CELLULAR NETWORKS OF OSCILLATORS

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Abstract. The problem of using dynamical equilibria of a Cellular Neural Network (CNN) of relaxation oscillators is faced. The behavior of a planar matrix and of a two-torus of connected oscillators is investigated by computer simulation. Some theorems are stated in the case of: i) two oscillators, ii) a tree of oscillators, iii) a loop of oscillators.

Introduction. It is well known that among neural networks, CNN's [1] exhibit the advantage of possessing only local connections. This property is important with respect to their implementation by means of dedicated integrated circuits.

On the other hand, some recent studies put into evidence that many biological systems behave as distributed oscillatory systems i.e. large arrays (generally planar) of coupled oscillators. A good conjecture, based on results of experiments performed on the visual cortex of cats and the olfactory cortex of rabbits [2,3], seems to be that synchronization on different attractors emerging from a chaotic substratum could be an efficient way of storing and processing information.

These two facts induced us to start studying a new architecture possessing these two distinctive features:

1) Its topology is similar to that of a planar Cellular Neural Network with connectivity 1, but with links only along rows and columns of a matrix (no diagonal connections);

2) Neurons are oscillators instead of nonlinear controlled sources. We chose to use relaxation oscillators [4] for two reasons: i) Computer simulations can result more exact as in this case integration is straightforward; ii) Hardware implementation can take advantage of using only digital circuits.

Therefore we study a matrix of connected oscillators, the behavior of which is of course dependent on many parameters: the natural frequency of each oscillator, the strength of each coupling, the initial state of the system. As a natural extension, we will consider also the same matrix wound on a two-torus, even if the analogy with biological structures becomes less evident in this case.

At our best knowledge it is possible to find in literature only studies on the existence of periodic attractors for systems of sinusoidal oscillators with random natural frequencies and equal coupling coefficients [5,6].

In the following, we shall denote as k_{ij} the strength of the coupling between oscillator i and j , i.e., the amplitude of the signal arriving to oscillator i from oscillator j . k_{ii} determines the natural frequency of oscillator i .

Experimental results. Due to the amount of parameters we have to deal with, that is very large even in the case of a small (4 by 4) matrix, as a first step we started using the so called "experimental mathematics". Therefore we performed the following experiments:

A) We considered a planar matrix of 16 identical oscillators (i.e. having the same natural frequency) and found the following types of behavior:

1) In the case when the coupling coefficients are all equal to each other, the behavior of the system is chaotic for every initial state. The Lyapunov exponent, computed by following trajectories in state space and applying Wolf's algorithm [7], seems to be everywhere positive, except when starting from the origin of state space, where Lyapunov exponent is zero.

2) In the case when the coupling coefficients satisfy condition

$$\text{sign}(k_{ij}) = -\text{sign}(k_{ji})$$

(opposite-sign condition), the system always reaches a stable oscillatory state at the common natural frequency of the oscillators, with every oscillator differing in phase for $1/4$ period from its neighbors (obvious consistency conditions apply). Magnitude of the coefficients is not important.

The stabilizing effect of the opposite-sign structure is analogous to the case of ordinary Cellular Neural Networks [8].

3) With random coupling coefficients, different stable limit cycles are observed, together with non-periodic evolutions: quasi-periodic or chaotic.

In order to inquire into the structure of the attraction basins of the said limit cycles, we performed simulations taking initial conditions along segments in the 16-dimensional state space. Structures of these sets look not at all trivial, and transitions between neighboring basins often seem to be fractal. However, the presence of a set of clearly distinct stable limit cycles, with attraction basins possessing at least a closed inside, makes us believe that they might lend themselves to be used as "codes" or "memory states" for a content-addressable memory (CAM).

The overall structure of attractors is more dependent on the mutual relations between coupling coefficients (e.g. sign, topology...) than on their magnitude. In next section this is proved on simpler structures.

Another set of experiments was performed in order to study the transition between a situation when a stable limit cycle is observed (case 3) and that described under case 1 (transition to chaos). For this purpose, the evolution of the network was simulated for different values of its parameters obtained by moving along a segment in the 48-dimensional space of free coefficients. The results of one of these simulations are shown in figure 1, where the ordinate is the normalized frequency attained at the end of the transient. Zero ordinate means non-periodic behavior.

B) We also considered matrices on a two-torus, which also showed trivial stable behavior under the opposite-sign condition.

In the case of identical coefficients, toroidal matrices show a much larger number of periodic states, with smaller, most probably zero-measure, attraction basins. Two kinds of behavior were observed, the prototypes of which are shown in figures 2 and 3 where the abscissa is taken over a segment, in the 16-dimensional space of initial states, joining two proper sets of them. Figure 2 shows continuous change among indifferent equilibrium limit cycles, while figure 3 shows complex kneading of periodic and non-periodic evolution. As an example of instability, the leftmost state of figures 2 and 3 (the same in the two cases) is periodic, but Lyapunov exponent estimation gives a value of about 1.2.

In the case of coefficients all different from each other, randomly chosen, no periodic evolution has ever been observed.

Theoretical results. We shall consider in this section simpler structures than matrices, in increasing order of complexity. Most of the results described can be straightforwardly extended to sinusoidal oscillator systems.

1) The simplest system consists of two mutually coupled oscillators. Such a system shows a wide spectrum of behaviors: (i) single stable limit cycle, with oscillators operating all at the same or (ii) at different frequencies; (iii) periodic evolution at a frequency dependent on initial conditions; (iv) quasi-periodic or non-periodic motion.

The following results are proved in [9]:

Theorem 1: Necessary and sufficient condition for the two oscillators to evolve at the same frequency is that

$$|k_{12} - k_{21}| \geq |k_{11} - k_{22}|$$

$$k_{22}k_{12} \neq k_{11}k_{21}$$

Duty cycle is 50%.

Theorem 2: If $k_{11}/k_{22} = k_{12}/k_{21} = r$, then the evolution of the system is periodic if r is rational, else quasi-periodic, and the rate of the frequencies of the two oscillators is r .

Theorem 3: If $k_{11} = k_{22}$ and $k_{12} = k_{21}$ (bifurcation condition) the systems evolves periodically with no transitory, because there is an infinity of periodic indifferent equilibrium states.

2) A more complex system is a tree of oscillators. For such a structure it is possible to compute easily a linear relation between the frequencies of the oscillators. The algorithm is described in [9]. It can be extended to a general structure, where also loops are present, but the relation becomes nonlinear, and computational difficulty increases rapidly. The following results are proved in [9]:

Theorem 4: Let

$$D_i^0 = 1 - \sum_{j \in \langle i \rangle} R_{ij} D_j^i$$

$$D_j^i = 1 - \sum_{\substack{k \in \langle j \rangle \\ k \neq i}} R_{jk} D_k^j$$

$$R_{ij} = k_{ij} / k_{ji}$$

If all oscillators evolve at the same frequency f , and $D_i^i = 0$, then f can be computed through the following recurrent expression, starting from any oscillator i of the tree:

$$f = 0.5 N_i^0 / D_i^0 \quad (1)$$

where

$$N_i^0 = k_{ii} - \sum_{j \in \langle i \rangle} R_{ij} N_j^i$$

$$N_j^i = k_{jj} - \sum_{\substack{k \in \langle j \rangle \\ k \neq i}} R_{jk} N_k^j$$

and $\langle i \rangle$ is the set of indices of all oscillators connected to oscillator i .

If all oscillators have the same natural frequency and $D_i^i = 0$

(bifurcation condition) then (1) takes the form $0/0$, and so loses significance; multiple attractors are present (as in the case of two oscillators), depending on initial conditions.

When a loop is considered, the bifurcation condition may be stated as the bifurcation condition on two open chains obtained from it.

It is reasonable to conjecture that the bifurcation condition for matrices corresponds to the fact that any tree that can be extracted from the matrix is in the bifurcation condition.

Conclusion. A planar array of relaxation oscillators has been considered in the view of using it as a new basic structure in the field of Neural Network architectures. Although this goal has not yet been attained, such a structure, never studied before, exhibits a rich and interesting behavior. A systematic research about it was started and some basic results were obtained.

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Fig. 1 - Normalized period vs. normalized abscissa on a segment (48-dimensional space) joining two different sets of coefficients, one corresponding to periodic behavior, the other to chaos (bottom ordinate)

Fig. 2 - Normalized period vs. normalized abscissa on a segment joining two different initial conditions in state space. Infinite periodic attractors of zero measure are shown.

Fig. 3 - Normalized period vs. normalized abscissa on a segment joining two different initial conditions in state space. Mixed behavior of chaos (bottom ordinate) and periodic oscillations is shown.