Ilirias Journal of Mathematics
ISSN: 2334-6574, URL: http://www.ilirias.com
Volume 4 Issue 1(2015), Pages 16-23.

LACUNARY GENERATING FUNCTIONS OF HERMITE POLYNOMIALS AND SYMBOLIC METHODS

G. DATTOLI, B. GERMANO, M.R. MARTINELLI, P.E. RICCI

Abstract. We employ an umbral formalism to reformulate the theory of Hermite polynomials and the derivation of the associated lacunary generating functions.

1. Introduction

In a paper of few years ago Gessel and Jayawant [6] have discussed a triple lacunary generating function for Hermite polynomials. The Authors employ two different methods, one of umbral nature, the other based on combinatorial arguments.

In this paper we comment on the umbral technique proposed in [6], discuss its link with previously developed formalisms and suggest extensions, allowing the umbral treatment of families of Hermite like polynomials.

The 2-variable polynomials, defined by the series [1]

$$H_n(x,\, y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2\,r} y^r}{(n-2\,r)!\, r!} \tag{1.1a}$$

belong to the Hermite-like polynomials. This family of polynomials has many generalization and sometimes there is some confusion in the literature, regarding the relevant notation. For reasons which will be clarified in the following, they should be denoted by $H_n^{(2)}(x,\, y)$ and should be referred to as "second order two variable Hermite polynomials", we will however keep the upper index only for polynomials with order $\geqslant 3$ or add it whenever strictly necessary to avoid confusion.

The function generating (1.1a) reads

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,\, y) = e^{x\,t + y\,t^2}\,. \tag{1.1b}$$

A remarkable property is the operational definition [3]

$$H_n(x,\, y) = e^{y\,\partial_x^2} x^n \tag{1.2}$$

2000 *Mathematics Subject Classification.* 33C45, 33F10.

Key words and phrases. Hermite polynomials, Lacunary generating functions, Symbolic and umbral methods.

Submitted May 29, 2015. Published Jul 21, 2015.

due to the fact that they are a solution of the partial differential equation

$$\partial_y F(x,y) = \partial_x^2 F(x,y), \tag{1.3}$$

with the "initial condition"

$$F(x,\, 0) = x^n\,. \tag{1.4}$$

For this reason they are also defined "heat polynomials" [10].

From equation (1.1a) we derive the boundary condition at $x = 0$

$$H_n(0,\, y) = n! \frac{y^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \left|\cos\left(n\,\frac{\pi}{2}\right)\right|\,. \tag{1.5}$$

The use of equation (1.2) is extremely useful, for example we can derive straightforwardly the double lacunary Hermite generating function, which can be formally written as

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2\,n}(x,\, y) = e^{y\,\partial_x^2} e^{x^2 t}\,. \tag{1.6}$$

A definite meaning to the rhs of equation (1.6) is obtained through the application of the Weierstrass Gauss transform [8]

$$e^{y\,\partial_x^2} f(x) = \frac{1}{2\sqrt{\pi\, y}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4\,y}} f(\xi)\, d\xi \tag{1.7}$$

which allows the derivation of the generating function (1.6) according to the following expression

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2\,n}(x,\, y) = \frac{1}{\sqrt{1-4\,y\,t}} e^{\frac{x^2\,t}{1-4\,y\,t}}\,, \\ &|t| < \tfrac{1}{4\,|y|} \end{aligned} \tag{1.8}$$

which is sometimes called Doetsch rule [5].

The procedure we have adopted to derive equation (1.8) can be generalized to get the following generalization of the Doetsch rule

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2\,n+l}(x,\, y) &= e^{y \partial_x^2} \left(e^{t\,x^2} x^l\right) \\ &= \frac{1}{\sqrt{1-4\,y\,t}} e^{\frac{x^2\,t}{1-4\,y\,t}} \frac{H_l\left(\frac{x}{\sqrt{1-4\,y\,t}},\, y\right)}{(1-4\,y\,t)^{\frac{l}{2}}}\,, \\ |t| &< \frac{1}{4\,|y|}\,. \end{aligned} \tag{1.9}$$

It is evident that the operational method exploited so far can be extended to higher order lacunary generating functions and we will show, in the forthcoming sections, that it may become a fairly powerful tool once complemented with a notation of umbral nature.

2. Umbra and Hermite polynomials

In a previous investigation we have shown that symbolic methods seem to be tailor suited to deal with the properties of Bessel functions and Laguerre polynomials as well [2]. The methods we have developed largely rely on techniques of umbral type and therefore in order of treating Hermite polynomials we introduce the following “umbra”

$$\hat{h}_y^r \varphi_0 = \frac{y^{\frac{r}{2}}\, r!}{\Gamma\left(\frac{r}{2}+1\right)} \left|\cos\left(r\frac{\pi}{2}\right)\right| . \tag{2.1}$$

Which reduces, for $y = \frac{1}{2}$, to an analogous quantity defined in [6]. By such a notation we can redefine the Hermite polynomials as

$$H_n(x,\, y) = \left(x + \hat{h}_y\right)^n \varphi_0 \,. \tag{2.2}$$

According to equation (2.2) the Hermite polynomials are reduced to the n-th power of a binomial. All the relevant properties can be obtained by handling equation (2.2) by means of elementary algebraic tools.

The exponentiation of the umbra $\hat{h}_y$ will be particularly important in the present context, we note therefore that

$$e^{\hat{h}_y\, z} \varphi_0 = \sum_{r=0}^{\infty} \frac{\left(\hat{h}_y\, z\right)^r}{r!} \varphi_0 = e^{y\, z^2} \tag{2.3a}$$

and

$$\begin{aligned} e^{\hat{h}_y^2\, z} \varphi_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2\,\xi \hat{h}_y\, \sqrt{z}} \varphi_0 d\xi = \frac{1}{\sqrt{1-4\,y\,z}}, \\ |z| < \frac{1}{4\,|y|} \,. \end{aligned} \tag{2.3b}$$

Which is just a consequence of equation (2.3a), if we note that

$$\begin{aligned} e^{\hat{h}_y^2 z} \varphi_0 &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2\,\xi\, \hat{h}\,_y \sqrt{z}}\, d\xi\, \varphi_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2\,\xi\, \hat{h}_y\, \sqrt{z}} \varphi_0\, d\xi \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 (1-4\,y\,z)}\, d\xi. \end{aligned} \tag{2.4}$$

The Doetsch formula and its extension can be therefore derived by the use of the identity

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2n}(x,\, y) = e^{(x+\hat{h}_y)^2 t} \varphi_0 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2 + 2\,(x+\hat{h}_y)\,\sqrt{t}\xi} \varphi_0\, d\xi \,. \tag{2.5}$$

Let us now consider the same problem from a different point of view and write

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2n}(x,\,y) &= e^{(x+\hat{h}_y)^2 t}\varphi_0 = e^{x^2 t}e^{\hat{h}_y^2\,t+2\,\hat{h}_y\;x\,t}\,\varphi_0 \\ &= e^{x^2 t}\sum_{r=0}^{\infty}\frac{\hat{h}_y^r}{r!}H_r(2\,x\,t,\,t)\,\varphi_0 = e^{x^2 t}\sum_{r=0}^{\infty}\frac{y^r}{r!}H_{2r}(2\,x\,t,\,t)\end{aligned} \tag{2.6}$$

which has been derived by using equation (1.1b), by comparison with the Doetsch rule equation (2.6) yields the operational identity

$$e^{\hat{h}_y^2\,t+2\,\hat{h}_y\;x\,t}\,\varphi_0 = \frac{1}{\sqrt{1-4\,y\,t}}e^{\frac{4\,y\,(t\,x)^2}{1-4\,y\,t}}\,. \tag{2.7}$$

Before going further let us note that the third order Hermite polynomials [3]

$$H_n^{(3)}(x,\,y,z) = n!\sum_{r=0}^{\left[\frac{n}{3}\right]}\frac{z^r H_{n-3\,r}(x,y)}{r!\,(n-3\,r)!} \tag{2.8}$$

can be defined through the generating function

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^{(3)}(x,\,y,\,z) = e^{x\,t+y\,t^2+z\,t^3}\,. \tag{2.9}$$

On account of equation (2.2) we can write the triple lacunary Hermite generating function as

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{3\,n}(x,\,y) = e^{t\,(x+\hat{h}_y)^3}\varphi_0\,. \tag{2.10}$$

Which, according to equation (2.9), allows the following conclusion

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{3\,n}(x,\,y) &= e^{t\,x^3}\sum_{r=0}^{\infty}\frac{\hat{h}_y^r}{r!}H_r^{(3)}(3\,x^2 t,\,3\,x\,t^2,t)\,\varphi_0 \\ &= e^{t\,x^3}\sum_{r=0}^{\infty}\frac{y^r}{r!}H_{2\,r}^{(3)}(3\,x^2 t,\,3\,x\,t^2,t)\,.\end{aligned} \tag{2.11}$$

Which can be worded as it follows: the triple lacunary generating function of second order Hermite polynomials, can be expressed in terms of double lacunary generating function of third order Hermite polynomials.

3. The associated Hermite polynomials

In analogy with the case of Laguerre polynomials [2] we introduce the associated Hermite polynomials, which, according to the present formalism, read

$$\begin{aligned}H_n(x,\,y|p) &= \hat{h}_y^p\left(x+\hat{h}_y\right)^n\,\varphi_0 \\ &= n!\sum_{r=0}^{n}\frac{(r+p)!y^{\frac{r+p}{2}}x^{n-r}}{\Gamma\left(\frac{r+p}{2}+1\right)\,(n-r)!}\left|\cos\left((r+p)\frac{\pi}{2}\right)\right|\,.\end{aligned} \tag{3.1}$$

They cannot be identified with the generalized heat polynomials [8], and, within the present context, deserve a separate treatment.

According to the previous definition we can "state" the following index-duplication formula

$$\begin{aligned} H_{2\,n}(x,\,y) &= (\hat{h}_y + x)^n (\hat{h}_y + x)^n \varphi_0 \\ &= n! \sum_{s=0}^{n} \frac{x^{n-s}}{(n-s)!\,s!} H_n(x,\,y|s) \end{aligned} \tag{3.2}$$

and argument-duplication formula

$$\begin{aligned} H_n(2\,x,\,y) &= \left[\left(\frac{\hat{h}_y}{2} + x \right) + \left(\frac{\hat{h}_y}{2} + x \right) \right]^n \varphi_0 \\ &= \sum_{s=0}^{n} \binom{n}{s} \sum_{r=0}^{s} \binom{s}{r} \frac{x^r}{2^{s-r}} H_{n-s} \left(x,\, \frac{y}{2} | s-r \right) . \end{aligned} \tag{3.3}$$

It is furthermore easily checked that

$$x^n = \left[\left(x + \hat{h}_y \right) - \hat{h}_y \right]^n \varphi_0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} H_{n-r}(x, y|r) \tag{3.4}$$

and that

$$\begin{aligned} H_{n+m}(\,x,\,y) &= (\hat{h}_y + x)^m \,(\hat{h}_y + x)^n \varphi_0 \\ &= \sum_{r=0}^{m} \binom{m}{r} x^{m-r} H_n(x,\,y|r)\,. \end{aligned} \tag{3.5}$$

The last identity is a reformulation of the Nilsen theorem, concerning the sum of the indices of Hermite polynomials [1]. The previous results (3.2)-(3.5) occur in the literature in different forms [4] and in the concluding section we will comment on the differences between the ordinary formulation and the ones presented in this paper.

Even though not explicitly mentioned the Hermite umbra can be raised to any real power, and this allows noticeable freedom in guessing possible generalizations. A fairly direct example is provided by the following extension

$$H_n(x,\,y\,|\beta;\alpha) = \hat{h}_y^\beta \left(x + \hat{h}_y^\alpha \right)^n \varphi_0\,. \tag{3.6}$$

Yielding a family of polynomials with generating function

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,\,y\,|\beta;\alpha) = e^{x\,t} y^{\frac{\beta}{2}} e_{\alpha,\,\beta}(y^{\frac{\alpha}{2}}\,t), \\ &e_{\alpha,\,\beta}(x) = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha\,r + \beta + 1)\,x^r}{\Gamma\left(\frac{\alpha\,r+\beta}{2} + 1\right)\,r!} \left| \cos\left(\frac{\alpha\,r + \beta}{2}\,\pi \right) \right| . \end{aligned} \tag{3.7}$$

Their properties can easily be studied and they are framed within the context of the Sheffer family, they can accordingly be defined through the operational rule

$$H_n(x,\,y\,|\beta;\alpha) = \, y^{\frac{\beta}{2}} e_{\alpha,\,\beta}(y^{\frac{\alpha}{2}}\,\partial_x)\,x^n \tag{3.8}$$

In the following part of the paper we will discuss further elements characterizing the usefulness of the umbral point of view to the theory of Hermite polynomials.

4. Umbra, Higher order Hermite polynomials and final comments

The Hermite polynomials, defined through the operational identity [3]

$$H_n^{(m)}(x,\, z) = e^{z\, \partial_x^m}\, x^n \tag{4.1}$$

are specified by the series

$$H_n^{(m)}(x,\, z) = n! \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{x^{n-m\,r}\, z^r}{(n-m\,r)!\, r!}\,. \tag{4.2}$$

They can reduced to the n-th power of a binomial by introducing the umbra

$$\begin{aligned} {}_m\hat{h}_y^r \varphi_0 &= \frac{y^{\frac{r}{m}}\, r!}{\Gamma\left(\frac{r}{m}+1\right)} A_{m,r} \\ A_{m,r} &= \begin{pmatrix} 1, & r = m\,p \\ 0, & \text{otherwhise} \end{pmatrix}, \quad p \equiv \text{integer}\,. \end{aligned} \tag{4.3}$$

Which allows to define them as

$$H_n^{(m)}(x,\, z) = ({}_m\hat{h}_z + x)^n \varphi_0\,. \tag{4.4}$$

It is clearly evident that not too much effort is necessary to study the relevant properties, which can be derived using the same procedure adopted for the second order case.

We can combine the Hermitian umbra to get further generalizations, as for the three variable third order Hermite polynomials, which, according to the previous formalism, can be defined as

$$H_n^{(3)}(x,\, y, z) = ({}_3\hat{h}_z + {}_2\hat{h}_y + x)^n \varphi_{0,z}\varphi_{0,y}\,. \tag{4.5}$$

Thereby we find

$$\begin{aligned} H_n^{(3)}(x,\, y, z) &= \sum_{s=0}^{n} \binom{n}{s}\, {}_{(3,2)}\hat{h}_{z,\,y}^s x^{n-s} \varphi_{0,z}\varphi_{0,y}, \\ {}_{(3,2)}\hat{h}_{z,\,y}^s &= \sum_{r=0}^{s} \binom{s}{r}\, {}_3\hat{h}_z^{s-r}\, {}_2\hat{h}_y^r\,. \end{aligned} \tag{4.6}$$

The extension of the method to bilateral generating functions is quite straightforward too. We consider indeed the generating function

$$\begin{aligned} G(x,\, y;\, z,\, w|\, t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,\, y)\, H_n(z,\, w) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (\hat{h}_y\ + x)^n H_n(z,\, w) \varphi_{y,0} \\ &= e^{(\hat{h}_y + x)\, z\, t + \left[(\hat{h}_y + x)\, t\right]^2 w} \varphi_{y,0} \end{aligned} \tag{4.7}$$

the use of our technique yields

$$G(x,\, y;\, z,\, w|\, t) = \frac{1}{\sqrt{1-4yt^2 w}} e^{\frac{(x^2 w + y\, z^2)\, t^2 + x\, t\, z}{1-4\, y\, t^2 w}}\,. \tag{4.8}$$

Exotic generating functions involving *e.g.* products of Laguerre and Hermite polynomials can also be obtained and will be discussed elsewhere.

The use of umbral methods looks much promising to develop a new point of view on the theory of special polynomials and of special functions as well.

Just to provide a flavor of the directions along which may develop future speculations, we consider the definition of the following umbra

$$\hat{H}_{x,y}^{n}\varphi_0 = H_n(x,\, y)\,. \tag{4.9}$$

Accordingly we can write

$$e^{t\,\hat{H}_{x,y}}\varphi_0 = e^{x\,t+y\,t^2} \tag{4.10}$$

and also

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2\,n}(x,\,y) &= e^{t\,\hat{H}_{x,y}^2}\varphi_0\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\xi^2+2\,\xi\,\hat{H}_{x,y}\sqrt{t}}d\xi\,\varphi_0 \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1-4\,y\ t)\,\xi^2+2\,\xi\,x\sqrt{t}}d\xi\ = \frac{1}{\sqrt{1-4\,y\,t}}e^{\frac{x^2\,t}{1-4\,y\,t}}\,.\end{aligned} \tag{4.11}$$

Let us now consider the following integral

$$I\,(x,y) = \int_{-\infty}^{+\infty} e^{-z^2\,\hat{H}_{x,-\,y}}\,dz\,\varphi_0 \tag{4.12a}$$

equivalent to

$$I(x,y) = \int_{-\infty}^{+\infty} e^{-y\,z^4-x\,z^2}\,d\,z\ = \int_{0}^{+\infty} e^{-s\,x\,-s^2 y}s^{-\,\frac{1}{2}}\,ds\,. \tag{4.12b}$$

The same result will be now derived using a symbolic procedure, based on the application of the Ramanujan master theorem [7], which allows to write the formal solution of the integral in equation (4.12a) according to the identity

$$I\,(x,y) = \sqrt{\pi}\,\hat{H}_{x,-y}^{-\frac{1}{2}}\varphi_0\,. \tag{4.13}$$

We are therefore faced with the necessity of specifying the meaning of the Hermite umbra raised to a negative fractional power.

The use of standard Laplace transform methods yields

$$\hat{H}_{x,\,-y}^{-\frac{1}{2}}\varphi_0 = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{0}^{+\infty} e^{-s\hat{H}_{x,\,-\,y}}s^{-\,\frac{1}{2}}ds\varphi_0 = \frac{1}{\sqrt{\pi}}\int_{0}^{+\infty} e^{-s\,x\,-s^2 y}s^{-\,\frac{1}{2}}ds \tag{4.14}$$

which correctly reproduces equation (4.12b). This is quite a significant result, which ensures that the formalism has a very high level of flexibility.

It is also interesting to note that the "Fourier" transform

$$\hat{F}(f;\,k,\,\beta)\,\varphi_0 = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{+\infty} f(x)\,e^{i\,x\hat{H}_{k,\,\beta}}dx\,\varphi_0 \tag{4.15}$$

can be viewed as a kind of Gabor transform.

The technique we have discussed in the paper is tightly bound to the method of quasi-monomials developed in [9], provided that we make the following identification

$$\left(x + \hat{h}_y\right) \to (x + 2\, y\, \partial_x)\,. \tag{4.16}$$

The differential operator on the left is used to define the Hermite polynomials as

$$H_n(x, y) = (x + 2\, y\, \partial_x)^n\, 1\,. \tag{4.17}$$

There are certain advantages offered by the umbra method with respect to the monomiality technique, which are all associated with the fact that in the former case one deals with commuting operators.

References

[1] P. Appell, J. Kampé de Fériet, "Fonctions Hypergéométriques et Hypersphériques : Polynômes d'Hermite", Gauthier-Villars, Paris, 1926.

[2] D. Babusci, G. Dattoli, K. Górska, *Symbolic calculus and integrals of Laguerre polynomials,* arXiv:1202.1976v1 [math.CA] 9 Feb. 2012.

[3] G. Dattoli, P. L. Ottaviani, A. Torre, L. Vázquez, *Evolution operator equations: Integration with algebraic and finite-difference methods.Applications to physical problems in classical and quantum mechanics and quantum field theory,* Riv. Nuovo Cimento Soc. Ital. Fis: (Ser. 4) **20** (1997), 1–113.

[4] G. Dattoli, A. Torre, M. Carpanese, *Operational rules and arbitrary order Hermite generating functions.* J. Math. Anal. Appl. **227** (1998), n. 1, 98–111.

[5] G. Doetsch, *Integraleigenschaften der Hermiteschen Polynome,* Math. Z. **32** (1930), 557–559.

[6] I. M. Gessel , P. Jaywant, *A triple lacunary generating function for Hermite polynomials*, arXiv: math/0403086v2 [math.CO] 22 Jul 2005.

[7] G. H. Hardy, "Ramanujan. Twelve Lectures on Subjects Suggested by his Life an Work", Cambridge University Press, Cambridge, 1940.

[8] I.I. Hirschman, D. V. Widder, "The Convolution Transform". Princenton University press, Princeton, N. J., 1955. x + 268 pp.

[9] P. E. Ricci, I. Tavkhelidze, *An introduction to operational techniques and special polynomials. Differential equations and their applications.* J. Math. Sci. (N. Y.) **157** (2009), n.1, 161–189.

[10] D. V. Widder, "The Heat Equation", Pure and Applied Mathematics, Vol.67, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. xiv + 267 pp.

Giuseppe Dattoli
Unità Tecnico Scientifica Tecnologie Avanzate, ENEA-Centro Ricerche Frascati, C.P. 65, Via Enrico Fermi 45, I-00044, Frascati, Roma, Italia
E-mail address: dattoli@frascati.enea.it

Bruna Germano
Sapienza - Università di Roma, Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Via A. Scarpa, 14 - 00161 Roma, Italia
E-mail address: bruna.germano@sbai.uniroma1.it

Maria Renata Martinelli
Sapienza - Università di Roma, Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Via A. Scarpa, 14 - 00161 Roma, Italia
E-mail address: martinelli@dmmm.uniroma1.it

Paolo Emilio Ricci
International Telematic University UniNettuno, Corso Vittorio Emanuele II, 39 - 00186 Roma, Italia
E-mail address: paoloemilioricci@gmail.com