J. Differential Equations 254 (2013) 2233–2260

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Endpoint estimates and global existence for the nonlinear Dirac equation with potential

Federico Cacciafesta, Piero D'Ancona *

SAPIENZA – Università di Roma, Dipartimento di Matematica, Piazzale A. Moro 2, I-00185 Roma, Italy

ARTICLE INFO

Article history:
Received 9 October 2012
Available online 20 December 2012

ABSTRACT

We prove endpoint estimates with angular regularity for the wave and Dirac equations perturbed with a small potential. The estimates are applied to prove global existence for the cubic Dirac equation perturbed with a small potential, for small initial H^1 data with additional angular regularity. This implies in particular global existence in the critical energy space H^1 for small radial data.

1. Introduction

The main topic of this paper is the cubic massless Dirac equation on $\mathbb{R}^{1+3}$ perturbed with a potential

$$iu_t = \mathcal{D}u + V(x)u + P_3(u, \bar{u}), \qquad u(0, x) = f(x), \tag{1.1}$$

where $u(t, x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4$ and $P_3(u, \bar{u})$ is any homogeneous, $\mathbb{C}^4$-valued cubic polynomial. We recall that the Dirac operator $\mathcal{D}$ is defined as

$$\mathcal{D} = i^{-1}(\alpha_1\partial_1 + \alpha_2\partial_2 + \alpha_3\partial_3)$$

where ∂_j are the partial derivatives on $\mathbb{R}^3_x$ and α_j are the Dirac matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

* Corresponding author.
E-mail addresses: cacciafe@mat.uniroma1.it (F. Cacciafesta), dancona@mat.uniroma1.it (P. D'Ancona).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.12.002

The anticommutation relations

$$\alpha_\ell \alpha_k + \alpha_k \alpha_\ell = 2\delta_{kl} I_4$$

imply that $\mathcal{D}^2 = -I_4 \Delta$ is a diagonal operator, showing the intimate connection of the massless Dirac system with the wave equation.

The unperturbed nonlinear Dirac equation

$$iu_t = \mathcal{D}u + F(u), \qquad u(0,x) = f(x) \tag{1.3}$$

is important in relativistic quantum mechanics, and was studied in a number of works (see e.g. [20,7,19,18,8,17,16] and for the more general Dirac–Klein–Gordon system see [6,5]). In particular, it is well known that the cubic nonlinearity is critical for solvability in the energy space H^1; global existence in H^1 is still an open problem even for small initial data, while the case of subcritical spaces H^s, $s > 1$ was settled in the positive in [8,17].

Criticality is better appreciated in terms of Strichartz estimates, which are the main tool in the study of nonlinear dispersive equations. The identity

$$e^{it\mathcal{D}} f = \cos\bigl(t|D|\bigr) f + i \frac{\sin(t|D|)}{|D|} \mathcal{D} f, \quad |D| = (-\Delta)^{1/2}$$

shows that the estimates for the Dirac flow $e^{it\mathcal{D}}$ are immediate consequences of the corresponding estimates for the wave flow $e^{it|D|}$, with the same indices, restricted to the special case of dimension $n = 3$. For the wave equation on $\mathbb{R}^{1+n}$, $n \geqslant 3$, Strichartz estimates can be combined with Sobolev embedding and take the general form

$$\bigl\| |D|^{\frac{n}{r} + \frac{1}{p} - \frac{n}{2}} e^{it|D|} f \bigr\|_{L^p L^r} \lesssim \|f\|_{L^2} \tag{1.4}$$

for all p, r such that

$$p \in [2, \infty], \quad 0 < \frac{1}{r} \leqslant \frac{1}{2} - \frac{2}{(n-1)p}.$$

We are using here the mixed time–space $L^p L^q$ norms defined by

$$\bigl\| u(t,x) \bigr\|_{L^p L^q} = \bigl\| \|u\|_{L^q_x} \bigr\|_{L^p_t}.$$

Notice that the limiting case $r = \infty$

$$\bigl\| e^{it|D|} f \bigr\|_{L^2 L^\infty} \lesssim \bigl\| |D|^{\frac{n-1}{2}} f \bigr\|_{L^2} \tag{1.5}$$

is always excluded and is indeed false for general data. See [11] and [13] for the general Strichartz estimates; concerning the limiting case $r = \infty$, see [14,9]. The corresponding estimates for the Dirac equation are given in [3].

In particular, the endpoint estimate

$$\bigl\| e^{it\mathcal{D}} f \bigr\|_{L^2 L^\infty} \lesssim \bigl\| |D| f \bigr\|_{L^2} \tag{1.6}$$

fails. To see the connection with the critical equation (1.3), we rewrite it as a fixed point problem for the map

$$v \mapsto \Phi(v) = e^{it\mathcal{D}} f + i \int_0^t e^{i(t-t')\mathcal{D}} P_3\big(v(t')\big)\, dt'.$$

If (1.6) were true one could write

$$\left\| \int_0^t e^{i(t-t')\mathcal{D}} v(t')^3\, dt \right\|_{L^2L^\infty} \lesssim \int_{-\infty}^{+\infty} \left\| e^{it\mathcal{D}} e^{-it'\mathcal{D}} P_3\big(v(t')\big)\right\|_{L^2L^\infty} dt' \lesssim \left\| v^3 \right\|_{L^1H^1}$$

and in conjunction with the conservation of H^1 energy this would imply

$$\left\|\Phi(v)\right\|_{L^\infty_t H^1_x} + \left\|\Phi(v)\right\|_{L^2_t L^\infty_x} \lesssim \|f\|_{H^1} + \|v\|_{L^\infty H^1} \|v\|^2_{L^2L^\infty}.$$

In other words, a contraction argument in the norm $\|\cdot\|_{L^2L^\infty} + \|\cdot\|_{L^\infty H^1}$ would be enough to prove global existence of small H^1 solutions to (1.3).

It was already noted in [14] that (1.5) is true for radial data when $n = 3$. This remark is not of immediate application for the Dirac equation, since solutions corresponding to radial data need not be radial (due to the fact that the operator $\mathcal{D}$ does not commute with rotations of $\mathbb{R}^3$). Nevertheless, for radial H^1 and even more general data, in [16] global existence was achieved via finer estimates, which separate radial from angular regularity. We introduce the natural notations

$$\|f\|_{L^a_r L^b_\omega} = \left(\int_0^\infty \left\| f(r\,\cdot)\right\|^a_{L^b(\mathbb{S}^{n-1})} r^{n-1}\, dr \right)^{\frac{1}{a}}$$

and

$$\|f\|_{L^\infty_r L^b_\omega} = \sup_{r \geqslant 0} \left\| f(r\,\cdot)\right\|_{L^b(\mathbb{S}^{n-1})}.$$

Then the following estimate is proved in [16]:

$$n = 3, \quad \left\| e^{it|D|} f \right\|_{L^2 L^\infty_r L^p_\omega} \lesssim \sqrt{p} \cdot \left\| |D| f \right\|_{L^2}, \quad \forall p < \infty. \tag{1.7}$$

This gives a bound for the standard L^2L^∞ norm via Sobolev embedding on the unit sphere $\mathbb{S}^2$

$$\left\| e^{it|D|} f \right\|_{L^2L^\infty} \lesssim \left\| \Lambda^\epsilon_\omega e^{it|D|} f \right\|_{L^2 L^\infty_r L^p_\omega} \lesssim \left\| |D| \Lambda^\epsilon_\omega f \right\|_{L^2}, \quad p > \frac{2}{\epsilon} \tag{1.8}$$

where the angular derivative operator Λ^s_ω is defined in terms of the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$ as

$$\Lambda^s_\omega = (1 - \Delta_{\mathbb{S}^{n-1}})^{s/2}.$$

Using (1.8) one can prove global existence for (1.3) provided the norm $\||D|\Lambda^s_\omega f\|_{L^2}$ of the data is small enough for some $s > 0$. In particular, this includes all radial data with a small H^1 norm.

Our main goal here is to extend this group of results to Eq. (1.1) perturbed with a small potential $V(x)$. We consider first the linear equation

$$iu_t = \mathcal{D}u + V(x)u + F(t,x). \tag{1.9}$$

The perturbative term Vu cannot be handled using the inhomogeneous version of (1.7) because of the loss of derivatives. Instead, we prove new mixed Strichartz-smoothing estimates (Theorem 2.3)

$$n \geqslant 3, \quad \left\| \int_0^t e^{i(t-s)|D|} F(s,x)\, ds \right\|_{L^2_t L^\infty_{|x|} L^2_\omega} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} |D|^{\frac{n-1}{2}} \Lambda^\sigma_\omega F \right\|_{L^2_t L^2_x} \tag{1.10}$$

where

$$\text{for } n = 3, \quad \sigma = 0,$$
$$\text{for } n \geqslant 4, \quad \sigma = 1 - \frac{n}{2}. \tag{1.11}$$

Remark 1.1. As a byproduct of our proof, we obtain the following endpoint estimates for the wave flow with gain of angular regularity (Theorem 2.1):

$$n \geqslant 3, \quad \left\| e^{it|D|} f \right\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \left\| \Lambda^\sigma_\omega f \right\|_{\dot H^{\frac{n-1}{2}}} \tag{1.12}$$

where σ is as in (1.11). Although this was not the main purpose of the paper, it is interesting to compare (1.12) with known results. In dimension $n = 3$, estimate (1.12) is just a special case of Theorem 1.1-III in [16] where (1.12) is proved with $\sigma = -\frac{3}{4}$; it is not known if this value is sharp, however in the same paper it is proved that the estimate is false for $\sigma < -\frac{5}{6}$. On the other hand, to our knowledge, estimate (1.12) for $n \geqslant 4$ and (1.10) for $n \geqslant 3$ are new. The literature on these kind of estimates is extensive and we refer to [10,12] and the references therein for further information.

Combining (1.10) with the techniques of [4] we obtain the following endpoint result for a 3D linear wave equation with singular potential. Analogous estimates can be proved for higher dimensions; here we chose to focus on the 3D case since the assumptions on V take a particular simple form:

Theorem 1.1. *Let $n = 3$ and consider the Cauchy problem for the wave equation*

$$u_{tt} - \Delta u + V(x)u = F, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x)$$

under the assumptions:

(i) *$V(x)$ is real valued and the positive and negative parts $V_\pm$ satisfy*

$$V_+ \leqslant \frac{C}{|x|^{\frac{1}{2}-\epsilon} + |x|^2}, \qquad V_- \leqslant \frac{\delta}{|x|^{\frac{1}{2}-\epsilon} + |x|^2} \tag{1.13}$$

for some δ, ϵ sufficiently small and some $C \geqslant 0$;

(ii) *$-\Delta + V$ is selfadjoint;*

(iii) *0 is not a resonance for $-\Delta + V_-$ (in the following sense: if f is such that $(-\Delta + V_-)f = 0$ and $\langle x \rangle^{-1} f \in L^2$, then $f \equiv 0$).*

Then the solution $u(t,x)$ satisfies the endpoint Strichartz estimate

$$\|u\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|f\|_{\dot{H}^1} + \|g\|_{L^2} + \big\| \langle x\rangle^{\frac{1}{2}+} F \big\|_{L_t^2 L_x^2}. \tag{1.14}$$

The next step is to prove suitable smoothing estimates for the Dirac equation with potential

$$iu_t = \mathcal{D}u + V(x)u + F(t,x)$$

(see Proposition 4.2 and Corollary 4.3). Then by a perturbative argument we obtain the following endpoint estimates for the linear flows:

Theorem 1.2. *Assume that the hermitian matrix $V(x)$ satisfies, for δ sufficiently small, C arbitrary and $\sigma > 1$, with $v(x) = |x|^{\frac{1}{2}} |\log|x||^{\frac{1}{2}+} + |x|^{1+}$,*

$$\big|V(x)\big| \leqslant \frac{\delta}{v(x)}, \qquad \big|\nabla V(x)\big| \leqslant \frac{C}{v(x)}. \tag{1.15}$$

Then the perturbed Dirac flow satisfies the endpoint Strichartz estimate

$$\big\|e^{it(\mathcal{D}+V)} f\big\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|f\|_{H^1}. \tag{1.16}$$

If the potential satisfies the stronger assumptions: for some $s > 1$,

$$\big\|\Lambda_\omega^s V\big(|x|\,\cdot\big)\big\|_{L^2(\mathbb{S}^2)} \leqslant \frac{\delta}{v(x)}, \qquad \big\|\Lambda_\omega^s \nabla V\big(|x|\,\cdot\big)\big\|_{L^2(\mathbb{S}^2)} \leqslant \frac{C}{v(x)}, \tag{1.17}$$

then we have the endpoint estimate with angular regularity

$$\big\|\Lambda_\omega^s e^{it(\mathcal{D}+V)} f\big\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \big\|\Lambda_\omega^s f\big\|_{H^1} \tag{1.18}$$

and the energy estimate with angular regularity

$$\big\|\Lambda_\omega^s e^{it(\mathcal{D}+V)} f\big\|_{L_t^\infty H^1} \lesssim \big\|\Lambda_\omega^s f\big\|_{H^1}. \tag{1.19}$$

We can finally apply Theorem 1.2 to the nonlinear equation (1.1) and we obtain:

Theorem 1.3. *Consider the perturbed Dirac system (1.1), where the 4×4 matrix valued potential $V(x)$ is hermitian and satisfies assumptions (1.17). Let $P_3(u,\bar{u})$ be a $\mathbb{C}^4$-valued homogeneous cubic polynomial. Then for any $s > 1$ there exists ϵ_0 such that for all initial data satisfying*

$$\big\|\Lambda_\omega^s f\big\|_{H^1} < \epsilon_0 \tag{1.20}$$

the Cauchy problem (5.1) admits a unique global solution $u \in CH^1 \cap L^2 L^\infty$ with $\Lambda_\omega^s u \in L^\infty H^1$.

In particular, problem (1.1) has a global unique solution for all radial data with sufficiently small H^1 norm.

Remark 1.2. It is clear that our methods can also be applied to nonlinear wave equations perturbed with potentials, and allow to prove global well posedness for some types of critical nonlinearities. This problem will be the object of a further note.

Remark 1.3. We did not strive for the sharpest condition on the potential V, which can be improved at the price of additional technicalities which we prefer to skip here. Moreover, the result can be extended to more general cubic nonlinearities $|F(u)| \sim |u|^3$.

Notice also that we need an angular regularity $s > 1$ on the data, higher than the $s > 0$ assumed in the result of [16]. It is possible to relax our assumptions to $s > 0$; the only additional tool we would need to prove is a Moser-type product estimate

$$\left\| \Lambda^s_\omega (uv) \right\|_{L^2_\omega} \lesssim \|u\|_{L^\infty_\omega} \left\| \Lambda^s_\omega v \right\|_{L^2_\omega} + \left\| \Lambda^s_\omega u \right\|_{L^2_\omega} \|v\|_{L^\infty_\omega}, \quad s > 0$$

and an analogous one for $\Lambda^s_\omega |D|(uv)$. This would require a fair amount of calculus on the sphere $\mathbb{S}^2$, and here we preferred to use the conceptually much simpler algebra property of $H^s(\mathbb{S}^{n-1})$ for $s > \frac{n-1}{2}$.

On the other hand, the extension of our results to the massive case

$$iu_t = \mathcal{D}u + V(x)u + m\beta u + F(u), \quad m \neq 0$$

requires a different approach and will be the object of further work.

2. Endpoint estimates for the free flows

To fix our notations, we recall some basic facts on spherical harmonics (see [22]) on $\mathbb{R}^n$, $n \geqslant 2$. For $k \geqslant 0$, we denote by $\mathcal{H}_k$ the space of harmonic polynomials homogeneous of degree k, restricted to the unit sphere $\mathbb{S}^{n-1}$. The dimension of $\mathcal{H}_k$ for $k \geqslant 2$ is

$$d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \simeq \langle k \rangle^{n-2}$$

while $d_0 = 1$ and $d_1 = n$. $\mathcal{H}_k$ is called the space of *spherical harmonics of degree* k, and we denote by Y^l_k, $1 \leqslant l \leqslant d_k$ an orthonormal basis. Since

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$$

every function $f(x) = f(r\omega)$, $r = |x|$, can be expanded as

$$f(r) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} f^l_k(r) Y^l_k(\omega) \tag{2.1}$$

and we have

$$\left\| f(r\omega) \right\|_{L^2_\omega} = \sum_{\substack{k \geqslant 0 \\ 1 \leqslant l \leqslant d_k}} \left| f^l_k \right|^2,$$

where we use the notation $L^2_\omega = L^2(\mathbb{S}^{n-1})$. More generally, if $\Delta_{\mathbb{S}}$ is the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$ and

$$\Lambda_\omega = (1 - \Delta_{\mathbb{S}})^{1/2},$$

we have the equivalence

$$\|\Lambda_\omega^\sigma f(r\omega)\|_{L^2_\omega} \simeq \sum_{\substack{k\geqslant 0\\ 1\leqslant l\leqslant d_k}} \langle k\rangle^{2\sigma} |f_k^l|^2, \quad \sigma \in \mathbb{R}.$$

As a consequence we have the equivalence

$$\|\Lambda_\omega^\sigma f\|^2_{L^2(\mathbb{R}^n)} \simeq \sum_{\substack{k\geqslant 0\\ 1\leqslant l\leqslant d_k}} \langle k\rangle^{2\sigma} \|f_k^l(r) r^{\frac{n-1}{2}}\|^2_{L^2_r(0,\infty)}. \tag{2.2}$$

In a similar way

$$\begin{aligned}\|\nabla f\|^2_{L^2(\mathbb{R}^n)} &= (-\Delta f, f)_{L^2}\\ &\simeq \sum_{\substack{k\geqslant 0\\ 1\leqslant l\leqslant d_k}} \big(\|r^{\frac{n-1}{2}} \partial_r f_k^l(r)\|^2_{L^2_r(0,\infty)} + k^2 \|r^{\frac{n-3}{2}} f_k^l(r)\|^2_{L^2_r(0,\infty)}\big)\end{aligned} \tag{2.3}$$

where we used the following representation of the action of Δ

$$-\Delta f(x) = \sum Y_k^l\left(\frac{x}{|x|}\right)\left[-r^{1-n}\partial_r\big(r^{n-1}\partial_r f_k^l\big) + \frac{k(k+n-2)}{r^2} f_k^l\right], \quad r = |x|.$$

More generally we have for integer m

$$-\Delta(1-\Delta_S)^m f(x) = \sum \big(1 + k(k+n-2)\big)^m Y_k^l\left[-r^{1-n}\partial_r\big(r^{n-1}\partial_r f_k^l\big) + \frac{k(k+n-2)}{r^2} f_k^l\right]$$

which implies

$$\begin{aligned}\|\nabla\Lambda_\omega^m f\|^2_{L^2(\mathbb{R}^n)} &= \big(-\Delta(1-\Delta_S)^m f, f\big)_{L^2}\\ &\simeq \sum_{\substack{k\geqslant 0\\ 1\leqslant l\leqslant d_k}} \langle k\rangle^{2m}\big(\|r^{\frac{n-1}{2}} \partial_r f_k^l(r)\|^2_{L^2_r(0,\infty)} + k^2 \|r^{\frac{n-3}{2}} f_k^l(r)\|^2_{L^2_r(0,\infty)}\big)\end{aligned} \tag{2.4}$$

and by interpolation and duality we see that (2.4) holds for all $m \in \mathbb{R}$.

We shall estimate the solution using the following norm:

$$\|f\|_{L^\infty_r L^2_\omega} = \sup_{r>0} \|f(r\omega)\|_{L^2_\omega(\mathbb{S}^{n-1})}.$$

Theorem 2.1. *For all $n \geqslant 4$ the following estimate holds*

$$\|e^{it|D|} f\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|\Lambda_\omega^{1-\frac{n}{2}} f\|_{\dot H^{\frac{n-1}{2}}}, \tag{2.5}$$

while for $n = 3$ we have

$$\big\|e^{it|D|}f\big\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot{H}^1}. \tag{2.6}$$

Remark 2.1. In dimension $n = 3$ the previous result is a special case of the stronger estimate proved in [16]:

$$\big\|e^{it|D|}f\big\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \big\|\Lambda_\omega^{-3/4} f\big\|_{\dot{H}^1}. \tag{2.7}$$

Notice that it is not known if estimate (2.7) is sharp. For higher dimension, estimate (2.5) seems to be new; it is reasonable to guess that this result is not sharp and might be improved at least to

$$\big\|e^{it|D|}f\big\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \big\|\Lambda_\omega^{\epsilon-\frac{n-1}{2}} f\big\|_{\dot{H}^{\frac{n-1}{2}}}, \quad \epsilon > 0. \tag{2.8}$$

Proof of Theorem 2.1. It is well known that the $\mathcal{H}_k$ spaces are invariant for the Fourier transform $\mathcal{F}$, and more precisely

$$\mathcal{F}\big(c(r)Y^l_k(\omega)\big)(\xi) = g\big(|\xi|\big)Y^l_k\left(\frac{\xi}{|\xi|}\right) \tag{2.9}$$

where g is given by the Hankel transform

$$g(r) = (2\pi)^{\frac{n}{2}} i^{-k} r^{-\frac{n-2}{2}} \int_0^\infty c(\rho) J_{k+\frac{n-2}{2}}(r\rho)\rho^{\frac{n}{2}}\, d\rho. \tag{2.10}$$

Here J_ν is the *Bessel function* of order ν which we shall represent using the Lommel integral form

$$J_\nu(y) = \frac{(y/2)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+1/2)} \int_{-1}^{1} e^{iy\lambda}\big(1-\lambda^2\big)^{\nu-\frac{1}{2}}\, d\lambda. \tag{2.11}$$

Now, given a function $f(x)$, we denote by $\check{f}$ its inverse Fourier transform and with $\check{f}^l_k(r)$ the coefficients of the expansion in spherical harmonics of $\check{f}$:

$$\check{f} = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} \check{f}^l_k(r)Y^l_k(\omega). \tag{2.12}$$

Recalling (2.9) we obtain the representation

$$f(x) = \sum (2\pi)^{\frac{n}{2}} i^{-k}|x|^{1-\frac{n}{2}} Y^l_k\left(\frac{x}{|x|}\right) \int_0^\infty \check{f}^l_k(\rho) J_{k+\frac{n-2}{2}}\big(|x|\rho\big)\rho^{\frac{n}{2}}\, d\rho \tag{2.13}$$

which implies

$$e^{it|D|}f = \sum (2\pi)^{\frac{n}{2}} i^{-k} |x|^{1-\frac{n}{2}} Y_k^l\left(\frac{x}{|x|}\right) \int_0^\infty e^{it\rho} \check{f}_k^l(\rho) J_{k+\frac{n-2}{2}}\bigl(|x|\rho\bigr) \rho^{\frac{n}{2}}\, d\rho. \tag{2.14}$$

Consider now Lommel's formula (2.11) for J_ν; since $e^{i\lambda y} = (iy)^{-k}\partial_\lambda^k(e^{i\lambda y})$, after k integration by parts we obtain

$$J_{k+\frac{n-2}{2}}(y) = c_k y^{\frac{n}{2}-1} \int_{-1}^{1} e^{i\lambda y} \partial_\lambda^k\bigl(\bigl(1-\lambda^2\bigr)^{k+\frac{n-3}{2}}\bigr)\, d\lambda \tag{2.15}$$

with

$$c_k = \frac{i^k 2^{-\frac{n}{2}-k+1}}{\pi^{\frac{1}{2}} \Gamma(\frac{n-1}{2}+k)}. \tag{2.16}$$

Thus we can write

$$\begin{aligned} &|x|^{1-\frac{n}{2}} \int_0^\infty e^{it\rho} \check{f}_k^l(\rho) J_{k+\frac{n-2}{2}}\bigl(|x|\rho\bigr) \rho^{\frac{n}{2}}\, d\rho \\ &\quad = c_k \int_{-1}^{1} \partial_\lambda^k\bigl(\bigl(1-\lambda^2\bigr)^{k+\frac{n-3}{2}}\bigr) \left[\int_{-\infty}^{+\infty} \mathbf{1}_+(\rho) \check{f}_k^l(\rho) \rho^{n-1} e^{i\rho(t+\lambda|x|)}\, d\rho\right] d\lambda \end{aligned} \tag{2.17}$$

where $\mathbf{1}_+(\rho)$ is the characteristic function of $(0,+\infty)$; regarding the inner integral as a Fourier transform we arrive at

$$= c_k \int_{-1}^{1} \partial_\lambda^k\bigl(\bigl(1-\lambda^2\bigr)^{k+\frac{n-3}{2}}\bigr) \hat{g}_k^l\bigl(t+\lambda|x|\bigr)\, d\lambda$$

where

$$g_k^l(\rho) = \mathbf{1}_+(\rho) \check{f}_k^l(\rho) \rho^{n-1}. \tag{2.18}$$

In conclusion, we have the following representation

$$e^{it|D|}f = \sum (2\pi)^{\frac{n}{2}} i^{-k} Y_k^l\left(\frac{x}{|x|}\right) c_k \int_{-1}^{1} \partial_\lambda^k\bigl(\bigl(1-\lambda^2\bigr)^{k+\frac{n-3}{2}}\bigr) \hat{g}_k^l\bigl(t+\lambda|x|\bigr)\, d\lambda \tag{2.19}$$

where the constants c_k are given by (2.16) and g_k^l by (2.18). Notice that similar representations play a fundamental role also in [10,12]. In particular this gives for the L^2_ω norm of the solution at t, $|x|$ fixed the formula

$$\left\| e^{it|D|} f(|x|\,\cdot)\right\|^2_{L^2_\omega} \simeq \sum |c_k|^2 \left| \int_{-1}^{1} \partial_\lambda^k\left(\left(1-\lambda^2\right)^{k+\frac{n-3}{2}}\right) \hat{g}_k^l\left(t+\lambda|x|\right) d\lambda \right|^2. \tag{2.20}$$

We now need the following estimate:

Lemma 2.2. *Let $Q_k(x)$ be the function*

$$Q_k(x) = \frac{\partial_x^k((1-x^2)^{k+\frac{n-3}{2}})}{2^k \Gamma(k+\frac{n-1}{2})}.$$

Then we have on $x \in [-1,1]$

$$\left|Q_k(x)\right| \lesssim \langle k\rangle^{1-\frac{n}{2}} \quad \text{if } n \geqslant 4, \qquad \left|Q_k(x)\right| \leqslant 1 \quad \text{if } n = 3. \tag{2.21}$$

Proof. We recall that the Jacobi polynomials are defined by

$$\mathbf{P}_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k k!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^k}{dx^k}\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right]. \tag{2.22}$$

We shall use some standard properties of these polynomial which can be found in [1]. The function Q_k can be expressed in terms of $\mathbf{P}_k^{(\alpha,\alpha)}(x)$ with $\alpha = (n-3)/2$ as

$$\left|Q_k(x)\right| = \frac{k!(1-x^2)^{\frac{n-3}{2}}}{\Gamma(k+\frac{n-1}{2})}\left|\mathbf{P}_k^{(\frac{n-3}{2},\frac{n-3}{2})}(x)\right|. \tag{2.23}$$

Thus in order to estimate Q_k we need a bound for the function

$$T_a(x) = \left(1-x^2\right)^a \mathbf{P}_k^{(a,a)}(x), \quad a = \frac{n-3}{2}. \tag{2.24}$$

The following approach was suggested by Ilia Krasikov, see [15]. Consider the second order differential equation

$$f''(x) + p(x)f'(x) + q(x)f(x) = 0$$

on the interval $(-1,1)$, and define the *Sonine function* as

$$S(f,x) = f(x)^2 + \frac{f'(x)^2}{q(x)}$$

under the assumption $q > 0$. It is easy to check that function S satisfies the relation

$$S' = -\left(2\frac{p}{q} + \frac{q'}{q^2}\right)f'^2.$$

The function $T_a(x)$ defined in (2.24) satisfies the differential equation

$$T_a''(x) + \frac{2(2a-1)}{1-x^2} x T_a'(x) + \frac{(k+1)(2a+k)}{1-x^2} T_a(x) = 0$$

so that the associated Sonine function

$$S_a(x) = T_a^2 + \frac{1-x^2}{(k+1)(2a+k)} {T_a'}^2$$

satisfies

$$S_a' = -\frac{2(2a-1)}{(k+1)(2a+k)} x {T_a'}^2. \tag{2.25}$$

From this identity it is clear that S_a has a maximum at $x = 0$ provided $a \geqslant 1/2$ i.e. $n \geqslant 4$. In this case we have

$$S_a(x) \leqslant S_a(0) = T_a(0)^2 + \frac{T_a'(0)^2}{(k+1)(2a+k)} = \mathbf{P}_k^{(a,a)}(0)^2 + \frac{{\mathbf{P}_k^{(a,a)}}'(0)^2}{(k+1)(2a+k)}.$$

Now we recall that, for *even* $k \geqslant 2$,

$$\mathbf{P}_k^{(a,a)}(0) = \frac{\Gamma(k+a+1)}{(-2)^k \Gamma(\frac{k}{2}+1)\Gamma(\frac{k}{2}+a+1)} \simeq (-1)^k k^{-\frac{1}{2}}, \qquad {\mathbf{P}_k^{(a,a)}}'(0) = 0$$

where we used the Stirling asymptotics

$$k! \simeq k^{k-1/2} e^{-k}, \qquad \Gamma(k+a+1) \simeq k^{k+a-1/2} e^{-k}.$$

In a similar way, for *odd* k,

$$\mathbf{P}_k^{(a,a)}(0) = 0, \qquad {\mathbf{P}_k^{(a,a)}}'(0) = \frac{\Gamma(k+a+1)}{(-2)^{k-1} \Gamma(\frac{k}{2}+\frac{1}{2})\Gamma(\frac{k}{2}+a+\frac{1}{2})} \simeq (-1)^{k-1} k^{\frac{1}{2}}.$$

Thus for all values of $k \geqslant 1$ we have

$$|T_a(x)| \leqslant \sqrt{S_a(x)} \lesssim \frac{1}{\sqrt{k}}$$

and by (2.23) we conclude that, for $k \geqslant 1$ and $|x| < 1$,

$$|Q_k(x)| \lesssim k^{1-\frac{n}{2}}$$

which is precisely (2.21) for $n \geqslant 4$.

In the remaining case $n = 3$ we have $a = 0$ and the best we can do is to use the sharp inequality $|\mathbf{P}_k^{(0,0)}| \leqslant 1$ to obtain

$$|Q_k(x)| = \frac{k!}{k!} |\mathbf{P}_k^{(0,0)}| \leqslant 1. \qquad \square$$

Using the lemma, we can continue estimate (2.20) as follows

$$\left\| e^{it|D|} f\left(|x|\,\cdot\right)\right\|^2_{L^2_\omega} \lesssim \sum \omega_k^2 \left(\int_{-1}^{1} \left|\hat{g}_k^l\left(t+\lambda|x|\right)\right| d\lambda \right)^2$$

where

$$\omega_k = 1 \quad \text{if } n=3, \qquad \omega_k = \langle k\rangle^{1-\frac{n}{2}} \quad \text{if } n \geqslant 4. \tag{2.26}$$

Since

$$\int_{-1}^{1} \left|\hat{g}_k^l\left(t+\lambda|x|\right)\right| d\lambda = \frac{1}{|x|} \int_{-|x|}^{|x|} \left|\hat{g}_k^l(t+\lambda)\right| d\lambda \leqslant M\left(\hat{g}_k^l\right)(t)$$

where $M(g)$ is the centered maximal function, we obtain

$$\left\| e^{it|D|} f\left(|x|\,\cdot\right)\right\|^2_{L^2_\omega} \lesssim \sum \omega_k^2 M\left(\hat{g}_k^l\right)(t)^2.$$

Now we can take the sup in $|x|$ which gives

$$\left\| e^{it|D|} f\right\|^2_{L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2 M\left(\hat{g}_k^l\right)(t)^2,$$

and integrating in time, by the L^2 boundedness of the maximal function, we obtain

$$\left\| e^{it|D|} f\right\|^2_{L^2_t L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2 \left\|\hat{g}_k^l\right\|^2_{L^2} \simeq \sum \omega_k^2 \left\|g_k^l\right\|^2_{L^2} \simeq \sum \omega_k^2 \left\|\check{f}_k^l(\rho)\rho^{n-1}\right\|^2_{L^2_\rho(0,\infty)}.$$

It is immediate to check that the last sum is equivalent to

$$\sum \omega_k^2 \left\|\check{f}_k^l(\rho)\rho^{n-1}\right\|^2_{L^2_\rho(0,\infty)} \simeq \left\| |D|^{\frac{n-1}{2}} \Lambda_\omega^\sigma f\right\|^2_{L^2(\mathbb{R}^n)}$$

where $\sigma = 1 - n/2$ for $n \geqslant 4$, which proves (2.5), and $\sigma = 0$ for $n = 3$, which proves (2.6). □

Although the method of proof of Theorem 2.1 is probably not sharp for the homogeneous operator, it has the advantage that it can be adapted to handle also the nonhomogeneous term and gives the following mixed Strichartz-smoothing estimate:

Theorem 2.3. *For any $n \geqslant 3$, the following estimate holds*

$$\left\| \int_0^t e^{i(t-s)|D|} F(s,x)\, ds \right\|_{L^2_t L^\infty_{|x|} L^2_\omega} \lesssim \left\| \langle x\rangle^{\frac{1}{2}+} |D|^{\frac{n-1}{2}} \Lambda_\omega^\sigma F\right\|_{L^2_t L^2_x} \tag{2.27}$$

where

$$\sigma = 1 - \frac{n}{2} \quad \textit{if } n \geqslant 4, \qquad \sigma = 0 \quad \textit{if } n = 3.$$

Proof. As in the proof of the previous theorem, we expand F in spherical harmonics and we obtain the representation

$$\int_0^t e^{i(t-s)|D|}F(s,x)\,ds$$
$$= \sum (2\pi)^{\frac{n}{2}} i^{-k} c_k Y_k^l\left(\frac{x}{|x|}\right) \int_{-1}^{1} \partial_\lambda^k\left(\left(1-\lambda^2\right)^{k+\frac{n-3}{2}}\right)\widehat{G}_k^l\left(s, t-s+\lambda|x|\right) d\lambda \tag{2.28}$$

with the constants c_k as in (2.16), where the functions G_k^l are defined as follows: denoting by $F_k^l(t,r)$ the coefficients of the expansion into spherical harmonics of the inverse Fourier transform $\check{F} = \mathcal{F}^{-1}(F)$

$$\check{F}(s,x) = \sum \check{F}_k^l\left(s,|x|\right)Y_k^l\left(\frac{x}{|x|}\right)$$

and by G_k^l the functions

$$G_k^l(s,\rho) = \mathbf{1}_+(\rho)\rho^{n-1}\check{F}_k^l(s,\rho), \tag{2.29}$$

the $\widehat{G}_k^l(s,r)$ are the Fourier transforms of G_k^l in the second variable:

$$\widehat{G}_k^l(s,r) = \int_{-\infty}^{+\infty} e^{ir\rho}G_k^l(s,\rho)\,d\rho.$$

Thus applying Lemma 2.2 we obtain

$$\left|\int_0^t e^{i(t-s)|D|}F(s,x)\,ds\right| \lesssim \sum |Y_k^l|\frac{\omega_k}{|x|}\int_{-|x|}^{|x|} d\lambda \int_0^t ds\left|\widehat{G}_k^l(s,t-s+\lambda)\right| \tag{2.30}$$

where ω_k is the same as in (2.26). We estimate the integral in s as follows

$$\int_0^t \left|\widehat{G}_k^l\right| ds \leqslant \int_{-\infty}^{+\infty} \langle\lambda+t-s\rangle^{\frac{1}{2}+}\langle\lambda+t-s\rangle^{-\frac{1}{2}-}\left|\widehat{G}_k^l(s,\lambda+t-s)\right| ds$$
$$\lesssim \left(\int \langle\lambda+t-s\rangle^{1+}\left|\widehat{G}_k^l(s,\lambda+t-s)\right|^2 ds\right)^{\frac{1}{2}} = Q_k^l(\lambda+t),$$

where

$$Q_k^l(\mu) = \left(\int_{-\infty}^{\infty} \left|\widehat{G}_k^l(s,\mu-s)\right|^2\langle\mu-s\rangle^{1+}\,ds\right)^{\frac{1}{2}}.$$

Thus we see that

$$\frac{1}{|x|}\int_{-|x|}^{|x|} d\lambda \int_0^t ds\,\big|\widehat{G}_k^l(s,t-s+\lambda)\big| \lesssim \frac{1}{|x|}\int_{-|x|}^{|x|} Q_k^l(\lambda+t)\,d\lambda \leqslant M\big(Q_k^l\big)(t).$$

Coming back to (2.30) we obtain

$$\left|\int_0^t e^{i(t-s)|D|}F(s,x)\,ds\right| \lesssim \sum \omega_k\big|Y_k^l\big|M\big(Q_k^l\big)(t)$$

and taking first the L^2_ω norm, then the sup in $|x|$, then the L^2_t norm, by the L^2 boundedness of the maximal function we have

$$\left\|\int_0^t e^{i(t-s)|D|}F(s,x)\,ds\right\|^2_{L^2_t L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2\big\|Q_k^l(t)\big\|^2_{L^2_t}.$$

The definition of Q_k^l implies

$$\int \big|Q_k^l(t)\big|^2\,dt = \iint \big|\widehat{G}_k^l(s,\mu-s)\big|^2\langle\mu-s\rangle^{1+}\,ds\,d\mu = \big\|\widehat{G}_k^l(t,r)\langle r\rangle^{\frac{1}{2}+}\big\|^2_{L^2_t L^2_r}$$

and hence

$$\left\|\int_0^t e^{i(t-s)|D|}F(s,x)\,ds\right\|^2_{L^2_t L^\infty_r L^2_\omega} \lesssim \sum \omega_k^2\big\|\widehat{G}_k^l(t,r)\langle r\rangle^{\frac{1}{2}+}\big\|^2_{L^2_t L^2_r}. \tag{2.31}$$

Recalling the definition (2.29) of G_k^l, we see that to obtain (2.27) it is sufficient to prove the following general inequality for $s=1/2+$ and arbitrary σ:

$$\sum \langle k\rangle^{2\sigma}\big\|\langle y\rangle^s\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{n-1}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_y} \lesssim \big\|\langle x\rangle^s\Lambda_\omega^\sigma|D|^{\frac{n-1}{2}}f\big\|^2_{L^2(\mathbb{R}^b)}. \tag{2.32}$$

Here as usual $\check{f}_k^l$ denote the coefficients in the expansion in spherical harmonics of the inverse Fourier transform $\check{f}=\mathcal{F}^{-1}f$.

First of all, since $\mathcal{F}^{-1}(|D|^{\frac{n-1}{2}}f)=|\xi|^{\frac{n-1}{2}}\check{f}$, we see that it is enough to prove, for $0\leqslant s\leqslant 1$ and arbitrary σ, the slightly simpler

$$\sum \langle k\rangle^{2\sigma}\big\|\langle y\rangle^s\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_y} \lesssim \big\|\langle x\rangle^s\Lambda_\omega^\sigma f\big\|^2_{L^2(\mathbb{R}^b)}. \tag{2.33}$$

The inequality will follow by interpolation between the cases $s=0$ and $s=1$; indeed, we can regard it as the statement that the operator T defined as

$$T: f\mapsto \big\{\langle y\rangle^s\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{g}_k^l\big)\big\}_{l,k},\quad g=\Lambda_\omega^{-\sigma}f$$

which associates to the function f the sequence of coefficients in the expansion of $\mathcal{F}^{-1}(\Lambda_\omega^{-\sigma} f)$, multiplied by $\lambda^{(n-1)/2}\mathbf{1}_+$, transformed again and multiplied by $\langle y\rangle^s$, is bounded between the weighted spaces

$$T : L^2\big(\langle x\rangle^{2s}\,dx\big) \to \ell^2_{\langle k\rangle^{2\sigma}}\big(L^2\big(\langle\lambda\rangle^{2s}\,d\lambda\big)\big).$$

When $s=0$ we have by Plancherel's Theorem and by (2.2)

$$\sum \langle k\rangle^{2\sigma}\big\|\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_y} \simeq \sum \langle k\rangle^{2\sigma}\big\|\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big\|^2_{L^2_\lambda(0,\lambda)} \simeq \big\|\Lambda_\omega^\sigma\check{f}\big\|^2_{L^2(\mathbb{R}^n)}.$$

Since Λ_ω commutes with the Fourier transform, indeed

$$\mathcal{F}(-\Delta_S f) = \mathcal{F}\sum (x_j\partial_k - x_j\partial_j)^2 f = \sum (\partial_j\xi_k - \partial_k\xi_j)^2\mathcal{F}f,$$

again by Plancherel we obtain (2.33) for $s=0$.

To handle the case $s=1$ we consider the quantity

$$\begin{aligned}\big\|y\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_y} &= \big\|\partial_\lambda\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_\lambda}\\ &\lesssim \big\|\lambda^{\frac{n-1}{2}}\partial_\lambda\check{f}_k^l(\lambda)\big\|^2_{L^2_\lambda(0,\infty)} + \big\|\lambda^{\frac{n-3}{2}}\check{f}_k^l(\lambda)\big\|^2_{L^2_\lambda(0,\infty)}.\end{aligned}$$

Multiplying by $\langle k\rangle^{2\sigma}$, summing over l, k and recalling (2.4), we obtain

$$\sum \langle k\rangle^{2\sigma}\big\|y\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_y} \lesssim \big\|\nabla\Lambda_\omega^\sigma\check{f}\big\|^2_{L^2(\mathbb{R}^n)} + \big\|\lambda^{\frac{n-3}{2}}\check{f}_0^0(\lambda)\big\|^2_{L^2_\lambda(0,\infty)}$$

where the last term cannot estimated by (2.4) because of the factor k^2 which vanishes when $k=0$. However we have

$$\check{f}_0^0(\lambda) = \int\limits_{|\omega|=1}\check{f}(\lambda\omega)\,d\lambda = \int\limits_{|\omega|=1}\Lambda_\omega^\sigma\check{f}(\lambda\omega)\,d\lambda$$

which implies, using Hardy's inequality

$$\big\|\lambda^{\frac{n-3}{2}}\check{f}_0^0(\lambda)\big\|^2_{L^2_\lambda(0,\infty)} \lesssim \left\|\frac{\Lambda_\omega^\sigma\check{f}}{|\xi|}\right\|_{L^2(\mathbb{R}^n)} \lesssim \big\|\nabla\Lambda_\omega^\sigma\check{f}\big\|_{L^2}.$$

Thus we have proved

$$\sum \langle k\rangle^{2\sigma}\big\|y\mathcal{F}_{\lambda\to y}\big(\mathbf{1}_+(\lambda)\lambda^{\frac{n-1}{2}}\check{f}_k^l(\lambda)\big)\big\|^2_{L^2_y} \lesssim \big\|\nabla\Lambda_\omega^\sigma\check{f}\big\|^2_{L^2(\mathbb{R}^n)} \simeq \big\||x|\Lambda_\omega^\sigma f\big\|^2_{L^2}$$

again by the commutation of Λ_ω with the Fourier transform. This gives (2.33) for $s=1$ and concludes the proof of the theorem. □

Remark 2.2. Since the operator Λ_ω commutes with $|D|$, estimates (2.5), (2.6) and (2.27) obviously generalize to the following; for any real $s \geqslant 0$,

$$\left\| \Lambda_\omega^s e^{it|D|} f \right\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \left\| \Lambda_\omega^{s+\sigma} f \right\|_{\dot{H}^{\frac{n-1}{2}}} \tag{2.34}$$

and

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-s)|D|} F(s,x)\, ds \right\|_{L_t^2 L_{|x|}^\infty L_\omega^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} |D|^{\frac{n-1}{2}} \Lambda_\omega^{s+\sigma} F \right\|_{L_t^2 L_x^2} \tag{2.35}$$

where

$$\sigma = 1 - \frac{n}{2} \quad \text{if } n \geqslant 4, \qquad \sigma = 0 \quad \text{if } n = 3.$$

From the previous estimate for the free wave equation it is not difficult to obtain analogous endpoint Strichartz and Strichartz-smoothing estimates for the 3D Dirac system:

Corollary 2.4. *Let $n = 3$. Then the flow $e^{it\mathcal{D}}$ satisfies, for all $s \geqslant 0$, the estimates*

$$\left\| \Lambda_\omega^s e^{it\mathcal{D}} f \right\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \left\| \Lambda_\omega^s f \right\|_{\dot{H}^1}, \tag{2.36}$$

and

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F\big(t',x\big)\, dt' \right\|_{L_t^2 L_{|x|}^\infty L_\omega^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} |D| \Lambda_\omega^s F \right\|_{L_t^2 L_x^2}. \tag{2.37}$$

Proof. If u solves the problem

$$iu_t + \mathcal{D}u = 0, \qquad u(0) = f(x), \tag{2.38}$$

by applying the operator $(i\partial_t - \mathcal{D})$, we see that u solves also

$$\Box u = 0, \qquad u(0) = f(x), \qquad u_t(0) = i\mathcal{D}f. \tag{2.39}$$

This gives the representation

$$e^{it\mathcal{D}} f = \cos\big(t|D|\big) f + i \frac{\sin(t|D|)}{|D|} \mathcal{D} f. \tag{2.40}$$

Moreover, we recall that the Riesz operators $|D|^{-1}\partial_j$ are bounded on weighted L^2 spaces with weight $\langle x \rangle^a$ for $a < n/2$. Thus in the case $s = 0$ estimates (2.36), (2.37) are immediate consequences of the corresponding estimates for the wave equation proved above.

In order to complete the proof in the case $s > 0$, we need analyze the structure of the Dirac operator $\mathcal{D}$ in greater detail. Following [23], we know that the space $L^2(\mathbb{R}^3)^4$ is isomorphic to an orthogonal direct sum

$$L^2(\mathbb{R}^3)^4 \simeq \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} L^2(0,+\infty;dr) \otimes H_{m_j,k_j}.$$

Each space H_{m_j,k_j} has dimension two and is generated by the orthonormal basis $\{\Phi^+_{m_j,k_j}, \Phi^-_{m_j,k_j}\}$, which can be explicitly written in terms of spherical harmonics: when $k_j = j + 1/2$ we have

$$\Phi^+_{m_j,k_j} = \frac{i}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j} \quad Y^{m_j-1/2}_{k_j} \\ -\sqrt{j+1+m_j} \quad Y^{m_j+1/2}_{k_j} \\ 0 \\ 0 \end{pmatrix},$$

$$\Phi^-_{m_j,k_j} = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j} \quad Y^{m_j-1/2}_{k_j-1} \\ \sqrt{j-m_j} \quad Y^{m_j+1/2}_{k_j-1} \\ 0 \\ 0 \end{pmatrix}$$

while when $k_j = -(j+1/2)$ we have

$$\Phi^+_{m_j,k_j} = \frac{i}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j} \quad Y^{m_j-1/2}_{1-k_j} \\ \sqrt{j-m_j} \quad Y^{m_j+1/2}_{1-k_j} \\ 0 \\ 0 \end{pmatrix},$$

$$\Phi^-_{m_j,k_j} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j} \quad Y^{m_j-1/2}_{-k_j} \\ -\sqrt{j+1+m_j} \quad Y^{m_j+1/2}_{-k_j} \\ 0 \\ 0 \end{pmatrix}.$$

The isomorphism is expressed by the explicit expansion

$$\Psi(x) = \sum \frac{1}{r}\psi^+_{m_j,k_j}(r)\Phi^+_{m_j,k_j} + \frac{1}{r}\psi^-_{m_j,k_j}(r)\Phi^-_{m_j,k_j} \tag{2.41}$$

with

$$\|\Psi\|^2_{L^2} = \sum \int_0^\infty \left[\left|\psi^+_{m_j,k_j}\right|^2 + \left|\psi^-_{m_j,k_j}\right|^2\right] dr. \tag{2.42}$$

Notice also that

$$\|\Psi\|^2_{L^2_\omega} = \sum \frac{1}{r^2}\left|\psi^+_{m_j,k_j}\right|^2 + \frac{1}{r^2}\left|\psi^-_{m_j,k_j}\right|^2. \tag{2.43}$$

Each $L^2(0,+\infty;dr)\otimes H_{m_j,k_j}$ is an eigenspace of the Dirac operator $\mathcal{D}=i^{-1}\sum\alpha_j\partial_j$ and the action of $\mathcal{D}$ can be written, in terms of the expansion (2.41), as

$$\mathcal{D}\Psi=\sum\left(-\frac{d}{dr}\psi^-_{m_j,k_j}+\frac{k_j}{r}\psi^-_{m_j,k_j}\right)\frac{\Phi^+_{m_j,k_j}}{r}+\left(\frac{d}{dr}\psi^+_{m_j,k_j}+\frac{k_j}{r}\psi^+_{m_j,k_j}\right)\frac{\Phi^-_{m_j,k_j}}{r}.$$

From decomposition (2.41) it is clear that the operator Λ^σ_ω, which acts on spherical harmonics as

$$\Lambda^\sigma_\omega Y^m_\ell=\big(1+\ell(\ell+1)\big)^{\frac{\sigma}{2}}\cdot Y^m_\ell,\tag{2.44}$$

does not commute with $\mathcal{D}$. Indeed, each space H_{m_j,k_j} involves two spherical harmonics Y^m_ℓ with two values of ℓ which differ by 1, and $\mathcal{D}$ swaps them. However, the modified operator $\widetilde{\Lambda}^\sigma_\omega$ defined by

$$\widetilde{\Lambda}^\sigma_\omega\Phi^\pm_{m_j,k_j}=|k_j|^\sigma\Phi^\pm_{m_j,k_j}\tag{2.45}$$

obviously commutes with $\mathcal{D}$, thus estimates (2.36), (2.37) are trivially true if we replace Λ with $\widetilde{\Lambda}$. It remains to show that we obtain equivalent norms. The equivalence

$$\big\|\widetilde{\Lambda}^\sigma_\omega f\big\|_{L^2_\omega}\simeq\big\|\Lambda^\sigma_\omega f\big\|_{L^2_\omega}$$

follows directly from (2.44), (2.45) and (2.43). Moreover, $\widetilde{\Lambda}$ and Λ commute with Δ, hence with $|D|$, and this implies

$$\big\||D|\Lambda^s_\omega f\big\|_{L^2}\simeq\big\||D|\widetilde{\Lambda}^s_\omega f\big\|_{L^2}$$

or, equivalently,

$$\big\|\Lambda^s_\omega f\big\|_{\dot H^1}\simeq\big\|\widetilde{\Lambda}^s_\omega f\big\|_{\dot H^1}.$$

This is sufficient to prove (2.36). Since $\widetilde{\Lambda}$ and Λ also commute with radial weights we have

$$\big\|\langle x\rangle^{\frac{1}{2}+}|D|\Lambda^s_\omega f\big\|_{L^2}\simeq\big\|\langle x\rangle^{\frac{1}{2}+}|D|\widetilde{\Lambda}^s_\omega f\big\|_{L^2}$$

which gives (2.37). □

3. The wave equations with potential

Our next goal is to extend the results of previous section to the case of perturbed flows. This will be obtained by a perturbative argument, relying on the smoothing estimates of [4] and the mixed Strichartz-smoothing estimates of the previous section. In [4] smoothing estimates were proved for several classes of dispersive equations perturbed with electromagnetic potentials (while the 1D case was analyzed in [2]). For the wave equation in dimension $n\geqslant 3$ the estimates are the following:

Proposition 3.1. *Let $n\geqslant 3$. Assume the operator*

$$-\Delta+W(x,D)=-\Delta+a(x)\cdot\nabla+b_1(x)+b_2(x)$$

is selfadjoint and its coefficients satisfy

$$\left|a(x)\right| \leqslant \frac{\delta}{|x|^{1-\epsilon} + |x|^2 |\log|x||^{\sigma}}, \tag{3.1}$$

$$\left|b_1(x)\right| \leqslant \frac{\delta}{|x|^{1-\epsilon} + |x|^2}, \qquad 0 \leqslant b_2(x) \leqslant \frac{C}{|x|^{1-\epsilon} + |x|^2} \tag{3.2}$$

for some $\delta, \epsilon > 0$ sufficiently small and some $\sigma > 1/2$, $C > 0$. Moreover assume that 0 is not a resonance for $-\Delta + b_2$. Then the following smoothing estimate holds

$$\left\| \left(|x|^{\frac{1}{2}-\epsilon} + |x|\right)^{-1} e^{it\sqrt{-\Delta+W}} f \right\|_{L^2 L^2} \lesssim \|f\|_{L^2}. \tag{3.3}$$

The assumption that 0 is not a resonance for $-\Delta + b_2(x)$ here means: if $(-\Delta + b_2) f = 0$ and $\langle x\rangle^{-1} f \in L^2$ then $f \equiv 0$.

Combining Proposition 3.1 with (2.27) we obtain the following Strichartz endpoint estimate for the 3D wave equation perturbed with an electric potential:

Theorem 3.2. *Let $n = 3$ and consider the Cauchy problem for the wave equation*

$$u_{tt} - \Delta u + V(x) u = F, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x)$$

under the assumptions:

(i) *$V(x)$ is real valued and the positive and negative parts $V_\pm$ satisfy*

$$V_+ \leqslant \frac{C}{|x|^{\frac{1}{2}-\epsilon} + |x|^2}, \qquad V_- \leqslant \frac{\delta}{|x|^{\frac{1}{2}-\epsilon} + |x|^2} \tag{3.4}$$

for some δ, ϵ sufficiently small and some $C \geqslant 0$;

(ii) *$-\Delta + V$ is selfadjoint*;

(iii) *0 is not a resonance for $-\Delta + V_-$.*

Then the solution $u(t,x)$ satisfies the endpoint Strichartz estimate

$$\|u\|_{L^2_t L^\infty_r L^2_\omega} \lesssim \|f\|_{\dot H^1} + \|g\|_{L^2} + \left\| \langle x\rangle^{\frac{1}{2}+} F \right\|_{L^2_t L^2_x}. \tag{3.5}$$

Proof. We represent $u(t,x)$ in the form

$$u(t,x) = I + II - III$$

where

$$I = \cos\bigl(t|D|\bigr) f + \frac{\sin(t|D|)}{|D|} g,$$

$$II = \int_0^t |D|^{-1} \sin\bigl((t-s)|D|\bigr) F \, ds,$$

and

$$III = \int_0^t |D|^{-1} \sin\bigl((t-s)|D|\bigr) V u \, ds.$$

We can use (2.5) to estimate I and (2.27) to estimate II in the norm $L_t^2 L_r^\infty L_\omega^2$ directly. On the other hand, applying (2.27) to III we get

$$\|III\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \bigl\| \langle x\rangle^{\frac{1}{2}+} V u \bigr\|_{L^2 L^2} \leqslant \bigl\| \langle x\rangle^{\frac{1}{2}+} \tau_\epsilon V \bigr\|_{L_x^\infty} \bigl\| \tau_\epsilon^{-1} u \bigr\|_{L^2 L^2}.$$

By assumption $\langle x\rangle^{\frac{1}{2}+} \tau_\epsilon V$ is bounded on $\mathbb{R}^n$, moreover we are allowed to use (3.3) since V satisfies the assumptions of Proposition 3.1. Notice that (3.3) implies

$$\bigl\| \tau_\epsilon^{-1} u \bigr\|_{L^2 L^2} \lesssim \|f\|_{L^2} + \bigl\| |D|^{-1} g \bigr\|_{L^2}$$

and in conclusion we have proved

$$\|III\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|f\|_{L^2} + \bigl\| |D|^{-1} g \bigr\|_{L^2}$$

which completes the proof of (3.5). □

Analogous estimates can be proved for the Klein–Gordon equation, or in higher dimension $n \geqslant 3$, for first order perturbations, and for angular derivatives of the solutions. We omit the details since we prefer to focus on the Dirac equation here.

4. The Dirac equation with potential

We consider now the perturbed Dirac operator $\mathcal{D} + V(x)$ where $V(x)$ is a small 4×4 hermitian matrix valued potential. We prove here more general versions of the estimates given in [4,3] in order to include angular regularity. We begin with the free Dirac equation:

Proposition 4.1. *The free Dirac flow satisfies, for all $\sigma > 1$ and $s \geqslant 0$, the smoothing estimates (with $w_\sigma(x) = |x|(1 + |\log|x||)^\sigma$)*

$$\bigl\| w_\sigma^{-1/2} \Lambda_\omega^s e^{it\mathcal{D}} f \bigr\|_{L_t^2 L_x^2} \lesssim \bigl\| \Lambda_\omega^s f \bigr\|_{L^2} \tag{4.1}$$

and

$$\left\| w_\sigma^{-1/2} \Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F(t')\, dt' \right\|_{L_t^2 L_x^2} \lesssim \bigl\| w_\sigma^{1/2} \Lambda_\omega^s F \bigr\|_{L_t^2 L_x^2}. \tag{4.2}$$

Proof. When $s = 0$, both estimates follow from the resolvent estimate

$$\bigl\| w_\sigma^{-1/2} R_{\mathcal{D}}(z) f \bigr\|_{L^2(\mathbb{R}^3)} \leqslant C \bigl\| w_\sigma^{1/2} f \bigr\|_{L^2(\mathbb{R}^3)}, \quad z \notin \mathbb{R},$$

with a constant uniform in z, proved in [3] using a standard application of Kato's theory (see also [4]). The case $s > 0$ is proved exactly as in Corollary 2.4, first by replacing Λ_ω with $\widetilde{\Lambda}_\omega$ which commutes with the flow, and then by using the equivalence of norms. □

We consider now the case of a perturbed Dirac system

$$iu_t = \mathcal{D}u + Vu$$

where $V(x)$ is a 4×4 matrix potential. If V is hermitian and its weak $L^{3,\infty}$ norm is small enough, the operator $\mathcal{D} + V$ is selfadjoint as proved in [3]. In all of the following results the assumptions on the potential are somewhat stronger than this, so in all cases the unitary flow $e^{it(\mathcal{D}+V)}$ will be well defined and continuous on $L^2(\mathbb{R}^3)^4$ by spectral theory.

Proposition 4.2. *Let $V(x)$ be a hermitian 4×4 matrix on $\mathbb{R}^3$ such that*

$$\left|V(x)\right| \leqslant \frac{\delta}{w_\sigma(|x|)}, \qquad w_\sigma(r) = r \cdot \left(1 + |\log r|\right)^\sigma \tag{4.3}$$

for some $\delta > 0$ sufficiently small and some $\sigma > 1$. Then the perturbed Dirac flow $e^{it(\mathcal{D}+V)}$ satisfies the smoothing estimates

$$\left\| w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f \right\|_{L_t^2 L_x^2} \lesssim \|f\|_{L^2}, \tag{4.4}$$

$$\left\| w_\sigma^{-1/2} \int_0^t e^{i(t-t')(\mathcal{D}+V)} F(t')\, dt' \right\|_{L_t^2 L_x^2} \lesssim \left\| w_\sigma^{1/2} F \right\|_{L_t^2 L_x^2}. \tag{4.5}$$

If in addition V satisfies for some $s > 1$ the condition

$$\left\| \Lambda_\omega^s V(r\,\cdot) \right\|_{L^2(\mathbb{S}^2)} \leqslant \frac{\delta}{w_\sigma(r)}, \tag{4.6}$$

then we have, for all $0 \leqslant s \leqslant 2$, the estimates with angular regularity

$$\left\| w_\sigma^{-1/2} \Lambda_\omega^s e^{it(\mathcal{D}+V)} f \right\|_{L_t^2 L_x^2} \lesssim \left\| \Lambda_\omega^s f \right\|_{L^2}, \tag{4.7}$$

$$\left\| w_\sigma^{-1/2} \Lambda_\omega^s \int_0^t e^{i(t-t')(\mathcal{D}+V)} F(t')\, dt' \right\|_{L_t^2 L_x^2} \lesssim \left\| w_\sigma^{1/2} \Lambda_\omega^s F \right\|_{L_t^2 L_x^2}. \tag{4.8}$$

Proof. If u solves

$$iu_t = \mathcal{D}u + Vu + F, \qquad u(0) = f$$

we can write

$$u = e^{it\mathcal{D}} f + i \int_0^t e^{i(t-t')\mathcal{D}} \left[Vu(t') + F(t') \right] dt'.$$

Using (4.1), (4.2) with $s = 0$ and assumption (4.3) we get

$$\begin{aligned} \left\| w_\sigma^{-1/2} u \right\|_{L_t^2 L_x^2} &\lesssim \|f\|_{L^2} + \left\| w_\sigma^{1/2} [Vu + F] \right\|_{L^2 L^2} \\ &\leqslant \|f\|_{L^2} + \delta \left\| w_\sigma^{-1/2} u \right\|_{L^2 L^2} + \left\| w_\sigma^{1/2} F \right\|_{L^2 L^2}. \end{aligned}$$

If δ is sufficiently small this implies both (4.4) and (4.5).

To prove (4.7), (4.8) we proceed in a similar way using again (4.1) and (4.2):

$$\left\| w_\sigma^{-1/2} \Lambda_\omega^s u \right\|_{L_t^2 L_x^2} \lesssim \left\| \Lambda_\omega^s f \right\|_{L^2} + \left\| w_\sigma^{1/2} \Lambda_\omega^s (Vu + F) \right\|_{L^2 L^2}.$$

We shall need the following fairly elementary product estimate involving the angular derivative operator Λ_ω

$$\left\| \Lambda_\omega^s (gh) \right\|_{L_\omega^2(\mathbb{S}^2)} \lesssim \left\| \Lambda_\omega^s g \right\|_{L_\omega^2(\mathbb{S}^2)} \left\| \Lambda_\omega^s h \right\|_{L_\omega^2(\mathbb{S}^2)} \tag{4.9}$$

which holds provided $s > 1$. This estimate can be proved e.g. by localizing the norm on the sphere via a finite partition of unity, and then applying in each coordinate patch a standard product estimate in the Sobolev space $H^s(\mathbb{R}^2)$, $s > 1$.

Applying (4.9), and using assumption (4.6), we have

$$\left\| w_\sigma^{1/2} \Lambda_\omega^s (Vu + F) \right\|_{L^2 L^2} \leqslant \delta \left\| w_\sigma^{-1/2} \Lambda_\omega^s u \right\|_{L^2 L^2} + \left\| w_\sigma^{1/2} \Lambda_\omega^s F \right\|_{L^2 L^2}$$

and the proof is concluded as above. □

We note the following consequence of (4.4):

Corollary 4.3. *Assume that the hermitian matrix $V(x)$ satisfies, for δ sufficiently small, C arbitrary and $\sigma > 1$ (with $w_\sigma(r) = r(1 + |\log r|)^\sigma$)*

$$\left| V(x) \right| \leqslant \frac{\delta}{w_\sigma(|x|)}, \qquad \left| \nabla V(x) \right| \leqslant \frac{C}{w_\sigma(|x|)}. \tag{4.10}$$

Then besides (4.4) we have the estimate for the derivatives of the flow

$$\left\| w_\sigma^{-1/2} \nabla e^{it(\mathcal{D}+V)} f \right\|_{L_t^2 L_x^2} \lesssim \| f \|_{H^1}. \tag{4.11}$$

If in addition we assume that, for some $s > 1$,

$$\left\| \Lambda_\omega^s V(r\,\cdot) \right\|_{L^2(\mathbb{S}^2)} \leqslant \frac{\delta}{w_\sigma(r)}, \qquad \left\| \Lambda_\omega^s \nabla V(r\,\cdot) \right\|_{L^2(\mathbb{S}^2)} \leqslant \frac{C}{w_\sigma(r)}, \tag{4.12}$$

then we have the following estimate with angular regularity

$$\left\| w_\sigma^{-1/2} \nabla \Lambda_\omega^s e^{it(\mathcal{D}+V)} f \right\|_{L_t^2 L_x^2} \lesssim \left\| \Lambda_\omega^s f \right\|_{H^1}. \tag{4.13}$$

Proof. Assume at first $s = 0$ and let $u = e^{it(\mathcal{D}+V)} f$. Each derivative $u_j = \partial_j u$ satisfies an equation like

$$i\partial_t u_j = \mathcal{D} u_j + V u_j + V_j u, \quad V_j = \partial_j V, \qquad u_j(0, x) = f_j = \partial_j f$$

so we can represent it in the form

$$u_j = e^{it(\mathcal{D}+V)} f_j + i \int_0^t e^{i(t-s)(\mathcal{D}+V)} V_j u \, ds.$$

To the first term at the r.h.s. we can apply estimate (4.4) obtaining

$$\left\| w_\sigma^{-1/2} e^{it(\mathcal{D}+V)} f_j \right\|_{L_t^2 L_x^2} \lesssim \|f\|_{\dot{H}^1}.$$

To handle the second term we use (4.5):

$$\left\| w_\sigma^{-1/2} \int e^{i(t-s)(\mathcal{D}+V)} V_j u \right\|_{L^2} \lesssim \left\| w_\sigma^{1/2} V_j u \right\|_{L_t^2 L_x^2} \leqslant \|w_\sigma V_j\|_{L_x^\infty} \left\| w_\sigma^{-1/2} u \right\|_{L_t^2 L_x^2}$$

and again by (4.4) and by the assumption on ∇V we conclude the proof of (4.11).

For the proof of (4.13) we apply to the equation for u the operator $|D|$ which commutes with $\mathcal{D}$:

$$i\partial_t\big(|D|u\big) = \mathcal{D}\big(|D|u\big) + |D|(Vu)$$

and we use estimates (4.7), (4.8), obtaining

$$\left\| w_\sigma^{-1/2} |D| \Lambda_w^s u \right\|_{L^2 L^2} \lesssim \left\| \Lambda_w^s f \right\|_{\dot{H}^1} + \left\| w_\sigma^{1/2} |D| \Lambda_w^s (Vu) \right\|_{L^2 L^2}.$$

Now in the last term we commute $|D|$ with Λ and we notice that we can replace $|D|$ by ∇ obtaining an equivalent norm. This gives

$$\left\| w_\sigma^{1/2} |D| \Lambda_w^s (Vu) \right\|_{L^2 L^2} \leqslant \left\| w_\sigma^{1/2} \Lambda_w^s (\nabla V) u \right\|_{L^2 L^2} + \left\| w_\sigma^{1/2} \Lambda_w^s V (\nabla u) \right\|_{L^2 L^2}.$$

We can now apply the product estimate (4.9) and assumptions (4.12); proceeding as in the first part of the proof we finally obtain (4.13). □

By a similar perturbative argument, we obtain the endpoint Strichartz estimates for the Dirac equation with potential. Notice that in the version of this theorem given in the Introduction (Theorem 1.2) we used an equivalent formulation in terms of the potential $v_\sigma(x) = |x|^{\frac{1}{2}} |\log|x||^\sigma + \langle x \rangle^{1+\sigma}$.

Theorem 4.4. *Assume that the hermitian matrix $V(x)$ satisfies, for δ sufficiently small, C arbitrary and $\sigma > 1$ (with $w_\sigma(r) = r(1 + |\log r|)^\sigma$)*

$$\big|V(x)\big| \leqslant \frac{\delta}{\langle x \rangle^{\frac{1}{2}+} w_\sigma(|x|)^{\frac{1}{2}}}, \qquad \big|\nabla V(x)\big| \leqslant \frac{C}{\langle x \rangle^{\frac{1}{2}+} w_\sigma(|x|)^{\frac{1}{2}}}. \tag{4.14}$$

Then the perturbed Dirac flow satisfies the endpoint Strichartz estimate

$$\left\| e^{it(\mathcal{D}+V)} f \right\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \|f\|_{H^1}. \tag{4.15}$$

If instead we make the following assumption (which implies (4.14)): for some $s > 1$,

$$\left\| \Lambda_\omega^s V(r\,\cdot) \right\|_{L^2(\mathbb{S}^2)} \leqslant \frac{\delta}{\langle r \rangle^{\frac{1}{2}+} w_\sigma(r)^{\frac{1}{2}}}, \qquad \left\| \Lambda_\omega^s \nabla V(r\,\cdot) \right\|_{L^2(\mathbb{S}^2)} \leqslant \frac{C}{\langle r \rangle^{\frac{1}{2}+} w_\sigma(r)^{\frac{1}{2}}}, \tag{4.16}$$

then we have the endpoint estimate with angular regularity

$$\left\| \Lambda_\omega^s e^{it(\mathcal{D}+V)} f \right\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \left\| \Lambda_\omega^s f \right\|_{H^1} \tag{4.17}$$

and the energy estimate with angular regularity

$$\left\| \Lambda_\omega^s e^{it(\mathcal{D}+V)} f \right\|_{L_t^\infty H^1} \lesssim \left\| \Lambda_\omega^s f \right\|_{H^1}. \tag{4.18}$$

Proof. Consider first (4.15). Notice that V satisfies in particular the assumptions of Corollary 4.3. We can write

$$e^{it(\mathcal{D}+V)} f = I + II \tag{4.19}$$

with

$$I = e^{it\mathcal{D}} f, \qquad II = i \int_0^t e^{i(t-s)\mathcal{D}} V u \, ds.$$

The term I is estimated directly using (2.36) with $s = 0$. On the other hand, applying (2.37) to the term II we get

$$\|II\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} |D| (V u) \right\|_{L^2 L^2}.$$

Now we recall that the Riesz operators $|D|^{-1}\nabla$ are bounded on weighted L^2 spaces with A_2 weights, and $\langle x \rangle^s$ belongs to this class provided $s < n/2$ (see [21]). Thus we can continue the chain of inequalities as follows:

$$= \left\| \langle x \rangle^{\frac{1}{2}+} |D|^{-1} \nabla |D| (V u) \right\|_{L^2 L^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} \nabla (V u) \right\|_{L^2 L^2} \lesssim A + B$$

where

$$A = \left\| \langle x \rangle^{\frac{1}{2}+} (\nabla V) u \right\|_{L^2 L^2}, \qquad B = \left\| \langle x \rangle^{\frac{1}{2}+} V \nabla u \right\|_{L^2 L^2}.$$

Then we have

$$A \leqslant \left\| \langle x \rangle^{\frac{1}{2}+} w_\sigma^{\frac{1}{2}} \nabla V \right\|_{L^\infty} \left\| w_\sigma^{-\frac{1}{2}} u \right\|_{L^2 L^2} \lesssim \| f \|_{L^2}$$

by the assumptions on ∇V and (4.4), while

$$B \leqslant \left\| \langle x \rangle^{\frac{1}{2}+} w_\sigma^{\frac{1}{2}} V \right\|_{L^\infty} \left\| w_\sigma^{-\frac{1}{2}} \nabla u \right\|_{L^2 L^2} \lesssim \| f \|_{H^1}$$

by (4.11). Summing up, we arrive at (4.15).

The proof of (4.17) is similar. We estimate I using (2.36). Applying (2.37) to the term II we get

$$\left\| \Lambda_\omega^s II \right\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} |D| \Lambda_\omega^s (V u) \right\|_{L^2 L^2}.$$

Then we commute $|D|$ with Λ_ω, and we can replace the operator $|D|$ with ∇ since the norm is equivalent; we arrive at

$$\left\| \Lambda_\omega^s II \right\|_{L_t^2 L_r^\infty L_\omega^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s (\nabla V) u \right\|_{L^2 L^2} + \left\| \langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s V (\nabla u) \right\|_{L^2 L^2}.$$

Now we use the product estimate (4.9) and assumptions (4.16) to obtain

$$\lesssim C\left\|w_\sigma^{-1/2}\Lambda_\omega^s u\right\|_{L^2L^2}+\delta\left\|w_\sigma^{-1/2}\Lambda_\omega^s\nabla u\right\|_{L^2L^2}$$

and recalling the smoothing estimates (4.7), (4.13) we conclude the proof of (4.17).

It remains to prove (4.18). Consider first the free case $V\equiv 0$. We have the conservation laws

$$\left\|e^{it\mathcal{D}}f\right\|_{L^\infty L^2}\equiv\|f\|_{L^2},\qquad\left\|\mathcal{D}e^{it\mathcal{D}}f\right\|_{L^\infty L^2}\equiv\|\mathcal{D}f\|_{L^2}\tag{4.20}$$

which imply

$$\left\|e^{it\mathcal{D}}f\right\|_{L^\infty H^1}\simeq\|f\|_{H^1}$$

since $\|\mathcal{D}f\|_{L^2}\simeq\|f\|_{\dot H^1}$. Moreover, the operator $\widetilde{\Lambda}_\omega$ introduced in (2.45) commutes with $\mathcal{D}$, so that we have for all $s\geqslant 0$

$$\left\|\widetilde{\Lambda}_\omega^s e^{it\mathcal{D}}f\right\|_{L^\infty H^1}\equiv\|\widetilde{\Lambda}_\omega f\|_{H^1}$$

and switching back to the equivalent operator Λ_ω as in the proof of Corollary 2.4 we obtain

$$\left\|\Lambda_\omega^s e^{it\mathcal{D}}f\right\|_{L^\infty H^1}\lesssim\|\Lambda_\omega f\|_{H^1}.\tag{4.21}$$

Consider now the case $V\not\equiv 0$. We start from

$$\left\|\langle x\rangle^{-\frac{1}{2}-}e^{it\mathcal{D}}f\right\|_{L^2L^2}\leqslant\|f\|_{L^2}\tag{4.22}$$

which is a consequence of (4.1) (we relaxed the weight). Taking the dual of (4.22) we get

$$\left\|\int e^{-it'\mathcal{D}}F(t')\,dt'\right\|_{L^2}\lesssim\left\|\langle x\rangle^{\frac{1}{2}+}F\right\|_{L^2L^2}$$

which together with (4.20) gives

$$\left\|\int e^{i(t-t')\mathcal{D}}F(t')\,dt'\right\|_{L^\infty L^2}\lesssim\left\|\langle x\rangle^{\frac{1}{2}+}F\right\|_{L^2L^2}.$$

Now a standard application of Christ–Kiselev' Lemma in the spirit of [13] (see also [4] for the case of Dirac equations) allows to replace the time integral with a truncated integral and we obtain

$$\left\|\int_0^t e^{i(t-t')\mathcal{D}}F(t')\,dt'\right\|_{L^\infty L^2}\lesssim\left\|\langle x\rangle^{\frac{1}{2}+}F\right\|_{L^2L^2}.\tag{4.23}$$

Recalling that the operator $\widetilde{\Lambda}_\omega$ introduced in (2.45) commutes with $\mathcal{D}$, and proceeding as in the proof of Corollary 2.4 we obtain for all $s\geqslant 0$

$$\left\|\Lambda_\omega^s\int_0^t e^{i(t-t')\mathcal{D}}F(t')\,dt'\right\|_{L^\infty L^2}\lesssim\left\|\langle x\rangle^{\frac{1}{2}+}\Lambda_\omega^s F\right\|_{L^2L^2}\tag{4.24}$$

and finally, applying $|D|$ which commutes both with $\mathcal{D}$ and Λ_ω, we have also

$$\left\| \Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F(t')\,dt' \right\|_{L^\infty \dot{H}^1} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s |D| F \right\|_{L^2 L^2}. \tag{4.25}$$

Now we use again the representation (4.19); by (4.21) and (4.25) we can write

$$\left\| \Lambda_\omega^s e^{it(\mathcal{D}+V)} f \right\|_{L^\infty \dot{H}^1} \lesssim \left\| \Lambda_\omega^s f \right\|_{H^1} + \left\| \langle x \rangle^{\frac{1}{2}+} \Lambda_\omega^s |D| (Vu) \right\|_{L^2 L^2}$$

and proceeding exactly as in the first part of the proof we arrive at (4.18). □

5. The nonlinear Dirac equation

Theorem 4.4 contains all the necessary tools to prove global well posedness for the cubic nonlinear Dirac equation

$$iu_t = \mathcal{D}u + Vu + P_3(u, \bar{u}), \qquad u(0,x) = f(x). \tag{5.1}$$

Our result is the following:

Theorem 5.1. *Consider the perturbed Dirac system (5.1), where the 4×4 matrix valued potential $V = V(|x|)$ is hermitian and satisfies assumptions (4.16). Let $P_3(u, \bar{u})$ be a $\mathbb{C}^4$-valued homogeneous cubic polynomial. Then for any $s > 1$ there exists ϵ_0 such that for all initial data satisfying*

$$\left\| \Lambda_\omega^s f \right\|_{H^1} < \epsilon_0 \tag{5.2}$$

the Cauchy problem (5.1) admits a unique global solution $u \in CH^1 \cap L^2 L^\infty$ with $\Lambda_\omega^s u \in L^\infty H^1$.

Proof. The proof is based on a fixed point argument in the space X defined by the norm

$$\|u\|_X := \left\| \Lambda_\omega^s u \right\|_{L_t^2 L_r^\infty L_\omega^2} + \left\| \Lambda_\omega^s u \right\|_{L_t^\infty H_x^1}. \tag{5.3}$$

Notice that in Theorem 4.4 we proved the estimate

$$\left\| e^{it(\mathcal{D}+V)} f \right\|_X \lesssim \left\| \Lambda_\omega^s \right\|_{H^1}. \tag{5.4}$$

Define $u = \Phi(v)$ for $v \in X$ as the solution of the linear problem

$$iu_t = \mathcal{D}u + Vu + P(v, \bar{v}), \qquad u(0,x) = f(x) \tag{5.5}$$

and represent u as

$$u = \Phi(v) = e^{it(\mathcal{D}+V)} f + i \int_0^t e^{i(t-t')(\mathcal{D}+V)} P\big(v(t'), \overline{v(t')}\big)\,dt'.$$

We recall now the product estimate

$$\left\| \Lambda_\omega^s(gh) \right\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \left\| \Lambda_\omega^s g \right\|_{L^2_\omega(\mathbb{S}^2)} \left\| \Lambda_\omega^s h \right\|_{L^2_\omega(\mathbb{S}^2)}$$

(see (4.9)). Then we have, by (5.4)

$$\|u\|_X \lesssim \left\| \Lambda_\omega^s f \right\|_{H^1} + \int_0^\infty \left\| e^{i(t-t')\mathcal{D}} P\big(v(t'), \overline{v(t')}\big) \right\|_X dt'$$
$$\lesssim \left\| \Lambda_\omega^s f \right\|_{H^1} + \int_0^\infty \left\| \Lambda_\omega^s P\big(v(t'), \overline{v(t')}\big) \right\|_{H^1} dt' \equiv \left\| \Lambda_\omega^s f \right\|_{H^1} + \left\| \Lambda_\omega^s P(v, \bar{v}) \right\|_{L^1 H^1}.$$

By (4.9) we have

$$\left\| \Lambda_\omega^s \big(v^3\big) \right\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \left\| \Lambda_\omega^s v \right\|^3_{L^2_\omega(\mathbb{S}^2)}$$

whence

$$\left\| \Lambda_\omega^s \big(v^3\big) \right\|_{L^2_x} \lesssim \left\| \Lambda_\omega^s v \right\|_{L^2_x} \left\| \Lambda_\omega^s v \right\|^2_{L^\infty_r L^2_\omega}$$

and

$$\left\| \Lambda_\omega^s \big(v^3\big) \right\|_{L^1_t L^2_x} \lesssim \left\| \Lambda_\omega^s v \right\|_{L^\infty_t L^2_x} \left\| \Lambda_\omega^s v \right\|^2_{L^2_t L^\infty_r L^2_\omega} \leqslant \|v\|^3_X. \tag{5.6}$$

In a similar way,

$$\left\| \Lambda_\omega^s \nabla \big(v^3\big) \right\|_{L^2_\omega(\mathbb{S}^2)} \lesssim \left\| \Lambda_\omega^s \nabla v \right\|_{L^2_\omega(\mathbb{S}^2)} \left\| \Lambda_\omega^s v \right\|^2_{L^2_\omega(\mathbb{S}^2)}$$

so that

$$\left\| \Lambda_\omega^s \nabla \big(v^3\big) \right\|_{L^2_x} \lesssim \left\| \Lambda_\omega^s \nabla v \right\|_{L^2_x} \left\| \Lambda_\omega^s v \right\|^2_{L^\infty_r L^2_\omega}$$

and

$$\left\| \Lambda_\omega^s \nabla \big(v^3\big) \right\|_{L^1_t L^2_x} \lesssim \left\| \Lambda_\omega^s \nabla v \right\|_{L^\infty_t L^2_x} \left\| \Lambda_\omega^s v \right\|^2_{L^2_t L^\infty_r L^2_\omega} \leqslant \|v\|^3_X. \tag{5.7}$$

In conclusion, (5.6) and (5.7) imply

$$\left\| \Lambda_\omega^s P(v, \bar{v}) \right\|_{L^1 H^1} \lesssim \|v\|^3_X$$

and the estimate for $u = \Phi(v)$ is

$$\|u\|_X \equiv \left\| \Phi(v) \right\|_X \lesssim \left\| \Lambda_\omega^s f \right\|_{H^1} + \|v\|^3_X.$$

An analogous computation gives the estimate

$$\left\| \Phi(v) - \Phi(w) \right\|_X \lesssim \|v - w\|_X \cdot \big(\|v\|_X + \|w\|_X\big)^2$$

and an application of the contraction mapping theorem concludes the proof. □

Acknowledgments

We are indebted with Ilia Krasikov and Jim Wright for invaluable conversations which helped shape up the proofs in Section 2.

References

[1] Milton Abramowitz, Irene A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Natl. Bur. Stand., Appl. Math. Ser., vol. 55, US Government Printing Office, Washington, DC, 1964, for sale by the Superintendent of Documents.
[2] Piero D'Ancona, Luca Fanelli, L^p-boundedness of the wave operator for the one dimensional Schrödinger operator, Comm. Math. Phys. 268 (2) (2006) 415–438.
[3] Piero D'Ancona, Luca Fanelli, Decay estimates for the wave and Dirac equations with a magnetic potential, Comm. Pure Appl. Math. 60 (3) (2007) 357–392.
[4] Piero D'Ancona, Luca Fanelli, Strichartz and smoothing estimates of dispersive equations with magnetic potentials, Comm. Partial Differential Equations 33 (4–6) (2008) 1082–1112.
[5] Piero D'Ancona, Damiano Foschi, Sigmund Selberg, Local well-posedness below the charge norm for the Dirac–Klein–Gordon system in two space dimensions, J. Hyperbolic Differ. Equ. 4 (2) (2007) 295–330.
[6] Piero D'Ancona, Damiano Foschi, Sigmund Selberg, Null structure and almost optimal local regularity for the Dirac–Klein–Gordon system, J. Eur. Math. Soc. (JEMS) 9 (4) (2007) 877–899.
[7] João-Paulo Dias, Mário Figueira, Global existence of solutions with small initial data in H^s for the massive nonlinear Dirac equations in three space dimensions, Boll. Unione Mat. Ital. Sez. B (7) 1 (3) (1987) 861–874.
[8] Miguel Escobedo, Luis Vega, A semilinear Dirac equation in $H^s(\mathbf{R}^3)$ for $s > 1$, SIAM J. Math. Anal. 28 (2) (1997) 338–362.
[9] Daoyuan Fang, Chengbo Wang, Some remarks on Strichartz estimates for homogeneous wave equation, Nonlinear Anal. 65 (3) (2006) 697–706.
[10] Daoyuan Fang, Chengbo Wang, Weighted Strichartz estimates with angular regularity and their applications, 2008.
[11] Jean Ginibre, Giorgio Velo, Generalized Strichartz inequalities for the wave equation, in: Partial Differential Operators and Mathematical Physics, Holzhau, 1994, in: Oper. Theory Adv. Appl., vol. 78, Birkhäuser, Basel, 1995, pp. 153–160.
[12] Chengbo Wang Jin-Cheng Jiang, Xin Yu, Generalized and weighted Strichartz estimates, 2010.
[13] Markus Keel, Terence Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (5) (1998) 955–980.
[14] Sergiu Klainerman, Matei Machedon, Space–time estimates for null forms and the local existence theorem, Comm. Pure Appl. Math. 46 (9) (1993) 1221–1268.
[15] Ilia Krasikov, On the maximum value of Jacobi polynomials, J. Approx. Theory 136 (1) (2005) 1–20.
[16] Shuji Machihara, Makoto Nakamura, Kenji Nakanishi, Tohru Ozawa, Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation, J. Funct. Anal. 219 (1) (2005) 1–20.
[17] Shuji Machihara, Makoto Nakamura, Tohru Ozawa, Small global solutions for nonlinear Dirac equations, Differential Integral Equations 17 (5–6) (2004) 623–636.
[18] Yves Moreau, Existence de solutions avec petite donnée initiale dans H^2 pour une équation de Dirac non linéaire, in: Workshop on Hyperbolic Systems and Mathematical Physics, Lisbon, 1988, Port. Math. 46 (Suppl.) (1989) 553–565.
[19] Branko Najman, The nonrelativistic limit of the nonlinear Dirac equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1) (1992) 3–12.
[20] Michael Reed, Abstract Non-Linear Wave Equations, Lecture Notes in Math., vol. 507, Springer-Verlag, Berlin, 1976.
[21] Elias M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, in: Princet. Math. Ser., vol. 43, Princeton University Press, Princeton, NJ, 1993, with the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[22] Elias M. Stein, Guido Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princet. Math. Ser., vol. 32, Princeton University Press, Princeton, NJ, 1971.
[23] Bernd Thaller, The Dirac equation, Texts Monogr. Phys., Springer-Verlag, Berlin, 1992.