\rm c}^+ & = & \upsilon_{\rm c} - 1 \\ w^+ & = & w \\ z^+ & = & z \\ e^+ & = & e \end{array}$$

when $(\upsilon_{\rm c}, w, z, e) \in [1, n_0] \times W \times \mathbb{R}^n \times \mathbb{R}$.

For this hybrid system we consider the Lyapunov function

$$V_{\rm h}(\upsilon_{\rm c}, w, z, e) = e^{N\upsilon_{\rm c}} V_{\rm cl}(w,z,e)$$

with $N \in (\alpha_{\rm d} T_0, \alpha_{\rm c}\tau)$, by taking

$$\tau^\star = \frac{\alpha_{\rm d} T_0}{\alpha_{\rm c}}\,.$$

During flows we have that

$$\begin{array}{rcl} D^+_{(35)} V_{\rm h} & = & N\dot{\upsilon}_{\rm c} e^{N\upsilon_{\rm c}} V_{\rm cl} + e^{N\upsilon_{\rm c}} D^+_{(22)} V_{\rm cl} \\ & \leq & \dfrac{N}{\tau} e^{N\upsilon_{\rm c}} V_{\rm cl} - \alpha_{\rm c} e^{N\upsilon_{\rm c}} V_{\rm cl} \\ & \leq & \dfrac{N}{\tau} V_{\rm h} - \alpha_{\rm c} V_{\rm h} \\ & \leq & -\alpha_{\rm c}' V_{\rm h} \end{array}$$

where $\alpha'_{\rm c} = \alpha_{\rm c} - N/\tau > 0$. On the other hand, during jumps,

$$\begin{aligned} V_{\rm h}^{+} &= e^{N v_{\rm c}^{+}} V_{\rm cl}^{+} \\ &\leq e^{N(v_{\rm c}-1)} e^{\alpha_{\rm d} T_0} V_{\rm cl} \\ &= e^{-(N-\alpha_{\rm d} T_0)} e^{N v_{\rm c}} V_{\rm cl} \\ &= \epsilon V_{\rm h} \end{aligned}$$

with $\epsilon := e^{-(N-\alpha_{\rm d}T_0)} < 1$. This Lyapunov function is thus decreasing both during flows and during jumps and it is positive definite with respect to the set $[0,\, n_0] \times \mathcal{A}^\star \times \{0\}$. This proves the proposition. ■

Proposition 2 provides a sufficient condition under which the outputs $y_k(t)$, with $k = 1, \ldots, N$, of the agents (2) asymptotically converge to the output $y_0(t)$ of the leader (3). A couple of remarks are in order. First of all it is observed that, if $t_{k+1,1} = t_{k,2}$ for all $k \in \mathbb{N}$, property (32) holds for all $(e,t) \in \mathbb{R}^N \times R_{\geq 0}$. This condition reflects the property the digraph induced by $A(t)$ is connected at each t. In this case, $T_0 = 0$ and, as seen from the proof of the Proposition by taking $\tau^* = 0$, if κ is large enough, the set $\mathcal{A}^\star \times \{0\}$ is globally asymptotically stable for system (22). Thus, under such (strong) connectivity assumption, the proposed scheme is able to solve the leader-follower coordination problem for a network of nonlinear agents, if the latter are strongly minimum-phase. In the (more challenging) case in which $T_0 > 0$, coordination can be achieved *provided that* the average dwell-time condition (33) holds for some n_0 and $\tau = \tau^*$, with τ^* a time proportional (as seen from the proof of the Proposition) to T_0. In this case, again, coordination is achieved by picking a large enough value of κ.

D. The case of higher relative degree

We conclude the analysis with a quick discussion on how to handle the general case of relative degree $r > 1$. Recalling the definition (9) of ϑ_k, for $k = 1, \ldots, N$, set also

$$\vartheta_0 = L_s^{r-1} h_0(w) + \sum_{j=1}^{r-1} c_j L_s^{j-1} h_0(w)\,.$$

The variable $\chi_k(t) = \vartheta_k(t) - \vartheta_0(t)$ can be expressed as

$$\chi_k(t) = e_k^{r-1}(t) + \sum_{j=1}^{r-1} c_j e_k^{j-1}(t) \tag{36}$$

and the control law (5) can be rewritten as

$$\nu_k(t) = \sum_{\substack{j=1\\ j\neq k}}^{N} a_{kj}(t)\chi_j(t) - [a_{k0}(t) + \sum_{\substack{j=1\\ j\neq k}}^{N} a_{kj}(t)]\chi_k(t)\,. \tag{37}$$

If all ϑ_k's were available for measurement, the implementation of such a control law would give rise to an overall closed-loop system modelled by equations of the form (compare with (22))

$$\begin{aligned}
\dot{w} &= s(w)\\
\dot{z} &= f(z,\chi,w)\\
\dot{\chi} &= q(z,\chi,w) + \kappa B M(t)\chi\,,
\end{aligned}$$

in which

$$\chi = \mathrm{col}(\chi_1,\ldots,\chi_N)\,.$$

If the sufficient conditions presented above are satisfied, $\chi(t)$ decays exponentially to zero and so is the vector $e(t)$ of tracking errors, which according to (36) can be seen as a component of the state of a stable linear system driven by $\chi(t)$.

If the ϑ_k's are not available, one could replace them by quantities of the form

$$\hat{\vartheta}_k = \hat{\xi}_{kr} + \sum_{j=1}^{r-1} c_j \hat{\xi}_{kj} := C\hat{\xi}_k\,,$$

in which the $\hat{\xi}_k$'s, for $k = 0,1,\ldots,N$, are estimates provided by rough high-gain observers, the kth one of which ($k = 0,1,\ldots,N$) is modeled by equations of the form

$$\begin{aligned}
\dot{\hat{\xi}}_{k1} &= \hat{\xi}_{k2} + g a_{r-1}(y_k - \hat{\xi}_{k1})\\
\dot{\hat{\xi}}_{k2} &= \hat{\xi}_{k3} + g^2 a_{r-2}(y_k - \hat{\xi}_{k1})\\
&\cdots\\
\dot{\hat{\xi}}_{k,r-1} &= \hat{\xi}_{kr} + g^{r-1} a_1(y_k - \hat{\xi}_{k1})\\
\dot{\hat{\xi}}_{k,r} &= g^r a_0(y_k - \hat{\xi}_{k1})\,,
\end{aligned} \tag{38}$$

in which the a_i's are such that the polynomial $p(\lambda) = \lambda^{r-1} + a_{r-1}\lambda^{r-2} + \cdots + a_2\lambda + a_0$ is Hurwitz and g is a design parameter.

Well-known results (see [6], [28], [7] and also [10]) can be invoked to conclude that, if the gain parameter g is large enough, in the resulting closed-loop system the tracking error $e(t)$ asymptotically decays to zero, i.e., the output $y_k(t)$ of each agent asymptotically tracks the output $y_0(t)$ of the leader. In this respect, it should also be borne in mind that, if one is interested in establishing convergence for a fixed (compact) set of initial conditions (in which case the linear bounds for the ISS gains and the linear growth assumptions considered in the previous analysis are required to hold only in fixed compact regions, with parameters depending on the size of these regions) the control provided by (37) should be appropriately saturated, so as to avoid the occurrence of finite escape times (see again [6], [28]).

V. EXAMPLES

A. The setup

We consider the case of a set of four ($N = 4$) agents following a leader, with information exchanged via a periodically time-varying communication graph. On each period, the graph is connected for a time $\tau_0 > 0$. The adjacency matrix $A(t)$ is defined as follows (see also Example 1 in [22]):

$$0 \le t < \frac{1}{3}\tau_0 \quad \Rightarrow \quad A(t) = A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{3}\tau_0 \le t < \frac{2}{3}\tau_0 \quad \Rightarrow \quad A(t) = A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\frac{2}{3}\tau_0 \le t < \tau_0 \quad \Rightarrow \quad A(t) = A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\tau_0 \le t < \tau_0 + T_0 \quad \Rightarrow \quad A(t) = A_4 = 0\,.$$

The leader is a harmonic oscillator, described by

$$\begin{aligned} \dot{w} &= Sw \\ y_0 &= C_0 w \end{aligned}$$

in which

$$S = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \qquad C_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

In the simulations, the frequency is $\omega_0 = 1$ and the initial condition is $w(0) = \mathrm{col}(1, 0)$.

Two of the followers are linear systems having relative degree 1, described by

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned}$$

in which

$$A_k = \begin{pmatrix} 0 & 1 & 0 \\ -a_{0k} & -a_{1k} & 1 \\ c_{0k} & c_{1k} & c_{2k} \end{pmatrix} \qquad B_k = \begin{pmatrix} 0 \\ 0 \\ b_k \end{pmatrix}$$

$$C_k = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

In the simulations, the parameters of agent 1 are $a_{01} = 1$, $a_{11} = 2$, $c_{01} = 0$, $c_{11} = 1$, $c_{21} = 0$, $b_1 = 1$, whereas the parameters of agent 2 are $a_{02} = 3$, $a_{12} = 1$, $c_{02} = 0$, $c_{12} = 1$, $c_{22} = 2$, $b_2 = 1$. The initial conditions are randomly chosen between -1 and 1.

The other two followers are nonlinear systems whose dynamics are that of a Van der Pol oscillator, with output map chosen so as to have relative degree 1

$$\begin{aligned} \dot{x}_{1k} &= x_{2k} \\ \dot{x}_{2k} &= -x_{1k} + \varepsilon_k x_{2k}(1 - x_{1k}^2) + b_k u_k \ , \\ y_k &= x_{2k} + c_k x_{1k} \end{aligned}$$

in which $\varepsilon_k, c_k > 0$.

In the simulations, the parameters of agent 3 are $\varepsilon_3 = 1$, $c_3 = 1$, $b_3 = 1$, whereas the parameters of agent 4 are $\varepsilon_4 = 10$, $c_4 = 2$, $b_4 = 1$. The initial conditions are randomly chosen between -1 and 1.

The normal forms of agents 1 and 2 read as

$$\begin{aligned} \dot{z}_k &= \begin{pmatrix} 0 & 1 \\ -a_{0k} & -a_{1k} \end{pmatrix} z_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_k \\ \dot{\xi}_k &= \begin{pmatrix} c_{0k} & c_{1k} \end{pmatrix} z_k + c_{2k}\xi_k + b_k u_k \\ y_k &= \xi_k \end{aligned}$$

while the normal forms of agents 3 and 4 read as

$$\begin{aligned} \dot{z}_k &= -c_k z_k + \xi_k \\ \dot{\xi}_k &= c_k \xi_k - (1 + c_k^2)\, z_k + \varepsilon_k\, (\xi_k - c_k z_k)\, (1 - z_k^2) \\ &\quad + b_k u_k \\ y_k &= \xi_k \,. \end{aligned}$$

From this, it is seen that all agents have relative degree 1 and Assumption 1 is fulfilled.

B. The localized internal models

According to the theory of output regulation, for the two linear agents we choose as internal model a system of the form

$$\begin{aligned}\dot{\eta}_k &= (F_k + G_k\Gamma_k)\eta_k + G_k v_k\\ u_k &= \Gamma_k\eta_k + v_k\end{aligned}$$

in which F_k is a Hurwitz matrix, (F_k, G_k) is controllable and Γ_k such that the matrix $F_k+G_k\Gamma_k$ is similar to the matrix S that describes the dynamics of the leader. In the simulations, we have

$$F_1 = F_2 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \qquad G_1 = G_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\Gamma_1 = \Gamma_2 = \begin{pmatrix} 0 & 2 \end{pmatrix}.$$

For the two nonlinear agents, according to the results of [2], the internal model can be chosen as a linear system of appropriate dimension. In particular, it can be chosen as

$$\begin{aligned}\dot{\eta}_k &= (F_k + G_k\Gamma_k)\eta_k + G_k v_k\\ u_k &= \Gamma_k\eta_k + v_k\end{aligned}$$

in which F_k is a Hurwitz matrix, (F_k, G_k) is controllable and Γ_k is such that the spectrum of $F_k+G_k\Gamma_k$ is $\{j\omega_0, -j\omega_0, j3\omega_0, -j3\omega_0\}$. In the simulations, we have

$$F_3 = F_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{pmatrix}. \quad G_3 = G_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\Gamma_3 = \Gamma_4 = \begin{pmatrix} -8 & 4 & -4 & 4 \end{pmatrix}.$$

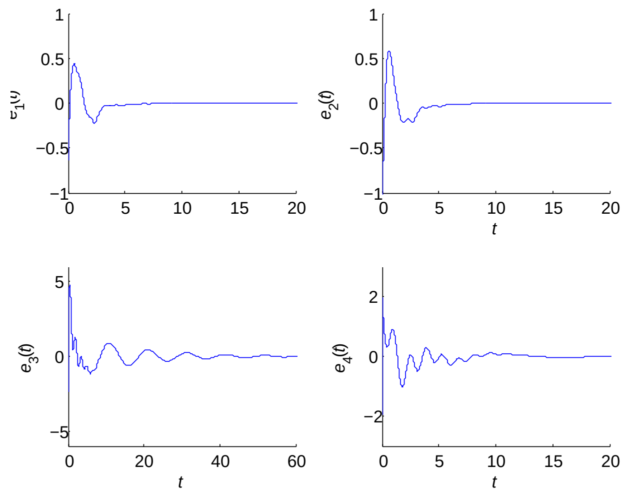

Fig. 1. Error dynamics in simulation 1 ($\tau_0 = 1.2$, $T_0 = 0$)

This yields

$$F_k + G_k\Gamma_k = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{pmatrix},$$

which, recalling that $\omega_0 = 1$, has the desired eigenvalues.

C. Simulation of the entire control system

The entire system was simulated, each agent being controlled by $u_k = \Gamma_k \eta_k + v_k$, in which $v_k = -\kappa\nu_k$, with ν_k defined as in (19) and κ a constant gain.

In the first simulation, we have considered the case in which the graph is connected for all time t, i.e., $T_0 = 0$. We have chosen $\tau_0 = \tau_0 + T_0 = 1.2$ and set the gain parameter $\kappa = 2$. Figure 1 shows the resulting error dynamics of the four agents, which asymptotically converge to zero; Figure 2 shows the control actions.

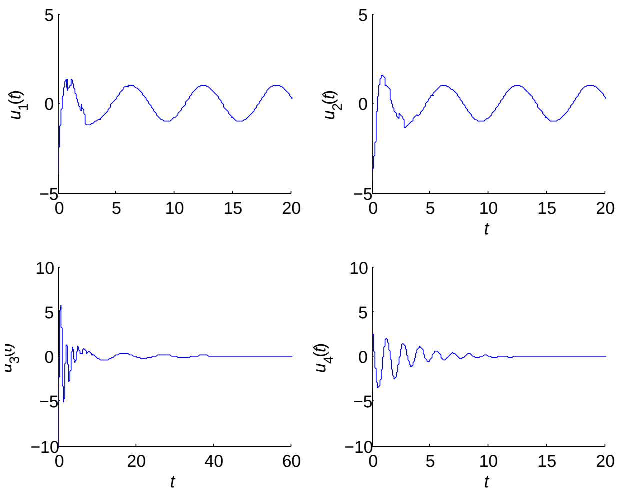

Fig. 2. Control actions in simulation 1 ($\tau_0 = 1.2$, $T_0 = 0$)

Then, in the second simulation, we have considered the case in which $\tau_0 = 0.3$, with the graph disconnected over an interval of duration $T_0 = 0.9$. In this setup, the value of the gain parameter κ selected in the previous simulation proved to be insufficient; therefore, κ was set to the larger value $\kappa = 20$. Figure 3 shows the error dynamics of the four agents, which, also in this case, asymptotically converge to zero; Figure 4 shows the control actions. By comparing Figure 2 and Figure 4, it can be noted that the control actions are stronger in the second simulation, since a larger value of κ is needed to guarantee the asymptotic convergence when $T_0 > 0$.

VI. Final remarks and conclusions

In this paper, we have addressed the problem of leader-follower coordination for a *heterogeneous* network of *nonlinear* agents exchanging relative *output* information through a *time-varying* communication network, extending in various directions a number of existing results, such as those of [19], dealing with the problem of achieving consensus in a set of first-order (integrator) systems, as well as those of [20], [22] and [33], dealing with the problem of leader-follower coordination for a homogeneous network of identical linear/nonlinear agents exchanging relative

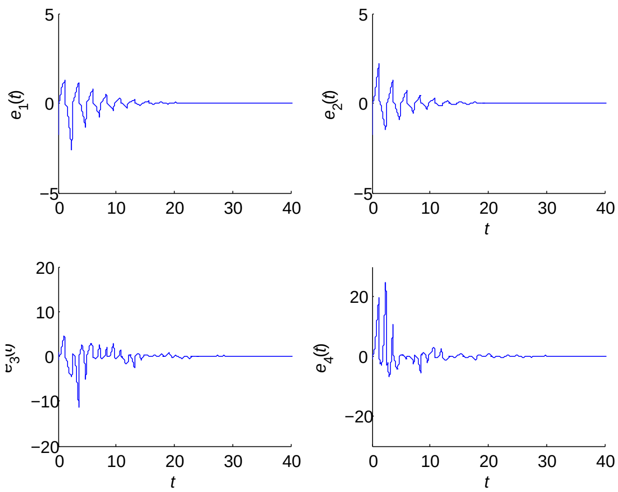

Fig. 3. Error dynamics in simulation 2 ($\tau_0 = 0.3$, $T_0 = 0.9$)

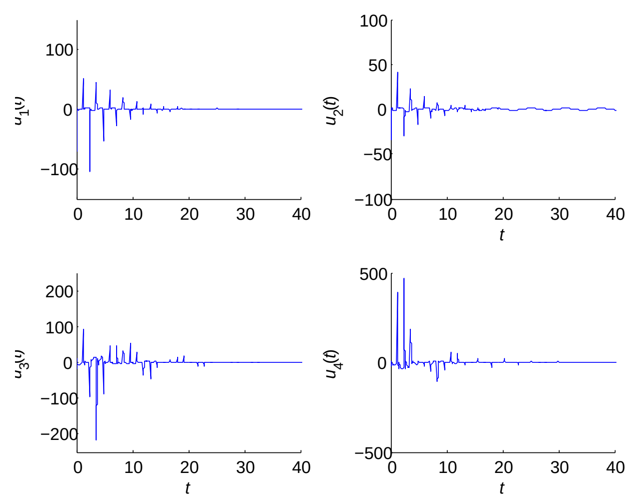

Fig. 4. Control actions in simulation 2 ($\tau_0 = 0.3$, $T_0 = 0.9$)

state information. In our approach, the individual followers are not assumed to be identical, the state-space model is nonlinear and not necessarily of first order, and the required information exchange between leader and (a limited fraction of the) followers as well as between neighboring followers only deals with relative values of the (output) variables that have to be synchronized (as opposed to the case in which full state information exchange is used).

We address and solve the problem under appropriate assumptions. One basic assumption is that the state-space model of each agent has a well-defined relative degree and possesses *a zero dynamics that is input-to-state stable*. This a standard assumptions under which, for each individual follower, the problem of asymptotically tracking a reference generated by an autonomous leader can be solved. What makes the current problem different from the standard problem of asymptotic tracking is the information pattern, which, in this case, is characterized by the exchange of only relative information between neighbors connected through a possibly time-varying graph. The other basic assumption is a *connectivity property* of the communication graph. The connectivity property expresses the guaranteed decay, over time intervals of variable length satisfying an average dwell-time condition, of a candidate Lyapunov function associated to an auxiliary network of integrator systems. The property in question implies the existence of a well-defined (average) bound between time intervals in which the graph is disconnected and time intervals in which the graph is connected. This, in turn, makes it possible to use high-gain output feedback to dominate the effects of the internal dynamics when the dimension of the followers is higher than one. If all followers were one-dimensional, a weaker connectivity condition – essentially equivalent to the one considered in [18] – would suffice.

Since the design depends on picking a sufficiently large value of the gain parameter κ, one may wonder how the minimal value of such parameter scales with the number N of agents. In this respect, it can be seen from the proof of the main result, that this minimal value (denoted κ^*) depends on certain bounds associated with the model of each follower (and as such, independent of the number N of agents so long as the same bounds can be uniformly established for all such agents) as well as on certain parameters (specifically the parameters a and $\underline{a}_e$ in Assumption 2) associated with the connectivity hypothesis. Thus, it can be claimed that the value of κ^* is independent of N so long as Assumption 2 holds for *fixed* pairs of such parameters.

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