Group rings whose skew elements are bounded Lie Engel*

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December 24, 2020

Abstract

Let FG be the group ring of a group G over a field F of characteristic different from 2, and let FG have an involution induced from one on G. Assuming that G has no elements of order 2 and no dihedral group involved, we determine the conditions under which the set of skew elements of FG is bounded Lie Engel. Furthermore, we make the determination with no restrictions upon G when the involution on FG is classical.

1 Introduction

Let R be a ring with involution $*$. Write R^- for the set of skew elements of R. That is, $R^- = \{r \in R : r^* = -r\}$. It is a natural question to ask if properties of R^- are inherited by R. For instance, a famous result due to Amitsur states that if R^- satisfies a polynomial identity, then so does R. Along this line, a general problem of interest is to discover if Lie properties of R^- are also satisfied by R.

In particular, let G be a group with involution $*$, and let F be a field of characteristic different from 2. Extending $*$ linearly to the group ring FG, we observe that $(FG)^-$ is the set of linear combinations of terms of the form $g - g^*$, with $g \in G$. We define the Lie product on FG via

$$[x_1, x_2] = x_1x_2 - x_2x_1$$

and, inductively,

$$[x_1, \dots, x_{n+1}] = [[x_1, \dots, x_n], x_{n+1}].$$

A subset S of FG is said to be Lie nilpotent if there exists an n such that

$$[s_1, \dots, s_n] = 0$$

*This paper was written while the first named author was visiting the Università degli Studi di Roma “La Sapienza”, and while both authors were visiting the Università del Salento and the University of Alberta. The authors thank the members of the mathematics departments at these institutions for their warm hospitality.

for all $s_i \in S$. On the other hand, S is Lie n-Engel if

$$[s_1, \underbrace{s_2, \ldots, s_2}_{n \text{ times}}] = 0$$

for all $s_1, s_2 \in S$, and bounded Lie Engel if it is Lie n-Engel for some n.

The conditions under which FG satisfies these properties were determined in classical papers. Passi, Passman and Sehgal classified the groups such that FG is Lie nilpotent, and Sehgal did the same for bounded Lie Engel group rings. These results are discussed in Chapter V of [16].

Quite a few papers have considered the classical involution on FG, induced from the map given by $g^* = g^{-1}$ for all $g \in G$. For example, in [8], Giambruno and Sehgal showed that if $(FG)^-$ is Lie nilpotent, and G has no 2-elements, then FG is also Lie nilpotent. Lee proved the analogous result for the bounded Lie Engel property in [11].

In recent years, other involutions on the group ring have begun to be considered. In [2], for instance, Broche Cristo, Jespers, Polcino Milies and Ruiz Marín determined when the skew elements of FG commute, for any involution induced from an involution on G. Subsequently, in [6], Giambruno, Polcino Milies and Sehgal determined when $(FG)^-$ is Lie nilpotent, if G is a torsion group having no elements of order 2. Catino, Lee and Spinelli [3] proved the corresponding result for the bounded Lie Engel property. It should be noted that these results turned out to be more involved than those for the symmetric elements (that is, those fixed by the involution). There are exceptional cases even when the group has no 2-elements.

In a recent paper, [7], Giambruno, Polcino Milies and Sehgal showed, using a different method of proof, that the hypothesis of [6] can be weakened from G being torsion to G having no dihedral group involved; that is, G has no subgroup H having a nonabelian dihedral group as a homomorphic image. (The assumption that G has no 2-elements must remain in place.) It seems only natural to ask if such a result holds true for the bounded Lie Engel property as well. The purpose of this paper is to show that it does. Our main theorem is the following. Recall that a group G is said to be p-abelian if G' is a finite p-group, and that 0-abelian means abelian.

Theorem 1. *Let F be a field of characteristic $p \neq 2$ and G a group having no elements of order 2, such that no dihedral group is involved in G. Let $*$ be an arbitrary involution on G, and extend it F-linearly to FG. Then $(FG)^-$ is bounded Lie Engel if and only if either*

1. *FG is bounded Lie Engel, or*
2. *$p > 2$, G has a p-abelian normal subgroup of finite index, and G has a normal $*$-invariant p-subgroup N of bounded exponent such that the induced involution on G/N is trivial.*

In the final section, we add a footnote to our result by classifying the groups G such that the set of skew elements of FG, with respect to the classical involution, is bounded Lie Engel, without any restriction upon G.

2 Background results

Let us gather some necessary results. Throughout, let G be a group with involution and F a field of characteristic $p \geq 0$. Let FG have the induced involution. First, we need to know the conditions under which a group ring is bounded Lie Engel. These were determined by Sehgal in [16, Theorem V.6.1].

Lemma 1. *If $p = 0$, then FG is bounded Lie Engel if and only if G is abelian. If $p > 0$, then FG is bounded Lie Engel if and only if G is nilpotent and G contains a p-abelian normal subgroup of p-power index.*

If R is an F-algebra, then R is said to satisfy a polynomial identity if there exists a nonzero polynomial $f(x_1, \dots, x_n)$ in the free algebra $F\{x_1, x_2, \dots\}$ on noncommuting indeterminates $x_1, x_2, \dots$ such that $f(r_1, \dots, r_n) = 0$ for all $r_i \in R$. The conditions under which FG satisfies a polynomial identity were determined by Isaacs and Passman.

Lemma 2. *The group ring FG satisfies a polynomial identity if and only if G has a p-abelian normal subgroup of finite index.*

Proof. See [14, Corollaries 5.3.8 and 5.3.10]. □

Now, if R is an F-algebra with involution, then we say that R satisfies a $*$-polynomial identity if there exists a nonzero polynomial $f(x_1, x_1^*, \dots, x_n, x_n^*)$ in the free algebra with involution $F\{x_1, x_1^*, \dots\}$ such that $f(r_1, r_1^*, \dots, r_n, r_n^*) = 0$ for all $r_i \in R$. For our purposes, $(FG)^-$ will be Lie n-Engel. Thus, FG will satisfy

$$[x_1 - x_1^*, \underbrace{x_2 - x_2^*, \dots, x_2 - x_2^*}_{n \text{ times}}].$$

By a theorem of Amitsur (see [1]), if R satisfies a $*$-polynomial identity, then R satisfies a polynomial identity. Thus, our group G will always have a p-abelian normal subgroup A of finite index. By replacing A with $A \cap A^*$, we may assume that A is $*$-invariant.

The starting point of our investigations will be the following lemma.

Lemma 3. *Let G be a group without 2-elements and such that no dihedral is involved. Suppose that $(FG)^-$ is bounded Lie Engel. If $p = 0$, then G is abelian. If $p > 2$, then the p-elements of G form a subgroup P, and G/P is abelian.*

Proof. In view of [3, Lemma 4], the $p = 0$ case follows just as in [7, Corollary 2.2]. When $p > 2$ and G is torsion, the fact that P is a subgroup comes from the main result of [3]. With this in mind, the proof of [7, Lemma 2.5] can be used for the bounded Lie Engel property as well. □

Thus, the $p = 0$ case is done, so we need only concern ourselves with fields of odd prime characteristic. Let us continue to write P for the group of p-elements in G. Since G' is a p-group, we see that the torsion elements of G form a subgroup, T. As we will need to restrict the form of T, let us state a result from the torsion case.

Lemma 4. *If G is a torsion group without 2-elements, then our main theorem holds. In particular, if $(FG)^-$ is bounded Lie Engel and $p > 2$, then P is nilpotent, and if FG is not bounded Lie Engel, then the subgroup of P/P' upon which $*$ acts as the classical involution has bounded exponent.*

Proof. See the main theorem as well as Lemmas 10 and 11 of [3]. □

We also need a few group-theoretic lemmas concerning dihedral involvement. Write ζ for the centre of G.

Lemma 5. *Let p be an odd prime. Suppose that $G = P \rtimes X$ is a finite group, where P is a p-group and X is an abelian p'-group of even order. Then either a dihedral group is involved in G, or there exists an element $z \in X \cap \zeta$ with $o(z) = 2$.*

Proof. See [7, Lemma 2.10]. □

Lemma 6. *Let p be an odd prime. Suppose that G' is a p-group, G has no 2-elements and $(G : \zeta) < \infty$. If no dihedral is involved in G, then $(G : \zeta)$ is odd.*

Proof. See [7, Corollary 2.12]. □

The next fact is mentioned in the discussion following Theorem 2.1 of [7].

Lemma 7. *Suppose that G is a group without 2-elements having an abelian subgroup of index 2. Then either G is abelian or there is a dihedral involved in G.*

Two lemmas concerning involutions on groups will be very helpful.

Lemma 8. *Let A be an abelian group without 2-elements, having an involution. Then $A^2 \leq A_1 \times A_2$, where $A_1 = \{a \in A : a^* = a\}$ and $A_2 = \{a \in A : a^* = a^{-1}\}$. In particular, if $A_2 = 1$, then $*$ acts trivially upon A.*

Proof. The first part is Lemma 2.9 of Giambruno, Polcino Milies and Sehgal [5]. For the second part, suppose that $A_2 = 1$. Then $A^2 \leq A_1$; that is, $*$ acts trivially upon A^2. But take any $a \in A$. Then $a^2 = (a^*)^2$, hence $(a^*a^{-1})^2 = 1$. As A has no 2-elements, $a^*a^{-1} = 1$, and hence $*$ acts trivially upon A. □

We will use the notation A_1 and A_2 throughout the paper.

Lemma 9. *Let G have an abelian normal $*$-invariant torsion subgroup A without 2-elements. Suppose that $x \in G$ satisfies $x^* \in x^{-1}A$. Then there exists $c \in A_1$ such that $(xc)^* = (xc)^{-1}$.*

Proof. See [5, Lemma 2.11]. □

One additional result is required. If $g \in G$ has finite order, then write $\hat{g} = \sum_{i=1}^{o(g)} g^i$.

Lemma 10. *Suppose that* $0 \neq \alpha \in FG$ *and* $1 \neq d \in G$*. If* $\alpha(1-d) = 0$*, then* d *has finite order,* $\alpha \in FG\hat{d}$*, and the support of* α *consists of a multiple of* $o(d)$ *elements. In particular, if* $p \neq 2$*,* G *has no* 2*-elements, and* $\alpha = [a, b-c]$*, for some* $a, b, c \in G$*, then* $p = o(d) = 3$ *and* $b^{-1}c \in \langle d \rangle$*.*

Proof. The first part is [7, Lemma 2.7]. In the case where $\alpha = [a, b-c]$, since only four group elements appear, we must have $o(d) = 3$ and the support of α consists of precisely three group elements. In particular, either $\alpha = 2ab - ac - ba$, where ab, ac and ba are distinct, or $\alpha = ab + ca - 2ac$, where ab, ca and ac are distinct. Either way, since $\alpha \in FG\hat{d}$, we can only have $p = 3$ and $\alpha = \pm g\hat{d}$, for some $g \in G$. In particular, letting $ab = gd^i$ and $ac = gd^j$, we have $b^{-1}c = d^{j-i}$, as required. □

3 The case where $*$ is trivial on G/P

Let G be a group without 2-elements such that no dihedral is involved in G. Let G have an arbitrary involution $*$, and extend it linearly to FG. As the characteristic zero case is finished, let char $F = p > 2$. We have already observed that if $(FG)^-$ is bounded Lie Engel, then the p-elements of G form a subgroup P, the torsion elements form a subgroup T, and G has a p-abelian $*$-invariant normal subgroup A of finite index. In this section, we will dispose of the situation where $*$ acts trivially upon G/P.

It should not be surprising that certain arguments from [3, 5, 7] will be useful to us here. Where we can cite lemmas from these papers, we will do so, but in some cases, we will use the arguments and adapt them to our purposes. In these instances, we will tend to give a full proof, in order to make the paper more readable.

We begin with a special case.

Lemma 11. *Let* G *be a group without* 2*-elements such that no dihedral group is involved in* G*. If* $*$ *is trivial on* G/P*,* $(P/P')_2$ *has bounded exponent and* $(FG)^-$ *is bounded Lie Engel, then* G *has a* $*$*-invariant normal* p*-subgroup* N *of bounded exponent such that* $*$ *is trivial on* G/N*.*

Proof. We claim that P' has bounded exponent. By Lemma 4, either FP is bounded Lie Engel or P has a $*$-invariant normal subgroup M of bounded exponent such that $*$ is trivial on P/M. In the first case, we note that FP satisfies the identity

$$[x_1, \underbrace{x_2, \ldots, x_2}_{p^m \text{ times}}] = [x_1, x_2^{p^m}],$$

for some suitable m. That is, $[u, v^{p^m}] = 0$ for all $u, v \in P$. In particular, P is a p-group of bounded exponent modulo its centre. By Lemma 4, P is nilpotent as well. Thus, by a theorem of Schur (see [16, Corollary I.4.3]), P' has bounded exponent, as claimed. In the second case, we simply note that if $*$ is trivial on P/M, then $P' \leq M$, and the claim is proved here as well.

Thus, let us factor out P'. Then P is abelian, and by assumption, P_2 has bounded exponent. As P is abelian and normal, this implies that the $*$-invariant normal subgroup N of G generated by P_2 is also a p-group of bounded exponent. Therefore, we factor it out as well, and conclude that $*$ is trivial on P. Of course, $*$ also acts trivially on G/P.

Take any $g \in G$. Then $g^* \in gP$, so let us say that $g^* = gc$, with $c \in P$. But then

$$g = (g^*)^* = (gc)^* = c^* g^* = cgc.$$

Therefore, $g^{-1}cg = c^{-1}$. It now follows that $\langle c, g^2 \rangle$ is an abelian subgroup of index at most 2 in $\langle c, g \rangle$. By Lemma 7, either $\langle c, g \rangle$ is abelian or a dihedral group is involved. As the latter option is impossible, $cg = gc$. Therefore, $g = gc^2$, and hence $c = 1$. That is, $g^* = g$ for all $g \in G$. $\square$

The following lemma was proved for torsion groups in [3], but by modifying the proof, we can allow an arbitrary group (even with 2-elements or a dihedral group involved). It also works for the set of symmetric elements in FG.

Lemma 12. *Let G have a central subgroup H of unbounded exponent upon which the involution is classical. If $(FG)^-$ is Lie p^m-Engel, then FG is Lie p^m-Engel.*

Proof. We know that FG satisfies

$$[x_1 - x_1^*, \underbrace{x_2 - x_2^*, \ldots, x_2 - x_2^*}_{p^m \text{ times}}].$$

As this identity is linear in x_1, we see from [8, Theorem 2] that FG satisfies the $*$-polynomial identity

$$[x_1, \underbrace{x_2 - x_2^*, \ldots, x_2 - x_2^*}_{p^m \text{ times}}];$$

that is,

$$[x_1, (x_2 - x_2^*)^{p^m}].$$

Suppose the theorem fails, and fix $\alpha, \beta \in FG$ such that $[\alpha, \beta^{p^m}] \neq 0$. Let g_i, $1 \leq i \leq r$, be the group elements appearing in the support of $[\alpha, \beta^{p^m}]$. Now,

$$[\alpha, (\beta - \beta^*)^{p^m}] = 0.$$

The left-hand side of this last equation is a linear combination of group elements. Naturally, they must all cancel, but let h_j, $1 \leq j \leq s$, be all of the group elements appearing in this linear combination. Surely each g_i is equal to some h_j. Furthermore, for any $z \in H$,

$$0 = [\alpha, (z\beta - (z\beta)^*)^{p^m}] = [\alpha, (z\beta - z^{-1}\beta^*)^{p^m}].$$

Now, $[\alpha, (z\beta)^{p^m}] = z^{p^m}[\alpha, \beta^{p^m}] \neq 0$, and we see that each $z^{p^m} g_i$ is equal to some $z^k h_j$, with $-p^m \leq k \leq p^m$. But this means that $z^{p^m - k} = h_j g_i^{-1}$. However, there are only finitely many h_j and g_i, so since H has unbounded exponent, we can choose z in such a way that $z^{p^m-k} \neq h_j g_i^{-1}$ for all i and j unless $k = p^m$. Thus, the group elements in the support of $z^{p^m}[\alpha, \beta^{p^m}]$ cannot cancel with any other terms in our calculation. Since $[\alpha, \beta^{p^m}] \neq 0$ but $[\alpha, (z\beta - (z\beta)^*)^{p^m}] = 0$, we have a contradiction. $\square$

In any group, write $(g, h) = g^{-1}h^{-1}gh$ and $g^h = h^{-1}gh$. The main result for this section is

Lemma 13. *Let G have no 2-elements and suppose that no dihedral group is involved in G. If $*$ is trivial on G/P, then our main theorem holds.*

Proof. Suppose that $(FG)^-$ is bounded Lie Engel. By Lemma 4, either FT is bounded Lie Engel or $(P/P')_2$ has bounded exponent. In the latter case, we are done, by Lemma 11 (since G has a p-abelian normal subgroup of finite index whenever FG satisfies a polynomial identity), so assume that FT is bounded Lie Engel. Then we know from Lemma 1 that T is nilpotent. Furthermore, we have $[u, v^{p^m}] = 0$ for all $u, v \in T$, hence T is a p-group of bounded exponent modulo its centre. By Schur's Theorem, T' is a p-group of bounded exponent. Suppose that we can prove our result for G/T'. If G/T' has a $*$-invariant normal p-subgroup N/T' of bounded exponent, and $*$ is trivial on $(G/T')/(N/T')$, then obviously $*$ is trivial on G/N, and since T' is a p-group of bounded exponent, so is N. As FG satisfies a polynomial identity, we have our p-abelian normal subgroup of finite index as well, hence our result holds for G.

If, on the other hand, we find that $F(G/T')$ is bounded Lie Engel, then we know that G/T' and T are both nilpotent. Hence, by Hall's criterion (see [15, 5.2.10]), G is nilpotent. Furthermore, since FG satisfies a polynomial identity, G has a p-abelian normal subgroup A of finite index. Also, since $F(G/T')$ is bounded Lie Engel, we see as before that $(G/T')'$ is a p-group of bounded exponent, and therefore, so is G'. By Lemma 18 of [13], $F(G/A')$ is bounded Lie Engel. Thus, G/A' has a p-abelian normal subgroup B/A' of p-power index, and therefore B is a p-abelian normal subgroup of p-power index in G. Thus, in this case, FG is bounded Lie Engel. Therefore, it costs us nothing to factor out T' and assume that T is abelian.

As we mentioned above, we are done if P_2 has bounded exponent, so let P_2 have unbounded exponent. We claim that there is a fixed n such that $a^{p^n} \in \zeta$ for all $a \in P_2$. Since $P_2^{p^n}$ also has unbounded exponent, this claim combined with Lemma 12 will complete the proof.

Since FG satisfies a polynomial identity, G has a $*$-invariant p-abelian normal subgroup A of finite index, say $|A'| = p^k$. Letting $\bar{G} = G/A'$, if we can show that $(\bar{a})^{p^m}$ is central in $\bar{G}$, then for any $g \in G$, $(a^{p^{m+k}}, g) = (a^{p^m}, g)^{p^k}$ (since P is abelian and normal), and as $(a^{p^m}, g) \in A'$, this is 1. Thus, we are free to factor out A' and assume that A is abelian.

Choose A in such a way that $(G : A)$ is minimal. Suppose that $(G : A)$ is even. Now, G' is a p-group and therefore, so is $(G/A)'$. Thus, in view of Lemma 5, either a dihedral group is involved in G/A (which is not allowed), or G/A has a central element gA of order 2. If $g^* \in g^{-1}A$, then let $h = g$. Otherwise, let

$h = gg^*$. Either way, hA is a central element of order 2 in G/A, and $\langle A, h\rangle$ is a $*$-invariant normal subgroup of G having A as a subgroup of index 2. Again, since dihedral groups are not permitted, Lemma 7 says that $\langle A, h\rangle$ is abelian. But this contradicts the minimality of $(G : A)$. Therefore, $(G : A)$ is odd.

As G/A is finite, choose r such that $P_2^{p^r} \subseteq A_2$. Let $(FG)^-$ be Lie p^m-Engel. We claim that $P_2^{p^{r+m}}$ is central. Fix any $c, d \in P_2$, and let $a = c^{p^r}$ and $b = d^{p^r}$. Take any $y \in G$. Then by assumption, $y^* \equiv y \pmod{P}$, say $y^* = yc$, with $c \in P$. We must show that $(b^{p^m}, y) = 1$. Let $x = yy^* = y^2c$. If $(b^{p^m}, x) = 1$, then since P is abelian, $(b^{p^m}, y^2) = 1$. But $y^k \in A$ for some odd k. Thus, $(b^{p^m}, y^k) = 1$, and therefore b^{p^m} commutes with y. Thus, we need only show that $(b^{p^m}, x) = 1$. Of course, x is symmetric.

As in the proof of [7, Key Lemma], we write

$$0 = [xa - a^{-1}x, b^{p^m} - b^{-p^m}] = [x, b^{p^m} - b^{-p^m}](1 - (a^{-1})^x a^{-1})a.$$

In particular,

$$[x, b^{p^m} - b^{-p^m}](1 - (a^{-1})^x a^{-1}) = 0.$$

If $(a^{-1})^x a^{-1} = 1$ for all such a, then $(a^{-1})^x = a$, hence $(x^2, a^{-1}) = 1$. But $x^k \in A$ for some odd k, hence $(x^k, a^{-1}) = 1$ as well, and therefore $(x, a) = 1$. That is, $P_2^{p^r}$ centralizes x, and so $(b, x) = 1$. Otherwise, Lemma 10 tells us that either $[x, b^{p^m} - b^{-p^m}] = 0$ or $p = 3$, $o((a^{-1})^x a^{-1}) = 3$ and $[x, b^{p^m} - b^{-p^m}]$ has precisely three elements in its support. In the first case, since $b^{2p^m} = 1$ implies $b^{p^m} = 1$, we can only have $xb^{p^m} = b^{p^m}x$, as required. In the second case, we must have either $xb^{p^m} = b^{-p^m}x$ or $xb^{-p^m} = b^{p^m}x$. Either way, $(x^2, b^{p^m}) = 1$. But as $x^k \in A$ for some odd k, we once again conclude that $(x, b^{p^m}) = 1$.

The sufficiency follows just as in [3]. □

4 The case where $*$ is not trivial on G/P

As before, we let F be a field of characteristic $p > 2$, G a group with involution $*$ having no 2-elements and no dihedral involved, P the set of p-elements of G and T the set of torsion elements of G. Assuming that $(FG)^-$ is bounded Lie Engel, we know that P and T are subgroups of G, G/P is abelian and G has a $*$-invariant p-abelian normal subgroup A of finite index. Also write Q for the set of p'-elements in T. In this section, we handle the case where $*$ is not trivial on G/P. Our goal is to prove that FG is bounded Lie Engel. This will certainly involve showing that G is nilpotent. Let us begin by assuming that fact and prove

Lemma 14. *Let G be a nilpotent group without 2-elements such that no dihedral is involved in G. Suppose that $(FG)^-$ is bounded Lie Engel. Then G' is a p-group of bounded exponent.*

Proof. Clearly, factoring out a $*$-invariant normal p-group of bounded exponent will not harm our conclusion, so we will do so freely. In particular, we know from

Lemma 4 that T' is a p-group of bounded exponent (as we have seen before, this follows from Schur's Theorem when FT is bounded Lie Engel), so we factor it out and assume that T is abelian. In particular, $T = P \times Q$. As G' is a p-group, we see that $(G/Q)' = G'Q/Q \simeq G'/(G' \cap Q) = G'$. Thus, we may factor out Q and assume that $T = P$. If $*$ is trivial on $G/P = G/T$, then we already know the answer. Therefore, by Lemma 8, we may assume that there exists a nontrivial $bP \in G/P$ (necessarily of infinite order) upon which $*$ acts classically. In view of Lemma 9, we may assume that $b^* = b^{-1}$. Replacing b with a suitable power, we may take $b \in A$.

Factoring out A', we assume that A is abelian. For the moment, suppose that G/A is an abelian p-group. In view of Schur's Theorem, it suffices to show that G/ζ is a p-group of bounded exponent. Since G/A is a finite p-group, it is sufficient to show that $A^{p^n} \leq \zeta$, for a fixed n.

Let $(FG)^-$ be Lie p^n-Engel. Take $x \in G$, $a \in A$. We claim that if $a^{p^{n+1}} \neq (a^*)^{p^{n+1}}$, then $(x, a^{p^n}) = 1$. Write $x = x_1x_2$ with $x_iA \in (G/A)_i$. It will suffice to show that a^{p^n} commutes with each x_i. That is, we may assume that $G = \langle A, x_i \rangle$. Let $x = x_1$. Now, if $(xc)^* = xc$ for all $c \in A$, then $c^*x = xc$ for all $c \in A$, so $xcx^{-1} = c^*$. Therefore, $c = (c^*)^* = x(xcx^{-1})x^{-1} = x^2cx^{-2}$. But then $\langle A, x^2 \rangle$ is an abelian subgroup of index at most 2 in G. Since no dihedrals are allowed, it now follows from Lemma 7 that G is abelian, and we are done. Therefore, some xc is not symmetric. Replacing x with xc, we have $x^* = xd$, for some $1 \neq d \in A$.

Now,

$$0 = [x - x^*, a^{p^n} - (a^*)^{p^n}] = [x, a^{p^n} - (a^*)^{p^n}](1 - d).$$

By Lemma 10, either $[x, a^{p^n} - (a^*)^{p^n}] = 0$ or $p = o(d) = 3$ and $(xa^{p^n})^{-1}(x(a^*)^{p^n}) \in \langle d \rangle$. In particular, the latter case gives us $(a^*a^{-1})^{p^n} \in \langle d \rangle$, and hence $(a^*)^{p^{n+1}} = a^{p^{n+1}}$, which is a contradiction. Thus,

$$[x, a^{p^n} - (a^*)^{p^n}] = 0.$$

In particular, either $xa^{p^n} = a^{p^n}x$ (as desired) or $a^{p^n} = (a^*)^{p^n}$ (which contradicts our assumption on a).

Now let $x = x_2$. Once again, we have

$$[x - x^*, a^{p^n} - (a^*)^{p^n}] = 0.$$

But xA and x^*A are different cosets of A. Hence,

$$[x, a^{p^n} - (a^*)^{p^n}] = 0.$$

Therefore, either x commutes with a^{p^n} (as desired), or $a^{p^n} = (a^*)^{p^n}$ (which is not allowed).

On the other hand, suppose that $a^{p^{n+1}} = (a^*)^{p^{n+1}}$. In this instance, by choice of b, we see that $b^{p^{n+1}} \neq (b^*)^{p^{n+1}}$ and $(ab)^{p^{n+1}} \neq ((ab)^*)^{p^{n+1}}$. Thus, x commutes with both b^{p^n} and $(ab)^{p^n}$, and hence with a^{p^n}. This case is complete.

Therefore, let us drop the assumption that G/A is an abelian p-group. Proceed by induction on $|G/A|$. As G/A is nilpotent, take a nontrivial yA in the centre of G/A. If y^* does not lie in $y^{-1}A$, then let $x = yy^*$. Otherwise, let $x = y$. Replacing x with a suitable power, we may assume that $o(Ax) = q$ is prime. Let $H = \langle A, x\rangle$. Suppose that $q \neq p$. We notice that $x^q \in A$, hence x^q is central in H. Letting $\bar{H} = H/\langle x^q\rangle$, we note that for any $h \in H$, (x, h) is a p-element (since G' is a p-group), but also, since $\bar{H}$ is nilpotent and $\bar{x}$ has order q, $(\bar{x}, \bar{h})$ is a q-element. That is, $(\bar{x}, \bar{h}) = 1$, which means that $(x, h) \in \langle x^q\rangle$. But again, (x, h) is a p-element. Since G contains only p-elements and elements of infinite order, and x^q is in A but x is not, we see that x^q has infinite order. Therefore, either $(x, h) = 1$ or (x, h) has infinite order. The latter gives a contradiction, so x centralizes A and H is abelian. Furthermore, H is $*$-invariant and normal in G, and as $|G/H| < |G/A|$, by our inductive hypothesis, we are done.

Now, let $q = p$. Then by our above considerations, H' is a p-group of bounded exponent. Again, since H is normal and $*$-invariant, we notice that G/H' has H/H' as an abelian normal $*$-invariant subgroup of index smaller than $|G/A|$, and by our inductive hypothesis, $(G/H')'$ has bounded exponent. Therefore, G' has bounded exponent. The proof is complete. □

Our remaining task is to show that G is nilpotent. Let us make some reductions. By Lemma 4, P is nilpotent. Thus, by Hall's criterion, it suffices to show that G/P' is nilpotent. We may therefore let P be abelian. As a first step, let us show that we may assume that $*$ is nontrivial on $A/A \cap P$. We will need to insist that $\zeta \leq A$, but as we can always replace A with $A\zeta$, this is not a problem.

Lemma 15. *Let G be a group without 2-elements or dihedral involvement, and let P be abelian. Suppose that $(FG)^-$ is bounded Lie Engel and A is a $*$-invariant p-abelian normal subgroup of finite index in G, containing ζ. If $*$ is not trivial on G/P, then $*$ is not trivial on $A/A \cap P$.*

Proof. Suppose that $*$ is trivial on $A/A \cap P$. Take any $x \in G$. We claim that $x^* \equiv x \pmod{P}$. As xA has finite order, write $x = yz$, where yA is a p-element and zA a p'-element. If we can show that $y^* \equiv y \pmod{P}$ and similarly for z, then since G is abelian modulo P, we obtain our conclusion. Therefore, we may assume that $o(xA) = q$, where q is either a power of p or relatively prime to p.

By assumption, since $x^q \in A$, we have $(x^q)^* \equiv x^q \pmod{P}$. Again, since G/P is abelian, this means $(x^*x^{-1})^q \in P$. If q is a p-power, then $x^*x^{-1} \in P$, so $x^*x^{-1} \equiv 1 \pmod{P}$. If q is relatively prime to p, then $x^*x^{-1} \equiv w \pmod{P}$, where $w \in Q$. As the former case is contained in the latter, we assume the latter. Thus, in any case, $x^* \equiv xw \pmod{P}$.

Now, $x = (x^*)^* \equiv w^*xw \equiv xww^* \pmod{P}$, as G/P is abelian. That is, $w^* \equiv w^{-1} \pmod{P}$.

There are two cases to consider. If T is nilpotent, then as G' is a p-group, Q is central. By our assumption, $Q \leq A$. Therefore, by hypothesis, $w^* \equiv w \pmod{P}$. Thus, $w^2 \in P \cap Q = 1$, and as G has no 2-elements, $w = 1$. If, on the other hand, T is not nilpotent, then by Lemma 4, $*$ is trivial on T/P, and we obtain the same conclusion. We are done. □

Let us make another useful reduction by showing that A is nilpotent. If so, then by another application of Hall's criterion, we can assume that A is abelian. To this end, we must adapt some arguments from [7] to our purposes and incorporate them into the next proof.

Lemma 16. *Let G be a p-abelian group without 2-elements or dihedral involvement, such that $*$ is not trivial on G/P. If $(FG)^-$ is bounded Lie Engel, then G is nilpotent.*

Proof. By Lemma 4, P is nilpotent. Therefore, in view of Hall's criterion, it is sufficient to consider G/P'. Thus, P is abelian. Since G' is finite, another theorem of Hall (see [10]) tells us that the second centre, H, is of finite index in G. Now, (H, G) is a central p-group, so we can factor it out without harming our hypotheses or conclusion. Thus, we assume that H is a central subgroup of finite index. In particular, $(G : \zeta) < \infty$. By Lemma 6, $(G : \zeta)$ is odd. Now, if G/ζ is nilpotent, then we are done, so suppose otherwise. Then by Lemma 4 and the Schur-Zassenhaus Theorem, $G/\zeta = (N/\zeta) \rtimes (X/\zeta)$, where N/ζ is the group of p-elements of G/ζ and X/ζ is a group of odd p'-order. Furthermore, $*$ is trivial on $(G/\zeta)/(N/\zeta) = G/N$.

Now, if $*$ is trivial on $\zeta/(\zeta \cap P)$, then the preceding lemma gives us a contradiction. Therefore, by Lemma 8, we may choose $a \in \zeta \backslash P$ such that $a^* \equiv a^{-1} \pmod{\zeta \cap P}$. By Lemma 9, we may as well assume that $a^* = a^{-1}$. Now, if $o(a) = \infty$, then by Lemma 12, we are done. Therefore, replacing a with a suitable power, we may assume that $o(a) = q$, where q is an odd prime different from p.

Take any $x \in X \backslash \zeta$. Then $x^* = xu$, for some $u \in N$. Letting $y = xx^*$, we have $y = x^2u$. Now, if $(FG)^-$ is Lie p^m-Engel, then $y^{p^m} = x^{2p^m}v$, for some $v \in N$. Let us say that $o(y^{p^m}\zeta) = p^l r$, where r is odd and relatively prime to p. Then letting $k = m + l$, we see that y^{p^k} is a p'-element modulo ζ. Furthermore, working in $\bar{G} = G/\zeta$, we see that $(\bar{y})^{p^k} = (\bar{x})^{2p^k}\bar{w}$, for some $w \in N$, and as $\bar{x}$ is a nontrivial p'-element, we find that $(\bar{y})^{p^k} \neq 1$; that is, $y^{p^k} \notin \zeta$.

Therefore, choose $h \in G$ with which y^{p^k} does not commute. First of all, suppose that h is not symmetric. As y is symmetric and a is central, we have

$$0 = [h - h^*, (ya - a^{-1}y)^{p^k}] = [h - h^*, y^{p^k}](a^{p^k} - a^{-p^k}).$$

In particular,

$$[h - h^*, y^{p^k}](a^{2p^k} - 1) = 0.$$

Now, $o(a^{2p^k}) = o(a) = q$. Taking into account Lemma 10 and the fact that $q \neq p$, we can only have $[h - h^*, y^{p^k}] = 0$. As $h \neq h^*$, we obtain $[h, y^{p^k}] = 0$, contradicting our choice of h.

Finally, let h be symmetric. Then

$$0 = [ha - a^{-1}h, (ya - a^{-1}y)^{p^k}] = [ha - a^{-1}h, y^{p^k}](a^{p^k} - a^{-p^k}).$$

As above, this implies that

$$0 = [ha - a^{-1}h, y^{p^k}] = [h, y^{p^k}](a - a^{-1}),$$

and applying the same argument again, we get $[h, y^{p^k}] = 0$. Thus, we conclude that no such x exists; that is, X/ζ is trivial. In particular, G/ζ is a finite p-group, and hence nilpotent. Thus, G is nilpotent. □

We need to handle one more special case before proving the main theorem.

Lemma 17. *Suppose that $G = \langle A, x\rangle$ where A is an abelian $*$-invariant normal subgroup, and $o(xA) = q$ is prime. Further suppose that P is abelian and $*$ is not trivial upon G/P. If G has no 2-elements, no dihedral group is involved in G, and $(FG)^-$ is Lie p^m-Engel, then $A^{p^m} \leq \zeta$.*

Proof. If $x^* \not\equiv x^{-1} \pmod{A}$, then replacing x with xx^*, we may assume that $x^* = x$. If $q = 2$, then by Lemma 7, G is abelian, as no dihedrals are involved. Thus, let q be odd. Also, by Lemma 15, $*$ does not act trivially upon $A/(A \cap P)$. Lemma 8 therefore allows us to choose $a \in A \backslash P$ such that $(aP)^* = a^{-1}P$. By Lemma 9, we may assume that $a^* = a^{-1}$.

We need to show that $(b^{p^m}, x) = 1$ for all $b \in A$. But we have

$$0 = [x - x^*, b^{p^m} - (b^*)^{p^m}].$$

If $x^* \equiv x^{-1} \pmod{A}$, then x and x^* belong to different cosets of A, hence

$$[x, b^{p^m} - (b^*)^{p^m}] = 0.$$

That is, b^{p^m} commutes with x unless b^{p^m} is symmetric. But if b^{p^m} is symmetric, then noting that neither a^{p^m} nor $(ab)^{p^m}$ is symmetric, we see that x commutes with a^{p^m} and $(ab)^{p^m}$, hence with b^{p^m}, as required.

Therefore, we may assume that $x^* = x$. Suppose that $(xc)^* = xc$ for all $c \in A$. Then $c^*x = xc$, hence $c^* = xcx^{-1}$ for all $c \in A$. Therefore, $c = (c^*)^* = xc^*x^{-1} = x^2cx^{-2}$. Thus, x^2 commutes with c, and since x^q does as well, we conclude that $(x, c) = 1$ for all $c \in A$. That is, G is abelian, and we are done. Thus, choose $c \in A$ such that $(xc)^* \neq xc$. Replacing x with xc, we have $x^* = xd$, with $1 \neq d \in A$.

We claim that if $b^{p^{m+1}} \neq (b^*)^{p^{m+1}}$, then x commutes with b^{p^m}. Indeed, we have

$$0 = [x - xd, b^{p^m} - (b^*)^{p^m}] = [x, b^{p^m} - (b^*)^{p^m}](1 - d).$$

In view of Lemma 10, we have either $[x, b^{p^m} - (b^*)^{p^m}] = 0$ or $p = o(d) = 3$ and $(xb^{p^m})^{-1}(x(b^*)^{p^m}) \in \langle d\rangle$, and hence $(b^*b^{-1})^{p^{m+1}} = 1$. But this contradicts the choice of b. Therefore,

$$[x, b^{p^m} - (b^*)^{p^m}] = 0$$

and proceed precisely as before.

If, on the other hand, $b^{p^{m+1}}$ is symmetric, then noting that $a^{p^{m+1}}$ and $(ab)^{p^{m+1}}$ are not symmetric, we find that x commutes with a^{p^m} and $(ab)^{p^m}$, hence with b^{p^m}. The proof is complete. □

We are now in a position to prove our main result.

Proof of Theorem 1. Suppose that $(FG)^-$ is bounded Lie Engel. If $p = 0$, then Lemma 3 does the job, so let p be an odd prime. By Lemma 3, G' is a p-group. If $*$ is trivial on G/P, then by Lemma 13, we are finished. Therefore, assume that $*$ is not trivial on G/P. Since FG satisfies a polynomial identity, let A be a normal $*$-invariant p-abelian subgroup of finite index.

We claim that G is nilpotent. By Lemma 4, P is nilpotent. Hence, by Hall's criterion, it suffices to show that G/P' is nilpotent. We therefore assume that P is abelian. Replacing A with $A\zeta$ if necessary, assume that $\zeta \leq A$. By Lemma 15, $*$ is not trivial on $A/(A \cap P)$. Thus, combining Lemmas 8 and 9, we know that there exists $a \in A \backslash P$ such that $a^* = a^{-1}$. If $o(a) < \infty$, then replacing a with a suitable power, we may assume that $a \in Q$.

By Lemma 16, A is nilpotent. Again applying Hall's criterion, we see that it suffices to assume that A is abelian. We prove our claim by induction on $(G : A)$. Of course, if this index is 1 there is nothing to do. Otherwise, we note that since $(G/A)'$ is a finite p-group, G/A is solvable. Therefore, let H/A be the last nontrivial term in the derived series of G/A. If we can show that H is nilpotent, then it suffices to show that G/H' is nilpotent. But G/H' has H/H' as an abelian normal $*$-invariant subgroup of index $(G : H) < (G : A)$. Thus, in this case, we are done. Therefore, it suffices to assume that G/A is abelian.

Furthermore, take $xA \in G/A$ of prime order. If $x^* \notin x^{-1}A$, then replace x with xx^*. Let $K = \langle A, x \rangle$. Then K is a $*$-invariant normal subgroup of G. If K is nilpotent, then it suffices to show that G/K' is nilpotent, and once again, by induction, we are done. Thus, we may assume that $G = \langle A, x \rangle$, where $o(xA) = q$ is prime. If $q = 2$, then by Lemma 7, G is abelian. Therefore, q is an odd prime. By Lemma 17, if $(FG)^-$ is Lie p^m-Engel, then $A^{p^m} \leq \zeta$. Now, if $o(a) = \infty$, then by Lemma 12, we are done. Therefore, we may assume that $a \in Q$.

We have two cases to consider. If $q = p$, then we have $g^{p^{m+1}} \in \zeta$ for all $g \in G$. That is, G/ζ is a p-group of bounded exponent, and $F(G/\zeta)$ satisfies a polynomial identity. Therefore, by [12, Lemma 3.2.7], G/ζ is nilpotent. Hence, G is nilpotent.

Now suppose that $q \neq p$. We claim that, in fact, G is abelian. To see this, we proceed in a similar manner to [5, Proposition 3.4]. If G is not abelian, then choose noncommuting $g, h \in G$. We may assume that $G = \langle g, h, g^*, h^*, a \rangle$. Thus, G is finitely generated, and as A has finite index, A is finitely generated abelian. Of course, g and h cannot both lie in A. Without loss of generality, say $g \notin A$. As $|G/A|$ is prime, it suffices to show that $(g, A) = 1$. Take any $v \in A$. Let us write $A = C \times D$, where C is a direct product of finitely many infinite cyclic groups, and D is finite, say $|D| = p^r s$, where s is an odd number relatively prime to p. Let $B = A^{p^r s} = C^{p^r s}$. Now, B is $*$-invariant and normal in G, and G/B is a torsion group with no 2-elements. Also, since we are only factoring out elements of infinite order, aB is still a nontrivial p'-element satisfying $(aB)^* = (aB)^{-1}$. Thus, by Lemma 4, $F(G/B)$ is bounded Lie Engel. In particular, G/B is nilpotent, and therefore p'-elements are central. Now, $g^{qp^r s} \in B$. Thus, the order of $g^{p^r}B$ divides qs, so it is a p'-element, and therefore central. That is, $(g^{p^r}, v) \in B$. But also $g^q \in A$, hence $(x^q, v) = 1$. As q and p^r are relatively prime, $(g, v) \in B$. But B is torsion-free and G' is torsion.

Therefore, $(g, v) = 1$, completing the proof of the claim.

We now know that G is nilpotent. By Lemma 14, G' is a p-group of bounded exponent. But then by [13, Lemma 18], $F(G/A')$ is bounded Lie Engel. Thus, by Lemma 1, G/A' has a p-abelian normal subgroup L/A' of p-power index. We now see that L is a p-abelian normal subgroup of p-power index in G. Thus, FG is bounded Lie Engel.

The sufficiency follows just as in [3]. $\square$

5 Group rings with the classical involution

Supose that FG has the classical involution induced from $g^* = g^{-1}$ for all $g \in G$. In this section, we classify the groups such that $(FG)^-$ is bounded Lie Engel, without any restrictions upon G. This can be done by making simple modifications to the proofs in Giambruno and Sehgal [9], where the corresponding results for Lie nilpotence were proved. We will omit the details contained in [9] that apply directly to our situation. Throughout this section, $*$ is classical. The result is

Theorem 2. *Let F be a field of characteristic $p \neq 2$ and G a group. With respect to the classical involution, the set of skew elements of FG is bounded Lie Engel if and only if one of the following occurs:*

1. *G has an elementary abelian 2-subgroup of index 2,*
2. *G has a normal subgroup H such that FH is bounded Lie Engel and $o(g) = 2$ for all $g \in G \backslash H$, or*
3. *$p > 2$, G has a p-abelian subgroup of finite index, the p-elements of G form a normal subgroup P of bounded exponent, and G/P is an elementary abelian 2-group.*

We need to consider some special cases.

Lemma 18. *Suppose that $p > 2$ and the p-elements of G form an abelian subgroup P of unbounded exponent. Furthermore, let $(G : P) = 2$. If P intersects the centre of G nontrivially and $(FG)^-$ is bounded Lie Engel, then G is abelian.*

Proof. Take any $x \in G$ of order 2. As in Lemma 8, let us write $P = P_1 \times P_2$, where we consider the involution to be the conjugation action of x on P. Let $1 \neq a \in P_1$ be central in G and let $(FG)^-$ be Lie p^n-Engel. Then for any $b \in P_2$, we have

$$[ax - xa^{-1}, b^{p^n} - b^{-p^n}] = 0.$$

If $axb^{p^n} = axb^{-p^n}$, then $b^{2p^n} = 1$, hence $b^{p^n} = 1$. If $axb^{p^n} = xa^{-1}b^{p^n}$, then $a^2 = 1$, which is impossible. If $axb^{p^n} = b^{p^n}ax$, then $(b^{p^n})^x = b^{p^n}$. But $b^x = b^{-1}$, so we can only conclude that $b^{p^n} = 1$. If $axb^{p^n} = b^{-p^n}xa^{-1}$, then $(b^{p^n}a)^x = (b^{p^n}a)^{-1}$. However, $(b^{p^n})^x = b^{-p^n}$ and $a^x = a \neq a^{-1}$, so this is impossible.

The only other possibility is that $p = 3$ and axb^{p^n} agrees with two added terms in our equation. But if $axb^{p^n} = xa^{-1}b^{-p^n}$, then $a = a^x = a^{-1}b^{-2p^n}$, and hence $a^2 = b^{-2p^n} \in P_1 \cap P_2 = 1$. Once again, $b^{p^n} = 1$. If $axb^{p^n} = b^{p^n}xa^{-1}$, then $b^{-p^n} = (b^{p^n})^x = a^{-2}b^{p^n}$, and we reach the same conclusion.

Therefore, we know that $P_2^{p^n} = 1$. Since P has unbounded exponent, this means that P_1 (which is central) has unbounded exponent. By Lemma 12, FG is bounded Lie Engel, and therefore G is nilpotent, hence abelian. □

This can easily be extended to

Lemma 19. *Suppose that $p > 2$, P is a subgroup of unbounded exponent and $(G : P) = 2$. If G/P' has a nontrivial centre and $(FG)^-$ is bounded Lie Engel, then G is nilpotent.*

Proof. By the previous lemma, G/P' is abelian. But Lemma 4 tells us that P is nilpotent. We are done. □

Lemma 20. *Suppose that $p > 2$, P is a subgroup of unbounded exponent and $(G : P) = 2$. Let $(FG)^-$ be bounded Lie Engel. If G/P' has a trivial centre, then $G = P \rtimes \langle x \rangle$, where P is abelian, $o(x) = 2$ and x acts dihedrally on P.*

Proof. Take $x \in G$ of order 2, and let $\bar{G} = G/P'$. Writing $\bar{P} = \bar{P}_1 \times \bar{P}_2$, where we regard the action of $\bar{x}$ on $\bar{P}$ as the involution, we see that by assumption, $\bar{P}_1 = 1$. That is, $\bar{x}$ acts dihedrally upon $\bar{P}$. We are done if we can show that P is abelian.

Supposing that $P' \neq 1$, we observe that just as in the proof of [9, Lemma 13], we may assume that P' is central in G. (Note that (P', P) gets factored out here. This is fine, because we know from our work with p-groups that P is nilpotent, and P' has bounded exponent.) Also, since $b^x \equiv b^{-1} \pmod{P'}$, for all $b \in P$, we see that x normalizes $\zeta(P)$. Therefore, $\zeta(P)$ is a normal subgroup of G.

Let $H = \langle \zeta(P), x \rangle$. Then $\zeta(P)$ contains a nontrivial central subgroup, P'. By Lemma 18, H is abelian. Thus, x centralizes $\zeta(P)$. But x inverts P modulo P'. We conclude that $\zeta(P) = P'$. However, we know from [11, Theorem 3] that FP is bounded Lie Engel, and therefore, as we have seen before, $P/\zeta(P)$ has bounded exponent, and so does P'. But then P has bounded exponent. This contradicts our assumption and completes the proof. □

Finally, we have the

Proof of Theorem 2. Let us verify the necessity. As in [9], the proof breaks down into cases.

Case I: G has an element of infinite order. Follow the proof of [9, Theorem 1] verbatim, replacing the reference to [8, Corollary] with an appeal to Lemma 12. We conclude that part 2 of the theorem must hold.

Case II: G is torsion and FG is semiprime. By [3, Lemma 4], in this case, if $(FG)^-$ is bounded Lie Engel, then it is commutative. In particular, $(FG)^-$ is Lie nilpotent, and we can apply Theorem 3 of Giambruno and Polcino Milies [4] to conclude that either part 1 or part 2 of our theorem holds. (In particular, the characteristic zero case is done, so assume now that $p > 2$.)

Case III: G is torsion, FG is not semiprime, and G has an element of order 4 or an odd prime different from p. Here, the proofs of [9, §2] carry over to the bounded Lie Engel property without change. The conclusion is that either we are in part 2 of our theorem, or $G = P \times Q$, where P is a p-group and Q is abelian. In the latter case, from [11, Theorem 3] we know that FP is bounded Lie Engel. In view of Lemma 1, so is FG, and again, part 2 of our theorem applies.

Case IV: Every element of G has order either a power of p or twice a power of p. By [3, Lemma 6], the p-elements of G form a subgroup P. (The assumption that G has no 2-elements was not needed for that part of the proof.) Thus, G/P is an elementary abelian 2-group. If P has bounded exponent, then we are in part 3 of our theorem, since FG satisfies a polynomial identity. Therefore, assume that P has unbounded exponent.

In view of the preceding lemmas, we apply the proof of [9, Proposition 14] to show that in this case, either $G = P \times K$, where K is abelian, or $G = L \times (P \rtimes \langle x \rangle)$, where L is an elementary abelian 2-group, P is an abelian p-group, $o(x) = 2$ and x acts dihedrally upon P. In the first case, since we already know that FP is bounded Lie Engel, so is FG, and we are in part 2 of our theorem. In the second case, letting $H = L \times P$, we observe that we are again in part 2.

Let us prove the sufficiency. The first part is the same as in [9], so there is nothing to do. For the second part, we note that $(FG)^- = (FH)^-$ and again, we are finished. For the third part, we observe that $(FG)^- \subseteq \Delta(G, P)$, where $\Delta(G, P)$ is the kernel of the natural homomorphism $FG \to F(G/P)$. But in this case, by [12, Lemma 1.3.14], $\Delta(G, P)$ is a nil ideal of bounded exponent, say p^k. Thus, if $\alpha, \beta \in (FG)^-$, then

$$[\alpha, \underbrace{\beta, \ldots, \beta}_{p^k \text{ times}}] = [\alpha, \beta^{p^k}] = 0.$$

The proof is complete. □

Acknowledgement

The authors thank Professor Sudarshan Sehgal for proposing the problem, and for his helpful suggestions.

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