SAPIENZA
UNIVERSITÀ DI ROMA

Alternative Compactifications of $\mathrm{M}_{g,n}$ via Cluster Algebras and their Birational Geometry

Scuola di Dottorato Vito Volterra
PhD in Mathematics (XXXVII cycle)

Davide Gori
ID number 2002502

Advisor
Prof. Filippo Viviani

Academic Year 2024/2025

Thesis defended on 19 January 2026
in front of a Board of Examiners composed by:

Prof. Luca Battistella
Prof. Giulio Codogni
Prof. Luca Tasin

Thesis reviewed by:

Prof. Jarod Alper
Prof. Maksym Fedorchuk

Alternative Compactifications of $\mathrm{M}_{g,n}$ via Cluster Algebras and their Birational Geometry
PhD thesis. Sapienza University of Rome

Author's email: davide.gori@uniroma1.it

Contents

Abstract

We construct new compactifications of $\mathrm{M}_{g,n}$ as good moduli spaces of moduli stacks of curves with singularities of type A_i for $i \leq 3$. These are all the partial $\mathbb{Q}$-factorizations of $\overline{\mathrm{M}}_{g,n}(7/10)$, the space appearing in the first flip of the Hassett–Keel program, providing a new instance of the modularity principle for the minimal model program of $\overline{\mathrm{M}}_{g,n}$.

We study the stack of curves with ample log dualizing sheaf and singularities of the above type, establishing a characterization of open substacks that admit a proper good moduli space. We then recover the compactifications via semistability with respect to suitable line bundles on $\overline{\mathcal{M}}_{g,n}(7/10)$, the stack of curves appearing in the first flip of the Hassett–Keel program.

Our approach develops a framework for studying semistability with respect to line bundles, revealing a wall-crossing phenomenon in a quotient of the Picard group. In the case of $\overline{\mathcal{M}}_{g,n}(7/10)$, this wall-crossing is given by the cluster fan of certain finite-type cluster algebras.

This work extends the results and answers open questions in a paper by Codogni, Tasin, and Viviani.

Acknowledgments

First of all, I am thankful to my supervisor, Filippo Viviani, for his guidance and constant support throughout these years in Rome. I am especially grateful for the freedom he gave me in pursuing my research, and for his positive mindset, which always helped me stay motivated.

I am grateful to Jarod Alper for inviting me to Seattle during the past year and for our many stimulating conversations. I also wish to thank Luca Francone, Salvatore Stella, and Giovanni Cerulli Irelli, from whom I learned about cluster algebras, whose combinatorial structure has been crucial to this thesis.

I would like to thank Luca Battistella and Andrea Petracci for spotting a gap in a proof appearing in an early draft of this thesis. Furthermore, I would like to thank Ludvig Modin, Michele Pernice, and Andres Fernandez Herrero for many interesting discussions.

I am grateful to the algebra and geometry communities of Tor Vergata and Roma Tre, which provided a welcoming and stimulating environment throughout these years. In particular, I would like to thank Roberto, Nelson, and Guido, who always kindly hosted me at Tor Vergata.

Finally, I would like to thank my university friends. In particular Andrea, Filippo, Frank, Furio, Giacomo, and Luca: your friendship is invaluable. A special thank you goes to my parents and grandparents for their unconditional support. It all began with your love and wouldn't have been possible otherwise.

Chapter 1

Introduction

In this thesis, we address questions arising from the study of the birational geometry of $\overline{\mathrm{M}}_{g,n}$, the Deligne–Mumford compactification of the moduli space of smooth curves. Our approach is inspired by the vague modularity principle, aiming to find modular interpretations for the steps of the minimal model program (MMP) of $\overline{\mathrm{M}}_{g,n}$. With this aim, we will construct many new alternative modular compactifications of $\mathrm{M}_{g,n}$.

In the past years, the birational geometry of moduli spaces of curves has been extensively studied in the literature. We focus on the Hassett–Keel program (HKP), introduced to study the canonical model of $\overline{\mathrm{M}}_{g,n}$. This program consists on the study the models associated with $K_{\overline{\mathcal{M}}_{g,n}} + \psi + \alpha(\delta - \psi)$ as α decreases from 1 to 0:

$$\overline{\mathrm{M}}_{g,n}(\alpha) := \operatorname{Proj} \bigoplus_{m \geq 0} H^0\left(\overline{\mathcal{M}}_{g,n}, \left\lfloor m(K_{\overline{\mathcal{M}}_{g,n}} + \psi + \alpha(\delta - \psi))\right\rfloor\right). \tag{1.1}$$

For $n = 0$, these models were introduced by Hassett, who posed the question of whether these spaces arise naturally as good moduli spaces of a classifying stack of curves [Has05]. A positive answer for the first steps was established by Hassett and Hyeon in [HH09; HH13]. These results were later generalized and extended for $n > 0$ by Alper, Fedorchuk, Smyth, and Van Der Wyck in a series of papers [Alp+17; AFS17a; AFS17b]. As α decreases, a wall-crossing phenomenon occurs, and for sufficiently large α, the spaces arise as good moduli spaces of classifying stacks of curves:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n} \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}^{\mathrm{wps}}_{g,n} \arrow[dd] & \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n} \arrow[l, hook'] \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}) \arrow[dd] & \overline{\mathcal{M}}_{g,n}((\tfrac{7}{10}, \tfrac{2}{3})) \arrow[l, hook'] \arrow[d] \\
\overline{\mathrm{M}}_{g,n} \arrow[dr, "j_1^+"] & & \overline{\mathrm{M}}_{g,n}((\tfrac{9}{11}, \tfrac{7}{10})) = \overline{\mathrm{M}}^{\mathrm{ps}}_{g,n} \arrow[dl, "\sim"'] \arrow[dr, "j_2^+"] & & \overline{\mathrm{M}}_{g,n}((\tfrac{7}{10}, \tfrac{2}{3})) \arrow[dl, "j_2^-"'] \\
& \overline{\mathrm{M}}_{g,n}(\tfrac{9}{11}) & & \overline{\mathrm{M}}_{g,n}(\tfrac{7}{10}) &
\end{tikzcd},$$

where an interval I in parentheses indicates that the space remains the same for all $\alpha \in I$, and the model is obtained at one of these values. Moreover, the vertical maps are good moduli spaces. The stacks $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ and $\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}$ are, respectively, the stacks of pseudostable and weakly pseudostable curves. The latter is the classifying stack of curves with nodal and cuspidal singularities and ample log-dualizing sheaf, while the former is given by the open substack of curves that do not admit an elliptic tail. The stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is defined as the stack of curves with ample log-dualizing sheaf and with nodal, cuspidal, and tacnodal singularities (i.e., of type A_i for $i = 1, 2, 3$) that do not contain nodally or A_3-attached elliptic tails. Moreover, the open substack $\overline{\mathcal{M}}_{g,n}((\frac{9}{11}, \frac{7}{10})) \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is described as the locus where certain combinatorial conditions on the curves are satisfied. For a comprehensive description and additional steps, we refer to [Alp+17].

In the diagram above, the first step is a divisorial contraction, and $\overline{\mathrm{M}}_{g,n}(9/11) = \overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$ is a projective normal $\mathbb{Q}$-factorial irreducible variety. The map j_1^+ corresponds to the contraction of a natural $K_{\overline{\mathrm{M}}_{g,n}}$-negative extremal ray of $\overline{\mathrm{NE}}(\overline{\mathrm{M}}_{g,n})$ generated by a curve C_{ell}, which parameterizes a moving 1-pointed elliptic curve (E, p) attached at p to a fixed $n + 1$-pointed smooth irreducible curve of genus $g - 1$.

Moreover, the maps $j_2^{+/-}$ have a specific interpretation in the context of the MMP: j_2^+ contracts a $K_{\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}}$-negative face of the Mori cone, while j_2^- corresponds to the flip relative to $K_{\overline{\mathrm{M}}_{g,n}} + \psi$.

In this work, we analyze the second step, focusing on the space $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$. This space is a projective normal irreducible variety, which is no longer $\mathbb{Q}$-factorial. This is due to the nature of the map $j_2^+\colon \overline{\mathrm{M}}^{\mathrm{ps}}_{g,n} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, indeed it factors as a divisorial contraction (for $n > 0$) followed by a small contraction: this force a flip in the minimal model program. The map j_2^+ contract an higher dimensional face of the Mori cone of $\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$ (cf. [CTV23, Thm. 6.1]) and the next space in the HKP, $\overline{\mathrm{M}}_{g,n}((\frac{7}{10}, \frac{2}{3}))$, is not $\mathbb{Q}$-factorial as well. This cause a discrepancy between MMP and HKP, that is the main motivation of our work. One of the main results consist in a complete description of all the (partial) $\mathbb{Q}$-factorializations of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, using techniques from the combinatorics of cluster algebras. As a byproduct result, we have that if the F-conjecture holds (see [GKM02, Conj. (0.2)]), then we describe all the possible first steps of the $(K_{\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}} + \psi)$-MMP for $\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}$.

In [CTV23], Codogni, Tasin, and Viviani introduced new compactifications of $\mathrm{M}_{g,n}$ arising as good moduli spaces of open substacks of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$. They described some contraction morphisms and flips between them. Using these results, they constructed several possible second steps of the minimal model program for $\overline{\mathrm{M}}_{g,n}$.

In this work, we provide complete answers to open questions 1, 2, and 3 from [CTV23] and extend some results of [CTV21] to the newly constructed compactifications. These are a consequence of our description of the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, that is carried out as explained in the next section.

1.1 Main Results

The results of this thesis are naturally divided into three parts. First, we establish a general framework for the study of Θ-semistability (or simply semistability) on algebraic stacks, with refined conclusions when they admit a good moduli space. Next, we shift our focus to the stack $\widetilde{\mathcal{M}}^i_{g,n}$, defined below, in two directions: we study the general problem of identifying open substacks that admit good moduli spaces, and we describe compactifications arising from the semistability of the stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$.

1.1.1 Semistability respect to line bundles

In this part of the work, we review the classical notion of Θ-(semi)stability, that we simply call (semi)stability, with respect to a line bundle on an algebraic stack $\mathcal{X}$ (cf. [Hei17, Def. 1.2]). Our analysis is carried out within the framework of *Beyond GIT*, as introduced in [Hal14], where the term was coined. We define a space of stability conditions $\mathbb{E}(\mathcal{X})$ as a $\mathbb{Q}$-vector space obtained as a quotient of $\operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$ (cf. Definition 2.1.29). We recall here the notion of Θ-covering, that is suggestive of our approach:

Definition 1.1.1 (Θ-Covering)**.** A map $f\colon \mathcal{X} \to \mathcal{Y}$ among algebraic stacks is said to be a Θ-covering if any map $\Theta := [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{Y}$ has a lift, up to an endomorphism given by $t \mapsto t^n$ $(n > 0)$ at the level of the smooth presentation $\mathbb{A}^1 \to \Theta$.

$$\begin{tikzcd} & & \mathcal{X} \arrow[d, "f"] \\ \Theta \arrow[r, "\cdot n"] \arrow[urr, dashed] & \Theta \arrow[r] & \mathcal{Y} \end{tikzcd}.$$

Then the first main result of the section is a stronger version of the following (cf. Theorem 2.1.36, Theorem 2.2.7)

Theorem A (Wall-Crossing)**.** *Let $\mathcal{X}$ be an algebraic stack admitting a Θ-covering $[\operatorname{Spec} A/G] \to \mathcal{X}$, where A is a finitely generated algebra over an algebraically closed field k, and G is affine linearly reductive. Then there exists a finite-dimensional $\mathbb{Q}$-vector space, that is a quotient of $\operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$, and which admits a decomposition into rational polyhedral regions $\{\mathcal{C}_i\}_{i\in I}$:*

$$\mathbb{E}(\mathcal{X}) := \bigsqcup_{i\in I} \mathcal{C}_i,$$

where two line bundles of $\mathcal{X}$ lie in the same region $\mathcal{C}_i$ if and only if the Θ-semistable points with respect to the two line bundles coincide.
Moreover, if $\mathcal{X}$ has affine diagonal and admits a good moduli space[1] $\pi\colon \mathcal{X} \to X$*, then*

$$\mathbb{E}(\mathcal{X}) = {}^{\operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})}\big/_{\pi^* \operatorname{Pic}_{\mathbb{Q}}(X)}$$

and the regions $\mathcal{C}_i$ are rational polyhedral cones that form a complete pseudofan (cf. Definition 2.1.45) $\Sigma(\mathcal{X})$ in $\mathbb{E}(\mathcal{X})$.

[1] In that case, such a Θ-covering always exists if $\mathcal{X}$ is of finite type over k and has affine stabilizers [AHR19, Sec. 6].

The result above opens at least two different directions. The first is a general study of the semistable loci with respect to line bundles, aiming to generalize results from [DH98; Tha96]. The second is to investigate what the data of $\Sigma(\mathcal{X})$ corresponds to in the more restrictive case where $\mathcal{X}$ admits a good moduli space $\mathcal{X} \to X$. We answer some natural questions in the first direction, for example we prove that under suitable assumption the semistable loci are constructible and, in this case, it is open if $\mathcal{X}$ is Θ-complete. We aim to adapt and generalize results in [DH98] in a future work. Regarding the second direction, we relate $\Sigma(\mathcal{X})$ with the fan $\Sigma^{\mathbb{Q}}(X)$ of partial $\mathbb{Q}$-factorialization of the good moduli space X: a birational gadget that we now explain. Let X be a projective normal variety, then a natural question arising from the minimal model perspective is to describe the $\mathbb{Q}$-factorialization of this space. That is, to classify the small contractions $Y \to X$, where Y is $\mathbb{Q}$-factorial. These always exists when X has klt type singularities over the complex numbers, then the cones in the fan $\Sigma^{\mathbb{Q}}(X)$ are given by the decomposition:

$$\mathrm{Cl}_{\mathbb{Q}}(X) \big/ \mathrm{Pic}_{\mathbb{Q}}(X) = \coprod_i \mathrm{Amp}(X_i/X)$$

where $X_i \to X$ are all the $\mathbb{Q}$-factorializations of X. Notice that thanks to [Bra+24], any projective normal variety over the complex numbers that is the good moduli space of a smooth algebraic stack has klt type singularities. The following theorem is a simplified version of Theorem 2.3.23:

Theorem B. *Let $\mathcal{X}$ be a smooth and finite type algebraic stack over an algebraically closed field k. Assume that $\mathcal{X}$ has affine diagonal and that admit a good moduli space $\pi\colon \mathcal{X} \to X$, where X is projective. Suppose there exists $L \in \mathbb{E}(\mathcal{X})$ lying in a maximal cone of $\Sigma(\mathcal{X})$ such that:*

- *the open inclusion $\mathcal{X}^{L-\mathrm{ss}} \subset \mathcal{X}$ is big, i.e. $\mathcal{X} \setminus \mathcal{X}^{L-\mathrm{ss}}$ has codimension at least 2 in $\mathcal{X}$;*
- *the induced map $X^{L-\mathrm{ss}} \to X$ is a $\mathbb{Q}$-factorialization of X.*

Then, under the identification of $\mathrm{Pic}_{\mathbb{Q}}\left(X^{L-\mathrm{ss}}\right)$ and $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$, the fan $\Sigma(\mathcal{X})$ is the $\mathbb{Q}$-factorialization fan of X. Furthermore, for any $M \in \mathbb{E}(\mathcal{X})$ the corresponding partial $\mathbb{Q}$-factorialization is given by $X^{M-\mathrm{ss}} \to X$.

As a byproduct, we obtain that the $\mathbb{Q}$-factorializations maps $X^{M-\mathrm{ss}} \to X$ in the theorem above are relative Mori dream spaces Corollary 2.3.24.

The proof of the theorem above relies on the correspondence between the finite generation of the graded algebra

$$\bigoplus_{n\geq 0} \mathcal{O}_X(\lfloor nD \rfloor) \qquad D \in \mathrm{Cl}_{\mathbb{Q}}(X),$$

which is a key input for the construction of $\mathbb{Q}$-factorializations (cf. Proposition 2.3.12), and Hilbert's fourteenth problem on the finite generation of the ring of invariants.

The above result will be crucial in the description of the partial $\mathbb{Q}$-factorializations of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$.

1.1.2 The stacks $\overline{\mathcal{M}}^S_{g,n}$ and the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$.

The results in this section holds over an algebraically closed field k, such that $\operatorname{char} k \gg (g,n)$, cf. Definition 4.1.1. The space $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ arises as the good moduli space of the smooth stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$; it has Cohen–Macaulay singularities and, if $\operatorname{char} k = 0$, it is of KLT type. The stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ has points with non-discrete stabilizers. This prevents the space from being $\mathbb{Q}$-factorial and raises a natural question:

Question. *Do the (partial) $\mathbb{Q}$-factorializations of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ arise naturally as alternative compactifications of $\mathrm{M}_{g,n}$?*

The answer is affirmative. In particular, we can apply Theorem 2.3.23 to reduce the above question to the description of the semistable loci with respect to the line bundles on the stack. For this purpose, we make use of cluster algebra combinatorics. More specifically we can construct full rank root system $\Phi_{g,n} = \prod_{I\in\mathcal{P}} \Phi^I_{g,n} \subset \mathbb{E}_{g,n}$ that comes with a natural basis $\{\alpha_{i,I} \mid 0 \le i \le g, I \subset [n]\} \sqcup \{\alpha_{\mathrm{irr}}\}$. We can then define cluster algebras (cf. Definition 4.2.1) of type

$$\begin{cases} A_1 \times (A_g)^{2^{n-1}-1} \times A_{g-2} & \text{if } n > 0, \\ A_1 \times A_{(g-4)/2} & \text{if } n = 0 \text{ and } 2 \mid g, \\ A_1 \times C_{(g-3)/2} & \text{if } n = 0 \text{ and } 2 \nmid g. \end{cases}$$

This give rise to a d-vector cluster fan in $\mathbb{E}_{g,n}$, that we call $\Sigma_{g,n}$. We can then describe the fan of semistable regions as follows (Theorem 4.2.2):

Theorem C. *Let (g,n) be an hyperbolic pair (i.e. $3g-3+n \ge 0$) with $g \ge 2$ and $(g,n) \ne (2,0)$. The fan of semistable regions $\Sigma(\overline{\mathcal{M}}_{g,n}(\frac{7}{10}))$ is a coarsening of the fan $\Sigma_{g,n}$. Where two cones $C_1 = \overline{\operatorname{Span}_+ S_1}$ and $C_2 = \overline{\operatorname{Span}_+ S_2}$ in $\Sigma_{g,n}$ correspond to the same semistable loci if and only if they are equivalent $S_1 \sim S_2$, according to Definition 4.2.48.*

The equivalence relation $\sim$ for $n = 0$ is trivial, hence the fan $\Sigma_{g,0}$ and $\Sigma(\overline{\mathcal{M}}_g(\frac{7}{10}))$ coincides.

We can identify the cones of $\Sigma_{g,n}$ using the minimal set S of vectors that generate them (which are elements of $\Phi_{\ge -1}$, in the above root systems). Therefore, given that the loci depend only on the cone where the line bundle lies, we can rename the semistable locus as $\overline{\mathcal{M}}^S_{g,n}$. These can be described using the stratification derived from the combinatorics of curves. In particular, we have a notion of dual graph, we write S_Γ for the stratum associated with a graph Γ. We have

$$\overline{\mathcal{M}}^S_{g,n} \coloneqq \overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right) \setminus \bigcup_{\langle \Gamma, \ell\rangle > 0} \overline{S_\Gamma},$$

where ℓ is an arbitrary element of $\operatorname{Span}_+ S$, Γ ranges over a distinguished class of graphs called *principal*, and $\langle \Gamma, \ell\rangle$ denotes the pairing defined in Definition 4.1.9. As a consequence of the previous section, we have

Theorem D. *Let (g,n) be an hyperbolic pair with $(g,n) \ne (2,0)$. Fix a set of generators S of a cone in $\Sigma_{g,n}$. Then:*

- *The stack $\overline{\mathcal{M}}^S_{g,n}$ is algebraic, smooth, irreducible, and of finite type over k. Moreover, there are natural inclusions*
$$\overline{\mathcal{M}}^S_{g,n} \subset \overline{\mathcal{M}}^{S'}_{g,n} \quad \textit{whenever} \quad S' \subset S.$$
- *The stacks $\overline{\mathcal{M}}^S_{g,n}$ admit good moduli spaces $\phi^S\colon \overline{\mathcal{M}}^S_{g,n} \to \overline{\mathrm{M}}^S_{g,n}$, which is integral, normal, and projective over k. Furthermore, for every $S' \subset S$, there is a commutative diagram:*
$$\begin{tikzcd} \overline{\mathcal{M}}^S_{g,n} \arrow[r, hook] \arrow[d, "\phi^S"] & \overline{\mathcal{M}}^{S'}_{g,n} \arrow[d, "\phi^{S'}"] \\ \overline{\mathrm{M}}^S_{g,n} \arrow[r, "f^{S'}_S"] & \overline{\mathrm{M}}^{S'}_{g,n} \end{tikzcd},$$
where $f^{S'}_S$ is a projective (and birational, if $(g,n) \neq (1,2)$).

Finally, let $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ be an open substack that admits a projective good moduli space. Then there exists an S such that $\mathcal{U} = \overline{\mathcal{M}}^S_{g,n}$.

As a byproduct of the previous theorem, we give a positive answer to the third open question in [CTV23]. Namely, with the notation of loc. cit., the spaces $\overline{\mathrm{M}}^T_{g,n}$ and $\overline{\mathrm{M}}^{T+}_{g,n}$ are projective over k, even in positive characteristic; see Section 4.3.2.

We can describe the contracted locus of $f^{S'}_S$ and deduce the regularity of the spaces $\overline{\mathrm{M}}^S_{g,n}$ (cf. Proposition 4.3.2, Theorem 4.3.11, and Theorem 4.4.9). The maps $f^{S'}_S$ are not always small contractions, but restricting to a subfan allows us to describe the $\mathbb{Q}$-factorializations. The fan $\Sigma^{\mathbb{Q}}(\overline{\mathrm{M}}_{g,n}(\frac{7}{10}))$ is again described by a cluster algebra depending on (g,n) as follows (cf. Theorem 4.4.14).

Theorem E. *Assume $(g,n) \neq (2,0)$, $g \geq 2$. The $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ is given by the restriction of $\Sigma_{g,n}$ to the subspace*
$$V_{g,n} := \{v \in \mathbb{E}_{g,n} | x_{1,\{i\}}(v) = 0 \textit{ for all } i \in [n]\}.$$
Where $x_{1,i} \in \mathbb{E}^*_{g,n}$ is the dual of $\alpha_{1,\{i\}}$ with respect to the basis $\Delta_{g,n}$. Any cone $\mathcal{C} \subset V_{g,n}$ corresponds to the partial $\mathbb{Q}$-factorialization given by $\overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, where $S \subset (\Phi_{g,n})_{\geq -1}$ is such that $\mathcal{C}^\circ = \mathrm{Span}_+ S$. As a byproduct, the $\mathbb{Q}$-factorializations $f\colon \overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ are relative Mori dream spaces.

For $n = 0$, the $\mathbb{Q}$-factorialization fan and $\Sigma_{g,n}$ coincide. The special case $g = 1$, not covered by the theorem above, is completely described in Section 4.3.1.

As a corollary, we have a description of wall-crossings as flips, we refer here to admissible subsets of $\Phi_{g,n}$ as in Definition 4.3.3. Maximal cones corresponds to cluster algebra seeds in our parallelism: then flips corresponds to mutations.

Corollary 1.1.2. *Let $S' \subset S$ be two admissible seeds, and fix a line bundle $L \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$ that is anti-ample relative to $\overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}^{S'}_{g,n}$. Then the L-flip of $\overline{\mathrm{M}}^S_{g,n} \to$*

$\overline{\mathrm{M}}_{g,n}^{S'}$ is given by

$$\begin{tikzcd} \overline{\mathrm{M}}_{g,n}^{S} \arrow[rr, dashed] \arrow[dr, "f_{S'}^{S}"'] & & \overline{\mathrm{M}}_{g,n}^{\widetilde{S}} \arrow[dl, "f_{S'}^{\widetilde{S}}"] \\ & \overline{\mathrm{M}}_{g,n}^{S'} & \end{tikzcd}$$

where $\widetilde{S} \supset S'$ corresponds to the unique cone whose generators contain S' and whose image under the map

$$p\colon \mathbb{E}(\mathcal{X}) \longrightarrow \mathbb{E}(\mathcal{X})/\mathrm{Span}\, S'$$

contains $p(-L)$. In other words, it is the cone that contains the projection of $-L$ in the star construction with respect to the cone corresponding to S' (cf. Definition 2.2.12).

Moreover, if S is a maximal seed and $S \setminus S' = \{\alpha\}$, the $-\alpha$-flip is given by the seed $\widetilde{S}$ corresponding to the mutation of S along α.

The above theorems provide a complete description of the ample cones of the morphisms $f_S^{S'}$ (cf. Proposition 4.4.21). This also allow us to establish in Proposition 4.4.22 which rational maps $\overline{\mathrm{M}}_{g,n}^{S} \to \overline{\mathrm{M}}_{g,n}^{S'}$ defined as the identity on $\overline{\mathrm{M}}_{g,n}$ extends to a morphism. Finally, in Corollary 4.4.23, we obtain some constraints on the ample cone of $\overline{\mathrm{M}}_{g,n}$ that we aim to generalize in the future.

1.1.3 Modular compactifications of $\mathrm{M}_{g,n}$ with A-type singularities

In the previous section, several algebraic stacks containing $\mathcal{M}_{g,n}$ that admit a proper good moduli space were introduced. In this context, we always consider classifying stacks of curves with A-type singularities, subject to the additional condition that the log-dualizing sheaf is ample. Let $\widetilde{\mathcal{M}}_{g,n}^{i}$ denote the classifying stack of log-stable curves with singularities of type A_j, $j \leq i$. The following question arises naturally:

Question. *Can we describe all the possible open substacks $\mathcal{U} \subset \widetilde{\mathcal{M}}_{g,n}^{i}$ that admit a proper good moduli space?*

We investigate the question for $i \leq 3$, using the Θ- and S-completeness criteria introduced in [AHH23]. We prove in Theorem 3.6.5 that these stacks can be described using the natural stratification of $\widetilde{\mathcal{M}}_{g,n}^{i}$ in terms of the combinatorics of curves. To address this combinatorics, we generalize the notion of dual graph for A-type singularities. Define the stack

$$\overline{\mathcal{M}}_{g,n}^{!} := \widetilde{\mathcal{M}}_{g,n}^{3} \setminus \overline{S_{\mathrm{ell}} \cup S_{\mathrm{tac}}},$$

which parameterizes curves that do not contain a nodally attached elliptic tail nor an irreducible tacnode (i.e. such that the partial normalization along it remains irreducible). We have the following characterization:

Theorem F. *Let $\mathcal{U} \subset \widetilde{\mathcal{M}}_{g,n}^{3}$ be an open substack with a proper good moduli space. Then one of the following occurs:*

- *$\mathcal{U} \subset \widetilde{\mathcal{M}}_{g,n}^{2}$, in which case $\mathcal{U}$ is necessarily $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g,n}^{\mathrm{ps}}$, or $\overline{\mathcal{M}}_{g,n}^{\mathrm{wps}}$;*

- $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}(7/10)$;
- $\mathcal{U} \subset \overline{\mathcal{M}}^{!}_{g,n}$.

The result is achieved using the Θ- and S-completeness criteria introduced in [AHH23]. As a byproduct of these techniques, we prove that the stack $\overline{\mathcal{M}}_2^{\mathrm{irr}}$, defined in [CTV21, Rmk. 3.11], does not have a good moduli space. This answer an open question in loc. cit. The latter question can also be solved utilizing the stack of coverings, as it will be explained in a forthcoming paper by Ludvig Modin, Michele Pernice and the author.

In a future work we will address the study of $\overline{\mathcal{M}}^{!}_{g,n}$ and describe the semistable loci with respect to line bundles on the stack. We conjecture the following:

Conjecture. *Every open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ that admits a proper good moduli space is either $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}$, or arises via semistability with respect to some line bundle in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ or $\overline{\mathcal{M}}^{!}_{g,n}$. The latter has a projective good moduli space, therefore any proper good moduli space of an open substack of $\widetilde{\mathcal{M}}^3_{g,n}$ is projective.*

With this, we could give a complete answer to the question of this section, for $i \leq 3$.

1.2 Outline of the chapters

The division into chapters reflects the subsections of Section 1.1. Chapter 2 introduces the notion of semistability and contain the characterizations of semistability regions for stacks. Chapter 3 is devoted mainly to the study of curves: we define a general notion of dual graph for curves with A-type singularities, describe the stratification of $\widetilde{\mathcal{M}}^i_{g,n}$, and prove the result stated in the last subsection above. Finally, Chapter 4 treats the stacks $\overline{\mathcal{M}}^S_{g,n}$ and the birational geometry of the newly constructed compactifications.

1.3 Notations and conventions

1. We denote $\mathbb{N}$ the natural number i.e. the non-negative integers.
2. For $n \in \mathbb{N}$ we denote $[n] \coloneqq \{1, \ldots, n\}$ if $n > 0$ and $[0] = \emptyset$.
3. For a subset $I \subset [n]$, we denote $I^c = [n] \setminus I$.
4. An hyperbolic pair (g, n) is a couple of natural numbers such that $3g-3+n \geq 0$.
5. Given a vector space V over k, we write $V_K \coloneqq V \otimes_k K$ for any field extension $k \subset K$.
6. Given a vector space V over $\mathbb{R}$ (or over $\mathbb{Q}$) and some vectors $v_1, \ldots, v_k$ we denote $\mathrm{Span}_+\{v_1, \ldots, v_k\}$ the set of combinations $\sum_{i=1}^k a_i v_i$ with $a_i > 0$. By convention $\mathrm{Span}_+ \emptyset = 0$.
7. Given a vector space V over $\mathbb{R}$ (or over $\mathbb{Q}$) say that $C \subset V$ is a cone if it is closed under positive scalar multiplication and sums.

8. Given a cone C in a real vector space V we say that $F \subset \overline{C}$ is a face if exists $f \in V^*$ such that $\overline{C} \subset H_f^+ = \{x | f(x) \geq 0\}$ and $F = \overline{C} \cap H_f^+$.
9. For a birational morphism $f : X \to Y$ among finite type normal schemes over an algebraically closed field k, the *exceptional locus* $\mathrm{Exc}(f) \subset X$ is the subscheme with reduced structure whose k-points are the locus where f is not biregular (that is, f^{-1} is not a morphism at $f(x)$).
10. Given a projective variety X over a field of characteristic 0, we say that X is of klt type if there exist a $\mathbb{Q}$-Cartier divisor Δ such that (X, Δ) is a klt pair.
11. For an algebraic stack $\mathcal{X}$, we denote with $|\mathcal{X}|$ the topological space associated to it.

Chapter 2

Variation of Geometric Invariant Theory

We recall here the notion of (semi)stability, defined using very close degenerations, following [Hei17]. The main object of this section is the stability condition space $\mathbb{E}(\mathcal{X})$, introduced for an algebraic stack $\mathcal{X}$ in Definition 2.1.29. Under mild hypotheses on $\mathcal{X}$, we show that $\mathbb{E}(\mathcal{X})$ is a finite-dimensional vector space. This space is partitioned into finitely many rational polyhedral chambers, with the property that two points are in the same chamber if and only if they yeld the same semistable locus. Moreover, when $\mathcal{X}$ admits a good moduli space $\mathcal{X} \to X$, these regions form a polyhedral fan in $\mathbb{E}(\mathcal{X})$. Finally, in the last section, we establish a connection between the fan in $\mathbb{E}(\mathcal{X})$ and the $\mathbb{Q}$-factorialization fan of the good moduli space X.

2.1 Line bundle stability

We recall some basic facts and establish notation to be used in the following sections.

A line bundle $\mathcal{L}$ over $\Theta \coloneqq [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to the datum of a line bundle over $\mathbb{A}^1$ together with a $\mathbb{G}_m$*-linearization*, i.e. the datum of $\pi^*\mathcal{L}$ and the descent data from the smooth presentation $\pi\colon \mathbb{A}^1 \to \Theta$.

Consider the affine map $f\colon \Theta \to B\mathbb{G}_m$. A line bundle $\mathcal{L}$ on Θ can be recovered from the data of the quasi-coherent sheaf $f_*\mathcal{L}$, together with the structure of an $f_*\mathcal{O}_\Theta$-module. This implies that the $\mathbb{G}_m$-linearization can be recovered from the $\mathbb{G}_m$-action on $H^0(\mathbb{A}^1, \pi^*\mathcal{L}) = \mathbb{Z}[x] \cdot e$, i.e. it is determined by the $\mathbb{G}_m$-weight on e. We define $\mathrm{wt}(\mathcal{L})$ to be that weight.

Notice that $\mathrm{wt}(\mathcal{L}) \leq 0$ if and only if $\mathcal{L}$ has global sections. Moreover, consider the pullback $i^*\mathcal{L}$ along the closed immersion

$$i\colon B\mathbb{G}_m \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m];$$

this corresponds to a rank-one free module generated by a single element having weight $\mathrm{wt}(\mathcal{L})$ with respect to the $\mathbb{G}_m$-action. We will often identify these two weights in the following sections.

Similarly, we define the notion of weight for a line bundle in the relative case, i.e. over $\Theta_T = \Theta \times T$, where T is a connected scheme. Finally, by linearity, we can extend our definition to any $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\Theta_T)$.

Definition 2.1.1 (Semistable points)**.** Let $\mathcal{X}$ be an algebraic stack and fix $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$. We say that a topological point $\overline{p} \in |\mathcal{X}|$ is $\mathcal{L}$-*unstable* if there exists a representative $p\colon \operatorname{Spec} K \to \mathcal{X}$ and a map

$$f\colon [\mathbb{A}^1_K/\mathbb{G}_{m,K}] = \Theta_K \to \mathcal{X}$$

such that $f(1) \simeq p$ and $\mathrm{wt}(f^*\mathcal{L}) > 0$.
We say that a point is $\mathcal{L}$-*semistable* if it is not $\mathcal{L}$-unstable. If the locus of semistable points is locally closed, we write $\mathcal{X}^{\mathcal{L}-\mathrm{ss}}$ for the substack of semistable points with the reduced structure.

Notice that the weight in the definition above is preserved under base change. Namely, given $g\colon \Theta_S \to \Theta_T$ induced by a base change of connected schemes $T \to S$, for any $f\colon \Theta_T \to \mathcal{X}$ and $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ we have $\mathrm{wt}(f^*\mathcal{L}) = \mathrm{wt}((f \circ g)^*\mathcal{L})$.

We refer to such a map f in the definition as a *very close degeneration*. We say that it is *non-degenerate* if $f(1) \neq f(0)$ as topological points. Furthermore, we say that a very close degeneration $\Theta_K \to \mathcal{X}$ is *trivial* if it factors through $\Theta_K \to \operatorname{Spec} K$.

Recall that a non-degenerate very close degeneration $f\colon \Theta \to \mathcal{X}$ induces a nontrivial map at the level of stabilizers $\mathbb{G}_m \simeq \mathrm{Stab}_\Theta(0) \to \mathrm{Stab}_{\mathcal{X}}(f(0))$ when $\mathcal{X}$ is a finite type algebraic stack over an algebraically closed field, with quasi-affine diagonal [Hei17, Lemma 1.5]. For completeness, we recall the notion of stability from [Hei17], even if it will not play a central role in the following chapters.

Definition 2.1.2 ((Poly)stable points)**.** Let $\mathcal{X}$ be an algebraic stack and fix $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$. We say that an $\mathcal{L}$-semistable point $p \in |\mathcal{X}|$ is $\mathcal{L}$-*polystable* (resp. $\mathcal{L}$-*stable*) if for every very close degeneration of p

$$f\colon [\mathbb{A}^1_K/\mathbb{G}_{m,K}] = \Theta_K \to \mathcal{X}$$

(for some field extension $k \subset K$) we have $\mathrm{wt}(f^{-1}\mathcal{L}) = 0$ if and only if f is degenerate (resp. trivial).

Remark 2.1.3. Let $\mathcal{X}$ be a finite-type, quasi-separated algebraic stack over a field k with affine stabilizers. Then (semi/poly)stability of a k-point can be verified via very close degenerations defined over a finite field extension $k \subset K$.

Proof. By [Hal14, Prop. 1.1.2], the stacks $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})$ and $\underline{\mathrm{Mor}}(B\mathbb{G}_m, \mathcal{X})$ are locally of finite type over k. Consider the evaluation map

$$ev_1\colon \underline{\mathrm{Mor}}(\Theta, \mathcal{X}) \longrightarrow \mathcal{X}$$

defined by precomposition with $\operatorname{Spec} k \xrightarrow{1} \Theta$. Let $x \in \mathcal{X}(k)$, and suppose we are given a very close degeneration of x, that is, a morphism $f\colon \operatorname{Spec}\Omega \to ev_1^{-1}(x)$, where the target is of finite type over k. Then there exists a point $\xi \in \overline{f(\Omega)}$ defined over some finite field extension $K \supset k$ such that $ev_1(\xi) = x$. This yields a very close degeneration of x over K with the same associated weight, since the weight is constant on connected components. This proves the claim for semistability. Now assume that x is semistable. Consider the cartesian diagram

$$\begin{tikzcd} U' \arrow[rr, "f"] \arrow[d] & & U \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "x"] \\ \mathcal{X} \simeq \underline{\mathrm{Mor}}(\operatorname{Spec} k, \mathcal{X})^{\mathrm{triv}} \arrow[rr, "s\circ -"] & & \underline{\mathrm{Mor}}(\Theta, \mathcal{X})^{\mathrm{triv}} \arrow[r, "ev_1"] & \mathcal{X}, \end{tikzcd}$$

where the superscript triv denotes the locus on which the weight with respect to the chosen line bundle is trivial and $s\colon \Theta \to \operatorname{Spec} k$ is the structure map. Then x is stable if and only if the map f is surjective. Since U' and U are of finite type over k, surjectivity may be checked on points with residue field $K \supset k$ finite. Finally, consider the map

$$\underline{\mathrm{Mor}}(\Theta, \mathcal{X}) \times_{ev_1, \mathcal{X}, x} \operatorname{Spec} k \xrightarrow{ev_0} \mathcal{X}.$$

A semistable point x is polystable if and only if the fiber over x of this map equals $\underline{\mathrm{Mor}}(\Theta, \mathcal{X})^{\mathrm{triv}} \times_{ev_1, \mathcal{X}, x} \operatorname{Spec} k$. This completes the proof. □

The notion of (semi)stability will coincides with the one given by Mumford in [GIT, Sec. 1.4] when we work on projective-over-affine schemes, as we now recall.

Let $G \curvearrowright X$ be an action of an affine linearly reductive group G/k on a finite-type, separated k-scheme X. Fix a G-linearized line bundle $\mathcal{L}$ on X.

For any one-parameter subgroup $\lambda\colon \mathbb{G}_{m,K} \to G_K$ and point $p\colon \operatorname{Spec} K \to X$ such that $\lim_{t\to 0} \lambda(t)p = q$, the line bundle $\mathcal{L}$ defines, via pullback, a $\mathbb{G}_m$-representation on the one-dimensional vector space $H^0(\operatorname{Spec} K, q^*\mathcal{L})$, determined by the weight n of a generator. The *Hilbert–Mumford weight* with respect to these data is defined to be

$$\mu_{\mathcal{L}}(p, \lambda) \coloneqq n.$$

We have the following comparison theorem. The equivalence between (2) and (3) is commonly known as the Hilbert–Mumford criterion.

Theorem 2.1.4 (Equivalence of stabilities)**.** *Let k be an algebraically closed field, and consider an action of an affine linearly reductive group G/k on a projective-over-affine, finite-type k-scheme U. Fix a line bundle $\mathcal{L}$ on the quotient stack $\mathcal{X} = [U/G]$, whose pullback via the presentation $\pi\colon U \to [U/G]$ is ample. Let $k \subset K$ be an extension of fields with K algebraically closed. Then, for a point $p\colon \operatorname{Spec} K \to U$, the following are equivalent:*

1. *p, viewed as a topological point in $|\mathcal{X}|$, is semistable with respect to $\mathcal{L}$.*
2. *For every one-parameter subgroup $\lambda\colon \mathbb{G}_{m,K} \to G_K$ such that $\lim_{t\to 0} \lambda(t)p$ exists, we have $\mu_{\mathcal{L}}(p, \lambda) \le 0$.*
3. *There exist $N > 0$ and a section in $H^0(U, \pi^*\mathcal{L}^N)^G$ that does not vanish at p.*

In particular, the semistable locus is an open substack. The following are equivalent:

1. *p, viewed as a topological point in $|\mathcal{X}|$, is stable with respect to $\mathcal{L}$.*

2. *For every one-parameter subgroup $\lambda\colon \mathbb{G}_{m,K} \to G_K$ such that $\lim_{t\to 0}\lambda(t)p$ exists, we have $\mu_{\mathcal{L}}(p,\lambda) \leq 0$, and equality holds if and only if λ is trivial.*

3. *$p \in \mathcal{X}^{\mathcal{L}-\mathrm{ss}}$, $\mathrm{Stab}_{\mathcal{X}}(p)$ is finite, and p is closed in $\mathcal{X}^{\mathcal{L}-\mathrm{ss}}$.*

Finally, for polystability, the following are equivalent:

1. *p, viewed as a topological point in $|\mathcal{X}|$, is polystable with respect to $\mathcal{L}$.*

2. *For every one-parameter subgroup $\lambda\colon \mathbb{G}_{m,K} \to G_K$ such that $\lim_{t\to 0}\lambda(t)p$ exists, we have $\mu_{\mathcal{L}}(p,\lambda) \leq 0$, and equality holds if and only if there exists $g \in P_\lambda$ (the parabolic subgroup associated to λ) such that $g\lambda g^{-1}$ has image in G_p.*

3. *$p \in \mathcal{X}^{\mathcal{L}-\mathrm{ss}}$ and p is closed in $\mathcal{X}^{\mathcal{L}-\mathrm{ss}}$.*

A key ingredient of the proof is [Alp25, Prop. 6.9.1], which we now recall.

Proposition 2.1.5. *If G is a smooth affine algebraic group over an algebraically closed field k acting on a separated algebraic space U of finite type over k, then there is an equivalence of groupoids*

$$\mathrm{Mor}_k(\Theta, [U/G]) \xrightarrow{\sim} \{(x \in U(k), \lambda\colon \mathbb{G}_m \to G) \mid \lim_{t\to 0}\lambda(t)\cdot x \in U(k) \ \textit{exists}\}.$$

Where a morphism $(x,\lambda) \to (x',\lambda')$ is an isomorphism class of pairs (g,h) with $g \in P_\lambda(k)$ and $h \in G(k)$ such that $x' = hgx$ and $\lambda' = h\lambda h^{-1}$, with $(g,h) \sim (c^{-1}g, hc)$ for $c \in C_\lambda$. Under this correspondence, the morphism $\Theta \to [U/G]$ sends 1 to x and 0 to $\lim_{t\to 0}\lambda(t)\cdot x$.

Lemma 2.1.6. *Let U and G be as in Theorem 2.1.4, and fix a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. Consider a point $x \in U(k)$ such that $\lim_{t\to 0}\lambda(t)\cdot x$ exists. Then $\lim_{t\to 0}\lambda(t)\cdot x \in Gx$ if and only if there exists $s \in P_\lambda(k)$ such that $s\lambda s^{-1} \subset G_x$.*

Proof. For a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ and any $y \in U(k)$ such that the limit $\lim_{t\to 0}\lambda(t)\cdot y$ exists, we can characterize the parabolic subgroup $P_\lambda \subset G$ by

$$P_\lambda(k) = \{\, h \in G(k) \mid \lim_{t\to 0}\lambda(t)h\cdot y \text{ exists}\,\}.$$

Suppose first that $\lim_{t\to 0}\lambda(t)\cdot x = g\cdot x$ for some $g \in G(k)$. Then we have $\lim_{t\to 0}\lambda(t)g\cdot x = g\cdot x$, by Białynicki-Birula stratification $g\cdot x$ is fixed by λ and hence $g^{-1}\lambda g \subset G_x$. By the characterization above, the existence of $\lim_{t\to 0}\lambda(t)g\cdot x$ implies $g \in P_\lambda(k)$. Thus, $s \coloneqq g^{-1}$ satisfies $s\lambda s^{-1} \subset G_x$. Conversely, if there exists $s \in P_\lambda(k)$ such that $s\lambda s^{-1} \subset G_x$, then

$$\lim_{t\to 0}\lambda(t)\cdot x = \lim_{t\to 0} s^{-1}(s\lambda(t)s^{-1})s\cdot x = s^{-1}x \in Gx,$$

so the limit lies in the orbit of x. $\square$

Remark 2.1.7. Let $\mathcal{X} = [U/G]$ be as in Theorem 2.1.4. Then the very close degeneration $\Theta \to \mathcal{X}$ is degenerate if and only if it factors through the map $\Theta \to B\mathbb{G}_m$ induced, at the level of presentations, by the $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to \mathrm{Spec}\, k$.

Proof. If the very close degeneration is degenerate, then under the identification of Proposition 2.1.5 we have $\lim_{t\to 0}\lambda(t)x \in Gx$. By Lemma 2.1.6, there exists $s \in P_\lambda(k)$ such that $s\cdot\lambda\cdot s^{-1} \in G_x$. Setting $g = s^{-1}$ and $h = s$ in Proposition 2.1.5, we obtain a pair (x', λ') representing the same morphism, with $\lambda' \in G_x$. The claim follows. □

Proof of Theorem 2.1.4. The equivalence between (1) and (2) in the (semi)stable case follows from Proposition 2.1.5. The case of polystability follows from Remark 2.1.7. The equivalence of (2) and (3) in the (semi)stable case is a relative version of the Hilbert–Mumford criterion [GHH15, Thm. 3.3]. We now prove the polystable case. By the equivalence for semistability, there exists a covering

$$[U/G]^{\mathcal{L}\text{-ss}} = \bigcup_{i\in I}[U_i/G]$$

where each $U_i \subset U$ is a G-invariant affine open subset. We may therefore assume that $U = \operatorname{Spec} A$ is affine and that $[\operatorname{Spec} A/G] = [\operatorname{Spec} A/G]^{\mathcal{L}\text{-ss}}$. Assume first that $p \in \operatorname{Spec} A$ does not have a closed orbit. By the destabilization theorem [Alp25, Thm. 7.3.6], there exists $\lambda\colon \mathbb{G}_m \to G$ such that the limit $p_0 = \lim_{t\to 0}\lambda(t)\cdot p$ exists and p_0 has a closed orbit. By Lemma 2.1.6, there does not exist $s \in P_\lambda(K)$ such that $s\cdot\lambda\cdot s^{-1} \in G_p$. Moreover, as a consequence of the Białynicki-Birula stratification, the image of λ lies in G_{p_0} and

$$\mu_{\mathcal{L}}(p, \lambda) = \mu_{\mathcal{L}}(p_0, \lambda).$$

Since $\lim_{t\to 0}\lambda^{-1}(t)p_0 = p_0$, we obtain $\mu_{\mathcal{L}}(p_0, \lambda) = 0$, and hence p is not polystable. Conversely, suppose that p has a closed orbit, and let $\lambda\colon \mathbb{G}_m \to G$ be such that the limit $p_0 = \lim_{t\to 0}\lambda(t)\cdot p$ exists (so in particular $p_0 \in Gp$). By Lemma 2.1.6, there exists $s \in P_\lambda(K)$ such that $s\cdot\lambda\cdot s^{-1} \in G_p$. As above, we have $\mu_{\mathcal{L}}(p,\lambda) = \mu_{\mathcal{L}}(p_0,\lambda)$, and the latter equals 0 since λ factors through G_{p_0}. Hence $\mu_{\mathcal{L}}(p,\lambda) = 0$, and p is polystable. □

For the sake of completeness, we note the following.

Remark 2.1.8. Our notion of semistability does not generally coincide with that of [Alp13, Def. 1.11]. The latter implies that the stable locus is always open, whereas our definition allows examples where this does not hold (cf. Example 2.1.20). Nevertheless, in the setting of Theorem 2.1.4, the two notions agree.

We aim to study the semistable locus and how it changes when the line bundle $\mathcal{L}$ varies. Our approach is inspired by [Alp+17, Sec. 3]; see also [DH98; Tha96] for the general theory of Variation of GIT (VGIT) for projective varieties. To carry out the constructions in this thesis, we focus on developing a theory of VGIT for stacks admitting a good moduli space, relying on their local structure. Interestingly, many of the results hold in a more general setting.

The following result holds more generally, but only the stated case will be used.

Proposition 2.1.9 (Local structure for good moduli spaces, [Alp25, Cor. 6.7.3])**.** *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field k with* *affine diagonal. A morphism $\mathcal{X} \to X$, where X is an algebraic space, is a good moduli space if and only if there exists a finite type k-algebra A, a linearly reductive group G/k, and a cartesian diagram*

$$\begin{tikzcd} {[\operatorname{Spec} A/G]} \arrow[r, "f'"] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} A^G \arrow[r, "f"] & X \end{tikzcd},$$

where f is étale and surjective.

We introduce the following nonstandard notation.

Definition 2.1.10. We say that an algebraic stack $\mathcal{X}$ over an algebraically closed field k is *geometric* if it is finite type over k with affine diagonal.

The following properties of maps will play a role in the next sections.

Definition 2.1.11. Let k be an algebraically closed field. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called:

- **$B\mathbb{G}_m$-covering** if any map $B\mathbb{G}_{m,k} \to \mathcal{X}$ has a lift, up to compose with an endomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by $t \mapsto t^n$, $n > 0$:

$$\begin{tikzcd} & & \mathcal{X} \arrow[d, "f"] \\ B\mathbb{G}_{m,k} \arrow[r, "\cdot n"] \arrow[rru, dashed] & B\mathbb{G}_{m,k} \arrow[r] & \mathcal{Y} \end{tikzcd}.$$

- **Θ-covering** if any diagram below has a lift, up to compose with an endomorphism of Θ, given by $t \mapsto t^n$ $(n > 0)$ at the level of the smooth presentation $\mathbb{A}^1_k \to \Theta_k$.

$$\begin{tikzcd} & & \mathcal{X} \arrow[d, "f"] \\ \Theta_k \arrow[r, "\cdot n"] \arrow[rru, dashed] & \Theta_k \arrow[r] & \mathcal{Y} \end{tikzcd}.$$

- **Weakly Θ-surjective** if any diagram below has a lift to compose with up to an endomorphism of Θ, given by $t \mapsto t^n$ at the level of the smooth presentation $\mathbb{A}^1 \to \Theta$.

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, hook, "1"'] & \operatorname{Spec} k \arrow[r, "x"] \arrow[d, hook, "1"'] & \mathcal{X} \arrow[d, "f"] \\ \Theta_k \arrow[r, "\cdot n"] \arrow[rru, dashed] & \Theta_k \arrow[r] & \mathcal{Y} \end{tikzcd}. \tag{2.1}$$

- **Θ-surjective** if it is weakly Θ-surjective and the map in the diagram above lifts for $n = 1$.

Remark 2.1.12. Any Θ-covering map is a $B\mathbb{G}_m$-covering. Moreover, any $B\mathbb{G}_m$-covering is necessarily surjective.

Any Θ-surjective map is clearly weakly Θ-surjective, but note that a Θ-surjective morphism is not necessarily surjective.

Remark 2.1.13. Any surjective and weakly Θ-surjective morphism is a Θ-covering.

Lemma 2.1.14. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, where $\mathcal{X}$ is noetherian with affine stabilizers. Then:*

- *if f is finite and surjective, then it is $B\mathbb{G}_m$-covering;*
- *if f is formally étale and surjective, then f is a Θ-covering;*
- *in the setting of Proposition 2.1.9, f' is surjective and Θ-surjective.*

Proof. Suppose f is finite and surjective. For any $q \in \mathcal{Y}$ there exists $w \in \mathcal{X}$ mapping to q such that the map between the residual gerbes $\mathcal{G}_w \to \mathcal{G}_q$ is finite. After a finite extension of the base field, the map between gerbes is a finite map $BG \to BH$, hence induced by a finite morphism of algebraic groups $G \to H$. Then any $\mathbb{G}_m \to H$ lifts to $\mathbb{G}_m \to G$ up to precomposition with a map $\Theta \to \Theta$ given by $t \mapsto t^n$, proving the first claim.

Define $\Theta^{(n)} := [\mathrm{Spec}(k[x]/x^n)/\mathbb{G}_m]$, a thickening of Θ around 0. If f is formally étale, then we can lift the map $\Theta^{(n)} \to \mathcal{Y}$ for any positive integer n. We conclude the second claim by coherent completeness of Θ ([AHR20, Example 2.4]).

The third claim is immediate because being Θ-surjective is invariant under base change, and $\mathrm{Spec}\, A^G \to X$ is Θ-surjective since it is a morphism of algebraic spaces. $\square$

Moreover, more concretely we have:

Lemma 2.1.15. *Let G be an affine smooth group acting on algebraic spaces X, Y, both separated and of finite type over an algebraically closed field k. For any proper G-equivariant morphism $f\colon X \to Y$, the induced map*

$$[X/G] \to [Y/G]$$

is Θ-surjective.

Proof. Let $q\colon \Theta_k \to [Y/G]$ be a morphism and consider a lift of the generic point $\mathrm{Spec}\, k \to [X/G]$. By [Alp25, Prop. 6.9.1], the morphism q is described by a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ and a map $\mathbb{A}^1 \to Y$ equivariant respect to λ. Let $p\colon \mathrm{Spec}\, k \to X$ be a representative for the lifted point. Then we have the following diagram, where the maps to X and Y are equivariant with respect to λ. This has a lift by properness of $X \to Y$.

$$\begin{tikzcd}
\mathbb{A}^1_k \setminus \{0\} \arrow[r, "\lambda(t)\cdot p"] \arrow[d, hook] & X \arrow[d, "f"] \\
\mathbb{A}^1_k \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}.$$

Since the maps are equivariant, this provides a lift for $n = 1$ in the diagram 2.1; hence $[X/G] \to [Y/G]$ is Θ-surjective. □

These notions interact with semistability.

Lemma 2.1.16. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of algebraic stacks, then for $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{Y})$ we have*

1. $f(\mathcal{X} \setminus \mathcal{X}^{f^*\mathcal{L}-\mathrm{ss}}) \subset \mathcal{Y} \setminus \mathcal{Y}^{\mathcal{L}-\mathrm{ss}}$*;*
2. *if f is a Θ-covering, then $f(\mathcal{X} \setminus \mathcal{X}^{f^*\mathcal{L}-\mathrm{ss}}) = \mathcal{Y} \setminus \mathcal{Y}^{\mathcal{L}-\mathrm{ss}}$;*
3. *if f is surjective and weakly Θ-surjective, then $f^{-1}(\mathcal{Y}^{\mathcal{L}-\mathrm{ss}}) = \mathcal{X}^{f^*\mathcal{L}-\mathrm{ss}}$.*

Proof. The first claim follows because if $\Theta \to \mathcal{X}$ destabilizes a point with respect to $f^*\mathcal{L}$, then $\Theta \to \mathcal{X} \xrightarrow{f} \mathcal{Y}$ destabilizes it with respect to $\mathcal{L}$.

The second claim follows similarly: if $\Theta \to \mathcal{Y}$ destabilizes a point $p \in |\mathcal{Y}|$, then we can lift the map to a morphism $\Theta \to \mathcal{X}$ that destabilizes $q \in |\mathcal{Y}|$, with $f(q) = p$.

For the third claim, fix $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $f(x) = y$. We must prove that x is $f^*\mathcal{L}$-unstable if and only if y is $\mathcal{L}$-unstable. If $\Theta \to \mathcal{X}$ destabilizes x, then $\Theta \to \mathcal{X} \to \mathcal{Y}$ destabilizes y. Conversely, let $\Theta \to \mathcal{Y}$ be destabilizing for y and consider a square diagram as in (2.1) given by $\mathrm{Spec}\, K \to \mathcal{X}$ whose image is x: the lifting provided by weak Θ-surjectivity destabilizes x. □

We will work in the following setting.

Definition 2.1.17 ((†))**.** We say that an algebraic stack $\mathcal{X}$ of finite type over a field k has property (†) if there exists a Θ-covering of the form $[\mathrm{Spec}\, A/G] \to \mathcal{X}$ where $\mathrm{Spec}\, A$ is of finite type over k and G is affine and linearly reductive.

That is true under mild assumptions

Proposition 2.1.18. *If $\mathcal{X}$ is of finite type over an algebraically closed field k, with affine stabilizers and linearly reductive stabilizers at closed points, then (†) is satisfied.*

Proof. We can apply [AHR20, Thm. 1.1] to produce an étale cover from $[\mathrm{Spec}\, A/G]$, as required in (†). The claim then follows from Lemma 2.1.14. □

As a first consequence, in the setting above we have:

Proposition 2.1.19 (Constructibility and openness of the semistable locus)**.** *Let $\mathcal{X}$ be a finite type stack over a field k satisfying (†). Then $\mathcal{X}^{\mathcal{L}-\mathrm{ss}}$ is constructible in $|\mathcal{X}|$ for any $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$.*

Moreover, if $\mathcal{X}$ is Θ-reductive, then $\mathcal{X}^{\mathcal{L}-\mathrm{ss}}$ is open.

Proof. Let $\pi\colon [\mathrm{Spec}\, A/G] \to \mathcal{X}$ be a Θ-covering as in hypothesis (†). Then, by Lemma 2.1.16,

$$\mathcal{X}^{\mathcal{L}-\mathrm{ss}} = \pi([\mathrm{Spec}\, A/G] \setminus [\mathrm{Spec}\, A/G]^{\pi^*\mathcal{L}-\mathrm{ss}}).$$

The unstable locus with respect to any line bundle is closed in $[\operatorname{Spec} A/G]$ by Theorem 2.1.4. The first claim then follows from Chevalley's theorem.

For the second claim, it suffices to show that the unstable locus is closed under specialization. Let $\eta\colon \operatorname{Spec} R \to \mathcal{X}$ be a map from a DVR (with fraction field K and residue field $k(s)$), and assume that the central fiber is unstable. Then, after an extension of DVRs, there exists a very close degeneration $f\colon \Theta_K \to \mathcal{X}$ such that $f(1)$ is the central fiber of η and $\operatorname{wt}(f^*\mathcal{L}) > 0$. We can glue η and f along the generic fiber, obtaining a map $\Theta_R \setminus 0 \to \mathcal{X}$ that, by hypothesis, extends to $\Theta_R \to \mathcal{X}$. In this way we obtain a very close degeneration $f'\colon \Theta_{k(s)} \to \mathcal{X}$ whose generic fiber is the closed fiber of η. By connectedness of $\operatorname{Spec} R$, we have $\operatorname{wt}(f'^*\mathcal{L}) = \operatorname{wt}(f^*\mathcal{L})$, which is positive.

□

The following is an example non-Θ-reductive stack with a non-open semistable locus.

Example 2.1.20. Consider the stack

$$\mathcal{X} = \left[(\operatorname{Spec} \mathbb{A}^2_k \setminus \{0\})/\mathbb{G}_m\right],$$

where the $\mathbb{G}_m$-action is given by the restriction of the action on $k[x, y]$ with weights 1 and 0 on x and y, respectively. Then the semistable locus with respect to the structure sheaf twisted by the identity character $\mathbb{G}_m \to \mathbb{G}_m$ is the image of the k-point $(1, 0)$ in the presentation $\mathbb{A}^2_k \setminus \{0\} \to \mathcal{X}$, hence it is not open, in this case it is actually closed.

The semistable locus with respect to a line bundle admits a good moduli space whenever the stack itself does, as we now show.

Proposition 2.1.21 ([GIT, Thm. 1.10])**.** *Let $\pi\colon [U/G] \to X$ be a good moduli space, where U is projective-over-affine and G is linearly reductive. Then*

$$[U/G]^{\mathcal{L}-\mathrm{ss}} \to \operatorname{Proj}_X \bigoplus_{n\in\mathbb{N}} \pi_*\mathcal{L}^n$$

is a good moduli space. In the special case where $\mathcal{L}$ is trivial over $U = \operatorname{Spec} A$, the datum of $\mathcal{L}$ is equivalent to a G-linearization of $\mathcal{O}_{\operatorname{Spec} A}$, which is given by a character $\chi\colon G \to \mathbb{G}_m$. In this case we have

$$[\operatorname{Spec} A/G]^{\mathcal{L}-\mathrm{ss}} = [(\operatorname{Spec} A \setminus V(I_\chi))/G] \to \operatorname{Proj} \bigoplus_{n\geq 0} A_n$$

as a good moduli space, where $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is the decomposition induced by χ, explicitly

$$A_n \coloneqq \{f \in A \mid \sigma^*(f) = \chi^*(t)^{-n} f\},$$

where σ denotes the action. Moreover, the ideal $I_\chi = (f \mid f \in A_n$ for some $n > 0) \subset A$.

To study the openness of the semistable locus and the existence of a good moduli space for it, a more general theory is developed in [Hal14], where the notions of Θ-monotonicity and numerical invariants are introduced (cf. [Hal14, Thm. 5.6.1]).

Theorem 2.1.22 (Variation of good moduli spaces)**.** *Let $\mathcal{X}$ be a geometric algebraic stack and assume that $\mathcal{X}$ admits a good moduli space $\pi\colon \mathcal{X} \to X$. Then, for any $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$,*

$$\mathcal{X}^{\mathcal{L}-\mathrm{ss}} \to \mathrm{Proj}_X \bigoplus_{n\geq 0} \pi_*(\mathcal{L}^n)$$

is a good moduli space, and the target is of finite type over k.

Proof. The claim can be checked étale-locally on the target, as we now explain. Let $f\colon Y \to X$ be an étale cover, and assume the claim holds for the good moduli space $\pi'\colon \mathcal{Y} := \mathcal{X} \times_X Y \to Y$. The induced map $f'\colon \mathcal{Y} \to \mathcal{X}$ is Θ-surjective; hence for any $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ the semistable locus (which is open by Proposition 2.1.19) satisfies $\mathcal{Y}^{f'^*\mathcal{L}-\mathrm{ss}} = f'^{-1}(\mathcal{X}^{\mathcal{L}-\mathrm{ss}})$. Moreover, by flat base change we have $f^*\pi_*\mathcal{L}^d \simeq \pi_* f'^*\mathcal{L}^d$, and similarly for the pullback along the structure map $p\colon Y \times_X Y \to X$. Writing $\mathcal{L}' := f'^*\mathcal{L}$ and $\mathcal{L}'' := p^*\mathcal{L}$, we obtain the following commutative diagram, where the bottom squares are cartesian:

$$\begin{tikzcd}
& \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & \mathcal{Y} \arrow[rr, "f'"] \arrow[dd, "\pi'"] & & \mathcal{X} \arrow[dd, "\pi"] \\
\mathcal{Y}^{\mathcal{L}'-\mathrm{ss}} \times_{\mathcal{X}^{\mathcal{L}-\mathrm{ss}}} \mathcal{Y}^{\mathcal{L}'-\mathrm{ss}} \arrow[ur, hook] \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, "\pi''"] & & \mathcal{Y}^{\mathcal{L}'-\mathrm{ss}} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & \mathcal{X}^{\mathcal{L}-\mathrm{ss}} \arrow[ur, hook] \arrow[dd, dashed] & \\
& Y \times_X Y \arrow[rr, shift left] \arrow[rr, shift right] & & Y \arrow[rr, "f"] & & X \\
\mathrm{Proj}_{Y\times_X Y}\, \pi''_*\mathcal{L}'' \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] & & \mathrm{Proj}_Y\, \pi'_*\mathcal{L}' \arrow[ur] \arrow[rr] & & \mathrm{Proj}_X\, \pi_*\mathcal{L} \arrow[ur] &
\end{tikzcd}.$$

We can then construct the dotted arrow by smooth descent. Since being a good moduli space is étale local on the target. Then, by the local structure theorem for good moduli spaces (Proposition 2.1.9), it suffices to prove the theorem for $\mathcal{X} = [\mathrm{Spec}\, A/G]$ with G linearly reductive and A of finite type over k. This particular case follows from Theorem 2.1.4. The final claim follows from [Alp13, Thm. 4.16]. □

Remark 2.1.23. If $\pi\colon \mathcal{X} \to X$ is a good moduli space, then $\bigoplus_{d\in\mathbb{N}} \pi_*\mathcal{L}^d$ is a finitely generated sheaf of algebras over X.

Proof. The claim can be checked étale-locally on X, hence we may assume X affine. In this case, it follows from [Alp13, Lemma 4.14]. □

In the special setting of the proposition above, we have:

Proposition 2.1.24. *Let $\mathcal{X}$ be a geometric stack with a good moduli space $\pi\colon \mathcal{X} \to X$. Then, for any $\mathcal{L} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$, we have the following diagram:*

$$\begin{tikzcd}
\mathcal{X}^{\mathcal{L}-\mathrm{s}} \arrow[r, hook] \arrow[d] & \mathcal{X}^{\mathcal{L}-\mathrm{ss}} \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
X^{\mathcal{L}-\mathrm{s}} \arrow[r, hook] & X^{\mathcal{L}-\mathrm{ss}} \arrow[r] & X
\end{tikzcd},$$

where the vertical arrows are good moduli spaces, the left square is cartesian, and the top horizontal arrows, as well as the map $X^{\mathcal{L}-\mathrm{s}} \hookrightarrow X^{\mathcal{L}-\mathrm{ss}}$, are open immersions.

Proof. Let $f' \colon [\operatorname{Spec} A/G] \to \mathcal{X}$ be an étale covering provided by Proposition 2.1.9:

$$\begin{tikzcd} {[\operatorname{Spec} A/G]} \arrow[r, "f'"] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} A^G \arrow[r, "f"] & X \end{tikzcd}.$$

We first prove that $f'^{-1}(\mathcal{X}^{\mathcal{L}-\mathrm{s}}) = [\operatorname{Spec} A/G]^{f'^*\mathcal{L}-\mathrm{s}}$. Since f' is a Θ-surjective covering, by Lemma 2.1.16 it suffices to show that

$$f'^{-1}(\mathcal{X}^{\mathcal{L}-\mathrm{ss}} \setminus \mathcal{X}^{\mathcal{L}-\mathrm{s}}) = [\operatorname{Spec} A/G]^{f'^*\mathcal{L}-\mathrm{ss}} \setminus [\operatorname{Spec} A/G]^{f'^*\mathcal{L}-\mathrm{s}}.$$

Assume that $q \in \mathcal{X}$ is $\mathcal{L}$-semistable but not $\mathcal{L}$-stable. Then there exists a non-degenerate very close specialization $g \colon \Theta \to \mathcal{X}$ with trivial weight with respect to $\mathcal{L}$. For any k-point $p \in f'^{-1}(q)$, we can consider the morphism $\Theta \to \operatorname{Spec} k \xrightarrow{p} [\operatorname{Spec} A/G]$. Since the diagram above is cartesian, the data of g and p determine a very close degeneration $\Theta \to [\operatorname{Spec} A/G]$, which has trivial weight with respect to $f'^*\mathcal{L}$ and is non-degenerate by construction. Therefore, we have

$$f'^{-1}(\mathcal{X}^{\mathcal{L}-\mathrm{ss}} \setminus \mathcal{X}^{\mathcal{L}-\mathrm{s}}) \subset [\operatorname{Spec} A/G]^{f'^*\mathcal{L}-\mathrm{ss}} \setminus [\operatorname{Spec} A/G]^{f'^*\mathcal{L}-\mathrm{s}}.$$

Conversely, if a very close degeneration $g \colon \Theta \to [\operatorname{Spec} A/G]$ is non-degenerate, then necessarily $f' \circ g \colon \Theta \to \mathcal{X}$ is also non-degenerate. Suppose, for contradiction, that it is not; then $f' \circ g(0) \simeq f' \circ g(1)$. By the cartesianity of the diagram above, this would imply $g(0) \simeq g(1)$, a contradiction.

This shows that the claim can be verified after the étale base change $f \colon \operatorname{Spec} A^G \to X$. The proof is then a direct application of [GIT, Amplification 1.11]. □

We can now define the notion of a saturated substack:

Definition 2.1.25. Let $\mathcal{X}$ be an algebraic stack admitting a good moduli space $\pi \colon \mathcal{X} \to X$. We say that an open substack $\mathcal{U} \subset \mathcal{X}$ is *saturated* if there exists an open subspace $U \hookrightarrow X$ such that the following diagram is cartesian:

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ U \arrow[r, hook] & X \end{tikzcd}.$$

The above definition is equivalent to requiring that $\mathcal{U} = \pi^{-1}(\pi(\mathcal{U}))$; moreover, this equality can be checked at the level of topological points.

Remark 2.1.26. Under the assumptions of Proposition 2.1.24, the substack $\mathcal{X}^{\mathcal{L}-\mathrm{s}} \subset \mathcal{X}^{\mathcal{L}-\mathrm{ss}}$ is saturated.

Proposition 2.1.27. *Let $\mathcal{X}$ be a geometric algebraic stack with a good moduli space $\pi\colon \mathcal{X} \to X$. Then an open substack $\mathcal{U} \subset \mathcal{X}$ is saturated if and only if every point $p \in \mathcal{U}(k)$ does not admit very close degenerations whose closed point lies in $\mathcal{X} \setminus \mathcal{U}$.*

Moreover, $\mathcal{U}$ is saturated if and only if $|\mathcal{U}| = \pi^{-1}(\pi(|\mathcal{U}|))$.

Proof. Since π is a closed morphism, the substack $\mathcal{U}$ is saturated if and only if the images $\pi(|\mathcal{U}|)$ and $\pi(|\mathcal{X} \setminus \mathcal{U}|)$ are disjoint in the topological space $|X|$, this shows the latter claim. As $\mathcal{X}$ is of finite type over an algebraically closed field k, it suffices to check this condition on k-points. If $\mathcal{U}$ is saturated, then for any $x \in \mathcal{U}(k)$ and $x_0 \in (\mathcal{X} \setminus \mathcal{U})(k)$, we have $\pi(x) \neq \pi(x_0)$. Hence, there cannot exist a very close degeneration of x to x_0, since such a degeneration would force $\pi(x) = \pi(x_0)$.

For the converse, we can check the claim after passing to an étale neighbourhood of the form $\mathcal{X} = [\operatorname{Spec} A/G]$ by the local structure theorem for good moduli spaces (Proposition 2.1.9). Assume by contradiction that there exist $x \in \mathcal{U}$ and $y \in Z$ with the same image under π. Then there exist two lifts to the presentation $\widetilde{x}, \widetilde{y} \in \operatorname{Spec} A(k)$ such that the orbits $G\widetilde{x}$ and $G\widetilde{y}$ both contain a closed orbit $G\widetilde{x}_0$ in their closure. In particular, the image of $\widetilde{x}_0$ under the presentation is contained in Z.

By the destabilization theorem [Alp25, Thm. 7.3.6], there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{x} \in G\widetilde{x}_0$. This provides a very close degeneration of a point of $\mathcal{U}$ to a point of Z, giving a contradiction and proving the claim. □

2.1.1 The space of stability conditions

We now study how the semistable locus changes as we vary the line bundle $\mathcal{L}$. Our approach is similar to that of [Alp+17, Sec. 3], and follows many ideas from classical VGIT [DH98; Tha96]. We obtain a wall-crossing decomposition that is finite, rational, and polyhedral under mild hypotheses. A main difference here is that we do not require $\mathcal{X}$ to be a quotient of a (projective) variety.

We rely on the local structure and ultimately develop stronger results in the case where $\mathcal{X}$ admits a good moduli space.

The following theorem motivates the main construction of this section.

Theorem 2.1.28. [Alp13, Thm. 10.3] *Let $\mathcal{X}$ be a locally noetherian stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. The pullback functor π^* induces an equivalence of categories between vector bundles on X and the full subcategory of vector bundles on $\mathcal{X}$ with trivial stabilizer action at closed points. The inverse is provided by the pushforward functor π_*.*

In particular, the map

$$\pi^*\colon \operatorname{Pic}_{\mathbb{Q}}(X) \to \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$$

is injective. We say that elements in the image of this map are the $\mathbb{Q}$-line bundles that descend *to the good moduli space.*

Definition 2.1.29 (Space of stability conditions)**.** For a stack $\mathcal{X}$, let $W \subset \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ denote the subspace of $\mathbb{Q}$-line bundles that have a power with trivial stabilizer action at closed points. W is a vector space. We define the *space of stability conditions* as

$$\mathbb{E}(\mathcal{X}) := {\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})}\big/{W}.$$

If $\mathcal{X}$ is a locally noetherian stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$, then

$$\mathbb{E}(\mathcal{X}) = {\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})}\big/{\pi^* \mathrm{Pic}_{\mathbb{Q}}(X)}.$$

The assignment $\mathcal{X} \mapsto \mathbb{E}(\mathcal{X})$ is functorial:

Definition-Proposition 2.1.30. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Then f induces a pullback

$$\begin{aligned} f^*\colon \mathbb{E}(\mathcal{Y}) &\to \mathbb{E}(\mathcal{X}), \\ L &\mapsto f^*L \end{aligned}$$

compatible with composition.

Proof. For any point $p \in \mathcal{X}$, restrict to the residual gerbe $\mathcal{G}_p \to \mathcal{X}$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{G}_p \arrow[r] \arrow[d, hook] & \mathcal{G}_{f(p)} \arrow[d, hook] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}.$$

If $L_{|\mathcal{G}_{f(p)}}$ is trivial, then necessarily $f^*L_{|\mathcal{G}_p}$ is also trivial, hence the claim. □

This space admits a natural decomposition.

Definition 2.1.31 (Semistability regions)**.** Define $\{C_i\}_{i\in I}$ to be the partition

$$\mathbb{E}(\mathcal{X}) = \bigsqcup_{i\in I} C_i,$$

where two points lie in the same region if and only if they define the same semistable locus in $|\mathcal{X}|$. The subsets $C_i \subset \mathbb{E}(\mathcal{X})$ are called *semistability regions.* For $L \in \mathbb{E}(\mathcal{X})$ we denote by $\mathcal{C}(L)$ the semistability region containing L. We denote by $\Omega(\mathcal{X}) \subset \mathbb{E}(\mathcal{X})$ the region consisting of those $\mathcal{L} \in \mathbb{E}(\mathcal{X})$ such that $\mathcal{X}^{\mathcal{L}\text{-ss}} \subset \mathcal{X}$ is dense.

Remark 2.1.32. For any algebraic stack $\mathcal{X}$, if $C_j \cap \overline{C_i} \neq \emptyset$, then $\mathcal{X}^{\mathcal{L}\text{-ss}} \subset \mathcal{X}^{\mathcal{N}\text{-ss}}$ for $\mathcal{N} \in C_j$ and $\mathcal{L} \in C_i$.

Remark 2.1.33. The semistability regions C_i and $\Omega(\mathcal{X})$ are cones (i.e., invariant under positive scalar multiplication), but not necessarily convex a priori. Moreover, $\Omega(\mathcal{X}) \subset \mathbb{E}(\mathcal{X})$ is always closed in the Euclidean topology. If $\mathcal{X}$ is irreducible, then $\Omega(\mathcal{X})$ is convex, since for $\mathcal{L}, \mathcal{N} \in \Omega(\mathcal{X})$ we have $\emptyset \neq \mathcal{X}^{\mathcal{L}\text{-ss}} \cap \mathcal{X}^{\mathcal{N}\text{-ss}} \subset \mathcal{X}^{(\mathcal{L}+\mathcal{N})\text{-ss}}$.

Remark 2.1.34. We have a natural isomorphism for disjoint unions:

$$\mathbb{E}(\mathcal{X} \sqcup \mathcal{Y}) \simeq \mathbb{E}(\mathcal{X}) \times \mathbb{E}(\mathcal{Y}),$$

and a natural inclusion in the case of products $\mathcal{X} \times \mathcal{Y}$, where p_i denotes the projection onto the i-th factor:

$$\begin{aligned} \mathbb{E}(\mathcal{X}) \times \mathbb{E}(\mathcal{Y}) &\hookrightarrow \mathbb{E}(\mathcal{X} \times \mathcal{Y}), \\ (L, N) &\mapsto p_1^* L \otimes p_2^* N. \end{aligned}$$

The following is an immediate consequence of Lemma 2.1.16.

Proposition 2.1.35. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the induced map $f^* \colon \mathbb{E}(\mathcal{Y}) \to \mathbb{E}(\mathcal{X})$, we have :*

1. *If f is a $B\mathbb{G}_m$-cover, f^* is injective.*
2. *If f is a Θ-cover, the inclusion f^* is a coarsening of semistable regions i.e., for each semistable region $C \subset \mathbb{E}(\mathcal{X})$, the preimage $f^{*,-1}(C)$ is contained in a unique semistable region of $\mathbb{E}(\mathcal{Y})$.*
3. *If f is surjective and weakly Θ-surjective, along the inclusion f^* we have that the semistable regions of $\mathcal{X}$ are given by the intersections of the semistable regions of $\mathcal{Y}$ with the image of f^*.*

The following result will be improved in the next sections under stronger hypotheses.

Theorem 2.1.36 (Wall-crossing)**.** *Let $\mathcal{X}$ be an algebraic stack over an algebraically closed field k satisfying (†). Then $\mathbb{E}(\mathcal{X})$ is finite-dimensional, and the semistable regions $\{\mathcal{C}_i\}_{i\in I}$ are finitely many, rational, and polyhedral (cf. Definition 2.1.40).*

Lemma 2.1.37. *Let G be an affine linearly reductive algebraic group over an algebraically closed field k, acting on an affine reduced scheme V finite type over k. Assume that every closed point of $[V/G]$ has stabilizer equal to the whole group G. Let $V_0 \subset V$ denote the closed subscheme consisting of closed points of $[V/G]$, and assume that V_0 is connected. Then*

$$\mathbb{E}([V/G]) \simeq \mathbb{X}^*(G)_{\mathbb{Q}}.$$

Proof. We have a diagram

$$\begin{tikzcd} BG \times V_0 \arrow[r, hook, "\iota"] \arrow[dr, "p_2"'] & {[V/G]} \arrow[d, "\pi"] \\ & V_0 \end{tikzcd},$$

where π and p_2 are good moduli spaces. This induces a pullback map

$$\mathbb{E}([V/G]) = \frac{\operatorname{Pic}_{\mathbb{Q}}([V/G])}{\pi^* \operatorname{Pic}_{\mathbb{Q}}(V_0)} \xrightarrow{\iota^*} \mathbb{E}(BG \times V_0) = \frac{\operatorname{Pic}_{\mathbb{Q}}(BG \times V_0)}{p_2^* \operatorname{Pic}_{\mathbb{Q}}(V_0)} \simeq \mathbb{X}^*(G)_{\mathbb{Q}},$$

where the last isomorphism follows from [Bri15, Prop. 2.10, Lemma 2.3], using that G is split.

Given a character $\rho \in \mathbb{X}^*(G)$, consider the element $\mathcal{O}^{\rho}_{[V/G]} \in \operatorname{Pic}_{\mathbb{Q}}([V/G])$ given by twisting the structure sheaf by ρ. Under the identification above, we have $\iota^*\mathcal{O}^{\rho}_{[V/G]} = \rho$. This shows that ι^* is surjective. Now assume that $\mathcal{L} \in \operatorname{Pic}_{\mathbb{Q}}([V/G])$ lies in the kernel of ι^*. Then $\iota^*\mathcal{L} \in p_2^* \operatorname{Pic}_{\mathbb{Q}}(V_0)$. By Theorem 2.1.28, the line bundle $\mathcal{L}$ descends to the good moduli space V_0, hence it is trivial in $\mathbb{E}([V/G])$. This proves the claim. $\square$

Proof of Theorem 2.1.36. By point (2) of Proposition 2.1.35, it suffices to prove the claim for $\mathcal{X} = [\operatorname{Spec} A/G]$, where A is a finitely generated k-algebra and G is an affine linearly reductive group over k. Moreover, let $G_0 \subset G$ be the connected component to the identity, by passing to the $B\mathbb{G}_m$-covering

$$[\operatorname{Spec} A/G_0] \longrightarrow [\operatorname{Spec} A/G],$$

we may assume that G is connected. We prove the finite generation of $\mathbb{E}([\operatorname{Spec} A/G])$ by induction on $r = \dim G$. If $r = 0$, there is nothing to prove. Assume now $r > 0$ and that the statement holds for groups of strictly smaller dimension. By Luna's étale slice theorem, for any closed point $p \in [\operatorname{Spec} A/G](k)$ whose stabilizer $G_p \subsetneq G$ is a proper subgroup, there exists a strongly étale morphism

$$f_p\colon ([W_p/G_p], 0) \to ([\operatorname{Spec} A/G], p)$$

with open image, which is a $B\mathbb{G}_m$-covering onto its image. Let $H \subset \mathcal{X}(k)$ denote the set of such points p, and set

$$V \coloneqq X \setminus \bigcup_{p\in H} f_p(W_p),$$

which is a G-invariant closed subset of X. After extracting a finite subcover, there exists a finite subset $F \subset \mathcal{X}(k)$ such that the induced morphism

$$\begin{array}{ccc} [V/G] \sqcup \bigsqcup_{p\in F}[W_p/G_p] & \longrightarrow & [X/G] \\ \downarrow & & \downarrow \\ V/\!/G \sqcup \bigsqcup_{p\in F} W_p/\!/G_p & \longrightarrow & X/\!/G \end{array}$$

is a $B\mathbb{G}_m$-covering. By the inductive hypothesis and again Proposition 2.1.35, it suffices to prove the claim for $[V/G]$. Let $V_0 \subset V$ denote the closed locus of points whose stabilizer is equal to G. The morphism $[V/G] \to V_0$ is a good moduli space. After decomposing $[V/G]$ into its finitely many connected components, we may assume that V_0 is connected. The conclusion now follows from Lemma 2.1.37 together with the finite generation of $\mathbb{X}^*(G)$.

The second claim is an application of Proposition 2.1.35 to Theorem 2.2.7 (proved in the following sections). $\square$

Finally, we introduce notation used in the following section. For a point $p \in \mathcal{X}$, denote by $\mathbb{X}_*(\operatorname{Stab}_{\mathcal{X}}(p))$ the space of one-parameter subgroups. Every $\lambda\colon \mathbb{G}_m \to \operatorname{Stab}_{\mathcal{X}}(p)$ defines a morphism $\iota_\lambda\colon B\mathbb{G}_m \to \mathcal{X}$ such that $\iota_\lambda(0) = p$, and the induced map $\operatorname{Stab}_{B\mathbb{G}_m}(0) \to \operatorname{Stab}_{\mathcal{X}}(p)$ is λ.

Definition 2.1.38 (Descent pairing)**.** Let $\mathcal{X}/k$ be an algebraic stack and fix a point $p \in |\mathcal{X}|$. We define a pairing

$$\begin{aligned} \langle \_, \_ \rangle \colon \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X}) \times \mathbb{X}_*(\operatorname{Stab}_{\mathcal{X}}(p)) &\to \mathbb{Q}, \\ (L, \lambda) &\mapsto \operatorname{wt}(\iota_\lambda^* L). \end{aligned}$$

For brevity, we write $\operatorname{wt}_\lambda = \operatorname{wt}(\iota_\lambda^* L)$. When $\operatorname{Stab}_{\mathcal{X}}(p)^\circ$ is commutative (e.g., when it is a torus), we can replace the second factor with $\mathbb{X}_*(\operatorname{Stab}_{\mathcal{X}}(p)) \otimes \mathbb{Q}$, as it has a $\mathbb{Z}$-module structure; the pairing then becomes bilinear over $\mathbb{Q}$.

Note that the L-semistable locus coincides with the entire stack $|\mathcal{X}|$ if and only if for every point $p \in |\mathcal{X}|$ and every $\rho\colon \mathbb{G}_m \to \operatorname{Stab}_{\mathcal{X}}(p)$, one has $\langle L, \rho \rangle = 0$.

2.1.2 VGIT for torus actions on affine spaces

We briefly recall some results on VGIT for varieties with a torus action; see [KSZ91; AH99; BH06].

We follow the setup of [BH06, Sec. 2]. Let k be an algebraically closed field, and let $T = \mathbb{G}_{m,k}^n$ be a split torus with character lattice M. Let $X = \operatorname{Spec} R$ be an integral affine scheme of finite type over k, equipped with a T-action. Then

$$R = \bigoplus_{m \in M} R_m,$$

which describes the T-action via the grading. We are interested in understanding the *VGIT chambers* arising from different linearizations of the structure sheaf. We write m-ss for the semistable locus with respect to the structure sheaf twisted by the character $m \in M$. This yields a natural map

$$\iota\colon M \hookrightarrow \operatorname{Pic}([\operatorname{Spec} R/T])\,.$$

In this setting we have (cf. Proposition 2.1.21)

$$[\operatorname{Spec} R/T]^{m\text{-ss}} = [(\operatorname{Spec} R \setminus V((f \mid f \in R_{nm}, n > 0)))/T]\,.$$

We define the *weight cone* as the convex cone in $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ generated by all $w \in M$ for which the stable locus is nonempty:

$$\Omega_T(X) \coloneqq \operatorname{cone}(w \in M \mid R_w \neq 0).$$

This is a closed, finitely generated convex cone because R is finitely generated and X is integral. The cone $\Omega_T(X)$ plays the role of $\Omega([\operatorname{Spec} R/T])$.

Definition 2.1.39 ([BH06], Def. 2.8)**.** For a character m, define the *GIT cone*

$$\sigma(m) = \bigcap_{x \in [X/T]^{m\text{-ss}}} \{v \in M_{\mathbb{Q}} \mid x \text{ is } v\text{-semistable}\}.$$

We denote by $\Sigma_T(X)$ the collection of these cones and set $\omega(x) \coloneqq \{v \in M_{\mathbb{Q}} \mid x \text{ is } v\text{-semistable}\}$.

By construction, the cones above are convex, but a priori their relative interiors do not necessarily define semistability regions in the sense of Definition 2.1.31. Moreover, [BH06, Prop. 2.5] guarantees that there are only finitely many GIT cones. We now introduce some basic notions from convex geometry.

Definition 2.1.40 (Faces and polyhedral cones)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$, and let $C \subset V$ be a convex cone. We say that $F \subset \overline{C}$ is a *face of* C if there exists $f \in V^*$ such that

$$\overline{C} \subset H_f^+ \coloneqq \{x \mid f(x) \geq 0\} \quad \text{and} \quad F = \overline{C} \cap \{x \mid f(x) = 0\}.$$

A cone is said to be *polyhedral* if it has finitely many faces.

Definition 2.1.41 (Rational cones)**.** Let V be a finite-dimensional vector space over $\mathbb{Q}$, and let $C \subset V$ be a cone. We define the *realification of* C as the cone

$$C_{\mathbb{R}} = \{x \mid x = \sum \lambda_i v_i \text{ for some } \lambda_i \in \mathbb{R},\, v_i \in V\} \subset V \otimes_{\mathbb{Q}} \mathbb{R}.$$

In this setting, a face of a cone $C \subset V$ is a face of $C_{\mathbb{R}}$. We say that a face is *rational* if it is defined by an element $f \in V^* \subset V_{\mathbb{R}}^*$. More generally, a cone $C \subset V$ is *rational* if all of its faces are rational.

Definition 2.1.42 (Relative interior)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{Q}$, and let $\sigma \subset V$ be a cone. We define the *relative interior of* σ as

$$\sigma^\circ \coloneqq \{\, x \in \sigma \mid \exists\, \varepsilon > 0 \text{ such that } x + B_{\operatorname{Span}\sigma}(0, \varepsilon) \subseteq \sigma \,\},$$

where $B_{\operatorname{Span}(\sigma)}(0, r)$ denotes the open ball of radius r in the subspace $\operatorname{Span}\sigma \subset V$.

Notice that the relative interior of the realification of a cone coincides with the realification of its relative interior.

Definition 2.1.43 (Fan and quasifan)**.** Let V be a finite-dimensional real vector space. A *quasifan* is a finite collection Σ of convex polyhedral cones in V such that, for every $\sigma \in \Sigma$, all faces of σ belong to Σ, and for any $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both.
The *support of* Σ is the cone $\cup\Sigma \subset V$. We say that Σ is *complete* if $V = \cup\Sigma$.
A *fan* is a quasifan Σ such that every cone is *strictly convex*, i.e., contains no line.
If V is a vector space over $\mathbb{Q}$, then a fan is a fan Σ in $V_{\mathbb{R}}$. In this case, we say that Σ is *rational polyhedral* if every cone in Σ is rational and polyhedral.

Finally, we say that a finite collection of mutually disjoint regions $\{\mathcal{C}_i\}_{i \in I}$ in V *define a fan* if $\{\overline{\mathcal{C}_i} \subset V \mid i \in I\}$ is a fan and $\mathcal{C}_i$ is the relative interior of $\overline{\mathcal{C}_i}$ for each $i \in I$. The same terminology applies when V is rational, upon replacing $\mathcal{C}_i$ with their realifications.

Remark 2.1.44. A collection of cones Σ is a fan if and only if it is a quasifan and $\{0\} \in \Sigma$.

We now introduce a slightly weaker notion of fan, required to state Theorem 2.2.7. We give an operative definition, tailored to our needs.

Definition 2.1.45 (Pseudofan)**.** Let Σ be a collection of cones in a finite-dimensional real vector space V. We say that Σ is a *pseudofan* if there exist fans Σ_i in real vector spaces V_i, for finitely many indices $i \in I$, and an injective map $f\colon V \to \prod_{i\in I} V_i$ such that the following construction holds. Set

$$\Sigma_i' = \Sigma_i \cup \{\overline{V_i \setminus \cup\Sigma_i}\},$$

and consider the fan in $\prod_{i\in I} V_i$ given by $\Sigma' = \prod_{i\in I} \Sigma_i'$. Then we require

$$\Sigma = \{\, C \subset V \mid C = f^{-1}(D) \text{ for some } D \in \Sigma' \,\}.$$

When V is rational, we define a *pseudofan* as a pseudofan Σ in $V_{\mathbb{R}}$. In that case, we say that Σ is *rational polyhedral* if $V_i = V'_{i,\mathbb{R}}$ for some subspaces $V_i' \subset V$, the Σ_i' are rational polyhedral, and f is induced by a map $V \to \bigoplus_{i\in I} V_i'$.

Notice that by design a pseudofan is always complete.

Theorem 2.1.46 ([BH06], Thm. 2.11)**.** *The collection of all GIT cones $\Sigma_T(X)$ forms a rational polyhedral quasifan in the vector space $M_{\mathbb{Q}}$ with support $\Omega_T(X)$.*

As a consequence of the previous theorem, we obtain a characterization of the semistability regions.

Proposition 2.1.47. *Consider the morphism $p\colon M_{\mathbb{Q}} \to \mathbb{E}([X/T])$ induced by ι. For any cone $\sigma \in \Sigma_T(X)$, the relative interior σ° is the inverse image of a semistability region in $M_{\mathbb{Q}}$ containing $p(m)$ for any $m \in \sigma^\circ$. The semistability regions corresponding to a nonempty semistable locus in $\mathbb{E}([X/T])$ form a rational polyhedral fan with support $\Omega([X/T])$. In particular, the semistability regions form a complete pseudofan in $\mathbb{E}([X/T])$.*

Proof. We first show that, for any $m \in M_{\mathbb{Q}}$, the cone $\sigma(m) \in \Sigma_T(X)$ satisfies that $\sigma(m)^\circ$ is contained in $p^{-1}(\mathcal{C})$ for a unique semistability region. Suppose this is not the case. Then there exists $m' \in \sigma(m)$ such that $\mathcal{X}^{m-\mathrm{ss}} \subsetneq \mathcal{X}^{m'-\mathrm{ss}}$. This implies $\sigma(m') \subsetneq \sigma(m)$, since for $x \in \mathcal{X}^{m'-\mathrm{ss}} \setminus \mathcal{X}^{m-\mathrm{ss}}$ we have $m \notin \omega(x)$. By Theorem 2.1.46, $\sigma(m') \cap \sigma(m)$ is a face of $\sigma(m)$, which is a contradiction because $m' \in \sigma(m') \cap \sigma(m)$ lies in the relative interior of $\sigma(m)$.
Finally, given two distinct cones $\sigma(m), \sigma(m') \in \Sigma_T(X)$, there exists (without loss of generality) a point $x \in \mathcal{X}$ such that $\sigma(m) \subset \omega(x)$ but $\sigma(m') \not\subset \omega(x)$. This shows that m and m' lie in different semistability regions. Since no semistability region contains a line in the stability condition space, the last claim follows. □

We can now restate [BH06, Prop. 2.9] in the following form.

Proposition 2.1.48. *We have:*

- $[X/T]^{m-\mathrm{ss}} \subset [X/T]^{m'-\mathrm{ss}}$ *if and only if* $\mathcal{C}(\mathcal{O}_{\mathcal{X}}^{m}) \subset \mathcal{C}(\mathcal{O}_{\mathcal{X}}^{m'})$.
- $[X/T]^{m-\mathrm{ss}} = [X/T]^{m'-\mathrm{ss}}$ *if and only if* $\mathcal{C}(\mathcal{O}_{\mathcal{X}}^{m}) = \mathcal{C}(\mathcal{O}_{\mathcal{X}}^{m'})$.

We next characterize GIT chambers using a theorem from [AH06].

Definition 2.1.49 (Generating pair)**.** We say that a pair $u, v \in \Omega_T(\operatorname{Spec} R) \cap M$ is *generating* if there exists an $\ell > 0$ such that, for any $k > 0$, the multiplication map

$$R_{k\ell u} \otimes_k R_{k\ell v} \to R_{k\ell(u+v)}, \qquad f \otimes g \mapsto fg$$

is surjective.

Theorem 2.1.50 ([AH06, Thm. 1.1])**.** *Let R and M be as above, and let $\Sigma_T(\operatorname{Spec} R)$ be the fan given by the GIT cones. Then:*

(i) If $u, v \in \Omega(\operatorname{Spec} R) \cap M$ form a generating pair, then the weights u and v lie in a common GIT cone $\sigma \in \Sigma_T(\operatorname{Spec} R)$.

(ii) If $u, v \in \Omega(\operatorname{Spec} R) \cap M$ lie in a common GIT cone $\sigma \in \Sigma_T(\operatorname{Spec} R)$ and u belongs to the relative interior $\sigma^\circ \subseteq \sigma$, then u, v form a generating pair.

This has a direct consequence for the variation of GIT.

Proposition 2.1.51. *Let $\sigma \in \Sigma_T(\operatorname{Spec} R)$ and let $u \in \sigma^\circ$ and $v \in \sigma \setminus \sigma^\circ$. Then*

$$[X/T]^{u-\mathrm{ss}} = ([X/T]^{v-\mathrm{ss}})^{u-\mathrm{ss}}.$$

Proof. For $m \in \Omega_T(X) \cap M_{\mathbb{Q}}$, the m-semistable locus on X is $\operatorname{Spec} R \setminus V(I_m)$, where $I_m = (f \mid f \in R_{nm}$ for some $n > 0)$. The inclusion $\supset$ holds for any choice of $v, w \in M_{\mathbb{Q}}$, so the claim is equivalent to showing that

$$(\operatorname{Spec} R \setminus V(I_v))^{u-\mathrm{ss}} \subset \operatorname{Spec} R \setminus V(I_u).$$

The u-semistability is determined by the nonvanishing of some section of the u-twisted structure sheaf on $\operatorname{Spec} R \setminus V(I_v)$. Consider the affine cover given by $\operatorname{Spec} R_f$ for $f \in R_{lv}$ with $l > 0$. Then any invariant section can be written as $g = {}^{h}\!/\!_{f^n}$, where $h \in R_{qu+lnv}$ for some $q, n \geq 0$. For a point $x \in \operatorname{Spec} R_f$ with $g(x) \neq 0$, after replacing g by a suitable power g^j, the following map is surjective by Theorem 2.1.50:

$$R_{jqu} \otimes_k R_{jlnv} \to R_{j(qu+lnv)}, \qquad f \otimes g \mapsto fg.$$

Hence, there exists a section $\ell \in R_{jqu}$ not vanishing at x, which proves the claim. □

The previous result does not hold if u, v do not both belong to $\Sigma_T(X)$.

Example 2.1.52. Consider $[\mathbb{A}^1/\mathbb{G}_m]$. The cocharacter lattice is $M_{\mathbb{Q}} = \langle \chi \rangle_{\mathbb{Q}} \simeq \mathbb{Q}$, where χ is the character given by the identity. There are three semistability regions: the origin and the two half-lines. We have $[\mathbb{A}^1/\mathbb{G}_m]^{\chi-\mathrm{ss}} = 1 \subset [\mathbb{A}^1/\mathbb{G}_m]$ and $[\mathbb{A}^1/\mathbb{G}_m]^{-\chi-\mathrm{ss}} = \emptyset$. In particular, $([\mathbb{A}^1/\mathbb{G}_m]^{\chi})^{-\chi} = \emptyset \neq [\mathbb{A}^1/\mathbb{G}_m]^{-\chi}$, providing the desired example.

Example 2.1.53. Consider $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m^2]$, where $\mathbb{G}_m^2$ acts with weights $(1,0)$ and $(0,1)$ on the coordinates (x, y). The space $\mathbb{E}(\mathcal{X})$ is two-dimensional, generated by the cocharacters given by the projections onto the coordinate factors $\mathbb{G}_m^2 \to \mathbb{G}_m$. The semistability regions define a quasifan, which is not a fan, as in the figure above. Where σ_2 correspond to the empty semistable locus.

In the next section, we extend the above results to the case of stacks admitting a good moduli space.

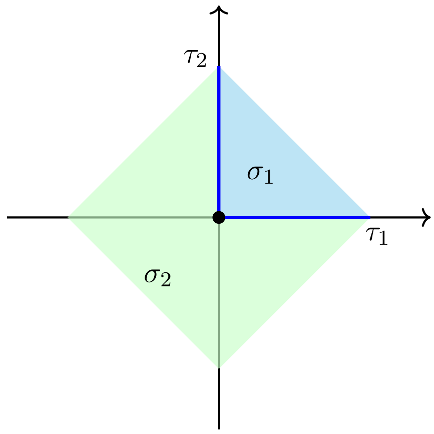


Figure 2.1. The cones $\sigma_1, \sigma_2, \tau_1, \tau_2$ correspond respectively to the semistable locus given by the set of points $\{(1,1)\}, \emptyset, \{(1,1),(0,1)\}, \{(1,1),(1,0)\}$.

2.2 Relative Cox ring

Let $\mathcal{X}$ be an algebraic stack, and fix a group morphism $\mathcal{L}\colon V \to \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$, where V is an abelian group. We define the $\mathcal{O}_{\mathcal{X}}$-module

$$\mathcal{R}_V(\mathcal{X}) \coloneqq \bigoplus_{t\in V} \mathcal{L}(t),$$

which comes with a structure morphism $\Psi_V\colon \mathcal{R}_V \to \mathcal{X}$, of finite type if V is finitely generated. Moreover, we have a natural action of the Cartier dual of V, given by multiplication:

$$T = \mathrm{Hom}_{\mathrm{grp}}(V, \mathbb{G}_m) \curvearrowright \mathcal{R}_V(\mathcal{X}).$$

When V is torsion-free, the group T is a split torus whose characters are naturally identified with V. We are mostly interested in this case. When $V = \mathbb{Z}$, then $\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}_V(\mathcal{X})$ is the *total space* of $\mathcal{L}(1)$. More generally Ψ_V is a T-torsor, and the structural morphism $[\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}_V/T] \to \mathcal{X}$ is an isomorphism.

Proposition 2.2.1. *Let $\mathcal{L}\colon V \to \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ be as above, and let $s \in V$. Then the line bundle $\mathcal{L}(s)$ is given, on the presentation $\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}_V \to [\mathcal{R}_V/T] = \mathcal{X}$, by the twist of the structure sheaf on $\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}_V$ corresponding to the character associated with $s \in V$.*

Proof. Let $\mathcal{L}(s) = L$, We can factor ψ_V as follows:

$$\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}_V = \mathrm{Spec}_{\mathcal{X}} \bigoplus_{t\in V} \mathcal{L}(t) \xrightarrow{\psi_1} [\mathrm{Spec}_{\mathcal{X}} \bigoplus_{i\in\mathbb{Z}} L^i/\mathbb{G}_m] \xrightarrow{\psi_2} \mathcal{X}.$$

The pullback $\psi_2^* L$ is given by the $\mathcal{O}_{\mathcal{X}}$-algebra $(\bigoplus_{i\in\mathbb{Z}} L^i) \otimes_{\mathcal{O}_{\mathcal{X}}} L$, equipped with a $\mathbb{G}_m$-action of weight 1 on the first factor and trivial on the second. Hence, $\psi_2^* L$ is the structure sheaf twisted by the character $\mathbb{G}_m \to \mathbb{G}_m$ given by the identity. The map ψ_1 is the natural inclusion $\bigoplus_{n\in\mathbb{Z}} L^i \subset \bigoplus_{t\in V} \mathcal{L}(t)$, which is equivariant with respect to the character $T \to \mathbb{G}_m$ associated with $\mathcal{L} \in V$. Therefore, $\psi_V^* L$ is the structure sheaf $\mathcal{O}_{\mathcal{R}_V}$ twisted by the character $s \in V$. $\square$

Definition 2.2.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism, and fix $\mathcal{L}\colon V \to \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ as above. We define

$$\mathrm{Cox}_V(\mathcal{X}/\mathcal{Y}) \coloneqq f_*\mathcal{R}_V = \bigoplus_{t\in V} f_*\mathcal{L}(t),$$

the *Cox ring of* $f\colon \mathcal{X} \to \mathcal{Y}$.

The action of T on $\mathcal{R}_V$ induces an action $T = \mathrm{Hom}(V, \mathbb{G}_m) \curvearrowright \mathrm{Cox}_V(f)$, yielding the morphism

$$\mathcal{X} = [\mathcal{R}_V/T] \to [\mathrm{Spec}_{\mathcal{Y}} \mathrm{Cox}_V(\mathcal{X}/\mathcal{Y})/T].$$

When $\pi\colon \mathcal{X} \to X$ is a good moduli space, we set

$$\underline{\mathrm{Cox}}_V(\mathcal{X}) \coloneqq \bigoplus_{t\in V} \pi_*\mathcal{L}(t).$$

We write $\underline{\mathrm{Cox}}(\mathcal{X})$ when the induced map $V \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\mathcal{L}_{\mathbb{Q}}} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}) \to \mathbb{E}(\mathcal{X})$ is an isomorphism.

Proposition 2.2.3. *Let $\pi\colon \mathcal{X} \to X$ be a good moduli space. Then:*

1. *If V is a finitely generated group, then $\underline{\mathrm{Cox}}_V(\mathcal{X})$ is a finitely generated $\mathcal{O}_X$-algebra.*
2. *The natural map $\eta_V\colon \mathrm{Spec}_{\mathcal{X}} \mathcal{R}_V \to \mathrm{Spec}_X \underline{\mathrm{Cox}}_V(\mathcal{X})$ is a good moduli space.*
3. *The morphism $[\mathrm{Spec}_X \underline{\mathrm{Cox}}_V(\mathcal{X})/T] \to X$ is a good moduli space.*
4. *For any morphism $f\colon \mathcal{X} \to S$ to an algebraic space S and $V \to \mathrm{Pic}(\mathcal{X})$, the induced map $[\mathcal{R}_V/T] \to [\mathrm{Cox}(\mathcal{X}/S)/T]$ factors through*

$$\mathcal{X} = [\mathrm{Spec}_{\mathcal{X}} \mathcal{R}_V/T] \to [\underline{\mathrm{Cox}}_V(\mathcal{X})/T] \to [\underline{\mathrm{Cox}}_V(\mathcal{X}/S)/T],$$

where the factorization is provided by the universal property of good moduli spaces.

Proof. The first claim can be checked étale-locally on X, hence we may assume X affine. In this case, it follows from [Alp13, Lemma 4.14]. The second claim follows from [Alp13, Lemma 4.14].
For point (3), it suffices to compute the T-invariants of the $\mathcal{O}_X$-algebra $\bigoplus_{t\in V} \mathcal{L}(t)$, which is the degree-zero subalgebra $\mathcal{O}_{\mathcal{X}}$, proving the claim.
Point (4) follows directly from the universal property of good moduli spaces. □

Definition 2.2.4. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, and fix $\mathcal{L}\colon V \to \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ as above. For any $t \in V$, we define $\underline{\mathcal{L}}(t) \in \mathrm{Pic}([\mathrm{Spec}_{\mathcal{Y}} \mathrm{Cox}_V(\mathcal{X}/\mathcal{Y})/T])$ to be the line bundle given by the structure sheaf of $\mathrm{Spec}_{\mathcal{Y}} \mathrm{Cox}_V(\mathcal{X})$ twisted by the character $T \to \mathbb{G}_m$ associated with t. In particular, this defines a map of abelian groups

$$\begin{aligned} V &\to \mathrm{Pic}([\mathrm{Spec}_{\mathcal{Y}} \underline{\mathrm{Cox}}_V(\mathcal{X}/\mathcal{Y})/T]),\\ t &\mapsto \underline{\mathcal{L}}(t). \end{aligned}$$

Proposition 2.2.5. *In the setting of Definition 2.2.4, we have* $\eta^*\underline{\mathcal{L}}(t) \simeq \mathcal{L}(t)$, *where*

$$\eta\colon \mathcal{X} \to [\operatorname{Spec}_{\mathcal{Y}} \operatorname{Cox}_V(\mathcal{X}/\mathcal{Y})/T].$$

Proof. The claim is a consequence of $\mathcal{X} \simeq [\operatorname{Spec}_{\mathcal{X}} \mathcal{R}_V/T]$ and the map $\operatorname{Spec}_{\mathcal{X}} \mathcal{R}_V \to \operatorname{Spec}_{\mathcal{Y}} \operatorname{Cox}_V(\mathcal{X}/\mathcal{Y})$ being T-equivariant. □

Proposition 2.2.6. *Let* $\mathcal{X}$ *be a geometric algebraic stack admitting a good moduli space* $\pi\colon \mathcal{X} \to X$. *The morphism*

$$p\colon [\operatorname{Spec}_{\mathcal{X}} \mathcal{R}_V/T] \to [\operatorname{Spec}_X \underline{\operatorname{Cox}}(\mathcal{X})/T]$$

satisfies

$$p^{-1}\left([\operatorname{Spec}_X \underline{\operatorname{Cox}}(\mathcal{X})/T]^{\underline{\mathcal{L}}(s)-\mathrm{ss}}\right) = [\operatorname{Spec}_{\mathcal{X}} \mathcal{R}_V/T]^{\mathcal{L}(s)-\mathrm{ss}},$$

for any $s \in V$.

Proof. It suffices to prove the statement after passing to an étale covering of X. By Proposition 2.1.9, we may assume $\mathcal{X} = [\operatorname{Spec} A/G]$. In this case, we have the commutative diagram

$$\begin{tikzcd}
{[\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t)/(G \times T)]} \arrow[r, "p"] \arrow[d] & {[\operatorname{Spec}_{A^G} \bigoplus_{t\in V} \pi_*\mathcal{L}(t)/T]} \arrow[d] \\
{[\operatorname{Spec} A/G]} \arrow[r, "\pi"] & \operatorname{Spec} A^G
\end{tikzcd}.$$

Denote by $\sigma\colon G \times \operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t) \to \operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t)$ the induced G-action. The action of $T \times G$ can be written as

$$\begin{aligned}\tau^*\colon \bigoplus_{t\in V} \mathcal{L}(t) &\to k[t \in V] \otimes_k \mathcal{O}(G) \otimes_k \bigoplus_{t\in V} \mathcal{L}(t),\\ \mathcal{L}(t) \ni l &\mapsto t \otimes \sigma^*(l).\end{aligned}$$

The commutativity of the T and G actions is intrinsic to the construction. Thanks to the description provided in Proposition 2.1.21, for a character $s \in V$, the $\underline{\mathcal{L}}(s)$-unstable locus of $[\operatorname{Spec}_{A^G} \bigoplus_{t\in V} \pi_*\mathcal{L}(t)/T]$ is defined by the vanishing of

$$I_s = (f \mid f \in \pi_*\mathcal{L}(ns) \text{ for some } n > 0).$$

Similarly, the $\mathcal{L}(s)$-unstable locus of $[\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t)/(G \times T)]$ is defined by the vanishing of

$$J_s = (f \in \bigoplus_{t\in V} \mathcal{L}(t) \mid \tau^*(f) = \chi_s^{*,-n}(q) f \text{ for some } n > 0),$$

where $\chi_s\colon G \times T \to T \to \mathbb{G}_m = \operatorname{Spec} k[q^{\pm 1}]$ is the character associated with s. For $f \in \bigoplus_{t\in V} \mathcal{L}(t)$, the equality $\tau^*(f) = \chi_s^{*,-n}(q) f$ holds if and only if $f \in \mathcal{L}(ns)$ and $\sigma(f) = f$, i.e. f is an invariant section $f \in H^0([\operatorname{Spec} A/G], \mathcal{L}(ns)) = \pi_*\mathcal{L}(ns)$. Therefore, $\pi^{-1}(V(I_s)) = V(J_s)$, and the claim follows by taking complements. □

Under the assumption of Proposition 2.2.6, the following commutative diagram can help to visualize the result of the previous proposition

$$\begin{tikzcd}
& {[\mathcal{R}_V/T]} \arrow[rr] \arrow[dd, "\sim"] & & {[\operatorname{Spec}_X \underline{\operatorname{Cox}}(\mathcal{X})/T]} \arrow[dd] \\
{[\mathcal{R}_V/T]^{\mathcal{L}(s)-\mathrm{ss}}} \arrow[ur] \arrow[rr] \arrow[dd, "\sim"] & & {[\operatorname{Spec}_X \underline{\operatorname{Cox}}(\mathcal{X})/T]^{\mathcal{L}(s)-\mathrm{ss}}} \arrow[ur] \arrow[dd] & \\
& \mathcal{X} \arrow[rr, "\pi"] & & X \\
\mathcal{X}^{\mathcal{L}(s)-\mathrm{ss}} \arrow[ur] \arrow[rr] & & \operatorname{Proj}_X \bigoplus_{n\geq 0} \pi_*\mathcal{L}(ns) \arrow[ur] &
\end{tikzcd},$$

where the top square is cartesian. This complete the reduction of VGIT to the case of an action of a split torus on an affine scheme. We are now ready for the main result of the section:

Theorem 2.2.7. *Let $\mathcal{X}$ be a geometric algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. Then the collection of semistability regions $\{C_i\}_{i\in I}$ are rational polyhedral cones and define a pseudofan in $\mathbb{E}(\mathcal{X})$. The rational polyhedral cone $\Omega(\mathcal{X})$ is a union of semistability regions, and these form a fan with support $\Omega(\mathcal{X})$. In particular, every region in $\Omega(\mathcal{X})$ is strictly convex.*

Proof. Thanks to Proposition 2.1.35, for the first claim it suffices to prove the statement after passing to a surjective and Θ-surjective cover of X. We can then apply Proposition 2.1.9 and reduce to the case $\mathcal{X} = [\operatorname{Spec} A/G]$. Since the morphism $[\operatorname{Spec} A/G] \to \mathcal{X}$ is étale and surjective, the preimage of a dense open is dense. This implies that, under the inclusion $i\colon \mathbb{E}(\mathcal{X}) \to \mathbb{E}([\operatorname{Spec} A/G])$, the image of $\Omega(\mathcal{X})$ is $\operatorname{Im} i \cap \Omega([\operatorname{Spec} A/G])$. Hence, the claim reduces to the case $\mathcal{X} = [\operatorname{Spec} A/G]$.

Moreover, we may further reduce to the case where $\operatorname{Spec} A$ is a disjoint union of G-invariant spectra of integral normal rings A_i, as we now explain. Since G is linearly reductive, we have $G = G_0 \rtimes \Gamma$, where G_0 is a smooth connected group and Γ is finite. Let $\operatorname{Spec} B \to \operatorname{Spec} A$ be the normalization map from the disjoint union of irreducible components with the reduced structure. Since k is algebraically closed, we can lift the action $G_0 \curvearrowright \operatorname{Spec} A$ to $\operatorname{Spec} B$. For each irreducible component of $\operatorname{Spec} B$, let $\operatorname{Spec} \widehat{B} \to \operatorname{Spec} B$ be its normalization. Given that G_0 is smooth, the normalization of $G_0 \times \operatorname{Spec} B$ is $G_0 \times \operatorname{Spec} \widehat{B}$ by [Sta25, Lemma 07TD], and the action lifts to the normalization. We then have morphisms

$$\left[\operatorname{Spec} \widehat{B} \middle/ G_0\right] \to \left[\operatorname{Spec} B \middle/ G_0\right] \to \left[\operatorname{Spec} A \middle/ G_0\right] \to \left[\operatorname{Spec} A \middle/ G\right].$$

The last map is surjective and Θ-surjective because Γ is discrete, while the other two are induced by proper morphisms $\operatorname{Spec} \widehat{B} \to \operatorname{Spec} A$ and hence are Θ-surjective by Lemma 2.1.15. Thus, we can then reduce to the case $\operatorname{Spec} A$ is a disjoint union of G-invariant spectra of integer normal rings A_i.

Assume $\mathcal{X} = [\operatorname{Spec} A/G]$ with $\operatorname{Spec} A$ normal and irreducible. Then the first claim of Theorem 2.1.36 ensures that $\mathbb{E}(\mathcal{X})$ is finite-dimensional. Let $V \simeq \mathbb{Q}^n \xrightarrow{\mathcal{L}} \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X}) \to \mathbb{E}(\mathcal{X})$ be an isomorphism induced by a chosen basis of $\mathbb{E}(\mathcal{X})$. Construct the morphism

p as in Proposition 2.2.6 using $\mathcal{L}$. The pullback p^* then induces an isomorphism of semistability regions:

$$\mathbb{E}(\operatorname{Spec}_{A^G} \underline{\operatorname{Cox}}([\operatorname{Spec} A/G])/T) \xrightarrow{\sim} \mathbb{E}([\operatorname{Spec} A/G]).$$

The claim then follows from Proposition 2.1.47 applied to the action of the affine torus T on $\underline{\operatorname{Cox}}([\operatorname{Spec} A/G])$. The general case $\mathcal{X} = \bigsqcup_i [\operatorname{Spec} A_i/G]$ follows from Remark 2.1.34. $\square$

Definition 2.2.8. Let $\mathcal{X}$ be a geometric algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. The *semistability fan* is the rational polyhedral fan

$$\Sigma(\mathcal{X}) \coloneqq \{\sigma \subset \Omega(\mathcal{X}) \mid \sigma \text{ is closed and } \sigma^\circ \text{ is a semistability region in } \mathbb{E}(\mathcal{X})\}\,.$$

For a cone $\sigma \in \Sigma(\mathcal{X})$ we write $\mathcal{X}(\sigma) \to X(\sigma)$ for the open substack $\mathcal{X}^{L-\mathrm{ss}} \subset \mathcal{X}$ and its good moduli space, for any line bundle $L \in \sigma^\circ$.

We have the following generalization of Proposition 2.1.51:

Theorem 2.2.9. *Let $\mathcal{X}$ be a geometric algebraic stack that admit a good moduli space $\pi\colon \mathcal{X} \to X$. Then for any $\sigma \in \Sigma(\mathcal{X})$ we have*

$$\mathcal{X}^{\mathcal{L}(s)-\mathrm{ss}} = (\mathcal{X}^{\mathcal{L}(v)-\mathrm{ss}})^{\mathcal{L}(s)-\mathrm{ss}}$$

for any $s \in \sigma^\circ$ and $v \in \sigma \setminus \sigma^\circ$.

Proof. It suffices to prove the statement after passing to a surjective and Θ-surjective cover of X. Arguing as in the proof of Theorem 2.2.7, we can assume that $\mathcal{X} = [\operatorname{Spec} A/G]$ with A normal. Let $\sigma\colon G \times \operatorname{Spec} A \to \operatorname{Spec} A$ denote the action. Fix an isomorphism $V \xrightarrow{\mathcal{L}} \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X}) \to \mathbb{E}(\mathcal{X})$ induced by a chosen basis of $\mathbb{E}(\mathcal{X})$. We then have the commutative diagram

$$\begin{array}{ccc} [\operatorname{Spec}_A(\bigoplus_{t\in V} \mathcal{L}(t))/(G\times T)] & \xrightarrow{p} & [\operatorname{Spec}_{A^G}(\bigoplus_{t\in V} \pi_*\mathcal{L}(t))/T] \\ \downarrow & & \downarrow \\ [\operatorname{Spec} A/G] & \xrightarrow{\pi} & \operatorname{Spec} A^G \end{array}.$$

Assume that the following equality holds:

$$p^{-1}\left(\left(\left[\operatorname{Spec}_{A^G} \bigoplus_{t\in V} \pi_*\mathcal{L}(t) \big/ T\right]^{\underline{\mathcal{L}(v)}-\mathrm{ss}}\right)^{\underline{\mathcal{L}(s)}-\mathrm{ss}}\right) = \left(\left[\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t) \big/ (G\times T)\right]^{\mathcal{L}(v)-\mathrm{ss}}\right)^{\mathcal{L}(s)-\mathrm{ss}},$$

Then we may assume that G is a torus by Proposition 2.2.6, and the toric case is established in Proposition 2.1.51. We now verify the equality above by writing the loci explicitly, using the same notation as in the proof of Theorem 2.2.7.

The $\mathcal{L}(v)$-unstable locus of $[\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t)/(G\times T)]$ is given by the vanishing of

$$J_v = (f \in \bigoplus_{t\in V} \mathcal{L}(t) \mid \tau^*(f) = \chi_v^{*,-n}(q) f \text{ for some } n > 0),$$

where χ_v denotes the character corresponding to v. As established in the proof of Theorem 2.2.7, we have $\tau^*(f) = \chi_t^{*,-n}(q)f$ when f is an invariant section $f \in H^0([\operatorname{Spec} A/G], \mathcal{L}(nt)) = \pi_*\mathcal{L}(nt)$. A similar description holds for the $\underline{\mathcal{L}(v)}$-invariant locus of $[\operatorname{Spec}_{A^G} \bigoplus_{t\in V} \pi_*\mathcal{L}(t)/T]$.

Therefore, We can cover the $\underline{\mathcal{L}(v)}$-semistable locus with affines given by the non-vanishing of G-invariants in J_v. Moreover, for such an $f \in J_V$ we have

$$p^{-1}((\operatorname{Spec}_{A^G} \bigoplus_{t\in V} \pi_*\mathcal{L}(t))_f) = (\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t))_f.$$

An $\mathcal{L}(s)$-stable point x in $(\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t))_f$ is then characterized by the non-vanishing of some section of the structure sheaf of $\operatorname{Spec}_A \bigoplus_{t\in V} \mathcal{L}(t) \setminus V(J_v)$. On our covering, this section can be written as an invariant section for the twisted action by χ_s, namely $g = {}^{h}\!/_{f^n}$ with $h \in \pi_*\mathcal{L}(qs + lnv)$ for some $q, n \geq 0$. The same description holds for $(\operatorname{Spec}_{A^G} \bigoplus_{t\in V} \pi_*\mathcal{L}(t))_f$, hence the claim. □

Proposition 2.2.10. *Let $\mathcal{X}$ be a geometric algebraic stack admitting a good moduli space, and let $\sigma, \tau \in \Sigma(\mathcal{X})$. Then*

$$\mathcal{X}(\sigma) \subset \mathcal{X}(\tau) \quad \Leftrightarrow \quad \tau \subset \overline{\sigma}.$$

In this case we have $\mathcal{X}(\sigma) = \mathcal{X}(\tau)^{L_{|\mathcal{X}(\tau)}-\mathrm{ss}}$, where $L \in \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$ has image in σ° under the quotient map $\operatorname{Pic}_{\mathbb{Q}}(\mathcal{X}) \to \mathbb{E}(\mathcal{X})$.

Proof. The equivalence follows from Proposition 2.1.48, after reducing to the case of a torus action on an affine scheme, as in the proof of Theorem 2.2.7. In that case, the final claim follows from Theorem 2.2.9. □

2.2.1 The semistability fan

We collect here some results about the semistability fan. We work with geometric algebraic stacks admitting a good moduli space.

Proposition 2.2.11 (Descent of line bundles)**.** *Let $\mathcal{X}$ be a geometric algebraic stack with a good moduli space. Fix a cone $\sigma \in \Sigma(\mathcal{X})$. Then, for any $L \in \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$,*

$$\overline{L} \in \operatorname{Span}_{\mathbb{Q}} \sigma \subset \mathbb{E}(\mathcal{X}) \implies L_{|\mathcal{X}(\sigma)} \text{ descends to the good moduli space } X(\sigma).$$

If the cone $\Sigma(\mathcal{X})$ is top dimensional, then the implication is an equivalence. When no confusion can arise, we will say that $\mathcal{L} \in \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$ descends *to $X(\sigma)$ to mean that $L_{|\mathcal{X}(\sigma)}$ descends to it.*

Proof. Let $p \in |\mathcal{X}|$ be an L-semistable closed point, where $L \in \operatorname{Pic}_{\mathbb{Q}}(\mathcal{X})$. The residual gerbe at p, is BG with G linearly reductive. The restriction of L corresponds to a character $\rho\colon G \to \mathbb{G}_m$. Assume by contradiction that it is nontrivial, then there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\rho \circ \lambda$ is a nontrivial endomorphism of $\mathbb{G}_m$, given by $t \mapsto t^n$; up to replacing λ by λ^{-1}, we assume $n > 0$. Consider the very close degeneration

$$f\colon \Theta \to B\mathbb{G}_m \xrightarrow{\lambda} BG \hookrightarrow \mathcal{X},$$

where the first map is induced on a presentation by $\mathbb{A}^1 \to \operatorname{Spec} k$. Then $\operatorname{wt}(f^*L) = n$, contradicting the L-semistability of p. Therefore, L restrict trivially on the residual gerbes of any closed point in $\mathcal{X}^L$, this proves that L descend on it. The last claim will be proved in Theorem 2.2.13. □

Definition 2.2.12 (Star construction)**.** Let Σ be a (pseudo)fan on a vector space W, with support $\Omega \subset W$. For any face $\tau \in \Sigma$, define the fan Σ_τ on the vector space ${W}/{\operatorname{Span}\langle\tau\rangle}$ by

$$\Sigma_\tau = \{\overline{\sigma} \mid \sigma \in \Sigma,\ \sigma \succeq \tau\},$$

where $\overline{\sigma}$ is the image of σ under the projection $W \to {W}/{\operatorname{Span}\langle\tau\rangle}$. This fan has support contained in $\overline{\Omega}$.

Theorem 2.2.13. *Let $\mathcal{X}$ be a geometric algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. For any $\tau \in \Sigma(\mathcal{X})$, restrict line bundles induces a morphism*

$$q_\tau \colon {\mathbb{E}(\mathcal{X})}/{\operatorname{Span}\langle\tau\rangle} \to \mathbb{E}(\mathcal{X}(\tau)).$$

This provides an injective map of cones

$$\begin{aligned} \Sigma(\mathcal{X})_\tau &\longrightarrow \Sigma(\mathcal{X}(\tau)), \\ \sigma &\longmapsto q_\tau(\sigma). \end{aligned}$$

Moreover,

$$\dim \operatorname{Ker} q_\tau \leq \dim \mathbb{E}(\mathcal{X}) - \dim \operatorname{Span} \Omega(\mathcal{X}).$$

In particular, if $\Omega(\mathcal{X})$ is top dimensional, then q_τ is injective.

Proof. The map q_τ is well-defined by Proposition 2.2.11. The existence of the injective map $\Sigma(\mathcal{X})_\tau \to \Sigma(\mathcal{X}(\tau))$ follows from Theorem 2.2.9. For the last claim, consider a sequence of proper inclusions of cones

$$\tau = \tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_j$$

with $\dim \operatorname{Span} \tau_i - \dim \operatorname{Span} \tau_{i-1} = 1$. This must remain a sequence of proper inclusions under q_τ, which gives the inequality. In particular, if $\Omega(\mathcal{X})$ is top dimensional, then q_τ is injective, proving the last claim of Proposition 2.2.11. □

Example 2.2.14. Consider

$$\mathcal{X} = \Theta \times B\mathbb{G}_m = [\mathbb{A}^1/\mathbb{G}_m^2].$$

The space $\mathbb{E}(\mathcal{X})$ is two-dimensional, generated by the cocharacters given by the projections onto the coordinate factors $\mathbb{G}_m^2 \to \mathbb{G}_m$. The semistability regions are described in Figure 2.2 and form a quasifan. Moreover, upon restricting to the one-dimensional cone $\tau \subset \Omega(\mathcal{X})$, the space $\mathbb{E}(\mathcal{X}(\tau))$ is the zero vector space, while the map q_τ has one-dimensional target; indeed, $\Omega(\mathcal{X})$ has codimension 1.

Many of the stacks of interest are smooth; in that case q_τ satisfies the following.

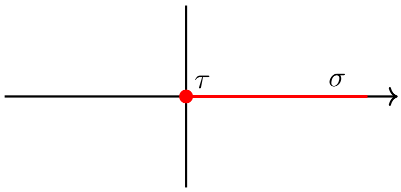


Figure 2.2. The cones τ and σ correspond respectively to the semistable loci $\mathcal{X}^{0-\mathrm{ss}} = \mathcal{X}$ and $\mathcal{X}^{L-\mathrm{ss}} = \mathcal{X} \setminus \{0\}$, for L in the relative interior of the one dimensional cone. The complement of the cones corresponds to the empty set.

Proposition 2.2.15. *With the notation of Theorem 2.2.13, assume moreover that $\mathcal{X}$ is regular. Then the map q_τ is surjective. In particular, if $\Omega(\mathcal{X})$ is top dimensional, then q_τ is an isomorphism.*

Proof. By [Alp13, Lemma 11.13], the restriction map $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{Pic}_{\mathbb{Q}}(\mathcal{U})$ is surjective for any open $\mathcal{U} \subset \mathcal{X}$, hence q_τ is surjective. If $\Omega(\mathcal{X})$ is full dimensional, then q_τ is injective by Theorem 2.2.13, and therefore an isomorphism. □

Under suitable hypotheses, any open $\mathcal{U} \subset \mathcal{X}$ admitting a quasi-projective good moduli space arises from semistability:

Proposition 2.2.16. *Let $\mathcal{X}$ be a regular geometric algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. For any $\mathcal{U} \subset \mathcal{X}$ that admits a good moduli space $\psi\colon \mathcal{U} \to U$ which is quasi-projective over X, there exists a line bundle $L \in \mathrm{Pic}(\mathcal{X})$ such that*

$$\begin{tikzcd}
\mathcal{U} \arrow[r, hook] \arrow[d, "\psi"] & \mathcal{X}^{L-\mathrm{ss}} \arrow[r, hook] \arrow[d, "\pi_L"] & \mathcal{X} \arrow[d, "\pi"] \\
U \arrow[r, hook] & X^{L-\mathrm{ss}} \arrow[r] & X
\end{tikzcd}$$

where $U \hookrightarrow X^{L-\mathrm{ss}}$ is an open immersion and $\mathcal{U}$ is saturated, i.e. $\mathcal{U} = \pi_L^{-1}(U)$.

Proof. Let N be a relatively ample line bundle over U. By [Alp13, Lemma 11.13], the line bundle $\psi^* N$ on $\mathcal{U}$ extends to a line bundle $\mathcal{N}$ on $\mathcal{X}$. Let $\mathcal{D}$ be the effective divisor given by the components of $\mathcal{X} \setminus \mathcal{U}$ in codimension 1. We claim that there exists n such that $\mathcal{U} \subset \mathcal{X}^{\mathcal{N}+\mathcal{O}(n\mathcal{D})-\mathrm{ss}}$.

Let $f\colon Y \to X$ be an étale cover. Then the preimage of $\mathcal{D}$ under the base change map $f'\colon Y \times_X \mathcal{X} \to \mathcal{X}$ is the codimension 1 component of $Y \times_X \mathcal{X} \setminus Y \times_X \mathcal{U}$. Therefore, it suffices to prove the claim after étale base change. By Proposition 2.1.9 we may assume $X = [\mathrm{Spec}\, A/G]$, and the claim then follows from [GIT, Converse 1.13]. □

2.3 Semistability and birational geometry

An interesting question, after constructing projective moduli spaces, concerns the study of their birational geometry and, more specifically, the possibility of running the minimal model program on them. For a smooth stack $\mathcal{X}$ admitting a good moduli space $\mathcal{X} \to X$, the smoothness (or even the $\mathbb{Q}$-factoriality) of X is not guaranteed.

From the perspective of the MMP, $\mathbb{Q}$-factoriality is essential in order to run the program. For this reason, we are primarily interested in mild desingularizations with $\mathbb{Q}$-factorial spaces, and we therefore recall the notion of $\mathbb{Q}$-factorializations. The main goal of this section is to construct such models as good moduli spaces of semistable loci with respect to line bundles. This approach ultimately provides a method to obtain desingularizations of the moduli spaces of curves of interest.

2.3.1 Flips

We recall the notion of flip as it appears in [CTV23]. The main difference from the classical definition is that, in the following definition, we do not require the map f to be small.

Definition 2.3.1. Let $f\colon X \to Y$ be a proper morphism between normal algebraic spaces of finite type over k and let D be an f-antiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. A *D-flip of f* is a proper morphism $f_D^+\colon X_D^+ \to Y$ of algebraic spaces fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\eta"] \arrow[dr, "f"'] & & X_D^+ \arrow[dl, "f_D^+"] \\ & Y & \end{tikzcd} \tag{2.2}$$

where η is a rational map, and such that

1. the algebraic space X_D^+ (which is automatically of finite type over k) is normal;
2. the morphism f_D^+ is a small contraction, i.e. a contraction whose exceptional locus $\mathrm{Exc}(f_D^+)$ has codimension at least two;
3. the $\mathbb{Q}$-divisor $D^+ := \eta_*(D)$ is $\mathbb{Q}$-Cartier and f_D^+-ample.

A D-flip is called *elementary* if f has relative Picard number 1. A D-flip is called *small* if f is a small birational contraction.

We have the following characterization of flips, which is standard.

Lemma 2.3.2. *Let $f\colon X \to Y$ be a proper morphism of normal algebraic spaces of finite type over k and let D be an f-antiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

1. *If the D-flip of f exists, then it is given by*
$$f_D^+ : X_D^+ = \mathrm{Proj} \bigoplus_{m\geq 0} \mathcal{O}_Y(\lfloor mf_*(D)\rfloor) \to Y. \tag{2.3}$$
In particular, the D-flip of f is unique.

   Moreover, the D-flip depends only on the $\mathbb{Q}$-line bundle $L = \mathcal{O}_X(D)$ associated to D and hence it will also be denoted by $f_L\colon X_L^+ \to Y$ and called the L-flip of f.

2. *If $\mathrm{char}(k) = 0$, X is klt, and K_X is f-antiample, then the coherent sheaf $\bigoplus_{m\geq 0} \mathcal{O}_Y(\lfloor mf_*(D)\rfloor)$ of $\mathcal{O}_Y$-algebras is finitely generated, hence the D-flip of f exists.*

Proof. [CTV23, Lemma 7.3] □

The following is a key result of this section.

Proposition 2.3.3. *Let $\mathcal{X}$ be a normal geometric stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. Fix a big open substack $\mathcal{X}_F \subset \mathcal{X}$ that admit a good moduli space $\pi_F\colon \mathcal{X}_F \to X_F$. Let L be a $\mathbb{Q}$-line bundle on $\mathcal{X}$ such that $L_{|\mathcal{X}_F}$ descends to X_F and is relatively antiample with respect to the induced map $q\colon X_F \to X$.*

$$\begin{tikzcd} \mathcal{X}_F \arrow[r, hook, "i"] \arrow[d, "\pi_F"] & \mathcal{X} \arrow[d, "\pi"] & \mathcal{X}^{L-\mathrm{ss}} \arrow[l, hook'] \arrow[d] \\ X_F \arrow[r, "q"] & X & X^{L-\mathrm{ss}} \arrow[l] \end{tikzcd}$$

Then the $\pi_{F,*}(L_{|\mathcal{X}_F})$-flip of q exists and is given by the induced map on good moduli spaces $X^{L-\mathrm{ss}} \to X$.

Proof. Let $i\colon \mathcal{X}_F \hookrightarrow \mathcal{X}$ be the inclusion. By Lemma 2.3.2, the $\pi_{F,*}(L_{|\mathcal{X}_F})$-flip of q is

$$\operatorname{Proj}_X q_* \bigoplus_{n\geq 0} (\pi_{F,*} i^* L)^{\otimes n} = \operatorname{Proj}_X \pi_* \bigoplus_{n\geq 0} i_*(i^*L)^{\otimes n} = \operatorname{Proj}_X \pi_* \bigoplus_{n\geq 0} L^{\otimes n}.$$

The first equality holds because $(\pi_{F,*} i^* L)^{\otimes n} = \pi_{F,*} i^* L^{\otimes n}$ (the good moduli space map π_F is Stein); the second equality holds because $i\colon \mathcal{X}_F \hookrightarrow \mathcal{X}$ is a big open immersion and $\mathcal{X}$ is normal. Finally, $X^{L-\mathrm{ss}} = \operatorname{Proj}_X \pi_* \bigoplus_{n\geq 0} L^{\otimes n}$ is the content of Theorem 2.1.22. □

As in [Tha96], the wall-crossing picture arising from VGIT produces many flips. In particular, we have:

Proposition 2.3.4. *Let $\mathcal{X}$ be a normal geometric algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$. Fix cones $\sigma > \tau$ in $\Sigma(\mathcal{X})$ such that the inclusion $\mathcal{X}(\sigma) \subset \mathcal{X}(\tau)$ has codimension 1. For any $L \in \operatorname{Pic}(\mathcal{X})$ that descends to a line bundle on $X(\sigma)$ and is antiample with respect to $X(\sigma) \to X(\tau)$, the following diagram is an L-flip:*

$$\begin{tikzcd} X(\sigma) \arrow[rr, dashed, "\eta"] \arrow[dr] & & X(\sigma') \arrow[dl] \\ & X(\tau) & \end{tikzcd} \tag{2.4}$$

where $\sigma' \subset \mathbb{E}(X)$ is the unique cone $\sigma' > \tau$ whose image under $p\colon \mathbb{E}(\mathcal{X}) \to \mathbb{E}(\mathcal{X})/\operatorname{Span}\tau$ contains $p(-L)$. In other words, σ' is the cone containing the projection of $-L$ in the star construction $\Sigma_\tau \subset V/\operatorname{Span}\langle\tau\rangle$ with respect to τ (cf. Definition 2.2.12).

Proof. The claim holds for $\tau = \{0\} \subset \mathbb{E}(X)$ by Proposition 2.3.3 applied to $\mathcal{X}_F = \mathcal{X}(\sigma)$. In general, we reduce to the previous case by applying Proposition 2.2.15 and then working in the stability space $\mathbb{E}(\mathcal{X}(\tau))$ via Theorem 2.2.13. □

In general, flips are related to a wall-crossing picture in birational geometry. The aim of the next section is to establish a correspondence with the stability condition fan.

2.3.2 $\mathbb{Q}$-factorialization fan

We present here some revisited classical results in birational geometry. We view $\mathbb{Q}$-factorializations as desingularizations of certain spaces of interest, which in the following chapters will be moduli spaces of curves.

Definition 2.3.5 (Partial $\mathbb{Q}$-factorialization)**.** Let X be a normal proper scheme over a field k. A morphism from a projective normal scheme over k

$$f\colon Y \to X$$

is a *partial* $\mathbb{Q}$*-factorialization* if f is a small birational projective morphism such that

$$\dim \frac{\operatorname{Pic}_{\mathbb{Q}}(Y)}{f^*\operatorname{Pic}_{\mathbb{Q}}(X)} > 1.$$

We say that f is a $\mathbb{Q}$*-factorialization* if Y is $\mathbb{Q}$-factorial.

Notice that partial $\mathbb{Q}$-factorializations are always Stein maps by Zariski's Main Theorem.

Remark 2.3.6 ([Kol08, Ex. 108])**.** Let X be a normal projective scheme over a field k with $\operatorname{char} k = 0$, and assume that X is of klt type. Then there exists a $\mathbb{Q}$-factorialization $Y \to X$.

Proof. Let Δ be an effective divisor such that (X, Δ) is klt. Take a log resolution $f\colon Y \to X$ with exceptional divisor E, and run the relative MMP for the pair (Y, Δ_Y), where $\Delta_Y = f_*^{-1}\Delta + (1-\epsilon)E$. Then

$$K_Y + f_*^{-1}\Delta_Y - f^*(K_X + \Delta) \equiv_f \sum_i (a(E_i, X, \Delta) + 1 - \epsilon)E_i = \sum_i b_i E_i,$$

where the $a(E_i, X, \Delta)$ are the discrepancies. Choose $\epsilon > 0$ such that

$$b_i \coloneqq a(E_i, X, \Delta) + 1 - \epsilon > 0$$

for all i. By [Bir+10, Thm. 1.2], the relative MMP terminates. Let $f_r\colon Y_r \to X$ be the final step; the variety Y_r is $\mathbb{Q}$-factorial, so it remains to show that f_r is a small contraction. The MMP ends with a pair (Y_r, Δ_{Y_r}) such that

$$K_{Y_r} + \Delta_{Y_r} \equiv_f \sum_i b_i E_i^{(r)},$$

where the $E_i^{(r)}$ are the transforms of the components of E that have not been contracted. The divisor $\sum_i b_i E_i^{(r)}$ is f_r-nef and effective, and satisfies $f_{r,*}E_i^{(r)} = 0$. By [KM98, Lemma 3.39(1)], this implies $\sum_i b_i E_i^{(r)} = 0$. Hence, f_r is small, completing the proof. ☐

Remark 2.3.7. For any partial $\mathbb{Q}$-factorialization $f\colon Y \to X$, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Pic}_{\mathbb{Q}}(Y) \arrow[d, hook] & \operatorname{Pic}_{\mathbb{Q}}(X) \arrow[l, hook', "f^*"'] \arrow[d, hook] \\ \operatorname{Cl}_{\mathbb{Q}}(Y) \arrow[r, "f_*"] & \operatorname{Cl}_{\mathbb{Q}}(X) \end{tikzcd},$$

where f_* is an isomorphism and the other three maps are inclusions. This follows from the fact that f is a proper birational small morphism between normal schemes.

Definition 2.3.8. For any normal and proper scheme X over a field, we define the space

$$V(X) := {\operatorname{Cl}_{\mathbb{Q}}(X)} \big/ {\operatorname{Pic}_{\mathbb{Q}}(X)}.$$

Let $f\colon Y \to X$ be a partial $\mathbb{Q}$-factorialization. We define the *relative ample cone*

$$\operatorname{Amp}(Y/X) \subset \frac{\operatorname{Pic}_{\mathbb{Q}}(Y)}{f^*\operatorname{Pic}_{\mathbb{Q}}(X)} \subset V(X)$$

as the convex cone consisting of f-ample $\mathbb{Q}$-line bundles. The second inclusion is induced by the identification provided by f_* in Remark 2.3.7; thus, we write $\operatorname{Amp}(Y/X) \subset V(X)$.

Note that when $f\colon Y \to X$ is a $\mathbb{Q}$-factorialization, then

$$V(X) \simeq \frac{\operatorname{Pic}_{\mathbb{Q}}(Y)}{f^*\operatorname{Pic}_{\mathbb{Q}}(X)}.$$

Remark 2.3.9. Ample line bundles are characterized by numerical conditions, and it is often convenient to consider divisor and cycle groups modulo numerical equivalence. Given a $\mathbb{Q}$-factorialization $f\colon Y \to X$, we have the following map of exact sequences, where Ψ is surjective:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Pic}_{\mathbb{Q}}(X) \arrow[r, "f^*"] \arrow[d] & \operatorname{Pic}_{\mathbb{Q}}(Y) \arrow[r] \arrow[d] & \dfrac{\operatorname{Pic}_{\mathbb{Q}}(Y)}{f^*\operatorname{Pic}_{\mathbb{Q}}(X)} \arrow[r] \arrow[d, "\Psi"] & 0 \\ 0 \arrow[r] & N^1(X)_{\mathbb{Q}} \arrow[r, "f^*"] & N^1(Y)_{\mathbb{Q}} \arrow[r] & N^1(Y/X)_{\mathbb{Q}} \arrow[r] & 0 \end{tikzcd}.$$

By [Kle05, Thm. 6.3(a)–(b)], the map Ψ is an isomorphism if and only if the cokernel of $f^*\colon \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(Y)$ is torsion. Over a field k with $\operatorname{char} k = 0$, this holds when $R^1 f_*\mathcal{O}_X = 0$ (for instance, if X is of klt type). In the situations considered in this thesis, Ψ will always be an isomorphism. However, this is not true in general, and in particular $V(X)$ need not be finite-dimensional.

We now recall some basic results in birational geometry. The following lemma is a classical consequence of [Kle66, Thm. 1.4.23].

Lemma 2.3.10. *With the notation of Definition 2.3.8, the closure of the relative ample cone is the relative* nef cone

$$\operatorname{Nef}(Y/X) = \overline{\operatorname{Amp}(Y/X)} \subset V(X).$$

The cone $\operatorname{Nef}(Y/X) \subset V(X)$ *is convex, and* $\operatorname{Amp}(Y/X) = \operatorname{Nef}(Y/X)^\circ$ *(the relative interior). Moreover,*

$$\operatorname{Span}\operatorname{Amp}(Y/X) = \frac{\operatorname{Pic}_{\mathbb{Q}}(Y)}{f^*\operatorname{Pic}_{\mathbb{Q}}(X)} \subset V(X).$$

Given a divisor in the relative ample cone, we can describe a partial $\mathbb{Q}$-factorialization as follows.

Proposition 2.3.11. *Let* X *be a normal proper scheme over* k*, and let* $f\colon Y \to X$ *be a partial* $\mathbb{Q}$*-factorialization. Then, for any* $\mathbb{Q}$*-Cartier divisor* D *on* Y *that is* f*-ample, the morphism* $f\colon Y \to X$ *is given by*

$$f_D\colon Y \simeq \operatorname{Proj}_X \bigoplus_{n\geq 0} f_*\mathcal{O}_Y(\lfloor nD \rfloor) \simeq \operatorname{Proj}_X \bigoplus_{n\geq 0} \mathcal{O}_X(\lfloor nD \rfloor) \to X,$$

where $\mathcal{O}_X(E)$ *denotes the reflexive sheaf associated with a cycle* $E \in \operatorname{Cl}(X)$*. In particular, the ample cones of distinct partial* $\mathbb{Q}$*-factorializations are disjoint in* $V(X)$*.*

Proof. The only nontrivial statement is the second isomorphism over X. Since X and Y are normal and f is small, there exists a common big regular open subset U with embeddings $j_1\colon U \hookrightarrow X$ and $j_2\colon U \hookrightarrow Y$. Then

$$j_{1,*}\mathcal{O}_U(\lfloor nD_{|U} \rfloor) = \mathcal{O}_X(\lfloor nD \rfloor), \qquad j_{2,*}\mathcal{O}_U(\lfloor nD_{|U} \rfloor) = \mathcal{O}_Y(\lfloor nD \rfloor).$$

Since $f_D \circ j_2 = j_1$, we have $f_*\mathcal{O}_Y(\lfloor nD \rfloor) \simeq \mathcal{O}_X(\lfloor nD \rfloor)$, and the claim follows. □

A natural question is whether these regions completely cover the vector space $V(X)$.

Proposition 2.3.12 ([KM98, Lemma 6.2])**.** *Let* X *be a normal proper scheme over* k*, and let* $D \in \operatorname{Cl}_{\mathbb{Q}}(X)$*. Then the following are equivalent:*

- *There exists a partial* $\mathbb{Q}$*-factorialization* $Y_D \to X$ *such that* $D \in \operatorname{Amp}(Y_D/X)$*;*
- *the* $\mathcal{O}_X$*-algebra* $\mathcal{R}_X(D) \coloneqq \bigoplus_{n\geq 0}\mathcal{O}_X(\lfloor nD \rfloor)$ *is finitely generated.*

The previous two propositions imply the following.

Corollary 2.3.13. *Let* $f\colon Y \to X$ *be a partial* $\mathbb{Q}$*-factorialization, and let* D *be a* $\mathbb{Q}$*-Cartier divisor on* Y *such that* $\bigoplus_{n\geq 0} f_*\mathcal{O}_Y(\lfloor nD \rfloor)$ *is a finitely generated* $\mathcal{O}_X$*-algebra. Then the map*

$$Y_D \coloneqq \operatorname{Proj}_X \bigoplus_{n\geq 0} f_*\mathcal{O}_Y(\lfloor nD \rfloor) \to X$$

is the unique partial $\mathbb{Q}$*-factorialization such that* $D \in \operatorname{Amp}(Y_D/X)$*.*

For varieties of klt type, [Bir+10] ensures that $\mathcal{R}_X(D)$ is finitely generated.

Proposition 2.3.14 ([Kol08, Thm. 92])**.** *Let* X *be a normal projective scheme over* k *with* $\operatorname{char} k = 0$*, and assume that* X *is of klt type. Then, for any* $D \in \operatorname{Pic}_{\mathbb{Q}}(X)$*, the* $\mathcal{O}_X$*-algebra* $\bigoplus_{n\geq 0}\mathcal{O}_X(\lfloor nD \rfloor)$ *is finitely generated.*
In particular, there exist partial $\mathbb{Q}$*-factorializations of* X *whose relative ample cones completely cover the rational vector space* $V(X)$*.*

The relative ample chambers in $V(X)$ intersect along their boundaries, as explained in the next propositions.

Proposition 2.3.15. *Let X be a normal proper scheme over k, and let $Y_D \to X$ and $Y_E \to X$ be partial $\mathbb{Q}$-factorializations such that $D \in \mathrm{Amp}(Y_D/X)$ and $E \in \mathrm{Amp}(Y_E/X)$. Then*

$$\overline{\mathrm{Amp}(Y_D/X)} \cap \mathrm{Amp}(Y_E/X) \neq \emptyset \quad \Leftrightarrow \quad Y_D \dashrightarrow Y_E \text{ extends to a morphism over } X.$$

In that case, we have $\mathrm{Amp}(Y_E/X) \subset \overline{\mathrm{Amp}(Y_D/X)} = \mathrm{Nef}(Y_D/X)$.

Proof. For $E \in \mathrm{Pic}_{\mathbb{Q}}(Y_D)$ big and nef, being semiample is equivalent to the finite generation of $\mathcal{R}_X(E)$; see [Laz17, Thm. 2.3.15]. The latter is equivalent to the existence of such a Y_E by Proposition 2.3.12. □

Proposition 2.3.16. *Let X be a normal proper scheme over k. Consider two partial $\mathbb{Q}$-factorializations $Y_D \to X$ and $Y_E \to X$ such that $D \in \mathrm{Amp}(Y_D/X)$ and $E \in \mathrm{Amp}(Y_E/X)$. Assume moreover that $\mathrm{Amp}(Y_E/X) \subset \overline{\mathrm{Amp}(Y_D/X)}$ is an open whether restricted to a face of $\overline{\mathrm{Amp}(Y_D/X)}$. Then*

$$\overline{\mathrm{Amp}(Y_E/X)} = \overline{\mathrm{Amp}(Y_D/X)} \cap \{x \mid f(x) = 0\},$$

where $f \in V(X)^*$ defines the face, i.e. $\overline{\mathrm{Amp}(Y_D/X)} \subset H_f^+ = \{x \mid f(x) \geq 0\}$ (cf. Definition 2.1.40).

Proof. The inclusion $\subset$ is immediate. By openness,

$$H_f^+ \cap \overline{\mathrm{Amp}(Y_D/X)} \subset \mathrm{Span}\,\mathrm{Amp}(Y_E/X),$$

hence any $G \in H_f^+ \cap \overline{\mathrm{Amp}(Y_D/X)}$ is $\mathbb{Q}$-Cartier over Y_E by Lemma 2.3.10. By Proposition 2.3.15, we have the factorization $Y_D \to Y_E \to X$. Thus, any such G is numerically trivial, and the claim follows from $\mathrm{Nef}(Y_E/X) = \overline{\mathrm{Amp}(Y_E/X)}$. □

Definition-Proposition 2.3.17 ($\mathbb{Q}$-factorialization fan)**.** Let X be a normal proper scheme over a field k. Consider the collection of relative ample cones in $V(X)$ of all the partial $\mathbb{Q}$-factorializations $f_i \colon X_i \to X$. We define

$$\Sigma^{\mathbb{Q}}(X) \coloneqq \{\text{face of } \overline{\mathrm{Amp}(X_i/X)}_{\mathbb{R}} \subset V(X)_{\mathbb{R}} \mid X_i \xrightarrow{f_i} X \text{ partial } \mathbb{Q}\text{-factorialization}\}.$$

Whenever this collection is finite and $V(X)$ is finite-dimensional, it forms a quasifan. Moreover, if in addition a $\mathbb{Q}$-factorialization exists and Ψ in Remark 2.3.9 is injective, then $\Sigma^{\mathbb{Q}}(X)$ is a fan. In these cases, we call $\Sigma^{\mathbb{Q}}(X)$ the *$\mathbb{Q}$-factorialization (quasi)fan of X.*

Proof. The cones in $\Sigma^{\mathbb{Q}}(X)$ are convex by Lemma 2.3.10. If $\Sigma^{\mathbb{Q}}(X)$ is finite, then every cone is polyhedral. This is a consequence of the following fact. Consider a finite collection of closed convex top-dimensional subsets $\{C_i\}_{i\in I}$ of $\mathbb{R}^n$ whose union is a convex set with pairwise disjoint interior and finitely many faces (i.e., a polyhedral set). Then each C_i is necessarily polyhedral. The proof proceeds by

induction on $|I|$. By convexity of the subsets, there exists a hyperplane $H \subset \mathbb{R}^n$ such that two of the C_i have interior part that lie in the two opposite half-spaces H^+ and H^-. The sets $\{C_i \cap H^+\}_{i \in I}$ and $\{C_i \cap H^-\}_{i \in I}$ are then closed and convex within a polyhedral convex set, and both collections contain strictly fewer top-dimensional elements.
Finally, note that $\Sigma^{\mathbb{Q}}(X)$ contains all faces (which are finitely many), and that intersections of faces are again faces by Proposition 2.3.16. This provides that $\Sigma^{\mathbb{Q}}(X)$ is a quasifan. For the last statement, if $\Psi\colon V(X) \to N^1(Y/X)$ is injective (hence an isomorphism), then any cone is strictly convex because if both D and $-D$ are nef, then D is numerically trivial. □

Notice that if Ψ in Remark 2.3.9 is injective, then $V(X) \simeq N^1(Y/X)$ is automatically finite-dimensional, cf. [Kle66]. In that case, if $\Sigma^{\mathbb{Q}}(X)$ consists of finitely many cones, it is a fan.

Remark 2.3.18. Even when the $\mathbb{Q}$-factorialization (quasi)fan exists, not every face $F \subset \mathrm{Nef}(Y/X)$ of a maximal cone corresponds to a partial $\mathbb{Q}$-factorialization. Two distinct phenomena may occur:

- The face F is not contained in a rational subspace. In this case, the $\mathcal{O}_X$-algebra $\mathcal{R}_X(D)$ for $D \in F$ is never finitely generated (cf. [Kol08, Ex. 94]), hence no corresponding partial $\mathbb{Q}$-factorialization exists.
- The face F is rational polyhedral, but any $D \in F \subset \mathrm{Pic}_{\mathbb{Q}}(Y)$ satisfies that $\bigoplus_{n \geq 0} \mathcal{O}_Y(\lfloor nD \rfloor)$ is not finitely generated, even if D is nef. In particular, D is not semiample [Laz17, Thm. 2.3.15].

When neither of the above phenomena occurs and the $\mathbb{Q}$-factorialization quasifan is a complete fan, then $\Sigma^{\mathbb{Q}}(X)$ is rational polyhedral, and

$$V(X) = \bigcup \{\mathrm{Amp}(X_i/X) \subset V(X) \mid f_i\colon X_i \to X \text{ partial } \mathbb{Q}\text{-factorialization}\}.$$

In this case, each $\mathbb{Q}$-factorialization $f_i\colon X_i \to X$ is a *relative Mori dream space* (cf. [Oht22, Def. 3.1], [AW14, Def. 2.5]). Conversely, any small birational relative Mori dream space $X \to Y$ admits a $\mathbb{Q}$-factorialization fan that is rational polyhedral and complete.

For varieties of klt type, the finiteness hypothesis can be verified only on $\mathbb{Q}$-factorializations, not including the partial ones.

Proposition 2.3.19. *Let X be a normal projective variety over k with $\mathrm{char}\, k = 0$ and assume X is of klt type. Then X has a $\mathbb{Q}$-factorialization fan if and only if it has finitely many $\mathbb{Q}$-factorializations. In such a case, the fan is complete.*

Proof. By Remark 2.3.9 and Remark 2.3.6, there exist a $\mathbb{Q}$-factorialization and $V(X) \simeq N^1(Y/X)$ is finite-dimensional. Therefore, whenever the quasifan of $\mathbb{Q}$-factorializations exists, it is a fan.
By Proposition 2.3.14, every element $D \in V(X)$ is relatively ample for some partial $\mathbb{Q}$-factorialization $Y_D \to X$. Moreover, there exists $E \in \mathrm{Cl}_{\mathbb{Q}}(X)$ such that $\mathrm{Amp}(Y_E/X)$ is open and contains D in its closure. Suppose not; then consider $E \in \mathrm{Cl}_{\mathbb{Q}}(X)$

such that $E \in \mathrm{Nef}(Y_E/X)$ and this nef cone has maximal dimension among those containing D. If $\mathrm{Nef}(Y_E/X) \subset V(X)$ is not full-dimensional, then Y_E is not $\mathbb{Q}$-factorial. Since Y_E is also of klt type, there exists a non-Cartier divisor $F \in \mathrm{Cl}_{\mathbb{Q}}(Y_E)$, yielding a partial $\mathbb{Q}$-factorialization $Y_F \to Y_E$. Then $Y_F \to X$ is a partial $\mathbb{Q}$-factorialization with $\mathrm{Nef}(Y_F/X)$ of strictly larger dimension than $\mathrm{Nef}(Y_E/X)$ and still containing D, a contradiction.
This implies that there are finitely many top-dimensional convex ample cones that do not overlap and whose closures cover $V(X)$. As in the proof of Definition-Proposition 2.3.17, these regions are polyhedral, hence with a finite amount of faces, and $\Sigma^{\mathbb{Q}}(X)$ is a complete fan. □

Remark 2.3.20. The cones in $\Sigma^{\mathbb{Q}}(X)$ encode all possible small projective flips over X. Indeed, let $f\colon Y \to X$ be a projective small morphism between normal proper schemes over k, and assume that $D \in \mathrm{Pic}_{\mathbb{Q}}(Y)$ is f-antiample. Then, if the D-flip $f^+\colon X^+ \to Y$ exists, it is given by the $\mathbb{Q}$-factorialization of X corresponding to the cone in $\Sigma^{\mathbb{Q}}(X)$ containing the class of $-D$ in $V(X)$.

Finally, notice that via the star construction we obtain the $\mathbb{Q}$-factorialization fan of partial $\mathbb{Q}$-factorializations.

Remark 2.3.21. Let X be a normal projective scheme over k admitting a $\mathbb{Q}$-factorialization (quasi)fan $\Sigma^{\mathbb{Q}}(X)$. Then any partial $\mathbb{Q}$-factorialization $f_i\colon X_i \to X$, corresponding to a cone $C \in \Sigma^{\mathbb{Q}}(X)$, admits a $\mathbb{Q}$-factorialization fan obtained via the star construction on $\Sigma^{\mathbb{Q}}(X)$ with respect to C (cf. Definition 2.2.12).

2.3.3 Comparison of fans in $\mathbb{E}(\mathcal{X})$ and $V(X)$

The main obstacle in constructing $\mathbb{Q}$-factorializations lies in controlling the finite generation of $\mathcal{R}_X(D)$ (cf. Proposition 2.3.12). When $\operatorname{char} k = 0$, this holds for varieties of klt type. We now introduce techniques, arising from VGIT, to establish the required finite generation for good moduli spaces. This approach not only provides an effective method for computing $\mathbb{Q}$-factorializations of moduli spaces, but also shows that these $\mathbb{Q}$-factorializations are relative Mori dream spaces. Additionally, this technique works in arbitrary characteristic and avoids reliance on [Bir+10].

For a smooth stack $\mathcal{X}$ admitting a good moduli space $\mathcal{X} \to X$, the $\mathbb{Q}$-factoriality of X is not guaranteed. In this subsection, we compare the spaces $V(X)$ and $\mathbb{E}(\mathcal{X})$, establishing a correspondence between the finite generation of $\mathcal{R}_X(D)$ and Hilbert's fourteenth problem on the finite generation of invariants.

Definition 2.3.22. Let $\mathcal{X}$ be a smooth geometric algebraic stack over a field k admitting a proper good moduli space $\pi\colon \mathcal{X} \to X$. Fix $L \in \mathbb{E}(\mathcal{X})$ and consider the diagram

$$\begin{tikzcd} \mathcal{X}^{L-\mathrm{ss}} \arrow[r, hook, "i_L"] \arrow[d, "\pi_L"] & \mathcal{X} \arrow[d, "\pi"] \\ X^{L-\mathrm{ss}} \arrow[r, "f_L"] & X \end{tikzcd} \tag{2.5}$$

We say that L satisfies condition $(\star)$ if:

- The open inclusion i_L is big (i.e., its complement has codimension at least two);
- The projective morphism f_L is a $\mathbb{Q}$-factorialization of X.

Then we have a diagram

$$\begin{tikzcd}
 & \mathrm{Pic}_{\mathbb{Q}}\left(\mathcal{X}^{L-\mathrm{ss}}\right) & & \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}) \arrow[ll, "\iota_L^*"'] \\
\mathrm{Pic}_{\mathbb{Q}}\left(X^{L-\mathrm{ss}}\right) \simeq & \mathrm{Cl}_{\mathbb{Q}}\left(X^{L-\mathrm{ss}}\right) \arrow[u, hook, "\pi_L^*"] \arrow[rr, "f_{L,*}"] & & \mathrm{Cl}_{\mathbb{Q}}(X)
\end{tikzcd}$$

where ι_L^* and $f_{L,*}$ are isomorphisms. This yields a morphism $\mathrm{Cl}_{\mathbb{Q}}(X) \to \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$, which induces an inclusion after factoring out by $\mathrm{Pic}_{\mathbb{Q}}(X)$:

$$j_L \colon V(X) \hookrightarrow \mathbb{E}(\mathcal{X}).$$

To recover the $\mathbb{Q}$-factorialization fan from the stability condition space, it suffices to find an $L \in \mathbb{E}(\mathcal{X})$ satisfying $(\star)$.

Theorem 2.3.23. *Let $\mathcal{X}$ be a smooth geometric algebraic stack admitting a good moduli space $\pi \colon \mathcal{X} \to X$. Assume that X is proper and that there exists $L \in \mathbb{E}(\mathcal{X})$ satisfying $(\star)$. Then*

$$\Sigma^{\mathbb{Q}}(X) = \{\sigma \cap j_L(V(X)) \mid \sigma \in \Sigma(\mathcal{X})\},$$

i.e., the $\mathbb{Q}$-factorialization fan of X exists and is obtained by restricting the semistability fan. In particular, $\Sigma^{\mathbb{Q}}(X)$ is complete and rational polyhedral. For any $D \in V(X)$, the corresponding $\mathbb{Q}$-factorialization is given by $X^{j_L(D)-\mathrm{ss}} \to X$.

Proof. We argue similarly to Proposition 2.3.3. For any $M \in \mathrm{Pic}_{\mathbb{Q}}\left(X^{L-\mathrm{ss}}\right)$, let $\widetilde{M} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ denote its image under the composition

$$\mathrm{Pic}_{\mathbb{Q}}\left(X^{L-\mathrm{ss}}\right) \xhookrightarrow[\phi_L^*]{} \mathrm{Pic}_{\mathbb{Q}}\left(\mathcal{X}^{L-\mathrm{ss}}\right) \xrightarrow[(i_L^*)^{-1}]{\sim} \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}),$$

where i_L^* is invertible because $\mathcal{X}^{L-\mathrm{ss}} \subset \mathcal{X}$ is big. Then

$$\begin{aligned}
\mathrm{Proj}_X \bigoplus_{n\geq 0} f_{L,*}M^n &= \mathrm{Proj}_X \bigoplus_{n\geq 0} f_{L,*}\phi_{L,*}\phi_L^* M^n = \mathrm{Proj}_X\, \pi_* \bigoplus_{n\geq 0} i_{L,*}(\phi_L^* M)^n \\
&= \mathrm{Proj}_X\, \pi_* \bigoplus_{n\geq 0} ((i_L^*)^{-1}\phi_L^* M)^n = \mathrm{Proj}_X\, \pi_* \bigoplus_{n\geq 0} \widetilde{M}^n = X^{\widetilde{M}-\mathrm{ss}}.
\end{aligned}$$

The first equality follows from Theorem 2.1.28, the third from the fact that i_L is a big open inclusion with $\mathcal{X}$ normal, and the last from Theorem 2.1.22. The claim is then a consequence of Corollary 2.3.13. Notice that $\Sigma^{\mathbb{Q}}(X)$ is strictly convex because $\Sigma(\mathcal{X})$ is. □

We have then a relative version of what obtained in [HK00] (cf. Remark 2.3.18)

Corollary 2.3.24 (Relative Mori dream spaces)**.** *Let $\mathcal{X}$ be a smooth geometric algebraic stack admitting a good moduli space $\pi\colon \mathcal{X} \to X$, and assume that there exists $L \in \mathbb{E}(X)$ satisfying $(\star)$. Then the morphism*

$$X^{L-\mathrm{ss}} \to X$$

is a relative Mori dream space.

Chapter 3

Moduli stacks of curves

In this chapter, we recall some background on curves and contribute with the following:

1. We define a category of dual graphs for curves with A_k singularities, which yields a stratification of $\widetilde{\mathcal{M}}_{g,n}$ determined by the combinatorics of curve singularities (Definition 3.2.9).
2. We prove that every universally closed open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ is a union of strata (Theorem 3.6.5).
3. We establish necessary conditions for an open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ to admit a proper good moduli space (Theorem 3.8.2).

We are confident that the first two results in the list above can be extended to broader classes of singularities (e.g. of type ADE). This lies beyond the scope of the present thesis and will be discussed in future work.

Throughout this chapter we work over a field k. As will be specified later, the results about $\widetilde{\mathcal{M}}^3_{g,n}$ works whenever $\operatorname{char} k \neq 2, 3, 5$.

3.1 Notations and preliminary notions

We introduce here some objects used in the following sections.

Definition 3.1.1 (Curve)**.** We say that $(\pi\colon \mathcal{C} \to S, \{p_i\colon S \to \mathcal{C}\}_{i=1,\dots,n})$ (for brevity, $(\mathcal{C}, p_i)$ over S) is an *n-pointed family of curves* if:

1. $\pi\colon \mathcal{C} \to S$ is a flat and proper morphism of relative pure dimension 1 with reduced geometric fibres (not necessarily connected);
2. $p_i\colon S \to \mathcal{C}$ are ordered sections of π whose schematic images are pairwise disjoint and contained in the smooth locus of $\pi\colon \mathcal{C} \to S$.

For brevity, we will simply call $(\mathcal{C}, p_i)$ a (family of) curves when the n-markings are clear from the context. A morphism of families $(\mathcal{C}, p_i) \to (\mathcal{C}', p'_i)$ is a morphism of schemes over S, $f\colon \mathcal{C} \to \mathcal{C}'$, such that $p'_i = f \circ p_i$ for every $i \in \{1, \dots, n\}$.

Notation 3.1.2. Given a family of pointed curves $(\mathcal{C}, p_i)$ over S and a topological point $s \in |S|$, we denote by $(\mathcal{C}_s, p_{i,s})$ its geometric fibre, i.e. the pointed curve obtained by base change $\operatorname{Spec} \overline{k(s)} \to S$.

When $S = \operatorname{Spec} R$ for a discrete valuation ring (DVR) R, we write $(\mathcal{C}_0, p_{i,0})$ for the pointed curve over $\operatorname{Spec} k(s)$ (the closed point of $\operatorname{Spec} R$), and $(\mathcal{C}_\eta, p_{i,\eta})$ for the pointed curve over $\operatorname{Spec} k(\eta)$ (the generic point of $\operatorname{Spec} R$). For a pointed curve over a field, we say that a point $p \in \mathcal{C}$ is *special* if it is either a marked point or a singular point.

Unless otherwise stated, the genus of a curve is always understood to be its arithmetic genus. We say that a marked curve is of type (g, n) if it is a n-marked genus g curve.

We will always work with Gorenstein families of curves $\pi \colon \mathcal{C} \to S$. In this case, the relative dualizing sheaf $\omega_{\mathcal{C}/S}$ is locally free. We say that:

- $(\mathcal{C}, p_i)$ is *stable* if $\omega_{\mathcal{C}/S}(\sum p_i)$ (the log-dualizing sheaf) is relatively ample;
- $(\mathcal{C}, p_i)$ is *semistable* if $\omega_{\mathcal{C}/S}(\sum p_i)$ is relatively nef. Over an algebraically closed field, an irreducible component of the curve is called *semistable* if the log-dualizing sheaf restricts trivially to it;
- $(\mathcal{C}, p_i)$ is *almost-stable* if it is semistable and, for every $s \in S$, the semistable components of $(\mathcal{C}_s, p_{i,s})$ contain at least one marked point.

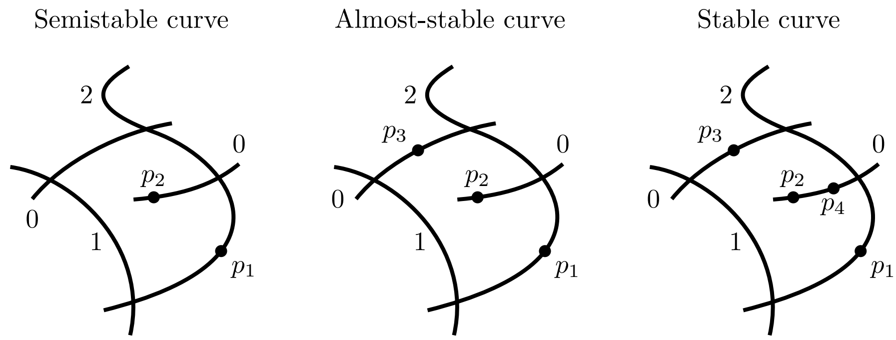


Figure 3.1. Example of (semi/almost-)stable curves

Remark 3.1.3. A family of pointed curves $(\mathcal{C}, p_i)$ is *stable* (resp. *semistable*, *almost-stable*) if and only if every geometric fiber of the family is.

Moreover, let $(C/k, p_i)$ be a pointed curve over an algebraically closed field of type $(g, n) \neq (1, 0)$. If it is *semistable* (resp. *almost-stable*), then for every irreducible component $D \subset C$, either the restriction $\omega_{C/k}(\sum p_i)_{|D}$ is ample, or D is a rational curve with at least one (resp. two) *special* points.

We say that a morphism $(\mathcal{C}, p_i) \to (\mathcal{C}', p_i')$ between semistable connected curves of type (g, n) (with $(g, n) \neq (1, 0)$) over a DVR R is a *stabilization map* if:

- On the generic fiber, it does contract the semistable rational irreducible components, and it is an isomorphism on the complementary of these;
- Contract all the semistable irreducible components of the central fiber $(\mathcal{C}_0, p_{i,0})$.

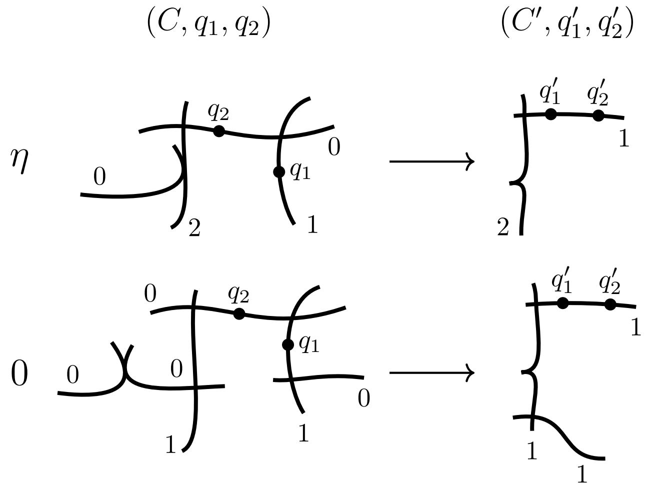


Figure 3.2. Example of stabilization map over a DVR with topological points $\{\eta, 0\}$.

3.1.1 Singularities of type A

Given a curve C over a field k, a singular point $p \in C$ is called an A_i singularity if, after a finite base change $k \subset k'$, there exists a point $q \in C_{k'}$ over p such that

$$\widehat{\mathcal{O}_{C_{k'},q}} \simeq k'[[x,y]]/(y(y-x^i)).$$

If $\operatorname{char} k > i+1$ or 0, we can replace $y(y-x^i)$ with $y^2 - x^{i+1}$ in the definition, via a change of coordinates. A-type singularities are local complete intersection, hence Gorenstein. Finally, for an A_i singularity $p \in C$, we have that the genus $g(C) = g(\widetilde{C}) + \lfloor i/2 \rfloor$, where $\widetilde{C} \to C$ is the seminormalization in a neighborhood of p (cf. [Sta25, Definition 0EUT]). The equation follows from the computation of the δ invariant, as explained in [Ser12, Prop. IV.7.3].

An A_1 (resp. A_2, A_3) singularity is called a *node* (resp. *cusp*, *tacnode*). If q is a k-point, then we say that the singularity is *rational.*

Given a family of curves $(\mathcal{C}, p_i)$ over a scheme S, we say that a section $f\colon S \to \mathcal{C}$ is an *equisingular* A_i section if, for every $s \in S$, $f(s) \in C_{k(s)}$ is an A_i singularity.

Fact 3.1.4. *([Tju69], [Ser06, Sec. 3.1]) Let $q \in C$ be a point on a curve with an A_i-type singularity. Then the space $T^1(\widehat{\mathcal{O}_{C,q}})$ is finite-dimensional over any field k with $\operatorname{char} k > i$. Therefore, a miniversal deformation space exists, in this case the deformation space is given by $k[a_0, \dots, a_{i-1}]$ and the versal deformation is*

$$y^2 - (x^{i+1} + a_i x^i + \dots + a_0) = 0.$$

This implies that an A_i-singularity can deform to a collection of $\{A_{i_1}, \dots, A_{i_r}\}$ singularities if and only if $\sum_{i=1}^r (i_i + 1) \leq i + 1$.

Let $\overline{\mathcal{M}}_{g,n}^{all}$ be the stack over $\mathrm{Sch}_{\text{ét}}/k$ whose fiber over a scheme S/k is given by the grupoid of n-pointed families of connected curves of genus g over S, where the maps are the isomorphisms of pointed curves.

Thanks to [Sta25, Tag 0E1I and Tag 0DSS], $\overline{\mathcal{M}}^{all}_{g,n}$ is an algebraic stack locally of finite type over k.

We then define two open substacks:

- $\widetilde{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}^{all}_{g,n}$, the open substack of stable curves (i.e. with log-dualizing ample) that have only singularities of type A (by Fact 3.1.4 this locus is open);
- $\widetilde{\mathcal{M}}^i_{g,n} \subset \widetilde{\mathcal{M}}_{g,n}$, the open substack of curves that have only singularities of type A_j for $j \leq i$.

We recall [Per23, Thm. 2.2]:

Proposition 3.1.5. *The algebraic stack $\widetilde{\mathcal{M}}^i_{g,n}$ is smooth, connected, and of finite type over k. Furthermore, it is a quotient stack: that is, there exists a smooth quasi-projective scheme X with an action of GL_N for some positive N, such that $\widetilde{\mathcal{M}}^i_{g,n} \simeq [X/\mathrm{GL}_N]$.*

Definition 3.1.6 (Inner/Outer/Lonely Singularities)**.** Let C be a curve over a field k, with algebraic closure $k' \supset k$. We say that an A_i-singularity $p \in C$ is *outer* if it lies on two distinct irreducible components of $C_{k'}$, and *inner* if it lies on a single irreducible component. We say that a singularity is *lonely* if it is outer and is the unique intersection of the two connected components of $C_{k'}$. (If i is even, then any A_i-singularity is necessarily inner).

For curves with A-type singularities we have a combinatorial description of the (semi)stability.

Proposition 3.1.7. *Let (C, p_i) a curve over an algebraically closed field k, whose singularities are all type A. Then for any irreducible component $D \subset C$ we can consider the composition with the normalization of D, $\pi\colon \widetilde{D} \to D \to C$. Let $\{q^i_j\}_{j\in J_i}$ be the inverse image of the A_i-singularities in C along the map π. We have:*

$$\pi^*\omega_{C/k} = \omega_{\widetilde{D}/k}(\sum_{2\nmid i} i \sum_{j\in J_i} q^i_j + \sum_{2\nmid i} \frac{i+1}{2} \sum_{j\in J_i} q^i_j).$$

Assume that (C, p_i) is of type $(g, n) \neq (1, 0)$. It is semistable if and only if for any rational irreducible component $D \in C$ either:

- *D contain two special points;*
- *D contain an A_i singularity with $i \geq 2$.*

Moreover, (C, p_i) is stable if and only if for any rational irreducible component $D \in C$ one of the following occur:

- *D contain at least one singularity of type A_i, $i \geq 4$;*
- *D contain an inner A_3 singularity;*
- *D contain two distinct special points, and one of them is a singularity of type A_i, $i \in \{2, 3\}$;*
- *D contain two distinct special points, one of which is an inner node;*

- *D contains at least 3 special points.*

Proof. The formula with dualizing sheaves follows from Noether Formula (cf. [Cat82, Prop. 1.2]), via a direct computation of the conductor. Consider now the normalization $f\colon \widetilde{C} \to C$, a line bundle L is ample on C if and only if f^*L is ample on $\widetilde{C}$. Hence, the second part of the claim follows from the formula for the dualiziang sheaf. □

Consider a family of curves $\mathcal{C} \to \operatorname{Spec} R$ with at worst A-type singularities, where R is a DVR. We are interested in how the singularities of $\mathcal{C}_\eta$ degenerate in $\mathcal{C}_0$. By Fact 3.1.4, an A_i-singularity can deform to a collection of $\{A_{i_1}, \dots, A_{i_r}\}$ singularities if and only if $\sum_{i=1}^r (i_i + 1) \leq i + 1$. For outer singularities on a curve we have constraints:

Proposition 3.1.8. *[Alp+17, Prop. 2.10]. Let $p \in C_0$ be an A_m-singularity, and suppose that p is the limit of an outer singularity $q \in C_\eta$. Then p is outer (in particular, m is odd), and each singularity of C_η that approaches p must be outer and must lie on the same two irreducible components of C_η as q. Moreover, the collection of singularities approaching p is necessarily of the form $\{A_{2i_1+1}, A_{2i_2+1}, \dots, A_{2i_r+1}\}$, where $\sum_{i=1}^r (2i_i + 2) = m + 1$, and there exists a simultaneous normalization of the family $C \to \Delta$ along this set of generic singularities.*

We recall a classical result on the local structure of nodes, as it appears in [TV10, Prop. 7.5]:

Proposition 3.1.9. *Suppose that $f : X \to S$ is a flat morphism of finite type, the fibre $X_0 = f^{-1}(s_0)$ over a point $s_0 \in S$ is a curve over $k(s_0)$ with isolated singularities, and $p \in X_0$ is a rational node. Then there exists $u \in \widehat{\mathcal{O}}_{S,s_0}$ and an isomorphism of $\widehat{\mathcal{O}}_{S,s_0}$-algebras*

$$\widehat{\mathcal{O}}_{X,p} \cong \widehat{\mathcal{O}}_{S,s_0}[[x, y]]/(xy - u).$$

This proposition allows us to define:

Definition 3.1.10 (Local parameter on a node)**.** Let $\mathcal{C} \to \operatorname{Spec} R$ be a curve over a DVR. Assume that $p \in \mathcal{C}_0$ is a rational node. Then we have:

$$\widehat{\mathcal{O}}_{\mathcal{C},p} = \widehat{R}[[x, y]]/(xy - a),$$

where $a \in \widehat{R}$ has valuation $v(a) \in \mathbb{Z}_{>0} \cup \{\infty\}$. This (unique) valuation is called the *local parameter* along the node p.

3.1.2 Normalization and gluing of singularities

We now discuss some local modifications of curves involving A_i singularities.

Definition 3.1.11 (Equisingular A_i section)**.** For a family of curves $(\mathcal{C} \to S, \sigma_i)$, we say that a seciton $s\colon S \to \mathcal{C}$ of the map $\mathcal{C} \to S$ is an *equisingular A_i section* if the set-theoretic image of any point $p \in S$ is an A_i singularity of $\mathcal{C}_p$. When $i = 1$ (resp 2, 3) we say that s is a nodal (resp. cuspidal, tacnodal) section.

Definition-Proposition 3.1.12 (Normalization along a section)**.** Consider a curve $(\mathcal{C}, p_i)$ over S and let $\sigma\colon S \to \mathcal{C}$ be an equisingular A_i section. Then there exists a curve $(\mathcal{C}', p_i')$ and a morphism

$$\nu\colon \mathcal{C}' \to \mathcal{C}$$

such that it restrict to an isomorphism $\mathcal{C}' \setminus \nu^{-1}(\operatorname{Im}\sigma) \to \mathcal{C} \setminus \operatorname{Im}\sigma$, on an open $\mathcal{U}$ containing $\operatorname{Im}\sigma$ is such that $\nu^{-1}(\mathcal{U}) \to \mathcal{U}$ is the normalization map, and $\nu(p_i') = p_i$. We call ν the *partial normalization along* σ.

Proof. The claim follow from the affine case, which is a special case of [CL06, Thm. 4.1]. □

The following three propositions are standard facts about A_i singularities for $i \geq 3$, and are consequence of [Fer03]. These create families of curves with equisingular section via gluing. . Further generalizations are possible but beyond our scope.

Proposition 3.1.13 (Nodal singularities as gluing of sections)**.** *Consider a pointed curve $(\mathcal{D}, q_i)$ over a scheme S and two smooth disjoint sections (disjoint from the markings) $\sigma_1\colon S \to \mathcal{D}$, $\sigma_2\colon S \to \mathcal{D}$. There exists a curve $\mathcal{C}$ over S that fits in the following pushout diagram:*

$$\begin{tikzcd} S \sqcup S \arrow[r] \arrow[d, "\sigma_1 \sqcup \sigma_2"] & S \arrow[d, "i_1"] \\ \mathcal{D} \arrow[r, "i_2"] & \mathcal{C} \end{tikzcd},$$

where the map $\sigma_1 \sqcup \sigma_2$ restricts to σ_1 and σ_2 on the two connected components. Furthermore, $(\mathcal{C}, i_2(q_i))$ is a family of pointed curves over S, i_1 is a nodal (rational) section, and i_2 induces an isomorphism $\mathcal{D} \setminus (\operatorname{Im}\sigma_1 \sqcup \operatorname{Im}\sigma_2) \to \mathcal{C} \setminus \operatorname{Im} i_1$.

Proposition 3.1.14 (Crimping of cusps)**.** *Consider a pointed family of curves $(\mathcal{D}, q_i)$ over a scheme S and a smooth section (disjoint from the markings) $\sigma\colon S \to \mathcal{D}$.*

Then there exists a pointed curve $(\mathcal{C}, p_i)$ and a cuspidal section $\widetilde{\sigma}\colon S \to \mathcal{C}$, such that there exists a morphism $\phi\colon \mathcal{D} \to \mathcal{C}$ that is an isomorphism restricted to $\mathcal{D} \setminus \operatorname{Im}\sigma \to \mathcal{C} \setminus \operatorname{Im}\widetilde{\sigma}$ and with $\phi(q_i) = p_i$.

On an affine open $\operatorname{Spec} A \subset S$, *$\mathcal{C}$ is given by the pushout of the following diagram:*

$$\begin{tikzcd} \operatorname{Spec} A[\epsilon]/(\epsilon^2) \arrow[r] \arrow[d, "t"] & \operatorname{Spec} A \arrow[d, "\widetilde{\sigma}_{|\operatorname{Spec} A}"] \\ \mathcal{D}_{\operatorname{Spec} A} \arrow[r, "i_2"] & \mathcal{C}_{\operatorname{Spec} A} \end{tikzcd},$$

where the top horizontal arrow is induced by the natural inclusion $A \subset A[\epsilon]/(\epsilon^2)$ and t corresponds to a section of the tangent bundle $T_\sigma\mathcal{D}$ over A, which generates $T_\sigma\mathcal{D}$ as a rank 1 free module over A. Moreover, i_2 restricts to $\phi_{\operatorname{Spec} A}$ outside the image of σ. We call this process crimping a cusp on the section f.

And finally for tacnodes we have (cf. [HH13, Sec. 4.1]):

Proposition 3.1.15 (Tacnodes as gluing of sections)**.** *Consider a pointed curve $(\mathcal{D}, q_i)$ over a scheme S and two smooth disjoint sections (disjoint from the markings) $\sigma_1 \colon S \to \mathcal{D}$, $\sigma_2 \colon S \to \mathcal{D}$. Fix, moreover, an isomorphism of the tangent bundles $gl \colon T_{\sigma_1}\mathcal{D} \simeq T_{\sigma_2}\mathcal{D}$ over S.*

Then there exists a pointed curve $(\mathcal{C}, p_i)$ and a tacnodal section $\widetilde{\sigma} \colon S \to \mathcal{C}$, such that there exists a morphism $\phi \colon \mathcal{D} \to \mathcal{C}$ that is an isomorphism restricted to $\mathcal{D} \setminus (\operatorname{Im}\sigma_1 \sqcup \operatorname{Im}\sigma_2) \to \mathcal{C} \setminus \operatorname{Im}\widetilde{\sigma}$ and with $\phi(q_i) = p_i$.

On an affine open $\operatorname{Spec} A \subset S$*, $\mathcal{C}$ is given by the pushout of the following diagram:*

$$\begin{tikzcd} \operatorname{Spec} A[\epsilon]/(\epsilon^2) \sqcup \operatorname{Spec} A[\epsilon]/(\epsilon^2) \arrow[r] \arrow[d, "t"] & \operatorname{Spec} A[\epsilon]/(\epsilon^2) \arrow[d, "i_1"] \\ \mathcal{D}_{\operatorname{Spec} A} \arrow[r, "i_2"] & \mathcal{C}_{\operatorname{Spec} A} \end{tikzcd},$$

where the top horizontal arrow is the identity on the connected components, and t corresponds to two non-vanishing sections τ_1, τ_2 on the tangents $T_{\sigma_1}\mathcal{D}$ and $T_{\sigma_2}\mathcal{D}$ over A. Any of these section generate $T_{\sigma_1}\mathcal{D}$ as a rank 1 free module over A, under this identification $gl_{|\operatorname{Spec} A}(\tau_1) = \tau_2$. Moreover, i_2 restricts to $\phi_{|\operatorname{Spec} A}$ outside the images of σ_1 and σ_2 and $\widetilde{\sigma}_{|\operatorname{Spec} A} = i_1 \circ \iota$ where ι is defined by $\epsilon \to 0$.

In this case, we say that $(\mathcal{D}, q_i)$ is created by gluing *the sections σ_1 and σ_2 along a tacnode.*

3.2 Stratification of $\widetilde{\mathcal{M}}_{g,n}$ via dual graphs

In this section, we define a stratification of $\widetilde{\mathcal{M}}_{g,n}$ determined by the combinatorics of curves with A-type singularities. This allows us to establish the notation that will later be used to treat $\widetilde{\mathcal{M}}^3_{g,n}$.

Our construction generalizes the well known case of nodal curves, extending the notion of dual graph to curves with A-type singularities. This framework will be useful for studying families over our standard test objects (i.e. $\overline{ST}_R$, Θ_R, and $\operatorname{Spec} R$).

A general stratification for arbitrary singularities is given in [BGS23]. Here, however, we adopt a different perspective: we construct a category whose morphisms correspond to degenerations of curves, in analogy with the classical case of nodal curves.

3.2.1 Edge-weighted n-marked graphs

Our notion of edge-weighted graph is obtained by modifying the definitions as they appear in [Cav+20].

Definition 3.2.1 (Graph)**.** A *Graph* G consists of the following data:

- A finite set $X(G) = V(G) \sqcup F(G)$, where $V(G)$ and $F(G)$ are called respectively the *vertices* and *flags* of G;
- A *root map* $r_G \colon X(G) \to X(G)$, which is idempotent and whose image is $V(G)$;
- An *involution* $\iota_G \colon X(G) \to X(G)$, whose fixed point set contains $V(G)$.

We picture the root map as associating a flag f to the vertex v from which it emanates. The involution ι_G partitions $F(G)$ into a collection of subsets $\{f, \iota_G(f)\}$ of size 1 or 2. We refer to the sets consisting of two distinct flags as the *edges* $E(G)$ of G. A *loop* is an edge $\{f, \iota_G(f)\}$ such that $r_G(f) = r_G(\iota_G(f))$. In this way we have a partition of the flags in *half edges* and *legs*:

$$H(G) \coloneqq \{f \in F(G) \mid \iota_G(f) \neq f\},$$
$$L(G) \coloneqq \{f \in F(G) \mid \iota_G(f) = f\}.$$

We have $F(G) = L(G) \sqcup H(G)$. Notice that the involution ι_G is equivalent to the datum of an action $S_2 \curvearrowright X(G)$.

There is a natural geometric realization functor taking a graph to a 1-complex (or a 1-complex with attached infinite legs). We say that a graph G is *connected* if its geometric realization is. Alternatively, G is connected if, for any two vertices $v, w \in V(G)$, there is a sequence of vertices and flags $v_0 = v, f_0, v_1, f_1, \ldots, v_n = w$ where $r(f_j) = v_j$ and $r(\iota(f_j)) = v_{j+1}$ for each j.

Definition 3.2.2 (Edge-Weighted n-Marked Graph)**.** An *edge-weighted n-marked graph* (G, h, m, s) consists of the data of a finite graph G together with:

- A weight on flags $s \colon F(G) \to \mathbb{Z}_{>0}$ such that $s \circ \iota = s$;
- A bijective map

$$\begin{aligned} m \colon \{1, \ldots, n\} &\to L(G) \cap s^{-1}(1) \\ i &\mapsto l_i, \end{aligned}$$

  called the *marking*;
- A function $h : V(G) \to \mathbb{Z}_{\geq 0}$, called the *vertex weighting.*

We say that two edge-weighted graphs $(G, h, m, s), (G', h', m', s')$ are isomorphic if there exist a bijection $X(G) \simeq X(G')$ under which identification the maps r_G, ι_G, h, m, s coincides respectively with r_G, ι_G, h', m', s'.

We refer to the set $L(G) \cap s^{-1}(1)$ as the set of *marked points* $M(G, h, m, s)$ of (G, h, m, s), and to the set $C(G, h, m, s) \coloneqq L(G) \setminus M(G, h, m, s)$ as the set of *cusps* of (G, h, m, s). To lighten the notation, we will simply write $C(G)$ and $M(G)$ when there is no risk of confusion on the edge-weighted n-marked graph (G, h, m, s). Notice that

$$L(G) = M(G) \sqcup C(G).$$

Remark 3.2.3. With the notation above, if $\operatorname{Im} s = \{1\}$, we recover the notion of weighted n-marked graph in [Cav+20] by forgetting the datum of s.

We adapt some classical notions to our case:

- The *valence* $\mathrm{val}(v)$ of a vertex v of G is defined as
$$\mathrm{val}(v) = \sum_{\substack{f\in F(G)\setminus C(G)\\ r_G(f)=v}} s(f) \ + \sum_{\substack{f\in C(G)\\ r_G(f)=v}} (s(f)-1).$$
The latter definition will be clarified in the next section.

- For an edge-weighted n-marked graph (G,h,m,s), we define its *genus* as
$$g(G,h,m,s) = \sum_{f\in F(G)} \frac{s(f)-1}{2} \ + \ b_1(G) \ + \sum_{v\in V(G)} h(v),$$
where $b_1(G)$ denotes the first Betti number of G. We say that (G,h,m,s) is *stable* if for every vertex v of G we have
$$2h(v) - 2 + \mathrm{val}(v) > 0.$$

We will picture dual graphs as in Figure 3.3.

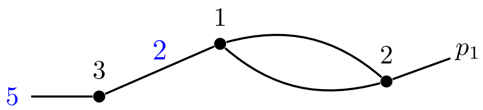


Figure 3.3. Example of a dual graph. The values of h are shown in black near the vertices (in black). $m(i)$ is denoted by p_i (these will correspond to marked points). The values of s are shown on the (half)edges (in blue), and are omitted when equal to 1.

We now define a notion of *covering graph*. This carries the same data as an edge-weighted marked graph, but will be used to define morphisms in our category.

Definition 3.2.4 (Covering graph)**.** A *covering graph* $\widetilde{G}$ consists of the following data:

- A finite set $X(\widetilde{G}) = V(\widetilde{G}) \sqcup M(\widetilde{G}) \sqcup C(\widetilde{G}) \sqcup H(\widetilde{G})$, whose elements are called *vertices*, *marks*, *cusps*, and *half-edges*, respectively;
- A *root map* $r_{\widetilde{G}}\colon X(\widetilde{G}) \to X(\widetilde{G})$, which is idempotent and whose image is $V(\widetilde{G})$;
- An *involution* $\iota_{\widetilde{G}}\colon X(\widetilde{G}) \to X(\widetilde{G})$ whose fixed points are $V(\widetilde{G}) \sqcup M(\widetilde{G}) \sqcup C(\widetilde{G})$;
- A bijection
$$m\colon \{1,\dots,n\} \xrightarrow{\ \sim\ } M(\widetilde{G}), \qquad i \mapsto l_i;$$
- A function $h\colon V(\widetilde{G}) \to \mathbb{Z}_{\geq 0}$, called the *vertex weighting*;
- A partition $H(\widetilde{G}) = \bigsqcup_{i\in I(\widetilde{G})} H_i$, for an index set $I(\widetilde{G})$, such that $|r_{\widetilde{G}}(H_i)| = 1$ and for every $i \in I(\widetilde{G})$ $\iota_{\widetilde{G}}(H_i) = H_j$ for some $j \in I(\widetilde{G})$;
- A partition $C(\widetilde{G}) = \bigsqcup_{i\in J(\widetilde{G})} C_i$, for an index set $J(\widetilde{G})$, such that $|r_{\widetilde{G}}(C_i)| = 1$.

We say that two covering graphs $\widetilde{G}, \widetilde{G}'$ are *isomorphic* if there exists a bijection $X(\widetilde{G}) \simeq X(\widetilde{G}')$ under which the root maps, involutions, and the functions h and m coincide.

We call the *flags* of $\widetilde{G}$ the elements of

$$F(\widetilde{G}) := M(\widetilde{G}) \sqcup C(\widetilde{G}) \sqcup H(\widetilde{G}).$$

Remark 3.2.5. Let $\widetilde{G}$ be a covering graph. There is a natural action of the product of symmetric groups

$$S_{\widetilde{G}} := \Big(\prod_{i \in I(\widetilde{G})} S_{|H_i|}\Big) \times \Big(\prod_{j \in J(\widetilde{G})} S_{|C_j|}\Big) \curvearrowright X(\widetilde{G}),$$

where $S_{|H_i|}$ (resp. $S_{|C_j|}$) acts by permuting the elements of H_i (resp. C_j). For brevity, set $S_{H(\widetilde{G})} = \prod_{i \in I(\widetilde{G})} S_{|H_i|}$ and $S_{C(\widetilde{G})} = \prod_{j \in J(\widetilde{G})} S_{|C_j|}$. As for usual graphs, giving the involution $\iota_{\widetilde{G}}$ is equivalent to giving an S_2-action on $X(\widetilde{G})$.

There is a correspondence between edge-weighted marked graphs and covering graphs, as we now explain. Roughly, the former is recovered as a quotient of the latter by the group in the remark above, remembering the sizes of stabilizers. Let CovGr be the set of covering graphs up to isomorphism and EGr the set of edge-weighted marked graphs up to isomorphism. Define

$$\sqsupset\colon \mathrm{CovGr} \longrightarrow \mathrm{EGr}, \qquad \widetilde{G} \longmapsto (G, h, m, s),$$

where (G, h, m, s) is given as we now explain. Set

$$X(G) := V(\widetilde{G}) \sqcup M(\widetilde{G}) \sqcup I(\widetilde{G}) \sqcup J(\widetilde{G}),$$

and let $\pi\colon X(\widetilde{G}) \to X(G)$ be the natural quotient map. The maps ι_G and r_G are the unique maps satisfying

$$\iota_G \circ \pi = \pi \circ \iota_{\widetilde{G}}, \qquad r_G \circ \pi = \pi \circ r_{\widetilde{G}}.$$

The vertex weighting h is the given one on $V(\widetilde{G}) = V(G)$; similarly for m, via $M(G) = M(\widetilde{G})$. Finally, define the edge weights $s\colon M(\widetilde{G}) \sqcup I(\widetilde{G}) \sqcup J(\widetilde{G}) \to \mathbb{Z}_{>0}$ by

$$s(i) = \begin{cases} |H_i| & \text{if } i \in I(\widetilde{G}), \\ |C_i| & \text{if } i \in J(\widetilde{G}), \\ 1 & \text{if } i \in M(\widetilde{G}). \end{cases}$$

Conversely, one constructs an inverse without much effort, as follows:

$$\begin{aligned} M(\widetilde{G}) &= M(G), I(\widetilde{G}) = F(G) \setminus (M(G) \cup C(G)), J(\widetilde{G}) = C(G), \\ H(\widetilde{G}) &= \{(a,b) | a \in I(\widetilde{G}), 1 \leq b \leq s(a)\}, \\ C(\widetilde{G}) &= \{(a,b) | a \in J(\widetilde{G}), 1 \leq b \leq s(a)\}. \end{aligned}$$

$$M(\widetilde{G}) = M(G),\ I(\widetilde{G}) = F(G) \setminus (M(G) \cup C(G)),\ J(\widetilde{G}) = C(G),$$

$$H(\widetilde{G}) = \{(a,b) \mid a \in I(\widetilde{G}),\ 1 \le b \le s(a)\}, \quad C(\widetilde{G}) = \{(a,b) \mid a \in J(\widetilde{G}),\ 1 \le b \le s(a)\}.$$

This defines $X(\widetilde{G})$, and the maps are constructed naturally. The outcome of this discussion is the following lemma.

Lemma 3.2.6. *The map* $\sqsupset$: CovGr $\to$ EGr *is a bijection.*

Definition 3.2.7 (Morphisms between covering graphs)**.** Let $\widetilde{G}$ and $\widetilde{G}'$ be covering graphs. A morphism $\widetilde{G} \to \widetilde{G}'$ is a map of sets $f\colon X(\widetilde{G}) \to X(\widetilde{G}')$ such that:

1. $r_{\widetilde{G}'} \circ f = f \circ r_{\widetilde{G}}$;
2. f sends $M(\widetilde{G})$ bijectively onto $M(\widetilde{G}')$ and preserves the markings, i.e. $f(l_i) = l_i'$;
3. For each flag $x' \in F(\widetilde{G}')$, its preimage $f^{-1}(x')$ consists of exactly one element, which lies in $F(\widetilde{G})$;
4. If $x \in H(\widetilde{G})$ and $f(x) \in H(\widetilde{G}')$, then $\iota_{\widetilde{G}'} \circ f(x) = f \circ \iota_{\widetilde{G}}(x)$;
5. For every $\sigma \in S_{H(\widetilde{G}')}$ and $x \in X(\widetilde{G})$, there exists $\tau \in S_{\widetilde{G}}$ such that $\sigma \circ f(x) = f(\tau \circ x)$;
6. For every $\sigma \in S_{C(\widetilde{G}')}$ and $x \in X(\widetilde{G})$, there exists $\tau \in S_{\widetilde{G}} \times S_2$ such that $\sigma \circ f(x) = f(\tau \circ x)$ (where the S_2-action is given by $\iota_{\widetilde{G}}$);
7. For every $v \in V(\widetilde{G}')$, the preimage

   $$f^{-1}(\{v\} \cup \{\, q \in C(\widetilde{G}') \mid r_{\widetilde{G}'}(q) = v \,\})$$

   is a connected subgraph of genus $h(v) + \sum_{r(C_j)=\{v\}}(|C_j| - 1)$.

Notice that if a morphism between two covering graphs exists, then the corresponding edge-weighted marked graphs (via the identification given by $\sqsupset$) have the same genus and number of markings. Moreover, given a covering $\widetilde{G}$, every $\tau \in S_{\widetilde{G}}$ defines an automorphism of $\widetilde{G}$.

Definition 3.2.8 (Morphisms between edge-weighted marked graphs)**.** Let G and G' be edge-weighted n-marked graphs. Define

$$\underline{\mathrm{Mor}}(G, G') \coloneqq S_{\widetilde{G}} \backslash \underline{\mathrm{Mor}}(\widetilde{G}, \widetilde{G}')/S_{\widetilde{G}'},$$

where the actions of $S_{\widetilde{G}}$ and $S_{\widetilde{G}'}$ permute the flags of $\widetilde{G}$ and $\widetilde{G}'$. This yields a well-defined notion of composition. Let $\mathrm{EGr}_{g,n}$ be the category whose objects are stable, connected, edge-weighted n-marked graphs of genus g, and whose morphisms are as above.

This category contains, as a full subcategory, the category of stable weighted marked graphs defined in [Cav+20, Sec. 3.1].

3.2.2 Dual graphs of curves and the stratification of $\widetilde{\mathcal{M}}_{g,n}$

We can associate a dual graph to points of $\widetilde{\mathcal{M}}_{g,n}$, yielding a natural stratification of this stack.

Definition 3.2.9 (Dual graph of a curve)**.** Fix a curve $(C, p_1, \dots, p_n)$ over an algebraically closed field k, with singularities of type A. Its associated *dual graph* (G, h, m, s) is the edge-weighted n-marked graph defined as follows:

1. The set of vertices $V(G)$ is in bijection with the irreducible components of C, equivalently with the connected components of the normalization $\nu\colon \widetilde{C} \to C$:
$$V(G) = \{C_v \mid C_v \text{ irreducible component of } C\}.$$
2. The function $h\colon V(G) \to \mathbb{Z}_{\geq 0}$ sends v to $g(\widetilde{C}_v)$, the genus of the corresponding normalized component.
3. The flags $F(G) = M(G) \sqcup C(G) \sqcup H(G)$ are constructed as follows:
   - $M(G)$ is in bijection with the marked points: $M(G) \coloneqq \{p_1, \dots, p_n\}$;
   - The half-edges $H(G)$ are in bijection with to the union of the preimages $q', q'' \in \widetilde{C}$ of bibranched singularities $q \in C$ under the normalization map $\nu\colon \widetilde{C} \to C$:
   $$H(G) \coloneqq \coprod_{\substack{q \text{ bibranched singularity in } C \\ \nu^{-1}(q) = \{q', q''\}}} \{q', q''\};$$
   - The cusps $C(G)$ correspond to the unibranched singularities in C:
   $$C(G) \coloneqq \{q \in C(k) \mid q \text{ unibranched singularity in } C\}.$$
4. The root map $r_G\colon X(G) \to X(G)$ fixes $V(G)$ and sends each $q \in F(G)$ to the connected component of $\widetilde{C}$ containing q.
5. The involution $\iota_G\colon X(G) \to X(G)$ fixes $V(G) \sqcup M(G) \sqcup C(G)$ and, for each bibranched singularity $q \in C$, exchanges the two elements of $H(G)$ corresponding to the preimage of a bibranched singularity in C.
6. The weight function $s\colon F(G) \to \mathbb{Z}_{>0}$ is defined by
$$s(f) = \begin{cases} 1 & \text{if } f \in M(G), \\ i+1 & \text{if } f \in H(G) \text{ corresponds to an } A_{2i+1} \text{ singularity}, \\ 2i+1 & \text{if } f \in C(G) \text{ corresponds to an } A_{2i} \text{ singularity}. \end{cases}$$
7. The map $m\colon \{1, \dots, n\} \to F(G)$ sends i to the flag corresponding to p_i.

The following remark follows from Proposition 3.1.7.

Remark 3.2.10. If the curve is stable, then the graph defined above is an object of $\mathrm{EGr}_{g,n}$, where g is the genus of (C, p_i) and n is the number of marked points.

Moreover, for an irreducible component $D \subset C$ corresponding to a vertex v, let $\pi\colon \widetilde{D} \to D$ denote its normalization. Then

$$\mathrm{val}(v) = \deg \pi^* \omega_C\Big(\sum p_i\Big) - \deg \omega_{\widetilde{D}}.$$

Example 3.2.11. In Figure 3.4, we illustrate a curve (and its dual graph) with three irreducible components: two of them meet at two nodes, and the third has an A_4 singularity and meets the rest of the curve in a tacnode. We omit the weight of flags when they are 1.

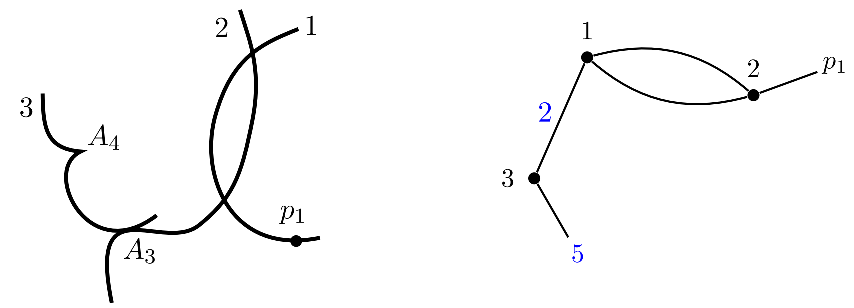


Figure 3.4. The marked point is denoted by p_1, while blue labels indicate (half)edge weights.

We recap in the table below the contribution of an A-type singularity that lies to a component corresponding to $v \in V(G)$ in the dual graph.

A-type	s	contribution to $\mathrm{val}(v)$	genus contribution
A_{2i}	$2i+1$	$2i$	i
A_{2i+1}	$i+1$	$i+1$	i

This combinatorial data allows us to construct a stratification of $\widetilde{\mathcal{M}}_{g,n}$ and $\widetilde{\mathcal{M}}^k_{g,n} \subset \widetilde{\mathcal{M}}_{g,n}$. With this purpose, we define:

Definition 3.2.12. Define $\mathrm{EGr}^i_{g,n}$ to be the full subcategory of $\mathrm{EGr}_{g,n}$ that contains the edge-weighted marked graphs (G, h, m, s) such that for every $f \in F(G)$

$$s(f) \leq \frac{i+1}{|\{f, \iota_G(f)\}|}.$$

Moreover, we define $\mathit{sing}(\Gamma) := \min\{k | \Gamma \in \mathrm{EGr}^k_{g,n}\}$.

The category $\mathrm{EGr}^i_{g,n}$ contains dual graphs of curves with singularities A_i with $i \leq k$. We have that $\mathrm{EGr}^1_{g,n}$ is the category of dual graphs of nodal stable curves. For genus reasons $\mathrm{EGr}^i\ g, n = \mathrm{EGr}_{g,n}$ if $i \geq 2g+1$, we then have proper inclusions

$$\mathrm{EGr}^1_{g,n} \subset \mathrm{EGr}^2_{g,n} \subset \cdots \subset \mathrm{EGr}^{2g+1}_{g,n} = \mathrm{EGr}_{g,n}\,.$$

Proposition 3.2.13. *Fix an element $\Gamma \in \mathrm{EGr}^i_{g,n}$ and assume $\mathrm{char}\, k > i$. Then the locus of curves in $|\widetilde{\mathcal{M}}^i_{g,n}|$ that have Γ as their dual graph is locally closed. The* *substack defined by this locus with the reduced structure is smooth and connected (hence, irreducible).*

Proof. Given a family of curve $\mathcal{C} \to S$ and an A_j singularity of a fiber over $s \in S$, the locus of S where this extends to an equisingular A_j section is locally closed thanks to Fact 3.1.4. This implies that the locus with dual graph Γ is locally closed, hence the claim. Let S_Γ such a locus with the reduced structure, this is the moduli stack of the curves with A-type singularities and dual graph Γ. Consider the map given by the normalization of the families of curves

$$S_\Gamma \to \prod_{v \in V(\Gamma)} \mathcal{M}_{h(v),n_v},$$

where $h(v)$ is the genus of the component associated to v and n_v is the number of the special points on it. Thanks to [Wyc10, Thm. 1.105], the map is an étale fibration on the crimping space. For A-type singularities, the crimping space is smooth and irreducible (cf. [Wyc10, Example 1.78 and Example 1.78]), hence the source space S_Γ is smooth and irreducible. □

Definition 3.2.14 (Strata)**.** Let k be a field with $\operatorname{char} k > g$. For every $\Gamma \in \mathrm{EGr}_{g,n}$, we define $S_\Gamma \hookrightarrow \widetilde{\mathcal{M}}_{g,n}$ to be the locally closed substack with the reduced structure whose points $|S_\Gamma| \subset |\widetilde{\mathcal{M}}_{g,n}|$ correspond to the curves with dual graph Γ.

We enhance this set with the natural poset structure $(\{S_\Gamma\}_{\Gamma\in\mathrm{EGr}_{g,n}}, \geq)$, where

$$S_\Gamma \geq S_{\Gamma'} \iff S_{\Gamma'} \subset \overline{S_\Gamma}.$$

Proposition 3.2.15 (Stratification of $\widetilde{\mathcal{M}}_{g,n}$)**.** *Let k be a perfect field. The poset $(\{S_\Gamma\}_{\Gamma\in\mathrm{EGr}_{g,n}}, \geq)$ provides a stratification of $\widetilde{\mathcal{M}}_{g,n}$:*

$$\widetilde{\mathcal{M}}_{g,n} = \bigsqcup_{\Gamma\in\mathrm{EGr}_{g,n}} S_\Gamma, \qquad \textit{and} \qquad \overline{S_\Gamma} = \bigcup_{\Gamma'\to\Gamma} S_{\Gamma'}.$$

The above proposition implies that there exists a functor

$$\begin{aligned} \mathrm{EGr}_{g,n}^{\mathrm{op}} &\longrightarrow (\{S_\Gamma\}_{\Gamma\in\mathrm{EGr}_{g,n}}, \geq), \\ \Gamma &\longmapsto S_\Gamma, \end{aligned}$$

which is full and essentially surjective.

Proof. The argument is analogous to the case of nodal stable curves. It suffices to show that, for $\Gamma, \Gamma' \in \mathrm{EGr}_{g,n}$, there exists a morphism $\Gamma \to \Gamma'$ if and only if $S_\Gamma \subset \overline{S_{\Gamma'}}$. Suppose $S_\Gamma \subset \overline{S_{\Gamma'}}$. Then there exists a morphism $\operatorname{Spec} R \to \widetilde{\mathcal{M}}_{g,n}$ from a DVR R whose generic fiber has dual graph Γ' and whose special fiber has dual graph Γ. We obtain a contraction of graphs $\Gamma \to \Gamma'$ by sending each vertex $v \in V(\Gamma)$, corresponding to an irreducible component D of the central fiber, to the vertex $v' \in V(\Gamma')$ corresponding to the irreducible component of the generic fiber that contains D in its closure. The map on flags at the level of covering graphs is induced by the specialization of singular points, which exists by the valuative criterion of properness. Thanks to Fact 3.1.4 and Proposition 3.1.8, the combinatorics of the singularities give a well-defined map of graphs. Conversely, assume there is a morphism of graphs $\phi\colon \Gamma \to \Gamma'$. For a point $(C, p_i) \in S_\Gamma$ we must show that there exists a family $\mathcal{C} \to \operatorname{Spec} R$ over a DVR R such that the special fiber is (C, p_i) and the generic fiber has dual graph Γ'. Assume first that Γ' consists of a single vertex. Then for each flag $f \in F(\Gamma')$ corresponding to an A_i singularity of C, we can deform them to the singularities prescribed in to the map ϕ, thanks to Fact 3.1.4, that have the same root thanks to the definition of morphism among dual graphs. This construct a deformation of C over a family $\mathcal{C} \to \operatorname{Spec} R$. Given that the marked points of (C, p_i) lie on the smooth locus of C, we can extend the marking up to change the DVR R with his completion.

We show the general case via deformation theory. We prove it in the case Γ' has only two vertices $\{v_1, v_2\} = V(\Gamma')$ and a unique outer singularity, the general case follows similarly. We have $V(\Gamma) = \phi^{-1}(v_1) \sqcup \phi^{-1}(v_2)$, let $C = C_1 \cap C_2$ a decomposition where C_j is the subcurve given by the union of components in $\phi^{-1}(v_j)$. Let $q = |C_1 \cap C_2|$ be the unique A_m outer singularity in the intersection, corresponding to $f_q \in \Gamma$. The map ϕ, defined via a map of covering graphs, force the existence of flags in Γ' that corresponds to singularities of the form $\{A_{2i_1+1}, A_{2i_2+1}, \dots, A_{2i_r+1}\}$, where $\sum_{i=1}^{r}(2i_i + 2) = m + 1$. We can then deform (C, p_i) in such a way the local picture around q is given by (cf. proof [Alp+17, Prop. 2.10] and Fact 3.1.4)

$$y^2 = \prod_{j=1}^{r} (x - a_j(t))^{2i_j+2},$$

where $\sum_{i=1}^{r}(2i_i + 2) = m + 1$. This concludes the claim. $\square$

As a byproduct of our discussion, we can also fix the maximum type of singularity.

Corollary 3.2.16. *Assume* $\operatorname{char} k > i$, *then the poset* $(\{S_\Gamma\}_{\Gamma \in \mathrm{EGr}^i_{g,n}}, \geq)$ *provides a stratification of* $\widetilde{\mathcal{M}}^i_{g,n}$.

3.3 Cut-and-paste along singularities

We introduce here some notation and results for the cut-and-paste modification of curves. For simplicity, we work over a field even if the constructions above works on a general base (up to replacing singularities with singular sections). These modifications have a clear combinatorial interpretation at the level of dual graphs.We start defining attached subcurves.

Definition 3.3.1 (Attached component of a curve)**.** Let (C, p_i) be an n-pointed curve over a field k. Given a pointed curve (C', q_i) over k, we say that $\gamma\colon (C', q_i) \to (C, p_i)$ is an *attached component* of (C, p_i) if:

- γ is a finite morphism over k, and it induces an open embedding of $C' \setminus \{q_i\}$ into $C \setminus \{p_i\}$;

- The image of $\{q_i\}$ consists of either marked points or singularities;
- Every marked point (resp. special point) of C in $\gamma(C')$ is the image of a marked point (resp. special point) of (C', q_i).

In this situation, we say that the marked points of C' sent via γ to singularities of C are called *attaching points*.

Notice that attached points are then sent to rational singularities or marked points. Moreover, we recall that for our definition of curve, the attaching curve is not necessarily connected.

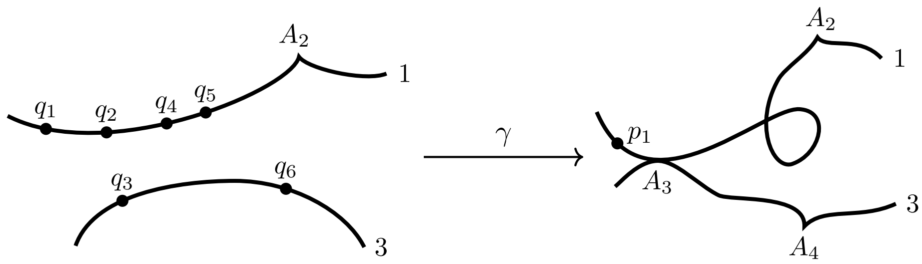


Figure 3.5. Example of attached curve, where the source is not connected and γ is surjective.

This notion has an analogous in terms of dual graphs.

Definition 3.3.2 (Subgraph)**.** Let $\Gamma = (G, h, m, s) \in \mathrm{EGr}_{g,n}$ be an edge-weighted marked graph. We define a *edge-weighted marked subgraph of* Γ, or simply *subgraph of* Γ, to be an edge-weighted marked graph (G', h', m', s'), together with an injective morphism $\gamma\colon X(G') \to X(G)$, such that:

- $\gamma(V(G')) \subset V(G)$, $\gamma(C(G')) \subset C(G)$, and $\gamma(H(G')) \subset H(G)$;
- $h' = h \circ \gamma$, and $\gamma \circ r_{G'} = r_G \circ \gamma$;
- For any vertex $v \in \operatorname{Im}\gamma$, we have that $\{q \in F(G) \mid r_G(q) = v\} \subset \operatorname{Im}\gamma$;
- For any $q \in H(G')$, $\gamma \circ \iota_{G'}(q) = \iota_G \circ \gamma(q)$;
- s' is defined as follows:

$$s'(x) = \begin{cases} s \circ \gamma(x) & \text{if } x \in C(G') \sqcup H(G'), \\ 1 & \text{otherwise.} \end{cases}$$

We say that the subgraph is *connected* if G' is connected, and that $f \in M(G')$ is an *attaching flag* if $\gamma(f) \notin M(G)$.

We point out that the disjoint union of subgraphs is still a subgraph, and that a subgraph is not necessarily stable. Notice moreover that being a subgraph does not carry a notion of morphism between graphs. In particular, with the notations above, we always have an inequality among genus $g(G') \leq g(G)$.

Remark 3.3.3. Given a curve $(C, p_i) \in \widetilde{\mathcal{M}}_{g,n}(k)$, over an algebraically closed field k, and with dual graph $\Gamma \in \mathrm{EGr}_{g,n}$, there is a bijection between dual attached subgraphs

$$\{\gamma\colon (C', q_i) \to (C, p_i) \text{ attached curve}\} \leftrightarrow \{\Gamma' \in \mathrm{EGr} \mid \Gamma' \text{ subgraph of } \Gamma\}.$$

Under this correspondence, connected attached curves correspond to connected subgraphs.

In Figure 3.5, the corresponding map among dual graph is as in Figure 3.6.

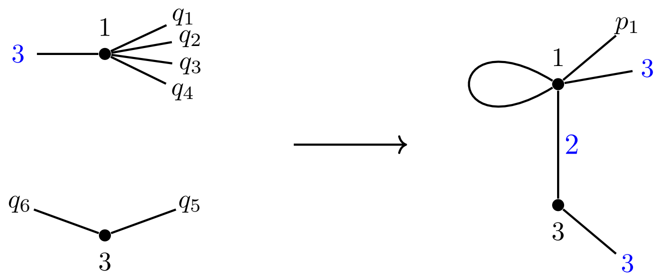


Figure 3.6. Marked points are denoted by q_i and p_i. Blue labels indicate (half)edge weights, which are omitted when the value is 1.

We now define a cut-and-paste modification for curves along A-type singularities.

Definition 3.3.4 (Replacement of an attached curve)**.** Let (C, p_i) be an n-pointed curve with A-type singularities over a field k. Fix an attached curve $\gamma : (C', q_i) \to (C, p_i)$ and consider another pointed curve $(\widetilde{C}', \widetilde{q}_i)$ over k and a bijective function $\phi : \{q_i\} \to \{\widetilde{q}_i\}$.

We say that a curve with A-type singularities $(\widetilde{C}, \widetilde{p}_i)$ is obtained by replacing $\gamma : (C', q_i) \to (C, p_i)$ with $(\widetilde{C}', \widetilde{q}_i)$ if there exists an attached curve $\widetilde{\gamma}\colon (\widetilde{C}', \widetilde{q}_i) \to (\widetilde{C}, \widetilde{p}_i)$ such that:

- There is an isomorphism
$$\Gamma\colon \overline{C \setminus \gamma(C')} \xrightarrow{\sim} \overline{\widetilde{C} \setminus \gamma(\widetilde{C}')};$$
- For every $q \in \{q_i\}$, $\gamma(q)$ and $\widetilde{\gamma}(\phi(q))$ are the same type of special point (i.e., node or A_i singularity for fixed i).

We can reformulate the notion of replacement with dual graphs, we omit the details because this notation won't be used in the following sections.

In general, the replacement could depend on a choice of the crimping data (cf. [Wyc10, Sec. 1.7]) of the singularities $\gamma'(q)$, $q \in \{q_i'\}$. Given that for A_i singularities with $i \leq 2$ the crimping space is trivial, we have the following:

Remark 3.3.5. With the same notations as in Definition 3.3.4, we have that if $\gamma(\{q_i\})$ consists of nodal, cuspidal, or marked points in (C, p_i), then the replacement is unique up to isomorphism. This means that, given another replacement $\widehat{\gamma} : (\widehat{C}', \widehat{q}_i) \to (\widehat{C}, \widehat{p}_i)$ and a map $\psi\colon \{q_i\} \to \{\widehat{q}_i\}$, there is a commutative diagram as follows:

$$\begin{array}{ccc} (\widetilde{C}', \widetilde{q}_i) & \xrightarrow{\widetilde{\gamma}} & (\widetilde{C}, \widetilde{p}_i) \\ \downarrow\simeq & & \downarrow\simeq \\ (\widehat{C}', \widehat{q}_i) & \xrightarrow{\widehat{\gamma}} & (\widehat{C}, \widehat{p}_i), \end{array}$$

where the left vertical morphism sends marked points according to $\psi^{-1} \circ \phi$, and the right vertical is a morphism of marked curves.

3.3.1 Descending isomorphisms from the normalization

The following proposition provides a way to control how isomorphisms can be glued in the context of these cut-and-paste modifications. The following standard results can be stated for any A, but that is not required for our scopes.

Proposition 3.3.6. *Let $(\mathcal{C}, p_i)$ and $(\mathcal{C}', p_i')$ be two families of pointed curves over a scheme S. Consider the following data:*

- *$\{\sigma_j\colon S \to \mathcal{C}\}_{j\in N}$ (resp. $\{\sigma_j'\colon S \to \mathcal{C}'\}_{j\in N'}$) are all the nodal equisingular sections of $(\mathcal{C}, p_i)$ (resp. of $(\mathcal{C}', p_i')$);*
- *$\{\beta_j\colon S \to \mathcal{C}\}_{j\in C}$ (resp. $\{\beta_j'\colon S \to \mathcal{C}'\}_{j\in C'}$) are all the cuspidal equisingular sections of $(\mathcal{C}, p_i)$ (resp. of $(\mathcal{C}', p_i')$);*
- *$\{\tau_j\colon S \to \mathcal{C}\}_{j\in T}$ (resp. $\{\tau_j'\colon S \to \mathcal{C}'\}_{j\in T'}$) are all the tacnodal equisingular sections of $(\mathcal{C}, p_i)$ (resp. of $(\mathcal{C}', p_i')$).*

Let $\pi\colon (\widetilde{\mathcal{C}}, \widetilde{p}_i) \to (\mathcal{C}, p_i)$ (resp. $\pi'\colon (\widetilde{\mathcal{C}}', \widetilde{p}_i') \to (\mathcal{C}', p_i')$) be the normalization along the sections σ_j, β_j and τ_j (resp. σ_j', β_j' and τ_j').

For every $j \in N$ (resp. $j \in N'$), we can lift this section to two possible sections $\widetilde{\sigma}_j^b\colon S \to \widetilde{\mathcal{C}}$ (resp. $\widetilde{\sigma}_j'^b\colon S \to \widetilde{\mathcal{C}}'$) for $b \in \{+,-\}$. We can define in an analogous way $\widetilde{\tau}_j^s\colon S \to \widetilde{\mathcal{C}}$ and $\widetilde{\tau}_j'^s\colon S \to \widetilde{\mathcal{C}}'$. Finally, for every $j \in C$ (resp. $j \in C'$), we can lift this section to a unique section $\widetilde{\beta}_j\colon S \to \widetilde{\mathcal{C}}$ (resp. $\widetilde{\beta}_j'\colon S \to \widetilde{\mathcal{C}}'$).
We have a natural inclusion of groups:

$$\iota\colon Isom_S((\mathcal{C}, p_i), (\mathcal{C}', p_i')) \to Isom_S((\widetilde{\mathcal{C}}, \widetilde{p}_i), (\widetilde{\mathcal{C}}', \widetilde{p}_i')),$$

where the image is given by the $\widetilde{\phi} \in Isom_S((\widetilde{\mathcal{C}}, \widetilde{p}_i), (\widetilde{\mathcal{C}}', \widetilde{p}_i'))$ such that:

- *There exists a bijection $sn\colon N \to N'$, such that for every lifting of a nodal section $\widetilde{\sigma}_l^s$, $s \in \{+,-\}$, we have that $\widetilde{\phi} \circ \widetilde{\sigma}_l^s$ is $\widetilde{\sigma}_{sn(l)}'^b$ for some $b \in \{+,-\}$;*
- *There exists a bijection $sc\colon C \to C'$, such that for every lifting of a cuspidal section $\widetilde{\beta}_l$, we have that $\widetilde{\phi} \circ \widetilde{\beta}_l$ is $\widetilde{\beta}_{sc(l)}'$;*
- *There exists a bijection $st\colon T \to T'$, such that for every lifting of a tacnodal section $\widetilde{\tau}_l^s$, $s \in \{+,-\}$, we have that $\widetilde{\phi} \circ \widetilde{\tau}_l^s$ is $\widetilde{\tau}_{st(l)}'^b$ for some $b \in \{+,-\}$;*

- *For every lift of a tacnodal singularity, the following diagram commutes:*

$$\begin{tikzcd} T_{\widetilde{\tau}_j^+}\widetilde{\mathcal{C}} \arrow[r, "gl_j"] \arrow[d, "d\widetilde{\phi}"] & T_{\widetilde{\tau}_j^-}\widetilde{\mathcal{C}} \arrow[d, "d\widetilde{\phi}"] \\ T_{\widetilde{\tau}_l^b}\widetilde{\mathcal{C}} \arrow[r, "gl_l"] & T_{\widetilde{\tau}_l^s}\widetilde{\mathcal{C}} \end{tikzcd}$$

where $\widetilde{\tau}_l^+ = \widetilde{\phi} \circ \widetilde{\tau}_j^b$, $s = \{+,-\} \setminus \{b\}$, and the horizontal isomorphisms are the gluing morphisms as in Proposition 3.1.15.

Proof. It is clear that we can lift the isomorphism to the partial normalization by the universal property. Then the proposition follows from Propositions 3.1.13, 3.1.14, and 3.1.15. □

We can extend this result as follows:

Remark 3.3.7. Using the same notations as in Proposition 3.3.6, suppose that S is an irreducible scheme with generic point $\eta \in S$. Let $(\mathcal{C}_\eta, p_{i,\eta})$ and $(\mathcal{C}'_\eta, p'_{i,\eta})$ be the generic fibers of the families, with dual graph given respectively by Γ, Γ'. Assume moreover that there exists an isomorphism of curves $(\mathcal{C}_\eta, p_{i,\eta}) \simeq (\mathcal{C}'_\eta, p'_{i,\eta})$, that induce an isomorphism $f\colon \Gamma \xrightarrow{\sim} \Gamma'$ at the level of dual graphs.

Fix now some subsets $\overline{N} \subset N$, $\overline{C} \subset C$, and $\overline{T} \subset T$ (resp. $\overline{N}' \subset N'$, $\overline{C}' \subset C'$, and $\overline{T}' \subset T'$) consisting of singularities that are invariant under automorphisms of Γ (resp. Γ') and such that $f(\overline{N}) = \overline{N}'$, $f(\overline{C}) = \overline{C}'$, and $f(\overline{T}) = \overline{T}'$. Then Proposition 3.3.6 holds by replacing N, N', C, C', T, T' with $\overline{N}$, $\overline{N}'$, $\overline{C}$, $\overline{C}'$, $\overline{T}$, and $\overline{T}'$.

Remark 3.3.8. Using the same notations as in Proposition 3.3.6, fix an element $\phi \in \mathrm{Isom}_S((\mathcal{C}, p_i), (\mathcal{C}', p'_i))$. Then there exists an ordering of the nodal, cuspidal, and tacnodal sections such that $\iota(\phi)$ lifts to an isomorphism of pointed curves:

$$(\widetilde{\mathcal{C}}, \widetilde{p}_i, \sigma_i^b, \beta_i, \tau_i^b) \to (\widetilde{\mathcal{C}}', \widetilde{p}'_i, \sigma_i'^b, \beta'_i, \tau_i'^b).$$

We can then re-index the equisingular sections so that $N = N'$, $C = C'$, $T = T'$, and the morphisms sn, sc, and st are the identity. Moreover, we can also assume that $\iota(\phi) \circ \widetilde{\sigma}_l^s = \widetilde{\sigma}_l'^s$ and $\iota(\phi) \circ \widetilde{\tau}_l^s = \widetilde{\tau}_l'^s$.

3.4 Local modification of families

In this section, we recall two results about local modification of the central fiber of families over a DVR.

3.4.1 Alternative limits

In this section we describe how to contract subcurves in the central fibre of a family of curves over the spectrum of a discrete valuation ring Δ, which we always assume to have algebraically closed residue field.

We recall [Smy13, Prop. 2.6] (the proof works without assuming the base to be a field), which is a consequence of the general theory of contractions (cf. [Art70, Sec. 6]).

Proposition 3.4.1 (Existence of Contractions)**.** *Let Δ be the spectrum of a discrete valuation ring with algebraically closed residue field, and let $\pi : \mathcal{C} \to \Delta$ be a generically smooth nodal curve over Δ. If $Z \subsetneq \mathcal{C}_0$ is a proper (possibly disconnected) subcurve of the special fibre $\mathcal{C}_0$, then there exists a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{C}' \arrow[dl] \\ & \Delta & \end{tikzcd}$$

such that:

1. *ϕ is proper, birational, $\phi_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}'}$, and $\mathrm{Exc}(\phi) = Z$;*
2. *$\mathcal{C}' \to \Delta$ is a flat family of geometrically reduced connected curves;*
3. *the restriction of ϕ to the special fibre induces a contraction of curves.*

Proposition 3.4.2. *Let $(\mathcal{C}, \sigma_i)$ be a geometrically reduced curve over a discrete valuation ring R. Consider a proper subcurve $Z \subsetneq \mathcal{C}_0$ of the special fibre $\mathcal{C}_0$ given by $Z = \bigsqcup Z_j$, where $(Z_j, q_{j,i})$ are disjoint connected attached curves $\gamma_j : (Z_j, q_{j,i}) \to \mathcal{C}_0$ satisfying one of the following:*

1. *$(Z_j, q_{j,1}, q_{j,2}) \in \overline{\mathcal{M}}^{\mathrm{ps}}_{1,2}(k)$, and $\gamma_j(q_{j,1}), \gamma_j(q_{j,2})$ are nodal singularities with the same local parameter, that is positive;*
2. *$(Z_j, q_{j,1}) \in \overline{\mathcal{M}}_{1,1}(k)$, and $\gamma_j(q_{j,1})$ is an isolated nodal singularity of $\mathcal{C}$.*

Then there exists a family $(\mathcal{C}', \sigma_i')$ over R and a contraction $\phi\colon \mathcal{C} \to \mathcal{C}'$ as in Proposition 3.4.1. Moreover, the image of Z_j is an A_3 singularity in case (1) and an A_2 singularity in case (2).

Proof. This follows from Proposition 3.4.1, using the same argument as in [FS, Prop. 2.22]. The n-marked family $(\mathcal{C}', \sigma_i')$ is obtained by taking the unique extensions of the generic sections $\sigma_{i,\eta}$ to the central fibre of $\mathcal{C}'$. □

Notice that if $(\mathcal{C}, \sigma_i)$ is semistable with singularities of type A_i for $i = 1, 2, 3$, then $(\mathcal{C}', \sigma_i')$ is also semistable with singularities of type A_i for $i = 1, 2, 3$. We also apply Proposition 3.4.1 to contract semistable rational components appearing in the central fibre.

Proposition 3.4.3 (Contraction of semistable subcurves)**.** *In the setting of Proposition 3.4.1, suppose that $\gamma\colon (Z, q_1, q_2) \to (\mathcal{C}_0, p_{i,0})$ is an attached curve such that $(Z, q_1, q_2) \simeq (\mathbb{P}^1, 0, \infty)$ and $\gamma(q_1), \gamma(q_2)$ are distinct rational nodal singularities of $\mathcal{C}_0$ with (positive) local parameters m_1 and m_2, respectively. Assume moreover that Δ is the spectrum of a discrete valuation ring over a field k with $\operatorname{char} k = 0$ or $\operatorname{char} k > 5$.*

Performing the contraction of Proposition 3.4.1, we obtain an n-marked family $(\mathcal{C}', p_i')$, where the p_i' are the unique extensions of $p_{i,\eta}$ to the central fibre of $\mathcal{C}'$.

Moreover, the set-theoretic image of $\mathbb{P}^1$ *is a rational nodal singularity on* $\mathcal{C}'$ *with local parameter* $m_1 + m_2$*.*

Fact 3.4.4. *Let* Δ *be the spectrum of a discrete valuation ring over a field* k *with* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > 5$*, and with algebraically closed residue field. Let* $\pi\colon \mathcal{C} \to \Delta$ *be a family of generically smooth curves, and let* $p \in \mathcal{C}_0$ *be an isolated nodal singularity of local parameter* $m+1$ *(for* $m \geq 0$*). Then* p *is an* A_m *singularity, and the minimal resolution around* p*,* $\widetilde{\phi}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$*, contains* m *exceptional rational curves* $\bigcup_{i=1}^m E_i$ *with dual graph* A_m*. Moreover,* $\widetilde{\mathcal{C}} \to \Delta$ *is a flat family of geometrically reduced curves.*

Proof. The description of the minimal resolution follows from [Bar+15, Sec. 3.7] when $\operatorname{char} k = 0$ and from [Art77] when $\operatorname{char} k > 5$. Flatness follows since the only associated point of $\widetilde{\mathcal{C}}$ is the generic one. For the final claim, write the cartier divisor $D = \pi^{-1}(0)$ and $\widetilde{\phi}^*(D) = \widetilde{D} + \sum_{i=1}^{m-1} E_i$, with $\widetilde{D}$ the strict transform. By projection formula we have:

$$0 = \widetilde{\phi}^*(D) \cdot E_i = -2E_i^2 + n_{i-1} + n_{i+1}$$

where we set $n_0 = 1 = \widetilde{D} \cdot E_1$ and $n_m = 1 = \widetilde{D} \cdot E_m$. For a resolution of A-type surface singularity $E_i^2 = -2$. Therefore, the sequence of numbers n_i is such that $n_i = \frac{n_{i-1}}{2}$ and $n_0 = n_m = 1$, this implies that $n_i = 1$ for all i. The central fiber is then geometrically reduced and Cohen–Macaulay (because $\widetilde{\mathcal{C}}$ is normal), hence we have the final claim. □

Proof of Proposition 3.4.3. By Proposition 3.4.2 we can contract Z, and the image $p = \phi(Z)$ is a bibranched singularity of genus zero, hence a node with some local parameter m'. We show that $m' = m_1 + m_2$.

Let $\widetilde{\phi}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ the minimal resolution of singularities of $\mathcal{C}$ around the isolated singularities $\gamma(q_1), \gamma(q_2)$, given as in Fact 3.4.4. We prove now that $\widetilde{\mathcal{C}} \to \mathcal{C}'$ is a minimal resolution of singularities around $p \in \mathcal{C}'$. This is equivalent to show that any contracted component is not a -1 curves. The exceptional curves are either exceptional for $\widetilde{\phi}$ or the strict transform of Z. Let E_i for $i = 1, \dot{;} m_1 + m_2 - 2$ be the former and call E the latter. We have $E_i^2 = -2$, is then enough to prove $E^2 \neq -1$. Consider the Cartier divisor $D = \pi^{-1}(0)$, where $0 \in \Delta$ is the closed point. By projection formula we have:

$$0 = (\phi \circ \widetilde{\phi})^* D \cdot E = (\widetilde{D} + E + \sum_{i=1}^{m_1+m_2-2} E_i) \cdot \widetilde{E} = 2 + E^2,$$

where $\widetilde{D}$ is the strict transform of D and the coefficient in the total transform are 1 because $\widetilde{\mathcal{C}} \to \Delta$ is a family of geometrically reduced curves. Then $E^2 = -2$, therefore the resolution $\widetilde{\mathcal{C}} \to \mathcal{C}'$ is minimal and then counting the number of the exceptional divisors we have $m' = m_1 + m_2 - 2 + 1 + 1 = m_1 + m_2$. □

We can reverse the process

Proposition 3.4.5 (Local parameters behavior)**.** *Let* Δ *be the spectrum of a discrete valuation ring over a field* k *with* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > 5$*, and with algebraically closed residue field. Let* $(\mathcal{C}, p_i)$ *be an* n*-pointed family of curves over* Δ*, and fix a* *node $p \in \mathcal{C}$ with local parameter $m \geq 1$. For any $m_1, m_2 \in \mathbb{Z}_{>0}$ with $m = m_1 + m_2$, there exists a diagram*

$$\begin{tikzcd}
(\widetilde{\mathcal{C}}, \widetilde{p}_i) \arrow[rr, "\phi"] \arrow[dr] & & (\mathcal{C}, p_i) \arrow[dl] \\
& \Delta &
\end{tikzcd}$$

such that:

1. *ϕ is proper and birational, satisfies $\phi_* \mathcal{O}_{\widetilde{\mathcal{C}}} = \mathcal{O}_{\mathcal{C}}$, and is a morphism of pointed curves restricting to an isomorphism on the generic fibre.*
2. *$\widetilde{\mathcal{C}} \to \Delta$ is a flat family of geometrically reduced connected curves.*
3. *The restriction of ϕ to the special fibre induces a contraction of pointed curves, and set-theoretically $\phi(\mathrm{Exc}(\phi)) = p$. The exceptional divisor $\mathrm{Exc}(\phi) \simeq \mathbb{P}^1$ has two special points p_1, p_2, which are rational nodes on $\widetilde{\mathcal{C}}_0$ with local parameters m_1 and m_2, respectively.*

Proof. Consider a minimal resolution $\mathcal{D} \to \mathcal{C}$ of the A_{m-1} surface singularity at p. By Fact 3.4.4, this introduces $m-1$ exceptional components with dual graph A_{m-1}. By applying Proposition 3.4.1, we first contract $m_1 - 1$ consecutive exceptional components. This yields a map $\mathcal{D} \to \mathcal{D}'$ in which the image point $p_1' \in \mathcal{D}'$ is a node with local parameter m_1, by Proposition 3.4.3. We then contract the $m_2 - 1$ exceptional components in $\mathcal{D}'$ that do not contain p_1'. This produces a morphism $\mathcal{D}' \to \widetilde{\mathcal{C}}$, and the resulting map $\widetilde{\mathcal{C}} \to \mathcal{C}$ follows from the universal property of contractions. By construction, the two new rational nodes on $\widetilde{\mathcal{C}}_0$ have local parameters m_1 and m_2, as required. □

3.4.2 Hassett Replacement

Conversely, to the previous section, we now analyze how we can replace the central fibre of families with less singular curves. The replacement is performed in the same spirit as the variety of stable limits introduced in [Has00]. Our notation is consistent with [Fed14].

The results are recalled using the proper stack of quasi-admissible hyperelliptic covers $\mathcal{H}_n[n-1]$, introduced in [Fed14], over a field of characteristic zero or such that $\mathrm{char}\, k > n+1$. Through the next section, we will be interested in the case $n \leq 3$, that is the case when parametrized curves only have singularities of type A_i, $i \leq 3$. As a consequence of [Fed14, Prop. 7.2], we have the following proposition:

Theorem 3.4.6 (Hassett Replacement)**.** *Let R be a DVR over a field k such that* $\mathrm{char}\, k = 0$ *or* $\mathrm{char}\, k > i+1$. *Consider a family of curves $(\mathcal{C}, p_i) \in \widetilde{\mathcal{M}}_{g,n}(R)$ with an isolated singularity $p \in \mathcal{C}_0$ of type A_i in the central fibre. Then, up to a finite DVR extension, there exists a family $(\mathcal{C}', p_i') \in \widetilde{\mathcal{M}}_{g,n}(R)$ and a morphism of pointed curves*

$$\phi\colon (\mathcal{C}', p_i') \to (\mathcal{C}, p_i),$$

that satisfies the following properties:

- *ϕ is a proper, birational map with $\phi_* \mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{C}}$ that is an isomorphism on the generic fibre.*
- *Let $E \subset \mathcal{C}_0$ be the exceptional locus of the map; we have $\phi(E) = p$, and the curve E correspond to a k point of $\mathcal{H}_i[i-1]$.*
- *E is nodally attached to $\mathcal{C}'_0$ via a Weierstrass point if $2 \nmid i$, or two points exchanged by the hyperelliptic involution if $2 \mid i$.*
- *If $2 \nmid i$, the attaching points of $(E, p, q) \to (\mathcal{C}_0, p_{i,0})$ are isolated singularities with equal local parameters.*

We will call such a family $(\mathcal{C}', p'_i)$, constructed in this way, a Hassett replacement *of p.*

The theorem follows from the analogous local statement [Fed14, Prop. 7.2]:

Proposition 3.4.7. *For a miniversal family $\mathcal{C} \to T$ of an A_k-singularity there is an alteration $f \colon T' \to T$ and a weighted blow-up $\mathcal{C}' \to \mathcal{C} \times_T T'$ of the A_k-locus in the fibers such that $\mathcal{C}' \to T'$ is a flat family of curves with at worst A_{k-1} singularities and such that $\mathcal{C}'_{|f^{-1}(0)} \simeq \mathcal{Y}_1 \cup \mathcal{Y}_2$ is a union of two irreducible components such that $\mathcal{Y}_1 \to f^{-1}(0)$ is an isotrivial family of normalizations of the central fiber $\mathcal{C}_0$ and $\mathcal{Y}_2 \to f^{-1}(0)$ is a family of curves in $\mathcal{H}_k[k-1]$.*

thanks to [Fed14, Prop. 7.2] there

Proof of Theorem 3.4.6. Consider a miniversal family $\pi \colon (\mathcal{V}, q) \to (T, 0)$ of an A_i singularity q, that exists thanks to Fact 3.1.4. Let $s \in \operatorname{Spec} R$ be the closed point, then exists a map $g \colon (\operatorname{Spec} R, s) \to (T, 0)$ such that $\mathcal{W} \colon = \mathcal{V} \times_T \operatorname{Spec} R$ (around a closed point $w \in (g \times \pi)^{-1}(q))$ is equivalent to $\mathcal{C}$ around p étale locally.

$$\begin{tikzcd}
 & (\mathcal{U}, u) \arrow[dl, "\text{étale}"', "c_1"] \arrow[dr, "\text{étale}", "c_2"'] & & \\
(\mathcal{C}, p) \arrow[drr] & & (\mathcal{W}, w) \arrow[r] \arrow[d] & (\mathcal{V}, q) \arrow[d, "\pi"] \\
 & & (\operatorname{Spec} R, s) \arrow[r, "g"'] & (T, 0)
\end{tikzcd} .$$

Let T' and $\mathcal{V}'$ be as in Proposition 3.4.7. Up to a finite DVR extension $(\operatorname{Spec} R', s') \to (\operatorname{Spec} R, s)$, we can lift the map g to a map $g' \colon (\operatorname{Spec} R', s') \to (T', 0')$, where $0' \in f^{-1}(0)$. Pulling back our families, we have the following solid diagram:

$$\begin{tikzcd}
 & \mathcal{U} \times_R \operatorname{Spec} R' \arrow[dl, "\text{étale}"', "c_{1,R'}"] \arrow[dr, "\text{étale}", "c_{2,R'}"'] & \mathcal{W}' \arrow[r] \arrow[d, "bl_{R'}"] & \mathcal{V}' \arrow[d, "bl"] \\
\mathcal{C} \times_R \operatorname{Spec} R' \arrow[drr] & & \mathcal{W} \times_R \operatorname{Spec} R' \arrow[r, "\widetilde{g'}"] \arrow[d] & \mathcal{V} \times_T T' \arrow[d, "\pi"] \\
 & & \operatorname{Spec} R' \arrow[r, "g'"] & T'
\end{tikzcd} ,$$

where $\mathcal{W}'$ is obtained by pullback. Let $\mathcal{W}'' \to \mathcal{U} \times_R \operatorname{Spec} R'$ be the pullback of $\mathcal{W}' \to \mathcal{W} \times_R \operatorname{Spec} R'$ along $c_{2,R'}$. By construction, $\mathcal{V}' \to \mathcal{V} \times T'$ over T' is an isomorphism outside $f^{-1}(0) \subset T'$, hence we recover an étale map over $\operatorname{Spec} R'$:

$$c' \colon \mathcal{U} \setminus (c_{2,R'} \circ \widetilde{g}')^{-1}(\mathcal{V}'_{|f^{-1}(0)}) \to \mathcal{W}''.$$

Now restrict $c_{1,R'}$ to the complement of the singularity p:

$$c'' \colon \mathcal{U} \setminus c_{1,R'}^{-1}(p) \to \mathcal{C} \times_R \operatorname{Spec} R' \setminus \{p\},$$

and then glue c' and c'' in the étale topology. In this way, we find a flat family of curves $\mathcal{C}' \to \operatorname{Spec} R'$ that, outside a subcurve over the closed point, coincides with $\mathcal{C} \setminus p$. Finally the existence of a map $\mathcal{W}'' \to \mathcal{C} \times_R \operatorname{Spec} R'$ gives the required map ϕ in the first claim. Thanks to [Fed14, p. 7.2] remain to verify that, in the case of a tacnode the two local parameters are the same, this follows via the same argument of [FS, Prop. 2.22]. □

Corollary 3.4.8. *Assuming that we are replacing an A_i singularity with $i = 2, 3$, we have an easier description of the replaced component:*

- *if $i = 2$, then $\mathcal{H}_2[1] \simeq \overline{\mathcal{M}}_{1,1}$ i.e. E is an elliptic tail nodally attached.*
- *if $i = 3$, then $\mathcal{H}_3[2] \simeq \overline{\mathcal{M}}^{\mathrm{ps}}_{1,2}$, the log-stable 2-pointed curves with at most A_2 singularities. In this case two attaching nodes have the same local parameters.*

3.4.3 Blow-up of families over Θ

We perform now a construction that will be used to determine the basins of attraction of $\widetilde{\mathcal{M}}^3_{g,n}$. Recall that a family of curves over Θ is the datum of a family of curves $\mathcal{C} \to \mathbb{A}^1$ endowed with a $\mathbb{G}_m$-equivariant action on the total space $\mathcal{C}$. We denote by $\mathcal{C}_0$ the central fiber of the family over $0 \in \mathbb{A}^1$.

Lemma 3.4.9 (Local structure for curves over Θ)**.** *Let $\pi \colon [\mathcal{C}/\mathbb{G}_m] \to \Theta$ be a family of curves over an algebraically closed field k. Let $p \in \mathcal{C}_0$ be a smooth point of the central fiber, fixed by the induced $\mathbb{G}_m$-action, and suppose that $\mathbb{G}_m$ acts on the one-dimensional vector space $T^1_p\mathcal{C}_0$ with weight a. Then there exist morphisms*

$$\begin{tikzcd}
([\mathcal{C}/\mathbb{G}_m],p) \arrow[dr] & ([\operatorname{Spec} A/\mathbb{G}_m],0) \arrow[l, "\iota"'] \arrow[r, "\psi"] & ([T_p\mathcal{C}/\mathbb{G}_m],0) \simeq ([\mathbb{A}^2/_{(1,-a)}\mathbb{G}_m],0) \arrow[dl, "f"] \\
& (\Theta,0) &
\end{tikzcd}$$

where the maps ι and ψ are étale at p, and f is the projection onto the factor with weight 1.

Proof. The morphisms ι is provided by the local structure theorem [Alp25, Thm. 6.7.1]. The morphism ψ is given by the étale slice theorem. Since $p \in \mathcal{C}$ is a smooth point on a surface, we have $\dim T_p\mathcal{C} = 2$. Consider the induced $\mathbb{G}_m$-equivariant morphism of tangent spaces $T_p\mathcal{C} \to T_0\mathbb{A}^1$. The action on the target has weight 1, hence by taking the kernel we obtain the desired decomposition. Moreover, this morphism of tangent spaces lifts to a morphism $g\colon [T_p\mathcal{C}/\mathbb{G}_m] \to \Theta$. It remains to verify that the diagram commutes. Consider the composition $[\operatorname{Spec} A/\mathbb{G}_m] \to [T_p\mathcal{C}/\mathbb{G}_m] \xrightarrow{g} \Theta$, and compare it with the morphism $[\operatorname{Spec} A/\mathbb{G}_m] \to \Theta$ induced by restricting the family. By construction, these two morphisms induce the same map at the level of tangent spaces at $0 \in \operatorname{Spec} A$. To conclude, we must show that two morphisms $([\operatorname{Spec} A/\mathbb{G}_m], 0) \to (\Theta, 0)$ inducing the same tangent map must coincide. The $\mathbb{G}_m$-action corresponds to decompositons $A = \bigoplus_{d\geq 0} A_d$ and $k[t] = \bigoplus_{d\geq 0} kt^d$, in this way the morphism is determined by $t \mapsto a \in A_1$. On tangent spaces, we have $A_1^* \to (kt^1)^*$ given by the dual of $t \mapsto a$. Hence, the two data coincide, proving the claim. □

Proposition 3.4.10 (Blow-up construction)**.** *Let $\pi\colon [\mathcal{C}/\mathbb{G}_m] \to \Theta$ be a family of curves over an algebraically closed field k. Let $p \in \mathcal{C}_0$ be a smooth $\mathbb{G}_m$-fixed point such that $\mathbb{G}_m$ acts on $T_p^1\mathcal{C}_0$ with weight a. For any positive integer n, there exists a family $[\mathcal{C}'/\mathbb{G}_m] \to \Theta$ and a $\mathbb{G}_m$-equivariant morphism $\phi\colon \mathcal{C}' \to \mathcal{C}$ that give a morphism of familes*

$$\begin{tikzcd}
{[\mathcal{C}'/\mathbb{G}_m]} \arrow[rr, "\widetilde{\phi}"] \arrow[dr, "\pi'"'] & & {[\mathcal{C}/\mathbb{G}_m]}, \arrow[dl, "\pi"] \\
& \Theta &
\end{tikzcd}$$

such that:

1. *ϕ is proper, birational, satisfies $\phi_*\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{C}}$, and restricts to an isomorphism over $\mathbb{A}^1 \setminus \{0\}$.*
2. *$\mathcal{C}' \to \mathbb{A}^1$ is a flat family of geometrically reduced and connected curves.*
3. *The restriction of ϕ to the special fiber induces a contraction of pointed curves, and set-theoretically $\phi(\operatorname{Exc}(\phi)) = p$. The exceptional divisor $\operatorname{Exc}(\phi) \simeq \mathbb{P}^1$ is $\mathbb{G}_m$-invariant, and the central fiber has a unique singularity on $\operatorname{Exc}(\phi)$, which is a rational node $p' \in \mathcal{C}'_0$ fixed by the $\mathbb{G}_m$-action with local parameter n.*
4. *The induced $\mathbb{G}_m$-action on $T^1_{p'}\operatorname{Exc}(\phi)$ has weight $-a-n$.*
5. *If $n + a \geq 0$, there exists a smooth section $\sigma\colon \Theta \to [\mathcal{C}'/\mathbb{G}_m]$ sending 0 to p'', the other $\mathbb{G}_m$-fixed point on $\operatorname{Exc}(\phi)$. In this case, the induced morphism on tangent spaces along the sections is given (as a morphism of $k[t]$-modules) by*

$$\begin{tikzcd}
T_\sigma\mathcal{C}' \arrow[r] \arrow[d, "\sim"] & T_{\phi\circ\sigma}\mathcal{C} \arrow[d, "\sim"] \\
k[t] \arrow[r, "1\mapsto t^{(n+a)}"] & k[t].
\end{tikzcd}$$

Proof. By Lemma 3.4.9, the construction can be carried out locally, as in Theorem 3.4.6. We perform the construction on the model family $[\mathbb{A}^2/_{(1,a)}\mathbb{G}_m] \to \Theta$, modifying the total space at the origin. Consider the blow-up

$$\begin{tikzcd} \operatorname{Spec} k[x,t] \arrow[d] & \operatorname{Bl}_{(x,t)} \operatorname{Spec} k[x,t] \arrow[l] \arrow[dl, "f'"] \\ \operatorname{Spec} k[t] & \end{tikzcd}$$

The family f' is flat with geometrically reduced fibers. Now perform the base change along $\mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^n$. By the compatibility of blow-up with flat base changes, we obtain the family

$$\begin{tikzcd} \operatorname{Spec} k[x,t] \arrow[d] & \operatorname{Bl}_{(x,t^n)} \operatorname{Spec} k[x,t] \arrow[l, "\phi"'] \arrow[dl, "f"] \\ \operatorname{Spec} k[t] & \end{tikzcd}$$

The morphism f remains flat with geometrically reduced fibers. The exceptional divisor $\operatorname{Exc}(\phi)$ is a rational curve with a unique nodal singularity p'. Locally near p', we have $\operatorname{Spec} k[x,y]/(xy - t^n)$, where y is the coordinate on the exceptional divisor.

Define the $\mathbb{G}_m$-action by assigning weight a to x and weight 1 to t. Since (x, t^n) is an invariant ideal, the action lifts to $\phi\colon \operatorname{Bl}_{(x,t^n)} \operatorname{Spec} k[x,t] \to \mathbb{A}^1_t$, yielding the morphism

$$[\operatorname{Bl}_{(x,t^n)} \operatorname{Spec} k[x,t]/\mathbb{G}_m] \to \Theta.$$

Locally at p', the equation of the family is $\operatorname{Spec} k[x,y]/(xy - t^n)$ where x, t have weight respectively $-a, 1$, then y has weight $n + a$, giving the claimed weight on $T_{p'} \operatorname{Exc}(\phi)$.

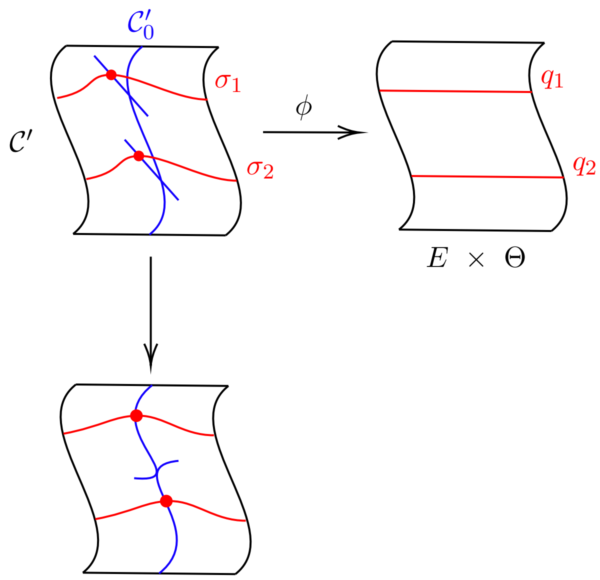
For (5), let $p'' \in \operatorname{Exc}(\phi)$ denote the other $\mathbb{G}_m$-fixed point. Near p'', the family can be written as $\operatorname{Spec} k[t,z] \to \operatorname{Spec} k[t]$, where $\mathbb{G}_m$ acts on z with weight $n + a$. Any section σ has the form $t \mapsto t$, $z \mapsto \alpha t^{n+a}$ (recall that $n + a \geq 0$ by assumption). Hence the induced map $T_\sigma \operatorname{Bl}_{(x,t^n)} \to T_{\phi\circ\sigma}\mathbb{A}^2$ is a morphism of $k[t]$-modules given by $t \mapsto t^{n+a}$.

The general construction of $\mathcal{C}'$ is then obtained by gluing this local model with $\mathcal{C} \setminus \{p'\}$. Properties (1)-(4) follows because can verified locally. For (5), note that the local

For (5), note that the local construction of σ defines a morphism $T_0\Theta \to T_0[\mathcal{C}'/\mathbb{G}_m]$. Since π' is smooth at p'', formal smoothness of Θ allows us to lift this morphism to a section $\Theta \to [\mathcal{C}'/\mathbb{G}_m]$, uniquely determined once $\alpha \in k$ is fixed. $\square$

Thanks to the previous proposition, we obtain a prototype of a very close degeneration in $\widetilde{\mathcal{M}}^3_{g,n}$.

Example 3.4.11. Let (E, q_1, q_2) be an elliptic bridge, and consider the trivial family $E \times \Theta \to \Theta$. Perform the construction of Proposition 3.4.10 at the marked points $(q_1, 0)$ and $(q_2, 0)$ using the same parameter n, that came equipped with two sections σ_1, σ_2. Consider the resulting family $(\mathcal{C}', \sigma_1, \sigma_2) \to \mathbb{A}^1$, by Proposition 3.4.2, we can contract the elliptic curve in the central fiber, so that $\mathcal{C}_0$ consists of two rational curves attached at a tacnodal singularity. Since the contracted locus is $\mathbb{G}_m$-invariant, the resulting family inherits a $\mathbb{G}_m$-action by the universal property of contractions.



3.5 Basins of attraction

The purpose of this section is to give an explicit description of the basins of attraction for curves in $\widetilde{\mathcal{M}}^3_{g,n}$, over a field k with $\operatorname{char} k \neq 2, 3$.

Proposition 3.5.1. *Let $\mathcal{X} \to S$ be an algebraic stack locally of finite presentation and quasi-separated over a quasi-separated scheme S, with affine stabilizers. Then $\underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X})$ and $\underline{\mathrm{Mor}}_S(\Theta, \mathcal{X})$ are algebraic stacks locally of finite presentation and quasi-separated over S.*

Proof. This follows from [Hal14, Prop. 1.1.2]. See also [AHH23, Page 7]. □

With the purpose to study very close degenerations, we define:

Definition 3.5.2 (Basin of attraction)**.** Given $\mathcal{X}/S$ be an algebraic stack as in Proposition 3.5.1. A point $x \in |\underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X})|$ is given by a point $p \in \mathcal{X}$ and a cocharacter $\rho\colon \mathbb{G}_m \to \mathrm{Aut}_{\mathcal{X}}(p)$. We have a diagram

$$\begin{tikzcd} \underline{\mathrm{Mor}}_S(\Theta, \mathcal{X}) \arrow[r, "ev_0"] \arrow[d, "ev_1"] & \underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X}) \\ \mathcal{X} & \end{tikzcd},$$

where ev_0 (resp. ev_1) is the restriction of a map to the closed substack $B\mathbb{G}_m$ (resp. to the open point). We define the *basin of attraction* $A_\rho(p)$ as $|ev_1(ev_0^{-1}(x))| \subset |\mathcal{X}|$.

Remark 3.5.3. Let $\mathcal{X}/S$ be as in Proposition 3.5.1 and assume that $\mathcal{X}/S$ has quasi-compact flag spaces (cf. [Hal14, Def. 3.8.1]), any basin of attraction is constructible in $|\mathcal{X}|$.

Proof. Thanks to Proposition 3.5.1 $\underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X})$ is quasi-separated, hence any point $x\colon \operatorname{Spec} K \to \underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X})$ is of finite type and thus $\{x\} \subset \underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X})$ is constructible (even locally closed). Given that $\underline{\mathrm{Mor}}_S(B\mathbb{G}_m, \mathcal{X})$ is a Θ-retract of $\underline{\mathrm{Mor}}_S(\Theta, \mathcal{X})$, we have that $ev_0^{-1}(x)$ is connected. Being a quasi-compact flag space means that ev_1 is quasi-compact on connected components, hence restricting to a connected component the map ev_1 is quasi-compact and locally of finite presentation, we conclude by Chevalley's theorem. □

A quotient stack via a reductive group has quasi-compact flag spaces, [Hal14, Example 3.8.3].

Remark 3.5.4. For a quotient stack $[X/G]$ where X is locally finite type scheme over a field and G is reductive, the basins of attraction are constructible.

For $\widetilde{\mathcal{M}}^3_{g,n}$, we will prove that the basins of attraction are union of strata, hence locally closed.

3.5.1 Special attached curves and principal strata in $\widetilde{\mathcal{M}}^3_{g,n}$

In this seciton we introduce some notation that will be useful in the description of basins of attraction and in Chapter 4. We recall definitions of rosaries, elliptic chains, and elliptic tails as in [CTV23].

Definition 3.5.5 (Tails, bridges and chains [CTV23, Def. 3.1], [Alp+17, Def. 2.1 and 2.3, Lem. 2.13])**.**

1. An *elliptic tail* is a 1-pointed irreducible curve (E, q) of arithmetic genus 1 (that is, E is either a smooth elliptic curve or a rational curve with one node or one cusp).
2. An *elliptic bridge* is a 2-pointed curve (E, q_1, q_2) of arithmetic genus 1 which either is irreducible or has two rational smooth components R_1 and R_2 that meet in either two nodes or one tacnode, and such that $q_i \in R_i$ for $i = 1, 2$.
3. An *elliptic chain of length r* is a 2-pointed curve (E, q_1, q_2) which admits a finite surjective morphism

$$\gamma\colon \bigsqcup_{i=1}^{r} (E_i, p_{2i-1}, p_{2i}) \to (E, q_1, q_2)$$

such that:

   (a) (E_i, p_{2i-1}, p_{2i}) is an elliptic bridge for $i = 1, \dots, r$;

   (b) γ induces an open embedding of $E_i \setminus \{p_{2i-1}, p_{2i}\}$ into $E \setminus \{q_1, q_2\}$ for $i = 1, \dots, r$;

   (c) $\gamma(p_{2i}) = \gamma(p_{2i+1})$ is a tacnode for $i = 1, \dots, r-1$;

   (d) $\gamma(p_1) = q_1$ and $\gamma(p_{2r}) = q_2$.

4. A *principal elliptic chain of length* r is an elliptic chain of length r where every elliptic bridge (E_i, p_{2i-1}, p_{2i}) is a smooth curve.

An elliptic tail that contain an A_2 singularity is a rational curve that contain only one singularity, we will call it a *rational elliptic tail.* Note that an elliptic chain of length r has arithmetic genus $2r-1$. An elliptic chain of length 1 is just an elliptic bridge.

The definitions above consists of genus 1 pseudostable curves $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ [Sch91] and weakly pseudostable curves $\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}$ introduced in [HM08]. More precisely:

Remark 3.5.6. Elliptic tails (E, q) consists of points in $\overline{\mathcal{M}}^{\mathrm{wps}}_{1,1}$. Elliptic bridges (E, q_1, q_2) are either points in $\overline{\mathcal{M}}^{\mathrm{ps}}_{1,2}$ or consists of two 1-marked smooth rational curves glued along a tacnodal singularity.

Definition 3.5.7 (Open and closed rosaries [CTV23, Def. 3.3], [Alp+17, Def. 2.26])**.**

1. An *open rosary* of length r, or simply a *rosary* of length r, is a 2-pointed curve (R, q_1, q_2) which admits a finite, surjective morphism
$$\gamma : \bigsqcup_{i=1}^{r} (L_i, p_{2i-1}, p_{2i}) \longrightarrow (R, q_1, q_2)$$
   such that:
   (a) (L_i, p_{2i-1}, p_{2i}) is a 2-pointed smooth rational curve for $i = 1, \dots, r$;
   (b) γ induces an open embedding of $L_i \setminus \{p_{2i-1}, p_{2i}\}$ into $R \setminus \{q_1, q_2\}$ for $i = 1, \dots, r$;
   (c) $a_i \coloneqq \gamma(p_{2i}) = \gamma(p_{2i+1})$ is a tacnode for $i = 1, \dots, r-1$;
   (d) $\gamma(p_1) = q_1$ and $\gamma(p_{2r}) = q_2$.
2. A *closed rosary* of length r is a (0-pointed) curve R which admits a finite, surjective morphism
$$\gamma : \bigsqcup_{i=1}^{r} (L_i, p_{2i-1}, p_{2i}) \longrightarrow R$$
   such that:
   (a) (L_i, p_{2i-1}, p_{2i}) is a 2-pointed smooth rational curve for $i = 1, \dots, r$;
   (b) γ induces an open embedding of $L_i \setminus \{p_{2i-1}, p_{2i}\}$ into R for $i = 1, \dots, r$;
   (c) $a_i \coloneqq \gamma(p_{2i}) = \gamma(p_{2i+1})$ for $i = 1, \dots, r-1$, and $a_r \coloneqq \gamma(p_1) = \gamma(p_{2r})$, are tacnodes.

Note that an open rosary (R, q_1, q_2) of length r has arithmetic genus $g(R) = r-1$, while a closed rosary R of length r has arithmetic genus $g(R) = r+1$. An open rosary of length 2 is an elliptic bridge and is the unique elliptic bridge containing a tacnode; for this reason, we will also call it the *tacnodal elliptic bridge.* More generally, any open rosary of even length r can be regarded as an elliptic chain of length $r/2$ in which all the elliptic bridges are tacnodal.

Definition 3.5.8 (Attached elliptic tails, chains and rosaries)**.** Let $(C, \{p_i\}_{i=1}^n)$ be an n-pointed curve of genus g. Let k, k_1, k_2 be in $\{1, 2, 3\}$.

1. (C, p_i) has an A_k-*attached elliptic tail* if there exists an attached subcurve $\gamma\colon (E, q) \to (C, p_i)$, where (E, q) is an elliptic tail and $\gamma(q)$ is an A_k-singularity or $k = 1$ and $\gamma(q)$ is a marked point.

2. (C, p_i) has an A_{k_1}/A_{k_2}-*attached elliptic chain* (of length r) if there exists attached subcurve $\gamma\colon (E, q_1, q_2) \to (C, p_i)$, where (E, q_1, q_2) is an elliptic chain (of length r) and $\gamma(q_i)$ is an A_{k_i}-singularity or $k_i = 1$ and $\gamma(q_i)$ is a marked point (for $i = 1, 2$).

3. (C, p_i) has an *attached rosary* (of length r) if there exists an attached subcurve $\gamma\colon (R, q_1, q_2) \to (C, p_i)$, where (R, q_1, q_2) is an open rosary (of length r) and $\gamma(q_i)$ is a node or a marked point (for $i = 1, 2$).

An A_{k_1}/A_{k_2}-attached elliptic chain of length 1 is also called an A_{k_1}/A_{k_2}-*attached elliptic bridge*. An A_k/A_k-attached elliptic chain $\gamma\colon (E, q_1, q_2) \to (C, p_i)$ of length r such that $\gamma(q_1) = \gamma(q_2)$ is called *closed*. In this case, γ is surjective and $(g, n) = (2r - 1 + \frac{k+1}{2}, 0)$.

The definition slightly differ from [CTV23, Def. 3.2, 3.5], [Alp+17, Def. 2.4, 2.26]. Indeed, we include among the attached chains and rosaries, the cases when an attaching point is a cusp.

Note finally that we could have an attached (open) rosary

$$\gamma : (R, q_1, q_2) \to (C, p_i)$$

of length r such that $\gamma(q_1) = \gamma(q_2)$ is a node: in this case we have $C = R$ and $(g, n) = (r, 0)$. Moreover, an attached rosary $(R, q_1, q_2) \to (C, p_i)$ is an attached elliptic chain if it has even length, otherwise there are exactly two subcurves of (R, q_1, q_2) maximal by inclusion that corresponds to attached elliptic chains.

We now define the notion of position for tacnodes, elliptic bridges, elliptic chains and rosaries. This is unfortunately a different notion respect to the *type* defined in [CTV23, Def. 3.6], but more suitable for our constructions. We first set the following:

$$P_{g,n} := (\{\text{irr}\} \cup \{(\tau, I) : 1 \le \tau \le g,\ I \subseteq [n] := \{1, \dots, n\}\})/\sim,$$

where $\sim$ is the equivalence relation such that irr is equivalent only to itself and $(\tau, I) \sim (\tau', I')$ if and only if $(\tau, I) = (\tau', I')$ or $(\tau', I') = (g - \tau + 1, I^c)$, where $I^c = [n] \setminus I$. We will denote the class of (τ, I) in $P_{g,n}$ by $[\tau, I]$ and the class of irr in $T_{g,n}$ again by irr.

Definition 3.5.9 (Position and support)**.** Let (C, p_i) be an n-pointed curve of genus g. We say that a tacnode $q \in C$ is in *position* $[i, I] \in P_{g,n}$ if the normalization of C at q consists of two connected components, one of which has arithmetic genus $i - 1$ and marked points $\{p_i\}_{i \in I}$ and the other of which has arithmetic genus $g - i$

and marked points $\{p_i\}_{i\in I^c}$. We say that it is in position irr if it is a non-separating tacnode. Let $\gamma\colon (E,q_1,q_2)\to (C,p_i)$ be an A_{k_1}/A_{k_2}-attached elliptic bridge, we say that it is in *position* $[i,I]\in P_{g,n}$ if the normalization of C at $\gamma(q_1)$ consists of two connected components, one of which has either arithmetic genus $i-1-\frac{k_1-1}{2}$ and marked points $\{p_i\}_{i\in I}$ or arithmetic genus $g-i$ and marked points $\{p_i\}_{i\in I^c}$. If the normalization at $\gamma(q_1)$ is connected, we have three cases:

- $k_1=1$ and $\gamma(q_1)$ is the marked point $i\in[n]$. In this case we set the position $[1,\{i\}]$;
- $k_1=2$, i.e. $\gamma(q_1)$ is a cusp. In this case we set the position $[2,\emptyset]$;
- $k_1\in\{1,3\}$ and $\gamma(q_1)$ is not a marked point. In this case $\gamma(q_1)$ is a not-separant singularity. In this we set the position irr.

Finally, for an elliptic attached chain $\gamma\colon (E,q_1,q_2)\to(C,p_i)$ we define the *support*

$$\mathrm{Supp}((E,q_1,q_2)\to(C,p_i))=\{[i,I],\dots,[j,I]\}\subset P_{g,n}$$

to be the union of all the tacnodes in the image and any attached bridge given by composing γ with an attached bridge $(E',q_1',q_2')\to(E,q_1,q_2)$. We set $\mathrm{Supp}((E,q_1,q_2)\to(C,p_i))=\{\mathrm{irr}\}$ if $\gamma(q_1)$ is a non-separating singularity.

We define the support of an attached rosary as the support of a maximal elliptic attached chain contained in it.

We say that an attached elliptic chain (or rosary) has *genus* l if the sum of the number of tacnodes and elliptic bridges in the image of γ is l.

We can easily go from our notion of position to the one of *type*. Using the notations in [CTV23, Def. 3.6], we have:

Remark 3.5.10. Let (C,p_i) be an n-pointed curve of genus g. A tacnode $q\in C$ in position $[i,I]$ (res. irr) is such that $\mathrm{type}(p)=\{[i-1,I],[i,I]\}$ (resp. irr). An elliptic attached chain (or rosary) $(E,q_1,q_2)\to(C,p_i)$ with support $\{[i,I],\dots,[j,I]\}\subset P_{g,n}$ (resp. irr) is of type $\{(i-1,I),\dots,(j,I)\}\in T_{g,n}$ (resp. irr).

Notice that the support of a chain is either $\{\mathrm{irr}\}$ or a subset $\{[i,I]|a\le i\le b\}$ for some $I\in[n]$, $1\le a,b\le n$.

Definition 3.5.11 (Principal strata)**.** Over a field of characteristic 0 or greater than 3, we define the stratum

$$S(i,k,I,\sigma)\subset\overline{\mathcal{M}}^{\mathrm{all}}_{g,n},$$

with $I\subset[n]$, $1\le i\le k\le g$, and $\sigma\in\{-1,+1\}$, to be the unique stratum parametrizing curves that:

- have an A_{k_1}/A_{k_2}-attached principal elliptic chain with support $\{[i,I],\dots,[k,I]\}$, where $k_s\in\{1,3\}$ (for $s=1,2$);
- have all singularities contained in the image of the attached principal elliptic chain;
- have a tacnode at the position $[i,I]$ if and only if either $\sigma=+1$ and $2\mid i$, or $\sigma=-1$ and $2\nmid i$ (in this case $k_1=3$);

- satisfy $k_1 = 1$ if and only if either $\sigma = +1$ and $2 \mid i$, or $\sigma = -1$ and $2 \nmid i$.

We write $S(l, \mathrm{irr}, \sigma)$ with $1 \leq l \leq g-1$ for the unique stratum parametrizing curves that:

- have an A_{k_1}/A_{k_2}-attached principal chain of irreducible type of genus l, with $k_s \in \{1, 3\}$ (for $s = 1, 2$);
- have all singularities contained in the image of the attached principal elliptic chain;
- satisfy $k_1 = 1$ if $\sigma = +1$, and $k_1 = 3$ if $\sigma = -1$.

We call the strata of the form $S(i, k, I, \sigma)$ the *separating principal strata*, and those of the form $S(l, \mathrm{irr}, \sigma)$ the *non-separating principal strata*. When $n = 0$, for simplicity we write

$$S(i, k, \sigma) := S(i, k, \emptyset, \sigma).$$

Notice that, by definition, if $\sigma = +1$ (resp. $\sigma = -1$), then the tacnodes in the chain occur at positions $[j, I]$ with j even (resp. odd).

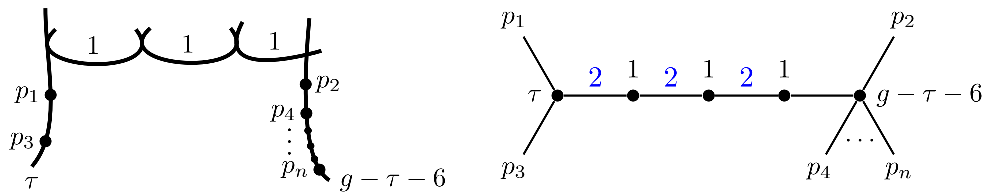


Figure 3.7. Example of a principal stratum in $\widetilde{\mathcal{M}}^3_{g,n}$: $S(\tau+1, \tau+6, \{1,3\}, (-1)^{\tau+3})$.

Remark 3.5.12. Notice that we have $S(i, k, I, \sigma) = S(g+1-k, g+1-i, I^c, (-1)^{g+1}\sigma)$ and that $S(l, \mathrm{irr}, +1) = \mathrm{S(l, irr, -1)}$ if and only if $2 \mid l$.
Moreover, $S(0, k, \emptyset, -1)$ parametrize curves that are not stable, hence not all the strata defined above are contained in $\widetilde{\mathcal{M}}^3_{g,n}$.

We give a definition that of a poset of principal strata that is more coarse than the relation provided by the stratificaiton.

Definition 3.5.13. Define $\mathbb{S}_{g,n}$ to be the set of principal strata. We define the poset structure $(\mathbb{S}_{g,n}, \geq)$ such that for $\mathcal{S}, \mathcal{S}' \in \mathbb{S}_{g,n}$ we have $\mathcal{S} \geq \mathcal{S}'$ if and only if $\mathcal{S}' \subset \overline{\mathcal{S}} \subset \overline{\mathcal{M}}^{\mathrm{all}}_{g,n}$ and the morphism of the attaching elliptic principal chain defining $\mathcal{S}$ factor through the attaching chain of $\mathcal{S}'$, for a point in $\mathcal{S}'$.

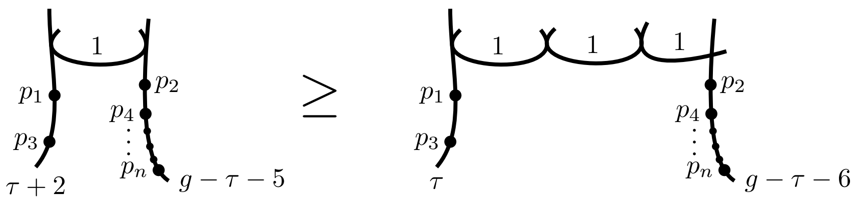


Figure 3.8. $S(\tau+3, \tau+5, \{1,3\}, (-1)^{\tau+3}) \geq S(\tau+1, \tau+6, \{1,3\}, (-1)^{\tau+3})$

We can explicitly write this relation:

Remark 3.5.14. In the poset structure $(\mathbb{S}_{g,n}, \geq)$ there are no relations among the separating and the non-separating strata. We have $S(a,b,I,\sigma) \geq S(a',b',I',\sigma')$ if $I = I'$, $\sigma = \sigma'$ and $a' \leq a \leq b \leq b'$ and:

- when $\sigma = 1$. We have either $a = a'$ or $2 \mid a$ and either $b = b'$ or $2 \mid b$;
- when $\sigma = -1$. We have either $a = a'$ or $2 \nmid a$ and either $b = b'$ or $2 \nmid b$.

Moreover, we have $S(l, \mathrm{irr}, \sigma) \geq S(l', \mathrm{irr}, \sigma')$ if $l = l'$ and $\sigma = \sigma'$ or $l < l'$ and one of the following occur:

- $2 \mid l'$ and $2 \mid l$;
- $2 \mid l'$, $2 \nmid l$ and $\sigma = -1$;
- $2 \nmid l'$, $2 \mid l$ and $\sigma' = +1$;
- $2 \nmid l'$, $2 \nmid l'$ and $\sigma = -1$.

The relation defined above is different from the one induced by the stratification

Example 3.5.15. We have $S(2,k,I,-1) \subset \overline{S(1,1,I,+1)}$ for $I = \emptyset$ or $I = \{j\}$, $j \in [n]$, but there is no relation among these strata.

Remark 3.5.16. Consider a hyperbolic pair (g,n) with $n = 0$. For the poset $(\mathbb{S}_g, \geq)$ restricted to the strata contained in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$, the order relation coincides with the one given by the stratification.

3.5.2 Basins of attraction in $\widetilde{\mathcal{M}}^3_{g,n}$

The basins of attraction for curves with A_i singularities, for $i = 2, 3$, first appeared in [HH13]. We develop this section starting from results in [AFS; CTV23]. This section is carried working over a field with $\operatorname{char} k \neq 2, 3$.

Proposition 3.5.17 (Automorphisms of curves)**.** *Consider an n-pointed curve $(C, p_i) \in \widetilde{\mathcal{M}}^3_{g,n}(k)$, where k is a field with $\operatorname{char} k \neq 2, 3$. Then $\operatorname{Aut}_k(C, p_i)^\circ$ is a torus (possibly non-split). Set $l = \dim \operatorname{Aut}_k(C, p_i)$, then up to a finite extension $k' \supset k$, we are in one of these two cases:*

1. *$l = 1$, $n = 0$ and $C_{k'}$ is a closed rosary of even length;*
2. *the total number of distinct attached curves to $(C_{k'}, p_{i,k'})$ that are either rosaries or elliptic rational tails is exactly l.*

In both cases, the action of $\operatorname{Aut}_{k'}(C_{k'}, p_{i,k'})^\circ$ is trivial outside the image of the attached rosaries and elliptic rational tails and fixes every special point. We have an exact sequence

$$0 \to \operatorname{Aut}(C, p_i)^\circ \to \operatorname{Aut}(C, p_i) \to G \to 0,$$

where G is a finite discrete group over k.

Proof. The proof is contained in [CTV23, Sec. 3]. □

The special points of a curve are invariants under the action of one-parameter subgroups. Thus, given $\lambda \colon \mathbb{G}_m \to \operatorname{Aut}(C, p_i)$, this induces a $\mathbb{G}_m$ action on any attached curve $\gamma\colon (C', q_i) \to (C, p_i)$. We define now a notion of weight looking to the action induced on tangent spaces. We will then explain the relation between this notion and deformation theory.

Definition 3.5.18 (Weight on special points)**.** Consider a point $(C, p_i) \in \widetilde{\mathcal{M}}^3_{g,n}(k)$, where k is a field with $\operatorname{char} k \neq 2, 3$. Fix a one parameter subgroup $\lambda\colon \mathbb{G}_m \to \operatorname{Aut}(C, p_i)^\circ$. For a special point $q \in (C, p_i)$ we define the *weight of* λ *on* q,

$$\mathrm{wt}^\lambda_q(C, p_i) \in \mathbb{Z}$$

according to the following two cases:

1. $q \in C$ is a cusp, tacnode or marked point. Then consider the normalization $(\widetilde{C}, \widetilde{q}) \to (C, q)$. λ lift to a one parameter subgroup $\widetilde{\lambda}$ of $\widetilde{C}$ that fixes $\widetilde{q}$. Then we have an induced action $\lambda \curvearrowleft T^1_{\widetilde{C},\widetilde{q}}$, defined by a weight e. Then we have $\mathrm{wt}^\lambda_q = e$.
2. $q \in C$ is a node. Then consider the normalization $\widetilde{C} \to C$ sending $\widetilde{q_1}, \widetilde{q_2}$ to q. λ lift to a one parameter subgroup $\widetilde{\lambda}$ of $\widetilde{C}$ that fixes $\widetilde{q_1}$, and $\widetilde{q_2}$. Then we have an induced action $\lambda \curvearrowleft T^1_{\widetilde{C},\widetilde{q_i}}$, defined by a weight e_1. Then we have $\mathrm{wt}^\lambda_q = e_1 + e_2$.

Finally, we define the *support of the action of* λ, denoted by $\mathrm{Supp}(\lambda)$, as the closed and reduced subscheme of C given by the irreducible components where λ acts non-trivially. Notice that this locus is a union of rosaries and rational tails.

Notice that, for a fixed field k, the action of $\lambda\colon \mathbb{G}_{m,k} \to \operatorname{Aut}_k(C, p_i)$ has support only on rational components whose normalization is isomorphic to $\mathbb{P}^1_k$. This follows from the fact that the support necessarily has a rational point.

We relate the above notion to the deformation space of the curve using a description from [Alp+17, Sec. 3.3], that we briefly recall.

1. Rational elliptic tail.
Suppose (E, p) is a rational elliptic tail, with the singular point $\xi \in E$. By Proposition 3.5.17, we may fix the isomorphisms $\operatorname{Aut}(E, q) \simeq \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, $\widehat{\mathcal{O}}_{E,\xi} \simeq k[[x, y]]/(y^2 - x^3)$, and $\widehat{\mathcal{O}}_{E,q} \simeq k[[n]]$ so that the action of $\operatorname{Aut}(E, q)$ is given as follows:

$$x \to t^2 x, \quad y \to t^3 y, \quad n \to t^{-1} n.$$

In this coordinate, $\operatorname{Aut}(E, p)$ act on the two-dimensional vector space $T^1(\widehat{\mathcal{O}}_{E,\xi})$ with weights $4, 6$ [Alp+17, Lemma 3.18]. Let $\gamma\colon (E, p) \to (C, p_i)$ be an attached curve, we have a restriction morphism $\operatorname{Aut}(C, p_i)^\circ \to \operatorname{Aut}(E, p)$ and an equivariant projection

$$\operatorname{Def}(C, p_i) \to T^1(\widehat{\mathcal{O}}_{E,\xi}).$$

2. Rosaries.
Suppose (R, r_1, r_2) is a length 2 rosary, with the singular point $\tau \in E$. We may fix the isomorphisms $\operatorname{Aut}(R, r_1, r_2) \simeq \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, $\widehat{\mathcal{O}}_{R,\tau} \simeq k[[x, y]]/(y^2 - x^4)$, and $\widehat{\mathcal{O}}_{R,r_1} \simeq k[[n]]$ so that the action of $\operatorname{Aut}(R, r_1, r_2)$ is given as follows:

$$x \to tx, \quad y \to t^2 y, \quad n \to t^{-1} n.$$

Under this base, $\mathrm{Aut}(R, r_1, r_2)$ act on the 3 dimensional vector space $T^1(\widehat{\mathcal{O}}_{R,\tau})$ with weights $2, 3, 4$ [Alp+17, Lemma 3.19]. Let $\gamma\colon (R, r_1, r_2) \to (C, p_i)$ be an attached curve, we have a restriction morphism $\mathrm{Aut}(C, p_i)^\circ \to \mathrm{Aut}(R, r_1, r_2)$ and an equivariant projection

$$\mathrm{Def}(C, p_i) \to T^1(\widehat{\mathcal{O}}_{R,\tau}).$$

3. Nodes lying on two rational curves.
Consider the curve $(\mathbb{P}^1 \cup_p \mathbb{P}^1, q_1, q_2)$ given by two 2-marked $\mathbb{P}^1$ attached via a node, we may fix the isomorphisms $\mathrm{Aut}(\mathbb{P}^1, q_1, q_2) \simeq \mathbb{G}_m^2 = \mathrm{Spec}\, k[t_1, t_2, t_1^{-1}, t_2^{-1}]$, $\widehat{\mathcal{O}}_{\mathbb{P}^1,p} \simeq k[[n, m]]/nm$ so that the action of $\mathrm{Aut}(\mathbb{P}^1, q_1, q_2)$ is given by

$$n \to tn, \quad m \to sm.$$

Let $\gamma\colon (\mathbb{P}^1 \cup_p \mathbb{P}^1, q_1, q_2) \to (C, p_i)$ be an attached curve, we have a restriction morphism $\mathrm{Aut}(C, p_i)^\circ \to \mathrm{Aut}(\mathbb{P}^1 \cup_p \mathbb{P}^1, q_1, q_2)$ and an equivariant projection

$$\mathrm{Def}(C, p_i) \to T^1(\widehat{\mathcal{O}}_{\mathbb{P}^1,p}).$$

Under this base, $\mathrm{Aut}(\mathbb{P}^1, q_1, q_2)$ act on the 1 dimensional vector space $T^1(\widehat{\mathcal{O}}_{\mathbb{P}^1 \cup_p \mathbb{P}^1,p})$ with weight $(1, 1)$.

4. Marked points and nodes on $\mathbb{P}^1$.
Finally, for the curve $(\mathbb{P}^1, q_1, q_2)$, we may fix the isomorphisms $\mathrm{Aut}(\mathbb{P}^1, q_1, q_2) \simeq \mathbb{G}_m = \mathrm{Spec}\, k[t, t^{-1}]$, $\widehat{\mathcal{O}}_{\mathbb{P}^1,q_1} \simeq k[[n]]$ so that the action of $\mathrm{Aut}(\mathbb{P}^1, q_1, q_2)$ is given by $n \to tn$. Let $\gamma\colon (\mathbb{P}^1, q_1, q_2) \to (C, p_i)$ be an attached curve such that $\gamma(q_1)$ is a marked point, we have a restriction morphism $\mathrm{Aut}(C, p_i)^\circ \to \mathrm{Aut}(\mathbb{P}^1, q_1, q_2)$ and an equivariant projection

$$\mathrm{Def}(C, p_i) \to T^1(\widehat{\mathcal{O}}_{\mathbb{P}^1,q_1}).$$

Under this base, $\mathrm{Aut}(\mathbb{P}^1, q_1, q_2)$ act on the 1 dimensional vector space $T^1(\widehat{\mathcal{O}}_{\mathbb{P}^1,q_1})$ with weight 1.

Remark 3.5.19. With the notation of Definition 3.5.18, we can describe $\mathrm{wt}_q^\lambda(C, p_i)$ as we now explain. When λ restrict to $\mathrm{Def}(C, p_i)$ trivially, we have $\mathrm{wt}_q^\lambda(C, p_i) = 0$. Otherwise, q is the image of a special point ξ, τ, p, q_1 through an attached curve as in the cases above. Denote with $\overline{\lambda}$ the restriction of λ to the corresponding attahced curve component, we have the following possibilities (where we express $\overline{\lambda}$ in the automorphism group coordinate above):

1. If q is a cusp, we are in case 1. Then $\overline{\lambda}$ is given by $t \to t^e$, and $\mathrm{wt}_q^\lambda(C, p_i) = e$.
2. If q is a tacnode, we are in case 2. Then $\overline{\lambda}$ is given by $t \to t^e$, and $\mathrm{wt}_q^\lambda(C, p_i) = e$.
3. If q is a node, we are either in case 3 or 4. If it is attached to two two-pointed rational components we are in case 3, then $\overline{\lambda}$ is given by $t \to t^{e_1} s^{e_2}$, and $\mathrm{wt}_q^\lambda(C, p_i) = e_1 + e_2$. Otherwise, we are in case 3 and $\overline{\lambda}$ is given by $t \to t^e$, and $\mathrm{wt}_q^\lambda(C, p_i) = e$.
4. If q is a marked point, we are in case 4. Then $\overline{\lambda}$ is given by $t \to t^e$, and $\mathrm{wt}_q^\lambda(C, p_i) = e$.

To give a characterization of the basins of attraction, we need two more definitions.

Definition 3.5.20 (Negative decomposition)**.** Let (C, p_i) be an n-pointed curve in $\widetilde{\mathcal{M}}^3_{g,n}$, and fix a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(C, p_i)$. We say that an attached curve $\gamma\colon (D, q_i) \to (C, p_i)$ is *the Negative decomposition with respect to* λ if γ is surjective and such that:

- Every singular point $q \in D$ on the support of the action satisfies $\mathrm{wt}^\lambda_{\gamma(q)} < 0$.
- Every marked point q_i such that $\gamma(q_i) \in C$ is a singularity satisfy $\mathrm{wt}^\lambda_{\gamma(q_1)} \geq 0$.
- Any singular point of C that does not lie in the support of λ is the image of a singularity of the same type.

Notice that such an attached curve is unique.

The action of λ induce an action of a one parameter subgroup $\overline{\lambda}\colon \mathbb{G}_m \to \mathrm{Aut}(C', q_i)$.

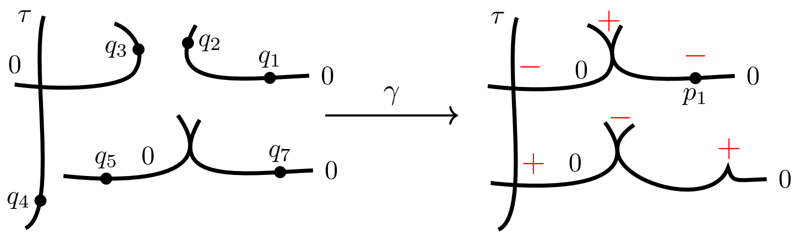


Figure 3.9. Example of negative decomposition. We picture the sign of the singularities' weights (in red).

Thanks to the discussion above, we have the following.

Proposition 3.5.21. *Fix a curve (C, p_i) in $\widetilde{\mathcal{M}}^3_{g,n}$ over an algebraically closed field k and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(C, p_i)$. Let $(D, q_i) \to (C, p_i)$ be the negative decomposition, then any connected component (D', q'_i) of (D, q_i) is either one of the following:*

(i) $\gamma(D') \not\subset \mathrm{Supp}(\lambda)$. *Then D' is almost-stable. In this case all the semistable components are contained in the support.*

(ii) $\gamma(D') \subset \mathrm{Supp}(\lambda)$ *and D' has genus* 0. *Then (D', q'_i) consists either in* $(\mathbb{P}^1, 0, \infty)$ *or in* $(\mathbb{P}^1 \cup_p \mathbb{P}^1, q_1, q_2)$*, two nodally attached rational components containig two special point each.*

(iii) $\gamma(D') \subset \mathrm{Supp}(\lambda)$ *and D' contain a cusp. Then (D', q'_i) consists of an elliptic rational tail or two nodally attached* 1*-marked rational components* $(\mathbb{P}^1 \cup_p E, q_1, q_2)$*, one of which is a elliptic rational tail.*

(iv) $\gamma(D') \subset \mathrm{Supp}(\lambda)$ *and D' contain a tacnode. Then (D', q'_i) consists of an elliptic tacnodal bridge, eventually with one or two nodally attached components. This means we could have* (E, p, q), $(\mathbb{P}^1 \cup_p E, q_1, p)$, $(\mathbb{P}^1 \cup_p E \cup_q \mathbb{P}^1, q_1, q_2)$*, where* (E, p, q) *is a nodally attached tacnodal bridge.*

Definition 3.5.22 (Isotrivial replacement)**.** Fix a curve (C, p_i) in $\widetilde{\mathcal{M}}^3_{g,n}$ over an algebraically closed field k and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(C, p_i)$. Let $\gamma\colon (D, q_i) \to (C, p_i)$ be the negative decomposition. We say that a replacement

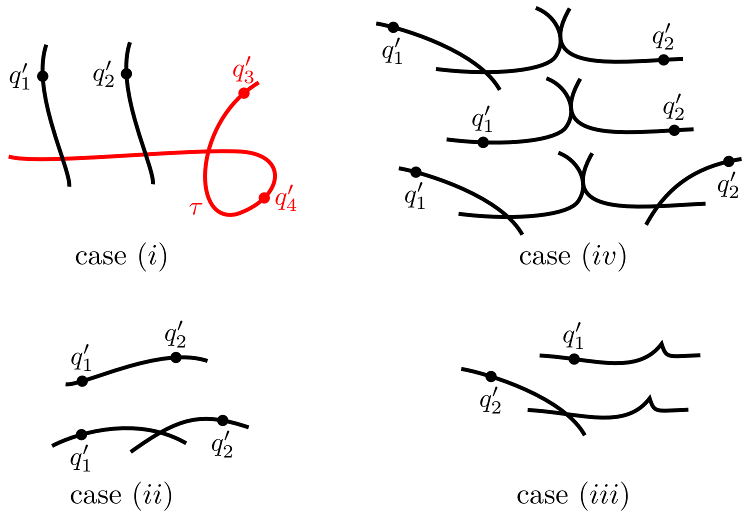


Figure 3.10. The possible connected components of the negative decomposition, we picture in black the support of λ.

$\widetilde{\gamma}\colon (\widetilde{D}, \widetilde{q}_i) \to (\widetilde{C}, \widetilde{p}_i)$ equipped with the bijection $\phi\colon \{q_i\} \to \{\widetilde{q}_i\}$ is an *isotrivial replacement with respect to* λ if for any connected component (D', q_i') of (D, q_i), the corresponding connected component $(\widetilde{D}', \widetilde{q}_i')$ of $(\widetilde{D}, \widetilde{q}_i)$ such that:

- The marked points corresponds, i.e. $\{\gamma(q_i')\} = \{\widetilde{q}_i'\}$;
- There exists a very close degeneration with generic fiber $(\widetilde{D}', \widetilde{q}_i')$ and whose fiber over $B\mathbb{G}_m$ is the datum of (D', q_i') and the action of λ restricted to it. Where $q_i' \in D'$ is the limit of $\phi(q_i') = \widetilde{q}_i' \in \widetilde{D}'$.

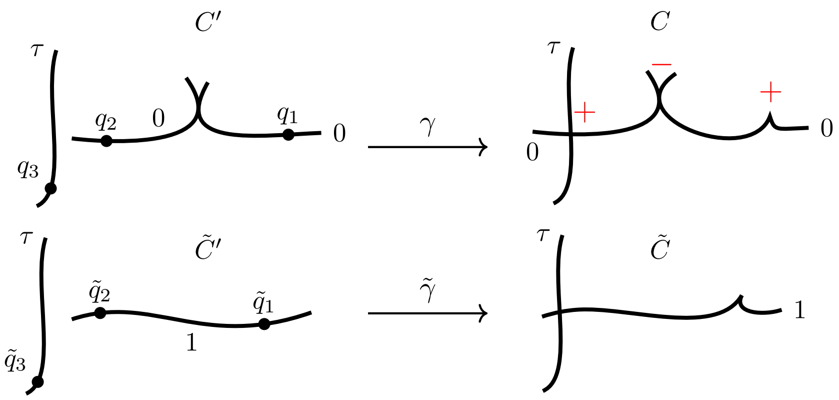


Figure 3.11. An example of isotrivial replacement, where we replace a tacnodal elliptic bridge with a smooth A_1/A_2-attached elliptic bridge (again we pitcure the weight signs in red)

We can now describe the very close degenerations in $\widetilde{\mathcal{M}}^3_{g,n}$.

Theorem 3.5.23 (Basins of attraction in $\widetilde{\mathcal{M}}^3_{g,n}$)**.** *Let k be an algebraically closed field. Fix a curve $(C, p_i) \in \widetilde{\mathcal{M}}^3_{g,n}(k)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}_k(C, p_i)^\circ$. The k-points of $A^\lambda(C, p_i)$ are given by the possible isotrivial replacements of (C, p_i).*

We have the following description for any irreducible component of an isotrivial replacement.

Proposition 3.5.24. *With the notation of Proposition 3.5.21, we classify the possible families of curves $(\mathcal{D}', \sigma_i')$ over Θ whose fiber over $B\mathbb{G}_m$ is the datum of (D', q_i') together with the restriction of λ to it. We describe the possible generic fibers according to the cases in* loc. cit.*:*

(i) It is obtained by contracting some (possibly zero) semistable $\mathbb{P}^1$ components contained in the support of $\overline{\lambda}$ in (D', q_i').

(ii) If $D' \simeq \mathbb{P}^1$, then the generic fiber is $(\mathbb{P}^1, q_1, q_2)$. Otherwise, the generic fiber is either $(\mathbb{P}^1 \cup_p \mathbb{P}^1, q_1, q_2)$ or $(\mathbb{P}^1, q_1, q_2)$.

(iii) If (D', q_i') is a rational elliptic tail, then the generic fiber is an elliptic tail. If it is of type $(\mathbb{P}^1 \cup_p E, q_1, q_2)$, then the generic fiber is either an elliptic tail or of the form $(\mathbb{P}^1 \cup_p F, q_1, q_2)$, where (F, q_2) is an elliptic tail and $q_1 \in \mathbb{P}^1$.

(iv) The generic fiber can be (F, p, q), $(\mathbb{P}^1 \cup_p F, q_1, p)$, or $(\mathbb{P}^1 \cup_p F \cup_q \mathbb{P}^1, q_1, q_2)$, where (F, p, q) is a nodally attached elliptic bridge. The first possibility always occurs; the second occurs when the central fiber is either $(\mathbb{P}^1 \cup_p E, q_1, p)$ or $(\mathbb{P}^1 \cup_p E \cup_q \mathbb{P}^1, q_1, q_2)$; and the third occurs only when the central fiber is $(\mathbb{P}^1 \cup_p F \cup_q \mathbb{P}^1, q_1, q_2)$.

Finally, let $(\widetilde{D}' \times \mathbb{A}^1, \widetilde{q}_i')$ be the constant family over $\mathbb{A}^1$ given by the generic fiber, and denote by $(\mathcal{E}, \tau_i)$ the pullback of $(\mathcal{D}', \sigma_i')$ along the presentation $\mathbb{A}^1 \to \Theta$. Then the morphism on tangent spaces over $\mathbb{A}^1 \setminus \{0\}$ extends to

$$\begin{tikzcd}
T_{\tau_i}\mathcal{E} \arrow[r] \arrow[d, "\sim"] & T_{\widetilde{q}_i'}(\widetilde{D}' \times \mathbb{A}^1) \arrow[d, "\sim"] \\
k[t] \arrow[r, "1 \mapsto t^{n_i}"] & k[t],
\end{tikzcd}$$

where n_i is the weight of the induced action of $\overline{\lambda}$ on the one-dimensional tangent space $T_{q_i'}D'$.

Proof. We first show that these are the only possibilities. For case (i), consider the nodal singularities in the central fiber of $(\mathcal{D}', \sigma_i')$ contained in the support of the action of λ. These are either isolated nodes or limits of nodal rational sections. Construct the family $(\mathcal{E}, \tau_i)$ over Θ by normalizing along such nodal sections and contracting the $\mathbb{P}^1$ components attached via isolated nodes in $\mathcal{D}'$. The resulting family over Θ has trivial action on the fiber over $B\mathbb{G}_m$, and hence, by [Hei17, Lemma 1.5], the family $(\mathcal{E}, \tau_i)$ is trivial. This forces the very close degeneration to be as in (i).

In case (ii), the form of the generic fiber follows from genus considerations and semicontinuity of the number of irreducible components. For cases (iii) and (iv), we reduce to the situation with no rational components in the central fiber, as in (i). For (iii), the central fiber is a one-marked genus 1 curve, hence necessarily an elliptic tail. For (iv), the central fiber is a two-marked genus 1 curve (F, p, q). The curve F cannot itself contain an elliptic tail; otherwise, by Proposition 3.1.8, the central fiber would contain a nodally attached elliptic tail. Thus (F, p, q) must be an elliptic bridge.

We now construct the degenerations case-by-case, applying Proposition 3.4.10 and Proposition 3.4.2. The constructions are as in Example 3.4.11: we first blow up the central fiber (possibly repeatedly), and then contract the elliptic bridge or tail when required. The final claim follows directly from point (5) of Proposition 3.4.10. □

We need a lemma in deformation theory

Lemma 3.5.25. *Let $f\colon \Theta_k \to \widetilde{\mathcal{M}}^3_{g,n}$ be a very close degeneration, that restricted to $B\mathbb{G}_m$ is the datum of the n-pointed curve (C, p_i) and $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}_k(C, p_i)^\circ$. Assume that there exists a singularity $q \in C$ with $\mathrm{wt}^\lambda_q(C, p_i) \geq 0$. Then there exists an equisingular global section $\sigma\colon \Theta \to C$ that restrict to q in $0 \in \Theta$.*

Proof. . We can write a local picture around $f(0)$ (cf. [Alp25, Rmk. 6.7.2]), and consider the projection π to the deformation space of the singularity q. We have a diagram as follows, where h, g are representable and étale around w.

$$\begin{tikzcd}
\left([\mathrm{Spec}\, A/G_{f(0)}], w\right) \arrow[r, "g"] \arrow[d, "h"] & \left(\left[T_{f(0)}\widetilde{\mathcal{M}}^3_{g,n}/G_{f(0)}\right], 0\right) \arrow[r, "\pi"] & \left(\left[T^1(\widehat{\mathcal{O}_{C,q}})/G_{f(0)}\right], 0\right) \\
(\widetilde{\mathcal{M}}^3_{g,n}, f(0)) & (\Theta, 0) \arrow[l, "f"] \arrow[ur, dashed] &
\end{tikzcd}$$

Where $G_{f(0)}$ is the automorphism group of $f(0)$. Given that h is étale, by coherently completeness of Θ we can lift f to a map $\widetilde{f}\colon \Theta \to [T^1(\widehat{\mathcal{O}}_{C,q})/G_{f(0)}]$. This is a map $\overline{f}\colon \mathbb{A}^1 \to T^1(\widehat{\mathcal{O}}_{C,q})$ equivariant respect to $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(C, p_i)$. Given that wt^λ_q is positive, then the weights on $T^1(\widehat{\mathcal{O}}_{C,q})$ are negative thanks to Remark 3.5.19. The map $\overline{f}$ is then the zero map, this imply that the singularity q does not deform, hence the claim. □

Proof of Theorem 3.5.23. Denote with $(\mathcal{C}, \sigma_i)$ a very close degeneration $\Theta \to \widetilde{\mathcal{M}}^3_{g,n}$ whose fiber over $B\mathbb{G}_m$ is given by (C, p_i) with one parameter subgroup λ. Thanks to Lemma 3.5.25, any singularity $q \in C$ with $\mathrm{wt}^\lambda_q(C, p_i) \geq 0$ extend to an equisingular section $\sigma\colon \Theta \to \mathcal{C}$. We perform now a partial normalization along all the section of such singularities that lies on $\mathrm{Supp}(\lambda)$, obtaining many connected components $\bigsqcup_{j \in J} \mathcal{C}_j \to \Theta$. The central fiber of any $\mathcal{C}_j$ consists of connected components of the negative decomposition, therefore any very close degeneration is given by an isotrivial replacement.

We prove now the converse, constructing a very close degeneration for any isotrivial replacement $\widetilde{\gamma}\colon (\widetilde{D}, \widetilde{q}_i) \to (\widetilde{C}, \widetilde{p}_i)$. By hypothesis there is a very close degeneration of $(\widetilde{D}, \widetilde{q}_i)$ to (D, q_i), let $(\mathcal{D}, \sigma_i)$ be the pullback of this over $\mathbb{A}^1$. We have to glue the families $(\mathcal{D}, \sigma_i)$ over $\mathbb{A}^1$ along some sections σ_i and crimping a cusp where required. In particular, the data of which section has to be glued is provided by the combinatorics of (C, p_i). We can then glue the nodes and crimp the cusps in a unique way, as prescribed by Propositions 3.1.13, 3.1.14, the action of $\mathbb{G}_m$ lift to an action of the glued family hence the family is constructed over Θ. Remain to glue the tacnodes. For any j_1, j_2 such that $\widetilde{q_{j_1}}$ and $\widetilde{q_{j_2}}$ are glued along a tacnode via an identification $T_{\widetilde{q_{j_1}}}\widetilde{D} \simeq T_{\widetilde{q_{j_2}}}\widetilde{D}$. We use now Proposition 3.1.15 to glue $(\mathcal{D}, \sigma_i)$ along the two corresponding sections such that it restrict on $t = 1$ to the prescribed gluing.

$$\begin{tikzcd}
k[t] \simeq T_{\sigma_{j_1}}\widetilde{\mathcal{D}} \arrow[r, "gl"] \arrow[d, "t=1"] & T_{\sigma_{j_2}}\widetilde{\mathcal{D}} \simeq k[t] \arrow[d, "t=1"] \\
T_{\widetilde{q_{j_1}}}\widetilde{D} \arrow[r] & T_{\widetilde{q_{j_2}}}\widetilde{D},
\end{tikzcd}$$

After gluing all the sections, we have a family over $\mathbb{A}^1$, remains to show that the family inherit a $\mathbb{G}_m$-action, that is equivalent to require that gl is $\mathbb{G}_m$-equivariant. From the last claim of Proposition 3.5.24 we have

$$\begin{tikzcd}
T_{\sigma_i}\widetilde{\mathcal{D}} \arrow[r] \arrow[d, "\sim"] & T_{\widetilde{q_i}}(\widetilde{D} \times \mathbb{A}^1) \arrow[d, "\sim"] \\
k[t] \arrow[r, "1 \mapsto t^{n_i}"] & k[t],
\end{tikzcd}$$

where n_i is the weight of the action of λ on $T_{q_i}D$. Given that $\gamma(q_{j_1}) = \gamma(q_{j_2})$ is a tacnode in D, we have $n_{j_i} = n_{j_2}$. This force gl to be given by $1 \to \alpha \in k^*$, hence the claim. $\square$

3.6 Every universally closed open substack of $\widetilde{\mathcal{M}}^3_{g,n}$ is a union of strata

In this chapter we work over a field k with $\operatorname{char} k \neq 2, 3, 5$.

Proposition 3.6.1 (Converging singularities)**.** *Let R be a discrete valuation ring over a field k with* $\operatorname{char} k \neq 2, 3$ *and fix an hyperbolic pair (g, n). Consider two families of curves $(\mathcal{C}, p_i)$ and $(\mathcal{C}', p'_i)$ in $\widetilde{\mathcal{M}}^3_{g,n}(R)$. Assume the following:*

- *the two families have the same generic fiber, i.e. $(\mathcal{C}_\eta, p_{i,\eta}) \simeq (\mathcal{C}'_\eta, p'_{i,\eta})$;*
- *$(\mathcal{C}, p_i) \in S_\Gamma(R)$ for $\Gamma \in \mathrm{EGr}_{g,n}$;*
- *$\mathcal{C}'$ does not have equisingular A_i sections.*

Then necessarily $(g, n) \in \{(1, 1), (1, 2)\}$, and:

- *if $(g, n) = (1, 1)$ then Γ is the dual graph of an elliptic tail with no cuspidal singularities. In this case, $(\mathcal{C}'_0, p'_{i,0})$ is either smooth or a rational elliptic tail (i.e. an elliptic tail with an A_2 singularity);*
- *if $(g, n) = (1, 2)$ then Γ is the dual graph of an elliptic bridge with at worse A_1 and A_2 singularities. In this case, $(\mathcal{C}'_0, p'_{i,0})$ is either smooth or a tacnodal elliptic bridge (i.e. contains an A_3 singularity).*

Proof. Assume first that $\mathcal{C}'_0$ is nodal, then Γ is the dual graph of a nodal curve and the thesis follows from stable reduction of $\overline{\mathcal{M}}_{g,n}$, We then assume that $\mathcal{C}'_0$ has at least one non-nodal singularity. Moreover, due to the non-existence of equisingular sections, the curve $(\mathcal{C}_\eta, p_{i,\eta})$ does not have tacnodal singularities and, thanks to Proposition 3.1.8, any cusp does not compare in a rational elliptic tail. This force Γ to correspond to a dual graph in $\overline{\mathcal{M}}^{\text{ps}}_{g,n}$. We split our proof in two.

Step 1. The curve $\mathcal{C}'_0$ has at most one singularity.
Assume by contradiction that this is not the case. Perform an Hassett replacement on every cusp and tacnode of the central fibre of $(\mathcal{C}', p'_i)$ (there is at least one) and then contract the unstable $\mathbb{P}^1$'s. Let $(\mathcal{C}'', p''_i)$ be the resulting family over the DVR, eventually after a finite extension of R. Notice that $(\mathcal{C}''_0, p''_{i,0})$ is a curve in $\overline{\mathcal{M}}^{\text{ps}}_{g,n}$: it does not have tacnodes or semistable components, and after performing the replacement we cannot see a rational elliptic tail. Finally, the central fiber $\mathcal{C}''_0$ has at least one elliptic tail or bridge, given that $\mathcal{C}'_0$ has at least two singularities, one of the two attaching point has to be a lonely node. We then have that the families $(\mathcal{C}, p_i)$ and $(\mathcal{C}'', p''_i)$ have the same generic fiber and lies inside $\overline{\mathcal{M}}^{\text{ps}}_{g,n}$. Thanks to the separatedness of this stack these coincides, hence $(\mathcal{C}''_0, p''_{i,0})$ has dual graph Γ. This implies that Γ has a lonely nodal singularity, by Proposition 3.1.8 this force the family $\mathcal{C}'$ has an A_1 to have an equisingular nodal section, this yelds a contradiction.

Step 2. We have three cases for the unique singularity $p \in \mathcal{C}'_0$:

1. p is an A_1 singularity. This implies that the generic fibre is smooth. The unicity of the stable limit assures us that this case is not possible.

2. p is an A_2 singularity. Then $(\mathcal{C}'', p''_i)$ is necessarily a nodal stable family, and the unicity of the limit forces Γ to be the dual graph of an elliptic tail. Hence, the first case of the claim holds.

3. p is an A_3 singularity. Then necessarily $(\mathcal{C}'', p''_i) \in \overline{\mathcal{M}}^{\text{ps}}_{g,n}$. From the fact that the generic fibre does not have lonely nodal singularities, it follows that $(g, n) = (1, 2)$, and the unicity of the limit in $\overline{\mathcal{M}}^{\text{ps}}_{g,n}$ tells us that the dual graph is that of an attached elliptic bridge. Hence, the second case of the claim holds.

□

Before proceeding with the main theorem of the section, we need two results about tangents along section of a family of curves.

Proposition 3.6.2 (Limit of tangent sections)**.** *Let R be a discrete valuation ring over a field k with* $\operatorname{char} k \neq 2, 3, 5$*. Consider a map of families of pointed curves over R as in the diagram*

$$\begin{tikzcd}
(\mathcal{C}', p'_i) \arrow[rr, "\phi"] \arrow[dr] & & (\mathcal{C}, p_i) \arrow[dl] \\
& \operatorname{Spec} R &
\end{tikzcd}.$$

Assume moreover that:

1. *ϕ is projective, birational, $\phi_*\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{C}}$, and the map is an isomorphism outside a subcurve of the central fiber $E = \mathrm{Exc}(\phi) \subset \mathcal{C}'_0$.*
2. *Assume that E is a rational smooth curve such that $E \cap \overline{\mathcal{C}'_0 \setminus E}$ consists on a unique nodal singularity $p \in \mathcal{C}'_0$ with local parameter m.*
3. *Assume $\phi(F) = p_{1,0}$, and let $p'_{1,0} \in F$ the unique marked point contained in that component.*

Then $\phi\colon (\mathcal{C}', p'_i) \to (\mathcal{C}, p_i)$ induces a morphism on tangent spaces along the smooth sections p_1 and p'_1 and we can write the morphism of rank 1 R-modules as

$$\begin{tikzcd} T^1_{p'_1}\mathcal{C}' \arrow[r] \arrow[d, "\simeq"] & T^1_{p_1}\mathcal{C} \arrow[d, "\simeq"] \\ R \arrow[r, "1\mapsto t^m"] & R. \end{tikzcd}$$

Proof. We resolve the A_{m-1} singularity of $\mathcal{C}'$ as explained in Fact 3.4.4 (hence the assumption on the characteristic). We then can extend the sections in order to have maps of pointed curves over $\operatorname{Spec} R$

$$(\widetilde{\mathcal{C}}, \widetilde{p}_i) \to (\mathcal{C}', p'_i) \xrightarrow{\phi} (\mathcal{C}, p_i).$$

The exceptional locus of the composition consists on m rational curves whose dual graph is A_m. This composition coincide with a sequence of m blow-ups on $\mathcal{C}'$ along smooth points with the reduced structure. We first blow-up on $p_{1,0} \in \mathcal{C}'$, then repeat to blow up $m-1$ times on smooth points of the central fiber (considering the total space as a family over R) contained in the exceptional locus. Given that we are only interested in the length of the cokernel of $T^1_{p'_1}\mathcal{C}' \to T^1_{p_1}\mathcal{C}$, we can perform this computation étale locally. Moreover, it is enough to prove that at any step of the procedure the map among the tangent spaecs is given by $R \to R$, $1 \mapsto t$. Performing the computation locally, we can blow up the model of the plane $\operatorname{Spec} k[x,t]$ around the origin. We then have that on the appropriate chart the morphism is given by $\operatorname{Spec} k[x,t] \to \operatorname{Spec} k[x,t]$ with $t \mapsto t$, $x \mapsto zt$, where the sections in the charts aregiven by $x = 0$ and $z = 0$. This concludes the claim. □

Remark 3.6.3. A similar computation holds where the exceptional fiber E is given by a chain of l rational components with local parameter m_i for $i = 1, \dots, l$. Then in this case the maps among tangents $T^1_{p'_1}\mathcal{C}' \to T^1_{p_1}\mathcal{C}$ is the maps of rank 1 free R modules given by $1 \mapsto t^{m_1+\cdots+m_l}$

We can control the behaviour of tangent section in the two cases of Proposition 3.6.1.

Proposition 3.6.4 (Tangent sections after Hassett Replacement)**.** *Consider two families of curves $(\mathcal{C}, p_i)$ and $(\mathcal{C}', p'_i)$ over a DVR R, and assume to be in one of the following settings:*

1. *$(\mathcal{C}, p_i), (\mathcal{C}', p'_i) \in \overline{\mathcal{M}}_{1,2}(\frac{7}{10})(R)$, such that:*
   - *$(\mathcal{C}_\eta, p_{i,\eta}) \simeq (\mathcal{C}'_\eta, p'_{i,\eta})$ is an elliptic bridge without A_3 singularities;*

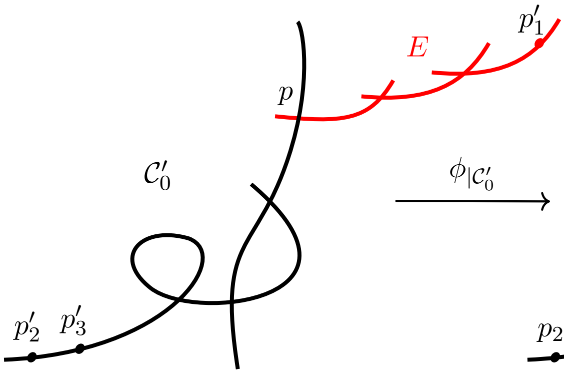


Figure 3.12. An example where E is a rational chain of length 3, we picture the map restricted to the central fiber.

- *$(\mathcal{C}_0', p_{i,0}')$ has an A_3 singularity, while $(\mathcal{C}_0, p_{i,0})$ does not have any A_3 singularity.*

2. $(\mathcal{C}, p_i), (\mathcal{C}', p_1') \in \widetilde{\mathcal{M}}^{\mathrm{wps}}_{1,1}(R)$, such that:

- *$(\mathcal{C}_\eta, p_{1,\eta}) \simeq (\mathcal{C}_\eta', p_{1,\eta}')$ is an elliptic tail without A_2 singularities;*
- *$(\mathcal{C}_0', p_{1,0}')$ has an A_2 singularity, while $(\mathcal{C}_0, p_{i,0})$ does not have any A_2 singularity.*

Then, up to a finite DVR extension, there exists a unique semistable family of curves $(\widetilde{\mathcal{C}}, \widetilde{p}_i)$ over $\operatorname{Spec} R$, whose stabilization is $(\mathcal{C}, p_i)$, such that we can construct a unique diagram:

$$\begin{tikzcd} (\widetilde{\mathcal{C}}, \widetilde{p}_i) \arrow[r, "q"] \arrow[d, "\pi"] & (\mathcal{C}', p_i') \\ (\mathcal{C}, p_i) & \end{tikzcd},$$

where π is the morphism given by the stabilization, and q is the Hassett replacement. Moreover, pulling back along $\widetilde{p}_i\colon \operatorname{Spec} R \to \widetilde{\mathcal{C}}$ and identifying $\widetilde{p}_i^* T^1_{\widetilde{\mathcal{C}}} \simeq (p_i')^* T^1_{\mathcal{C}'}$ via q, we have maps of R-modules:

$$\begin{tikzcd} T^1_{p_i'}\mathcal{C}' \arrow[r] \arrow[d, "\simeq"] & T^1_{p_i}\mathcal{C} \arrow[d, "\simeq"] \\ R \arrow[r, "t \mapsto t^{n_i}"] & R \end{tikzcd}$$

where $n_i > 0$ is the local parameter of the nodes in $\widetilde{\mathcal{C}}$ that lie on the semistable component of $(\widetilde{\mathcal{C}}_0, p_{i,0})$ that contains p_i.

Proof. The construction follows from Proposition 3.6.2 and Hassett replacement. $\square$

Theorem 3.6.5. *Let $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}/k$ be an open substack that is universally closed, and assume* $\operatorname{char} k \neq 2, 3, 5$*. Then $\mathcal{U}$ is a union of strata, i.e. $|\mathcal{U}| = \bigsqcup_{\alpha \in I} |S_\alpha|$, for some subset $I \subset \mathrm{EGr}_{g,n}$.*

Proof. Let $\mathcal{U}$ be a substack that satisfies the hypotheses. Since the strata are connected, the claim is equivalent to prove that for every $\alpha \in \mathrm{EGr}_{g,n}$, the open immersion $S_\alpha \cap \mathcal{U} \subset S_\alpha$ is closed by specialization. In this way, the theorem reduces to show that for every map from a DVR $f\colon \operatorname{Spec} R \to S_\alpha$, such that $f(\eta) \in |S_\alpha \cap \mathcal{U}|$, we have that $f(0) \in |\mathcal{U}|$. This can be verified after an extension of DVRs $R \subset R'$, hence we can assume:

- The residue field of R is algebraically closed;
- $f_{|k(\eta)}\colon \operatorname{Spec} k(\eta) \to S_\alpha \cap \mathcal{U}$ can be extended to a map $f'\colon \operatorname{Spec} R \to \mathcal{U}$, because $\mathcal{U}$ is universally closed;
- Every singularity in the curve induced by f over the generic point of $\operatorname{Spec} R$ is rational, i.e. a $k(\eta)$ point.

We will construct a map $\Theta \to \widetilde{\mathcal{M}}^3_{g,n}$ whose generic point is $f(0)$, and the special one is $f'(0)$. Given that $f'(0) \in |\mathcal{U}|$ and $\mathcal{U}$ is open, we will have that $f(0) \in |\mathcal{U}|$. We split the proof into three steps.

Step 1. Let $(\mathcal{C}, p_i)$ and $(\mathcal{C}', p'_i)$ be the families over $\operatorname{Spec} R$ induced respectively by f and f'. By assumption, there exists an isomorphism $\phi\colon (\mathcal{C}'_\eta, p'_{i,\eta}) \to (\mathcal{C}_\eta, p_{i,\eta})$. We want to choose some sections to rephrase the datum of ϕ using Remark 3.3.7.

Fix the following singularities $\{\theta_j\}_{j\in J}$ on $(\mathcal{C}'_\eta, p'_{i,\eta})$:

- The A_3 sections of $(\mathcal{C}'_\eta, p'_\eta)$.
- The cuspidal sections of $(\mathcal{C}'_\eta, p'_\eta)$ that are not the image of a cusp via an attached elliptic bridge $(E, q_1, q_2) \to \mathcal{C}'_\eta, p'_\eta)$.
- The nodal sections of $(\mathcal{C}'_\eta, p'_\eta)$ that are not the image of a node via an attached bridge or tail $(E, q_i) \to (\mathcal{C}'_\eta, p'_\eta)$.

All these sections extend equisingularly on $(\mathcal{C}', p'_i)$. Indeed, if we partially normalize $\mathcal{C}'$ along all its equisingular sections, thanks to Proposition 3.6.1, we obtain a curve whose connected components are generically: smooth, elliptic bridges with singularities A_1 and A_2, or elliptic tails with A_1 singularities. Moreover, the assumption $(\mathcal{C}, p_i) \in S_\Gamma(R)$ assures us that the chosen sections extend equisingularly on $(\mathcal{C}_\eta, p_{i,\eta})$ as well.

Let $\pi\colon (\widetilde{\mathcal{C}}, \widetilde{p}_i) \to (\mathcal{C}, p_i)$ (resp. $\pi'\colon (\widetilde{\mathcal{C}}', \widetilde{p}'_i) \to (\mathcal{C}', p'_i)$) be the partial normalization along the extensions of $\{\theta_j\}_{j\in J}$. Given that these sections are clearly invariant by the automorphism of the dual graph of $(\mathcal{C}'_\eta, p'_{i,\eta})$, we can apply Remark 3.3.8. We can name the lift to the normalization of the θ_j with the same notation of the loc. cit. Remark. In this way, ϕ is defined by an isomorphism of pointed curves:

$$\widetilde{\phi}\colon (\widetilde{\mathcal{C}}'_\eta, \widetilde{p}'_{i,\eta}, \sigma'^b_{i,\eta}, \beta'_{i,\eta}, \tau'^b_{i,\eta}) \to (\widetilde{\mathcal{C}}_\eta, \widetilde{p}_{i,\eta}, \sigma^b_{i,\eta}, \beta_{i,\eta}, \tau^b_{i,\eta}),$$

such that the following diagram commutes:

$$\begin{tikzcd} T_{\widetilde{\tau}'^+_{i,\eta}}\widetilde{\mathcal{C}}'_\eta \arrow[r, "d\widetilde{\phi}"] \arrow[d, "gl'_{i,\eta}"] & T_{\widetilde{\tau}^+_{i,\eta}}\widetilde{\mathcal{C}}_\eta \arrow[d, "gl_{i,\eta}"] \\ T_{\widetilde{\tau}'^-_{i,\eta}}\widetilde{\mathcal{C}}'_\eta \arrow[r, "d\widetilde{\phi}"] & T_{\widetilde{\tau}^-_{i,\eta}}\widetilde{\mathcal{C}}_\eta \end{tikzcd}.$$

To fix the notations, we call σ_i^b, β_i and τ_i^b (resp. $\sigma_i'^b$, β_i' and $\tau_i'^b$) the extensions of the sections $\sigma_{i,\eta}^b$, $\beta_{i,\eta}$ and $\tau_{i,\eta}^b$ on $\widetilde{\mathcal{C}}$ (resp. $\sigma_{i,\eta}'^b$, $\beta_{i,\eta}'$ and $\tau_{i,\eta}'^b$ on $\widetilde{\mathcal{C}}'$). We have the following commutative diagram:

$$\begin{tikzcd} T_{\tau_i'^+}\widetilde{\mathcal{C}}' \arrow[r, hook] \arrow[d, "gl_i'"] & T_{\tau_{i,\eta}'^+}\widetilde{\mathcal{C}}'_\eta \arrow[r, "d\widetilde{\phi}"] \arrow[d, "gl'_{i,\eta}"] & T_{\tau_{i,\eta}^+}\widetilde{\mathcal{C}}_\eta \arrow[d, "gl_{i,\eta}"] & T_{\tau_i^+}\widetilde{\mathcal{C}} \arrow[l, hook'] \arrow[d, "gl_i"] \\ T_{\tau_i'^-}\widetilde{\mathcal{C}}' \arrow[r, hook] & T_{\tau_{i,\eta}'^-}\widetilde{\mathcal{C}}'_\eta \arrow[r, "d\widetilde{\phi}"] & T_{\tau_{i,\eta}^-}\widetilde{\mathcal{C}}_\eta & T_{\tau_i^-}\widetilde{\mathcal{C}} \arrow[l, hook'] \end{tikzcd}, \tag{3.1}$$

where the horizontal inclusion arrows are the natural restriction of the tangent space to the generic fibre.

Step 2. We give now a numerical condition for $d\widetilde{\phi}$.
We will use Proposition 3.6.2 and Lemma 3.6.4, in order to write the diagram of R-modules 3.1 as:

$$\begin{tikzcd} t^{n_q}R \arrow[r, hook] \arrow[d, "\simeq"] & K \arrow[r, "d\widetilde{\phi}_q"] \arrow[d, "\simeq"] & K \arrow[d, "\simeq"] & R \arrow[l, hook'] \arrow[d, "\simeq"] \\ t^{n_s}R \arrow[r, hook] & K \arrow[r, "d\widetilde{\phi}_s"] & K & R \arrow[l, hook'] \end{tikzcd},$$

where, to keep the notation simpler, we write p, q for two sections τ_i^+, τ_i^- (for some index i) that glue tacnodally in $\widetilde{\mathcal{C}}'$, and n_p, n_q are defined as described below.

For every section τ_i^b, we have that $T_{\tau_i^b}\widetilde{\mathcal{C}} \simeq R$, and we choose bases on $T_{\tau_i^b}\widetilde{\mathcal{C}}$ such that gl_i is the identity map. The choice of such a basis describes the right half of the diagram.

Let $(\mathcal{D}, d_i)$ and $(\mathcal{D}', d_i')$ be two connected components respectively of $(\widetilde{\mathcal{C}}, \widetilde{p}_i, \sigma_i^b, \beta_i, \tau_i^b)$ and $(\widetilde{\mathcal{C}}', \widetilde{p}_i', \sigma_i'^b, \beta_i', \tau_i'^b)$ such that $\widetilde{\phi}$ induces an isomorphism between $(\mathcal{D}_\eta, d_\eta)$ and $(\mathcal{D}'_\eta, d'_{i,\eta})$. We prove the description of diagram 3.1 by cases on the connected components:

1. $(\mathcal{D}'_\eta, d'_{i,\eta})$ is an elliptic bridge without singularities of type A_3. If the stabilization of $(\mathcal{D}, d_1, d_2)$ lies in $\overline{\mathcal{M}}^{\mathrm{ps}}_{1,2}$, then we have a contraction map $(\mathcal{D}', d_1', d_2') \to (\mathcal{D}, d_1, d_2)$ (because $\overline{\mathcal{M}}^{\mathrm{ps}}_{1,2}$ is separated), and applying Proposition 3.6.2, we have our claim with $n_{d_i'}$ as the local parameter of the node on the semistable component that contains d_i'. If this is not the case, then the central fibre of $(\mathcal{D}', d_1', d_2')$ has stabilization as in Lemma 3.6.4, and using this result and again

Proposition 3.6.2, $n_{d'_i} = m_{d'_i,1} + m_{d'_i,2}$, where the first corresponds to n in the Lemma and the second to $m_{d'_i}$ as the local parameter of the Proposition loc. cit. (eventually, $m_{d'_i} = 0$).

2. $(\mathcal{D}'_\eta, d'_{i,\eta})$ is an elliptic tail without singularities of type A_2 or A_3. Analogous to the previous case, we can define in the same way $n_{d'_1}$, $m_{d'_i,1}$, and $m_{d'_i,2}$.

3. $(\mathcal{D}'_\eta, d'_{i,\eta})$ is smooth such that (g', n') is hyperbolic, where g' is the genus and n' are the number of marked points. Again we apply Proposition 3.6.2 as in the first part of the previous cases.

4. $(\mathcal{D}'_\eta, d'_{i,\eta}) \simeq (\mathbb{P}^1, 0, \infty)$. Then $(\mathcal{D}, d_i)$ corresponds to a map $R \to B\mathbb{G}_m$, hence it is the trivial family. After blowing down the negative curves of the family $(\mathcal{D}', d'_i)$, we have a smooth family of 2-marked genus 0 curves, so as before it is trivial. We can then construct a morphism $(\mathcal{D}', d'_i) \to (\mathcal{D}, d_i)$ that satisfies the hypotheses of Propositon 3.6.2 and conclude as in the previous case.

The choice of the θ_j assures us that there are no more cases. This description implies, with the notation of the diagram above, that $\widetilde{\phi}$ has to satisfy the following:

$$v(d\widetilde{\phi}_q) + n_q = v(d\widetilde{\phi}_s) + n_s, \tag{3.2}$$

where q and s as before glue along a tacnode, and v is the valuation in R. We analyze now the possibilities for Fixed a connected component $(\mathcal{D}', d'_i)$ of $(\widetilde{\mathcal{C}}', \widetilde{p}'_i, \sigma_i'^b, \beta'_i, \tau_i'^b)$, there are two possibilities for $v(d\widetilde{\phi}_{d'_i})$:

- if $(\mathcal{D}'_\eta, d'_{i,\eta}) \not\simeq (\mathbb{P}^1_\eta, 0, \infty)$, then $v(d\widetilde{\phi}_{d'_i}) = 0$. This follows from the finiteness of the inertia of $\overline{\mathcal{M}}^{\text{ps}}_{g,n}$ and $\overline{\mathcal{M}}_{1,1}$: indeed, using the family $(\widetilde{\mathcal{C}}, \widetilde{p}_i, \sigma_i^b, \beta_i, \tau_i^b)$, $(\mathcal{D}'_\eta, d'_{i,\eta})$ can be completed to a family over R, hence every automorphism of the generic fibre extend to an automorphism of the hole family, then the differential of this cannot be zero on the tangent space of the special fibre.
- if $(\mathcal{D}'_\eta, d'_{i,\eta}) \simeq (\mathbb{P}^1_\eta, 0, \infty)$, then $v(d'\widetilde{\phi}_{d'_1}) + v(d\widetilde{\phi}_{d'_2}) = 0$. This is a consequence of the explicit action of $\mathrm{Aut}(\mathbb{P}^1, 0, \infty) = \mathbb{G}_m$ on the tangent of 0 and ∞, that has opposite weights.

Step 3. In the last step of the proof, we show that there exists a map over Θ with generic fibre $(\mathcal{C}_0, p_{i,0})$ and special fibre $(\mathcal{C}'_0, p'_{i,0})$. We will prescribe an action of $\mathbb{G}_m$ on $(\mathcal{C}'_0, p'_{i,0})$ that makes possible the isotrivial specialization according to Theorem 3.5.23.

We first define an action on the central fibre of the partial normalization. Let $(\mathcal{D}', d'_i)$ be a connected component of $(\widetilde{\mathcal{C}}', \widetilde{p}'_i, \sigma_i'^b, \beta'_i, \tau_i'^b)$. According to the possible cases in the previous section, we have the following possibilities:

1. $(\mathcal{D}'_\eta, d'_{i,\eta})$ is an elliptic bridge without singularities of type A_3, and the stabilization of $(\mathcal{D}', d'_1, d'_2)$ lies in $\overline{\mathcal{M}}^{\text{ps}}_{1,2}$. Then we define the $\mathbb{G}_m$ action on $(\mathcal{D}'_0, d'_{1,0}, d'_{2,0})$ as the unique one that has weights $n_{d'_i}$ on $T_{d'_{i,0}}\mathcal{D}'_0$.

   Otherwise, if the central fibre of $(\mathcal{D}', d'_1, d'_2)$ has an A_3 singularity $t \in \mathcal{D}'_0$, we define the unique action $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(\mathcal{D}'_0, d'_1, d'_2)$ such that, on $T_{d'_{i,0}}\mathcal{D}'$, it has weight $n_{d'_i} = m_{d'_i,1} + m_{d'_i,2}$, and $\mathrm{wt}^\lambda_t(\mathcal{D}'_0, d'_{i,0}) = -m_{d'_i,1}$.

2. $(\mathcal{D}'_\eta, d'_{i,\eta})$ is an elliptic tail without singularities of type A_2 or A_3. Again, there are two cases. If there are no cuspidal singularities on the central fibre, then we define the $\mathbb{G}_m$ action on $(\mathcal{D}'_0, d'_{1,0})$ as the unique one that has weights $n_{d'_1}$ on $T_{d'_{1,0}}\mathcal{D}'_0$. Otherwise, if the central fibre of $(\mathcal{D}', d'_1)$ has an A_2 singularity $c \in \mathcal{D}'_0$, then we define the unique action $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(\mathcal{D}'_0, d'_1)$ such that, on $T_{d'_{1,0}}\mathcal{D}'$, it has weight $n_{d'_1} = m_{d'_1,1} + m_{d'_1,2}$, and $\mathrm{wt}^\lambda_c(\mathcal{D}'_0, d'_{1,0}) = -m_{d'_1,1}$.

3. $(\mathcal{D}'_\eta, d'_{i,\eta})$ is smooth such that (g', n') is hyperbolic, where g' is the genus and n' are the number of marked points. Then we define the $\mathbb{G}_m$ action on $(\mathcal{D}'_0, d'_{i,0})$ as the unique one that has weights $n_{d'_i}$ on $T_{d'_{i,0}}\mathcal{D}'_0$. The support of the action is on every semistable $\mathbb{P}^1$ of the central fibre (if any) and trivial elsewhere.

4. $(\mathcal{D}'_\eta, d'_{i,\eta}) \simeq (\mathbb{P}^1, 0, \infty)$. Then we define the $\mathbb{G}_m$ action on $(\mathcal{D}'_0, d'_{i,0})$ such that the weight on $T_{d'_{i,0}}\mathcal{D}'_0$ is $n_{d'_i} + v(d\widetilde{\phi}_{d'_i})$. The central fibre can have one or two irreducible components. In both cases, the action is defined uniquely: if $\mathcal{D}'_0$ has only one component, $v(d\widetilde{\phi}_{d'_1}) = -v(d\widetilde{\phi}_{d'_2})$.

To descend the $\mathbb{G}_m$ action to $(\mathcal{C}'_0, p'_{i,0})$, we only need to verify that we can glue it along the A_i singularities. For A_1 and A_2 singularities, it is enough that the attaching points are fixed by $\mathbb{G}_m$. This is always the case, given that the action fixes the marked points of $(\widetilde{\mathcal{C}}'_0, \widetilde{p}'_{i,0}, \sigma'^b_{i,0}, \beta'_{i,0}, \tau'^b_{i,0})$. For A_3 singularities, we need to verify that the action on the tangent at the attaching point has the same weight. By definition of the action, this last requirement is exactly the content of equation 3.2. Thus, we have constructed $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(\mathcal{C}'_0, p'_{i,0})$.

A straightforward computation shows that $\mathrm{wt}^\lambda_p(\mathcal{C}'_0, p_{i,0}) < 0$ for any singularity $p \in (\mathcal{C}'_0, p_{i,0})$ that lifts to a singularity of $\widetilde{\mathcal{C}}$ that lies on a semistable $\mathbb{P}^1$ of $(\widetilde{\mathcal{C}}'_0, \widetilde{p}'_{i,0}, \sigma'^b_{i,0}, \beta'_{i,0}, \tau'^b_{i,0})$. Applying Theorem 3.5.23, we conclude the existence of the family over Θ_k as required. □

We list here there possible steps to generalize the result to other type of singularities (e.g. ADE type):

1. classification of curves with non-discrete automorphism group and description of the basins of attraction;

2. Generalization of Proposition 3.3.6, concerning descend of automorphisms from the normalization (for A_i singularities, we can use Jet bundles);

3. Generalization of Proposition 3.6.1, via an analysis of the variety of stable limits (cf. [Has00] for ADE singularities).

3.7 A different proof

It is also possible to prove a weaker version of the main theorem from the previous section, provided that the substack is, in addition, S and Θ-complete.

We recall that for a stack that admit a separated good moduli space $\mathcal{X} \to X$, $\mathcal{X}$ is universally closed if and only if X proper [AHH23, Prop. 3.48 (3)].

Theorem 3.7.1. *Let $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})/k$ be an open substack that admit a proper good moduli space, we have that $\mathcal{U} = \bigsqcup_{\alpha \in I} S_\alpha$ is a union of strata for some subset $I \subset \mathrm{EGr}_{g,n}$.*

We will use the valuative criteria introduced in [AHH23], for a discussion of the test objects $\overline{ST}_R$, Θ_R we refer to [Alp25, Sec. 6.9]. When there is no risk of confusion, we will write $0 \in \overline{ST}_R$ and $0 \in \Theta_R$ to refer to the closed points of the two objects.

Remark 3.7.2. Given a map $f \colon \overline{ST}_R \to \mathcal{X}$, we can construct a morphism $g \colon \operatorname{Spec} R' \to \mathcal{X}$ such that $g_{|k(\eta)} \simeq f_{|k(\eta)}$ and $g_{|k(s)} \simeq f_{|0}$, where $s \in \operatorname{Spec} R$ is the closed point. This follows via precomposition with

$$\operatorname{Spec} R[t]/(t^2 - \pi) \to \left[\operatorname{Spec}\left(\frac{R[x,y]}{(xy - \pi)}\right)/\mathbb{G}_m\right] \simeq \overline{ST}_R$$

given by $x \mapsto t$, $y \mapsto t$.

The following two remarks are a straightforward consequence of the classification of Theorem 3.5.23.

Remark 3.7.3. Given a very close specialization $f \colon \Theta_k \to \widetilde{\mathcal{M}}^3_{g,n}$ such that no singularities in the central fibre are smoothed, then the central and generic fibres are isomorphic as curves over $\operatorname{Spec} k$.

Remark 3.7.4. Given a very close specialization $f \colon \Theta_k \to \widetilde{\mathcal{M}}^3_{g,n}$, corresponding to a family $(\mathcal{C}, p_i)$, and consider a special point $p \in \mathcal{C}_0$ such that $\mathrm{wt}^\lambda_p(\mathcal{C}, p_i) = 0$. Then, there exists a singularity q in the generic fibre that has limit p in the central fibre. Moreover, the two special points q and q are of the same type.

Proof of Theorem 3.7.1. Let $\mathcal{U}$ be a substack that satisfies the hypotheses. Since the strata are connected, the claim is equivalent to show that, for every $\alpha \in \mathrm{EGr}_{g,n}$, the open immersion $S_\alpha \cap \mathcal{U} \subset S_\alpha$ is closed by specialization. We can test this with maps from a DVR. Fix a map $f \colon \operatorname{Spec} R \to S_\alpha$ such that the generic point is in $|S_\alpha \cap \mathcal{U}|$, and we will prove that also the image of the special point is in $|\mathcal{U}|$.

This can be verified after an extension of DVRs $R \subset R'$, hence we can assume that:

- The residue field k of R is algebraically closed;
- $f_{|k(\eta)} \colon \operatorname{Spec} k(\eta) \to S_\alpha \cap \mathcal{U}$ can be extended to a map $f' \colon \operatorname{Spec} R \to \mathcal{U}$. This follows because $\mathcal{U}$ has a proper good moduli space, and hence satisfies the existence of the valuative criterion [AHH23, Prop. 3.48 (3)];
- Every singularity in the curve induced by f over the generic point of $\operatorname{Spec} R$ is rational, i.e. a $k(\eta)$ point.

Let $(\mathcal{C}, p_i)$ and $(\mathcal{C}', p'_i)$ be the families over $\operatorname{Spec} R$ induced respectively by f and f'. Moreover, let $\{\sigma_j \colon \operatorname{Spec} R \to \mathcal{C}\}_{j \in J}$ and $\{\sigma'_j \colon \operatorname{Spec} R \to \mathcal{C}'\}_{j \in J'}$ be the singular sections obtained as the limit of the special point of the curves over the generic point. The two families have isomorphic general fibres $(\mathcal{C}_\eta, p_{i,\eta})$, and hence they can be glued to construct a curve over $\overline{ST}_R \setminus \{0\} = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R$. Given that $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ has a good moduli space, we can complete the family over $\overline{ST}_R$. Let

(C, s_i) be the fibre over $0 \in \overline{ST}_R$. With the notation of Remark 3.7.2, we assume that the family restricted to $y \neq 0$ (resp. $x \neq 0$) is $(\mathcal{C}', p_i')$ (resp. $(\mathcal{C}, p_i)$).

The key point relies on proving that the family induced by pullback

$$\Theta_k = [\operatorname{Spec} k[y]/\mathbb{G}_m] \hookrightarrow \overline{ST}_R$$

has generic and special fibres isomorphic as pointed curves over $\operatorname{Spec} k$.
Let $\lambda\colon \mathbb{G}_m \to \operatorname{Aut}(C, s_i)$ be the one-parameter subgroup induced by the above very close specialization, and consider a singularity $p \in C$. We prove that this is the limit of a singular section of the same type on the generic fibre (i.e., it cannot be smoothed). There are two cases:

- $p \in C$ is such that $\operatorname{wt}_p^\lambda(C, s_i) \geq 0$. In this case, Remark 3.7.4 says that the singularity cannot be smoothed on the central fibre of Θ_k.
- $p \in C$ is such that $\operatorname{wt}_p^\lambda(C, s_i) < 0$. This implies that the weight of the action on p with respect to the family induced by $[\operatorname{Spec} k[x]/\mathbb{G}_m] \hookrightarrow \overline{ST}_R$ is positive. Hence, by remark 3.7.4, it is the limit of a singularity $p' \in \mathcal{C}_k$ of the same type, given that the central fibre of this pullback is $(\mathcal{C}_0, p_{i,0})$. Moreover, $(\mathcal{C}, p_i) \in S_\alpha(\operatorname{Spec} R)$ implies that p' is the limit of an equisingular section on the generic fibre $\sigma_i\colon \operatorname{Spec} K \to (\mathcal{C}_\eta, p_{i,\eta})$. By properness of the families of curves and Remark 3.7.2, this has a limit also in the family $(\mathcal{C}', p_i')$ that necessarily specializes to p on the fibre over $0 \in \overline{ST}_R$. Notice, moreover, that σ_i is a lonely singularity, and hence thanks to Proposition 3.1.8, the limit is of the same type, and hence p is not smoothed.

We can now apply remark 3.7.3. The central fibre of the family over $[\operatorname{Spec} k[x]/\mathbb{G}_m]$ is (C, s_i), and by openness of $\mathcal{U}$, we have $(\mathcal{C}_k, p_{i,k}) \in |\mathcal{U}|$, hence the thesis. □

3.8 A condition on opens with proper good moduli spaces

In this section, we give some necessary conditions for an open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ to admit a proper good moduli space. Thanks to [AHH23, Prop. 3.48 (3)], such an open $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ is universally closed over k. Under these assumptions, the main result of the previous section establishes that $\mathcal{U} = \sqcup_{\alpha \in I} S_\alpha$ for some $I \subset \mathrm{EGr}_{g,n}$. We now give some preliminary conditions on I. We approach this question using the Θ- and S-completeness criteria introduced in [AHH23].

Definition 3.8.1. We define the following open substack of $\widetilde{\mathcal{M}}^3_{g,n}$:

$$\overline{\mathcal{M}}^!_{g,n} := \widetilde{\mathcal{M}}^3_{g,n} \setminus \overline{S_{\mathrm{ell}} \cup S_{\mathrm{tac}}}.$$

Here S_{ell} is the stratum of curves with a nodally attached elliptic tail, and S_{tac} is the stratum of irreducible curves with exactly one singularity that is a tacnode. Finally, we define $\Gamma^i_{\mathrm{ell}} \in \mathrm{EGr}^3_{g,n}$ to be the stratum parametrizing curves with i nodally attached smooth elliptic tails, as pictured in Figure 3.13.

We can now state the main result of the section:

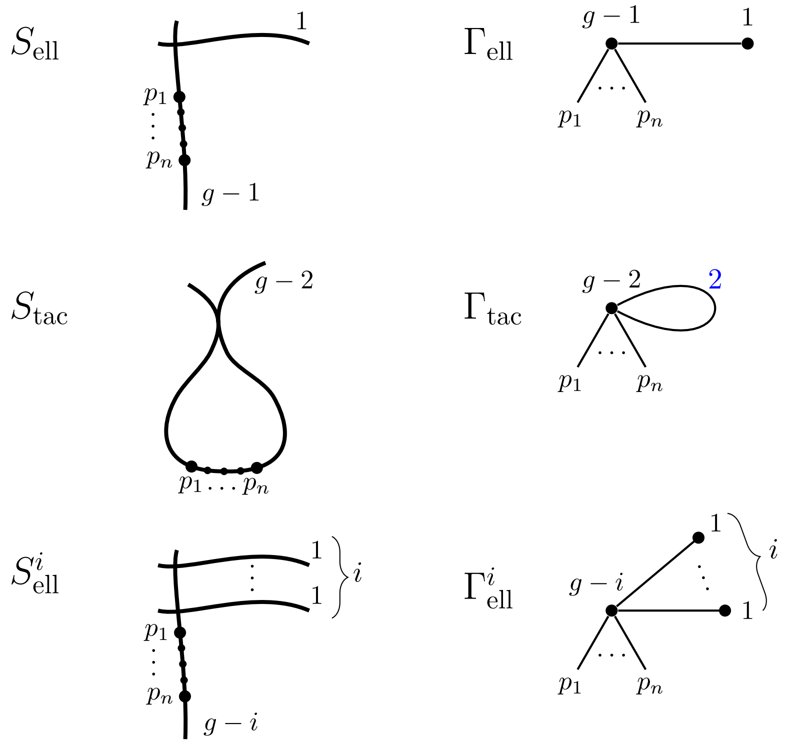


Figure 3.13. On the right the dual graph, on the left a combinatorial picture of the curve.

Theorem 3.8.2. *Fix a hyperbolic pair (g, n). Let $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ be an open substack with a proper good moduli space. Then one of the following occurs:*

(i) $\mathcal{U} \subset \widetilde{\mathcal{M}}^2_{g,n}$*, in which case* $\mathcal{U}$ *is necessarily* $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$*, or* $\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}$*;*

(ii) $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}(7/10)$*;*

(iii) $\mathcal{U} \subset \overline{\mathcal{M}}^{!}_{g,n}$.

The theorem will be proved using the fact that a stack admitting a separated good moduli space is Θ- and S-complete [AHH23, Thm. 5.4]. More specifically, we will use the techniques explained below. Fix the following notation for the test objects (in parentheses the weights of the $\mathbb{G}_m$-action):

$$\overline{ST}_R \simeq \left[\mathrm{Spec}\left(\frac{R[x,y]}{(xy-\pi)}\right) /_{(1,-1)} \mathbb{G}_m\right],$$
$$\Theta_R \simeq \left[\mathrm{Spec}\left(R[x]\right) /_{(1)} \mathbb{G}_m\right].$$

Technique 1 (Specialization)**.** *Fix a universally closed open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ and a discrete valuation ring R with residue field K. Consider a specialization of graphs $\Gamma_1 \to \Gamma_2$ in $\mathrm{EGr}^3_{g,n}$ such that $S_{\Gamma_2} \subset \mathcal{U}$. Then there exists a family of curves $(\mathcal{C}, p_i)$ over $\mathrm{Spec}\, R$ such that the generic fiber has dual graph Γ_2 and the special fiber has dual graph Γ_1. By the universal closedness of $\mathcal{U}$, there exists a family of curves $(\mathcal{C}', p'_i)$ corresponding to a $\mathrm{Spec}\, R$-point of $\mathcal{U}$ such that over the generic point*

$\eta \in \operatorname{Spec} R$ we have $(\mathcal{C}'_\eta, p'_{i,\eta}) \simeq (\mathcal{C}_\eta, p_{i,\eta})$. We can then use Hassett replacement and the uniqueness of the nodal stable limit (or the limit inside $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$) to deduce the combinatorics of the dual graph Γ' of $(\mathcal{C}'_0, p'_{i,0})$. In this way, we obtain constraints on which strata can appear in $\mathcal{U}$.

Technique 2 (S-completeness)**.** *Fix an S-complete open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ and a discrete valuation ring R with residue field K. We can test S-completeness via two families $(\mathcal{C}, p_i), (\mathcal{D}, q_i) \in \mathcal{U}(R)$ with isomorphic generic fibers over $K = \operatorname{Frac}(R)$. These define a map $\overline{ST}_R \setminus \{0\} \to \mathcal{U}$ that, by hypothesis, extends to a map $\overline{ST}_R \to \mathcal{U}$; this corresponds to a family of curves, and we denote by $(\mathcal{E}, \sigma_i)$ its pullback over $\operatorname{Spec} \frac{R[x,y]}{xy-t}$.*
Any singular section $\sigma\colon \operatorname{Spec} K \to \mathcal{C}_K$ extends to $\operatorname{Spec} R \to \mathcal{C}$ and to $\operatorname{Spec} R \to \mathcal{D}$ by properness of the families. This defines a section of the family over $\overline{ST}_R \setminus \{0\}$ whose image has closure $Z \subset \mathcal{E}$ in the total space of the family. If the singularities in Z are isolated (i.e. the unique singularities of an open subset of $\mathcal{E}$), then we can perform a partial normalization along Z and obtain constraints from the combinatorics of the curve.
We now explain a variation of the previous strategy. Instead of considering a section on the general fiber of $\overline{ST}_R$, we may first complete the family over $\overline{ST}_R$ and then consider the closed obtained by removing the general fiber. Given two sections on the closed fibers of $(\mathcal{C}, p_i)$ and $(\mathcal{D}, q_i)$, by properness these extend to the family obtained by pullback along $\Theta \to \overline{ST}_R$ (given by $x = 0$ or $y = 0$), cf. Example 3.8.3. In both cases the strategy is as follows:

1. *Fix some singular section of the generic fiber (or sections on $\mathcal{C}_0$ and $\mathcal{D}_0$), possibly after a finite DVR extension;*
2. *Use Proposition 3.1.8 to conclude that the extension of the singularity is equisingular (in particular, the singularities are isolated);*
3. *Use openness of $\mathcal{U}$ (which is a union of strata by Theorem 3.6.5) to deduce that a stratum with prescribed singularities lies in $\mathcal{U}$.*

Technique 3 (Θ-completeness)**.** *Fix a Θ-complete open substack $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ and a discrete valuation ring R with residue field K. We can test Θ-completeness via two families $(\mathcal{C}, p_i) \in \mathcal{U}(R)$ and $(\mathcal{D}, q_i) \in \mathcal{U}(\Theta_K)$ with isomorphic generic fibers over $K = \operatorname{Frac}(R)$. These define a map $\Theta_R \setminus \{0\} \to \mathcal{U}$ that, by hypothesis, extends to a map $\Theta_R \to \mathcal{U}$. Then the strategy follows verbatim Technique 2.*

The following example illustrates the *modus operandi* of the techniques.

Example 3.8.3. Let $(g, n) \neq (3, 0)$ be a hyperbolic pair with $g + n \geq 3$ and $g \geq 2$, and consider an S-complete open $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_g$. Assume that $\mathcal{U}$ contains two families of curves $\mathcal{C}, \mathcal{D}$ over a DVR R such that:

- the generic fibers of $\mathcal{C}$ and $\mathcal{D}$ are smooth and isomorphic over $K = \operatorname{Frac}(R)$;
- the special fiber of $\mathcal{C}$ consists of two irreducible smooth components of genus 1 and $g - 2$ meeting in two nodes;
- the special fiber of $\mathcal{D}$ consists of a unique irreducible component with a unique

singularity that is a tacnode.

Then we can apply Technique 2. The constraints on (g, n) ensure the existence of $\mathcal{C}$ and $\mathcal{D}$ and guarantee that over $0 \in \overline{ST}_R$ the two nodal singularities on $\mathcal{C}_0$ have distinct limits.

The completion of the S-diagram over $\widetilde{\mathcal{M}}^3_g$ is given by a curve contained in the closure of the stratum consisting of a non-separating elliptic tacnodal bridge, as in Figure 3.14. Notice that two such families can be constructed in $\widetilde{\mathcal{M}}^3_g$ thanks to Theorem 3.4.6.

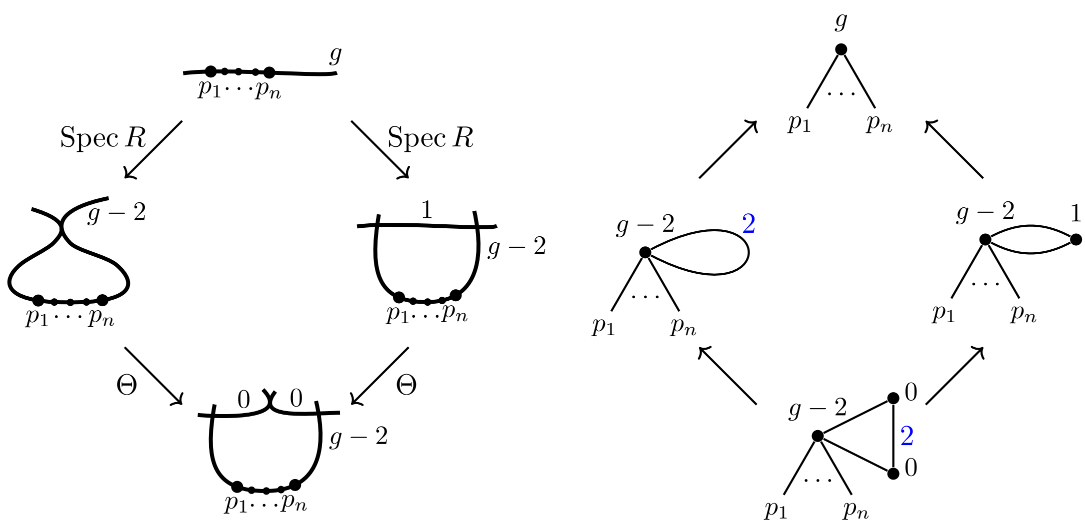


Figure 3.14. We picture with arrows the specializations (over Spec R and Θ), we include both the combinatorial picture and the maps at the level of dual graphs.

Lemma 3.8.4. *Let (g, n) be a hyperbolic pair with $g \geq 1$ and $g + n \geq 3$. Consider an open*

$$\mathcal{U} = \bigsqcup_{\alpha \in I} S_\alpha \subset \widetilde{\mathcal{M}}^3_{g,n}$$

that admits a proper good moduli space and assume that $\Gamma_{\text{ell}} \in I$. Then also $\Gamma^g_{\text{ell}} \in I$.

Proof. Denote with $\Xi^h \in \text{EGr}_{g,n}$, for $h \leq g$, the dual graph of a curve with h elliptic tails and $g - h$ cusps, as in Figure 3.15.

Apply Technique 1 to any contraction $\Gamma^g_{\text{ell}} \to \Gamma_{\text{ell}}$. We obtain that there exists $1 \leq h \leq g$ such that $\Xi^h \in I$. Let $h \in \mathbb{N}$ be the maximal integer such that $\Xi^h \in I$. Assume, by contradiction, that $h < g$.

For the proof, fix two 1-pointed smooth elliptic curves over k, (E, q) and (E', q'), which we assume to be non-isomorphic as pointed curves over $\overline{k}$.

Construct a family of curves over a DVR R, $(\mathcal{C}, p_i) \in \widetilde{\mathcal{M}}^3_{g,n}(R)$, such that $\mathcal{C}_\eta$ is smooth and $(\mathcal{C}_0, p_{i,0})$ has dual graph Ξ^{h+1}. Suppose moreover that, after a DVR

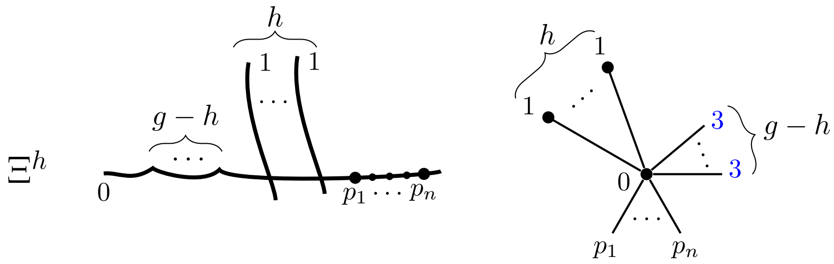


Figure 3.15. On the right the dual graph, on the left a combinatorial picture of the curve.

extension of R, $(\mathcal{C}_0, p_{i,0})$ has h attached elliptic tails isomorphic to (E, q) and one attached elliptic tail isomorphic to (E', q') (in both cases after a finite field extension).

Apply the contraction Lemma 3.4.2 to the family $(\mathcal{C}, p_i)$. After possibly a finite extension of the DVR R, we can contract an attached elliptic tail to a cusp. In this way we construct two different families $(\mathcal{C}', p_i')$ and $(\mathcal{C}'', p_i'')$, which we may assume are defined over R, such that:

- $(\mathcal{C}', p_i')$ and $(\mathcal{C}'', p_i'')$ have the same generic fiber as $(\mathcal{C}, p_i)$, and every singularity in the central fiber is rational;
- $(\mathcal{C}', p_i')$ and $(\mathcal{C}'', p_i'')$ have dual graph Ξ^h, hence are families in $\mathcal{U}(R)$;
- $(\mathcal{C}_0', p_{0,i}')$ has h attached elliptic curves isomorphic to (E, q);
- $(\mathcal{C}_0'', p_{0,i}'')$ has $h-1$ attached elliptic curves isomorphic to (E, q) and one attached elliptic curve isomorphic to (E', q').

Apply Technique 2 to the families $(\mathcal{C}', p_i')$ and $(\mathcal{C}'', p_i'')$, and let $(D, d_i) \in \mathcal{U}(k)$ be the central fiber of the S-diagram. A straightforward application of Proposition 3.1.8 ensures that (D, d_i) has at least h nodally attached elliptic chains; by maximality, these are exactly h. In this setting, it is clear that the attaching points of these elliptic tails are precisely the limits of the h attaching points of both $(\mathcal{C}_0', p_{i,0}')$ and $(\mathcal{C}_0'', p_{i,0}'')$. If we restrict the family to

$$\overline{ST}_R \setminus \{\eta\} = \left[\operatorname{Spec} \frac{k[x,y]}{xy} /_{(1,-1)} \mathbb{G}_m\right] \subset \overline{ST}_R,$$

we can equinormalize along these rational nodal sections, obtaining h families of 1-marked elliptic curves over $\overline{ST}_R \setminus \{\eta\}$.

One of these families is such that over the two open points of $\overline{ST}_R \setminus \{\eta\}$ the curves are isomorphic to (E, q) and (E', q'). By Proposition 3.5.23, the central fiber cannot be a rational elliptic tail with a non-trivial action; otherwise one of the two generic curves would have a cuspidal singularity. This forces the central fiber to be a nodal elliptic tail. The properness of $\overline{\mathcal{M}}_{1,1}$ forces $(E, q) \simeq (E', q')$, yielding a contradiction. $\square$

We now classify open subsets of $\widetilde{\mathcal{M}}^2_{g,n}$ that admit a proper good moduli space.

Proposition 3.8.5. *Let (g, n) be a hyperbolic pair. Consider an open*

$$\mathcal{U} = \bigsqcup_{\alpha \in I} S_\alpha \subset \overline{\mathcal{M}}^{\mathrm{wps}}_{g,n} = \widetilde{\mathcal{M}}^2_{g,n}$$

that admits a proper good moduli space. Then $\mathcal{U}$ is either $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$, or $\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}$.

Proof. We first prove the statement assuming that $g \geq 1$ and $g + n \geq 3$. Define the strata Ξ^h_{rat}, $\Gamma_{\mathrm{rat.ell}}$, and Γ_{cusp} as in Figure 3.16.

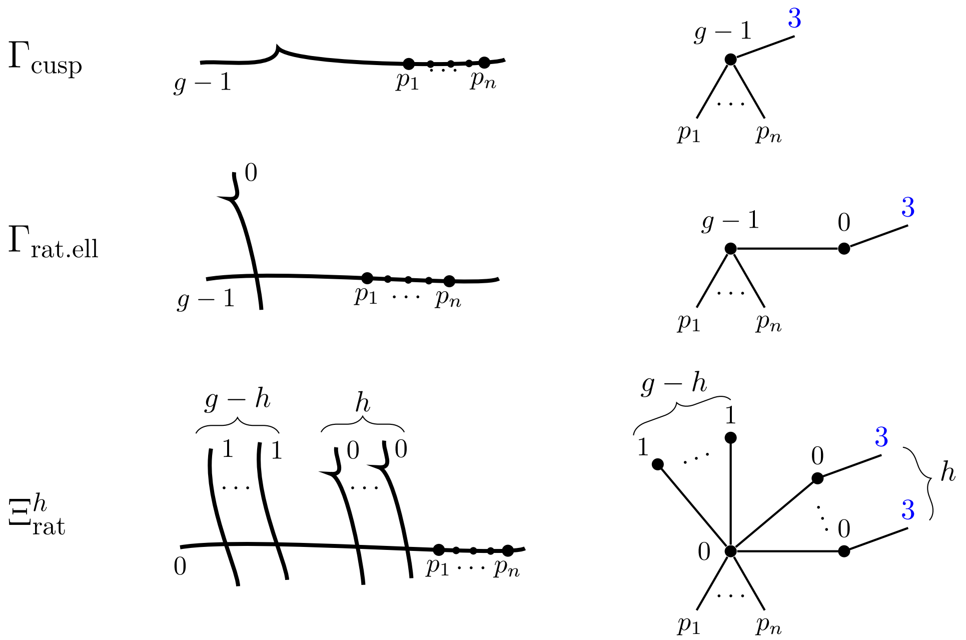


Figure 3.16. On the right, the dual graphs Ξ^h_{rat}, $\Gamma_{\mathrm{rat.ell}}$, and Γ_{cusp}; on the left, combinatorial pictures of the curves.

There are three cases:

Case 1. $\Gamma_{\mathrm{cusp}} \notin I$. In this case $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}$. Since this stack is proper, we necessarily have $\mathcal{U} = \overline{\mathcal{M}}_{g,n}$.

Case 2. $\Gamma_{tail} \notin I$. In this case $\mathcal{U} \subset \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$. Since this stack is proper (char $k \neq 2, 3$), we necessarily have $\mathcal{U} = \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$.

Case 3. $\Gamma_{tail} \in I$ and $\Gamma_{\mathrm{cusp}} \in I$. In this case, by Technique 2 we have $\Gamma_{\mathrm{rat.ell}} \in I$. Indeed, consider the family over $\operatorname{Spec} R$ whose generic fiber is smooth and whose special fiber has dual graph $\Gamma_{\mathrm{cusp}} \in I$. After a finite base change we can replace the special fiber with an elliptic tail. Completing the S-diagram over the closed point shows that $\Gamma_{\mathrm{rat.ell}} \in I$.

We can now use Technique 3 as in Figure 3.17. The degeneration over $\operatorname{Spec} R$ is given by $\Gamma^g_{\mathrm{ell}} \to \Gamma_{\mathrm{ell}}$, and the very close degeneration is given by $\Gamma_{\mathrm{rat.ell}} \to \Gamma_{\mathrm{ell}}$. Since these dual graphs lie in I (by Lemma 3.8.4), we deduce that $\Xi^1_{\mathrm{rat}} \in I$. Let now $1 \leq h \leq g$ be the maximal integer such that $\Xi^h_{\mathrm{rat}} \in I$, and assume, for a contradiction,

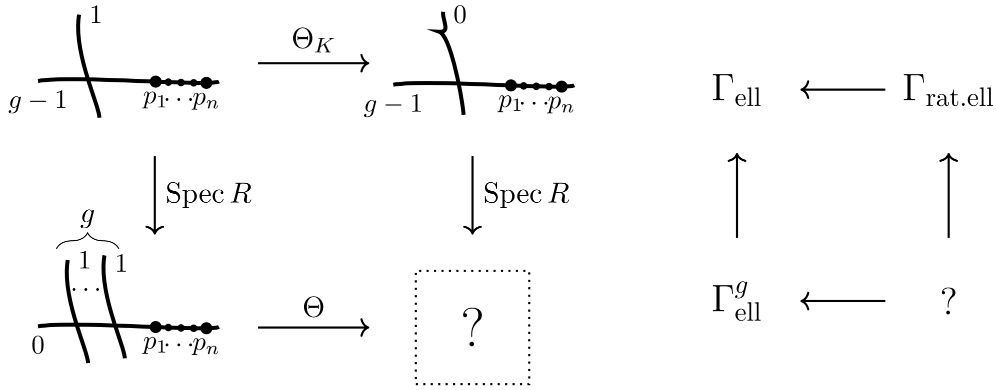


Figure 3.17. We picture with arrows the specializations (over Spec R, Θ_K and Θ), we include both the combinatorial picture and the maps at the level of dual graphs.

that $h \neq g$. We can use Technique 2 with two isotrivial families over Spec R given by two contractions of graphs $\Xi^h_{\text{rat}} \to \Gamma^g_{\text{ell}}$ that correspond to smoothing cusps on different components of the two special fibers. We obtain a diagram as Figure 3.18. Then the central fiber is Ξ^l_{rat} with $l > h$, a contradiction. Hence, $\Xi^g_{\text{rat}} \in I$.

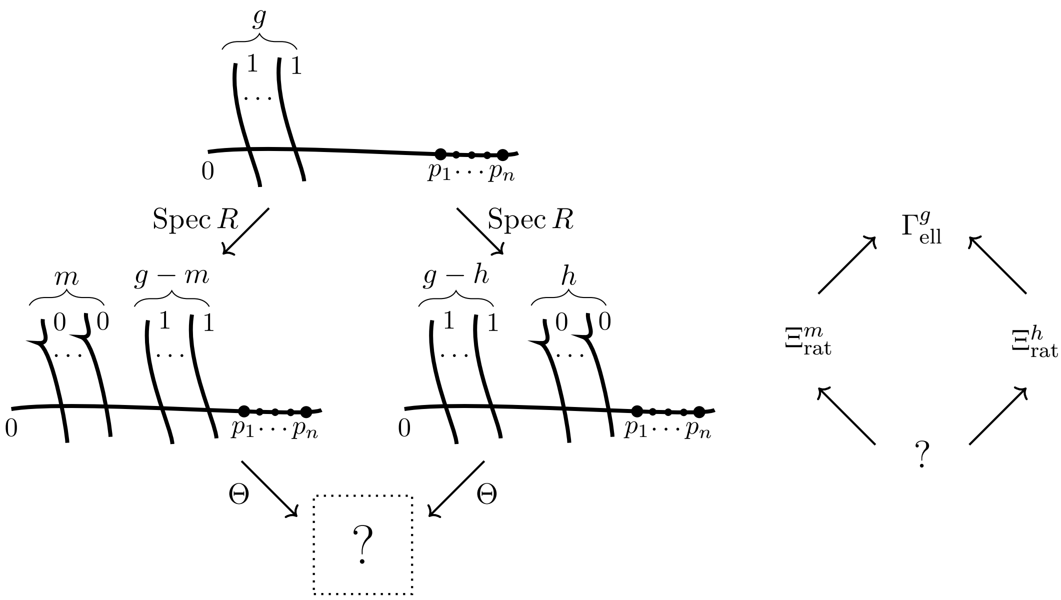


Figure 3.18. On the right: the combinatorial picture, where the maps are specializations (over Spec R and Θ). On the left: the corresponding dual graphs.

Consider now a dual graph $\Gamma \in \mathrm{EGr}^2_{g,n}$ that has h rational elliptic tails, no attached nodal elliptic tails and no cuspidal singularities lying outside an attached rational elliptic tail. Apply Technique 1 to the contraction $\Gamma \to \Xi^h_{\text{rat}}$; we obtain $\Gamma \in I$. By openness, $\mathrm{EGr}^2_{g,n} \subset I$, hence $\mathcal{U} = \overline{\mathcal{M}}^{\text{wps}}_{g,n}$. This stack indeed has a proper and separated good moduli space.

It remains to treat the cases $g = 0$ and $g + n < 3$. The case $g = 0$ is immediate: no

A_2 singularities can appear for genus reasons. Thus, only two cases remain:

- $(g, n) = (1, 1)$. The set $\mathrm{EGr}^2_{1,1}$ contains a unique stratum Γ' with a cusp (a rational curve with a marked point and a cusp). If $\Gamma' \in I$, then by openness $\mathcal{U} = \overline{\mathcal{M}}^{\mathrm{wps}}_{1,1}$. Otherwise, $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}$ and hence $\mathcal{U} = \overline{\mathcal{M}}_{g,n}$.

- $(g, n) = (2, 0)$. If $\mathcal{U}$ does not contain cuspidal singularities, then as above $\mathcal{U} = \overline{\mathcal{M}}_2$. Assume that $\mathcal{U}$ contains a stratum with a rational tail and consider a nodal stable family over a DVR whose generic fiber is smooth and whose special fiber is obtained by attaching two 1-pointed elliptic curves with different moduli. Apply Technique 2 with two families obtained by replacing one of the two elliptic tails with a rational cuspidal tail: this implies that the stratum with two attached rational elliptic tails is contained in $\mathcal{U}$, and then by openness $\mathcal{U} = \overline{\mathcal{M}}^{\mathrm{wps}}_2$. It remains the case when $\mathcal{U}$ has no curves with attached rational cuspidal elliptic tails but contains curves with cusps. Arguing as before with Technique 2, $\mathcal{U}$ does not contain curves with elliptic tails. Hence $\mathcal{U} \subset \overline{\mathcal{M}}^{\mathrm{ps}}_2$, and as before equality holds, proving the claim.

□

We now show that nodally attached elliptic tails and tacnodes cannot coexist in an open substack of $\widetilde{\mathcal{M}}^3_{g,n}$ that admits a proper good moduli space.

Proposition 3.8.6. *Let (g, n) be a hyperbolic pair with $g \geq 1$ and $g + n \geq 3$. Consider an open substack*

$$\mathcal{U} = \bigsqcup_{\alpha \in I} S_\alpha \subset \widetilde{\mathcal{M}}^3_{g,n}$$

that admits a proper good moduli space. Assume that $\Gamma_{\mathrm{ell}} \in I$. Then necessarily $\mathcal{U} \subset \overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}$.

Proof. The claim is clear if $\mathcal{U}$ does not contain strata parametrizing curves with a tacnodal singularity. Assume, for a contradiction, that this is not the case. Then, by openness of $\mathcal{U}$, it contains at least one stratum with dual graph $\Gamma^{h,J}_{\mathrm{tac}}$ for some $0 \leq h \leq g$ and $J \subset \{1, \ldots, n\}$ (as defined in Figure 3.19) or Γ_{tac}.

$\Gamma^{h,J}_{\mathrm{tac}}$

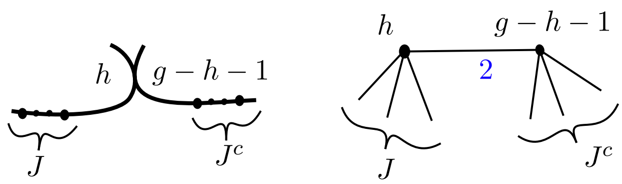


Figure 3.19. On the right, the dual graphs $\Gamma^{h,J}_{\mathrm{tac}}$; on the left, combinatorial picture of the curves. Where in the curly brackets there are marked points corresponding to $J \subset [n]$ or $J^c = [n] \setminus J$.

Case 1. Assume $\Gamma^{h,J}_{\mathrm{tac}} \in I$. Consider a family of curves over $\operatorname{Spec} R$ as in Figure 3.20, where the generic fiber has dual graph Γ^g_{ell} (which belongs to I by Lemma 3.8.4). Applying Technique 1 to this contraction, we find that the graph of the special fiber also belongs to I. Define $\Gamma^{h,J}_{\mathrm{ell}}$, $\Gamma^{h,J}_{\mathrm{bridge}}$, and $\Gamma^{h,J}_{\mathrm{tac.bridge}}$ as in Figure 3.21. By openness

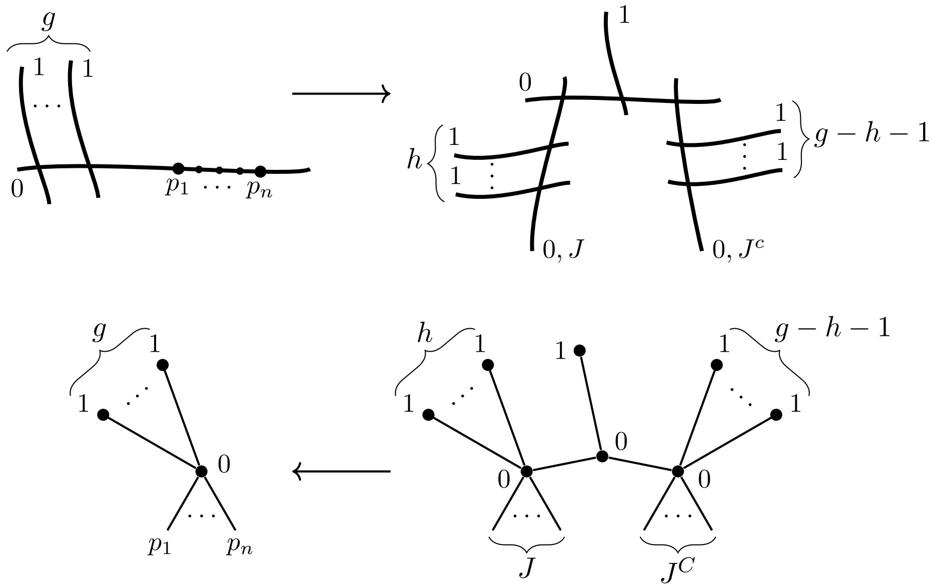


Figure 3.20. Family over $\operatorname{Spec} R$, on which we apply Technique 1. Where in the curly bracket there are marked points corresponding to $J \subset [n]$ or $J^c = [n] \setminus J$.

of $\mathcal{U}$, the graphs $\Gamma^{h,J}_{\text{ell}}$ and $\Gamma^{h,J}_{\text{bridge}}$ lies in I. Given that $\Gamma^{h,J}_{\text{bridge}}, \Gamma^{h,J}_{\text{tac}} \in I$, we can argue via Technique 2 analogously to Example 3.8.3, which shows that $\Gamma^{h,J}_{\text{tac.bridge}} \in I$. We are now ready to apply Technique 3. With the same notation, set $f\colon \operatorname{Spec} R \to \mathcal{U}$ to be a family corresponding to the contraction of graphs $\Gamma^{h,J}_{\text{ell}} \to \Gamma^{h,J}_{\text{bridge}}$, and let $v\colon \Theta_{k(\eta)} \to \mathcal{U}$ be a very close degeneration whose generic fiber is $f_{|k(\eta)}$ and whose closed fibers have dual graph $\Gamma^{h,J}_{\text{tac.bridge}}$, we have a diagram as in Figure 3.22. This yields a map $\Theta \setminus \{0\} \to \mathcal{U}$, which extends to $\Theta \to \mathcal{U}$ because $\mathcal{U}$ has a good moduli space by hypothesis. By Proposition 3.1.8, the attaching nodes of the elliptic bridge extend to two nodal sections over the whole Θ. Normalizing along these two sections and considering the 2-marked genus-1 component, we see that the curve over $0 \in \Theta$ is forced to have both an elliptic tail and a tacnode. A direct inspection of dual graphs shows that any such a curve has non-ample log dualizing sheaf, a contradiction.

Case 2. Assume $\Gamma_{\text{tac}} \in I$. Consider the specialization over a DVR as Figure 3.23. By hypothesis the generic fiber lies in $\mathcal{U}$; applying Technique 1 we have two possibilities. If $(g, n) \neq (3, 0)$, then the special fiber necessarily lies in $\mathcal{U}$. We can then apply Technique 2 as in Example 3.8.3; the claim follows by repeating verbatim the previous strategy.

It remains to prove the claim for $(g, n) = (3, 0)$. In this case, Technique 1 prescribes that either $\Gamma_{\text{narw}} \in I$ (defined in the picture) or $\Gamma^{ns}_{\text{ell}} \in I$.

If $\Gamma^{ns}_{\text{ell}} \in I$, then the proof follows as in Case 1. Assume now that $\Gamma_{\text{narw}} \in I$. For the rest of the proof, fix two 2-pointed smooth elliptic curves over k, (E, q_1, q_2) and (E', q'_1, q'_2), which we assume to be non-isomorphic as pointed curves over $\overline{k}$.

Construct a family of curves over a DVR R, $\mathcal{C} \in \widetilde{\mathcal{M}}^3_{3,0}(R)$, such that $\mathcal{C}_\eta$ is smooth

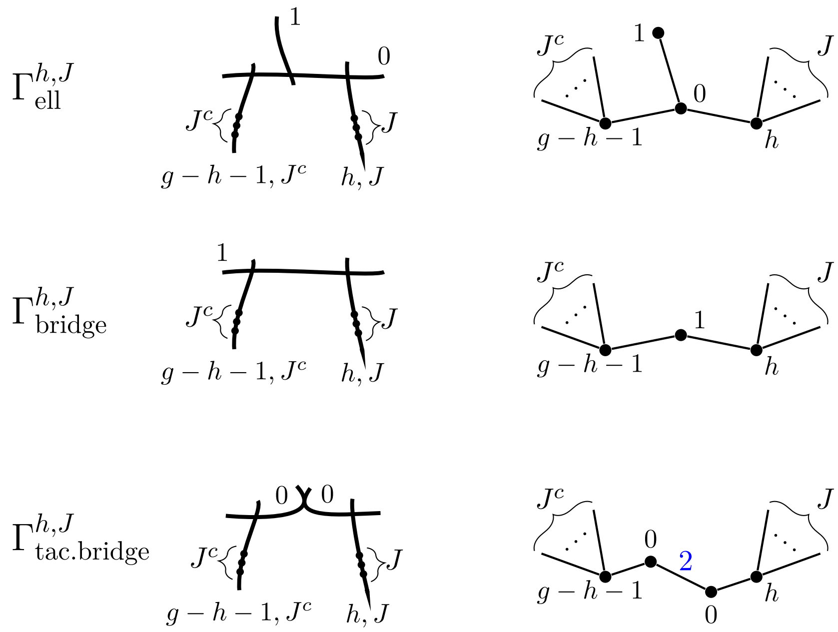


Figure 3.21. Here h, J near an irreducible component indicate its genus h and the set of marked points indexed by J. Again in the curly brackets there are marked points corresponding to $J \subset [n]$ or $J^c = [n] \setminus J$.

and $\mathcal{C}_0$ is obtained by gluing (E, q_1, q_2) and (E', q_1', q_2') along the marked points:

$$\mathcal{C}_0 = (E, q_1, q_2) \cup_{q_1 \sim q_1', q_2 \sim q_2'} (E', q_1', q_2').$$

Assume moreover that the local parameters at the two nodes of $\mathcal{C}_0$ are the same.

Apply the contraction Lemma 3.4.2 to the family $\mathcal{C}$. After possibly a finite extension of the DVR R, we can contract an attached elliptic bridge to a tacnode. In this way we construct two different families $\mathcal{C}'$, $\mathcal{C}''$, which we may assume are defined over R, such that:

- $\mathcal{C}'$ and $\mathcal{C}''$ have the same generic fiber as $\mathcal{C}$, and every singularity in the central fiber is rational;
- $\mathcal{C}_0'$ and $\mathcal{C}_0''$ have dual graph Γ_{tac}, hence are families in $\mathcal{U}(R)$;
- $\mathcal{C}_0'$ has an A_3/A_3-attached elliptic bridge isomorphic to (E, q_1, q_2);
- $\mathcal{C}_0''$ has an A_3/A_3-attached elliptic bridge isomorphic to (E', q_1', q_2').

Apply Technique 2 to the families $\mathcal{C}'$ and $\mathcal{C}''$, and let $D \in \mathcal{U}(k)$ be the central fiber of the S-diagram. We show that D has two tacnodal singularities.

By semicontinuity of the δ-invariant, D has at least one tacnode, which is necessarily non-separating. Assume, for a contradiction, that this is the only tacnode on D. In this setting, it is clear that the tacnodal singularity of D is the limit of the tacnodal singularities on both $\mathcal{C}_0'$ and $\mathcal{C}_0''$. If we restrict the family to

$$\overline{ST}_R \setminus \eta = \left[\mathrm{Spec}\left(\frac{k[x,y]}{xy}\right) /_{(1,-1)} \mathbb{G}_m\right] \subset \overline{ST}_R,$$

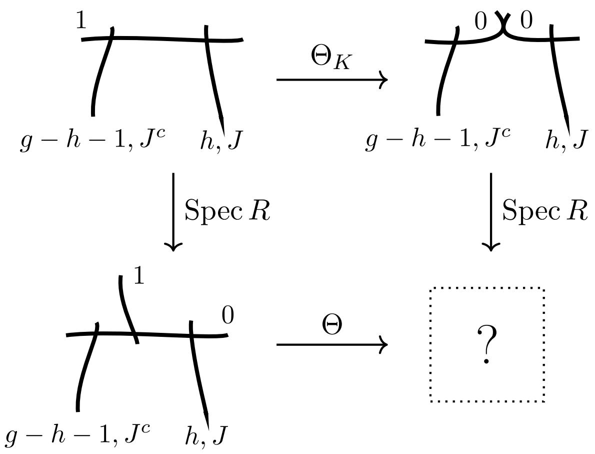


Figure 3.22. Family over $\Theta_R \setminus \{0\}$, on which we apply Technique 3.

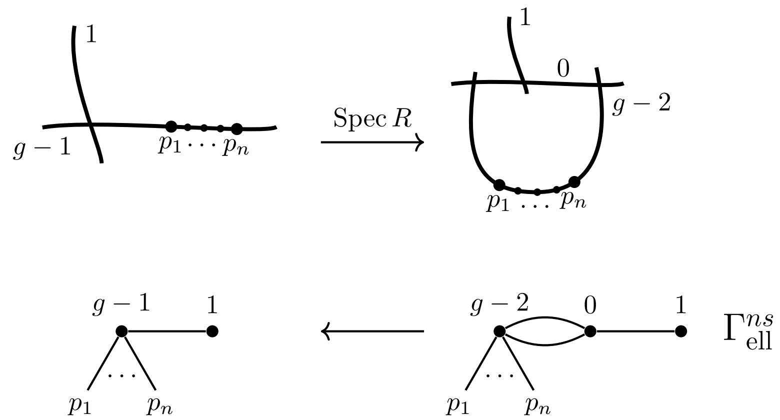


Figure 3.23. Specialization over Spec R. We define the dual graph Γ^{ns}_{ell} the rightmost graph.

we can perform a relative normalization along the tacnodal section and, after stabilization, obtain a 2-marked elliptic curve over $\overline{ST}_R \setminus \eta$. Let (D', d'_1, d'_2) denote the central fiber.

One such family satisfies that over the two open points of $\overline{ST}_R \setminus \eta$ the curves are isomorphic to (E, q_1, q_2) and (E', q'_1, q'_2). By Proposition 3.5.23 and the assumption that there is a unique tacnode on D, it follows that (D', d'_1, d'_2) is a k-point in $\overline{\mathcal{M}}^{\text{ps}}_{1,2}$. The properness of the latter forces $(E, q_1, q_2) \simeq (E', q'_1, q'_2)$ or $(E, q_2, q_1) \simeq (E', q'_1, q'_2)$. For elliptic curves over an algebraically closed field we have $(E, q_2, q_1) \simeq (E, q_1, q_2)$, so we obtain a contradiction because the two 2-marked elliptic curves chosen were not isomorphic. This shows that D has two (necessarily non-separating) tacnodes; hence $\Xi \in I$, where the graph is defined in the image above.

We are now ready to apply Technique 3, similarly to Case 1. With the same notation, set $f\colon \operatorname{Spec} R \to \mathcal{U}$ to be a family corresponding to the contraction of graphs $\Gamma_{\text{narw}} \to \Gamma_{\text{tac}}$, and let $v\colon \Theta_{k(\eta)} \to \mathcal{U}$ be a very close degeneration whose generic fiber is $f_{|k(\eta)}$ and whose closed fibers have dual graph Ξ, cf. Figure 3.25. This yields a

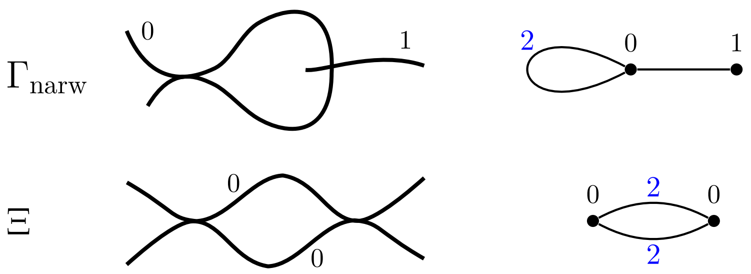


Figure 3.24. On the right, the dual graphs Ξ and Γ_{narw}; on the left, combinatorial picture of the curves.

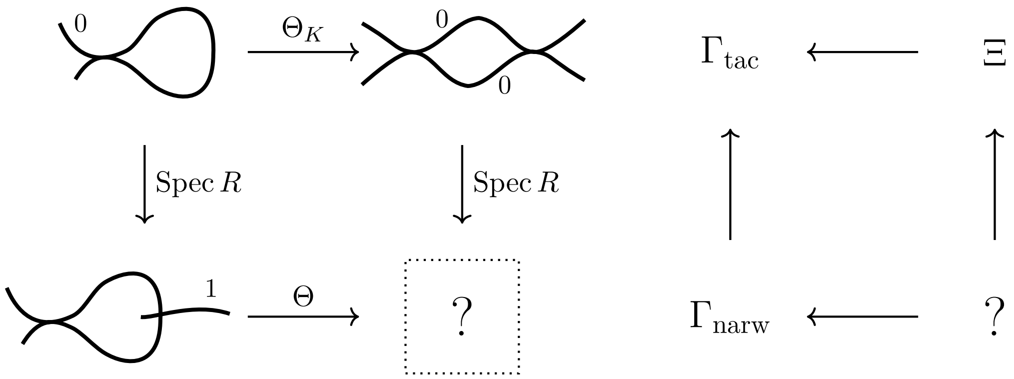


Figure 3.25. Θ-diagram, on the left we picture the specializations at the level of strata.

map $\Theta \setminus \{0\} \to \mathcal{U}$, which extends to $\Theta \to \mathcal{U}$ because $\mathcal{U}$ has a good moduli space by hypothesis. By Proposition 3.1.8, the central fiber has a nodally attached elliptic tail and two tacnodes. This is impossible for a curve lying in $\widetilde{\mathcal{M}}^3_{3,0}$, yielding the final contradiction. $\square$

We need a lemma for $(g, n) = (2, 0)$, which case is excluded in the proposition above. The following lemma answers the open question in [CTV21, Rmk. 3.11], showing that $\overline{\mathcal{M}}_2^{\text{irr}}$ does not admit a good moduli space.

Lemma 3.8.7. *Let $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{2,0}$ be an open substack that admits a proper good moduli space. Then it does not contain points corresponding to curves with A_3 singularities.*

Proof. Let R be a DVR over a field k. Consider the trivial family $(\mathcal{C}, p_1, p_2)$ over R given by the base change of $(\mathbb{P}^1, 0, \infty)$ over k. We now use Proposition 3.1.15 to glue the generic fiber of the family along the two marked points to obtain a tacnode. We use an isomorphism $K := \operatorname{Frac}(R) = T^1_{p_1}\mathcal{C}_\eta \simeq T^1_{p_2}\mathcal{C}_\eta$ that can be read as the multiplication by the uniformizer t of R on a basis of $T^1_{p_1}\mathcal{C}$ and $T^1_{p_2}\mathcal{C}$. Thanks to the loc. cit. Proposition, this does not extend to an isomorphism of the tangent spaces over $\operatorname{Spec} R$ (even after DVR extensions); cf. [HH13, Prop. 4.1].

If there is a curve with a tacnodal singularity in $\mathcal{U}$, then the curve over K obtained by the gluing above necessarily lies in $\mathcal{U}$. Since the latter is universally closed, after a base change, the gluing constructed above has a limit. The only possibility is that the limit is given by a family $\mathcal{D}$ over a DVR whose special fiber consists of two rational components meeting in a node and a tacnode. Define the dual graphs Γ', Γ_{cusp}, and Γ_{nod} as Figure 3.26. Since $\mathcal{U}$ contains the stratum with a unique

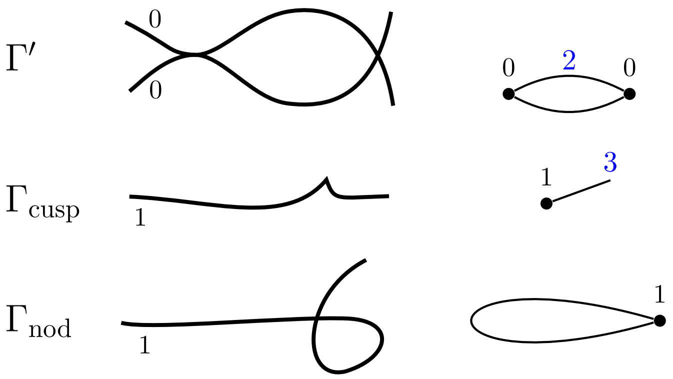


Figure 3.26. On the right, the dual graphs Γ', Γ_{cusp} and Γ_{nod}; on the left, combinatorial picture of the curves. We recall that near the component we specify the geometric genus.

tacnodal singularity, by openness it contains the strata given by Γ_{cusp} and Γ_{nod}. We can now apply Technique 3, where the family over the DVR $\operatorname{Spec} R$ is given by the contraction of dual graphs $\Gamma_{\text{cusp}} \to \Gamma_{\text{nod}}$, and we have a very close degeneration over Θ_K given at the level of graphs by $\Gamma' \to \Gamma_{\text{nod}}$, as in Figure 3.27. Then the fiber over 0 necessarily has dual graph Γ', but by Theorem 3.5.23 this does not admit a very close degeneration from a curve with dual graph Γ_{cusp}. $\square$

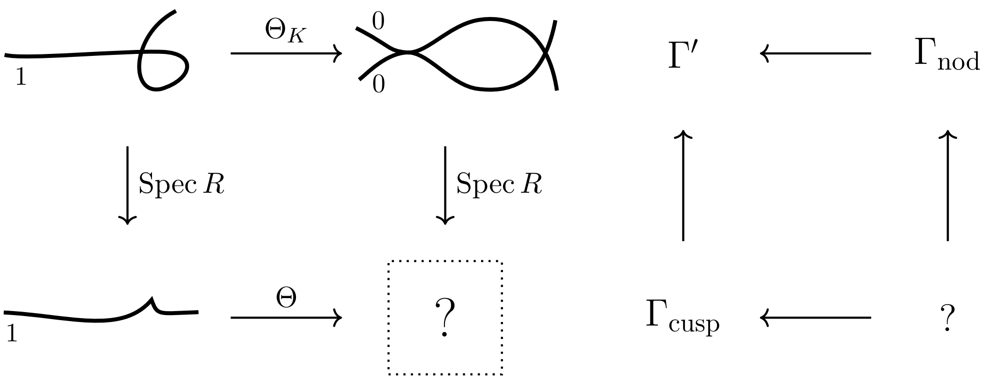


Figure 3.27. Θ-diagram, on the left we picture the specializations at the level of strata.

In the proof of Lemma 3.8.7, we answer the following open question posed in [CTV21, Rmk. 3.11].

Remark 3.8.8. The stack $\overline{\mathcal{M}}_2^{\text{irr}}$ (as defined in [CTV21]) is not Θ-complete, hence it does not admit a good moduli space. This is a consequence of Technique 3 applied to the diagram in Figure 3.27.

We are finally ready to prove the main result of this section.

Proof of Theorem 3.8.2. Assume first that $g \geq 1$ and $g+n \geq 3$.

If $\Gamma_{\text{ell}} \in I$, then the claim follows from Proposition 3.8.6 and Proposition 3.8.5. If $\Gamma_{\text{ell}} \notin I$, then the claim is equivalent to proving that it is impossible to have both $\Gamma_{\text{A3/ell}}$ and Γ_{tac} lying in I (as defined in Figure 3.28).

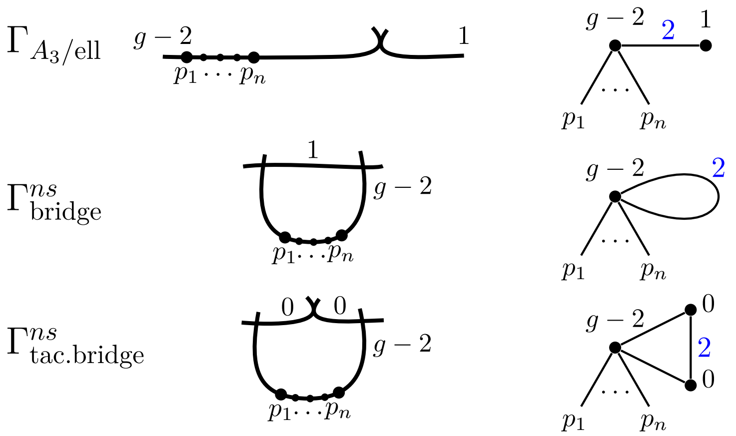


Figure 3.28. On the right, the dual graphs $\Gamma_{\text{A3/ell}}$, $\Gamma^{ns}_{\text{bridge}}$, and $\Gamma^{ns}_{\text{tac.bridge}}$; on the left, combinatorial picture of the curves.

Assume, by contradiction, that $\Gamma_{\text{A3/ell}}$ and Γ_{tac} lie in I. By openness of $\mathcal{U}$, necessarily $\Gamma^{ns}_{\text{bridge}} \in I$. We have two cases.

Case 1. $(g,n) \neq (3,0)$. Applying Example 3.8.3 verbatim, we obtain $\Gamma^{ns}_{\text{tac.bridge}} \in I$. We can now use Technique 3 as in Figure 3.29. The degeneration over Spec R is given by $\Gamma_{\text{A3/ell}} \to \Gamma^{ns}_{\text{bridge}}$, and the very close degeneration is given by $\Gamma^{ns}_{\text{tac.bridge}} \to \Gamma^{ns}_{\text{bridge}}$.

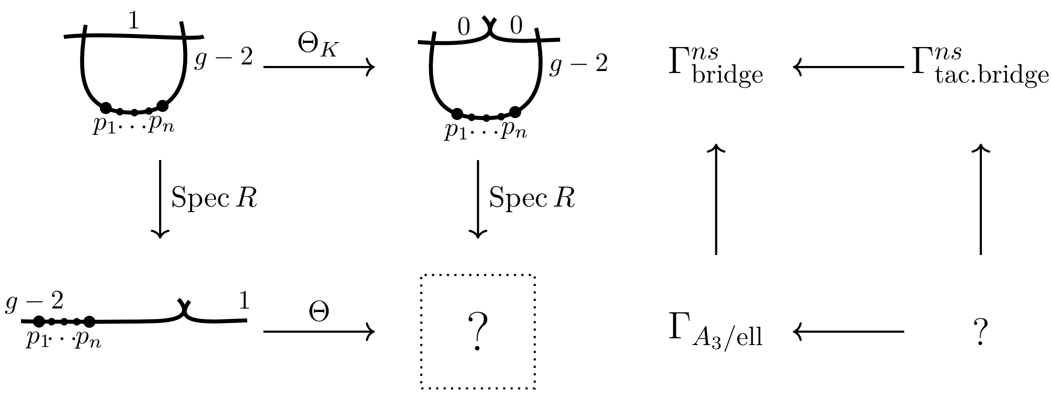


Figure 3.29. Θ-diagram, on the left we picture the specializations at the level of strata.

Then a singularity on the central fiber limits over $0 \in \Theta_R$ to either a separating tacnode or to a lonely nodal singularity (thanks to Proposition 3.1.8), which is impossible; hence we obtain a contradiction.

Case 2. $(g,n) = (3,0)$. We again apply the strategy of Example 3.8.3. There are two possible outcomes: either $\Gamma^{ns}_{\text{tac.bridge}} \in I$ or there exists $\Gamma \in I$ and a very close degeneration given by contraction of graphs $\Gamma \to \Gamma^{ns}_{\text{bridge}}$. Notice that in such case the stratum defined by Γ is in the closure of the one defined py Ξ (defined in Proposition 3.8.6). In the first case we conclude as in Case 1. Otherwise, argue via Technique 3. The degeneration over $\operatorname{Spec} R$ is given by $\Gamma_{\mathrm{A3/ell}} \to \Gamma^{ns}_{\mathrm{bridge}}$, and the very close degeneration is given by $\Gamma \to \Gamma^{ns}_{\mathrm{bridge}}$, as in Figure 3.30. Then the

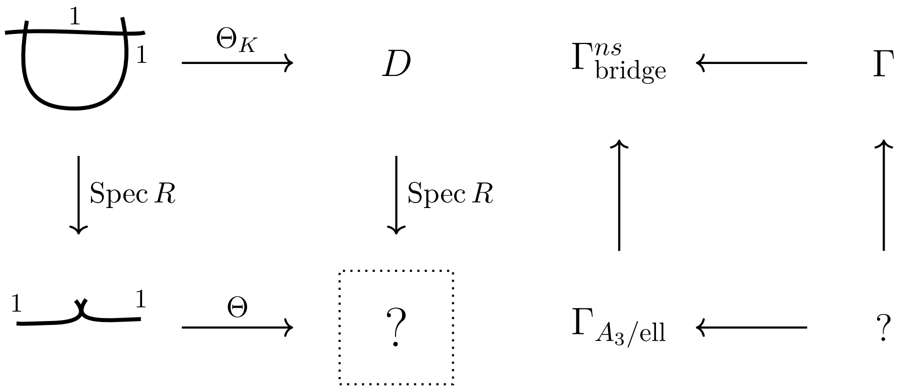


Figure 3.30. Θ-diagram, where we write D for a curve with dual graph Γ.

central fiber is a stable curve with two tacnodes whose dual graph is a degeneration of $\Gamma_{\mathrm{A3/ell}}$; by Proposition 3.1.8 we obtain a contradiction.

It remains to treat the cases $g = 0$ and $g + n < 3$. The case $g = 0$ is immediate: no A_2 or A_3 singularities can appear for genus reasons. Thus only two cases remain:

- $(g, n) = (1, 1)$. The set $\mathrm{EGr}^3_{1,1}$ does not contain a dual graph with a tacnode, hence the claim follows from Proposition 3.8.5.
- $(g, n) = (2, 0)$. By Lemma 3.8.7, necessarily $\mathcal{U} \subset \overline{\mathcal{M}}_2^{\mathrm{wps}}$. The claim then follows from Proposition 3.8.5.

$\square$

For small (g, n) the previous propositions give a complete classification as follows.

Proposition 3.8.9. *Let (g, n) be a hyperbolic pair. Consider an open $\mathcal{U} \subset \widetilde{\mathcal{M}}^3_{g,n}$ that admits a good moduli space. Then in the following cases we have:*

- $(g, n) = (1, 1)$. *Then $\mathcal{U}$ is either $\overline{\mathcal{M}}_{1,1}$ or $\overline{\mathcal{M}}_{1,1}^{\mathrm{wps}}$.*
- $(g, n) = (1, 2)$. *Then $\mathcal{U}$ is either $\overline{\mathcal{M}}_{1,2}$, $\overline{\mathcal{M}}_{1,2}^{\mathrm{ps}}$, $\overline{\mathcal{M}}_{1,2}^{\mathrm{wps}}$, or $\overline{\mathcal{M}}_{1,2}(\frac{7}{10})$.*
- $(g, n) = (2, 0)$. *Then $\mathcal{U}$ is either $\overline{\mathcal{M}}_2$, $\overline{\mathcal{M}}_2^{\mathrm{ps}}$, or $\overline{\mathcal{M}}_2^{\mathrm{wps}}$.*

Proof. The case $(g, n) = (1, 1)$ follows from Proposition 3.8.5 and the fact that ampleness of the log dualizing sheaf excludes tacnodes.

The case $(g, n) = (1, 2)$ follows from Proposition 3.8.5 and the fact that there is only one stratum with a tacnodal singularity. This implies that if $\mathcal{U}$ contains a stratum with a tacnode, then $\overline{\mathcal{M}}_{1,2}(\frac{7}{10}) \subset \mathcal{U}$, and hence equality holds.

The case $(g, n) = (2, 0)$ follows from Lemma 3.8.7 and Theorem 3.8.2. $\square$

Chapter 4

The Stacks $\overline{\mathcal{M}}^S_{g,n}$

This chapter is devoted to the construction of the stacks $\overline{\mathcal{M}}^S_{g,n}$ and their good moduli spaces. Throughout, we require (g,n) to be a hyperbolic pair (i.e. such that $2g-2+n>0$) with $g \geq 2$ and $(g,n) \neq (2,0)$, unless otherwise stated. All possible compactifications for $(2,0)$ have already been classified in Proposition 3.8.9, while the case $(1,n)$ is addressed in Section 4.3.1. The case $g=0$ is trivial, in this case we can assume $\Sigma_{0,n}$ to be the unique nonempty fan in the zero dimensional vector space.

We work over an algebraically closed field k such that $\operatorname{char} k$ is either zero or sufficiently large with respect to (g,n), cf. Definition 4.1.1. All results in this chapter are stated under this assumption, unless explicitly noted otherwise.

4.1 Semistability on $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$

We apply the results established in Chapters 2 and 3 to the stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$. The main goal of the following two sections is to describe the wall-crossing decomposition of $\mathbb{E}(\overline{\mathcal{M}}_{g,n}(\frac{7}{10}))$, starting from a description of the pairing introduced in Definition 2.1.38 using the modular interpretation of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$.

Definition 4.1.1 (Characteristic big enough with respect to (g,n))**.** Given an hyperbolic pair (g,n), we say that the base field k has *characteristic big enough with respect to* (g,n), and we write

$$\operatorname{char}(k) \gg (g,n),$$

either if $\operatorname{char}(k)=0$ or if $\operatorname{char}(k)>0$ and does not divide the order of the étale group scheme of connected components of the automorphism group schemes of every k-point of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$.

Thanks to [CTV23, Thm. 3.19 and Remark 3.20] and [Per23, Thm. 2.2], we have the following fact.

Fact 4.1.2. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$, and let k be an algebraically closed field with* $\mathrm{char} \gg (g,n)$*. Then the stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ over k is a smooth, connected, geometric stack that admits a good moduli space.*

Since $\overline{\mathrm{M}}_{g,n}$ is irreducible and, for $g \geq 2$, every smooth curve admits no nontrivial very close degeneration (cf. Theorem 3.5.23), we obtain the following.

Remark 4.1.3. For any hyperbolic pair (g,n) with $g \geq 2$, we have

$$\Omega\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right) = \mathbb{E}\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right).$$

The Picard group of the stack is described in [CTV23, Corollary 3.29], as we now briefly recall. Given a hyperbolic pair (g,n), define the set

$$T_{g,n} := \{\mathrm{irr}\} \cup \{(\tau, I) : 0 \leq \tau \leq g,\ I \subseteq [n] := \{1, \ldots, n\}\} \setminus \{(0,\emptyset), (g,[n])\}/\sim \quad (4.1)$$

where $\sim$ is the equivalence relation such that irr is equivalent only to itself, and $(\tau, I) \sim (\tau', I')$ if and only if either $(\tau, I) = (\tau', I')$ or $(\tau', I') = (g - \tau, I^c)$, where $I^c = [n] \setminus I$. We denote the class of (τ, I) in $T_{g,n}$ by $[\tau, I]$ and the class of irr in $T_{g,n}$ again by irr.

Proposition 4.1.4 ([CTV23, Fact 3.28, Corollary 3.29])**.** *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Then every $\mathbb{Q}$-line bundle $L \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\frac{7}{10})\right)$ can be written uniquely as*

$$L = a\lambda + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{[i,I] \in T_{g,n} \setminus \{[1,\emptyset], \mathrm{irr}\}} b_{i,I}\, \delta_{i,I} \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right),$$

where we set $\delta_{0,\{i\}} = -\psi_i$.

Following the notation of [CTV23, Sec. 3.4], we set

$$\begin{cases} \delta := \delta_{\mathrm{irr}} + \displaystyle\sum_{\substack{[i,I] \in T_{g,n} \setminus \{\mathrm{irr}\} \\ |I| \geq 2 \text{ if } i=0}} \delta_{i,I}, \\ \psi := \displaystyle\sum_{i=1}^{n} \psi_i, \\ \widehat{\delta} := \delta - \psi = \delta_{\mathrm{irr}} + \displaystyle\sum_{[i,I] \in T_{g,n} \setminus \{\mathrm{irr}\}} \delta_{i,I}. \end{cases}$$

We now compute the weight of a one-parameter subgroup using the classification of curves in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ that have a positive-dimensional automorphism group, given in [CTV23, Sec. 3.1], adopting the same notation.

Proposition 4.1.5 (Weights for attached rosaries)**.** *Let L be a $\mathbb{Q}$-line bundle on $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ written as in Proposition 4.1.4.*

1. *Let $(R, q_1, q_2) \to (C, \{p_i\})$ be an attached rosary of length r, and consider the one-parameter subgroup $\rho_R : \mathbb{G}_m \xrightarrow{\sim} \mathrm{Aut}(R, q_1, q_2)^o \subset \mathrm{Aut}(C, \{p_i\})$ normalized*

so that $\mathrm{wt}_{\rho_R}(T_{q_1}(R)) = 1$. *Then we have*

$$\langle L, \rho_R \rangle = \begin{cases} 0 & \textit{if } 2 \nmid r \textit{ and } \mathrm{Supp}(R, q_1, q_2) = \{\mathrm{irr}\}, \\ a + 10b_{\mathrm{irr}} & \textit{if } 2 \mid r \textit{ and } \mathrm{Supp}(R, q_1, q_2) = \{\mathrm{irr}\}, \\ \sum_{i=0}^{r-2} (-1)^i f_{\tau+i,I}(L) & \textit{if } \mathrm{Supp}(R, q_1, q_2) = \{[\tau+1, I], \ldots, [\tau+r-1, I]\}, \end{cases}$$

where

$$\begin{aligned} f_{\tau,I} \colon \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right)\right) &\longrightarrow \mathbb{Q}, \\ L &\longmapsto a + 12b_{\mathrm{irr}} - b_{\tau,I} - b_{\tau+1,I}. \end{aligned}$$

2. *Let* $R \in \overline{\mathcal{M}}_{r+1,0}(\frac{7}{10})(k)$ *be a closed rosary of even length, and consider the one-parameter subgroup* $\rho_R : \mathbb{G}_m \xrightarrow{\cong} \mathrm{Aut}(R)^o$. *Then*

$$\langle L, \rho_R \rangle = 0.$$

Proof. The only case not contained in Lemma 7.8 of [CTV23] is the formula for rosaries of length greater than 3 that are not of type $\{\mathrm{irr}\}$. The computation carried out in [CTV23] for length 3 applies directly to longer rosaries. $\square$

By linearity of the pairing, the weight can then be computed for any one-parameter subgroup of $\mathrm{Aut}(C, \{p_i\})$.

4.1.1 The vector spaces $\mathbb{E}_{g,n}$

Here and in the next sections, we denote the vector space

$$\mathbb{E}_{g,n} \coloneqq \mathbb{E}\left(\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right)\right),$$

we will write $\mathbb{E}_g$ when $n = 0$.

Proposition 4.1.6. *Let* (g, n) *be a hyperbolic pair with* $g \geq 2$ *and* $(g, n) \neq (2, 0)$. *With the notation of Proposition 4.1.4, a* $\mathbb{Q}$*-line bundle* $L \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\frac{7}{10})\right)$ *descends to the good moduli space* $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ *if and only if*

$$\begin{cases} a + 10b_{\mathrm{irr}} = 0, \\ a + 12b_{\mathrm{irr}} - b_{\tau,I} - b_{\tau+1,I} = 0, \textit{for any } [\tau, I] \in T_{g,n} \setminus \{[1, \emptyset], \mathrm{irr}\}. \end{cases} \tag{4.2}$$

Proof. Follows from Proposition 4.1.5 and Theorem 2.1.28. $\square$

We now give an explicit description of $\mathbb{E}_{g,n}$. Define

$$[n] \coloneqq \{1, \ldots, n\}, \qquad \mathcal{P} \coloneqq \mathcal{P}([n])/\sim, \tag{4.3}$$

where $\mathcal{P}([n])$ is the power set and $\sim$ is the equivalence relation $I \sim I'$ if $I = I'$ or $I^c = I'$. For $n > 0$, the solution space has dimension $2^{n-1}+1$, generated by

$$\begin{cases} \lambda - \frac{1}{10}\left(\delta_{\mathrm{irr}} + \displaystyle\sum_{[i,I]\in T_{g,n}\setminus\{[1,\emptyset],\mathrm{irr}\}} \delta_{i,I}\right) = \lambda - \frac{1}{10}\widehat{\delta}, \\ \displaystyle\sum_{i=0}^{g}(-1)^i\delta_{i,I} \qquad \text{for any } \emptyset \subsetneq I \subsetneq [n], \\ \displaystyle\sum_{i=2}^{g}(-1)^i\delta_{i,\emptyset}. \end{cases}$$

Therefore,

$$\begin{aligned} \mathbb{E}_{g,n} &= \operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right) \Big/ \operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}\left(\tfrac{7}{10}\right)\right) = \langle\overline{\delta_{\mathrm{irr}}}\rangle \oplus \\ &\oplus \langle\overline{\delta_{i,\emptyset}}\colon 2 \le i \le g\rangle \Big/ \sum_{i=2}^{g}(-1)^i\overline{\delta_{i,\emptyset}} \oplus \bigoplus_{\substack{[I]\in\mathcal{P} \\ [I]\neq[\emptyset]}} \langle\overline{\delta_{i,I}}\colon i \in [g]\rangle \Big/ \sum_{i=0}^{g}(-1)^i\overline{\delta_{i,I}}. \end{aligned}$$

For $n = 0$ and $g \ge 5$, the solution space has dimension 2 if g is even and 1 otherwise, generated by

$$\begin{cases} \lambda - \frac{1}{10}\left(\delta_{\mathrm{irr}} + \displaystyle\sum_{i=2}^{\lfloor (g+1)/2\rfloor} \delta_i\right) = \lambda - \frac{1}{10}\delta, \\ \displaystyle\sum_{i=2}^{g-2}(-1)^i\delta_i \qquad \text{if } 2 \mid g. \end{cases}$$

Hence, for $g \ge 5$ we have

$$\mathbb{E}_g = \begin{cases} \langle\overline{\delta_{\mathrm{irr}}}\rangle \oplus \langle\overline{\delta_i} : 2 \le i \le \frac{g-2}{2}\rangle \Big/ \displaystyle\sum_{i=2}^{g-2}(-1)^i\overline{\delta_i}, & \text{if } g \text{ is even}, \\ \langle\overline{\delta_{\mathrm{irr}}}\rangle \oplus \langle\overline{\delta_i} : 2 \le i \le \frac{g-1}{2}\rangle, & \text{if } g \text{ is odd}. \end{cases}$$

The remaining cases are $(g,n) \in \{(3,0),(4,0)\}$, where any tacnode is non-separating; hence

$$\operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_3\left(\tfrac{7}{10}\right)\right) = \langle\lambda - \frac{1}{10}\delta_{\mathrm{irr}}, \delta_1\rangle, \qquad \operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_4\left(\tfrac{7}{10}\right)\right) = \langle\lambda - \frac{1}{10}\delta_{\mathrm{irr}}, \delta_1, \delta_2\rangle.$$

Consequently, $\mathbb{E}_3 = \langle\overline{\delta_{\mathrm{irr}}}\rangle$ and $\mathbb{E}_4 = \langle\overline{\delta_{\mathrm{irr}}}\rangle$.

We next write a basis of $\mathbb{E}_{g,n}$ dual to the equations in Proposition 4.1.6. We write $\alpha_{i,I}$ for the dual vector to the equation corresponding to a length-two rosary of type

$\{[i-1, I], [i, I]\}$. In this notation, $\alpha_{i,I} = \alpha_{i',I'}$ if $[i, I] = [i', I']$ in $T_{g,n}$. For $n > 0$ set

$$\begin{aligned}
\alpha_{\mathrm{irr}} &\coloneqq \delta_{\mathrm{irr}} - 6 \sum_{[i,I]\in T_{g,n}\setminus\{[1,\emptyset],\mathrm{irr}\}} \delta_{i,I},\\
\alpha_{k,I} &\coloneqq (-1)^k \sum_{i=0}^{k-1} (-1)^i \delta_{i,I} \quad \text{for any } 1 \le k \le g \text{ and } \emptyset \subsetneq I \subsetneq [n],\\
\alpha_{k,\emptyset} &\coloneqq (-1)^k \sum_{i=2}^{k-1} (-1)^i \delta_{i,\emptyset} \quad \text{for any } 3 \le k \le g.
\end{aligned}$$

For $n = 0$ ($g \ge 3$), we set

$$\begin{aligned}
\alpha_{\mathrm{irr}} &\coloneqq \delta_{\mathrm{irr}} - 6 \sum_{2\le i\le \lfloor (g+1)/2\rfloor} \delta_i,\\
\alpha_k &\coloneqq (-1)^k \sum_{i=2}^{k-1} (-1)^i \delta_i \quad \text{for any } 3 \le k \le \lfloor (g+1)/2\rfloor.
\end{aligned}$$

Finally, we define a simple basis for a suitable root system to be used in the next sections. This is an ad hoc construction that will be clarified later.

Definition 4.1.7 (Root system in $\mathbb{E}_{g,n}$)**.** Let (g, n) be a hyperbolic pair with $g \ge 2$ and $(g, n) \ne (2, 0)$. Define

$$\Delta_{g,n} \coloneqq \Delta^{\mathrm{irr}}_{g,n} \cup \bigcup_{[I]\in\mathcal{P}} \Delta^{[I]}_{g,n}$$

as follows:

- if $n > 0$ and $[I] \ne [\emptyset]$, then $\Delta^{[I]}_{g,n} = (\alpha_{i,I})_{i=0,\dots,g}$ as an ordered basis;
- if $n > 0$ and $[I] = [\emptyset]$, then $\Delta^{[\emptyset]}_{g,n} = (\alpha_{i,\emptyset})_{i=3,\dots,g}$ as an ordered basis;
- if $n > 0$, then $\Delta^{\mathrm{irr}}_{g,n} = (\alpha_{\mathrm{irr}})$ as an ordered basis;
- if $n = 0$, then $\Delta^{[\emptyset]}_{g,0} = (\alpha_i)_{i=3,\dots,\lfloor (g-3)/2\rfloor}$ as an ordered basis.

By construction, in the basis $\Delta_{g,n}$ the pairing can be written as

$$\langle L, \rho_R\rangle = \begin{cases} 0 & \text{if } 2 \nmid r \text{ and } \mathrm{Supp}(R, q_1, q_2) = \{\mathrm{irr}\},\\ 10\, x_{\mathrm{irr}} & \text{if } 2 \mid r \text{ and } \mathrm{Supp}(R, q_1, q_2) = \{\mathrm{irr}\},\\ (-1)^\tau \displaystyle\sum_{i=\tau+1}^{r-1} (-1)^i x_{i,I} & \text{if } \mathrm{Supp}(R, q_1, q_2) = \{[\tau+1, I], \dots, [\tau+r-1, I]\}, \end{cases}$$

and for $n = 0$ we have (defining $x_{g-i+1} \coloneqq x_i \in \mathbb{E}^*_g$)

$$\langle L, \rho_R\rangle = \begin{cases} 0 & \text{if } 2 \nmid r \text{ and } \mathrm{Supp}(R, q_1, q_2) = \{\mathrm{irr}\},\\ 10\, x_{\mathrm{irr}} & \text{if } 2 \mid r \text{ and } \mathrm{Supp}(R, q_1, q_2) = \{\mathrm{irr}\},\\ (-1)^\tau \displaystyle\sum_{i=\tau+1}^{r-1} (-1)^i x_i & \text{if } \mathrm{Supp}(R, q_1, q_2) = \{[\tau+1, \emptyset], \dots, [\tau+r-1, \emptyset]\}. \end{cases}$$

When $n = 0$, these equations behave differently depending on the parity of g.

Remark 4.1.8. With the notation above, suppose $2 \mid g$, and fix a rosary of type $\{[\tau,\emptyset],\dots,[\tau',\emptyset]\}$ with $\tau < g/2 < \tau'$ and $\tau+\tau' \leq g$. Then

$$\langle L, \rho_R\rangle = (-1)^{\tau} \sum_{i=\tau+1}^{g-\tau'} (-1)^i x_i.$$

If $2 \nmid g$, and we fix a rosary of type $\{[\tau,\emptyset],\dots,[\tau',\emptyset]\}$ with $\tau < (g+1)/2 \leq \tau'$ and $\tau+\tau' \leq g$, then

$$\langle L, \rho_R\rangle = (-1)^{\tau} \sum_{i=\tau+1}^{g-\tau'} (-1)^i x_i \; + \; 2(-1)^{\tau} \sum_{i=g-\tau'}^{(g+1)/2} (-1)^i x_i.$$

Although the computations in this section are notationally heavy, the underlying idea is simple. We now explain a direct method to compute the product $\langle \_, \_ \rangle$.

Let $L = \sum x_{i,I}\alpha_{i,I}$ and fix a very close degeneration $\Theta \to \widetilde{\mathcal{M}}^3_{g,n}$ whose generic fiber is given by a principal chain S as in Figure 4.1. Then we can compute $\langle L, \rho_R\rangle$, where ρ_R

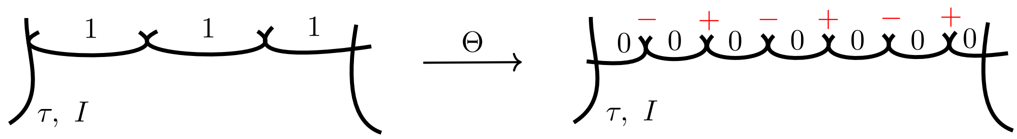


Figure 4.1. Example of very close degeneration, we picture in red the signs for the computation of the weight.

is the one-parameter subgroup induced on the central fiber of Θ, as follows. For each (i,I): if S has a tacnode in that position, assign the sign -1 to the corresponding tacnode on the central fiber; if it has a genus 1 component, assign the sign $+1$. The value of the pairing is obtained by summing, over all tacnodes on the central fiber, $+x_{i,I}$ if the sign at (i,I) is positive and $-x_{i,I}$ if it is negative. This rule will be useful in the next sections; we will often compute it for $L = \sum_{i=a}^{b} \alpha_{i,I}$: in this case, it is an alternating sum of ± 1.

4.1.2 A first description of semistable loci via principal strata

The result in this section translates the study of semistable loci into a completely combinatorial problem.

Definition 4.1.9 (Pairing with principal strata)**.** We define the pairing

$$\begin{aligned} \langle \_, \_ \rangle \colon \mathbb{S}_{g,n} \times \mathbb{E}_{g,n} &\to \mathbb{Q}, \\ (S, L) &\longmapsto \langle \rho_S, L\rangle, \end{aligned}$$

where ρ_S is as above. This pairing coincides with that of Definition 2.1.38 in the case of principal strata.

With the notation above, we have the following.

Theorem 4.1.10. *Fix $L \in \mathbb{E}_{g,n}$. Then*

$$C \in \left|\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right)\right| \text{ is } L\text{-unstable} \iff \exists S \in \mathbb{S}_{g,n} \text{ such that } C \in \overline{S} \text{ and } \langle S, L\rangle > 0.$$

This implies that

$$\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right)^{L-\mathrm{ss}} = \overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right) \setminus \bigcup_{S\in\mathbb{S}_{g,n} \,|\, \langle S,L\rangle>0} \overline{S}.$$

Roughly speaking, the proof proceeds by modifying the family over Θ in order to simplify the cocharacter that destabilizes the curve.

Proof. Let $f\colon \Theta \to \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ be a destabilizing very close degeneration of C, and consider the induced cocharacter $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(\Gamma)$, where Γ is the central fiber. After a finite base change, we can assume that the torus $\mathrm{Aut}(\Gamma)^0$ is split. Fix coordinates so that the coordinate cocharacters have support exactly on a rosary,

$$\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(\Gamma)^0 \simeq \mathbb{G}_m^k.$$

We identify a cocharacter λ with $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ and define $||\lambda|| := \sum_{i=1}^k n_i^2$. This norm is independent of the chosen coordinates. We say that $f\colon \Theta \to \mathcal{X}$ is a *minimal destabilizer* if $||\lambda||$ is minimal among all very close degenerations of C with $\langle \lambda, L\rangle > 0$.

We prove by induction on the genus that a minimal destabilizing cocharacter has norm 1. For $g = 0$ there is nothing to prove, since $\mathrm{Aut}(\Gamma)^0$ is trivial.

For the inductive step, we use the description of very close degenerations in Theorem 3.5.23. Let $f\colon \Theta \to \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ be a minimal destabilizer with $f(0) = \Gamma$. Then:

- The support of λ on Γ is connected. Otherwise, $\lambda = \lambda_1 \cdot \lambda_2$, where the factors have disjoint supports and $||\lambda_i|| < ||\lambda||$. Since $\langle \lambda_1, L\rangle + \langle \lambda_2, L\rangle = \langle \lambda, L\rangle > 0$, without loss of generality $\langle \lambda_1, L\rangle > 0$, and by Theorem 3.5.23 we obtain a destabilizing very close degeneration with norm $||\lambda_1||$, contradicting minimality.
- Let $p \in \Gamma$ be a node lying on two rational components on which λ acts with nontrivial weight. Then the induced action $\lambda \curvearrowright T^1_p\Gamma$ has negative weight. Otherwise, by Theorem 3.5.23, p is the limit of a nodal section on C, and we can perform the pointed normalization along this section. As above, we eventually obtain two connected components and cocharacters λ_1, λ_2, and we apply the inductive hypothesis to a component of strictly smaller genus. Gluing back along the section (for the other component, if present, we consider the trivial family over $\mathbb{A}^1$) yields a very close degeneration with strictly smaller norm, contradicting minimality.
- If Γ is a closed rosary, then $\mathrm{Aut}(\Gamma)^0 \simeq \mathbb{G}_m$, so there is a unique cocharacter up to positive multiple. We can construct a very close degeneration with $\langle \lambda, L\rangle > 0$ and hence $||\lambda|| = 1$. Thus, we can assume that Γ is not a closed rosary.

Assume by contradiction that $||\lambda|| > 1$. By the points above, Γ has attached rosaries (R_i, p_i, q_i) for $i = 1, \dots, l$ such that $\bigcup_i R_i$ is connected and consists of the support of the action, with $q_i = p_{i+1}$ for $i = 1, \dots, l$, and λ acts with negative weights on $T^1_{p_i}\Gamma$ for $i \geq 2$. See Figure 4.2 (possibly $p_1 = q_l$).

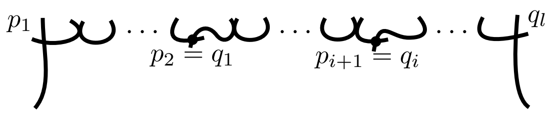


Figure 4.2. We omit specifying the genus. Every component, except the first and the last, has genus 0.

We first treat the case $p_1 \neq q_l$. Consider a point $q_i \in R_i \cup R_{i+1}$ where two consecutive rosaries meet, and consider the partial normalization $\widetilde{\Gamma} \to \Gamma$ along q_i. Since $p_1 \neq q_l$, the preimage of $\bigcup_i R_i$ in $\widetilde{\Gamma}$ is not connected, and thus consists of two connected components corresponding to subcurves Γ_1, Γ_2 on C. Then there exist cocharacters λ_1, λ_2 of $\mathrm{Aut}(\Gamma)$ such that $\lambda = \lambda_1 \cdot \lambda_2$ and whose supports are Γ_1, Γ_2, respectively (cf. example in Figure 4.3). As above, $\langle \lambda, L \rangle > 0$, hence without loss of generality we may assume $\langle \lambda_1, L \rangle > 0$.

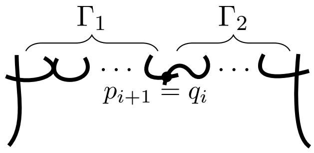


Figure 4.3

Assume that the actions of λ_1 and λ_2 on $T^1_{q_i}R_i$ and $T^1_{p_{i+1}}R_{i+1}$, respectively, are both negative. We can apply Theorem 3.5.23 to construct a very close degeneration of C whose central fiber Γ' is obtained by smoothing the singularities in Γ_2 different from p_{i+1} and with negative weight. We can construct such a degeneration whose cocharacter has support on Γ_1 with weights prescribed by λ_1 (where Γ_1 is naturally identified with a subcurve of Γ'). Then the inequality $||\lambda_1|| < ||\lambda||$ contradicts minimality and shows that the actions of λ_1 and λ_2 on $T^1_{q_i}R_i$ and $T^1_{p_{i+1}}R_{i+1}$ cannot both be negative. This implies that the two actions have opposite weights; they cannot both be positive because, as established above, the total action on $T_{p_i}\Gamma$ is negative.

What has been proven above implies that a minimal destabilizing cocharacter λ acts with negative weight on every internal node (i.e., neither p_1 nor q_l) of $\bigcup R_i$ and with opposite weight on any pair of consecutive tacnodes (i.e., lying on the same rational component or on two different rational components meeting at a node). A straightforward application of Theorem 3.5.23 allows us to construct a very close degeneration of C where the central fiber Γ' is obtained by smoothing the nodal singularities p_i for $i = 2, \dots, l$ in Γ, and the associated cocharacter ρ' is the unique one that acts with the same sign on the tacnodes of the rosaries and has norm 1. This yields a group morphism $\mathrm{Aut}(\Gamma')^0 \to \mathrm{Aut}(\Gamma)^0$. Then we define $\rho\colon \mathbb{G}_m \xrightarrow{\rho'} \mathrm{Aut}(\Gamma')^0 \to \mathrm{Aut}(\Gamma)^0$, that has norm 1.

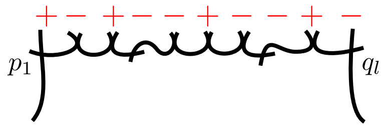


Figure 4.4. To any singularity is assigned a weight. In the picture we denote the sign above the corresponding singularity

If $\langle \rho', L\rangle > 0$ the claim follows. Otherwise, necessarily $\langle \lambda\cdot\rho^{-1}, L\rangle = \langle \lambda, L\rangle - \langle \rho', L\rangle > 0$. The weight of $\lambda\cdot\rho^{-1}$ on $T^1_{p_1}C$ and $T^1_{q_l}C$ changes by ± 1 and does not change for any other node in $\bigcup R_i$. The weight of $\lambda\cdot\rho^{-1}$ on the tacnodes of $\bigcup R_i$ either remains the same or vanishes. Smoothing the singularities on Γ on which the action of $\lambda\cdot\rho^{-1}$ is trivial, we construct a very close degeneration of C whose character on the central fiber is given by $\lambda\cdot\rho^{-1}$. Since $||\lambda|| > ||\lambda\cdot\rho^{-1}||$, we obtain a contradiction to the minimality of λ.

It remains to treat the case $p_1 = q_l$. In this case $n = 0$ and $\Gamma = \bigcup R_i$; in particular, every node appears as the intersection of two rosaries. As above, one proves that the action of λ has negative weight at every node. Consider two nodes $q_i \in R_i \cup R_{i+1}$ and $q_j \in R_j \cup R_{j+1}$ and construct the partial normalization $\widetilde{\Gamma} \to \Gamma$ along q_i and q_j. The curve $\widetilde{\Gamma}$ consists of two connected components, corresponding to attached subcurves Γ_1, Γ_2 of Γ. Then there exist cocharacters λ_1, λ_2 of $\mathrm{Aut}(\Gamma)$ such that $\lambda = \lambda_1 \cdot \lambda_2$ and whose supports are Γ_1, Γ_2, respectively. As in the corresponding case above, when the actions of λ_1 and λ_2 are both negative on $T^1_{q_i}(\Gamma)$ and $T^1_{q_j}(\Gamma)$, we can construct a very close degeneration of C using the cocharacter λ_1 or λ_2, both of strictly smaller norm. We may then assume that there is at most one node $p \in \Gamma$ where the action behaves in this way. In other words, the action of λ on $\bigcup R_i$ is negative at every node and has opposite weight on any pair of consecutive tacnodes except possibly one pair, separated by p. Arguing as in the case $p_1 \neq q_l$, we construct a very close degeneration of C where the central fiber Γ' is obtained by smoothing all nodal singularities except possibly one, which in that case corresponds to p. The proof then proceeds as before.

The reverse implication follows from the openness of the L-stable locus, given that $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ has a good moduli space. □

4.2 Semistable Loci Via Cluster Algebras

In this section we give a complete description of the semistable locus with respect to line bundles in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$. The main results are Theorems 4.2.2 and 4.2.33, that describe the fan $\Sigma(\overline{\mathcal{M}}_{g,n}(\frac{7}{10}))$ using the combinatorics of cluster algebras. In our analogy, d-vectors correspond to stability conditions (hence elements of $\mathbb{E}_{g,n}$), and c-vectors correspond to principal strata in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ (except for $n = 0$, where the description is slightly different). Before recalling some useful facts on cluster algebras (without coefficients) in the first subsection, we state the results for hyperbolic pairs (g,n) such that $g \geq 2$ and $(g,n) \neq (2,0)$.

Definition 4.2.1. Let (g,n) be a hyperbolic pair such that $g \geq 2$ and $(g,n) \neq (2,0)$. For $I \in \mathcal{P} \cup \{\mathrm{irr}\}$ (cf. Equation (4.3)) define the root systems $\Phi^I_{g,n} \subset \mathbb{E}_{g,n}$ and the cluster algebras $\mathcal{A}^I_{g,n}$ as follows:

- Case $I = \mathrm{irr}$. Let $\Phi^{\mathrm{irr}}_{g,n} \subset \mathbb{E}_{g,n}$ be the root system of type A_1 given by the simple base $\Delta^{\mathrm{irr}}_{g,n}$. Define $\mathcal{A}^{\mathrm{irr}}_{g,n}$ to be the cluster algebra with initial quiver of type A_1. We identify the associated root system with $\Phi^{\mathrm{irr}}_{g,n}$.

- Case $n > 0$ and $I \neq [\emptyset]$. Let $\Phi^I_{g,n} \subset \mathbb{E}_{g,n}$ be the root system of type A_g given by the simple base $\Delta^I_{g,n}$. Define $\mathcal{A}^I_{g,n}$ to be the cluster algebra with initial quiver of type A_g with alternating orientation (where the odd positions are sinks). We identify the associated root system with $\Phi^I_{g,n}$.

- Case $n > 0$ and $I = [\emptyset]$. Let $\Phi^{[\emptyset]}_{g,n} \subset \mathbb{E}_{g,n}$ be the root system of type A_{g-2} given by the simple base $\Delta^{[\emptyset]}_{g,n}$. Define $\mathcal{A}^{[\emptyset]}_{g,n}$ to be the cluster algebra with initial quiver of type A_{g-2} with alternating orientation (where the odd positions are sinks). We identify the associated root system with $\Phi^{[\emptyset]}_{g,n}$.

- Case $n = 0$ and $2 \mid g$. Let $\Phi^{[\emptyset]}_{g,0} \subset \mathbb{E}_{g,0}$ be the root system of type $A_{(g-4)/2}$ given by the simple base $\Delta^{[\emptyset]}_{g,0}$. Define $\mathcal{A}^{[\emptyset]}_{g,0}$ to be the cluster algebra with initial quiver of type $A_{(g-4)/2}$ with alternating orientation (where the odd positions are sinks). We identify the associated root system with $\Phi^{[\emptyset]}_{g,0}$.

- Case $n = 0$ and $2 \nmid g$. Let $\Phi^{[\emptyset]}_{g,0} \subset \mathbb{E}_{g,0}$ be the root system of type $C_{(g-3)/2}$ given by the simple base $\Delta^{[\emptyset]}_{g,0}$. Define $\mathcal{A}^{[\emptyset]}_{g,0}$ to be the folding of a cluster algebra with initial quiver of type A_{g-4} with alternating orientation (where the odd positions are sinks) and automorphism given by the permutation $k \mapsto g-3-k$, as described in [NS14, Sec. 5]. We identify the associated root system with $\Phi^{[\emptyset]}_{g,0}$.

Where $\Delta^{[I]}_{g,n} = \{\alpha_{j,I}\} \subset \mathbb{E}_{g,n}$ for $I \subset [n]$ and $\Delta^{\mathrm{irr}}_{g,n} = \{\alpha_{\mathrm{irr}}\}$ as in Definition 4.1.7.

Finally, we construct the cluster algebra given by the product of the types above. It comes with the associated full-rank root system $\Phi_{g,n} = \bigsqcup_{I \in \mathcal{P} \cup \{\mathrm{irr}\}} \Phi^I_{g,n} \subset \mathbb{E}_{g,n}$. We call $\Sigma_{g,n}$ the d-vector cluster fan in $\mathbb{E}_{g,n}$. It is of type:

$$\begin{cases} A_1 \times (A_g)^{2^{n-1}-1} \times A_{g-2}, & \text{if } n > 0, \\ A_1 \times A_{(g-4)/2}, & \text{if } n = 0 \text{ and } 2 \mid g, \\ A_1 \times C_{(g-3)/2}, & \text{if } n = 0 \text{ and } 2 \nmid g. \end{cases}$$

For $n > 0$, the root system on $\mathbb{E}_{g,n}$ defined above consists of the following almost positive roots, which are the rays of the fan $\Sigma_{g,n}$ as explained in Proposition 4.2.19.

For $n > 0$ we have

$$(\Phi_{g,n})_{\geq -1} = \bigsqcup_{\emptyset \subsetneq I \subsetneq [n]} \left(\left\{ \sum_{i=l}^{m} \alpha_{i,I} \,\middle|\, 1 \leq l \leq m \leq g-1 \right\} \sqcup \{-\alpha_{i,I} \,|\, 1 \leq i \leq g-1\} \right) \sqcup$$
$$\sqcup \left\{ \sum_{i=l}^{m} \alpha_{i,\emptyset} \,\middle|\, 2 \leq l \leq m \leq g-1 \right\} \sqcup \{-\alpha_{i,\emptyset} \,|\, 2 \leq i \leq g-1\} \sqcup \{\alpha_{\mathrm{irr}}\} \sqcup \{-\alpha_{\mathrm{irr}}\}.$$

When (g, n) are clear from the context, we will write $\Phi_{\geq -1}$ (resp. $\Phi^{[I]}, \Delta^{[I]}$) instead of $(\Phi_{g,n})_{\geq -1}$ (resp. $\Phi^{[I]}, \Delta^{[I]}_{g,n}$) to simplify the notation.

The fan of semistable regions defined in Definition 2.2.8 is a coarsening of $\Sigma_{g,n}$. Via the equivalence relation that we will define in Definition 4.2.48, we will prove the following:

Theorem 4.2.2. *Assume (g, n) is a hyperbolic pair with $g \geq 2$ and $(g, n) \neq (2, 0)$. With the notations above we have a well-defined map of posets (that reverse inclusions)*

$$(\Sigma_{g,n}, \supset) \to \left(\left\{ \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{L-\mathrm{ss}} \mid L \in \mathbb{E}_{g,n} \right\}, \subset \right)$$
$$C \mapsto \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{L-\mathrm{ss}} \quad \textit{for any } L \in C^{\circ},$$

that sends two cones $C_1 = \mathrm{Span}_+ S_1$ and $C_2 = \mathrm{Span}_+ S_2$ to the same open substack if and only if the two cones are equivalent $S_1 \sim S_2$, according to Definition 4.2.48.

The previous theorem establishes that the locus depends only on the cones of $\Sigma_{g,n}$, we then define:

Definition 4.2.3. Given a subset $S \subset \Phi_{\geq -1}$ of compatible roots (cf. Definition 4.2.17), we set

$$\overline{\mathcal{M}}^{S}_{g,n} \coloneqq \overline{\mathcal{M}}^{\ell-\mathrm{ss}}_{g,n} \quad \text{for an element } \ell \in \mathrm{Span}_+ S.$$

The following proposition, which we will prove later in the section, gives a first description of the equivalence relation in Definition 4.2.48.

Proposition 4.2.4. *Fix a hyperbolic pair (g, n) with $g \geq 2$ and $(g, n) \neq (2, 0)$. The map of posets defined in Theorem 4.2.2 is an isomorphism when $n = 0$. If $n > 0$, the map restricts to a bijection between cones lying in $H^- \coloneqq \{\beta \mid x_{1,\{i\}}(\beta) \leq 0\}$ and big open substacks of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ arising as stable loci for line bundles, namely:*

$$\{C \in \Sigma_{g,n} \mid C \subset H^-\} \to \left\{ \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{L-\mathrm{ss}} \mid \textit{big open of } \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right),\, L \in \mathbb{E}_{g,n} \right\}$$
$$C \mapsto \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{L-\mathrm{ss}} \quad \textit{for any } L \in C$$

describes an isomorphism of posets.

4.2.1 Generalities on Cluster Algebras

In this subsection, we briefly introduce cluster fans and recall some results that will be used in the following sections. For a systematic introduction to cluster algebras, we refer to [FZ02a, Sec. 2]. In this work, we restrict our attention to cluster algebras of finite type. In this context, [FZ02b] established a complete classification of cluster algebras of finite type via the classification of crystallographic finite root systems (the ADE classification).

For completeness, we recall here the definition of a finite type cluster algebra (without coefficients) that we use. We will not rely on this definition directly, as our focus lies mainly on the associated combinatorics. The definitions below are the coefficient free version of those appearing in [FWZ24, Sec. 3.1].

Definition 4.2.5 (Seed)**.** Let $\mathcal{F} \supset \mathbb{Q}$ be a field extension. A pair (X, B) is called a *seed of size* n if:

- $X = (x_1, \ldots, x_n)$ is an n-tuple of elements of $\mathcal{F}$ algebraically independent over $\mathbb{Q}$. The elements of X are called *cluster variables*;
- $B = (b_{i,j})$ is an $n \times n$ matrix with integer coefficients such that there exists an n-tuple of integers $(d_i)_{1 \le i \le n}$ satisfying $d_i b_{i,j} = -b_{j,i}$ (i.e., B is *skew-symmetrizable*).

We say that the seed (X, B) is defined over $\mathcal{F}$.

Definition 4.2.6 (Seed mutation)**.** Let (X, B) be a seed of size n over $\mathcal{F}$. For $1 \le k \le n$, the *mutation in direction* k of (X, B) is the seed (X', B') defined as follows:

- $B' = (b'_{i,j})$ is given by (note that B' is again skew-symmetrizable):

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0, \\ b_{ij} & \text{otherwise;} \end{cases}$$

- $X' = (x'_1, \ldots, x'_n)$, where $x'_i = x_i$ if $i \neq k$, and x'_k is the unique element such that

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

We write $\mu_k(X, B)$ for the seed (X', B'). Two seeds are said to be *mutation equivalent* if one can be obtained from the other by a (finite) sequence of mutations.

The composition of mutations does not commute in general.

Remark 4.2.7. [FWZ24, Exercise 2.7.7] We have $\mu_k(\mu_k(X, B)) = (X, B)$.

Definition 4.2.8 (Cluster algebra, [FWZ24, Def. 3.1.9])**.** A *cluster algebra* $\mathcal{A}$ of rank n is the data of a set of seeds $\mathcal{S}$ of size n over a field $\mathcal{F}$, and the subalgebra

$$\mathbb{Q}\left[x \mid x \text{ is a cluster variable for some seed } (X,B) \in \mathcal{S}\right] \subset \mathcal{F},$$

such that for any two seeds $S_1, S_2 \in \mathcal{S}$, there exists a sequence of mutations on S_1 that give the seed S_2.

Given a seed (X,B), the cluster algebra $\mathcal{A}(X,B)$ is the one generated by all seeds mutation equivalent to (X,B). In this case (X,B) is called the *initial seed*.

When the cluster variables are omitted, we write $\mathcal{A}(B)$ for $\mathcal{A}((x_1,\dots,x_n),B)$, where $\mathcal{F} = \mathbb{Q}(x_1,\dots,x_n)$.

Remark 4.2.9. The field $\mathcal{F}$ in the definition above plays no essential role. For seeds of size n, we may assume $\mathcal{F} = \mathbb{Q}(x_1,\dots,x_n)$.

Definition 4.2.10 (Finite type cluster algebra)**.** A cluster algebra $\mathcal{A}$ is said to be of *finite type* if it contains only finitely many seeds.

Definition 4.2.11 (Product of cluster algebras)**.** Let $\mathcal{A}$ and $\mathcal{A}'$ be cluster algebras of ranks n and n', respectively. We define their *product* $\mathcal{A} \times \mathcal{A}'$ as the cluster algebra of rank $n+n'$, whose seeds are defined over the compositum field over $\mathbb{Q}$, and such that, for any seeds $(X,B) \in \mathcal{A}$ and $(X',B') \in \mathcal{A}'$, the seed (Y,C) is given by:

- $Y = (x_1,\dots,x_n,x'_1,\dots,x'_{n'})$, where $X = (x_1,\dots,x_n)$ and $X' = (x'_1,\dots,x'_{n'})$;
- C is the $(n+n') \times (n+n')$ block-diagonal matrix with diagonal blocks B and B'.

Definition 4.2.12 (Cluster algebra from a quiver)**.** Let Q be an oriented graph with n labelled vertices $\{1,\dots,n\}$ and no loops. Define the $n \times n$ matrix $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if there is an arrow } i \to j, \\ -1 & \text{if there is an arrow } j \to i, \\ 0 & \text{otherwise.} \end{cases}$$

We define $\mathcal{A}(Q) \coloneqq \mathcal{A}(B)$.

Remark 4.2.13. For two quivers Q_1, Q_2 as above, we have

$$\mathcal{A}(Q_1 \sqcup Q_2) = \mathcal{A}(Q_1) \times \mathcal{A}(Q_2).$$

When the quiver in the above definition is of Dynkin type, the associated cluster algebra is of finite type. More generally:

Theorem 4.2.14. *[FZ02b, Thm. 1.5] Let $A = (a_{ij})$ be the Cartan matrix of a finite type crystallographic root system Φ of rank n, written with respect to a fixed simple basis Δ. Let $B = (b_{ij})_{1\le i,j\le n}$ be a matrix such that (so that A is the generalized*

Cartan matrix associated with B)

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

Then the cluster algebra $\mathcal{A}(B)$ is of finite type.

The previous theorem is, in fact, an equivalence, which strongly connects the study of finite type cluster algebras with the theory of root systems. By Gabriel's theorem, we have:

Corollary 4.2.15. *For any quiver Q of finite representation type, $\mathcal{A}(Q)$ is of finite type.*

By the following result, we can identify almost positive roots in Φ with cluster variables.

Theorem 4.2.16. *[FZ02b, Thm. 1.19] Fix any crystallographic root system Φ, and any B as in Theorem 4.2.14. There exists a unique bijection $\alpha \mapsto x[\alpha]$ between the almost positive roots $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Delta)$ and the cluster variables in $\mathcal{A}(B)$ such that, for any $\alpha \in \Phi_{\geq -1}$, the cluster variable $x[\alpha]$ can be expressed in terms of the initial cluster variables $\{x_i \mid 1 \leq i \leq n\}$ as*

$$x[\alpha] = \frac{P_\alpha(x_1, \ldots, x_n)}{x^\alpha},$$

where P_α is a polynomial with integer coefficients and nonzero constant term. Under this bijection, $x[-\alpha_i] = x_i$.

Hence, the almost positive roots $\Phi_{\geq -1} = \Phi^+ \cup (-\Delta)$ are often referred to as *denominator vectors* or simply *d-vectors*.

Definition 4.2.17 (Compatible roots)**.** Let $\mathcal{A}$ be a finite type cluster algebra with root system $\Delta \subset \Phi$. Two distinct cluster variables x, x' are *compatible* if there exists a seed of $\mathcal{A}$ containing both.

Under the identification with $\Phi_{\geq -1}$, two almost positive roots are said to be compatible if they correspond to compatible cluster variables.

A subset $S \subset \Phi_{\geq -1}$ is called *compatible* if every pair of distinct elements of S is compatible.

We now give an explicit description for the Dynkin type A_k, using cluster categories as explained in [Bua+06].

Given an acyclic oriented quiver Q, we denote by $\mathrm{rep}(Q)$ the category of finitely dimensional representations of Q over a field (that we may assume to be $\mathbb{C}$). When the quiver Q of finite representation type (i.e., it has only finitely many isomorphism classes of indecomposable) we can associate to it a root system Φ in a finitely dimensional vector space E. In this setting, there is a bijection between the positive roots Φ^+ and the indecomposable representations of Q. We write $M[\gamma]$ for the representation corresponding to γ.

Proposition 4.2.18 (Compatibility relation, [Bua+06, Cor. 4.3])**.** *Let Q be a quiver of finite representation type with associated root system Φ and simple basis $\Delta \subset \Phi$. Two distinct almost positive roots γ_1, γ_2 are compatible if and only if one of the following holds:*

- $\gamma_1, \gamma_2 \in \Phi_{>0}$ *and* $\mathrm{Ext}^1(\mathrm{M}[\gamma_1], \mathrm{M}[\gamma_2]) = 0$*;*
- $\gamma_1 \in -\Delta$ *and* $\gamma_2 \in \mathrm{Span}(\Delta \setminus \{\gamma_1\})$ *(i.e., γ_1 does not appear in the expression of γ_2 in the basis Δ);*
- *similarly,* $\gamma_2 \in -\Delta$ *and* $\gamma_1 \in \mathrm{Span}(\Delta \setminus \{\gamma_2\})$*.*

Note that this relation is symmetric (since $\mathrm{Ext}^1(\mathrm{A}, \mathrm{B}) \simeq \mathrm{Ext}^1(\mathrm{B}, \mathrm{A})$), but neither transitive nor reflexive. We can now define the following fan.

Proposition 4.2.19 (Cluster fan, [FZ02b, Thm. 3.2])**.** *Let $\mathcal{A}$ be a finite type cluster algebra with root system $\Delta \subset \Phi$, where $\mathrm{Span}\,\Phi = \mathbb{E}$. The collection of cones*

$$\Sigma(\mathcal{A}) := \{\overline{\mathrm{Span}_+(S)} \mid S \subset \Phi_{\geq -1} \text{ is a compatible subset}\}$$

is a rational, complete, polyhedral, and simplicial fan in $\mathbb{E}$, called the (d-vector) cluster fan of $\mathcal{A}$. In particular, any compatible subset $S \subset \Phi_{\geq -1}$ is linearly independent.

Definition 4.2.20 (c-vectors)**.** Let $\Sigma(\mathcal{A})$ be a cluster fan as above, and let $C \in \Sigma(\mathcal{A})$ be a top dimensional cone corresponding to a compatible subset $S \subset \Phi_{\geq -1}$. The *c-vectors of C* (or of S) is the dual base to S in E^*.

c-vectors are the rays of the dual cone of C in $\mathbb{E}^*$, these corresponds to the facets of C.

We are interested in root systems of types A and C, and we collect below some results needed for our construction.

Definition 4.2.21 (Support)**.** Let Q be a quiver of type A_n, and denote with Φ a root system of type A_n, with fixed simple basis $\Delta = \{\alpha_i\}$. For any $\beta = \pm \sum_{i=a}^{b} \alpha_i \in \Phi$ set

$$\mathrm{Supp}(M[\beta]) := \mathrm{Supp}(\beta) := \{a, \dots, b\} \subset [n].$$

Moreover, if $\beta \in \Phi_{>0}$, we denote $M[a,b] := M[\beta]$. Finally, for a set of roots Γ, we define $\mathrm{Supp}(\Gamma) = \bigcup_{\beta \in \Gamma} \mathrm{Supp}(\beta)$.

Proposition 4.2.22 (Compatibility for A_k)**.** *Let Q be a quiver of type A_k with alternating orientation. For $M[a,b], M[c,d]$ with $a \leq c$, we have:*

- *if $a = c$ or $b = d$, then* $\mathrm{Ext}^1(\mathrm{M[a,b]}, \mathrm{M[c,d]}) = 0$*;*
- *if $a \leq b < c \leq d$, then* $\mathrm{Ext}^1(\mathrm{M[a,b]}, \mathrm{M[c,d]}) \neq 0 \iff b = c - 1$*;*
- *if $a < c \leq b < d$, then* $\mathrm{Ext}^1(\mathrm{M[a,b]}, \mathrm{M[c,d]}) = 0 \iff 2 \mid (b + c)$*;*
- *if $a < d \leq c < b$, then* $\mathrm{Ext}^1(\mathrm{M[a,b]}, \mathrm{M[c,d]}) = 0 \iff 2 \nmid (c + d)$*.*

Proof. For a quiver Q, there is an Euler form

$$\langle \_, \_ \rangle_Q \colon kQ_0 \times kQ_0 \to k,$$

where kQ_0 is the vector space (over a field k) spanned by the vertices of Q, as defined in [Den08, Sec. 1.2]. In our case, Q is of type A_k, and kQ_0 has a basis indexed by $[k]$. For vectors $\mathbf{x}, \mathbf{y} \in kQ_0$, the Euler form can be written as

$$\langle \mathbf{x}, \mathbf{y} \rangle_Q \;=\; \sum_{\text{vertices } i} x_i y_i \;-\; \sum_{\text{arrows } i \to j} x_i y_j .$$

By [Den08, Prop. 1.9],

$$\langle M[a,b], M[c,d] \rangle_Q \;=\; \dim \mathrm{Hom}_Q(M[a,b], M[c,d]) \;-\; \dim \mathrm{Ext}^1_{\mathrm{Q}}(\mathrm{M[a,b]}, \mathrm{M[c,d]}).$$

The space $\mathrm{Hom}_Q(M[a,b], M[c,d])$ is either zero or one dimensional, and it can be determined in the various cases considered here. A case-by-case computation of the Euler form then yields the claim. □

By the representation theory of quivers of type A, we have the following.

Definition-Proposition 4.2.23. Let Q be a quiver of type A_k and let Q' be the full subquiver obtained by selecting some vertices of Q (where full means that all arrows between chosen vertices in Q' are included). There is a truncation functor on representations, given by restriction:

$$\operatorname{rep} Q \longrightarrow \operatorname{rep} Q'.$$

If Q' is obtained by choosing consecutive vertices, this functor sends indecomposable representations of Q to indecomposable representations of Q'. Under the identification of indecomposables with positive roots, it sends compatible roots to compatible roots.

The following pairing for quivers of type A will be used in what follows.

Definition 4.2.24 (Pairing)**.** Let Φ be a root system of type A_n and fix a simple basis $\Delta = (\alpha_i)_{i \in [n]}$. Define a pairing in the basis Δ on $\mathbb{E} := \mathrm{Span}_{\mathbb{Q}} \Phi$ by

$$\begin{gathered} [\_,\_] \colon \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{Q}, \\ \left(\sum_{i \in [n]} x_i \alpha_i, \sum_{i \in [n]} y_i \alpha_i \right) \longmapsto \sum_{i \in [n]} (-1)^i x_i y_i . \end{gathered}$$

This pairing is naturally defined on representations of the quiver via the above identification.

Remark 4.2.25. With the notation of the preceding definition, for any positive roots $\beta_1, \beta_2 \in \Phi^+$ we have $[\beta_1, \beta_2] \in \{1, 0, -1\}$.

We will use the following propositions.

Proposition 4.2.26. *Fix an interval $[a,b] \subset \{1, \dots, n\}$, and let $\beta = \sum_{j \in J} \lambda_j \beta_j$ be a strictly positive linear combination of compatible roots $\{\beta_j\}_{j \in J} \subset \Phi_{\geq -1}$. Assume that for some i we have $[\beta_i, M[a,b]] > 0$. Then there exists a subrepresentation $M[c,d] \subset M[a,b]$ such that $[\beta, M[c,d]] > 0$.*

Proof. By the restriction functor of Definition-Proposition 4.2.23, we may assume $(a,b) = (1,n)$. In that case, necessarily $|\operatorname{Supp}(\beta_i)|$ is odd.

We use the explicit combinatorial description of compatible roots given in Proposition 4.2.22. Let $M[c,d] \subset M[1,n]$ be a minimal irreducible subrepresentation (by inclusion) containing the support of β_i, and assume for a contradiction that there exists β_j (for some $j \in J$) such that $[\beta_j, M[c,d]] < 0$. The roots β_j and the representation $M[c,d]$ are determined by their supports. We distinguish several cases according to the relative positions of $\operatorname{Supp}(\beta_i)$ and $\operatorname{Supp}(\beta_j)$:

1. $\operatorname{Supp}(\beta_i) \cap \operatorname{Supp}(\beta_j) = \emptyset$. By the minimality of $M[c,d]$, we necessarily have $\operatorname{Supp}(M[c,d]) \cap \operatorname{Supp}(\beta_j) = \emptyset$, hence $[\beta_j, M[c,d]] = 0$, a contradiction.
2. The supports intersect and $\operatorname{Supp}(\beta_j) \setminus \operatorname{Supp}(\beta_i) = \{x, \dots, y\}$ is an interval. Then $2 = |[\beta_i, M[c,d]] - [\beta_j, M[c,d]]| = |[M[x,y], M[c,d]]|$ which is impossible by Remark 4.2.25.
3. $\operatorname{Supp}(\beta_i) \subset \operatorname{Supp}(\beta_j)$. Since $|\operatorname{Supp}(\beta_i)|$ is odd, the compatibility of β_i and β_j forces us into case (2).
4. $\operatorname{Supp}(\beta_j) \subset \operatorname{Supp}(\beta_i)$. Then either we are in case (2), or β_j has even support. In the latter situation, $\operatorname{Supp}(\beta_j) \subset \operatorname{Supp}(M[c,d])$, hence $[\beta_j, M[c,d]] = 0$, a contradiction.
5. $\operatorname{Supp}(\beta_j) \cap \operatorname{Supp}(\beta_i) \neq \emptyset$ and none of the above cases occurs. From $[\beta_i, M[c,d]] > 0$ it follows that the odd interval $\operatorname{Supp}(\beta_j) \cap \operatorname{Supp}(\beta_i) = \{x, \dots, y\}$ has x and y both even. Since $\{x, y\} \subset \operatorname{Supp}(M[c,d])$ and one of x, y is an endpoint of $\operatorname{Supp}(\beta_j)$, we get $[\beta_j, M[c,d]] \in \{0, 1\}$, again a contradiction.

□

Some dual result holds as well:

Proposition 4.2.27. *Fix an interval $[a,b] \subset \{1, \dots, n\}$, and let $\beta = \sum_{j \in J} \lambda_j \beta_j$ be a strictly positive linear combination of compatible roots $\{\beta_j\}_{j \in J} \subset \Phi_{\geq -1}$. Assume that for some i we have $[\beta_i, M[a,b]] < 0$. Then there exists a quotient $M[a,b] \twoheadrightarrow M[c,d]$ such that $[\beta, M[c,d]] < 0$.*

Proof. The proof is analogous to that of Proposition 4.2.26. □

Via a direct computation of the Euler form for quivers of type A_k (cf. [DW17, Exercise 6.4.2]), we obtain:

Remark 4.2.28. Let Q be the quiver of type A_k with alternating orientation (with sinks at the odd vertices), and let $Q_{|[a,b]}$ be the subquiver on the vertices in positions $a, \dots, b$. Denote the restriction of a Q-representation N to $Q_{|[a,b]}$ by $N \mapsto N_{|[a,b]}$. If N is a representation of Q with $\beta = \underline{\dim}\, N$, then

$$[\beta, M[a,b]] \;=\; \dim \mathrm{Ext}^1_{|[a,b]}(\mathrm{N}_{|[a,b]}, \mathrm{M[a,b]}),$$

where $\mathrm{Ext}^1_{|[a,b]}$ denotes extensions in the category $\operatorname{rep} Q_{|[a,b]}$.

4.2.2 Folding cluster algebras

We recall here a possible folding construction for cluster algebras of type C_k (and A_k), this will be used to prove Theorem 4.2.33 in the case without marked points.

Proposition 4.2.29 ([NS14, Sec. 5])**.** *Let $\mathcal{A}(Q)$ be a cluster algebra where Q is a quiver of type A_{2k+1} with alternating orientation (with sinks at the odd positions). Fix $\sigma \in \mathfrak{S}_{2k+1}$ an* automorphism *of Q, as in [NS14, Def. 5.1], which exchanges the vertices $i \mapsto 2k+2-i$. Denote by $\mathcal{A}_\sigma(Q)$ the cluster algebra obtained by folding Q with respect to σ, as in [NS14, Prop. 5.6]. Then $\mathcal{A}_\sigma(Q)$ is a cluster algebra of type C_{k+1}, and there is a natural inclusion of cluster fans*

$$\Sigma(\mathcal{A}_\sigma(Q)) \longrightarrow \Sigma(\mathcal{A}(Q)),$$

which can be read (in a basis of the root system) as

$$\begin{aligned} \phi_{2k+1}\colon \Phi(\mathcal{A}_\sigma(Q)) &\longrightarrow \Phi(\mathcal{A}(Q)),\\ \gamma_i &\longmapsto \begin{cases} \alpha_i + \alpha_{2k+2-i} & \text{if } i \neq k+1,\\ \alpha_i & \text{if } i = k+1. \end{cases} \end{aligned}$$

Here the image of the support of $\Sigma(\mathcal{A}_\sigma(Q))$ has equations $\alpha_i^* - \alpha_{2k+2-i}^*$ for $1 \leq i \leq k$ in $\operatorname{Span}\Phi$, where α_i^* denotes the basis dual to the simple roots.

Definition 4.2.30 (Support, C_n case)**.** With the notation of Proposition 4.2.29, let β be a root of type C_{k+1}. We define its support to be

$$\operatorname{Supp}(\beta) \coloneqq \operatorname{Supp}(\phi_{2k+1}(\beta)) \subset [2k+1].$$

Similarly to the A_n case, for a set of roots Γ, we define $\operatorname{Supp}(\Gamma) = \bigcup_{\beta\in\Gamma} \operatorname{Supp}(\beta)$.

Moreover, a phenomenon of a different nature occurs for A_{2k}.

Proposition 4.2.31. *Let $\mathcal{A}(Q_k)$ be the finite type cluster algebra where Q_k is a quiver of type A_k with alternating orientation (with sinks at the odd positions). There is a natural inclusion of cluster fans*

$$\Sigma(\mathcal{A}(Q_k)) \longrightarrow \Sigma(\mathcal{A}(Q_{2k}))$$

given by the linear map (in a basis of the root system)

$$\begin{aligned} \phi_{2k}\colon \Phi(A_k) &\longrightarrow \Phi(A_{2k}),\\ \alpha_i &\longmapsto \alpha_i + \alpha_{2k+1-i}. \end{aligned}$$

The subspace spanned by $\Sigma(\mathcal{A}(Q_k))$ is cut out by the equations $\alpha_i^* - \alpha_{2k+1-i}^*$ for $1 \leq i \leq k$ in $\Sigma(\mathcal{A}(Q_{2k}))$, where α_i^* is the dual of the simple basis.

Proof. Let $f\colon \Phi(A_k) \to \Phi(A_{2k})$ be the inclusion $\Delta_k \hookrightarrow \Delta_{2k}$ with $\alpha_i \mapsto \alpha_i$. Consider $\beta \in \Phi(A_k)_{\geq -1}$. If $\alpha_k^*(\beta) = 1$, then $\phi_{2k}(\beta)$ is a positive root in Φ^{2k}; otherwise, we can write it as the sum of two almost positive roots $f(\beta)$ and $\phi_{2k}(\beta)-f(\beta)$. The claim follows if, for every compatible set $S=\{\beta_i\}\subset\Phi^k_{\geq -1}$, the set

$$\begin{aligned}T(S) &\coloneqq \{\phi_{2k}(\beta_i)\mid \alpha_k^*(\beta_i)=1\}\\ &\quad\sqcup\ \{f(\beta_i)\mid \alpha_k^*(\beta_i)\neq 1\}\\ &\quad\sqcup\ \{\phi_{2k}(\beta_i)-f(\beta_i)\mid \alpha_k^*(\beta_i)\neq 1\}\ \subset\ \Phi(A_{2k})_{\geq -1}\end{aligned}$$

is a compatible set of roots. Indeed, in this case

$$\mathrm{Span}_+ S \ =\ \mathrm{Span}_+ T(S)\ \cap\ \bigcap_{i=1}^{k}\{\alpha_i^*-\alpha_{2k+1-i}^*=0\},$$

and hence ϕ_{2k} preserves the fan structure. The compatibility of $T(S)$ follows from a direct inspection of the conditions in Proposition 4.2.22. □

Remark 4.2.32. In the case $n=0$ of Definition 4.2.1, the cluster fan of $\mathcal{A}_{g,0}^{[\emptyset]}$ embeds via ϕ_{g-2} into $\Sigma(\mathcal{A}(Q))$, where Q is a quiver of type A_{g-2}, as explained in Propositions 4.2.29 and 4.2.31 (according to the parity of g).

4.2.3 The semistability regions are a coarsening of $\Sigma_{g,n}$

The aim of this section is to prove the following theorem. Throughout this subsection, to simplify notation we write $\Phi_{\geq -1}$ (resp. $\Phi^{[I]},\Delta^{[I]}$) instead of $(\Phi_{g,n})_{\geq -1}$ (resp. $\Phi^{[I]},\Delta^{[I]}_{g,n}$).

Theorem 4.2.33. *Let (g,n) be a hyperbolic pair with $g\geq 2$ and $(g,n)\neq(2,0)$. Fix a cone $C\in\Sigma_{g,n}$, given by the positive span of its rays $S\subset\Phi_{\geq -1}$. Then, for any $\ell\in C^\circ\subset\mathbb{E}_{g,n}$, we have*

$$\overline{\mathcal{M}}_{g,n}\Big(\frac{7}{10}\Big)^{\ell-\mathrm{ss}} \ =\ \bigcap_{\beta\in S}\overline{\mathcal{M}}_{g,n}\Big(\frac{7}{10}\Big)^{\beta-\mathrm{ss}}.$$

We will use Theorem 4.1.10 to prove the main result of this section. We defined a pairing between line bundles and principal strata in Definition 4.1.9, this coincides with the pairing of Definition 2.1.38 for principal degenerations. We now rewrite it in terms of the pairing from Definition 4.2.24.

Proposition 4.2.34. *Consider $\beta\in\Phi_{\geq -1}\subset\mathbb{E}_{g,n}$. Then, for $n>0$, we have*

$$\langle S(a,b,I,\sigma),\beta\rangle=\begin{cases}0 & \text{if } \beta\notin\Phi^{[I]},\\ \sigma\,[M[a,b],\beta] & \text{if } [I]\notin\{[\emptyset],\mathrm{irr}\} \text{ and } \beta\in\Phi^{[I]},\\ \sigma\,[M[a-1,b-1],\beta] & \text{if } I=\emptyset \text{ and } \beta\in\Phi^{[\emptyset]},\\ \sigma & \text{if } I=\mathrm{irr},\ 2\nmid a-b,\ \beta\in\Phi^{\mathrm{irr}},\\ -\sigma & \text{if } I=\mathrm{irr},\ 2\mid a-b,\ \beta\in\Phi^{\mathrm{irr}},\end{cases}$$

under the identification $\Phi^{[I]}\simeq\Phi(A_k)$, where $k=g$ if $[I]\neq[\emptyset]$, $k=g-2$ if $[I]=[\emptyset]$, and $k=1$ if $I=\mathrm{irr}$.

For $n = 0$, we have

$$\langle S(a,b,I,\sigma),\beta\rangle = \begin{cases} \sigma\,[M[a-1,b-1],\phi_{g-2}(\beta)] & \textit{if } 2 \nmid g,\ [I] \neq \mathrm{irr},\ \beta \in \Phi^{[I]}, \\ \sigma & \textit{if } I = \mathrm{irr},\ 2 \nmid a-b,\ \beta \in \Phi^{\mathrm{irr}}, \\ -\sigma & \textit{if } I = \mathrm{irr},\ 2 \mid a-b,\ \beta \in \Phi^{\mathrm{irr}}, \end{cases}$$

where $\phi_{g-2}(\beta)$ is the immersion from Remark 4.2.32.

Proof. Direct consequence of the computations in Section 4.1.1. □

Thanks to this description of the pairing, we have:

Remark 4.2.35. Given $\beta \in \Phi_{\geq -1}$ and $S \in \mathbb{S}_{g,n}$, we have $\langle S,\beta\rangle \in \{-1,0,1\}$.

The root systems $\Phi_{g,n}$ (and hence also the fan $\Sigma_{g,n}$) decompose as a product of smaller root systems depending on a subdivision of $[n]$; this is reflected in the following proposition.

Proposition 4.2.36. *Let $\ell = \ell_{\mathrm{irr}} + \sum_{[I]\in\mathcal{P}} \ell_{[I]}$, where $\ell_{[I]} \in \operatorname{Span}\Phi^{[I]}_{g,n}$. Then*

$$\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{\ell-\mathrm{ss}} = \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{\ell_{\mathrm{irr}}-\mathrm{ss}} \cap \bigcap_{[I]\in\mathcal{P}} \overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{\ell_{[I]}-rmss}.$$

Moreover (with the notation of Definition 3.8.1),

$$\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)^{\ell_{\mathrm{irr}}-\mathrm{ss}} = \begin{cases} \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}) \setminus \overline{S_{\mathrm{tac}}} & \textit{if } \ell_{\mathrm{irr}} \in \mathbb{Q}_{<0}\,\alpha_{\mathrm{irr}}, \\ \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}) & \textit{if } \ell_{\mathrm{irr}} = 0, \\ \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}) \setminus \mathcal{B}_{\mathrm{A1/A1\,chains}} & \textit{if } \ell_{\mathrm{irr}} \in \mathbb{Q}_{>0}\,\alpha_{\mathrm{irr}}, \end{cases}$$

where $\mathcal{B}_{\mathrm{A1/A1\,chains}}$ is the closed substack parametrizing curves containing an A_1/A_1-attached elliptic chain.

Proof. The first claim follows from Theorem 4.1.10 and the fact that $\langle S(I,a,b,\sigma),\alpha_{i,I}\rangle = 0$ if $[I] \neq [J]$ (cf. Proposition 4.2.34). The second claim follows from the direct computation of $\langle S(\{\mathrm{irr}\},a,b,\sigma),\ell_{\mathrm{irr}}\rangle$ given in Proposition 4.2.34. □

Proof of Theorem 4.2.33. Let ℓ lie in the relative interior of the cone generated by the compatible roots $\{\beta_i\}_{i\in J} = S \subset \Phi_{\geq -1}$. By Theorem 4.1.10, the claim is equivalent to

$$\bigcup_{\mathcal{S}\in\mathbb{S}_{g,n}\,:\,\langle\mathcal{S},\ell\rangle>0} \overline{\mathcal{S}} = \bigcup_{\mathcal{S}\in\mathbb{S}_{g,n}\,:\,\exists\,\beta_i \text{ with } \langle\mathcal{S},\beta_i\rangle>0} \overline{\mathcal{S}}.$$

Write $\ell = \ell_{\mathrm{irr}} + \sum_{[I]\in\mathcal{P}} \ell_{[I]}$ with $\ell_{[I]} \in \operatorname{Span}\Phi^{[I]}_{g,n}$. By Proposition 4.2.36, it suffices to prove the claim when $\ell = \ell_{[I]}$ for some $I \subset [n]$.

The left-hand side is always contained in the right-hand side since $\ell \in \operatorname{Span}_+ S$. For the converse inclusion, we use a parallelism between representations of quivers and strata that we now explain. The claim is established if, whenever $\langle\mathcal{S},\ell_{[I]}\rangle > 0$, there exist $\mathcal{S}' \in \mathbb{S}_{g,n}$ and $\beta_i \in S$ such that $\langle\mathcal{S}',\beta_i\rangle > 0$ and $\mathcal{S} \subset \overline{\mathcal{S}'}$.

First assume $n > 0$ and $[I] \neq [\emptyset]$. Let Q be a quiver of type A_g with alternating orientation (sources at the odd positions). For any stratum $S(I, a, b, \sigma)$, consider the irreducible representation $M[a, b]$. Then $S(I, a, b, \sigma) \geq S(I, a', b', \sigma')$ in the poset $\mathbb{S}_{g,n}$ (so the latter lies in the closure of the former) if and only if one of the following holds:

- $\sigma = \sigma' = 1$ and there is an injection $M[a, b] \hookrightarrow M[c, d]$ of irreducible representations of Q;
- $\sigma = \sigma' = -1$ and there is a surjection $M[a, b] \twoheadrightarrow M[c, d]$ of irreducible representations of Q.

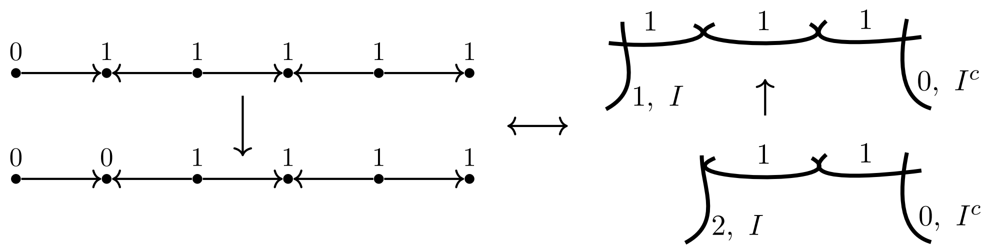


Figure 4.5. Example of the correspondenece for $g = 6$, $\sigma = \sigma' = -1$.

The claim then follows from Propositions 4.2.26 and 4.2.27. The case $I = \emptyset$ is analogous, using a quiver of type A_{g-2} (the dimension is $g - 2$ because a tacnode cannot occur in the first or second position).

It remains to consider the case $n = 0$. We use the description in Remark 4.2.32. By Proposition 4.2.34 we have

$$\langle S(I, a, b, \sigma), \beta\rangle \;=\; \sigma \cdot [M[a-1, b-1], \phi_{g-4}(\beta)],$$

where $\phi_{g-4}(\beta) \in \Phi(A_{g-4})$. The argument then proceeds verbatim as in the case $n > 0$. $\square$

Remark 4.2.37. Notice that Theorem 4.2.33 is not true if we do not assume the roots β_i to be pairwise compatible. For example:

- $\overline{\mathcal{M}}_{g,n}(\frac{7}{10}) \neq \overline{\mathcal{M}}_{g,n}(\frac{7}{10})^{\ell-\mathrm{ss}} \cap \overline{\mathcal{M}}_{g,n}(\frac{7}{10})^{-\ell-\mathrm{ss}}$ whenever $\ell \neq 0$; for instance, take $\ell = \alpha_{i,I}$.
- $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})^{\alpha_{i,I}+\alpha_{i+1,I}-\mathrm{ss}} \neq \overline{\mathcal{M}}_{g,n}(\frac{7}{10})^{\alpha_{i,I}-\mathrm{ss}} \cap \overline{\mathcal{M}}_{g,n}(\frac{7}{10})^{\alpha_{i+1,I}-\mathrm{ss}}$: the left-hand side contains the generic point of $S(i, I, i+1, \sigma)$, whereas the right-hand side does not.

The following remark gives a first insight into the stable locus; it is an immediate consequence of Theorem 4.2.33, Theorem 4.1.10, and the computation of the pairing.

Remark 4.2.38. Let $S \subset (\Phi_{g,n})_{\geq -1}$ be a compatible subset of roots. Then

$$\overline{\mathcal{M}}^S_{g,n} = \begin{cases} \overline{\mathcal{M}}^{S\cap\Phi^+_{g,n}}_{g,n} \setminus \left(\overline{S_{\mathrm{tac}}} \cup \bigcup_{-\alpha_{i,I}\in S} \overline{S(i,i,I,-1)} \right) & \text{if } \alpha_{\mathrm{irr}} \in S, \\ \overline{\mathcal{M}}^{S\cap\Phi^+_{g,n}}_{g,n} \setminus \bigcup_{-\alpha_{i,I}\in S} \overline{S(i,i,I,-1)} & \text{if } \alpha_{\mathrm{irr}} \notin S. \end{cases}$$

In other words, the negative roots in S prescribe the positions where a tacnode cannot appear on a semistable point.

4.2.4 Principal strata as c-vectors

For a maximal cone, we can interpret the principal strata that correspond to stable points as walls for the cones of the fan $\Sigma_{g,n}$ (i.e., as c-vectors), as we now explain.

To any principal stratum we associate a function.

Definition 4.2.39. Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Define

$$\begin{aligned} c\colon \mathbb{S}_{g,n} &\longrightarrow \mathbb{E}^\vee_{g,n}, \\ S(I,a,b,\sigma) &\longmapsto \langle S(I,a,b,\sigma), \_\rangle. \end{aligned}$$

For $I \subset [n]$, we can write this explicitly as

$$\begin{aligned} \langle S(I,a,b,\sigma), \_\rangle \colon \mathbb{E}_{g,n} \longrightarrow \operatorname{Span} \Phi^{[I]} &\longrightarrow \mathbb{Q}, \\ \sum_j b_j \alpha_{j,I} &\longmapsto \sigma \sum_{j=a}^{b} (-1)^j b_j, \end{aligned}$$

where the first map is the projection onto the subspace with respect to the basis Δ. In this section, we always assume $I \neq \{\mathrm{irr}\}$; the remaining case is completely described in Proposition 4.2.36. As a consequence of the discussion above and a direct analysis of the map, we obtain:

Proposition 4.2.40. *The semistability regions in $\mathbb{E}_{g,n}$ are defined by intersections of half-spaces determined by the hyperplanes $\langle S(I,a,b,\sigma), \_\rangle$.*

If $n > 0$, or if $n = 0$ and $2 \nmid g$, the map c restricted to the principal strata with $I \neq \mathrm{irr}$ is injective.

Proof. The first claim follows from Theorem 4.1.10. The second claim is clear for $n \neq 0$, and for $n = 0$ it follows from Remark 4.1.8. □

Remark 4.2.41. This is not the case for $n = 0$, $2 \nmid g$: if $b > g/2$, then $S(a,b,\sigma)$ and $S(a,b-a,\sigma)$ define the same equation.

For an element $\Gamma \in \mathbb{S}_{g,n}$, we have a direct criterion to determine whether $S_\Gamma \subset \overline{\mathcal{M}}^S_{g,n}$.

Proposition 4.2.42. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Consider $\beta \in (\Phi_{g,n})_{\geq -1}$ and fix a stratum $S \in \mathbb{S}_{g,n}$. If $n > 0$, then*

$$\langle S, \beta \rangle \leq 0 \iff \langle S', \beta \rangle \leq 0 \text{ for all } S' \geq S.$$

For $n = 0$, the equivalence holds if $S = S(a,b,\sigma)$ with $a, b < \lfloor \frac{g+1}{2} \rfloor$.

Proof. By Remark 4.2.35, we have $\{\langle S', \beta\rangle \mid S' \geq S\} \subset \{1, 0, -1\}$. We argue by contradiction. Let $S = S(a,b,I,\sigma)$ and suppose there exists $S' = S(a', b', I, \sigma) \geq S$ with $\langle S', \beta \rangle = 1$. Two cases can occur.

Case 1: $a' = a$ or $b' = b$. Without loss of generality, assume $a' = a$. Then $b' < b$, and both S and S' have a tacnode in position (b', I). Since $\langle S', \beta\rangle = 1$, we must have $b' \notin \mathrm{Supp}(\beta)$; consequently $\mathrm{Supp}(\beta) \cap \mathrm{Supp}(S) \subset \mathrm{Supp}(S')$. When $n > 0$, this implies $\langle S', \beta \rangle = \langle S, \beta \rangle$. For $n = 0$, the inclusion of supports yields equality provided $a, b < \lfloor \frac{g+1}{2} \rfloor$.

Case 2: $a < a' < b' < b$. In this situation S has tacnodes in position (a', I) and (b', I). Since $\langle S', \beta \rangle = 1$, we have $a' \notin \mathrm{Supp}(\beta)$ and $b' \notin \mathrm{Supp}(\beta)$, so $\mathrm{Supp}(\beta) \subset \mathrm{Supp}(S')$. When $n > 0$, this forces $\langle S', \beta\rangle = \langle S, \beta \rangle$. Again, for $n = 0$, equality holds provided $a, b < \lfloor \frac{g+1}{2} \rfloor$. □

The analogous statement for the converse inequality does not hold; this explains the non-symmetry of $\Sigma_{g,n}$ with respect to the origin.

Remark 4.2.43. Fix a root $\beta \in (\Phi_{g,n})_{\geq -1}$ and let $S \in \mathbb{S}_{g,n}$ with $\langle S, \beta \rangle > 0$. Consider a tacnode in S at a position (i, I) with $(i, I) \in \mathrm{Supp}(\beta)$. Then $S(i, i, I, (-1)^{i+1}) \geq S$ and $\langle S(i, i, I, (-1)^{i+1}), \beta \rangle = -1$.

Proposition 4.2.44. *Let $S \subset \Phi_{\geq -1}$ be a compatible subset and fix I with $[I] \neq \mathrm{irr}$. Let $\Gamma = S(I, a, b, \sigma) \in \mathbb{S}_{g,n}$. Then $S_\Gamma \subset \overline{\mathcal{M}}^S_{g,n}$ if and only if the following holds:*

1. *if $[I] = [\{i\}]$ and $a = 2$, then $x_{1,\{i\}}(\beta_i) \leq 0$ for every $\beta_i \in S$;*
2. *$\langle S(I, a, b, \sigma), \beta \rangle \leq 0$ for every $\beta \in S$.*

Proof. A stratum $\mathcal{S}$ is removed if and only if there exists a removed principal stratum containing it. Two strata $\mathcal{S}, \mathcal{S}'$ satisfy $\overline{\mathcal{S}'} \supset \mathcal{S}$ if and only if $\mathcal{S}' \geq \mathcal{S}$ in the poset $\mathbb{S}_{g,n}$, or $\mathcal{S}$ contains an elliptic bridge at $(1, \{i\})$ and $\mathcal{S}'$ is the generic stratum determined by such an elliptic bridge. The latter condition is precisely (1); applying

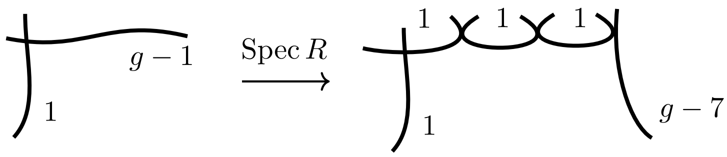


Figure 4.6. Example of specialization of strata with elliptic tail.

Proposition 4.2.42 shows that the former is equivalent to (2). □

We now relate principal strata and c-vectors.

Remark 4.2.45. Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Let $S \subset (\Phi_{g,n})_{\geq -1}$ be a set of compatible roots, and assume $[I] \notin \{[\emptyset], \mathrm{irr}\} \cup \{\{i\} \mid i \in [n]\}$. Suppose that $C = \mathrm{Span}_+ S \cap \mathrm{Span}\, \Phi^{[I]}_{g,n}$ is top-dimensional in $\mathrm{Span}\, \Phi^{[I]}_{g,n}$.

The c-vectors of the seed corresponding to C are of the form $\langle S(I,a,b,\sigma), \_\rangle$ for suitable a,b,σ. Moreover, a stratum $S(I,a,b,\sigma)$ is contained in the open $\overline{\mathcal{M}}^S_{g,n}$ if and only if $\langle S(I,a,b,\sigma), \_\rangle$ lies in the dual cone of C in $\mathbb{E}^\vee_{g,n}$, i.e., is the positive span of some c-vector of $\mathcal{A}^{[I]}_{g,n}$.

We can also write conditions for a stratum with a $\mathbb{G}_m$ action is included or not, in terms of cluster algebras.

Proposition 4.2.46 (Principal very close degenerations)**.** *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Fix a principal stratum $S(I,a,b,\sigma)$ and let $\mathcal{S}$ be the stratum containing the central fiber of the principal degeneration associated to $S(I,a,b,\sigma)$. For a compatible set of roots $S = \{\beta_i\} \subset \Phi_{\geq -1}$, we have $\mathcal{S} \subset \overline{\mathcal{M}}^S_{g,n}$ if and only if the two condition above are satisfied:*

1. *if $[I] = [\{i\}]$ and $a = 2$, then $x_{1,\{i\}}(\beta_i) \leq 0$ for every $\beta_i \in S$;*
2. *$\langle S(I,a,b,\sigma), \beta_i \rangle = 0$ for every $\beta_i \in S$.*

Proof. If $\mathcal{S} \subset \overline{\mathcal{M}}^S_{g,n}$, then by openness we have $S(I,a,b,\sigma) \cup S(I,a,b,-\sigma) \subset \overline{\mathcal{M}}^S_{g,n}$. Conversely, if $\mathcal{S} \not\subset \overline{\mathcal{M}}^S_{g,n}$, there exists $S(x,y,I,\tau) \not\subset \overline{\mathcal{M}}^S_{g,n}$ with $\mathcal{S} \subset \overline{S(x,y,I,\tau)}$. Hence, either $S(I,a,b,\sigma) \subset \overline{S(x,y,I,\tau)}$ or $S(I,a,b,-\sigma) \subset \overline{S(x,y,I,\tau)}$, and thus $S(I,a,b,\sigma) \cup S(I,a,b,-\sigma) \not\subset \overline{\mathcal{M}}^S_{g,n}$.

Therefore, $\mathcal{S} \subset \overline{\mathcal{M}}^S_{g,n}$ if and only if both $S(I,a,b,\sigma)$ and $S(I,a,b,-\sigma)$ lie in $\overline{\mathcal{M}}^S_{g,n}$. The claim then follows from Proposition 4.2.44 applied to $S(I,a,b,\sigma)$ and $S(I,a,b,-\sigma)$. $\square$

Again, there is a corresponding interpretation in terms of c-vectors.

Remark 4.2.47. Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Let $S \subset (\Phi_{g,n})_{\geq -1}$ be a set of compatible roots with $[I] \notin \{[\emptyset], \mathrm{irr}\} \cup \{\{i\} \mid i \in [n]\}$. Then the stratum consisting of curves with an attached rosary with support (I,a,b) belongs to $\overline{\mathcal{M}}^S_{g,n}$ only if S is contained in the zero locus of the functional $c(S(I,a,b,\sigma))$.

4.2.5 Cones in $\Sigma_{g,n}$ with the same semistable locus

The theorem established in the previous section shows that any point lying in a cone of the cluster fan $\Sigma_{g,n}$ yields the same semistable locus. We now give an explicit description of when different cones give the same stable locus, in terms of an equivalence relation on cones that we construct. This is the missing piece needed to establish Theorem 4.2.2.

Definition 4.2.48 (Equivalence of cones)**.** Let $S \subset (\Phi_{g,n})_{\geq -1}$ be a compatible subset of roots, and let $\beta \in \Phi^{\{i\}}_{\geq -1} \setminus S$ be compatible with the elements of S. We say that S is *equivalent* to $S \cup \{\beta\}$ if one of the following holds:

1. $\beta = \sum_{j=3}^{k} \alpha_{j,\{i\}}$ and $\alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \sum_{j=3}^{k} \alpha_{j,\{i\}} \in S$.
2. $\beta = \alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \sum_{j=3}^{k} \alpha_{j,\{i\}}$, $\sum_{j=3}^{k} \alpha_{j,\{i\}} \in S \cup \{0\}$, and there exists $\gamma \in S$ with $x_{1,\{i\}}(\gamma) \neq 0$.
3. $\beta = -\alpha_{2,\{i\}}$ and $\alpha_{1,\{i\}} \in S$.

Let $\sim$ denote the equivalence relation among compatible subsets of $\Phi_{\geq -1}$ generated by the relation above. For two subsets S', S'', we write $S' \sim S''$ if they are equivalent.

The definition immediately implies:

- *Convexity:* if $S \subset S_1$ and $S \subset S_2$ are equivalent, then any stability condition S' with $S \subset S' \subset S_1 \cup S_2$ belongs to the same equivalence class.
- If $S_1 \sim S_2$ are compatible subsets of roots, then the symmetric difference $S_1 \,\Delta\, S_2 \subset \bigcup_{i\in[n]} (\Phi_{g,n}^{[\{i\}]})_{\geq -1}$. Moreover, if $(\Phi_{g,n}^{[\{i\}]})_{\geq -1} \cap S_1 \neq (\Phi_{g,n}^{[\{i\}]})_{\geq -1} \cap S_2$, then $\mathrm{Span}_+ S_1$ lies in the half-space $\{\beta \mid x_{1,\{i\}}(\beta) > 0\}$ (and similarly for S_2).

Lemma 4.2.49. *Let S be a compatible subset of roots. Assume that S and $S \cup \{\beta\}$ define the same stable locus in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$. Then necessarily there exists $i \in [n]$ such that:*

- $\beta \in (\Phi_{g,n}^{[\{i\}]})_{\geq -1}$*;*
- $\pi(\beta) \in \pi(S) \cup \{0\}$*, where* $\pi\colon \mathbb{E}_{g,n} \to \mathrm{Span}\{\alpha_{j,\{i\}} \mid j \geq 3\}$ *is the projection with respect to the basis* Δ*;*
- $\pi'(\beta) \in \{0, \alpha_{1,\{i\}} + \alpha_{2,\{i\}}, -\alpha_{2,\{i\}}\}$*, where* $\pi'\colon \mathbb{E}_{g,n} \to \mathrm{Span}\{\alpha_{1,\{i\}}, \alpha_{2,\{i\}}\}$ *is the projection with respect to the basis* Δ*.*

Proof. We prove the three properties in order. Assume by contradiction, that $\beta \in (\Phi_{g,n}^{[I]})_{\geq -1}$ with $[I] \neq [\{i\}]$. Then there exists a c-vector corresponding to a maximal face of a seed containing S in $\mathcal{A}_{g,n}^{[I]}$ which is negative on β. By Proposition 4.2.44, this defines a stratum $S(I,a,b,\sigma)$ with $\langle S(I,a,b,\sigma), \beta\rangle < 0$, hence $S(I,a,b,\sigma) \subset \overline{\mathcal{M}}_{g,n}^{S}$ but $S(I,a,b,\sigma) \not\subset \overline{\mathcal{M}}_{g,n}^{S\cup\{\beta\}}$, a contradiction. This proves the first claim.

Consider the cluster algebra $\mathcal{A}_{g,n}^{[\{i\}]}$ associated to a quiver Q, and let Q' be the full subquiver obtained by deleting the vertices in positions 1 and 2. The restriction functor $\operatorname{rep} Q \to \operatorname{rep} Q'$ sends compatible roots to compatible roots (cf. Definition-Proposition 4.2.23). If $\pi(\beta) \notin \pi(S) \cup \{0\}$, then there exists a c-vector of a seed containing $\pi(S)$ in the cluster fan of $\mathcal{A}(Q')$ that is negative on β. This c-vector lifts via $\mathbb{E}_{g,n} \to \mathrm{Span}\{\alpha_{j,\{i\}} \mid j \geq 3\}$ to a c-vector of $\mathcal{A}_{g,n}^{[\{i\}]}$ corresponding to a stratum $S(\{i\}, a, b, \sigma)$ with $a > 2$, contradicting Proposition 4.2.44. This proves the second claim.

For the last claim, observe that

$$\pi'(\beta) \in \{0, \alpha_{1,\{i\}}, -\alpha_{1,\{i\}}, \alpha_{1,\{i\}} + \alpha_{2,\{i\}}, \alpha_{2,\{i\}}, -\alpha_{2,\{i\}}\}.$$

We exclude the values that cannot occur. If $\pi'(\beta) = -\alpha_{1,\{i\}}$, then $\overline{\mathcal{M}}_{g,n}^{S\cup\{\beta\}}$ would fail to contain the stratum with an elliptic bridge, while $\overline{\mathcal{M}}_{g,n}^{S}$ does. If $\pi'(\beta) =$

$\alpha_{1,\{i\}}$, then $\beta = \alpha_{1,\{i\}}$ and the chain $S(1,2,\{i\},+1)$ is removed from $\overline{\mathcal{M}}^{S\cup\{\beta\}}_{g,n}$. By hypothesis there exists $\gamma \in S$ with $\langle S(1,2,\{i\},+1),\gamma\rangle > 0$, and compatibility forces $\gamma = -\alpha_{2,\{i\}}$. The roots $\delta \in (\Phi_{g,n})_{\geq -1}$ compatible with $-\alpha_{2,\{i\}}$ and satisfying $\langle S(1,1,\{i\},+1),\delta\rangle \neq 0$ (the elliptic bridge at $(1,\{i\})$) are $\pm\alpha_{1,\{i\}}$, contradicting that S and $S\cup\{\beta\}$ define the same locus, since $S(1,1,\{i\},+1) \subset \overline{\mathcal{M}}^S_{g,n} \setminus \overline{\mathcal{M}}^{S\cup\{\beta\}}_{g,n}$.

It remains to exclude $\pi'(\beta) = \alpha_{2,\{i\}}$. Suppose this is not the case, then $\beta = \sum_{j=2}^k \alpha_{j,\{i\}}$. From the previous paragraph, either $\sum_{j=1}^k \alpha_{j,\{i\}}$ or $\sum_{j=3}^k \alpha_{j,\{i\}}$ lies in S; call this root α. Assume first that $2 \mid k$ and consider the stratum $S(1,k+1,\{i\},-1)$ (resp. $S(1,k,\{i\},-1)$ if $g = k$). We have $\langle S(1,k+1,\{i\},-1),\beta\rangle > 0$ (resp. $\langle S(1,k,\{i\},-1),\beta\rangle > 0$), so there exists $\gamma \in S$ with $\langle S(\{i\},1,k+1,-1),\gamma\rangle > 0$ (resp. $\langle S(\{i\},1,k,-1),\gamma\rangle > 0$). Compatibility with β and α forces $\gamma = \sum_{j=t}^k \alpha_{j,\{i\}}$ with $2 < t \leq k$ and $2 \mid t$; take t minimal with this property.

Now consider $S(\{i\},1,t-2,-1)$. Since $\langle S(\{i\},1,t-2,-1),\beta\rangle > 0$, there exists $\gamma' \in S$ with $\langle S(\{i\},1,t-2,-1),\gamma'\rangle > 0$. Compatibility with β gives $x_{1,\{i\}}(\gamma') = 0$, while compatibility with α and γ forces $\gamma' = \sum_{j=t'}^k \alpha_{j,\{i\}}$ with $2 \mid t'$, contradicting the minimality of t. When $2 \nmid k$, the same argument works with $S(1,k-1,\{i\},-1)$ in place of $S(1,k,\{i\},-1)$. $\square$

Proposition 4.2.50. *Let S and $S\cup\{\beta\}$ be compatible subsets of $(\Phi_{g,n})_{\geq -1}$. Then $\overline{\mathcal{M}}^S_{g,n} = \overline{\mathcal{M}}^{S\cup\{\beta\}}_{g,n}$ if and only if β satisfies the conditions in Definition 4.2.48.*

Proof. Assume first that S and $S\cup\{\beta\}$ define the same locus. By Lemma 4.2.49, $\beta \in (\Phi^{[\{i\}]}_{g,n})_{\geq -1}$. Let π,π' denote the projections as in that lemma. We distinguish three cases according to $\pi'(\beta)$.

1. $\beta = -\alpha_{2,\{i\}}$. Since $\langle S(1,2,\{i\},+1),\beta\rangle > 0$, there exists $\alpha \in S$ such that
$$\langle S(\{i\},1,2,+1),\alpha\rangle > 0.$$
Compatibility with β forces $\alpha = \alpha_{1,\{i\}}$.

2. $\beta = \alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \sum_{j=3}^k \alpha_{j,\{i\}}$. Since $\langle S(1,1,\{i\},+1),\beta\rangle > 0$ (i.e., curves with an elliptic bridge of type $(1,\{i\})$ are removed), there exists $\gamma \in S$ such that $x_{1,\{i\}}(\gamma) = 1$. Thanks to Lemma 4.2.49, there exists $\alpha \in S\cup\{0\}$ with $\pi(\alpha) = \pi(\beta) = \sum_{j=3}^k \alpha_{j,\{i\}}$. We now prove that such an α can be chosen with $\pi'(\alpha) = 0$. If this were not the case, then $\alpha = \sum_{j=2}^k \alpha_{j,\{i\}}$. The remainder of the argument follows analogously to the last claim of the previous lemma. Assume first that $2 \nmid k$ and consider the stratum $S(1,k+1,\{i\},+1)$ (resp. $S(1,k,\{i\},+1)$ if $g = k$): we have $\langle S(1,k+1,\{i\},+1),\beta\rangle > 0$ (resp. $\langle S(1,k,\{i\},-1),\beta\rangle > 0$), hence there exists $\gamma \in S$ such that $\langle S(\{i\},1,k+1,+1),\gamma\rangle > 0$ (resp. $\langle S(\{i\},1,k,+1),\gamma\rangle > 0$). The compatibility of γ with β and α forces $\gamma = \sum_{j=t}^k \alpha_{j,\{i\}}$ with $2 < t \leq k$ and $2 \mid t$. Let t be minimal with this property. Now consider the stratum $S(\{i\},1,t-2,+1)$. Since $\langle S(\{i\},1,t-2,+1),\beta\rangle > 0$, there exists $\gamma' \in S$ such that $\langle S(\{i\},1,t-2,+1),\gamma'\rangle > 0$. Compatibility of γ' with β forces $x_{1,\{i\}}(\gamma') = 0$. Then compatibility of γ' with α and γ forces $\gamma' = \sum_{j=t'}^k \alpha_{j,\{i\}}$ with $2 \mid t'$, contradicting the minimality of t. For the case

$2 \mid k$, the proof proceeds by the same strategy, considering $S(1, k-1, \{i\}, +1)$ instead of $S(1, k, \{i\}, +1)$.

3. $\beta = \sum_{j=3}^{k} \alpha_{j,\{i\}}$. Again by Lemma 4.2.49, there exists $\alpha \in S \cup \{0\}$ with $\pi(\alpha) = \sum_{j=3}^{k} \alpha_{j,\{i\}}$, and the same argument as in (2) shows that one may choose α with $\pi'(\alpha) = \alpha_{1,\{i\}} + \alpha_{2,\{i\}}$.

For the converse implication, suppose S and β satisfy Definition 4.2.48. It suffices to show: if $\langle S(a,b,I,\sigma), \beta\rangle > 0$, then there exist $\alpha \in S$ and $\mathcal{S} \in \mathbb{S}_{g,n}$ with $S(a,b,I,\sigma) \subset \overline{\mathcal{S}}$ and $\langle \mathcal{S}, \alpha\rangle > 0$. If $I \neq \{i\}$ or $I = \{i\}$ with $a > 2$, there is nothing to prove. Thus, only the following cases remain:

1. $\beta = -\alpha_{2,\{i\}}$ and $\alpha_{1,\{i\}} \in S$. If $\langle S(a,b,\{i\}), \beta\rangle > 0$, the stratum has a tacnode at $(2,\{i\})$ and therefore lies in the closure of $S(1,2,\{i\},+1)$, which pairs positively with $\alpha_{1,\{i\}}$.

2. $\beta = \alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \sum_{j=3}^{k} \alpha_{j,\{i\}}$, with $\sum_{j=3}^{k} \alpha_{j,\{i\}} \in S \cup \{0\}$ and some $\alpha \in S$ satisfying $x_{1,\{i\}}(\alpha) = 1$. If $\langle S(a,b,\{i\}), \beta\rangle > 0$, then either $a = 1$ and
$$\langle S(a,b,\{i\},\sigma), \sum_{j=3}^{k} \alpha_{j,\{i\}}\rangle = \langle S(a,b,\{i\},\sigma), \beta\rangle > 0,$$
so the stratum is removed in $\overline{\mathcal{M}}_{g,n}^{S}$, or $a = 2$ and $\sigma = -1$. In the latter case $S(2,b,\{i\},-1) \subset \overline{S(1,1,\{i\},+1)}$, which is removed from $\overline{\mathcal{M}}_{g,n}^{S}$ because $\alpha \in S$.

3. $\beta = \sum_{j=3}^{k} \alpha_{j,\{i\}}$ and $\alpha = \alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \sum_{j=3}^{k} \alpha_{j,\{i\}} \in S$. If $\langle S(a,b,\{i\}), \beta\rangle > 0$, then either $a = 1$ and $\langle S(a,b,\{i\},\sigma), \sum_{j=3}^{k} \alpha_{j,\{i\}}\rangle = \langle S(a,b,\{i\},\sigma), \beta\rangle > 0$ (and hence the stratum is removed in $\overline{\mathcal{M}}_{g,n}^{S}$), or $a = 2$. If $\sigma = -1$ we again use $S(2,b,\{i\},-1) \subset \overline{S(1,1,\{i\},+1)}$; if $\sigma = +1$, then the constructible sets defined by $S(1,b,\{i\},+1)$ and $S(2,b,\{i\},+1)$ in $\overline{\mathcal{M}}_{g,n}^{S}$ coincide, and we conclude from $\langle S(1,b,\{i\},+1), \beta\rangle$.

□

Proof of Theorem 4.2.2. By Theorem 4.2.33, the assignment is well defined. Thus, it suffices to show that for any two compatible subsets $S_1, S_2 \subset \Phi_{\geq -1}$, we have $\overline{\mathcal{M}}_{g,n}^{S_1} = \overline{\mathcal{M}}_{g,n}^{S_2}$ if and only if $S_1 \sim S_2$. The if implication follows from Proposition 4.2.50; we prove the only if implication.

The stack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ satisfies the hypotheses of Theorem 2.2.7, thanks to the first two results in Section 4.1. Hence, the semistability regions are convex and form a fan.

Assume $\overline{\mathcal{M}}_{g,n}^{S_1} = \overline{\mathcal{M}}_{g,n}^{S_2}$. The observation above implies that there exists $S_3 \subset S_1$ and $S_3 \subset S_2$ with $\overline{\mathcal{M}}_{g,n}^{S_1} = \overline{\mathcal{M}}_{g,n}^{S_2} = \overline{\mathcal{M}}_{g,n}^{S_3}$. It is therefore enough to show $S_1 \sim S_3$ and $S_2 \sim S_3$, so we reduce to the case $S_1 \subset S_2$. Every compatible set T with $S_1 \subset T \subset S_2$ defines the same semistable locus. The claim then follows from Proposition 4.2.50 by induction on $|T \setminus S_1|$. □

Proof of Proposition 4.2.4. For $n = 0$ there is nothing to prove, since the generic point of any boundary divisor is semistable. We prove the theorem for $n > 0$. Assume that S_1, S_2 are contained in

$$H^- := \{\zeta \in \mathbb{E}_{g,n} \mid x_{1,\{i\}}(\zeta) \leq 0 \text{ for all } i \in [n]\},$$

then necessarily $S_1 \sim S_2$ if and only if $S_1 = S_2$. Hence, the assignment

$$\{C \in \Sigma_{g,n} \mid C \subset H^-\} \longrightarrow \left\{\overline{\mathcal{M}}_{g,n}(\tfrac{7}{10})^{L-\mathrm{ss}} \,\middle|\, L \in \mathbb{E}_{g,n}\right\},$$
$$C \longmapsto \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10})^{L-\mathrm{ss}} \quad \text{for any } L \in C,$$

is injective. An open $\overline{\mathcal{M}}^{L-\mathrm{ss}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is big if and only if no unstable stratum is a divisor. The stratum $S(a, b, I, \sigma)$ is a divisor if and only if $a = b = 1$, $\sigma = +1$, and $[I] = [\{i\}]$. For $n = 0$ the claim follows. For $n > 0$ we use Proposition 4.2.34, $\langle S(1, 1, \{i\}, +1), \beta\rangle \leq 0$ if and only if $x_{1,\{i\}}(\beta) \geq 0$, which proves the statement. □

In this section, the definition of $\overline{\mathcal{M}}^S_{g,n}$ via the pairing $\langle \_, \_ \rangle$ may seem rather cryptic, but providing a direct description in terms of the strata that are removed is quite involved. For the sake of completeness, we treat an easy case, namely $|S| = 1$ (which corresponds to the rays of the fan). We emphasize that the combinatorial description of the stable locus is in general intricate, and the cluster algebra formalism provides a clear way to deal with it. The following remark is a consequence of Proposition 4.2.34.

Remark 4.2.51 (Description of rays)**.** Let $S = \{\beta\} \subset (\Phi^{[I]}_{g,n})_{\geq -1}$ with $[I] \neq \mathrm{irr}$[1]. Then

$$\overline{\mathcal{M}}^{\{\beta\}}_{g,n} = \overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}) \setminus \bigcup_{\mathcal{S} \in J} \overline{\mathcal{S}},$$

where $J \subset \mathbb{S}_{g,n}$ is as follows:

- if $\beta = -\alpha_{i,I}$, then $J = \{S(I, i, i, (-1)^i)\}$, i.e. we remove only curves with a tacnode at (I, i);
- if $\beta = \sum_{j=i}^k \alpha_{I,j}$, then

$$\begin{aligned} J = &\{S(r, s, I, (-1)^{r+1}) \mid 2 \mid s - r,\ i \leq r \leq s \leq k\} \\ &\sqcup\ \{S(i-1, s, I, (-1)^{i+1}) \mid 2 \mid s - i,\ i > 1,\ i \leq s \leq k\} \\ &\sqcup\ \{S(r, k+1, I, (-1)^{k+1}) \mid 2 \mid k - r,\ k < g,\ i \leq r \leq k\} \\ &\sqcup\ \{S(i-1, k+1, I, (-1)^{i+1}) \mid 2 \mid i + k,\ i > 1,\ k < g\}; \end{aligned}$$

- if $n = 0$, $\beta = \sum_{j=i}^k \alpha_j$ and $k < g/2$, then

$$\begin{aligned} J = &\{S(r, s, (-1)^r) \mid 2 \mid s - r,\ i \leq r \leq s \leq k\} \\ &\sqcup\ \{S(i-1, s, (-1)^{i+1}) \mid 2 \mid s - i,\ i > 1,\ i \leq s \leq k\} \\ &\sqcup\ \{S(r, k+1, (-1)^{k+1}) \mid 2 \mid k - r,\ k < g,\ i \leq r \leq k\} \\ &\sqcup\ \{S(i-1, k+1, (-1)^{i+1}) \mid 2 \mid i + k,\ i > 1,\ k < g\}; \end{aligned}$$

[1] The case $[I] = \mathrm{irr}$ is already described in Proposition 4.2.36.

- if $n = 0$, $\beta = \sum_{j=i}^{\lfloor (g+1)/2 \rfloor} \alpha_j$, then

$$\begin{aligned} J = &\{S(r, s, (-1)^r) \mid 2 \mid s - r,\ i \le r \le s \le g - i\} \\ &\sqcup \{S(i-1, s, (-1)^{i+1}) \mid 2 \mid s - i,\ i > 1,\ i \le s \le g - i\} \\ &\sqcup \{S(i-1, g-i+1, (-1)^{i+1}) \mid 2 \mid g+1,\ i > 1\}. \end{aligned}$$

We can moreover characterize which opens are Deligne–Mumford.

Proposition 4.2.52. *Let (g, n) be a hyperbolic pair with $g \ge 2$ and $(g, n) \ne (2, 0)$. The stack $\overline{\mathcal{M}}_{g,n}^S$ is Deligne–Mumford if and only if:*

- *When (g, n) is such that ($n > 0$ and $g > 2$) or ($n = 0$ and $2 \nmid g$): $-\alpha_{\mathrm{irr}} \in S$ and S is equivalent to a maximal set of almost positive roots in $(\Phi_{g,n})_{\ge -1}$ (cf. Definition 4.2.48).*
- *When $(g, n) = (g, 0)$ and $2 \mid g$: $\{-\alpha_{\mathrm{irr}}, -\alpha_{(g-4)/2}\} \subset S$ and S is maximal.*
- *When $(g, n) = (2, n)$ with $n > 0$: $\{\alpha_{\mathrm{irr}}, -\alpha_{\mathrm{irr}}\} \cap S \ne \emptyset$ and S is equivalent to a maximal set of compatible roots.*

Here a compatible set $S \subset (\Phi_{g,n})_{\ge -1}$ is called maximal *if it is maximal with respect to inclusion.*

Proof. The open $\overline{\mathcal{M}}_{g,n}^S$ is Deligne–Mumford if and only if for every pointed curve C corresponding to a topological point of $\overline{\mathcal{M}}_{g,n}^S$, the connected component $\mathrm{Aut}(C)^\circ$ is trivial.

We first show necessity. If S is not equivalent to a maximal cone, then there exists a strict inclusion $\overline{\mathcal{M}}_{g,n}^{S'} \subsetneq \overline{\mathcal{M}}_{g,n}^S$ with $S' \subset S$. By Proposition 4.2.46, there is a c-vector relative to the fan $\Sigma_{g,n}$ whose hyperplane contains S but not S'. This yields a curve C which is a very close degeneration of a principal stratum with $\mathrm{Aut}(C)^\circ \ne 0$, and the point corresponding to C lies in $\overline{\mathcal{M}}_{g,n}^S$.

Furthermore, if $-\alpha_{\mathrm{irr}} \notin S$, then the stratum defined by the length-three irreducible rosary lies in $\overline{\mathcal{M}}_{g,n}^S$ whenever $g > 2$. For $g = 2$, the stated condition is necessary to exclude curves with a length-two attached rosary of type irr.

Assume now $n = 0$ and $2 \mid g$. If $\overline{\mathcal{M}}_g^S$ is DM, then S is maximal (since $\sim$ is trivial for $n = 0$), so there exists $\beta \in S$ with nonzero $\alpha_{(g-4)/2}$-coordinate. If $\beta \ne -\alpha_{(g-4)/2}$, then by Remark 4.1.8 the generic point of the stratum parametrizes a curve with a length-three attached rosary of type $\{((g-6)/2), ((g-4)/2), ((g-2)/2)\}$; hence the stack cannot be DM.

For sufficiency, assume S satisfies the listed conditions. Suppose, by contradiction, that $\overline{\mathcal{M}}_{g,n}^S$ contains a curve C with positive-dimensional stabilizer. Then C admits an attached rosary $(R, p, q) \to C$ corresponding to a very close degeneration of a stratum $S(a, b, I, \sigma)$. By hypothesis $I \ne \mathrm{irr}$ for $g > 2$. Consequently, every set $S' \sim S$ is contained in the zero locus of $\langle S(a, b, I, \sigma), \_\_ \rangle$. Since this functional cannot be identically zero when $n > 0$, S cannot be equivalent to a maximal set of compatible roots (i.e. to a seed), a contradiction for $n > 0$ and $g \neq 2$. When $g = 2$ and $n > 0$, it remains to analyze $\tau = \mathrm{irr}$, in which case (R, p, q) must have length 2. The condition $\{\alpha_{\mathrm{irr}}, -\alpha_{\mathrm{irr}}\} \cap S \neq \emptyset$ precludes such an attached rosary.

Remain the case when $n = 0$ and $\langle S(a, b, I, \sigma), \_\rangle$ is identically zero. This can occur only if $2 \mid g$ and $b = g - a$ (cf. Proposition 4.2.34). The very close degeneration associated to $S(a, b, I, \sigma)$ has a tacnode at position $(g-4)/2$. By openness of $\overline{\mathcal{M}}_g^S$, the stratum with a tacnode at $(g-4)/2$ lies in $\overline{\mathcal{M}}_g^S$; since $-\alpha_{(g-4)/2} \in S$, we reach a contradiction. □

4.3 The spaces $\overline{\mathrm{M}}_{g,n}^S$ and the maps $f_S^{S'}$

We continue our description of $\overline{\mathcal{M}}_{g,n}^S$ by analyzing the corresponding moduli spaces and the morphisms among them. The statements are mainly a consequence of the previous section and Chapter 2.

Theorem 4.3.1 (Theorem D for $g \geq 2$)**.** *Let (g, n) be a hyperbolic pair with $g \geq 2$ and $(g, n) \neq (2, 0)$. For any compatible subset $S \subset \Phi_{\geq -1}$ we have:*

- *The stack $\overline{\mathcal{M}}_{g,n}^S$ is algebraic, smooth, irreducible, and of finite type over k, and it contains the locus of smooth curves $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}^S$. Moreover, the partial order on the data induces a reverse inclusion at the level of stacks, i.e.,*

$$\overline{\mathcal{M}}_{g,n}^S \subset \overline{\mathcal{M}}_{g,n}^{S'} \qquad \textit{whenever } S \supset S'.$$

- *The stacks $\overline{\mathcal{M}}_{g,n}^S$ admit good moduli spaces $\phi^S \colon \overline{\mathcal{M}}_{g,n}^S \to \overline{\mathrm{M}}_{g,n}^S$, where $\overline{\mathrm{M}}_{g,n}^S$ is an integral, normal, projective k-scheme. For every $S \supset S'$ there are commutative diagrams*

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}^S \arrow[r, hook] \arrow[d, "\phi^S"] & \overline{\mathcal{M}}_{g,n}^{S'} \arrow[d, "\phi^{S'}"] \\ \overline{\mathrm{M}}_{g,n}^S \arrow[r, "f_S^{S'}"] & \overline{\mathrm{M}}_{g,n}^{S'} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{M}_{g,n} \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{g,n}^S \arrow[d, "\phi^S"] \\ \mathrm{M}_{g,n} \arrow[r, hook] & \overline{\mathrm{M}}_{g,n}^S \end{tikzcd},$$

  where $\overline{\mathrm{M}}_{g,n} \hookrightarrow \overline{\mathrm{M}}_{g,n}^S$ is an open immersion, and the maps $f_S^{S'}$ are projective birational Stein morphisms.

Finally, let $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ be an open substack that admits a projective good moduli space. Then there exists an S such that $\mathcal{U} = \overline{\mathcal{M}}_{g,n}^S$.

Proof. Since each $\overline{\mathcal{M}}_{g,n}^S$ is an open substack of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$, it is algebraic, smooth, irreducible, and of finite type over k. The final part of the first claim follows from Theorem 4.2.33.

The existence and projectivity of the good moduli space ϕ^S are obtained by applying Theorem 2.1.22 to the good moduli space map $\overline{\mathcal{M}}_{g,n}(\frac{7}{10}) \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$. Each $\overline{\mathrm{M}}_{g,n}^S$ is integral and normal because it is the good moduli space of an integral, normal stack.

The map $\mathrm{M}_{g,n} \to \overline{\mathrm{M}}^S_{g,n}$ is induced by the universal property of good moduli spaces. Every very close degeneration of a smooth curve is trivial, hence the locus of smooth curves is always stable; by Proposition 2.1.24, the map is an open immersion.

For the last point, the morphisms $f_S^{S'}$ are projective, birational (they are isomorphisms on the locus of smooth curves), and satisfy $(f_S^{S'})_*\mathcal{O}_{\overline{\mathrm{M}}^S_{g,n}} = \mathcal{O}_{\overline{\mathrm{M}}^{S'}_{g,n}}$ by Zariski's Main Theorem. Finally, the last claim is an immediate consequence of Proposition 2.2.16, given that every open immersion between connected proper schemes is an isomorphism. □

Notice that the maps $f_S^{S'}$ correspond to face maps in the fan $\Sigma_{g,n}$.

The proper spaces $\overline{\mathrm{M}}^S_{g,n}$ arise as good moduli spaces of stacks of curves that naturally contain $\mathrm{M}_{g,n}$ as an open dense subspace. These spaces are often referred to as *alternative modular compactifications of* $\mathrm{M}_{g,n}$.

We can describe the singularities appearing on $\overline{\mathrm{M}}^S_{g,n}$.

Proposition 4.3.2. *The normal spaces $\overline{\mathrm{M}}^S_{g,n}$ are Cohen–Macaulay. If* $\operatorname{char} k = 0$*, they are of KLT type. When $\overline{\mathcal{M}}^S_{g,n}$ is Deligne–Mumford (cf. Proposition 4.2.52), then $\overline{\mathrm{M}}^S_{g,n}$ has finite quotient singularities.*

Proof. Since $\overline{\mathcal{M}}^S_{g,n}$ is smooth and of finite type over k with affine diagonal, by the local structure theorem (Proposition 2.1.9) and [HR74, Main Theorem], $\overline{\mathrm{M}}^S_{g,n}$ is Cohen–Macaulay in any characteristic.
The second claim follows directly from [Bra+24, Thm. 5].
The last claim again follows from the local structure of $\overline{\mathcal{M}}^S_{g,n} \to \overline{\mathrm{M}}^S_{g,n}$ and from the smoothness of $\overline{\mathcal{M}}^S_{g,n}$. □

We observed in the previous sections that distinct cones of $\Sigma_{g,n}$ may yield the same semistable locus. There are even examples of cones with different semistable loci $\overline{\mathcal{M}}^S_{g,n}$ but isomorphic good moduli spaces. To describe this phenomenon, we introduce the following definition.

Definition 4.3.3 (Admissible)**.** Let $S \subset \Phi_{\geq -1}$ be compatible roots. Define $q_i \coloneqq \alpha_{1,\{i\}} + \alpha_{2,\{i\}}$ for any $i \in [n]$. We set:

$$
\begin{aligned}
S^{\mathrm{adm}} \coloneqq & \left(\bigcup_{i\in[n]} \{\beta - q_i \mid \beta \in S \text{ and } x_{1,\{i\}}(\beta) = 1\} \right) \setminus \{0\} \cup \\
& \bigcup_{i\in[n]} \{\beta \mid \beta \in S \cap \Phi^{[\{i\}]}_{g,n} \text{ and } x_{1,\{i\}}(\beta) \in \{0,-1\}\}.
\end{aligned}
$$

If $S = S^{\mathrm{adm}}$, we say that *S is admissible.*

Notice that $q_i = \alpha_{1,\{i\}} + \alpha_{2,\{i\}}$ is the class of $\delta_{1,\{i\}}$ in $\mathbb{E}_{g,n}$.

Remark 4.3.4. For any set of compatible roots $S \subset (\Phi_{g,n})_{\geq -1}$, S is admissible if and only if $S \subset H^-$. Hence, S is admissible if and only if $\overline{\mathcal{M}}^S_{g,n} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is a big open subset (cf. Proposition 4.2.4). In particular, if $n = 0$, every set of compatible roots S is admissible.

In this section we prove that $\overline{\mathrm{M}}^S_{g,n} = \overline{\mathrm{M}}^{S^{\mathrm{adm}}}_{g,n}$.

Lemma 4.3.5. *Let S_1, S_2 be two sets of compatible roots with $S_1 \subset S_2$. Then $S_1^{\mathrm{adm}} \subset S_2^{\mathrm{adm}}$. Moreover, if two sets of compatible roots are equivalent $S_1 \sim S_2$, then $S_1^{\mathrm{adm}} = S_2^{\mathrm{adm}}$.*

Proof. The first claim follows directly from the definition of admissibility. For the second, it suffices to prove the statement for $S_2 = S_1 \cup \{\beta\}$, where β is as in Definition 4.2.48. The claim then follows by a direct inspection of the three cases. □

Proposition 4.3.6. *Let S_1, S_2 be two sets of compatible roots such that $S_1^{\mathrm{adm}} = S_2^{\mathrm{adm}}$. Then the rational map induced by the open common dense substack $\mathrm{M}_{g,n}$*

$$\begin{tikzcd} \overline{\mathrm{M}}^{S_1}_{g,n} \arrow[r, dashed] & \overline{\mathrm{M}}^{S_2}_{g,n} \\ \mathrm{M}_{g,n} \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{M}_{g,n} \arrow[u, hook] \end{tikzcd}$$

extends to an isomorphism $\overline{\mathrm{M}}^{S_1}_{g,n} \xrightarrow{\sim} \overline{\mathrm{M}}^{S_2}_{g,n}$.

Additionally, we will show that the birational map $\overline{\mathrm{M}}^{S_1}_{g,n} \dashrightarrow \overline{\mathrm{M}}^{S_2}_{g,n}$ extends to a morphism if and only if $S_1^{\mathrm{adm}} \subset S_2^{\mathrm{adm}}$ (see Proposition 4.4.22). In that case, we obtain a description of the contracted loci in Theorem 4.3.11.

Example 4.3.7. Consider $\overline{\mathcal{M}}_{2,3}(\frac{7}{10})$. The fan $\Sigma_{2,3}$ is the cluster fan of an algebra of type $A_1 \times (A_2)^3$. We describe the A_2-type factor given by $\mathcal{A}^{[\{1\}]}_{2,3}$ in Figure 4.9: we picture the removed strata for every cone; cf. Remark 4.2.38.

Lemma 4.3.8. *Let (g, n) be a hyperbolic pair with $g \geq 2$ and $n > 0$. Fix a subset of compatible roots $S \subset (\Phi_{g,n})_{\geq -1}$. Then there exists $S' \sim S$ such that:*

1. *$|\{\beta \mid \beta \in S'$ and $x_{1,i}(\beta) = 1\}| \leq 1$ for all $i \in [n]$.*
2. *If $\beta \in S'$ and $x_{1,i}(\beta) = 1$, then $\beta - q_i \in S'$.*
3. *If there exists $\beta \in S'$ with $x_{1,i}(\beta) = 1$, then $2 \nmid |\mathrm{Supp}(\beta)|$ (i.e., β is the sum of an odd number of simple roots $\alpha_{k,\{i\}}$ with $1 \leq k \leq g$).*

Proof. Let $S' \sim S$ be a set of compatible roots that minimizes the cardinality of $E_i \coloneqq \{\beta \mid \beta \in S'$ and $x_{1,i}(\beta) = 1\}$. Assume, for contradiction, that $|E_i| > 1$. Then necessarily $-\alpha_{1,\{i\}} \notin E_i$, and we have two cases.

Case 1. There exists an element $\beta \in E_i$ such that $2 \mid |\mathrm{Supp}(\beta)|$. Let β be such an element with minimal support. We have $\beta \neq \alpha_{1,\{i\}} + \alpha_{2,\{i\}}$; otherwise, we could use (1) in Definition 4.2.48 to contradict the minimality of $|E_i|$. Then necessarily $\beta - q_i \in (\Phi_{g,n})_{\geq -1}$. We now show that $\beta - q_i$ is compatible with any root in $S' \setminus \{\beta - q_i\}$. Assume, for contradiction, that there exists $\beta' \in S'$ not compatible with $\beta - q_i$. There are two possibilities:

- $x_{1,i}(\beta') = 1$. By direct inspection of the compatibility conditions (cf. Proposition 4.2.22), we then have $x_{2,i}(\beta') = 1$, and β' has even support (by compatibility with β), strictly smaller than the support of β. This contradicts the minimality of β.
- $x_{1,i}(\beta') = 0$. Again, by direct inspection of the compatibility conditions (cf. Proposition 4.2.22), we see that β' and β are not compatible. This is a contradiction.

Now we have that $S' \cup \{\beta - q_i\}$ is a set of compatible roots and $|E_i| > 1$: thanks to (2) in Definition 4.2.48 we have $S \sim S' \cup \{\beta - q_i\} \sim (S' \cup \{\beta - q_i\}) \setminus \{\beta\}$. This contradicts the minimality assumption on $|E_i|$.

Case 2. All the elements in E_i have odd support. Let $\beta \in E_i$ be such an element with maximal norm. We have $\beta \neq \alpha_{1,\{i\}}$, otherwise $|E_i| = 1$. Then necessarily $\beta - q_i \in (\Phi_{g,n})_{\geq -1}$. We show again that $\beta - q_i$ is compatible with any root in $S' \setminus \{\beta - q_i\}$. We follow the same cases as for even support. If $x_{1,i}(\beta') = 1$, then necessarily β' has odd support, strictly larger than the support of β, a contradiction. If $x_{1,i}(\beta') = 0$, we conclude again by direct inspection of compatibility with β. We then argue as before.

Part (3) is now a direct consequence of the proof of Case 1: a set of compatible roots S' that realizes the minimum of $|E_i|$ does not contain any root of even support with $x_{1,i}(\beta) = 1$.

Part (2) then follows similarly. Let $S' \sim S$ with $E_i = \{\beta\}$. If $\beta = \alpha_{1,\{i\}}$, then for all $\gamma \in S'$, $x_{2,i}(\gamma) \in \{0, -1\}$. This implies that $-\alpha_{2,\{i\}}$ is compatible with any root in $S' \setminus \{-\alpha_{2,\{i\}}\}$. By condition (2) in Definition 4.2.48, we can then assume $-\alpha_{2,\{i\}} \in S'$, hence the claim. If $\beta \neq \alpha_{1,\{i\}}$, then, again by direct inspection of compatibility, $\beta - q_i$ is compatible with any different root of S'. Thanks to (1) in Definition 4.2.48, we can assume $\beta - q_i \in S'$. This concludes the proof. □

Proposition 4.3.9. *Let (g, n) be a hyperbolic pair with $g \geq 2$ and $n > 0$. Fix a subset of compatible roots $S \subset (\Phi_{g,n})_{\geq -1}$. Then there exists a set of compatible roots $S' \sim S$ such that $S^{\mathrm{adm}} = (S')^{\mathrm{adm}} \subset S'$. In that case,*

$$\overline{\mathcal{M}}_{g,n}^{S'} = \overline{\mathcal{M}}_{g,n}^{S} = \overline{\mathcal{M}}_{g,n}^{S^{\mathrm{adm}}} \setminus \bigcup_{i \in E} \overline{\delta_{1,\{i\}}}$$

where $E = \{i \in [n] \mid \exists \beta \in S$ such that $x_{1,i}(\beta) = 1\}$.

Proof. Let S' be a set of compatible roots equivalent to S as in Lemma 4.3.8. Then, by definition of admissible, we have $(S')^{\mathrm{adm}} \subset S'$. Lemma 4.3.5 implies that $S^{\mathrm{adm}} = (S')^{\mathrm{adm}} \subset S'$, hence the first claim.

For the second claim, we may assume that $S = S'$ satisfies Lemma 4.3.8. Since $\overline{S(1,1,\{i\},1)} = \overline{\delta_{1,\{i\}}}$, the inclusion $\overline{\mathcal{M}}^S_{g,n} \subset \overline{\mathcal{M}}^{S^{\mathrm{adm}}}_{g,n} \setminus \bigcup_{i\in E} \overline{\delta_{1,\{i\}}}$ is straightforward. We now prove the converse inclusion. Consider a point $q \in |\overline{\mathcal{M}}^{S^{\mathrm{adm}}}_{g,n}| \setminus |\overline{\mathcal{M}}^S_{g,n}|$. Then there exists $\beta \in S \setminus S^{\mathrm{adm}}$ and a stratum $S(a,b,I,\sigma)$ such that $\langle S(a,b,I,\sigma), \beta\rangle > 0$, $q \in |\overline{S(a,b,I,\sigma)}|$, and $\langle S(a,b,I,\sigma), \alpha\rangle \leq 0$ for any other $\alpha \in S^{\mathrm{adm}}$. Since $\beta \in S \setminus S^{\mathrm{adm}}$, there exists $i \in E$ with $x_{1,i}(\beta) = 1$: we then have $\langle S(a,b,I,\sigma), \beta\rangle > 0$ and $\langle S(a,b,I,\sigma), \beta - q_i\rangle \leq 0$. The latter conditions force $I = \{i\}$ and $a \leq 2$. If $a = 1$, then necessarily $b = 1$, hence $S(a,b,I,\sigma) = \overline{S(1,1,\{i\},1)}$. The remaining case is $a = 2$. Assume, for contradiction, that $q \notin \overline{S(1,1,\{i\},1)}$; then necessarily $\sigma = 1$. In this case $q \in \overline{S(2,b,\{i\},+1)}$ corresponds to the same substack $\overline{S(1,b,\{i\},+1)}$ in $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$. Since $\langle S(1,b,\{i\},+1), \beta\rangle = \langle S(2,b,\{i\},+1), \beta - q_i\rangle$, we have that $q \notin \overline{\mathcal{M}}^S_{g,n}$, and hence a contradiction. □

In a spirit opposite to Proposition 4.3.9, for a set of compatible roots S we can construct a minimal stack, by inclusion, among those having moduli space $\overline{\mathrm{M}}^S_{g,n}$.

Proposition 4.3.10. *Assume $n > 0$ and let S be an admissible set of compatible roots. Then there exists a $S' \subset \Phi_{\geq -1}$ such that the following holds: $S \subset S'$, $S'^{\mathrm{adm}} = S$, and $\overline{\mathcal{M}}^{S'}_{g,n}$ does not contain the stratum $S(1,1,\{i\},+1)$ for any $i \in [n]$ with $-\alpha_{1,\{i\}} \notin S$.*

Proof. We provide a construction of S'. Let $E = \{i \in [n] \mid -\alpha_{1,\{i\}} \notin S\}$, and denote by $V = S \setminus \bigcup_{i\in E} \Phi^{\{i\}}_{g,n}$ the roots that lie outside the root systems $\Phi^{\{i\}}_{g,n}$. If $\mathcal{A}^{\{i\}}_{g,n}$ is a cluster algebra of type A_2, the claim can be verified by direct inspection (cf. Example 4.3.7). Otherwise, denote

$$F_i = \{\beta \in S \mid x_{1,\{i\}}(\beta) = x_{2,\{i\}}(\beta) = 0 \text{ and } x_{3,\{i\}}(\beta) = 1\}$$

for $i \in E$. We define V_i for $i \in E$ according to the following cases:

- If $-\alpha_{2,\{i\}} \in S$, then $V_i = (S \cap \Phi^{\{i\}}_{g,n} \cup \{\alpha_{1,\{i\}}\}) \setminus \{-\alpha_{2,\{i\}}\}$.
- If $-\alpha_{2,\{i\}} \notin S$ and there exists $\beta \in F_i$ such that $2 \nmid |\mathrm{Supp}(\beta)|$, denote by γ the element of F_i with odd support that is minimal, and set $V_i = S \cap \Phi^{\{i\}}_{g,n} \cup \{\alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \gamma\}$.
- If $-\alpha_{2,\{i\}} \notin S$, $F_i \neq \emptyset$, and for every $\beta \in F_i$ we have $2 \mid |\mathrm{Supp}(\beta)|$, denote by γ the element of F_i with maximal (even) support. Define $V_i = S \cap \Phi^{\{i\}}_{g,n} \cup \{\alpha_{1,\{i\}} + \alpha_{2,\{i\}} + \gamma\}$.
- If none of the above occur (i.e., $-\alpha_{2,\{i\}} \notin S$ and $F_i = \emptyset$), then $V_i = S \cap \Phi^{\{i\}}_{g,n} \cup \{\alpha_{1,\{i\}} + \alpha_{2,\{i\}}\}$.

By direct inspection of the conditions in Proposition 4.2.22, in all four cases V_i is a compatible set. Then $S' = V \cup \bigcup_{i\in E} V_i$ satisfies $S'^{\mathrm{adm}} = S^{\mathrm{adm}}$ and it is a compatible set of roots, hence the claim follows. □

Proof of Proposition 4.3.6. It is enough to treat the case $S_1 = S$ and $S_2 = S^{\mathrm{adm}}$. The claim then follows from the isomorphisms $\overline{\mathrm{M}}^{S_1}_{g,n} \simeq \overline{\mathrm{M}}^{S_1^{\mathrm{adm}}}_{g,n} \simeq \overline{\mathrm{M}}^{S_2^{\mathrm{adm}}}_{g,n} \simeq \overline{\mathrm{M}}^{S_2}_{g,n}$. Thanks to Proposition 4.3.9, we may assume that $S^{\mathrm{adm}} \subset S$. The map described in the statement is the face map

$$f^{S^{\mathrm{adm}}}_S : \overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}^{S^{\mathrm{adm}}}_{g,n} .$$

We prove that this is an isomorphism. The map $f^{S^{\mathrm{adm}}}_S$ is birational between normal schemes of finite type over a field, thanks to Stein factorization, the claim follows if the exceptional locus is empty. It is therefore enough to show that $f^{S^{\mathrm{adm}}}_S$ induces a bijection on k-points (k is algebraically closed). Since $\overline{\mathcal{M}}^S_{g,n}$ admits a good moduli space, we have a bijection between k-points of $\overline{\mathrm{M}}^S_{g,n}$ and closed k-points of $\overline{\mathcal{M}}^S_{g,n}$. The claim is then equivalent to proving that, for any closed point $C \in \overline{\mathcal{M}}^{S^{\mathrm{adm}}}_{g,n}(k)$, there exists exactly one closed point $D \in \overline{\mathcal{M}}^S_{g,n}(k)$ that admits a very close degeneration $f\colon \Theta_k \to \overline{\mathcal{M}}^{S^{\mathrm{adm}}}_{g,n}$, where $f(1) \simeq D$ and $f(0) \simeq C$. It suffices to prove this when $C \in \overline{\mathcal{M}}^{S^{\mathrm{adm}}}_{g,n} \setminus \overline{\mathcal{M}}^S_{g,n}$.

Proposition 4.3.9 forces C to have at least one attached elliptic bridge in position $(1, \{i\})$ for $i \in E$ (with the notation of the proposition). Since C is closed in $\overline{\mathcal{M}}^{S^{\mathrm{adm}}}_{g,n}$, it has attached length-2 rosaries $R_j \subset C$ with a tacnode in position $(1, \{j\})$ for any $j \in E$ such that $C \in S(1, 1, \{j\}, +1)$. We denote by p_j these attaching nodes of the rosaries.

We can construct such a closed point $D \in \overline{\mathcal{M}}^S_{g,n}(k)$ by smoothing the nodes p_j on C, in such a way we remove the corresponding rational component of the attached rosary while leaving the rest of the curve unchanged. Let C' be such a modification of C. Then the very close degeneration restricts over $B\mathbb{G}_m$ to C and the one-parameter subgroup $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}^0(C)$ whose action has support on the attached rosaries $R_j \subset C$ and acts with weight -1 on p_j. This shows the existence of D.

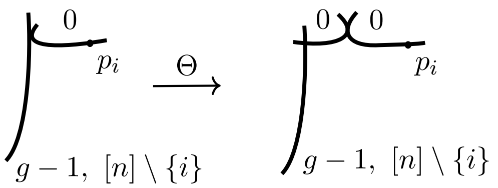


Figure 4.7. Example of a very close degeneration for a point D, when $E = \{i\}$.

It remains to prove uniqueness. Assume by contradiction that there exists another very close specialization to C whose generic fiber is a k-point $D' \neq D$ with D' a closed point in $\overline{\mathcal{M}}^S_{g,n}(k)$. In this very close degeneration the singularities p_j are all smoothed, and then the stratum containing D lies in the closure of the stratum containing D'. We can then apply Theorem 3.5.23 to deduce that there exists a very close degeneration of D' into D. This yields a contradiction because D and D' are both closed points of $\overline{\mathcal{M}}^S_{g,n}$. □

Theorem 4.3.11. *Let $S' \subset S$ be admissible sets of compatible roots. Then the exceptional locus of the map $f^{S'}_S \colon \overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}^{S'}_{g,n}$ is*

$$\mathrm{Exc}(f^{S'}_S) = \bigcup_{\Gamma \in U^{S'}_S} \overline{\phi^S(S_\Gamma)} \subset \overline{\mathrm{M}}^S_{g,n},$$

where $U^{S'}_S$ is a finite set of principal strata of $\mathbb{S}_{g,n}$ such that $\mathcal{S} \in U^{S'}_S$ if and only if:

- *$\langle \mathcal{S}, \beta \rangle = 0$ for all $\beta \in S'$;*
- *$\langle \mathcal{S}, \beta \rangle \geq 0$ for all $\beta \in S$;*
- *there exists $\beta \in S \setminus S'$ such that $\langle \mathcal{S}, \beta \rangle < 0$.*

The following remark will be used in the proof of the theorem.

Remark 4.3.12. The locus $\mathrm{Exc}(f^{S'}_S)$ is described as a redundant union of many closed subschemes. It would suffice to take the union over those strata $\mathcal{S}$ that satisfy the conditions above and for which $\overline{\phi^S(\mathcal{S})}$ is maximal by inclusion.

We use the following general characterization.

Remark 4.3.13. Let $\mathcal{U} \subset \mathcal{X}$ be an open inclusion of normal stacks of finite type over a field. Assume that the stacks admit good moduli spaces $\pi_U \colon \mathcal{U} \to U$ and $\pi \colon \mathcal{X} \to X$. Let $\phi \colon U \to X$ be the induced map between good moduli spaces, and assume that ϕ is birational and proper (and hence a Stein map).

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d, "\pi_U"] & \mathcal{X} \arrow[d, "\pi"] \\ U \arrow[r, "\phi"] & X \end{tikzcd}.$$

Then the k-points of the exceptional locus of ϕ are contained in the image of the k-points that are closed in $\mathcal{U}$ but not in $\mathcal{X}$. More generally, we have a closed immersion

$$\mathrm{Exc}(\phi) \hookrightarrow \pi_U(\pi^{-1}(\pi(\mathcal{X} \setminus \mathcal{U})) \cap \mathcal{U}),$$

which follows via base change with geometric points of $\phi(\mathrm{Exc})$.

Proof of Theorem 4.3.11. Thanks to Remark 4.3.13, any k-point in the exceptional locus is the image of a point in $\overline{\mathcal{M}}^S_{g,n}$ that does not lie in $\overline{\mathcal{M}}^{S'}_{g,n}$. By construction of the stacks $\overline{\mathcal{M}}^S_{g,n}$ and $\overline{\mathcal{M}}^{S'}_{g,n}$, we have

$$\mathrm{Exc}(f^{S'}_S) \subset \bigcup_{\Gamma \in U^{S'}_S} \overline{\phi^S(S_\Gamma)}^{\overline{\mathrm{M}}^S_{g,n}}.$$

It remains to prove the converse inclusion. We first show that the k-points of a stratum $\mathcal{S} \in U^{S'}_S$ are closed in $\overline{\mathcal{M}}^S_{g,n}$ whenever $\overline{\phi^S(\mathcal{S})}$ is maximal by inclusion. Assume, by contradiction, that this is not the case. Then there exists a stratum $\mathcal{S} \in U^{S'}_S$ and a curve $C \in \mathcal{S}(k)$ that admits a non-degenerate very close degeneration to a curve D. Thanks to Theorem 3.5.23, the curve D has some attached rosaries.

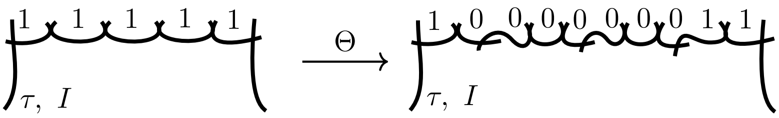


Figure 4.8. Example of a very close degeneration of a point in a principal stratum.

The curve D has at least one attached elliptic chain $D' \to D$ with support contained in the support of the stratum $\mathcal{S}$ and with support disjoint from the rosaries. When $\mathcal{S} \not\subset \overline{S(2,2,\{i\},-1)}$ for any $i \in [n]$, this follows from the fact that $\langle \mathcal{S}, \beta \rangle \neq 0$ for at least one $\beta \in S$. If instead $\mathcal{S} \subset \overline{S(2,2,\{i\},-1)}$, we use that $S(1,1,\{i\}) \subset \overline{\mathcal{M}}^{S}_{g,n}$ because S is admissible. In this case, by Θ-completeness (cf. Technique 3), we can assume $D \in \overline{S(1,1,\{i\})}$ and the claim follows similarly.
In both cases we can assume that there exists $\beta \in S$ with $\langle \mathcal{S}_1, \beta \rangle > 0$, where $\mathcal{S}_1 \in \mathbb{S}_{g,n}$ is the principal stratum defined by an attached elliptic chain in the same position as $D' \to D$. A direct inspection of the three conditions shows that $\mathcal{S}_1 \in U^{S'}_{S}$. Thanks to Theorem 3.5.23, we can use the very close degeneration in $\overline{\mathcal{M}}^{S}_{g,n}$ constructed above to show that the generic point of $\mathcal{S}$ has a very close degeneration to a stratum corresponding to the dual graph of D. Therefore, we have $\phi^S(\mathcal{S}) \subset \overline{\phi^S(\mathcal{S}_1)}$, which provides a contradiction.

It remains to prove that, for any stratum $\mathcal{S} = S(a,b,I,\sigma) \in U^{S'}_{S}$ such that $\overline{\phi^S(\mathcal{S})}$ is maximal by inclusion, the locus $\overline{\phi^S(\mathcal{S})}$ is contracted by $f^{S'}_{S}$. By hypothesis, $S(a,b,I,-\sigma) \subset \overline{\mathcal{M}}^{S'}_{g,n}$, hence the strata given by an attached rosary of type (a,b,I) is contained in $\overline{\mathcal{M}}^{S'}_{g,n}$. We conclude because two k-points of the stratum $S(a,b,I)$ constructed by attaching an elliptic chain with different moduli are sent to the same point via $f^{S'}_{S}$. Hence, the map contracts $\overline{\phi^S(S(a,b,I,\sigma))}$. Notice again that there are at least two such different k-points because $S(a,b,I,\sigma) \neq S(1,1,\{i\},-1)$, which follows from S admissible. □

Remark 4.3.14. Let $S' \subset S$ be admissible sets of almost positive roots. Then, among the contracted strata in $U^{S'}_{S}$ (with the notation of Theorem 4.3.11), the stratum $S(1,1,\{i\},-1)$ for $i \in [n]$ never appears.

Example 4.3.15. Let $S \subset \Phi^+$ be compatible roots. Then the contraction

$$\overline{\mathrm{M}}^{S'}_{g,n} \to \overline{\mathrm{M}}^{\emptyset}_{g,n} = \overline{\mathrm{M}}_{g,n}(\tfrac{7}{10})$$

has exceptional locus given by the closure of principal strata $S(k,k,I,+1)$ for those (k,I) such that $x_{k,I}(\beta) = 1$ for some $\beta \in S'$.

Via an analysis of the dimension of $\overline{\phi^S(S_\Gamma)}$, it is easy to determine when the contractions $f^{S'}_{S}$ are small or divisorial, as we now explain. For this purpose, we introduce the following.

Definition 4.3.16 (Divisorial part)**.** Let $S \subset \Phi_{\geq -1}$ be compatible roots. We define

$$S^{\mathrm{div}} = S^{\mathrm{adm}} \setminus \{-\alpha_{1,\{i\}} \mid i \in [n]\}.$$

Moreover, define $V_{g,n} := \bigcap_{i \in [n]} \{x_{1,i} = 0\}$.

Notice that for any set of compatible roots S, $S^{\mathrm{div}} \subset V_{g,n}$. If S is admissible, then the assignment $S \mapsto S^{\mathrm{div}}$ corresponds to the projection along the coordinates $x_{1,i}$. If $n = 0$, then $S = S^{\mathrm{div}}$ and $V_{g,n} = \mathbb{E}_{g,n}$.

In the next sections, we will see that the sets of compatible roots with $S = S^{\mathrm{div}}$ provide the partial $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$.

Theorem 4.3.17 (Factorization of $f^{S'}_S$)**.** *Let $S' \subset S$ be two sets of compatible roots. Then the induced morphism at the level of good moduli spaces factors through*

$$f^{S'}_S : \overline{\mathrm{M}}^S_{g,n} \xrightarrow{f^V_S} \overline{\mathrm{M}}^V_{g,n} \xrightarrow{f^{S'}_V} \overline{\mathrm{M}}^{S'}_{g,n},$$

where $V = S \setminus \{-\alpha_{1,\{i\}} \mid -\alpha_{1,\{i\}} \in S \setminus S'\} \subset \Phi_{\geq -1}$; notice that $S' \subset V \subset S$.

Here f^V_S is a divisorial contraction and $f^{S'}_V$ is a small contraction. The map f^V_S contracts the divisors $\phi^S(\delta_{1,\{i\}})$ for those i with $-\alpha_{1,\{i\}} \in S \setminus S'$. In particular:

- *$f^{S'}_S$ is a divisorial contraction if and only if ${S'}^{\mathrm{div}} = S^{\mathrm{div}}$.*
- *$f^{S'}_S$ is a small contraction if and only if $S \cap \{-\alpha_{1,i}\}_{i\in[n]} = S' \cap \{-\alpha_{1,i}\}_{i\in[n]}$.*

Proof. Thanks to Proposition 4.3.6, we can assume that S, S', and V are admissible. The claim follows from the computation of the dimensions of $\mathrm{Exc}(f^V_S)$ and $\mathrm{Exc}(f^{S'}_V)$ using Theorem 4.3.11, as we now explain.
The closed subscheme $\overline{\phi^S(\mathcal{S})} \subset \overline{\mathrm{M}}^S_{g,n}$ has dimension strictly less than $3g - 3 - 1$ for any $\mathcal{S} \in \mathbb{S}_{g,n} \setminus \{S(1, 1, \{i\}, +1) \mid i \in [n]\}$: this follows by a computation using the crimping space (cf. [Wyc10, Example 1.112]), observing that any such stratum contains a tacnode.
Let U^V_S be as in Theorem 4.3.11. We have

$$U^V_S \cap \{S(1, 1, \{i\}, +1) \mid i \in [n]\} = \{S(1, 1, \{i\}, +1) \mid -\alpha_{1,\{i\}} \in S \setminus S'\}.$$

Define $E \coloneqq \{i \in [n] \mid -\alpha_{1,\{i\}} \in S \setminus S'\}$. We have an inclusion $S(1, 1, \{i\}, +1) \subset \overline{\mathcal{M}}^S_{g,n}$ for any $i \in E$. Notice that $\overline{S(1, 1, \{i\}, +1)}^{\overline{\mathcal{M}}^S_{g,n}} = (\delta_{1,\{i\}})_{|\overline{\mathcal{M}}^S_{g,n}}$.
Thanks to Theorem 3.5.23, any point in $S(1, 1, \{i\}, +1)$ does not admit a non-trivial very close specialization in $\overline{\mathcal{M}}^S_{g,n}$ for $i \in E$. This implies that $\dim \overline{\phi^S(S(1, 1, \{i\}, +1))} = \dim(\delta_{1,\{i\}})_{|\overline{\mathcal{M}}^S_{g,n}} = 3g - 3 + n - 1$, hence it has codimension 1 in $\overline{\mathrm{M}}^S_{g,n}$ and is a contracted divisor. The last two claims follow from the above description. $\square$

Corollary 4.3.18. *Given two set of compatible roots $S' \subset S \subset \Phi_{\geq -1}$, there exists a natural commutative diagram*

$$\begin{tikzcd}
\overline{\mathrm{M}}^S_{g,n} \arrow[r, "f^{S'}_S"] \arrow[d] & \overline{\mathrm{M}}^{S'}_{g,n} \arrow[d] \\
\overline{\mathrm{M}}^{S^{\mathrm{div}}}_{g,n} \arrow[r] & \overline{\mathrm{M}}^{{S'}^{\mathrm{div}}}_{g,n}
\end{tikzcd}$$

where the vertical arrows are divisorial contractions, and the bottom horizontal arrow is a small contraction.

We now summarize some properties of S^{adm} and S^{div} according to the following cases:

- If $S \subset V_{g,n} = \bigcap_{i\in[n]}\{x_{1,i} = 0\}$, then $S = S^{\mathrm{adm}} = S^{\mathrm{div}}$. In this case $\overline{\mathcal{M}}^{S}_{g,n} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is big, and the induced map of good moduli spaces $\overline{\mathrm{M}}^{S}_{g,n} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ is small. We will see in Section 4.4.2 that this is the support of the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$.
- If $S \neq S^{\mathrm{adm}}$, then the open substack $\overline{\mathcal{M}}^{S}_{g,n} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is not big; in particular, it does not contain the generic point of $\delta_{1,\{i\}}$ whenever $\Phi^{[I]}_{g,n} \cap S \neq \Phi^{[I]}_{g,n} \cap S^{\mathrm{adm}}$.
- If $P = S^{\mathrm{adm}} \setminus S^{\mathrm{div}} \subset \{-\alpha_{1,\{i\}} \mid i \in [n]\}$ is nonempty, then for any $i \in [n]$ such that $-\alpha_{1,\{i\}} \in P$, the divisor $\delta_{1,\{i\}}$ is contracted along the map $\overline{\mathrm{M}}^{S}_{g,n} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$. In particular, the map is not small.

4.3.1 The case $g = 1$

The techniques above also apply to the case $(g, n) = (1, n)$ for $n \geq 1$, although the picture degenerates as we now explain. Any tacnode in $\widetilde{\mathcal{M}}^{3}_{1,n}$ is separating (for genus reasons), and the Picard group of the stack differs.

Throughout this section we work over an algebraically closed field k with $\operatorname{char} k = 0$ or $\operatorname{char} k > 3$. The latter condition corresponds to a characteristic large enough with respect to $(1, n)$ (cf. [CTV23, Def. 4.1]).

When $n = 1$, $\overline{\mathcal{M}}_{1,1}(\frac{7}{10}) = \emptyset$, because any curve has an attached elliptic tail. We therefore assume $n \geq 2$. We start by describing the Picard group.

Proposition 4.3.19. *[CTV23, Fact 3.28, Cor.3.29] The group* $\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{1,n}(\frac{7}{10})\right)$ *has dimension* $2^n - n - 1$*, and it is generated by* δ_{irr} *and* $\delta_{i,I}$ *for* $[i, I] \in T_{g,n} \setminus \{\mathrm{irr}, [1, \emptyset]\}$ *with the following relations:*

$$\delta_{\mathrm{irr}} + 12 \sum_{\substack{[0,I]\in T_{g,n}\setminus\{[1,\emptyset],\mathrm{irr}\} \\ p\in I}} \delta_{0,I} = 0, \quad \textit{for any } 1 \leq p \leq n.$$

In particular, any $L \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{1,n}(\frac{7}{10})\right)$ *can be written (non-uniquely) as*

$$L = \sum_{\emptyset\subsetneq I\subsetneq[n]} b_I\, \delta_{0,I}.$$

We can compute the weights of one-parameter subgroups analogously to Proposition 4.1.5. The proposition simplifies because there are no non-separating tacnodes, and any attached rosary has length 2 for genus reasons.

Proposition 4.3.20 (Weights for attached rosaries)**.** *Fix an integer* $n \geq 2$*, and consider a* $\mathbb{Q}$*-line bundle on* $\overline{\mathcal{M}}_{1,n}(\frac{7}{10})$ *written as in Proposition 4.3.19. Let* $(R, q_1, q_2) \to$

$(C,\{p_i\})$ be an attached rosary of length 2, and consider the one-parameter subgroup $\rho_R : \mathbb{G}_m \xrightarrow{\sim} \mathrm{Aut}(R,q_1,q_2)^o \subset \mathrm{Aut}(C,\{p_i\})$, normalized so that $\mathrm{wt}_{\rho_R}(T_{q_1}(R)) = 1$. We must have $\mathrm{Supp}(R,q_1,q_2) = \{[1,I]\}$ for $\emptyset \subsetneq I \subsetneq [n]$, and then:

$$\langle L, \rho_R \rangle = -b_I - b_{I^c}.$$

Similarly to Section 4.1.1, we have the following:

Proposition 4.3.21. *The rational Picard group of $\overline{\mathrm{M}}_{1,n}(\frac{7}{10})$ has dimension $2^{n-1}-n$, and it is given by*

$$\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{1,n}\left(\frac{7}{10}\right)\right) = \langle \delta_{0,I} - \delta_{0,I^c} \mid \emptyset \subsetneq I \subsetneq [n] \rangle \subset \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{1,n}\left(\frac{7}{10}\right)\right).$$

The space $\mathbb{E}_{1,n}$ has dimension $2^{n-1}-1$ with base

$$\mathbb{E}_{1,n} \simeq \bigoplus_{I \in \mathcal{P}\setminus\{[n]\}} \mathbb{Q}\,\alpha_I,$$

where α_I is the image of $-\delta_{0,I}$ under the quotient map, and $\mathcal{P}$ is defined as in Equation (4.3).

Proof. The only non-trivial claim is that the α_I are independent in $\mathbb{E}_{1,n}$. For any pair of integers $1 \le p, q \le n$, we have

$$\sum_{\substack{[0,I]\in T_{g,n}\setminus\{[1,\emptyset],\mathrm{irr}\} \\ p \in I}} \delta_{0,I} = \sum_{\substack{[0,I]\in T_{g,n}\setminus\{[1,\emptyset],\mathrm{irr}\} \\ q \in I}} \delta_{0,I}$$

in $\mathbb{E}_{1,n}$, by replacing each $\delta_{0,I}$ on the left-hand side with δ_{0,I^c} whenever $q \notin I$ on the right-hand side. This proves the independence of the α_I, and the dimension count follows. $\square$

Analogously to Definition 4.2.1, we define a root system on a basis of $\mathbb{E}_{1,n}$ such that the $\{\alpha_I\}_{I\in\mathcal{P}}$ in Proposition 4.3.20 are its coordinates.

Definition 4.3.22. Fix an integer $n \ge 2$. For any $I \in \mathcal{P}\setminus\{[n]\}$ (cf. Equation (4.3)), define:

- $\Phi^I_{1,n} \subset \mathbb{E}_{1,n}$ to be the root system of type A_1 with simple base $\{\alpha_I\} = \Delta^I_{1,n}$, where α_I is as in Proposition 4.3.21;
- $\mathcal{A}^I_{1,n}$ to be the cluster algebra constructed from the initial quiver of type A_1 (with odd vertices as sinks), identifying the associated root system with $\Phi^I_{1,n}$.

Finally, define the cluster algebra given by the product of the above types. It comes with the associated full-rank root system $\Phi_{1,n} = \bigsqcup_{I\in\mathcal{P}\setminus\{[n]\}} \Phi^I_{1,n} \subset \mathbb{E}_{1,n}$. We denote by $\Sigma_{1,n}$ the d-vector cluster fan in $\mathbb{E}_{1,n}$, which is of type

$$(A_1)^{2^{n-1}-1}.$$

To describe the semistability fan in the case $(1,n)$, we do not need any additional equivalence relation on the cones of $\Sigma_{1,n}$.

Theorem 4.3.23. *Fix an integer $n \geq 2$. The fan $\Sigma_{1,n}$ with support in $\mathbb{E}_{1,n}$ is the semistability fan for $\overline{\mathcal{M}}_{1,n}(\frac{7}{10})$. There is a well-defined bijection of posets*

$$(\Sigma_{1,n}, \supset) \longrightarrow \left(\left\{\overline{\mathcal{M}}_{1,n}\left(\frac{7}{10}\right)^{L-\mathrm{ss}} \mid L \in \mathbb{E}_{1,n}\right\}, \subset\right)$$
$$C \longmapsto \overline{\mathcal{M}}_{1,n}\left(\frac{7}{10}\right)^{L-\mathrm{ss}}, \quad \text{for any } L \in C^{\circ}.$$

For a set of compatible roots $S \subset (\Phi_{1,n})_{\geq -1}$, we denote by $\overline{\mathcal{M}}_{1,n}^{S}$ the corresponding stable locus.

Proof. The proof that $\Sigma_{1,n}$ is a coarsening of the stability condition fan proceeds as in Theorem 4.2.33. The claim then follows by directly analyzing which principal strata are included in the semistable locus associated with each cone, in this case the proof is extremely simplified from the fact that any principal strata has genus 1. □

We can give the following explicit description of our stacks, where we denote $\Phi := \Phi_{1,n}$.

Proposition 4.3.24. *Let $S \subset \Phi_{\geq -1}$ be a set of compatible roots. Then:*

$$\overline{\mathcal{M}}_{1,n}^{S} = \overline{\mathcal{M}}_{1,n}\left(\frac{7}{10}\right) \setminus \left(\bigcup_{\alpha_I \in S} \overline{S(1,1,I,+1)} \cup \bigcup_{-\alpha_I \in S} \overline{S(1,1,I,-1)}\right).$$

In particular, $\overline{\mathcal{M}}_{1,n}^{S}$ is Deligne–Mumford if and only if $\mathrm{Span}_{+} S \in \Sigma_{1,n}$ *is a maximal cone.*

The walls in a cluster fan are defined by c-vectors; in this case, the analogy with strata is trivial, since each wall in $\Delta_{1,n}^{I}$ is just the origin.

We have the following proposition, whose proof is analogous to the case $g > 1$.

Proposition 4.3.25. *There exist good moduli spaces $\overline{\mathcal{M}}_{1,n}^{S} \to \overline{\mathrm{M}}_{1,n}^{S}$ and maps $f_{S}^{S'}$ induced by inclusions. For $(1,n)$ with $n \geq 3$, an analogue of Theorem 4.3.1 holds (completing the statement of Theorem D).*

For $n = 2$, the fan $\Sigma_{1,2}$ has one-dimensional support. We have:

$$\overline{\mathcal{M}}_{1,2}^{S} = \begin{cases} \overline{\mathcal{M}}_{1,2}^{\mathrm{ps}}, & \text{if } S = \{-\alpha_{\{1\}}\}, \\ \overline{\mathcal{M}}_{1,2}(\frac{7}{10}), & \text{if } S = \emptyset, \\ \emptyset, & \text{if } S = \{\alpha_{\{1\}}\}. \end{cases}$$

Finally, the projective normal schemes $\overline{\mathrm{M}}_{1,n}^{S}$ are Cohen–Macaulay. If $\mathrm{char}\, k = 0$*, they are of KLT type. When $\overline{\mathcal{M}}_{g,n}^{S}$ is Deligne–Mumford (cf. Proposition 4.2.52), then $\overline{\mathrm{M}}_{g,n}^{S}$ has finite quotient singularities (cf. Proposition 4.3.2).*

We now analyze the spaces $\overline{\mathrm{M}}_{1,n}^{S}$ and the maps among them. We can define admissibility in a simpler way. The proof of the proposition above follows verbatim from Proposition 4.3.6.

Definition-Proposition 4.3.26 (Admissible)**.** Fix an integer $n \geq 3$. Let $S \subset \Phi_{\geq -1}$ be a set of compatible roots, and set

$$S^{\mathrm{adm}} := S \cap -\Delta_{1,n}.$$

If $S = S^{\mathrm{adm}}$, we say that S *is admissible*. Let S_1, S_2 be two sets of compatible roots such that $S_1^{\mathrm{adm}} = S_2^{\mathrm{adm}}$. Then the rational map induced by the common open dense $\mathrm{M}_{1,n}$

$$\begin{tikzcd} \overline{\mathrm{M}}^{S_1}_{1,n} \arrow[r, dashed] & \overline{\mathrm{M}}^{S_2}_{1,n} \\ \mathrm{M}_{1,n} \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{M}_{1,n} \arrow[u, hook] \end{tikzcd}$$

extends to an isomorphism $\overline{\mathrm{M}}^{S_1}_{1,n} \xrightarrow{\sim} \overline{\mathrm{M}}^{S_2}_{1,n}$.

We now provide a simpler description of the contracted locus.

Theorem 4.3.27. *Fix an integer $n \geq 3$ and consider $S' \subset S$ admissible. Then the exceptional locus of the map $f_S^{S'} \colon \overline{\mathrm{M}}^S_{1,n} \to \overline{\mathrm{M}}^{S'}_{1,n}$ is given by*

$$\mathrm{Exc}(f_S^{S'}) = \bigcup_{-\alpha_I \in S' \setminus S} \overline{\phi^S(\delta_{0,I})} \subset \overline{\mathrm{M}}^S_{1,n}.$$

In this case, $f_S^{S'}$ is a divisorial contraction of $\delta_{0,I} \in \mathrm{Pic}(\overline{\mathcal{M}}^{S'}_{1,n})$ (notice that these divisors descend).

Proof. The first claim follows analogously to Theorem 4.3.11, and the contracted locus is a divisor. ☐

Corollary 4.3.28. *The projective normal spaces $\overline{\mathrm{M}}^S_{1,n}$ are $\mathbb{Q}$-factorial for any set of compatible roots $S \subset \Phi_{\geq -1}$. In particular,*

$$\dim \mathrm{Pic}_{\mathbb{Q}} \left(\overline{\mathrm{M}}^S_{1,n}\right) = |S \cap -\Delta_{1,n}| + 2^{n-1} - n.$$

4.3.2 Comparison with previously known compactifications

We can now compare our construction with the compactifications established in [CTV23]. In what follows, we denote by $\overline{\mathcal{M}}^T_{g,n}$ and $\overline{\mathcal{M}}^{T+}_{g,n}$, for $T \subset T_{g,n}$, the stacks defined in loc. cit. In this context, T^{adm} is defined as in [CTV23, Def. 3.21].

Proposition 4.3.29. *Let (g,n) be a hyperbolic pair with $g \geq 1$ and $(g,n) \neq (2,0)$. We have:*

1. $\overline{\mathcal{M}}^T_{g,n} = \overline{\mathcal{M}}^S_{g,n}$, *where* $S = \{-\alpha_{i,I} \mid \{[i,I],[i+1,I]\} \not\subset T^{\mathrm{adm}}\}$;
2. $\overline{\mathcal{M}}^{T+}_{g,n} = \overline{\mathcal{M}}^S_{g,n}$, *where*

$$S = \{-\alpha_{i,I} \mid \{[i,I],[i+1,I]\} \not\subset T^{\mathrm{adm}}\} \cup$$
$$\cup \left\{ \sum_{i=k}^{j} \alpha_{i,I} \;\middle|\; \{[k-1,I],\ldots,[j,I]\} \subset T \textit{ is maximal by inclusion} \right\}.$$

Where T^{adm}, $\overline{\mathcal{M}}^T_{g,n}$, and $\overline{\mathcal{M}}^{T+}_{g,n}$ are defined as in [CTV23].

Proof. The first statement follows since $\overline{\mathcal{M}}^T_{g,n}$ parametrizes curves having a tacnode only in position (i, I) for which $\{[i, I], [i+1, I]\} \not\subset T^{\mathrm{adm}}$. The second follows because $\overline{\mathcal{M}}^{T+}_{g,n} \subset \overline{\mathcal{M}}^T_{g,n}$ is obtained by removing the locus containing A_1/A_1 attached elliptic chains whose bridges and tacnodes are positioned in T. Such A_1/A_1 chains correspond to strata $S(a, b, I, (-1)^{a+1})$ with $2 \nmid b-a$, hence $\langle S(a, b, I, (-1)^{a+1}), \sum_{i=k}^{j} \alpha_{i,I} \rangle > 0$. These are the unique principal chains with tacnodes of type in T whose pairing with $\sum_{i=k}^{j} \alpha_{i,I}$ is positive. $\square$

Notice that the contracted loci among these compactifications, described in [CTV23], are recovered as special cases of Theorem 4.3.11 for the maps $f^{S'}_S$. This also recovers the contracted loci of the first flip in the Hassett–Keel program [HH13]. The previous remark, together with Theorem 4.3.1, provides as a byproduct an answer to the third open question in [CTV23].

Corollary 4.3.30. *Let k be an algebraically closed field with* $\mathrm{char}\, k \gg (g, n)$*. For any hyperbolic pair (g, n) with $(g, n) \neq (2, 0)$, the spaces $\overline{\mathrm{M}}^T_{g,n}$ and $\overline{\mathrm{M}}^{T+}_{g,n}$ are projective over k, even in positive characteristic.*

4.4 Birational geometry of $\overline{\mathrm{M}}^S_{g,n}$

In this section, we generalize results from [CTV23] and [CTV21]. In particular, we construct the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, this answer open questions (1) and (2) in [CTV23].

4.4.1 Picard and Class Groups of $\overline{\mathrm{M}}^S_{g,n}$

We now describe the Picard and class groups of the spaces $\overline{\mathrm{M}}^S_{g,n}$. This will allow us to further analyze the singularities of the spaces and some relevant properties from a birational prospective. The following cube helps visualize the various groups, where $S' \subset S$ are sets of compatible almost positive roots.

$$\begin{tikzcd}
 & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \arrow[dd, dashed, "\phi^S_*"'] & & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^{S'}_{g,n}\right) \arrow[ll, two heads] \arrow[dd, dashed, "\phi^{S'}_*"'] \\
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \arrow[ur, "\sim"] & & \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^{S'}_{g,n}\right) \arrow[ll, two heads, "(i^{S'}_S)^*"'] \arrow[ur, "\sim"] & \\
 & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \arrow[rr, two heads] & & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S'}_{g,n}\right) \\
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \arrow[uu, hook, "\phi^{S*}"] \arrow[ur, hook] & & \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S'}_{g,n}\right) \arrow[ll, hook', "(f^{S'}_S)^*"] \arrow[uu, hook, "\phi^{S'*}"'] \arrow[ur, hook] &
\end{tikzcd} \tag{4.4}$$

If we ignore the two dashed arrows ϕ^S_* and $\phi^{S'}_*$, then all three square diagrams commute. The bottom square commutes by the projection formula (cf. [Ful13, Prop. 2.3(c)]); the vertical square commutes by functoriality of Picard groups; and the top one commutes because open immersions are flat (cf. [Ful13, Prop. 2.3(d)]).

The surjectivity of $(i^{S'}_S)^*$ follows from [Alp13, Lemma 11.13]. The injections and surjections in the bottom diagram follow because the corresponding maps are proper birational morphisms between normal schemes (cf. [EGAIV, p. IV.21.6]).

The pushforwards of cycles along ϕ^S and $\phi^{S'}$ are not defined in general, since these maps are not separated and hence not proper. The stack $\overline{\mathcal{M}}^S_{g,n}$ is a properly stable smooth algebraic stack (cf. [ES16, Def. 1.7]), as it always contains the saturated Deligne–Mumford substack $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}^S_{g,n}$. Theorem 2.1 loc. cit. provides a possible notion of pushforward to the good moduli spaces $\phi^S_*\colon \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \to \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$, although commutativity of the cube is not guaranteed. This is related to [ES16, Conj. 3.2]. We will therefore construct an ad hoc pushforward making the above diagram commutative.

We now compute all the groups appearing in Diagram 4.4.

Recall that for $S = \{-\alpha \mid \alpha \in \Delta_{g,n}\}$ we have $\overline{\mathcal{M}}^S_{g,n} = \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ (cf. Remark 4.2.38). The inclusion $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is a big open immersion, and we will leverage on this to compute the groups.

Over any field k with $\mathrm{char}\, k \neq 2$, the main result of [FV20] describes $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ as follows. Using the notation of loc. cit., define the set

$$\Lambda_{g,n} = \{\lambda, \delta_{\mathrm{irr}}\} \cup \{\delta_{a,A}\colon 0 \le a \le g,\, \emptyset \subseteq A \subseteq [n],\, (a,A) \neq (0,\emptyset), (g,[n])\}.$$

For $g \ge 2$ and $g+n > 2$, the Picard group is expressed as the quotient of the free $\mathbb{Z}$-module generated by $\Lambda_{g,n}$:

$$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \simeq {\mathrm{Span}_{\mathbb{Z}}\, \Lambda_{g,n}}\Big/{(\delta_{a,A} - \delta_{g-a,A^c} \mid \delta_{a,A} \in \Lambda_{g,n})}.$$

Consider the inclusions

$$\overline{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}^{\mathrm{wps}}_{g,n} \supset \overline{\mathcal{M}}^{\mathrm{ps}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right).$$

By excision for the class group (cf. [Kre98, Prop. 2.4.1]) and since class and Picard groups agree for smooth stacks, we obtain the induced maps

$$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \xleftarrow{\sim} \mathrm{Pic}(\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n}) \to {\mathrm{Pic}(\overline{\mathcal{M}}^{\mathrm{wps}}_{g,n})}\Big/{\langle \delta_{1,\emptyset}\rangle} \simeq \mathrm{Pic}(\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}) \xleftarrow{\sim} \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right)\right).$$

The isomorphisms and quotients above follow by dimension computations for the complements of the open immersions (cf. [CTV23, Corollary 3.29]).In the following propositions, we identify the Picard groups of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ and $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ with the quotients prescribed above.

Moreover, by Definition-Proposition 2.1.30, we have the diagram

$$\begin{tikzcd} \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\tfrac{7}{10})\right) \arrow[r, two heads, "i_S^*"] \arrow[d, two heads, "\pi"] & \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}^S\right) \arrow[d, two heads] \\ \mathbb{E}_{g,n} \arrow[r, two heads] & \mathbb{E}\left(\overline{\mathcal{M}}_{g,n}^S\right) \end{tikzcd} \tag{4.5}$$

where, again by excision, both horizontal maps are surjective.

Proposition 4.4.1 (Picard groups)**.** *For any set of compatible roots S, the inclusion $i_S\colon \overline{\mathcal{M}}_{g,n}^S \hookrightarrow \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ induces isomorphisms*

$$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}^S) \simeq {\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}(\tfrac{7}{10}))}\big/{\mathrm{Span}_{\mathbb{Z}}\{\delta_{1,\{i\}} \mid \exists\, \alpha \in S \ s.t.\ x_{1,\{i\}}(\alpha) = 1\}},$$

where the relations are empty when $n = 0$. Moreover, we can describe the rational line bundle that descend via

$$\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}^S\right) = i_S^*(\pi^{-1}(\mathrm{Span}_{\mathbb{Q}}\, S)) \subset \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}^S\right)$$

Where the morphisms are given in the diagram 4.5.

These identifications are compatible with the pullback maps in Diagram 4.4:

$$\begin{aligned} (i_S^{S'})^*\colon\ \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}^{S'}\right) &\to \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}^S\right), \\ (f_S^{S'})^*\colon\ \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}^{S'}\right) &\to \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}^{S'}\right). \end{aligned}$$

Proof. For the first claim, apply excision (cf. [Kre98, Prop. 2.4.1]):

$$A_*\left(\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right) \setminus \overline{\mathcal{M}}_{g,n}^S\right) \longrightarrow A_*\left(\overline{\mathcal{M}}_{g,n}\left(\frac{7}{10}\right)\right) \xrightarrow{i^*} A_*\left(\overline{\mathcal{M}}_{g,n}^S\right) \longrightarrow 0.$$

The closed substack $\overline{\mathcal{M}}_{g,n}(\frac{7}{10}) \setminus \overline{\mathcal{M}}_{g,n}^S$ has irreducible components of dimension $3g - 3 + n - 1$, given by the divisors

$$\{\delta_{1,\{i\}} \mid \exists\, \alpha \in S \text{ such that } x_{1,\{i\}}(\alpha) = -1\}.$$

Since $\overline{\mathcal{M}}_{g,n}^S$ is smooth, the claim follows. Compatibility follows from the functoriality of the pullback on class groups induced by $i_S^{S'}$.

The second claim follows from Proposition 2.2.11. Finally, compatibility for $f_S^{S'}$ follows from the commutative diagram above and the functoriality of $i_S^{S'}$:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}^S \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{g,n}^{S'} \arrow[d] \\ \overline{\mathrm{M}}_{g,n}^S \arrow[r, "f_S^{S'}"] & \overline{\mathrm{M}}_{g,n}^{S'}. \end{tikzcd}$$

∎

Theorem 4.4.2 (Class groups)**.** *For any set of compatible roots S, the class group of $\overline{\mathrm{M}}^S_{g,n}$ satisfies*

$$\mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) = {\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right)}\Big/{\mathrm{Span}_{\mathbb{Q}}\{\delta_{1,\{i\}} \mid -\alpha_{1,\{i\}} \notin S\}}.$$

These identifications are compatible with the pushforwards of class groups induced by the face maps $f_S^{S'}$. Furthermore, the inclusion $\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \hookrightarrow \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$ is given by the natural map

$$i_S^*(\pi^{-1}(\mathrm{Span}_{\mathbb{Q}} S)) \subset \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \longrightarrow {\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right)}\Big/{\mathrm{Span}_{\mathbb{Q}}\{\delta_{1,\{i\}} \mid -\alpha_{1,\{i\}} \notin S\}}.$$

Proof. We first compute the dimension of $\mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}(\frac{7}{10})\right)$, following an argument analogous to that of [CTV23, Prop. 3.16]. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}\right) & & \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\frac{7}{10})\right) \arrow[ll, "\sim"'] \\
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}\right) \arrow[u, "\wr"] \arrow[r, "\sim"] & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}\right) \arrow[r, two heads, "f_*"] & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}(\frac{7}{10})\right)
\end{tikzcd}.$$

By Theorem 4.3.17, the map f_* contracts precisely the independent cycles $\delta_{1,\{i\}}$ for $i \in [n]$. The vertical arrow is an isomorphism because $\overline{\mathcal{M}}^{\mathrm{ps}}_{g,n}$ is Deligne–Mumford. Hence, the claim follows for $S = \emptyset$.

Assume now that S is a maximal set of compatible roots (by inclusion) among the admissible ones. This implies that S contains $-\alpha_{i,I}$ for every $i \in [n]$. We have the following diagram, where the top horizontal map is an isomorphism by Remark 4.3.4:

$$\begin{tikzcd}
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^{S}_{g,n}\right) & & \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\frac{7}{10})\right) \arrow[ll, "\sim"'] \\
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S}_{g,n}\right) \arrow[u, "\wr"] \arrow[r, hook] & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S}_{g,n}\right) \arrow[r, two heads] & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g,n}(\frac{7}{10})\right)
\end{tikzcd}.$$

By Proposition 4.4.1 and the maximality of S, the vertical arrow is an isomorphism. Once again, the map among class groups is surjective and contracts exactly the independent cycles $\delta_{1,\{i\}}$ for all $i \in [n]$. Thus, comparing dimensions shows that the inclusion in the bottom row is an isomorphism, so $\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) = \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$, both vector spaces of dimension $\dim_{\mathbb{Q}} \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right)$.

For the general case, we can reduce to S admissible by Proposition 4.3.6. Consider then a maximal admissible set of almost positive roots $S' \supset S$. By Theorem 4.3.17, the surjective pushforward morphism

$$\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S'}_{g,n}\right) = \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S'}_{g,n}\right) \longrightarrow \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S}_{g,n}\right)$$

contracts exactly the independent divisors $\{\delta_{1,\{i\}} \mid -\alpha_{1,\{i\}} \in S\}$, yielding the claim. The presentation is compatible with pushforward by construction. Finally, for S admissible and maximal, the inclusion $\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \hookrightarrow \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$ is obvious. The general case follows from the projection formula (cf. [Ful13, Prop. 2.3(c)]). $\square$

We can now define the pushforward of codimension 1 cycles as follows.

Definition 4.4.3. We define the map ϕ^S_* as the natural projection in the diagram

$$\begin{tikzcd}
\mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \arrow[rr, "\phi^S_*"] & & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \\
\dfrac{\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\frac{7}{10})\right)}{\mathrm{Span}_{\mathbb{Q}}\{\delta_{1,\{i\}} \mid \exists \alpha \in S \text{ s.t. } x_{1,\{i\}}(\alpha) = 1\}} \arrow[u, "\wr"] & & \dfrac{\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}(\frac{7}{10})\right)}{\mathrm{Span}_{\mathbb{Q}}\{\delta_{1,\{i\}} \mid -\alpha_{1,\{i\}} \notin S\}} \arrow[u, "\wr"]
\end{tikzcd}.$$

By direct inspection, the map ϕ^S_* makes Diagram 4.4 commute, and in particular, the squares commute:

$$\begin{tikzcd}
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \arrow[r, "\sim"] & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \arrow[d, "\phi^S_*"] \\
\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \arrow[u, hook] \arrow[r, hook] & \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)
\end{tikzcd}.$$

Remark 4.4.4. For the surjection ϕ^S_* we have

$$\mathrm{Ker}\, \phi^S_* = \mathrm{Span}\{\delta_{1,\{i\}} \mid \forall \alpha \in S,\, x_{1,\{i\}}(\alpha) = 0\}.$$

An element $\phi^S_*(D) \in \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$ is $\mathbb{Q}$-Cartier if and only if there exists $E \in \mathrm{Ker}\, \phi^S_*$ such that $D + E \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$ under the identifications in the diagram above.

We now express the canonical divisors in terms of the above divisors.

Proposition 4.4.5. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Consider a compatible subset $S \subset \Phi_{\geq -1}$, and denote by $\delta := \delta_{\mathrm{irr}} + \sum_{[i,I] \in T'_{g,n}} \delta_{i,I}$ the total boundary, where*

$$T'_{g,n} := T_{g,n} \setminus (\{[i,I] \mid i = 0,\, |I| = 1\} \cup \{[1,\emptyset], \mathrm{irr}\})$$

(cf. Proposition 4.1.4). Then

$$\begin{aligned}
K_{\overline{\mathcal{M}}^S_{g,n}} &= 13\lambda - 2\delta + \psi \in \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right), \\
K_{\overline{\mathrm{M}}^S_{g,n}} &= 13\lambda - 2\delta + \psi \in \mathrm{Cl}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right), \quad \textit{if } (g,n) \neq (2,1), (3,0),
\end{aligned}$$

where the formulae are considered in the quotients described in Theorem 4.4.2 and Proposition 4.4.1. Notice that ψ vanishes when $n = 0$.

Proof. We have open inclusions of smooth algebraic stacks $\overline{\mathcal{M}}^S_{g,n} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$, so the formula for the stacks follows by pulling back the Mumford formula from [CTV23, Fact 3.28(2)].

The moduli spaces are normal, so their canonical classes exist as Weil divisors. Let $S' \supset S^{\mathrm{adm}}$ be the divisor provided in Proposition 4.3.10 for S^{adm}. Since $S^{\mathrm{adm}} = S'^{\mathrm{adm}}$, we have $\overline{\mathrm{M}}^S_{g,n} \simeq \overline{\mathrm{M}}^{S'}_{g,n}$, and it suffices to prove the claim for $S = S'$. The claim holds if $\overline{\mathcal{M}}^S_{g,n} \to \overline{\mathrm{M}}^S_{g,n}$ is an isomorphism in codimension 1. For this, it is enough to find a saturated big open $\mathcal{U} \subset \overline{\mathcal{M}}^S_{g,n}$ such that $\phi^S_{|\mathcal{U}} \colon \mathcal{U} \to \phi^S(\mathcal{U})$ is an isomorphism. By definition of saturated, $\phi^S(\mathcal{U})$ is open (cf. Definition 2.1.25) and in that case big. Let $\mathcal{U} \subset \overline{\mathcal{M}}^S_{g,n}$ be the locus of nodal curves with at most one singularity. This is a big open subset $\mathcal{U} \subset \overline{\mathcal{M}}^S_{g,n}$ since it contains the generic points of all boundary divisors. The open substack of $\mathcal{M}_{g,n}$ consisting of curves with trivial automorphism group has codimension 2 for $g \geq 4$, $(2,n)$ with $n \geq 2$, and $(3,n)$ with $n \geq 1$ (cf. [Cor87], [ACG+11, Prop. 2.5]). Moreover, the generic points of the boundary divisors of $\overline{\mathcal{M}}^S_{g,n}$ are trivial because $\delta_{1,\{i\}} \cap \overline{\mathcal{M}}^S_{g,n} = \emptyset$. Hence, the open substack $\mathcal{U}' \subset \mathcal{U}$ consisting of curves with trivial automorphism group has codimension 2 for $(g,n) \neq (2,1), (3,0)$. We now prove that $\mathcal{U}' \subset \overline{\mathcal{M}}^S_{g,n}$ is saturated. By Proposition 2.1.27, it suffices to show that every very close degeneration in $\overline{\mathcal{M}}^S_{g,n}$ with generic fiber in $\mathcal{U}'$ is trivial. This follows from the assumptions $g \geq 2$, $(g,n) \neq (2,0)$, and the fact that $\delta_{1,\{i\}} \cap \mathcal{U} = \emptyset$. □

Corollary 4.4.6. *Consider an hyperbolic pair (g,n) such that $g \geq 2$ and $(g,n) \neq (2,0), (2,1), (3,0)$. Then*

$$\phi^S_*(K_{\overline{\mathcal{M}}^S_{g,n}}) = K_{\overline{\mathrm{M}}^S_{g,n}}.$$

Remark 4.4.7. If $(g,n) = (2,1)$ or $(3,0)$, then $\Sigma_{g,n}$ is a cluster fan of type A_1, and the possible values for S are $\emptyset$, $\{\alpha_{\mathrm{irr}}\}$, and $\{-\alpha_{\mathrm{irr}}\}$ (cf. Definition 4.2.1). All these compactifications already appear in [CTV23], as explained in Proposition 4.3.29. A partial description of $K_{\overline{\mathrm{M}}^{\mathrm{ps}}_{g,n}}$ in these cases is given in [CTV23, Rmk. 5.4].

By direct computation, we can express the canonical classes in the basis $\Delta_{g,n}$ introduced in Definition 4.1.7.

Corollary 4.4.8. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Consider the map $\pi_S \colon \mathrm{Pic}(\overline{\mathcal{M}}^S_{g,n}) \to \mathbb{E}(\overline{\mathcal{M}}^S_{g,n})$ from Proposition-Definition 2.1.30. Then*

$$\pi_S(K_{\overline{\mathcal{M}}^S_{g,n}}) = \begin{cases} 7(\alpha_{\mathrm{irr}} + \sum_{(i,I)\in T_{g,n}} \alpha_{i,I}) - \sum_{i\in[n]} \alpha_{1,\{i\}}, & \text{if } n \neq 0, \\ 7(\alpha_{\mathrm{irr}} + \sum_{i\in T_g} \alpha_i), & \text{if } n = 0, \end{cases}$$

where the expressions are written in terms of the basis $\Delta_{g,n}$ under the image of $\mathbb{E}_{g,n} \to \mathbb{E}(\overline{\mathcal{M}}^S_{g,n})$. Moreover, for $n > 0$ we have

$$\pi_S(K_{\overline{\mathcal{M}}^S_{g,n}} + \psi) = 7(\alpha_{\mathrm{irr}} + \sum_{[i,I]\in T_{g,n}\setminus\{\mathrm{irr},[1,\emptyset]\}} \alpha_i).$$

The formula above highlights the naturality of considering the minimal model program with respect to the divisor $K_{\overline{\mathcal{M}}_{g,n}} + \psi$ (cf. [CTV23, Corollary 7.30]). In our setting, this class corresponds to a positive multiple of the longest root in $\Phi^{+}_{g,n}$.

Theorem 4.4.9. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. Then the projective and Cohen–Macaulay spaces $\overline{\mathrm{M}}_{g,n}^{S}$ over k (cf. Proposition 4.3.2) satisfy:*

1. *$\overline{\mathrm{M}}_{g,n}^{S}$ is $\mathbb{Q}$-factorial if and only if $S^{\mathrm{div}} = (\mathrm{Span}_{+} S^{\mathrm{adm}}) \cap V_{g,n}$ is a full-dimensional cone in $V_{g,n}$ (cf. Definition 4.3.16).*
2. *$\overline{\mathrm{M}}_{g,n}^{S}$ is $\mathbb{Q}$-Gorenstein if and only if $\Delta^{\mathrm{irr}}_{g,n} \cap S \neq \emptyset$ and, for any $I \subset [n]$, there exists a subset of roots $\{\beta_j\}_{j\in J} \subset S^{\mathrm{adm}} \cap \Phi^{[I]}_{\geq -1}$ whose supports $\mathrm{Supp}(\beta_j)$ (cf. Definition 4.2.21) are pairwise disjoint and satisfy:*
   - ***Case 1:** If $I = [\{i\}]$ and $-\alpha_{1,\{i\}} \notin S^{\mathrm{adm}}$, then $\bigcup_{j\in J} \mathrm{Supp}(\beta_j) = [2,n]$ and $\{\alpha_{2,i}, -\alpha_{2,i}\} \cap S^{\mathrm{adm}} \neq \emptyset$.*
   - ***Case 2:** If the previous condition does not hold, then $\bigcup_{j\in J} \mathrm{Supp}(\beta_j) = \mathrm{Supp}\left(\Delta^{[I]}_{g,n}\right)$.*
3. *When $n \geq 1$ and $(g,n) \neq (2,1)$, the divisor $K_{\overline{\mathrm{M}}_{g,n}^{S}} + \psi$ is $\mathbb{Q}$-Cartier if and only if $\Delta^{\mathrm{irr}} \cap S \neq \emptyset$ and, for any $I \subset [n]$, there exists a subset of roots $\{\beta_j\}_{j\in J} \subset S^{\mathrm{adm}} \cap \Phi^{[I]}_{\geq -1}$ with pairwise disjoint supports such that one of the following holds:*
   - *$\bigcup_{j\in J} \mathrm{Supp}(\beta_j) = [1,n]$;*
   - *$\bigcup_{j\in J} \mathrm{Supp}(\beta_j) = [3,n]$;*
   - *$I = [\{i\}]$, $\alpha_{2,\{i\}} \in S^{\mathrm{adm}}$, and $\bigcup_{j\in J} \mathrm{Supp}(\beta_j) = [2,n]$.*

Theorem 4.4.9 generalizes [CTV21, Prop. 3.16] and [CTV23, Corollary 7.9] to the spaces $\overline{\mathrm{M}}_{g,n}^{S}$.

We first establish a combinatorial lemma on cluster algebras.

Lemma 4.4.10. *Let $\mathcal{A}$ be a cluster algebra with initial quiver of type A_n and alternating orientation (where the odd vertices are sinks), or defined by folding as in Proposition 4.2.29. Denote $\Delta = \{\alpha_i\}_{i\in[n]}$ the simple basis and consider a set of compatible roots $S \subset \Phi_{\geq -1}$. Then $\alpha = \sum_{i\in[n]} \alpha_i \in \mathrm{Span}_{\mathbb{Q}} S$ if and only if there exists a subset $B = \{\beta_j\}_{j\in J} \subset S$ such that*

$$\alpha = \sum_{j\in J,\, -\beta_j \notin \Delta} \beta_j - \sum_{j\in J,\, -\beta_j\in\Delta} \beta_j.$$

In this case, the combination is unique, the supports of the β_j are pairwise disjoint, and $\bigcup_{j\in J} \mathrm{Supp}(\beta_j) = [n]$.

Proof. For the type A_n case, we argue by induction on n. The uniqueness statement follows from the linear independence of compatible roots (cf. Proposition 4.2.19).

For $n = 1$, the claim is trivial. Assume it holds for A_n. Let $\Phi_n \subset E_n$ be the root system with simple basis $\{\alpha_i\}_{i\in[n]} = \Delta_n$ in the $\mathbb{Q}$-vector space E_n.

Consider the projection $f\colon E_{n+1} \to E_n$ given by forgetting the coordinate α_{n+1}. By Definition-Proposition 4.2.23, any compatible set of roots $S \subset (\Phi_{n+1})_{\geq -1}$ maps to a compatible set $f(S) \setminus \{0\} \subset (\Phi_n)_{\geq -1}$. Let $S \subset (\Phi_{n+1})_{\geq -1}$ be such that $\alpha = \sum_{i\in[n+1]} \alpha_i \in \mathrm{Span}_{\mathbb{Q}}(S)$. Therefore,

$$\alpha = \sum_{\gamma\in R} \lambda_\gamma \gamma,$$

for some $R \subset S$ and $\lambda_\gamma \in \mathbb{Q} \setminus \{0\}$. By the inductive hypothesis applied to $f(\alpha)$, there exist $\{\beta_j\}_{j\in J} \subset \Phi_n$ such that

$$\sum_{i\in[n]} \alpha_i = f(\alpha) = \sum_{\gamma\in S} \lambda_\gamma f(\gamma) = \sum_{\substack{j\in J\\ -\beta_j\notin\Delta_n}} \beta_j - \sum_{\substack{j\in J\\ -\beta_j\in\Delta_n}} \beta_j.$$

Assume first that the roots $\{f(\gamma)\}_{\gamma\in R}$ are linearly independent. Then they correspond exactly to the set $\{\beta_j\}_{j\in J}$ and the claim follows. We now prove that the only possible source of linear dependence is the presence of some $\gamma \in R$ with $f(\gamma) = 0$. Assume by contradiction, that $f(\gamma) \neq 0$ for all $\gamma \in R$. Since any compatible set of almost positive roots is linearly independent, this would imply the existence of $\gamma_1, \gamma_2 \in R$ with the same image $f(\gamma_1) = f(\gamma_2)$. This forces

$$\gamma_1 = \sum_{i=k}^{n+1} \alpha_i, \qquad \gamma_2 = \sum_{i=k}^{n} \alpha_i$$

for some $1 \leq k \leq n$. By the inductive hypothesis, we obtain $\lambda_{\gamma_1} + \lambda_{\gamma_2} = 1$. Moreover, γ_1 and γ_2 are the only roots whose support contains n. By compatibility, the roots $\pm\alpha_{n+1}$ cannot lie in R. Thus, γ_1 is the unique root in R whose support contains $n+1$, which forces $\lambda_{\gamma_1} = 1$ and hence $\lambda_{\gamma_2} = 0$, contradicting the assumption $\lambda_{\gamma_2} \neq 0$. Therefore, some $\gamma \in R$ must satisfy $f(\gamma) = 0$.

An element $\gamma \in R$ such that $f(\gamma) = 0$ is $\pm\alpha_{n+1}$. If $-\alpha_{n+1} \in S$, then every other $\gamma \in R$ has support contained in $[1, n]$, which forces $\lambda_{-\alpha_{n+1}} = -1$, yielding the desired expression and uniqueness.

If instead $\alpha_{n+1} \in S$, then by compatibility any $\beta \in S \setminus (-\Delta_{n+1})$ satisfies either $\mathrm{Supp}(\beta) \subset [1, n-1]$ or $[n, n+1] \subset \mathrm{Supp}(\beta)$. If there is $\gamma \in R$ satisfies $[n, n+1] \subset \mathrm{Supp}(\gamma)$, then projecting

$$\sum_{\gamma\in R} \lambda_\gamma \gamma$$

onto $\langle \alpha_n, \alpha_{n+1} \rangle$ in the basis Δ_{n+1} forces the coefficient of α_{n+1} to vanish, which is impossible. Hence,

$$\mathrm{Supp}(\gamma) \subset [1, n-1] \cup \{n+1\}, \qquad \forall\, \gamma \in R \setminus (-\Delta_{n+1}),$$

which implies $-\alpha_n \in R$. Then $R \setminus \{-\alpha_{n+1}\} \subset E_n$, and the conclusion follows by the inductive hypothesis.

The case of type C_n is analogous, using the explicit compatibility conditions described in Proposition 4.2.29. $\square$

Proof of Theorem 4.4.9. The first two points for $(g,n)=(2,1)$ and $(3,0)$ follow from [CTV21, Prop. 3.16] and Remark 4.4.7. We can assume S admissible. The space $\overline{\mathrm{M}}^S_{g,n}$ is $\mathbb{Q}$-factorial if and only if the map

$$i_S^*(\pi^{-1}(\operatorname{Span}_{\mathbb{Q}} S)) \subset \operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}^S_{g,n}\right) \to \operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathcal{M}}_{g,n}\left(\tfrac{7}{10}\right)\right) \big/ \operatorname{Span}_{\mathbb{Q}}\{\delta_{1,\{i\}} \mid -\alpha_{1,\{i\}} \notin S\}$$

is surjective. $\operatorname{Ker}\pi$ always descends to a line bundle on $\overline{\mathrm{M}}^S_{g,n}$. Moreover, since S is admissible, i_S^* is an isomorphism. Therefore, $\overline{\mathrm{M}}^S_{g,n}$ is $\mathbb{Q}$-factorial if and only if the inclusion

$$\operatorname{Span}_{\mathbb{Q}} S \hookrightarrow \mathbb{E}_{g,n} \big/ \operatorname{Span}_{\mathbb{Q}}\{\delta_{1,\{i\}} \mid -\alpha_{1,\{i\}} \notin S\} \tag{4.6}$$

is an isomorphism. Part (1) for $n=0$ is then immediate. For $n>0$, it follows since for every compatible subset S and $\beta \in S$, either $x_{1,i}(\beta)=0$ or $\beta=-\alpha_{1,\{i\}}$.

For part (2), we must verify whether $\pi_S(K_{\overline{\mathrm{M}}^S_{g,n}})$ lies in the image of (4.6). Using the linear independence of $\Delta^{[I]}$ and Δ^{irr}, this is equivalent to the following conditions (for the third point we use $\delta_{1,\{i\}}=\alpha_{1,\{i\}}+\alpha_{2,\{i\}}$):

(a) $\alpha_{\mathrm{irr}} \in \operatorname{Span}_{\mathbb{Q}} S$;

(b) for $[I] \neq [\{i\}]$, $\sum_{\alpha_{j,I}\in\Delta^{[I]}_{g,n}} \alpha_{j,I} \in \operatorname{Span}_{\mathbb{Q}} S$;

(c) for $[I]=[\{i\}]$ and $-\alpha_{1,\{i\}} \in S$, the class $\alpha_{1,\{i\}}$ is then $\mathbb{Q}$-Cartier, we require $\sum_{\alpha_{j,\{i\}}\in\Delta^{[\{i\}]}_{g,n}} \alpha_{j,\{i\}} \in \operatorname{Span}_{\mathbb{Q}} S$;

(d) for $[I]=[\{i\}]$ and $-\alpha_{1,\{i\}} \notin S$, there exists $\lambda \in \mathbb{Q}$ such that

$$7 \sum_{\alpha_{j,\{i\}}\in\Delta^{[\{i\}]}_{g,n}} \alpha_{j,\{i\}} + (\lambda-1)\alpha_{1,\{i\}} + \lambda\alpha_{2,\{i\}} \in \operatorname{Span}_{\mathbb{Q}} S.$$

The condition (b) follows from Lemma 4.4.10. For condition (c), we argue similarly after projecting

$$\mathbb{E}_{g,n} \longrightarrow \operatorname{Span}_{\mathbb{Q}}(\Delta^{[\{i\}]} \setminus \{\alpha_{1,\{i\}}\}).$$

For condition (d), since S is admissible and $-\alpha_{1,\{i\}} \notin S$, we necessarily have $\lambda=1$. Projecting

$$\mathbb{E}_{g,n} \longrightarrow \operatorname{Span}_{\mathbb{Q}}(\Delta^{[\{i\}]}_{g,n} \setminus \{\alpha_{1,\{i\}}, \alpha_{2,\{i\}}\})$$

and applying Lemma 4.4.10, we obtain the existence of $\{\beta_j\}_{j\in J}$ with

$$[3,n] \subset \bigcup_{j\in J} \operatorname{Supp}(\beta_j).$$

Moreover, since $x_{1,i}(S)=\{0\}$, either $\alpha_{2,\{i\}}$ or $-\alpha_{2,\{i\}}$ must belong to S. This proves point (2) of the claim.

Point (3) follows analogously. □

For completeness, we state the simplified case $n=0$.

Corollary 4.4.11. *Assume $g > 2$ and let $S \subset \Phi_{\geq -1}$ be admissible. Then:*

1. $\overline{\mathrm{M}}^S_g$ *is $\mathbb{Q}$-factorial if and only if* $\operatorname{Span}_+ S$ *is a full-dimensional cone in* $\mathbb{E}_g$.
2. $\overline{\mathrm{M}}^S_g$ *is $\mathbb{Q}$-Gorenstein if and only if* $\Delta^{\mathrm{irr}}_g \cap S \neq \emptyset$ *and there exists* $\{\beta_j\}_{j \in J} \subset S^{\mathrm{adm}} \cap \Phi_{\geq -1}$ *with pairwise disjoint supports and* $\bigcup_{j \in J} \operatorname{Supp}(\beta_j) = \operatorname{Supp}(\Delta^{[\emptyset]}_g)$.

The compactifications constructed in [CTV23] have a special role

Remark 4.4.12. For any admissible set of compatible roots S, the cycle $K_{\overline{\mathrm{M}}^S_{g,n}} + \psi$ is $\mathbb{Q}$-Cartier if and only if there exists $T \subset T_{g,n}$ as in [CTV23, Sec. 1.2] such that (cf. Proposition 4.3.29)

$$\overline{\mathcal{M}}^S_{g,n} \subset \overline{\mathcal{M}}^{T+}_{g,n} .$$

4.4.2 Flips and $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}^S_{g,n}$

We now apply the general results of Section 2.3. This yields a description of all the possible flips in the second step of the Hassett–Keel program for $\overline{\mathrm{M}}_{g,n}$.

Theorem 4.4.13. *Consider admissible subsets of compatible roots $S' \subset S$ and fix a line bundle $L \in \operatorname{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right)$ that is antiample relative to the morphism $\overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}^{S'}_{g,n}$. Then the L-flip of $\overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}^{S'}_{g,n}$ is given by*

$$\begin{tikzcd}
\overline{\mathrm{M}}^S_{g,n} \arrow[rr, dashed, "\eta"] \arrow[dr, "f^S_{S'}"'] & & \overline{\mathrm{M}}^{\widetilde{S}}_{g,n} \arrow[dl, "f^{\widetilde{S}}_{S'}"] \\
& \overline{\mathrm{M}}^{S'}_{g,n} &
\end{tikzcd}$$

where $\widetilde{S} \supset S'$ is the unique set of compatible roots corresponding to the cone whose generating set contains S' and whose image under the projection

$$p \colon \mathbb{E}(\mathcal{X}) \longrightarrow \frac{\mathbb{E}(\mathcal{X})}{\operatorname{Span}\langle S' \rangle}$$

contains $p(-L)$. Equivalently, $\widetilde{S}$ corresponds to the cone in the star construction, with respect to $\operatorname{Span}_+ S'$, containing the projection of $-L$ (cf. Definition 2.2.12).

Moreover, if S is maximal and $S \setminus S' = \{\alpha\}$, then the $(-\alpha)$-flip is given by the seed (i.e. a maximal set of compatible roots) $\widetilde{S}$ obtained by mutating S along α.

Proof. Since S and S' are admissible, the open immersion $\overline{\mathcal{M}}^S_{g,n} \subset \overline{\mathcal{M}}^{S'}_{g,n}$ is big (cf. Remark 4.3.4). The first claim follows from Proposition 2.3.4. The second follows from the structure of the cluster fan, cf. Proposition 4.2.19 and Definition 4.2.6. □

Let $n > 0$. Then the subspace

$$V_{g,n} = \{\beta \in \mathbb{E}_{g,n} \mid x_{1,\{i\}}(\beta) = 0 \text{ for all } i\} \subset \mathbb{E}_{g,n}$$

is generated by $\Delta'_{g,n} = \Delta_{g,n} \setminus \{\alpha_{1,\{i\}} \mid i \in [n]\}$. If Q is the A-type quiver used to construct $\mathcal{A}_{g,n}$, the restriction functor $\operatorname{rep} Q \to \operatorname{rep} Q'$ forgetting the vertices of index 1 in each connected component defines a cluster algebra $\mathcal{A}'_{g,n}$ associated to the root system with basis $\Delta'_{g,n}$ (cf. Definition-Proposition 4.2.23). The fan $\Sigma_{g,n}$ restricted to $V_{g,n}$ is the cluster fan of $\mathcal{A}'_{g,n}$. We now recover the $\mathbb{Q}$-factorialization fan of $\overline{\mathcal{M}}_{g,n}(\frac{7}{10})$:

Theorem 4.4.14. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $(g,n) \neq (2,0)$. The restriction of $\Sigma_{g,n}$ to $V_{g,n} \subset \mathbb{E}_{g,n}$ is the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, denoted $\Sigma^{\mathbb{Q}}(\overline{\mathrm{M}}_{g,n}(\frac{7}{10}))$. Where a cone $\mathcal{C} \subset V_{g,n}$ corresponds to the partial $\mathbb{Q}$-factorialization $\overline{\mathrm{M}}_{g,n}^{S} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$, where $S \subset \Phi_{\geq -1}$ is the set of compatible roots such that $\mathcal{C} = \operatorname{Span}_+ S$.*

Proof. Let $\operatorname{Span}_+ S \subset V_{g,n}$ be a maximal cone in $V_{g,n}$. By Theorem 4.4.9 the space $\overline{\mathrm{M}}_{g,n}^{S}$ is $\mathbb{Q}$-factorial, the morphism $\overline{\mathrm{M}}_{g,n}^{S} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ is small, and the inclusion $\overline{\mathcal{M}}_{g,n}^{S} \subset \overline{\mathcal{M}}_{g,n}(\frac{7}{10})$ is big (cf. Theorem 4.3.17 and Remark 4.3.4). Therefore, the claim follows from Theorem 2.3.23 with $L \in \operatorname{Span}_+ S$. $\square$

Remark 4.4.15. For $n = 0$, Theorem 4.4.14 shows that the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g}(\frac{7}{10})$ is Σ_g.

In both cases $n = 0$ and $n > 0$, mutations correspond to elementary flips.

Corollary 4.4.16. *Let $S = S' \cup \{\alpha\} \subset V_{g,n}$ be a set of compatible roots maximal in $V_{g,n}$. Then the $(-\alpha)$-flip is given by the mutation of S along α with respect to the cluster algebra $\mathcal{A}'_{g,n}$. These are precisely the elementary flips.*

When $n > 0$, the $\mathbb{Q}$-factorialization fan can also be described via a star construction:

Proposition 4.4.17. *Let (g,n) be a hyperbolic pair with $g \geq 2$ and $n \geq 1$. Set $S = \{\alpha_{1,\{i\}} + \alpha_{2,\{i\}} \mid i \in [n]\}$. Then $\overline{\mathrm{M}}_{g,n}^{S} \simeq \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$. Under this identification, the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ is given by the star construction of $\Sigma_{g,n}$ with respect to the cone $\operatorname{Span}_+ S$ (cf. Definition 2.2.12).*

Proof. Since $S^{\mathrm{adm}} = \emptyset$, we have $\overline{\mathrm{M}}_{g,n}^{S} \simeq \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$. The strata $S(1,1,\{i\},1)$ are not semistable, hence $\overline{\mathcal{M}}_{g,n}^{S}$ does not contain any generic point of $\delta_{1,\{i\}}$. Thus, for any $S' \supset S$, the inclusion $\overline{\mathcal{M}}_{g,n}^{S'} \subset \overline{\mathcal{M}}_{g,n}^{S}$ is big, and for S' maximal the morphism $\overline{\mathrm{M}}_{g,n}^{S'} \to \overline{\mathrm{M}}_{g,n}^{S}$ is a $\mathbb{Q}$-factorialization. Applying Theorem 2.3.23 with $L \in \operatorname{Span}_+ S'$ gives the claim. $\square$

Thanks to Remark 2.3.21, we can construct the $\mathbb{Q}$-factorialization fans for any $\overline{\mathrm{M}}_{g,n}^{S}$. For $n = 0$:

Proposition 4.4.18. *Consider $g > 2$ and fix compatible roots $S \subset \Phi_{\geq 1}$. Then the $\mathbb{Q}$-factorialization fan $\Sigma^{\mathbb{Q}}\left(\overline{\mathrm{M}}_{g}^{S}\right)$ is given by the star construction of Σ_g with respect to $\operatorname{Span}_+ S$. Where the partial $\mathbb{Q}$-factorialization are $f_{S'}^{S} \colon \overline{\mathrm{M}}_{g}^{S'} \to \overline{\mathrm{M}}_{g}^{S}$ for compatible set of roots $S' \supset S$.*

For $n > 0$, we have:

Proposition 4.4.19. *Let (g,n) be hyperbolic with $g \geq 2$ and $(g,n) \neq (2,0)$. Fix $S \subset \Phi_{\geq 1}$ and let S' be the compatible subset constructed in Proposition 4.3.10. Then $\overline{\mathrm{M}}^{S'}_{g,n} \simeq \overline{\mathrm{M}}^S_{g,n}$, and under this identification the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}^S_{g,n}$ is given by the star construction of $\Sigma_{g,n}$ with respect to $\mathrm{Span}_+ S'$.*

Proof. The argument is the same as in Theorem 4.4.14, $S'' \supset S'$ the inclusion $\overline{\mathcal{M}}^{S''}_{g,n} \subset \overline{\mathcal{M}}^{S'}_{g,n}$ is big because $\overline{\mathcal{M}}^{S''}_{g,n}$ excludes the strata $S(1,1,i,+1)$ for all $i \in [n]$ with $-\alpha_{1,\{i\}} \notin S$. ☐

The $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}^S_{g,n}$ we can also be recovered in the spirit of Theorem 4.4.14 and Proposition 4.4.18 as follows.

Remark 4.4.20. Let $S \subset \Phi_{\geq -1}$ be admissible, and let Σ_S be the star construction of $\Sigma_{g,n}$ with respect to $\mathrm{Span}_+ S$. Then the $\mathbb{Q}$-factorialization fan of $\overline{\mathrm{M}}^S_{g,n}$ is the restriction of Σ_S to the subspace generated by

$$\mathrm{Span}\left(\Delta_{g,n} \setminus \{\alpha_{1,\{i\}} \mid -\alpha_{1,\{i\}} \notin S\}\right).$$

As a consequence of Proposition 2.3.3 and Definition-Proposition 2.3.17, we have:

Proposition 4.4.21 (Relative ample cone)**.** *Let $S' \subset S$ be admissible. Then the relative ample cone of $f^{S'}_S : \overline{\mathrm{M}}^S_{g,n} \to \overline{\mathrm{M}}^{S'}_{g,n}$ is the image of $\mathrm{Span}_+ S$ along the map*

$$\mathrm{Span}_+ S \subset \mathbb{E}_{g,n} \longrightarrow \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^S_{g,n}\right) \Big/ \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^{S'}_{g,n}\right).$$

The relative nef cone is the closure of this image.

As a consequence, we can describe all the possible morphism among our compactifications, generalizing Proposition 4.3.6.

Proposition 4.4.22. *Let S_1, S_2 be compatible sets of almost positive roots. The rational map*

$$\begin{tikzcd} \overline{\mathrm{M}}^{S_1}_{g,n} \arrow[r, dashed] & \overline{\mathrm{M}}^{S_2}_{g,n} \\ \mathrm{M}_{g,n} \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{M}_{g,n} \arrow[u, hook] \end{tikzcd}$$

extends to a morphism $\overline{\mathrm{M}}^{S_1}_{g,n} \to \overline{\mathrm{M}}^{S_2}_{g,n}$ if and only if $S_1^{\mathrm{adm}} \subset S_2^{\mathrm{adm}}$, in which case it is the face morphism $f^{S_2^{\mathrm{adm}}}_{S_1^{\mathrm{adm}}}$.

Proof. By Proposition 4.3.6, we may assume S_1 and S_2 are admissible. Since the spaces $\overline{\mathrm{M}}^S_{g,n}$ are separated and reduced, if $S_1 \subset S_2$ the map extends uniquely to $f^{S_2}_{S_1}$. Conversely, if the map extends, then composing with $\overline{\mathrm{M}}^{S_i}_{g,n} \to \overline{\mathrm{M}}_{g,n}(\frac{7}{10})$ gives an inclusion of relative nef cones. By Proposition 4.4.21, this implies $\mathrm{Span}_+ S_2 \subset \overline{\mathrm{Span}_+ S_1}$, hence $S_1^{\mathrm{adm}} \subset S_2^{\mathrm{adm}}$. $\square$

As byproduct of this section, we obtain some constraints on the ample and nef cones of $\overline{\mathrm{M}}_{g,n}$, that we formulate for $n = 0$.

Corollary 4.4.23. *Fix $g \geq 3$. The image of the cone* $\mathrm{Nef}\left(\overline{\mathrm{M}}_g\right) \cap \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_g^{\mathrm{ps}}\right)$ *along the projection*

$$\mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_g^{\mathrm{ps}}\right) \longrightarrow \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_g^{\mathrm{ps}}\right) \big/ \mathrm{Pic}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_g\left(\frac{7}{10}\right)\right)$$

is $\overline{\mathrm{Span}_+ \Delta_g}$.

Proof. This is a consequence of considering the morphisms $\overline{\mathrm{M}}_g \to \overline{\mathrm{M}}_g^{\mathrm{ps}} \to \overline{\mathrm{M}}_g(\frac{7}{10})$, where the second has relative ample cone $\mathrm{Span}_+ \Delta_g$. $\square$

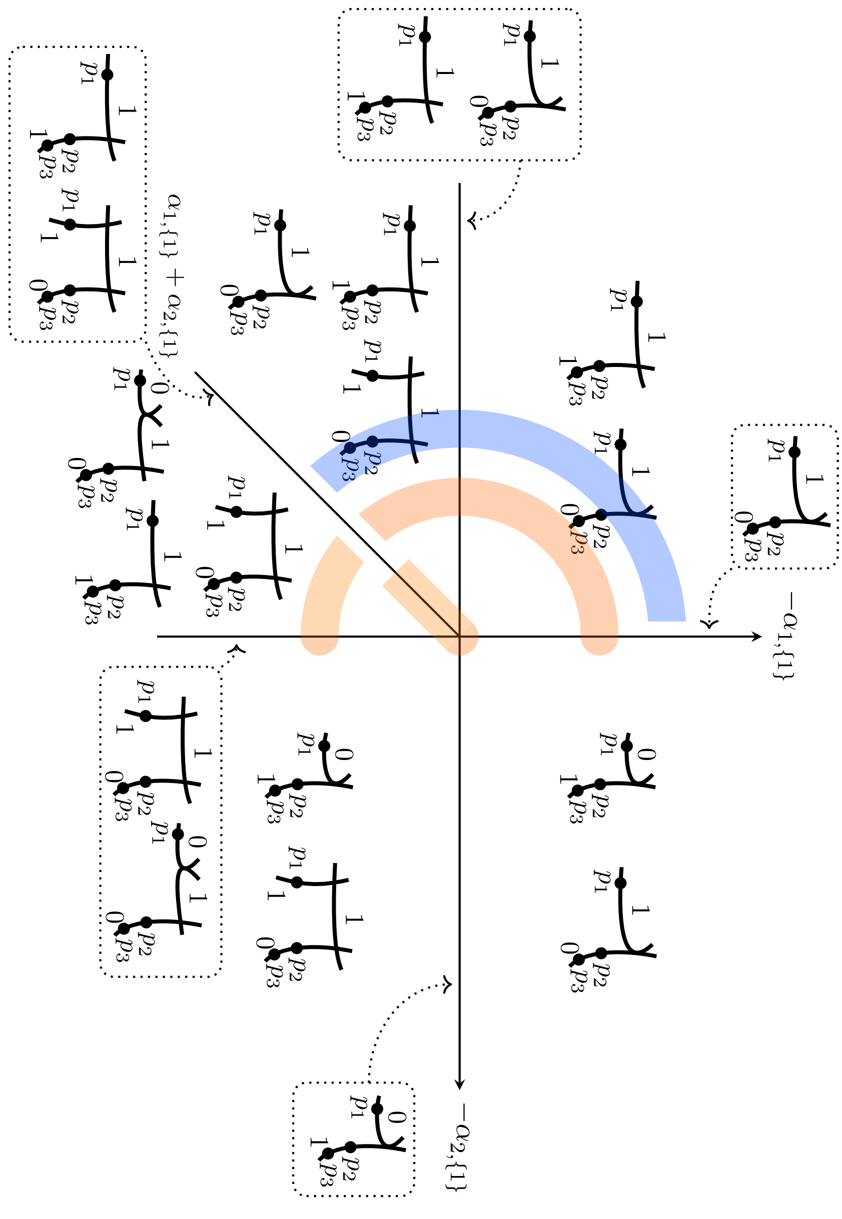


Figure 4.9. Cones equivalent with respect to $\sim$ are shown in blue (outer annular sector), while those with the same S^{adm} are shown in orange (the other highlighted parts).

Bibliography

[ACG+11] Enrico Arbarello, Maurizio Cornalba, Phillip A Griffiths, et al. *Geometry of algebraic curves: volume II with a contribution by Joseph Daniel Harris*. Springer, 2011.

[AFS] J Alper, M Fedorchuk, and DI Smyth. “Singularities with Gm-action and the log minimal model program for Mg, ArXiv e-prints (2010)”. In: *arXiv preprint arXiv:1010.3751* ().

[AFS17a] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. “Second flip in the Hassett–Keel program: existence of good moduli spaces”. In: *Compositio Mathematica* 153.8 (2017), pp. 1584–1609.

[AFS17b] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. “Second flip in the Hassett–Keel program: projectivity”. In: *International Mathematics Research Notices* 2017.24 (2017), pp. 7375–7419.

[AH06] Ivan V Arzhantsev and Jürgen Hausen. “On the multiplication map of a multigraded algebra”. In: *arXiv preprint math/0607819* (2006).

[AH99] Annette A’Campo-Neuen and Jurgen Hausen. “Quotients of toric varieties by the action of a subtorus”. In: *Tohoku Mathematical Journal, Second Series* 51.1 (1999), pp. 1–12.

[AHH23] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. “Existence of moduli spaces for algebraic stacks”. In: *Inventiones mathematicae* 234.3 (2023), pp. 949–1038.

[AHR19] Jarod Alper, Jack Hall, and David Rydh. “The\’étale local structure of algebraic stacks”. In: *arXiv preprint arXiv:1912.06162* (2019).

[AHR20] Jarod Alper, Jack Hall, and David Rydh. “A Luna étale slice theorem for algebraic stacks”. In: *Annals of mathematics* 191.3 (2020), pp. 675–738.

[Alp+17] Jarod Alper et al. “Second flip in the Hassett–Keel program: a local description”. In: *Compositio Mathematica* 153.8 (2017), pp. 1547–1583.

[Alp13] Jarod Alper. “Good moduli spaces for Artin stacks”. In: *Annales de l’Institut Fourier*. Vol. 63. 6. 2013, pp. 2349–2402.

[Alp25] Jarod Alper. *Stacks and Moduli*. Book. 2025. URL: https://sites.math.washington.edu//~jarod/moduli-versions/moduli-7-7-25.pdf.

[Art70] Michael Artin. "Algebraization of formal moduli: II. Existence of modifications". In: *Annals of Mathematics* 91.1 (1970), pp. 88–135.

[Art77] M Artin[1]. "Coverings of the rational double points in characteristic p". In: *Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira* (1977), p. 11.

[AW14] Marco Andreatta and Jarosław A Wiśniewski. "4-dimensional symplectic contractions". In: *Geometriae Dedicata* 168.1 (2014), pp. 311–337.

[Bar+15] Wolf Barth et al. *Compact complex surfaces.* Vol. 4. Springer, 2015.

[BGS23] Sebastian Bozlee, Christopher Guevara, and David Smyth. "A stratification of moduli of arbitrarily singular curves". In: *arXiv preprint arXiv:2307.10013* (2023).

[BH06] Florian Berchtold and Jürgen Hausen. "GIT-equivalence beyond the ample cone". In: *Michigan Mathematical Journal* 54.3 (2006), pp. 483–516.

[Bir+10] Caucher Birkar et al. "Existence of minimal models for varieties of log general type". In: *Journal of the American Mathematical Society* 23.2 (2010), pp. 405–468.

[Bra+24] Lukas Braun et al. "Reductive quotients of klt singularities". In: *Inventiones mathematicae* 237.3 (2024), pp. 1643–1682.

[Bri15] Michel Brion. "On linearization of line bundles". In: *J. Math. Sci. Univ. Tokyo* 22.1 (2015), pp. 113–147. ISSN: 1340-5705.

[Bua+06] Aslak Bakke Buan et al. "Tilting theory and cluster combinatorics". In: *Advances in mathematics* 204.2 (2006), pp. 572–618.

[Cat82] Fabrizio Catanese. "Pluricanonical—Gorenstein—curves". In: *Enumerative geometry and classical algebraic geometry.* Springer, 1982, pp. 51–95.

[Cav+20] Renzo Cavalieri et al. "A moduli stack of tropical curves". In: *Forum of Mathematics, Sigma.* Vol. 8. Cambridge University Press. 2020, e23.

[CL06] Hung-Jen Chiang-Hsieh and Joseph Lipman. "A numerical criterion for simultaneous normalization". In: (2006).

[Cor87] Maurizio Cornalba. "On the locus of curves with automorphisms". In: *Annali di Matematica pura ed applicata* 149.1 (1987), pp. 135–151.

[CTV21] Giulio Codogni, Luca Tasin, and Filippo Viviani. "On some modular contractions of the moduli space of stable pointed curves". In: *Algebra & Number Theory* 15.5 (2021), pp. 1245–1281.

[CTV23] Giulio Codogni, Luca Tasin, and Filippo Viviani. "On the first steps of the minimal model program for the moduli space of stable pointed curves". In: *Journal of the Institute of Mathematics of Jussieu* 22.1 (2023), pp. 145–211.

[Den08] Bangming Deng. *Finite dimensional algebras and quantum groups.* 150. American Mathematical Soc., 2008.

[DH98] Igor V Dolgachev and Yi Hu. "Variation of geometric invariant theory quotients". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 87 (1998), pp. 5–51.

[DW17] Harm Derksen and Jerzy Weyman. *An introduction to quiver representations*. Vol. 184. American Mathematical Soc., 2017.

[EGAIV] A. Grothendieck. "Eléménts de Géométrie Algébrique. IV." In: *Publ. Math. IHES* 28 (1966).

[ES16] Dan Edidin and Matthew Satriano. "Strong cycles and intersection products on good moduli spaces". In: *arXiv preprint arXiv:1609.08162* (2016).

[Fed14] Maksym Fedorchuk. "Moduli spaces of hyperelliptic curves with A and D singularities". In: *Mathematische Zeitschrift* 276.1 (2014), pp. 299–328.

[Fer03] Daniel Ferrand. "Conducteur, descente et pincement". In: *Bulletin de la société mathématique de France* 131.4 (2003), pp. 553–585.

[FS] Maksym Fedorchuk and David Smyth. "Alternate compactifications of moduli spaces of curves, 2010". In: *Handbook of Moduli, edited by G. Farkas and I. Morrison* ().

[Ful13] William Fulton. *Intersection theory*. Vol. 2. Springer Science & Business Media, 2013.

[FV20] Roberto Fringuelli and Filippo Viviani. "On the Picard group scheme of the moduli stack of stable pointed curves". In: *arXiv preprint arXiv:2005.06920* (2020).

[FWZ24] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras. Chapters 1-3*. 2024. arXiv: 1608.05735 [math.CO]. URL: https://arxiv.org/abs/1608.05735.

[FZ02a] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras I: foundations". In: *Journal of the American mathematical society* 15.2 (2002), pp. 497–529.

[FZ02b] Sergey Fomin and Andrei Zelevinsky. "Cluster algebras II: Finite type classification". In: *arXiv preprint math/0208229* (2002).

[GHH15] Martin G Gulbrandsen, Lars H Halle, and Klaus Hulek. "A relative Hilbert–Mumford criterion". In: *Manuscripta Mathematica* 148 (2015), pp. 283–301.

[GIT] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*. Vol. 34. Springer Science & Business Media, 1994.

[GKM02] Angela Gibney, Sean Keel, and Ian Morrison. "Towards the ample cone of $\overline{M}_{g,n}$". In: *Journal of the American Mathematical Society* 15.2 (2002), pp. 273–294.

[Hal14] Daniel Halpern-Leistner. "On the structure of instability in moduli theory". In: *arXiv preprint arXiv:1411.0627* (2014).

[Has00] Brendan Hassett. “Local stable reduction of plane curve singularities”. In: (2000).

[Has05] Brendan Hassett. “Classical and minimal models of the moduli space of curves of genus two”. In: *Geometric methods in algebra and number theory*. Springer, 2005, pp. 169–192.

[Hei17] J Heinloth. *Hilbert-Mumford stability on algebraic stacks and applications to G-bundles on curves, Épijournal Geom.* 2017.

[HH09] Brendan Hassett and Donghoon Hyeon. “Log canonical models for the moduli space of curves: the first divisorial contraction”. In: *Transactions of the American Mathematical Society* 361.8 (2009), pp. 4471–4489.

[HH13] Brendan Hassett and Donghoon Hyeon. “Log minimal model program for the moduli space of stable curves: the first flip”. In: *Annals of Mathematics* (2013), pp. 911–968.

[HK00] Yi Hu and Sean Keel. “Mori dream spaces and GIT.” In: *Michigan Mathematical Journal* 48.1 (2000), pp. 331–348.

[HM08] Donghoon Hyeon and Ian A Morrison. “Stability of Tails and 4-Canonical Models”. In: *arXiv: Algebraic Geometry* (2008). URL: https://api.semanticscholar.org/CorpusID:9925066.

[HR74] Melvin Hochster and Joel L Roberts. “Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay”. In: *Advances in mathematics* 13.2 (1974), pp. 115–175.

[Kle05] Steven L Kleiman. “The picard scheme”. In: *arXiv preprint math/0504020* (2005).

[Kle66] Steven L Kleiman. “Toward a numerical theory of ampleness”. In: *Annals of Mathematics* 84.3 (1966), pp. 293–344.

[KM98] János Kollár and Shigefumi Mori. “Birational geometry of algebraic varieties”. In: *(No Title)* 134 (1998).

[Kol08] János Kollár. “Exercises in the birational geometry of algebraic varieties”. In: *arXiv preprint arXiv:0809.2579* (2008).

[Kre98] Andrew Kresch. “Cycle groups for Artin stacks”. In: *arXiv preprint math/9810166* (1998).

[KSZ91] Mikhail M Kapranov, Bernd Sturmfels, and Andrei V Zelevinsky. “Quotients of toric varieties”. In: *Mathematische Annalen* 290 (1991), pp. 643–655.

[Laz17] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*. Vol. 48. Springer, 2017.

[NS14] Tomoki Nakanishi and Salvatore Stella. “Diagrammatic Description of c-Vectors and d-Vectors of Cluster Algebras of Finite Type”. In: *The Electronic Journal of Combinatorics* 21.1 (2014).

[Oht22] Rikito Ohta. “On the relative version of Mori dream spaces”. In: *European Journal of Mathematics* 8.Suppl 1 (2022), pp. 147–181.

[Per23] Michele Pernice. “The moduli stack of A_r-stable curves”. In: *arXiv preprint arXiv:2302.10877* (2023).

[Sch91] David Schubert. “A new compactification of the moduli space of curves”. In: *Compositio Mathematica* 78.3 (1991), pp. 297–313.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*. Springer, 2006.

[Ser12] Jean-Pierre Serre. *Algebraic groups and class fields*. Vol. 117. Springer Science & Business Media, 2012.

[Smy13] David Ishii Smyth. “Towards a classification of modular compactifications of”. In: *Inventiones mathematicae* 192.2 (2013), pp. 459–503.

[Sta25] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2025.

[Tha96] Michael Thaddeus. “Geometric invariant theory and flips”. In: *Journal of the American Mathematical Society* 9.3 (1996), pp. 691–723.

[Tju69] Galina N Tjurina. “Locally semiuniversal flat deformations of isolated singularities ofcomplex spaces”. In: *Mathematics of the USSR-Izvestiya* 3.5 (1969), p. 967.

[TV10] Mattia Talpo and Angelo Vistoli. “Deformation theory from the point of view of fibered categories”. In: *arXiv preprint arXiv:1006.0497* (2010).

[Wyc10] F.D.W. van der Wyck. *Moduli of Singular Curves and Crimping*. Harvard University, 2010. URL: https://books.google.it/books?id=gv5UewAACAAJ.