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The M -Layer Construction: A Diagrammatic Framework for Critical Behavior in Lattice Models

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Abstract

This thesis presents and applies the M -layer construction, a recent diagrammatic method for studying the critical behavior of lattice models in finite dimensions. The construction provides a unified tool to compute correlation functions and critical exponents through a perturbative expansion in topological loops, whose leading order corresponds to the behavior on a Bethe lattice. The first application of the method is on well-known problems, such as the Ising model and the percolation problem, where it successfully reproduces standard results, thus validating its efficiency. It is then applied to the Edwards–Anderson spin glass model in a magnetic field at zero temperature, a problem whose finite-dimensional physics remains widely debated. The remarkable new result of the M -layer construction is the identification, for the first time through analytical computations, of a phase transition in a field below the upper critical dimension. The results, obtained through analytical computations and partially checked numerically, show that this approach provides a promising alternative to standard Renormalization Group techniques for investigating phase transitions in disordered and frustrated systems.

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Introduction

In the framework of Statistical Mechanics, the so-called *critical phenomena* are characterized by the emergence of long-range correlations that appear when control parameters are tuned. At the critical point, a phase transition occurs, characterized by the nontrivial behavior of the so-called *order parameter*. The description of some of these phenomena can be achieved through the study of *ad hoc* lattice models, whose microscopic structure is determined by two essential ingredients: the interactions and the underlying lattice on which these interactions are defined. Interactions are usually restricted to nearest neighbors. Consequently, the geometry of the lattice plays a fundamental role, and two features are particularly significant: the number of neighbors associated with each site, also called *connectivity*, and the presence of topological loops. Both interaction and lattice features can enhance or suppress the correlations between microscopic degrees of freedom, leading to a macroscopic collective behavior.

Due to the intrinsic difficulties in solving these models, even if minimal, it is often necessary to rely on approximations. The most common one is the *mean-field approximation*, which amounts to neglecting the fluctuations of the order parameter. From the lattice perspective, this corresponds to considering an effective topology of infinite connectivity. This approximation usually captures the qualitative behavior of these models, but fails to make proper quantitative predictions on finite-dimensional systems. In finite-dimensional lattices, such fluctuations are crucial: they can be incorporated systematically using the Renormalization Group, which provides a framework for characterizing the scaling of observables at criticality.

When quenched disorder is introduced in the interactions, the behavior at criticality becomes strongly affected by the local environment. In such cases, standard mean-field approaches, which rely on the assumption of a homogeneous order parameter, become problematic and often provide an incomplete description of the system.

In recent decades, considerable effort has been devoted to the study of disordered models defined on finite-connectivity topologies. Among these structures, *locally tree-like* graphs, also known as Bethe lattices, played a particularly prominent role. A model defined on a Bethe lattice is mean-field in nature due to the absence of finite-length loops, and can often be solved through the so-called Bethe–Peierls approximation. However, despite their mean-field characterization, local observables on a Bethe lattice do not necessarily coincide with global ones: local fluctuations and heterogeneities remain present due to the finite connectivity. The main difficulty in the analysis of finite-connectivity systems arises when incorporating the contribution of topological loops with finite length, which are absent in the purely tree-like structure of the Bethe lattice.

This thesis is based on the M -layer construction, a recent approach designed to systematically develop a topological loop expansion around the Bethe–Peierls approx-

imation. Within this framework, observables can be computed starting from their values on a Bethe lattice, with perturbative corrections accounting for the contributions of finite-length loops. This method provides a controlled way to interpolate between mean-field behavior and the finite-dimensional properties of realistic systems.

The thesis is organized as follows. [Part I](#) and [Part II](#) provide the conceptual and methodological background required to understand the framework developed throughout this work. They both rely on well-established results from the literature. [Part III](#) constitutes the core of the original contributions of the present work, where the M -layer construction is applied to different models, leading to novel results. Finally, [Part IV](#) includes final remarks and outlines possible directions for future research.

- **Part I: Spin Glasses: Models and Background**

- [Chapter 1: Introduction to Spin Glasses](#)

The first part of this introductory chapter presents a historical overview together with the main motivations for the study of disordered systems. In the second part, the models used to reproduce the experimental observations on spin glass materials are defined. Then, the basic tools employed throughout this thesis for the analysis of such models are introduced, including a brief review of the main concepts from graph theory.

- [Chapter 2: Mean-Field Spin Glasses](#)

This chapter reviews the main results on the mean-field characterization of the spin glass models introduced in the previous chapter. In particular, the mean-field behavior on both the complete graph and the Bethe lattice is discussed. These results provide the foundation for the subsequent application of the M -layer construction to spin glass models.

- [Chapter 3: Beyond Mean-Field: Finite-Dimensional Systems](#)

Before introducing the new method at the core of this thesis, this chapter reviews the standard tools used to analyze finite-dimensional models. The most relevant results produced by these methods in the context of spin glass models are summarized, with particular attention to the complications that are addressed by the M -layer construction in the following chapters. In the final section, possible scenarios discussed in the literature are presented.

- **Part II: The M -layer Construction: General Framework**

- [Chapter 4: The Bethe \$M\$ -layer Construction](#)

This chapter presents the theoretical foundations of the Bethe M -layer construction. All the relevant notions are introduced explaining how to generally perform the computations that lead to the perturbative expansion for a generic observable. Some more recent features of the construction, identified in [\[Ang24\]](#), are inserted in the framework. Finally, the rules to perform the computations are explained in a general framework together with some relevant remarks.

- **Part III: Applications of the M -Layer Construction**

- [Chapter 5: The Ising Model: A Benchmark Case](#)

In this chapter, the first application of the M -layer construction is discussed

in detail. The method is employed to reproduce the results of the ϕ^4 Field Theory associated with the Ising model in finite dimensions. This analysis serves both to validate the method and to provide a methodological example. The new features that emerge from this application are used in the following chapters. The results presented in this chapter are based on [Ang24].

- **Chapter 6: *Percolation: Extracting Critical Exponents***

This chapter reviews the application of the Bethe M -layer construction to site and bond percolation problems. The critical exponents are extracted within the ϵ -expansion for the first time using this new approach. The results precisely reproduce the one-loop predictions of the corresponding Field Theory, namely the q -state Potts model in the limit $q \rightarrow 1$. These results are based on [Ang25c].

- **Chapter 7: *Spin Glass in a Field at Zero Temperature***

The technique described in the previous chapter is applied to the spin glass model in a field at zero temperature. A phase transition, with the corresponding critical exponents, is predicted both for the finite-dimensional and for the one-dimensional long-range associated model. The observables of interest are carefully defined and the computations are presented in detail from [Ang25a] and [Ang25b].

- **Part IV: Conclusions and Perspectives**

- **Chapter 8: *Conclusions***

In this final chapter, all the results are summarized and discussed together. Moreover, possible perspectives and future directions are debated.

Part I

Spin Glasses: Models and Background

Chapter 1

Introduction to Spin Glasses

1.1 A Historical and Conceptual Overview

Since the 1950s, experimental interest has grown in a certain class of materials consisting of magnetic impurities embedded in a non-magnetic environment [Myd93]. The peculiar transport properties of these compounds attracted the attention of the application-oriented scientific community, while questions regarding the formation of magnetic moments also generated theoretical interest. In the 1960s, the virtual-bound-state model provided an explanation for the latter [Fri62], partially containing research interest. However, between the late 1960s and early 1970s, further experiments revealed unusual behaviors in specific heat and magnetic susceptibility, differing significantly from those observed in more conventional ferromagnets. This led to the emerging hypothesis that these enigmatic materials were governed by the *interplay between ferromagnetic and antiferromagnetic domains*. A turning point came in the early 1970s, when the first theoretical models of these interactions inspired new experimental studies on magnetic susceptibilities. From then on, the defining characteristics of these materials (already referred to as “spin glasses” at the time) became increasingly clear and paved the way for a very rich branch of research. Over the past fifty years, “spin glass-related” investigations have extended beyond physics, influencing fields ranging from biology and computer science to epidemiology, standing out as a rich interdisciplinary subject.

The next two sections address in some detail two natural questions: i) “*what defines a spin glass?*” and ii) “*why is this subject of scientific interest?*”. The former question will be answered from the experimental point of view, that is where the name and the features of the spin glass were first identified. The latter is more related to the practical implementations and the theory behind them.

1.1.1 From Experiments to Theory

The term “spin glass” first appeared in 1970 in Anderson’s seminal paper [And70], where he credited Coles for drawing the initial parallel between magnetic spin materials and structural glasses [Mac97].

Several different materials, sharing some typical features, were classified under the name *spin glass*. The prototypical spin glass materials are noble metals weakly diluted with transition metal ions (CuMn, AuCo, PtMn, AuFe) [Can72, Mul81]. The scattering of noble metal’s conduction electrons produces indirect exchange interactions among

the spins of the magnetic impurities, the celebrated [Ruderman–Kittel–Kasuya–Yosida \(RKKY\)](#) interaction [[Rud54](#), [Kas56](#), [Yos57](#)]. The random spatial distribution of impurities, together with the interaction's oscillatory dependence on the distance, leads to an effective inter-moment coupling that varies randomly in both magnitude and sign. Apparently very different compounds, insulators with competing ferro- and antiferromagnetic exchange interactions ($\text{Eu}_x\text{Sr}_{1-x}\text{S}$, $\text{CdCr}_x\text{In}_{1-x}\text{S}_4$) [[Mal79](#), [Vin87](#)], showed the same rather sharp cusp in the frequency-dependent susceptibility.

Both kinds of materials seemed to share two features, identified as *randomness* and *frustration*. The former can be identified in the non-structured interactions, both in sign and magnitude, while the latter can be identified as the impossibility of some system components to simultaneously satisfy all the interactions between them [[Tou77](#)]. These two concepts, usually condensed in the word *disorder*, represent the fundamental building blocks of the associated theoretical framework, where they are rigorously defined (see [Section 1.2](#)).

The disorder can also be experimentally obtained by non-crystallinity, as for instance in the case of amorphous materials, metallic ($\text{Al}_{0.63}\text{Gd}_{0.37}$) [[Miz77](#), [Mal81](#)] or insulators ($\text{MnO} \cdot \text{Al}_2\text{O}_3 \cdot \text{SiO}_2$, $\text{CoO} \cdot \text{Al}_2\text{O}_3 \cdot \text{SiO}_2$) [[Lan](#), [Kru79](#), [Mor82](#)]. When investigating spin systems, anisotropy [[Sta68](#)] plays a crucial role in determining the nature of the spin glass phase, if any. To establish whether this phase is universal across different systems, comparative studies were performed on compounds of two elements where a third atom provides spin-orbit scattering, causing anisotropy [[Alb82](#)]: Ising-like spins, ZnMn ; planar spins described by the XY model, CdMn ; three-dimensional spins described by the Heisenberg model, MgMn .

Looking for a Phase Transition

The investigation began with modeling the exchange interactions, which then served as a framework to guide experimental research [[Sou69](#), [Sou71](#)]. Even if the complete modeling of the interactions for metallic materials was not possible [[Mor83](#)], what emerged was that, below a certain temperature T_g , the interaction of clusters of spins with *frozen* magnetization could (or not) characterize a peculiar phase, different from both paramagnetic and ferromagnetic ones [[Can72](#), [Tho74](#), [Tho77](#)]. The real part of the [alternating current \(ac\)](#) susceptibility, χ' , showed a rather sharp cusp at a finite temperature and the weak dependence of the maximum on the frequency of the measuring magnetic field suggested a phase transition, see [Figure 1.1](#). Edwards and Anderson mimicked the experimental systems with the homonymous model on a lattice [[Edw75](#)] and recognized that in this putative phase, individual spins $\{s_i\}_i$ (where $i = 1, \dots, N$ denotes the lattice site) become frozen in random orientations. Consequently, the relevant order parameter could not be the conventional global magnetization $M = \sum_{i=1}^N s_i$, but rather the following temperature-dependent quantity

$$q(T) = \frac{1}{N} \sum_{i=1}^N \overline{\langle s_i^2 \rangle}, \quad (1.1)$$

also known as *overlap*, where $\langle \cdot \rangle$ denotes the average over a time much longer than the equilibration time (or over the equilibrium configurations in a Classical Statistical Mechanics setting) and $\overline{}$ the average over different samples (or over different realizations of the disorder). In particular, one can expect that $q = 0$ above T_g , in the paramagnetic

phase, while $q > 0$ below. If the transition is a continuous one, approaching T_g from below the following relation holds with some *critical exponent* β

$$q(T) \sim |T - T_g|^\beta. \quad (1.2)$$

The connection between $q(T)$ and measurable quantities emerges through susceptibility analysis. Linear response theory reveals:

$$\chi(T) \sim \chi_0(T) - q(T), \quad (1.3)$$

where $\chi_0(T) \propto 1/T$ is the leading order of the expansion of M/H in even powers of the measuring field H [Bin86]

$$\frac{M}{H} = \chi_0(T) - \chi_{nl}(T) H^2 + \mathcal{O}(H^4), \quad (1.4)$$

and $\chi_{nl}(T)$ is the so-called *nonlinear* susceptibility. The key insight comes from defining the spin glass susceptibility

$$\chi_{SG} \equiv \frac{1}{N} \sum_{ij} \overline{\langle s_i s_j \rangle_c^2}, \quad (1.5)$$

where $\langle s_i s_j \rangle_c \equiv \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$ indicates connected correlation function. Remarkably, χ_{SG} is connected to nonlinear response through [Cha77, You98]:

$$\chi_{nl}(T) \propto \chi_{SG}(T). \quad (1.6)$$

This fundamental relationship provided crucial experimental access to spin glass ordering, as χ_{SG} could be measured through an *ac* magnetic field [You98]. If these materials exhibit a continuous phase transition at T_g with order parameter q , the spin glass susceptibility χ_{SG} (measuring order parameter fluctuations) must diverge as $T \rightarrow T_g$ with some critical exponent γ

$$\chi_{SG} \sim |T - T_g|^{-\gamma}, \quad (1.7)$$

and this divergence should be experimentally observable in the nonlinear susceptibility $\chi_{nl}(T)$. Although the divergence was experimentally verified [Mon82, Mal82, Oma83], the extracted critical exponents γ showed substantial variations ($1 \lesssim \gamma \lesssim 2$ from [Mon82] and $3 \lesssim \gamma \lesssim 4$ from [Oma83]) with notable material-dependent differences.

In order to look for a transition, researchers also investigated the behavior of the magnetic contribution to the specific heat, C_M . However, experimental determination of C_M proved challenging: rather than showing an anomaly at the freezing temperature T_g , it displayed a broad maximum above the critical temperature followed by a linear temperature dependence below it, as one can see from Figure 1.2 [Bro83, Coe77]. Being the specific heat a thermodynamic quantity, equilibration problems of these materials emerged. For these reasons, the results for C_M were not taken as evidence of the absence of the spin glass phase transition.

Off-equilibrium features

Apart from equilibration problems, for a thermodynamic transition occurring at a well-defined temperature T_g , a precise experimental determination of this critical

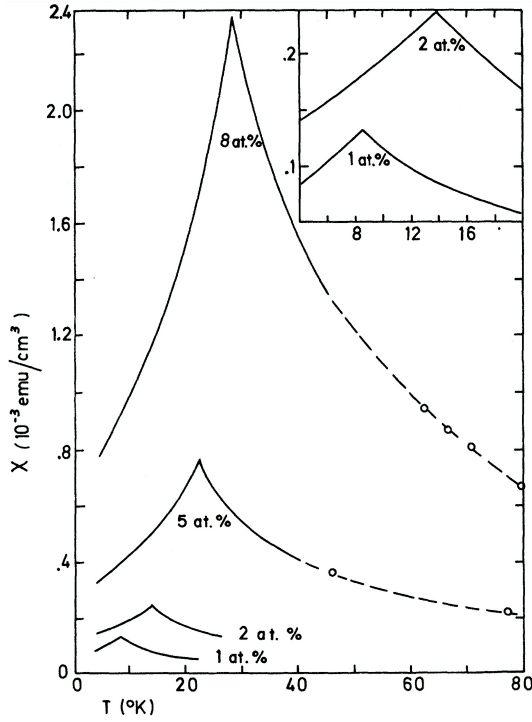


Figure 1.1. Static susceptibilities of AuFe with Fe concentrations from 1% to 8%. The experimental data points lie on the line and open circles indicate isolated points. From Ref. [Can72].

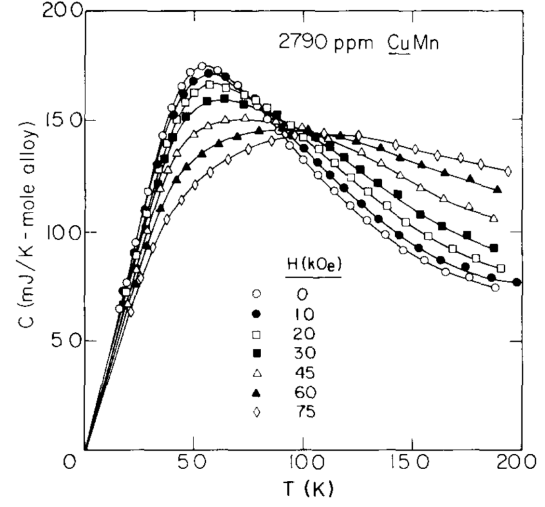


Fig 1 Typical specific heat data for CuMn ($T_g = 3.89 \text{ K}$)

Figure 1.2. Magnetic specific heat of CuMn spin glass with 2.79% dilution of Mn in different external magnetic fields. From Ref. [Bro83].

temperature is essential. This requirement was addressed through two different measurement protocols for linear susceptibility [Nag79]: **field cooled (fc)** where the sample is cooled below T_g under constant field, and **zero field cooled (zfc)** where the sample is first cooled below T_g , without external field, and then the field is applied to measure the linear response. The **fc** susceptibility exhibited time-independent, reversible behavior below T_g : keeping a constant field and changing the temperature above and below T_g , the same curve was obtained. On the contrary, **zfc** measurements showed slow, irreversible relaxation toward **fc** values: when the sample is heated above T_g and subsequently cooled again, the magnetization follows the **fc** curve, see Figure 1.3. This irreversibility onset at T_g provides a clear signature of the spin glass transition.

Off-equilibrium features emerged from these experiments. Supporting this hypothetical behavior, an experiment has been done aiming to estimate the distribution of relaxation times. From the distribution function of τ it was possible to estimate the order of magnitude of the maximum value $\tau_{max} \approx 10^{20} \text{ s}$, a few Kelvin degrees below the estimate of the critical temperature [Lun83].

1.1.2 From Theory to Applications

The results from the experimental community inspired theoretical insights into spin glass behavior. From **Mean-Field (MF)** theory [She75, Par79, Méz87c], it emerged

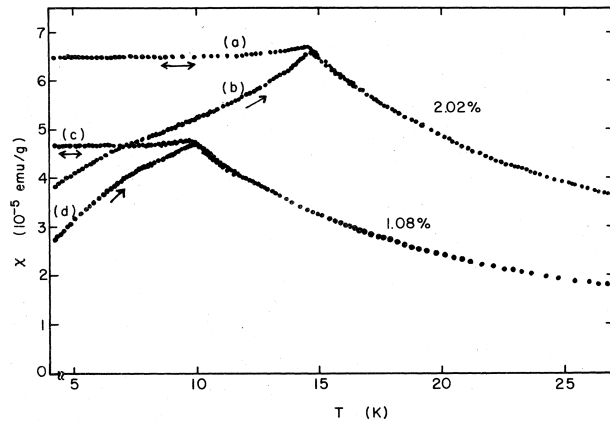


Figure 1.3. CuMn static susceptibilities measured in two regimes: *fc* and *zfc*. (b) and (d) are measured only by increasing temperature, while (a) and (c) curves are reversible. (a) and (b) are measured with 2.02% dilution of Mn, while (c) and (d) 1.08%. (a) and (c) are *fc* curves, while (b) and (d) are *zfc* ones. From Ref. [Nag79].

that the order parameter is the probability distribution of the overlap, describing a corrugated, coarse-grained free energy landscape as a function of some suitably defined phase space coordinates. In this landscape, high barriers separate numerous valleys with a hierarchical structure that depends chaotically on temperature. The probability distribution characterizes not only the valleys, but also their correlations. These ideas, along with the analytical techniques developed within this framework, became common tools across multiple disciplines.

From the rugged landscape picture of the free energy emerged the fundamental concept of “complexity”, which in the MF theory of spin glasses is understood as the exponential proliferation of pure states at low temperatures. The theoretical framework developed in this context soon revealed its broader potential, as its fundamental building blocks showed potential applications to seemingly distant disciplines. In a 1989 paper, Anderson recognized that the *randomly rugged landscape* paradigm underlay numerous optimization problems, prophetically declaring spin glasses and related models to be merely a “jumping-off point” for far-reaching applications [And89].

By rephrasing optimization and constraint satisfaction problems in the language of Statistical Mechanics, researchers could apply the machinery developed for spin glasses [Méz87b, Mon96]. While the most important results for spin glass models require computing the free energy to leading order in the thermodynamic limit, for optimization problems it is also interesting to calculate the first corrections to the analog of the free energy: the cost function [Par02]. Moreover, the theory generated numerous algorithms that proved to be fundamental not only in physics, but across multiple disciplines [Kir83, Mar92, Huk96, Mézo2, Krz12].

Parallel to these developments, significant work emerged establishing connections between the spin glass scenario and both neural networks [Ami85, Gar88, Kra89, Mato2], beginning with Hopfield’s seminal 1982 contribution [Hop82], and structural glasses [Kir87, Cav99, Fra11]. Furthermore, spin glass theory has found fruitful applications in biological systems [Pagoo, Shao6] and ecological modeling [Bis95]. It is noteworthy that research activity in all these domains continues to flourish today [Kab16, Alb16, Bir18, Alt21], evolving into a versatile toolkit for addressing complexity across disciplines [Cha23].

1.2 Essential Models and Notions

In order to reproduce the rich experimental results on these materials, universally called *spin glasses*, researchers understood that two ingredients were fundamental: frustration and randomness. Since its origin, the easiest and standard way to treat analytically magnetic systems is to deal with the Ising model. In its simplest version, no frustration and randomness are present:

$$\mathcal{H}_I(\{s_i\}_i) = -J \sum_{(i,j)} s_i s_j - H \sum_i s_i, \quad (1.8)$$

where $s_{i=1,\dots,N}$ are N Ising variables, $s_i = \pm 1$, the sum over (i,j) runs over nearest neighbors on a lattice, $J > 0$ is the intensity of the coupling between them and H/J is an external magnetic field. For some lattices, below a certain temperature, in zero external field, the global magnetization spontaneously assumes a macroscopic positive or negative value, long range order is created in the ferromagnetic phase. The order parameter can be identified by the magnetization per site:

$$m = \frac{1}{N} \sum_{i=1}^N s_i. \quad (1.9)$$

The thermodynamics of this system is described by the Helmholtz free energy (per site), $f(\beta, H)$, as a function of the temperature $T = 1/\beta$ (using $k_B = 1$) and the field, from the partition function $Z_N(\beta, H)$ in the thermodynamic limit

$$f(\beta, H) = - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \ln(Z_N(\beta, H)) \equiv - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \ln \left(\prod_{i=1}^N \sum_{s_i=\pm 1} e^{-\beta \mathcal{H}_I(\{s_i\}_i)} \right). \quad (1.10)$$

Since this computation is usually analytically unfeasible, it is standard to describe the equilibrium behavior of a system with the Gibbs free energy,

$$g(\{\phi_j\}_j) = - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \text{Tr}_{\{s_i\}|\{\phi_j\}} \left(e^{-\beta(\mathcal{H}_I + H \sum_i s_i)} \right), \quad (1.11)$$

effective function of some relevant phase-space variables $\{\phi_j\}_j$, where the trace is taken fixing the values for ϕ_j . For the Ising model, these variables reduce to the single global magnetization per site, given in Eq. 1.9. Within this framework, a simple picture can be given: $g(m)$ is convex above the critical temperature, while two minima at non-zero magnetization emerge and characterize the spontaneous symmetry breaking (the associated symmetry group is $\mathbb{Z}_2$).

By contrast, the analogous picture for spin glass systems is less clear. The phase-space variables ϕ_j should be, in this case, related to the quantity that seems to best characterize the occurrence of the spin glass transition: the overlap, as defined in Eq. 1.1. First of all, it is necessary to introduce the models used to reproduce the experimental systems. As stated in Section 1.1.1, the two ingredients to mimic a spin glass are *randomness* and *frustration*. In 1975, Edwards and Anderson identified these two features in the randomness of the couplings between nearest neighbors¹ [Edw75]

$$\mathcal{H}_{EA}(\{s_i\}_i; \mathbf{J}) = - \sum_{(i,j)} J_{ij} s_i s_j - H \sum_i s_i, \quad (1.12)$$

¹The original model was defined for vector spins [Edw75], but for simplicity Ising spins are considered here.

with random couplings drawn independently from a *normal* distribution: $J_{ij} \sim \mathcal{N}(J_0, J^2) \forall (i, j)$ nearest neighbors on a finite-dimensional hypercubic lattice. While the [Edwards–Anderson \(EA\)](#) model is originally defined in finite dimensions, when implemented on different lattices or graphs (as in the following) it will be referred to as a (generic) spin glass model.

Different distributions, such as a bimodal with $\pm J$ values, can be considered for the couplings, without altering the qualitative picture [[Méz87c](#), [Niso1](#)]. The vector $\mathbf{J}$ contains the values of the couplings corresponding to a specific realization. Three control parameters identify the thermodynamic state of the model: T/J , J_0/J and H/J . Frustration is present due to possible negative values of the couplings: consider a *plaquette* on a 2D square lattice, namely 4 spins on the vertices of a square, connected by nearest neighbors interactions; if the sign of the product of the four couplings is negative the configuration with the lowest energy is non-unique and it is impossible to satisfy all the couplings. The randomness of the EA model’s couplings, with its frustration effects is considered a kind of *quenched disorder*. The term “quenching” comes from metallurgy; it is a procedure where a workpiece is rapidly cooled down. In this sense, the couplings are frozen, i.e., their dynamics is much slower than the one of the microscopic degrees of freedom: the spins. The opposite procedure is called *annealing* and the term is also used in the framework of Statistical Mechanics.

Note that the quenched disorder in the EA model represents just one possible modeling approach. Consider the Hamiltonian in [Eq. 1.12](#) defined on a generic graph and suppose, for instance, that the couplings are all equal and positive (pure ferromagnet) while assigning to each site an independent random field H_i , drawn from $\mathcal{N}(0, \Delta H^2)$. The model just defined is called [Random Field Ising Model \(RFIM\)](#) [[Nat98](#)]. These minimal changes in the definition of the parameters, with respect to the spin glass model, lead to drastic changes in the corresponding critical behavior. However, whenever frustration is not relevant (i.e., at high temperature or on a loop-less graph with free boundary conditions) one can map the RFIM Hamiltonian into the spin glass one (in the $J_0 = 0$ case) [[Mat11](#)]. The differences between the two models arise approaching the corresponding critical temperatures: no spin glass phase is present in the low-temperature behavior of the RFIM, except under specific conditions [[Krz10](#)].

1.2.1 How to treat Disorder

From the Hamiltonian in [Eq. 1.12](#) it is clear that all macroscopic observables, such as the free energy, depend on $\mathbf{J}$, i.e., on the specific realization of the disorder. Since spin glass materials exhibit macroscopic properties, independent of specific realizations (e.g., particular Mn atom configurations in the Cu substratum), it is interesting to make predictions valid for all samples. Analogously, in the thermodynamic limit, it would be desirable that macroscopic observables of the EA model are independent of individual coupling realizations. This property is called *self-averaging* and it holds for all observables whose sample-to-sample fluctuations vanish in the large-system limit [[Méz87c](#), [Cas05](#)]. An important example is the free energy

$$f_N(\beta, H; \mathbf{J}) \equiv -\frac{1}{\beta N} \ln \left(\prod_{i=1}^N \sum_{s_i=\pm 1} e^{-\beta \mathcal{H}_{EA}(\{s_i\}; \mathbf{J})} \right), \quad (1.13)$$

where the dependence on the system size is explicit in the subscript. The interesting quantity is the disorder-average of the free energy, which will reproduce macroscopic

features of experimental systems. Here and in the following chapters (if not explicitly written) the average over the couplings (or *disordered average*) is denoted with $\overline{\cdot}$. The next relation holds for $f_N(\beta, H; \mathbf{J})$:

$$\overline{f_N(\beta, H; \mathbf{J})^2} - \left(\overline{f_N(\beta, H; \mathbf{J})} \right)^2 = \mathcal{O} \left(\frac{1}{N} \right), \quad (1.14)$$

and thus small deviations separate the disorder average value of the intensive free energy from $f_N(\beta, H; \mathbf{J})$, in the thermodynamic limit. From this property, one understands that both quantities should give the same result. In [Chapter 2](#) the average over the couplings' realizations for the MF version of the EA model is computed, since in this way the explicit dependence on $\mathbf{J}$ will analytically disappear.

The quantity one needs to compute is the so-called “quenched” average

$$\overline{f_N(\beta, H; \mathbf{J})} \equiv -\frac{1}{\beta N} \ln \left(\overline{\prod_{i=1}^N \sum_{s_i=\pm 1} e^{-\beta \mathcal{H}_{EA}(\{s_i\}; \mathbf{J})}} \right), \quad (1.15)$$

which implies that the couplings are fixed on the microscopic timescale, since the sum over the configurations of the spin is done inside the logarithm. This is also the reason why this computation is not easy to do. Going back to the metallurgy, one could think to let the couplings evolve at the same timescale as the spins,

$$f_N^{\text{ann}}(\beta, H) \equiv -\frac{1}{\beta N} \ln \left(\overline{\prod_{i=1}^N \sum_{s_i=\pm 1} e^{-\beta \mathcal{H}_{EA}(\{s_i\}; \mathbf{J})}} \right), \quad (1.16)$$

obtaining the so-called “annealed” average, which is way easier, since in this case the average over the disorder is done inside the logarithm, at the same level as the spin configurations. However, the annealed average is only an approximation of the quenched one, which is the needed object. Moreover, the Jensen inequality holds

$$\overline{f_N(\beta, H; \mathbf{J})} \leq f_N^{\text{ann}}(\beta, H), \quad (1.17)$$

with equality achieved in the paramagnetic phase (high T), where frustration, inherently accounted for in the quenched calculation but absent in the annealed approximation, does not dominate the thermodynamics.

1.2.2 The glassy phase

To properly address the question “what characterizes the glassy phase?”, one must ultimately refer to the results of MF theory, as presented in [Chapter 2](#). However, some key notions are introduced here to guide the following discussion.

In ferromagnets, “ergodicity breaking” is straightforwardly understood via the behavior of the Gibbs free energy at low temperatures. Below the critical point, two symmetry-broken states emerge with an effectively infinite barrier between them in the thermodynamic limit. An infinitesimal uniform field selects one, provided the zero-field limit is taken before the thermodynamic one. This enables the definition of thermal averages within a single *pure state*.

In spin glasses, the situation is more subtle. Below a certain temperature, the system freezes into disordered states with non-zero local magnetizations,

$$m_i = \langle s_i \rangle \neq 0, \quad (1.18)$$

yet no simple global field can select one of these states. To describe this complexity, an accurate definition of *pure state* must be introduced [Hus87]. One can picture the Gibbs measure as a sum over components, each with its own weight. Observables are then averaged as

$$\langle \cdot \rangle = \sum_a w_a \langle \cdot \rangle_a, \quad (1.19)$$

where the sum runs over all possible pure states; w_a is defined as the weight relative to state a : $w_a \equiv Z_a/Z$, where Z_a is the partition function of the system restricted to configurations belonging to state a and Z is the complete one. This structure is essential even in simpler models like the Ising ferromagnet, where $\langle \cdot \rangle = \frac{1}{2} \langle \cdot \rangle_+ + \frac{1}{2} \langle \cdot \rangle_-$, since the two pure states, with opposite non-zero magnetization, $\langle \cdot \rangle_+$ and $\langle \cdot \rangle_-$, are equally weighted in zero field, below the critical temperature.

In finite dimensions, only equilibrium states (those minimizing the free energy density) can truly break ergodicity. Metastable states are separated from the stable ones by finite barriers and decay over long times. The nucleation cost of a droplet of the stable phase with linear size r on a D -dimensional lattice in the Ising model scales as $\Delta F = \sigma r^{D-1} - \delta f r^{D/2}$, implying finite escape times unless the phases coexist. Hence, decomposing the Gibbs measure into metastable components only makes sense on intermediate timescales. In contrast, MF models allow for infinite barriers, and metastable states can behave as pure states with well-defined thermodynamic weights.

Moreover, as already stated in Section 1.1.1, a natural order parameter is the *overlap*, which quantifies the similarity between configurations or states. For two configurations $\{s_i\}_{i=1,\dots,N}$ and $\{\tilde{s}_i\}_{i=1,\dots,N}$, the overlap is

$$q_{s\tilde{s}} = \frac{1}{N} \sum_{i=1}^N s_i \tilde{s}_i, \quad (1.20)$$

ranging from 1 (identical configurations) to -1 (globally flipped configurations). The overlap between two states, a and b is

$$q_{ab} = \frac{1}{N} \sum_{i=1}^N \langle s_i \rangle_a \langle s_i \rangle_b, \quad (1.21)$$

and it captures the geometry of the space of pure states. The connection with the quantity $q_{s\tilde{s}}$ can be found making the average over configurations in states a and b explicit for a generic model with Hamiltonian $\mathcal{H}$:

$$q_{ab} = \frac{1}{Z_a Z_b} \sum_{\{s_i\}_{i \in a}} e^{-\beta \mathcal{H}(\{s_i\}_i)} \sum_{\{\tilde{s}_i\}_{i \in b}} e^{-\beta \mathcal{H}(\{\tilde{s}_i\}_i)} q_{s\tilde{s}}. \quad (1.22)$$

In the paramagnetic phase, at high temperature and/or high external field only one state will be present, characterized by zero local magnetization $\langle s_i \rangle_a = 0$ and, consequently, zero overlap. In the ferromagnetic state a local magnetization arises, $\langle s_i \rangle_a \neq 0$, implying non-zero overlap. On the other hand, a spin glass state, characterized by more than a single pure state, is characterized by frozen local magnetizations with random signs, such that $\sum_i \langle s_i \rangle \approx 0$, but with non-zero overlap. As an analogy, a single snapshot of the molecular configuration of a real glass is indistinguishable from that of

²With σ and δf two constant parameters.

a liquid; only by comparing two snapshots at very different times does the difference emerge: overlaps vanish in liquids but remain finite in glasses, reflecting persistent “memory”.

After this discussion, one might ask whether one valley (or more, if some symmetry is present), corresponding to fully ordered “monodomain” configurations, is deeper than others (which represent metastable states with large clusters of spins flipped relative to the monodomain). For comparison, in a ferromagnet or antiferromagnet, such configurations exist but are not separated by high free-energy barriers from the stable states, so they do not dominate the dynamics. The answer to this long-standing question for finite-dimensional spin glasses remains unclear and will be addressed in later sections.

1.2.3 Further Building Blocks: Graphs

Before starting with more precise computations for the models of interest, a brief introduction to notions and notation from graph theory is needed.

Given a set of *nodes*, $\mathcal{V}$, and a set of *edges*, $\mathcal{E} \subseteq \mathcal{V}^2$, a *graph* is the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ [Die18]. The cardinality of $\mathcal{V}$ and $\mathcal{E}$ is indicated, respectively, with $|\mathcal{V}| = N$ and $|\mathcal{E}| = N_{\mathcal{E}}$. An element of $\mathcal{E}$ is denoted with (i, j) , where i and j are said *ends* of (i, j) . Two nodes, $i, j \in \mathcal{V}$ are *neighbors* (or *adjacent*) if $(i, j) \in \mathcal{E}$, the number of neighbors of a node is its *degree* or *connectivity* (in the following denoted by z if it is equal for all nodes). The *neighborhood* of node i , that is the set of all neighbors of i , is denoted by ∂i .

A specific example of a graph, widely considered in this thesis, is the *hypercubic lattice* in D dimensions, for which the nodes are embedded in the D dimensional Euclidean space and the edges connect nearest neighbor sites plus, eventually, periodic boundary conditions.

Two edges are *incident* if they share one end. Given two graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$ then $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is referred to as a *subgraph* of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

A *path* from i_0 to i_k , of length k , is a graph with $\mathcal{V} = \{i_0, \dots, i_k\}$ (with all distinct nodes) and $\mathcal{E} = \{(i_0, i_1), \dots, (i_{k-1}, i_k)\}$, where i_0 and i_k are the *ends* and $\{i_1, \dots, i_{k-1}\}$ the *inner vertices* of the path. A *cycle* (or *loop*) is a path where i_0 and i_k coincide. A graph is called *connected* if for any two of its vertices there exists at least one path between them. If all the vertices of the graph are pairwise adjacent, the graph is called *complete* or **Fully Connected (FC)**. A connected graph without cycles is called a *tree*. In such a graph, nodes with connectivity equal to 1 are called *leaves*.

While **FC** graphs are generically useful, since **MF** theory is exact for models defined on them, the work presented in this thesis mostly deals with the Bethe M -layer construction, that allows to compute, for generic observables, a perturbative expansion in the number of (topological) loops, whose leading order is the behavior on a specific example of *random graph*, that is introduced here.

An *ensemble* of random graphs, $\mathcal{G}_N$, is a probability space over the subgraphs, $\mathcal{G}$, of the **FC** graph with N nodes [Bol98]. As for non-random graphs, these ensembles are characterized by the number of edges $N_{\mathcal{E}}$ and the degree of each node, which are both random variables in this case. Two key distributions characterize connectivity in random graphs: (i) the standard degree distribution and (ii) the “excess degree” distribution, which gives the degree distribution for nodes at the ends of randomly

selected edges [New18]. The excess degree is denoted by r in the following. Two ensembles that will be considered in this thesis are

- **Random Regular Graph (RRG)**, $\mathcal{G}_{N,N_{\mathcal{E}}}$: the probability of drawing a graph from this ensemble is uniform over the set of graphs with N nodes, each having the same connectivity $z = c$, such that the number of edges, which is not random, is:

$$N_{\mathcal{E}} = \frac{c N}{2}, \quad (1.23)$$

where $c N$ is even. In this case, the distributions of both the degree and the excess degree are Kronecker deltas peaked on c and $c - 1$, respectively.

- **Erdős–Rényi (ER)**, $\mathcal{G}_{N,\varphi}$: a graph belonging to this ensemble is constructed by including every potential edge independently, with probability $\varphi \in [0, 1]$. In this case, the probability of drawing a specific graph, with $N_{\mathcal{E}}$ edges, is binomial

$$\mathbb{P}[\mathcal{G}_{N,\varphi}] = \binom{N}{N_{\mathcal{E}}} \varphi^{N_{\mathcal{E}}} (1 - \varphi)^{\binom{N}{2} - N_{\mathcal{E}}}. \quad (1.24)$$

In this case, the average number of edges over the distribution of the graphs in the ER ensemble is

$$\varphi \binom{N}{2}. \quad (1.25)$$

Other characteristics of the resulting graphs depend on the scaling of the probability φ with N . In order to ensure an average finite connectivity c , even for $N \rightarrow \infty$, the following scaling is considered here and in the following: $\varphi = c/N$. The probability that the degree of a node is an integer value, k , denoted with $p_c(z = k)$, is

$$p_c(z = k) = \binom{N-1}{k} \left(\frac{c}{N}\right)^k \left(1 - \frac{c}{N}\right)^{N-1-k}, \quad (1.26)$$

whose asymptotic behavior is the Poisson distribution

$$p_c(z = k) \xrightarrow{N \rightarrow \infty} \frac{c^k}{k!} e^{-c}. \quad (1.27)$$

The excess degree distribution, $q_c(r = k)$, is related to $p_c(z = k)$ as follows [New18]

$$q_c(r = k) = \frac{(k+1) p_c(z = k+1)}{\sum_{l=0}^{\infty} (l+1) p_c(z = l+1)}, \quad (1.28)$$

notice that the denominator, which serves as a normalization constant, corresponds to the average value of the connectivity, c . Considering the Poisson distribution given in Eq. 1.27 it can be verified that $q_c(r = k) = p_c(z = k)$, with the same average value c :

$$q_c(r = k) = \frac{c^k}{k!} e^{-c}. \quad (1.29)$$

This is one of the main differences between ER and RRG, for which the excess degree is fixed to $c - 1$.

In the framework of complex networks, these two ensembles go under the name of *uncorrelated networks* [Dor22], characterized by the important feature that the joint probability distribution of the degree of two adjacent nodes factorizes for each couple of nodes. Another important property of these ensembles is that the graphs generated turn out to be *locally tree-like*, meaning that considering a generic node, its neighborhood at distance l (that can be large, but less than extensive in N) looks like a tree graph with high probability in the limit $N \rightarrow \infty$. More precisely, consider a realization of the ER random graph³ which contains a subgraph with s vertices that is a tree. Since the number of edges of the subgraph is $s - 1$, the probability for such a realization scales as

$$\frac{N!}{(N-s)!} \left(\frac{c}{N}\right)^{s-1} \sim N. \quad (1.30)$$

This means that in the large N limit the resulting graph has no loops as long as the number of nodes s becomes of order N , which happens at a distance [Mal19]

$$l \sim \frac{\ln N}{\ln c}, \quad (1.31)$$

where, considering the tree-like structure, the number of nodes within distance l scales as c^l on an ER random graph (and as $(c-1)^l$ on an RRG). At this distance, loops can appear.

Due to these properties, these graphs turn out to be particularly convenient for analyzing lattice models, allowing for simple and tractable analytical approaches. Of particular importance is the *Cayley tree* with generic connectivity z , constructed as follows. Starting from a node, called *root*, it is connected to z distinct nodes. Each of these z nodes is then connected to $z-1$ new nodes, again distinct from each other and from the z ones before, and this process is repeated l times until the final “generation” of nodes forms the boundary of the graph. Being all nodes different, the resulting structure is a tree. Consequently, the boundary comprises a finite fraction of the total nodes, a property that makes the Cayley tree unsuitable as the underlying graph for Statistical Mechanics models. Instead, one typically considers a subgraph including only generations up to $l' < l$. In the limit of large l and l' , with $l/l' \rightarrow \infty$, this subgraph is known as the *Bethe lattice*. However, for the spin glass models of interest, this graph remains problematic because boundary conditions play a significant role due to frustration [Mézo1]. Therefore, in the following, the term “Bethe lattice” will be used to refer to an RRG.

In Part II, different kinds of “ M -layer graphs”, widely used for the main results of this thesis, are defined in detail.

³The RRG ensemble shares the same property in the regime $N \rightarrow \infty$.

Chapter 2

Mean-Field Spin Glasses

2.1 The Sherrington–Kirkpatrick Model

The [Sherrington–Kirkpatrick \(SK\)](#) model is defined as the “infinite-dimensional” version of the [EA](#) model, as it is formulated on a [FC](#) graph [[She75](#)]. Models on [FC](#) graphs can be easily solved using the naive [MF](#) approximation. The model is described by the following Hamiltonian:

$$\mathcal{H}_{\text{SK}}(\{s_i\}_i) = - \sum_{i < j}^N J_{ij} s_i s_j - H \sum_{i=1}^N s_i, \quad (2.1)$$

where s_i for $i = 1, \dots, N$ are N Ising variables, H is a constant external field, and the couplings J_{ij} are independent and identically distributed random variables with the following Gaussian distribution:

$$P_J(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}} \exp \left\{ -\frac{N}{2J^2} \left(J_{ij} - \frac{J_0}{N} \right)^2 \right\}, \quad (2.2)$$

where the $1/N$ scaling of the mean (J_0/N) and variance (J^2/N) ensures the correct scaling with N of quantities such as the energy and the free energy, i.e., extensivity. Though minimal, the [SK](#) model proves to be analytically intriguing. To derive the correct equation of state of such quenched disordered models, the computation of the quenched free energy density

$$\overline{f(\beta, H)} = - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \int \prod_{i < j} dJ_{ij} P_J(J_{ij}) \ln Z_N(\beta, H) \quad (2.3)$$

is needed. As anticipated in [1.2.1](#), while the disorder average of the logarithm of the partition function $Z_N(\beta, H)$ is nontrivial, computing the average of the partition function itself, or even its moments, is relatively straightforward. In this context, the *Replica method* was used to tackle this problem.

2.1.1 The Replica Method

The mathematics behind the basic idea of this method is simply the following limit

$$\overline{\ln Z_N} = \lim_{n \rightarrow 0} \frac{\overline{Z_N^n} - 1}{n}. \quad (2.4)$$

The key point is to interpret Z_N^n as the partition function of a system composed of n independent *replicas* of the original SK model. The idea is, from Eq. 2.4, to compute the following quantity

$$\overline{Z_N^n(\beta, H)} = \int \prod_{i < j} dJ_{ij} P_J(J_{ij}) \operatorname{Tr} \exp \left(\beta \sum_{i < j} J_{ij} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha + \beta H \sum_{i=1}^N \sum_{\alpha=1}^n s_i^\alpha \right), \quad (2.5)$$

where greek letters identify replica indices and the trace Tr is over the spins of all replicas configurations: $\prod_{i=1}^N \prod_{\alpha=1}^n \sum_{s_i^\alpha = \pm 1}$. Ignoring irrelevant constants and subleading (in N) terms, standard procedures yield to the following result [Niso1]

$$\begin{aligned} \overline{Z_N^n(\beta, H)} = \exp \left(\frac{N \beta^2 J^2 n}{4} \right) & \int \prod_{\alpha < \beta} dQ_{\alpha\beta} \int \prod_{\alpha} dm_{\alpha} \times \\ & \exp \left(-\frac{N \beta^2 J^2}{2} \sum_{\alpha < \beta} Q_{\alpha\beta}^2 - \frac{N \beta J_0}{2} \sum_{\alpha} m_{\alpha}^2 + N \ln \prod_{\alpha} \sum_{s^{\alpha} = \pm 1} e^{\mathcal{L}} \right), \end{aligned} \quad (2.6)$$

where

$$\mathcal{L} \equiv \beta^2 J^2 \sum_{\alpha < \beta} Q_{\alpha\beta} s^{\alpha} s^{\beta} + \beta \sum_{\alpha} (J_0 m_{\alpha} + H) s^{\alpha}. \quad (2.7)$$

Notice that the exponents in Eq. 2.6 are all proportional to N , thus one could think to perform the integrals over $Q_{\alpha\beta}$ and m_{α} with the saddle point method¹, neglecting corrections

$$\overline{Z_N^n(\beta, H)} = 1 + N n \left\{ -\frac{\beta^2 J^2}{4 n} \sum_{\alpha \neq \beta} Q_{\alpha\beta}^2 - \frac{\beta J_0}{2 n} \sum_{\alpha} m_{\alpha}^2 + \frac{1}{n} \ln \prod_{\alpha} \sum_{s^{\alpha}} e^{\mathcal{L}} + \frac{1}{4} \beta^2 J^2 \right\}, \quad (2.8)$$

where the values of $Q_{\alpha\beta}$ ($\alpha \neq \beta$) and m_{α} are fixed by the saddle-point conditions

$$Q_{\alpha\beta} = \frac{1}{\beta^2 J^2} \frac{\partial}{\partial Q_{\alpha\beta}} \ln \prod_{\alpha} \sum_{s^{\alpha}} e^{\mathcal{L}} = \langle s^{\alpha} s^{\beta} \rangle_{\mathcal{L}}, \quad (2.9)$$

and

$$m_{\alpha} = \frac{1}{\beta J_0} \frac{\partial}{\partial m_{\alpha}} \ln \prod_{\alpha} \sum_{s^{\alpha}} e^{\mathcal{L}} = \langle s^{\alpha} \rangle_{\mathcal{L}}, \quad (2.10)$$

and $\langle \cdot \rangle_{\mathcal{L}}$ represents the average with the weight $e^{\mathcal{L}}$.

Before proceeding with the computations, some comments can be made. First of all, from Eq. 2.3 and Eq. 2.4 the quenched free energy reads

$$\lim_{n \rightarrow 0} A_n(\{Q_{\alpha\beta}\}, \{m_{\alpha}\}) \equiv \lim_{n \rightarrow 0} \left\{ \frac{\beta J^2}{4 n} \sum_{\alpha \neq \beta} Q_{\alpha\beta}^2 + \frac{J_0}{2 n} \sum_{\alpha} m_{\alpha}^2 - \frac{1}{\beta n} \ln \prod_{\alpha} \sum_{s^{\alpha}} e^{\mathcal{L}} - \frac{1}{4} \beta J^2 \right\}. \quad (2.11)$$

Moreover the saddle-point conditions for $Q_{\alpha\beta}$ and m_{α} are equivalent to

$$Q_{\alpha\beta} = \overline{\langle s_i^{\alpha} s_i^{\beta} \rangle}, \quad (2.12)$$

¹To do so, the limits $n \rightarrow 0$ and $N \rightarrow \infty$ should be exchanged, even if from Eq. 2.3 it is not obvious that it can be done.

and

$$m_\alpha = \overline{\langle s_i^\alpha \rangle}, \quad (2.13)$$

respectively. This connection allows one to identify $Q_{\alpha\beta}$ and m_α as order parameters. While the interpretation of m_α as the magnetization of replica α is straightforward (from the theory of ferromagnets), the identification of $Q_{\alpha\beta}$ as the *overlap* between replicas α and β is a specific feature of the SK model. These order parameters, that arise directly from the Replica method, are good candidates to characterize the putative spin glass phase in which spins are frozen in random directions, thus giving $\overline{\langle s_i^\alpha \rangle} = 0$, for a generic replica α . In this case, the overlap between two replicas α and β , with the same disorder instance, averaged over all possible instances will be different from zero, indicating the presence of a new phase.

2.1.2 Replica Symmetric Ansatz

Since it is not possible to solve the saddle-point equation for a generic $Q_{\alpha\beta}$ and m_α , a hypothesis on their structure is needed. As for the pure Ising model, where, due to homogeneity, one can assume that the magnetization is not dependent on the specific site, in this case one assumes it to be independent of the specific replica, once the disorder average is taken. Similarly, it is intuitive to think of a simple situation in which $Q_{\alpha\beta}$ is not dependent on the specific replicas α or β , since, *a priori*, there is no physical reason to expect a different replica-dependence. While the condition $Q_{\alpha\beta} = Q_{\beta\alpha}$ always holds by definition, a possible simple assumption is

$$Q_{\alpha\beta} = q \quad \forall \alpha \neq \beta. \quad (2.14)$$

In this context, this assumption is referred to as **Replica Symmetry (RS)**. The overlap $Q_{\alpha\beta}$ can be represented as a matrix and the RS hypothesis results in a symmetric $n \times n$ matrix with $Q_{\alpha\beta} = q$ for all $\alpha \neq \beta$ ². The free energy given this condition and in the limit $n \rightarrow 0$ is

$$\overline{f_{\text{RS}}(\beta, H)} = -\frac{\beta J^2}{4}(1-q)^2 + \frac{J_0}{2}m^2 - \frac{1}{\beta} \int \text{D}z \ln 2 \cosh [\beta(J\sqrt{q}z + J_0 m + H)], \quad (2.15)$$

where the Gaussian integral is introduced to treat the sums over the replicas in the overlap term in Eq. 2.11, with the following notation

$$\text{D}x = \text{d}x \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}. \quad (2.16)$$

The saddle-point equations can now be written in self-consistent forms for the two global parameters m and q :

$$m = \int \text{D}z \tanh [\beta(J\sqrt{q}z + J_0 m + H)] \quad (2.17)$$

and

$$q = \int \text{D}z \tanh^2 [\beta(J\sqrt{q}z + J_0 m + H)]. \quad (2.18)$$

²Specifically, in Eq. 2.6 the parameter $Q_{\alpha\beta}$ with $\alpha = \beta$ is not present, for this reason the diagonal of the matrix $Q_{\alpha\beta}$ is not relevant in the computation.

In this form, it is clear that the randomness of the couplings reflects in the randomness of the effective (mean-)field acting on a spin

$$h^{\text{loc}}(z) \equiv J\sqrt{q}z + J_0 m + H, \quad (2.19)$$

given that $z \sim \mathcal{N}(0,1)$. From Eq. 2.17 and Eq. 2.18 the transition lines from the paramagnetic phase to ferromagnetic and spin glass phases respectively, can be derived, as shown in Figure 2.1.

To verify the validity of the RS hypothesis, it is useful to compute thermodynamic observables, such as the free energy. This requires solving the equations of state, Eq. 2.17 and Eq. 2.18, for magnetization and overlap, respectively. While numerical methods are generally needed for arbitrary parameters J , J_0 , H , and T , analytical solutions become tractable in simplified regimes, specifically at low temperatures with $J_0 = 0 = H$. The low temperature limit can be computed for the overlap in this case

$$q = \int Dz \tanh^2 [\beta(J\sqrt{q}z)] = 1 - \sqrt{\frac{2}{\pi}} \frac{T}{J} + \mathcal{O}(T^2), \quad (2.20)$$

while $m = 0$ for symmetry reasons. Using this asymptotic expression for the overlap in Eq. 2.15, and neglecting $\mathcal{O}(T^2)$

$$\overline{f_{\text{RS}}(\beta, 0)} = -\sqrt{\frac{2}{\pi}} J + \frac{T}{2\pi}, \quad (2.21)$$

that leads to negative zero-temperature entropy. However, given the discrete nature of the microscopic variables, Ising spins, the thermodynamic entropy must be non-negative, invalidating the RS Ansatz.

To overcome this problem, a different approach should be considered. Since the RS Ansatz is based on the assumption that the overlap $Q_{\alpha\beta}$ is independent of the specific couples of replicas α and β , some attempts, by Blandin *et al.*, have been made to “break” this symmetry before the discovery of the solution [Bra78, Par80a].

The concept of **Replica Symmetry Breaking (RSB)** emerges from the analysis of the overlap matrix $Q_{\alpha\beta}$ in disordered systems, with two complementary aspects. The first is based on the following physical intuition. The ruggedness of the free energy landscape suggests the existence of multiple pure states. When two replicas, α and β , occupy the same state, their overlap $Q_{\alpha\beta}$ is larger than when they occupy different states. This dependence “breaks replica symmetry”, as $Q_{\alpha\beta}$ acquires a nontrivial dependence on the replica indices. The second aspect is based on “replica equivalence”. Quantities with single replica indices, such as the magnetization m_α , must be index-independent. Similarly, $\sum_\beta Q_{\alpha\beta}$ must lose its α -dependence after summation on β . This feature should always hold, regardless of the presence or absence of RS, because there is no a priori reason for any specific replica to be privileged.

RSB allows for a hierarchical structure of the matrix $Q_{\alpha\beta}$ that preserves replica equivalence while simultaneously permitting nontrivial overlaps between different states. However, the precise form of the overlap matrix in the RSB Ansatz can be guessed from the specific perturbation that makes the RS solution unstable.

Stability Analysis

A generic solution is valid if the resulting free energy is stable under small perturbations of the order parameter(s). In this case $A_n(\{Q_{\alpha\beta}\}, m_\alpha)$, in Eq. 2.11, should be expanded

to second order near the RS solution

$$A_n(\{Q_{\alpha\beta}\}, \{m_\alpha\}) = A_n(q, m) + \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 A_n(\{Q_{\alpha\beta}\}, \{m_\alpha\})}{\partial m_\alpha \partial m_\beta} \varepsilon_\alpha \varepsilon_\beta \\ + \frac{1}{2} \sum_{\alpha < \beta, \gamma} \frac{\partial^2 A_n(\{Q_{\alpha\beta}\}, \{m_\alpha\})}{\partial Q_{\alpha\beta} \partial m_\gamma} \eta_{\alpha, \beta} \varepsilon_\gamma + \frac{1}{2} \sum_{\alpha < \beta, \gamma < \delta} \frac{\partial^2 A_n(\{Q_{\alpha\beta}\}, \{m_\alpha\})}{\partial Q_{\alpha\beta} \partial Q_{\gamma\delta}} \eta_{\alpha\beta} \eta_{\gamma\delta}, \quad (2.22)$$

where

$$Q_{\alpha\beta} = q + \eta_{\alpha\beta} \quad \text{and} \quad m_\alpha = m + \varepsilon_\alpha. \quad (2.23)$$

Since all replicas must be equivalent, the perturbation of the RS solution for the magnetization must be of the same RS form: $\varepsilon_\alpha = \varepsilon$. Moreover, the magnetization doesn't undergo RSB, as can be verified with the stability computations [dAlm78]. Here, for simplicity, the stability analysis is done in the case $J_0 = 0$, thus no magnetization is involved. Notice also that the case $J_0 \neq 0$ and $H = 0$ can be deduced from the complementary case $J_0 = 0$ and $H \neq 0$, identifying $J_0 m$ with H .

Next, the diagonalization of the Hessian matrix is needed. Analyzing the replica indices, it can only have three distinct values

$$\frac{\partial^2 A_n(\{Q_{\alpha\beta}\})}{\partial Q_{\alpha\beta} \partial Q_{\gamma\delta}} = \begin{cases} P & \text{if } \alpha = \gamma \text{ and } \beta = \delta, \\ R & \text{if } \alpha = \gamma \text{ and } \beta \neq \delta, \\ Q & \text{if } \alpha \neq \gamma \text{ and } \beta \neq \delta, \end{cases} \quad (2.24)$$

which yields three distinct types of eigenvectors $\mu_{\alpha\beta}$ (and the corresponding eigenvalues λ):

- **Longitudinal sector**

The simplest eigenvector has the same structure as the RS solution and it is called *longitudinal*, with degeneracy 1:

$$\mu_{\alpha\beta}^L : \begin{pmatrix} 0 & & & \\ & 0 & & a \\ & & \ddots & \\ & a & & 0 \\ & & & & 0 \end{pmatrix} \quad \lambda^L = P + 2(n-2)Q + \frac{(n-2)(n-3)}{2}R.$$

The value a is arbitrary and can be fixed by imposing normalization of the vector.

- **Anomalous sector**

The so-called *anomalous* eigenvector singles out one replica index (say, $\alpha = 1$), leading to a matrix with a different value, b , in the first row and column, and c elsewhere. Its degeneracy is $n - 1$, the structure is:

$$\mu_{\alpha\beta}^A : \begin{pmatrix} 0 & b & \cdots & b \\ b & 0 & & c \\ \vdots & & \ddots & \\ b & c & & 0 \end{pmatrix} \quad \lambda^A = P + (n-4)Q - (n-3)R.$$

Notice that, at variance with the longitudinal eigenvector, this kind of perturbation doesn't respect equivalence of replicas, i.e., $\sum_{\beta} \mu_{\alpha\beta}^A$ does depend on the specific replica α . Orthogonality with the longitudinal eigenvector (that is a $n(n-1)/2$ dimensional vector with all equal entries) fixes the value b (or c):

$$b = \frac{2-n}{2}c. \quad (2.25)$$

It is important to notice, from this last equation and the eigenvalues λ^L and λ^A , that in the limit $n \rightarrow 0$ these two sectors become degenerate.

- **Replicon sector**

The last type of perturbation is the so-called *replicon* eigenvector. In this case, two replica indices are singled out (say, $\alpha = 1, \beta = 2$). The degeneracy of the replicon sector is $n(n-3)/2$, so that the sum of the degeneracies of the three sectors is $n(n-1)/2$, as expected. Generalizing the anomalous eigenvector case with three different values d, e and f , the structure is:

$$\mu_{\alpha\beta}^R : \begin{pmatrix} 0 & d & e & \cdots & e \\ d & 0 & e & \cdots & e \\ e & e & 0 & \begin{array}{c} \diagup f \diagdown \\ \end{array} \\ \vdots & \vdots & \begin{array}{c} \diagdown f \diagup \\ \end{array} & \ddots & \\ e & e & \begin{array}{c} \diagdown f \diagup \\ \end{array} & & 0 \end{pmatrix} \quad \lambda^R = P - 2Q + R.$$

Imposing orthogonality with the other two sectors, relations between the values d, e and f arise

$$\begin{cases} d = (2-n)e \\ e = \frac{3-n}{2}f \end{cases} \quad (2.26)$$

that ensure an important feature: the replicon perturbation respects equivalence between replicas, i.e., $\sum_{\beta} \mu_{\alpha\beta}^R$ doesn't depend on the replica index α .

Computing the quantities P , Q , and R from Eq. 2.24 reveals that, above a certain temperature, identified as the critical temperature for the spin glass phase, T_g , and for a fixed external field H , all three eigenvalues are equal and positive. Below this temperature, the only eigenvalue that turns negative is the replicon. For $J_0 = 0$ the condition $\lambda^R = 0$ leads to

$$\int Dz \operatorname{sech}^4[\beta(J\sqrt{q}z + h_{\text{dAT}}(T/J))] = \frac{T^2}{J^2}, \quad (2.27)$$

where $h_{\text{dAT}}(T/J)$ is the value of the external field, as a function of T/J , such that the RS solution becomes unstable, the so-called [de Almeida–Thouless \(dAT\) line](#) [[dAlm78](#)], represented in [Figure 2.1](#) as line (1). To better understand the main results of [Chapter 7](#) it is important to remark that the dAT line diverges at zero temperature, as can be seen from [Eq. 2.27](#).

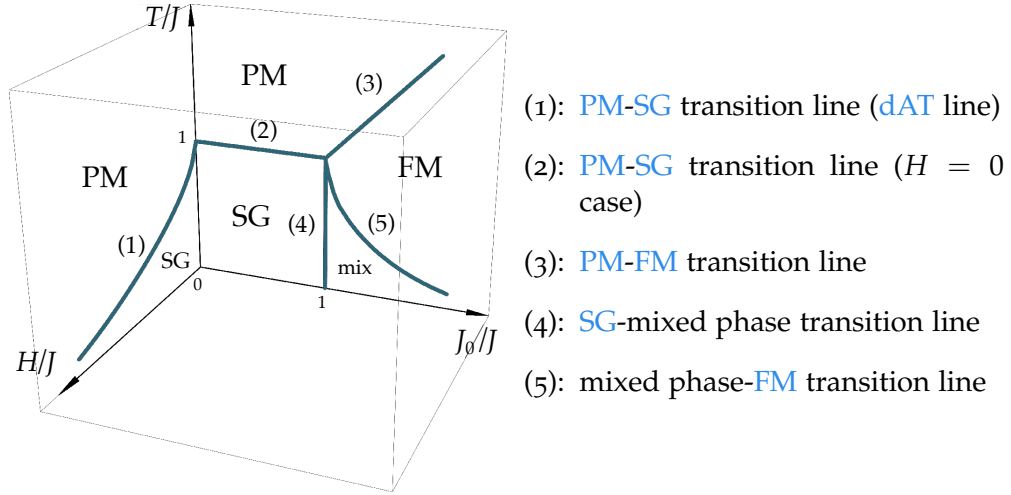


Figure 2.1. Phase diagram of the SK model for the three relevant control parameters T/J , H/J and J_0/J in the two planes T/J vs J_0/J and T/J vs H/J .

2.1.3 Replica Symmetry Breaking

This analysis indicates that the overlap matrix becomes unstable along the “replicon direction”. Instead of analyzing perturbations involving only a pair of singled-out replicas, one can consider similar perturbations extended to multiple pairs. This perspective naturally leads to the RSB structure, which was previously anticipated based on physical reasoning:

$$Q_{\alpha\beta} = \begin{cases} q_1 & \text{if } I\left(\frac{\alpha}{m_1}\right) = I\left(\frac{\beta}{m_1}\right), \\ q_0 & \text{if } I\left(\frac{\alpha}{m_1}\right) \neq I\left(\frac{\beta}{m_1}\right), \\ 0 & \text{if } \alpha = \beta, \end{cases} \quad (2.28)$$

where $I(x)$ is the smallest integer larger than or equal to x and the parameters are now three: q_1 is the overlap between two replicas in the same state, q_0 is the one for two replicas in different states and m_1 ($1 \leq m_1 \leq n$) is the number of replicas inside the same state. Notice that in the limit $n \rightarrow 0$ the inequality becomes $0 \leq m_1 \leq 1$. An example of the typical RSB structure is shown in Figure 2.2. Square blocks of size $m_1 \times m_1$ are present on the diagonal, with elements q_1 , the overlap between spins in the same state. Outside the blocks, all elements are q_0 ($q_0 < q_1$), which corresponds to the overlap between spins in different states.

It is possible to interpret the RSB Ansatz in terms of spontaneous symmetry breaking. It is indeed possible to check that the quenched free energy of Eq. 2.11 is invariant under the action of the group of permutations of the n replicas indices, denoted with $S^{(n)}$:

$$A_n(\{Q_{\alpha\beta}\}) = A_n(\{Q_{\pi(\alpha)\pi(\beta)}\}), \quad (2.29)$$

where the case $J_0 = 0$ is considered; π represents a given permutation of the replica indices. In this sense, the RS Ansatz matrix, defined in Eq. 2.14, is symmetric under the action of $S^{(n)}$. On the other hand, the RSB Ansatz matrix, given in Eq. 2.28, explicitly breaks this symmetry, in the same spirit as the positively and negatively magnetized

states break the spin flip symmetry, given by the group $\mathbb{Z}_2$, in the low-temperature regime of the Ising model.

The results of this new *Ansatz* for the overlap matrix $Q_{\alpha\beta}$, for generic values of the control parameters T , J_0 and H , can be found similarly to what shown for the [RS](#) case:

$$\overline{f_{1\text{RSB}}(\beta, H)} = \frac{\beta J^2}{4} [(m_1 - 1)q_1^2 - m_1 q_0^2 + 2q_1 - 1] + \frac{J_0}{2} m^2 - \frac{\ln 2}{\beta} - \frac{1}{\beta m} \int Du \ln \int Dv \cosh^{m_1} \Xi, \quad (2.30)$$

$$m = \int Du \frac{\int Dv \cosh^{m_1} \Xi \tanh \Xi}{\int Dv \cosh^{m_1} \Xi}, \quad (2.31)$$

$$q_0 = \int Du \left(\frac{\int Dv \cosh^{m_1} \Xi \tanh \Xi}{\int Dv \cosh^{m_1} \Xi} \right)^2, \quad (2.32)$$

$$q_1 = \int Du \frac{\int Dv \cosh^{m_1} \Xi \tanh^2 \Xi}{\int Dv \cosh^{m_1} \Xi}, \quad (2.33)$$

where

$$\Xi = \beta (J q_0 u + J \sqrt{q_1 - q_0} v + J_0 m + H). \quad (2.34)$$

In this case as well, the argument of the hyperbolic functions can be interpreted as a random effective field. However, now there are two distinct contributions: the random variable u describes the effective field acting on a spin due to fluctuations within a single valley, while v accounts for the contribution from different valleys. As for the [RS](#) case, with these results it is possible to compute the low-temperature entropy. Again it is negative, but the value is improved from $-1/(2\pi)$ to ≈ -0.01 , signaling that, even if this [RSB](#) structure cannot be the solution, the breaking of [RS](#) is the right direction. To check the validity of the last heuristic argument, the stability analysis of this new solution is needed. More terms arise in this case, as two replicas can be in the same or in two different blocks. In particular, the solution becomes unstable due to the replicon mode between two replicas in the same block that becomes negative for some values of the control parameter. This is an evidence that [RSB](#) should be considered inside a single block.

The configuration described above corresponds to the so-called *one-step RSB*; however, this hierarchical structure can be recursively applied within each block, identifying valleys within valleys. In this way, a general k -step [RSB](#) scheme emerges. An instance of the [2RSB](#) structure is depicted in [Figure 2.2](#). Interestingly, the [1RSB](#) matrix preserves the [RS](#) inside each block, given by the direct product of permutation groups $S^{(n/m_1)}$. Indeed, what the [2RSB Ansatz](#) does is to explicitly break this symmetry. This symmetry breaking is iterated in the generic k -step [RSB](#).

What turns out is that the stable solution of the [SK](#) model is given by the k [RSB](#) structure in the limit $k \rightarrow \infty$, the so-called [full Replica Symmetry Breaking](#) ([fRSB](#)) solution [[De 83](#)]³. In this regime, the overlap should be considered as a function

$$q(x) = q_i \quad m_{i+1} < x \leq m_i, \quad (2.35)$$

³In [[De 83](#)] the authors show that this solution is *marginally* stable, meaning that the critical mode, the replicon, goes to zero at criticality as $k \rightarrow \infty$

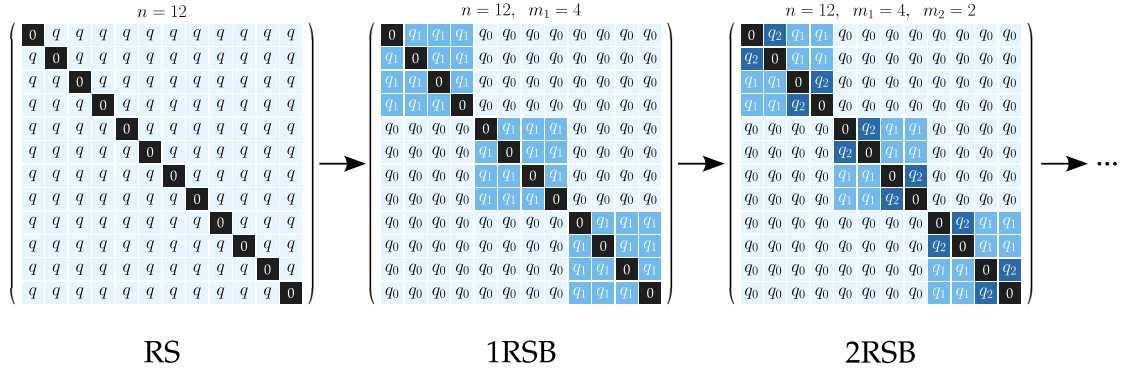


Figure 2.2. Pictorial representation of the iteration of the **RSB** scheme. The color gradient indicates the value of the overlaps, with higher overlaps ($q_2 > q_1 > q_0$) represented by increasingly darker shades of blue. The diagonal elements (black) correspond to zero overlap. Adapted from Ref. [Mal19].

and in the $n \rightarrow 0$ limit $0 \leq x \leq 1$, thus $q(x)$ is a real function in the interval $[0, 1]$. Nontrivial computations lead to the **fRSB** free energy [Par80a]

$$\overline{f_{\text{fRSB}}(\beta, H)} = -\frac{\beta J^2}{4} \left[1 + \int_0^1 dx q^2(x) - 2q(1) \right] + \frac{J_0}{2} m^2 - \frac{1}{\beta} \int Du f \left(0, J \sqrt{q(0)} u + H \right), \quad (2.36)$$

where the function $f(x, y)$ satisfies the following nonlinear differential equation

$$\frac{\partial f}{\partial x} = -\frac{J^2}{2} \frac{dq}{dx} \left[\frac{\partial^2 f}{\partial y^2} + x \left(\frac{\partial f}{\partial y} \right)^2 \right], \quad (2.37)$$

with the boundary condition

$$f(1, H) = \ln 2 \cosh[\beta(J_0 m + H)]. \quad (2.38)$$

As in previous cases, at this point, the computation of $q(x)$ from the extremization of the free energy is needed. One can perform this task near the critical point, that is

$$\mu \equiv \frac{T_g - T}{T_g} \ll 1, \quad (2.39)$$

where the overlap function is expected to be small. In this regime, the free energy Eq. 2.11 can be expanded for $Q_{\alpha\beta} \ll 1$ (see Ref. [Doto0]) and leads to a simpler form whose extremization is found for $J_0 = 0$ and $0 < H < h_{\text{dAT}}(T/J)$

$$q(x) = \begin{cases} q_m & 0 \leq x \leq x_m \\ \frac{x}{2} & x_m \leq x \leq x_M \\ q_M & x_M \leq x \leq 1 \end{cases} \quad (2.40)$$

where $x_m/2 = q_m \sim H^{2/3}$ and $x_M/2 = q_M \sim \mu$, as depicted in Figure 2.3. Notice that the **RS** solution is recovered in the limit $H \rightarrow h_{\text{dAT}}$ for which $q_m \rightarrow q_M$.

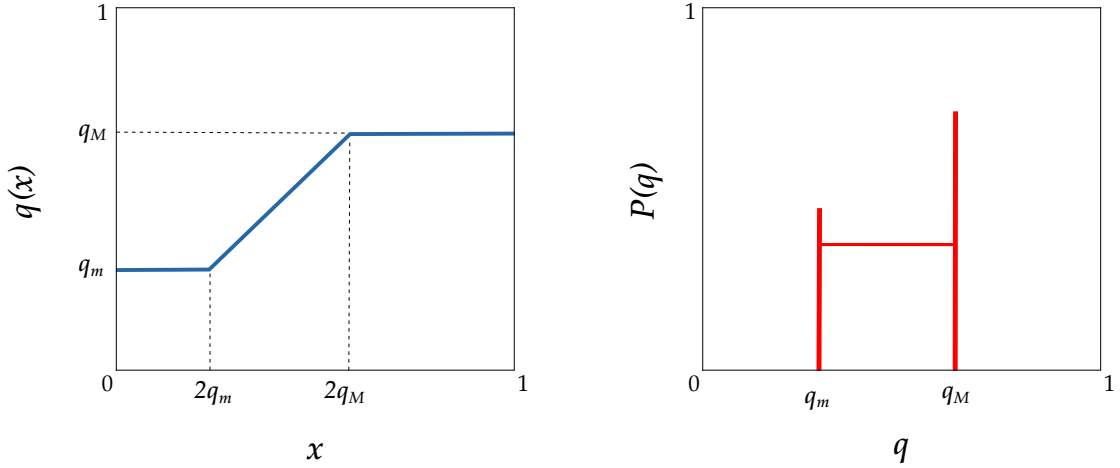


Figure 2.3. *Left:* schematic representation of the overlap function in the low-temperature regime. *Right:* schematic representation of the probability distribution of the overlap; Dirac delta functions with different weights are represented as vertical lines with different heights.

2.1.4 Physical interpretation of replicas

With the solution in hand, it is possible to interpret results and connect theory to experiments. To do so, a connection between the replica overlap matrix, $Q_{\alpha\beta}$, and the early definition of overlap between two states, given in Eq. 1.21, is needed.

First of all, what the [FRSB](#) solution for the overlap matrix suggests is the presence of many different pure states, identified also as valleys. For this reason, the values that the overlap assumes is considered as a function of the parameter x , which runs through different replicas and thus, possibly, over different states. As anticipated in [Section 1.2.2](#), with Eq. 1.19, this scenario can be described by decomposing the Gibbs measure as a sum over single states. An interesting quantity to be studied is the probability distribution of the overlap between pure states, q_{ab} , defined as

$$P(q) \equiv \overline{\sum_{a,b} w_a w_b \delta(q - q_{ab})}, \quad (2.41)$$

where the mean value is

$$q^{(1)} = \int dq q P(q) = \overline{\sum_{a,b} w_a w_b q_{ab}}. \quad (2.42)$$

Using the definition of q_{ab} , given in Eq. 1.21, with the decomposition of the Gibbs measure, Eq. 1.19, it is possible to rewrite this mean value

$$q^{(1)} = \overline{\sum_{a,b} w_a w_b q_{ab}} = \overline{\frac{1}{N} \sum_{i=1}^N \langle s_i \rangle^2}. \quad (2.43)$$

Interestingly, $q^{(1)}$ can be related to the saddle-point value of the overlap matrix $Q_{\alpha\beta}$. Indeed, Eq. 2.12 can be rewritten as

$$Q_{\alpha\beta} = \overline{\langle s_i^\alpha s_i^\beta \rangle} = \overline{\frac{1}{N} \sum_{i=1}^N \langle s_i \rangle^2} = q^{(1)}. \quad (2.44)$$

What one would expect is that, in general, $q^{(1)}$ doesn't depend on the specific replicas α and β . This could be true in the [RS](#) scenario, where $Q_{\alpha\beta}$ is the same for each couple $\alpha \neq \beta$, but not in [RSB](#) scenarios. For this reason, the following way of computing the mean value with the overlap matrix is considered to be correct

$$q^{(1)} = \lim_{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} Q_{\alpha\beta}. \quad (2.45)$$

With the Replica method it is possible to generalize the previous relation to an arbitrary moment (k -th) of the overlap distribution

$$q^{(k)} = \lim_{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} Q_{\alpha\beta}^k, \quad (2.46)$$

to finally write

$$P(q) = \lim_{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \delta(q - Q_{\alpha\beta}), \quad (2.47)$$

that allows one to evaluate the overlap distribution from the overlap matrix for whatever the matrix structure is. This physically means that the elements $Q_{\alpha\beta}$ are the overlap values between two different states. More precisely, the number of such values (divided by $(n-1)n$) is related to their probability of occurrence, through [Eq. 2.47](#).

In the [fRSB](#) case

$$P(q) = \lim_{k \rightarrow \infty} \sum_{j=0}^k (m_{j+1} - m_j) \delta(q - q_j) = \int_0^1 dx \delta(q - q(x)), \quad (2.48)$$

where $m_{j+1} - m_j \xrightarrow{k \rightarrow \infty} dx$ and $m_{k+1} = 1$, $m_0 = 0$. The last equation can be rewritten as

$$P(q) = \frac{dx(q)}{dq}, \quad (2.49)$$

if $x(q)$, the inverse of the function $q(x)$, exists.

The results shown in [Figure 2.3](#) reveal that the spin glass phase of the [SK](#) model is characterized by the emergence of a multitude of pure states arranged in a highly nontrivial, hierarchical manner. This complex organization is captured by the [fRSB](#) solution, in which the overlap matrix, $Q_{\alpha\beta}$, encodes an *ultrametric* structure among the states [[Mé84](#)]: taking three replica indices α , β and γ and computing the three overlaps $Q_{\alpha\beta}$, $Q_{\alpha\gamma}$ and $Q_{\beta\gamma}$, at least two of them are equal. Specifically, configurations that belong to the same pure state (or valley) have a high mutual overlap, denoted by q_k . These valleys, separated by infinitely high free energy barriers, are themselves grouped into larger valleys where configurations share a lower overlap q_{k-1} , and this nesting continues recursively down to the lowest level, where the overlap reaches q_0 . In this sense, the free energy landscape, in the spin glass phase, becomes extremely rugged, a large number of metastable states (separated by infinite barriers) appear, the signature of [MF](#) spin glasses.

With this result, it is also possible to recover some of the experimental results cited in [Chapter 1](#). To characterize the overlap distribution, not only is the mean value

important (notice that $q^{(1)}$ is equal to the overlap defined experimentally, in Eq. 1.1), but also the maximum, which is usually named after Edwards and Anderson

$$q_{\text{EA}} \equiv \max_{\alpha, \beta} Q_{\alpha\beta} = \max_x q(x). \quad (2.50)$$

These two values allow for the computation of two kinds of susceptibilities. The first one is the linear susceptibility averaged over the disorder

$$\chi = \frac{1}{N} \sum_{i=1}^N \frac{\overline{\partial \langle s_i \rangle}}{\partial H} = \beta \left(1 - \frac{1}{N} \sum_{i=1}^N \overline{\langle s_i \rangle^2} \right) = \beta(1 - q^{(1)}). \quad (2.51)$$

From the experimental point of view, this quantity is the **fc** susceptibility, which theoretically implies that the system has equilibrated among all states. On the contrary, one could be interested in the “linear response” inside a single valley

$$\chi_a = \beta \left(1 - \frac{1}{N} \sum_{i=1}^N \overline{\langle s_i \rangle_a^2} \right), \quad (2.52)$$

and then average over the pure states and the disorder

$$\chi_{\text{LR}} = \overline{\sum_a w_a \chi_a} = \beta(1 - q_{\text{EA}}). \quad (2.53)$$

This last quantity can be identified as the **zfc** susceptibility: the system is constrained to equilibrate in a specific state, since zero field is applied during the cooling. In other words, χ_{LR} measures the response of the system in a time such that the microscopic degrees of freedom cannot escape from a single valley, while χ is measured on a longer time scale, allowing the system to explore all pure states. What is interesting is that using the methods explained above, it is possible to find the behavior of both $q^{(1)}$ and q_{EA} below the critical temperature T_g [Bin86]

$$1 - q^{(1)} \propto T \quad \text{and} \quad 1 - q_{\text{EA}} \propto T^2, \quad (2.54)$$

that used in Eq. 2.51 and Eq. 2.53 reproduce qualitatively the experimental results for **fc** and **zfc** susceptibilities, see Figure 1.3. For this reason a non-zero difference $q_{\text{EA}} - q^{(1)}$ implies ergodicity breaking.

2.2 Spin Glass on the Bethe Lattice

Random graphs have attracted significant interest soon after the Replica method solution of the **SK** model was discovered. These structures provide natural topologies for optimization problems [Mé85, Mé87c] and they also better approximate finite-dimensional lattices than **FC** systems due to their finite connectivity. Thus, it is interesting to characterize the critical behavior of the same spin glass model on the Bethe lattice, in the following considered as a **RRG**, see Section 1.2.3. The model, originally introduced by Viana–Bray (**VB**) [Via85], shares the Hamiltonian of the **EA** model from Eq. 1.12:

$$\mathcal{H}_{\text{VB}}(\{s_i\}_i) = - \sum_{(i,j)} J_{ij} s_i s_j - \sum_i H_i s_i, \quad (2.55)$$

where $J_{ij} \sim \mathcal{N}(J_0, J)$ and the site dependence on the external field is made explicit for the sake of clarity in the following computations, in the end $H_i = H, \forall i = 1, \dots, N$. The relevant difference is that the sum over (i, j) runs over edges of a Bethe lattice with connectivity z and N nodes. Moreover, in the following, the case $J_0 = 0$ is considered.

Despite lacking rigorous foundations, the Replica method has proven highly effective for analyzing critical behavior in quenched disordered systems. Specifically, in [Section 2.1.2](#) it is explained that the solution of the [SK](#) model, in the case of zero mean Gaussian couplings, below the [dAT](#) line implies the [RSB](#) scenario. The Bethe lattice spin glass exhibits qualitatively similar critical behavior at non-zero temperatures, even if with a more complex order parameter characterizing the [RSB](#) phase [\[Gol90\]](#). Moreover, it has been shown that an effective theory for the overlap $Q_{\alpha\beta}$ describes the fluctuations of the [RS](#) phase, implying that the critical properties are the same as the [SK](#) model [\[Par13, Par14\]](#). At zero temperature, however, qualitative differences emerge. Considering again the case where the Gaussian couplings have zero mean, the main difference is that a transition at a finite value of the external field occurs, while the [dAT](#) line diverges for $T = 0$, see [Figure 2.1](#). Moreover, the critical phase is characterized by a different behavior of the microscopic degrees of freedom, an unusual [RSB](#) phase is observed and qualitatively described below in this section [\[Per22b\]](#).

The fundamental tool used in this framework is the *Cavity method*. Developed as an alternative to the Replica method, it enabled rigorous confirmation of the [SK](#) solution [\[Talo3\]](#) and was later applied to finite-connectivity tree-like lattices [\[Méz87a, Mézo1\]](#). In this section, the Cavity method is first described in general and then applied to study the spin glass on the Bethe lattice.

2.2.1 The Cavity Method

The simplest application of this method is on tree graphs, which are considered here to then analyse the case of random graphs with a simple generalization. Tree graphs are characterized by the absence of loops. Thanks to this property, recursion relations can be used to exactly compute the free energy of a model defined on this kind of graph. These recursion relations constitute one formulation of the *Cavity method* that, in its simplest form, corresponds to the [RS](#) scenario in the language of replicas. As stated before, the Cavity method can also be applied to graphs where short loops are present. Its validity is guaranteed by some assumptions, stated below.

Recursion relations can be written for the following quantities. Consider two adjacent nodes i and j in a tree, $Z_{i \rightarrow j}(s_i)$ is defined as the partition function of the tree with a fixed configuration of s_i and where the edge connecting i and j is cut. On the other hand, $Z_i(s_i)$ is the partition function of the whole tree, given the configuration of s_i . Given the absence of loops, cutting the edge (i, j) and being the configuration on node i fixed to s_i all the partial partition functions on nodes adjacent to i are statistically independent, thus it is possible to compute the free energy of a generic spin system, $Z_N(\beta, \{H_i\}_i)$, with pairwise interactions mediated by J_{ij} and with an external field H_i (that is with an Hamiltonian of the same form as in [Eq. 2.55](#))

$$Z_N(\beta, \{H_i\}_i) = \sum_{s_i = \pm 1} Z_i(s_i) \equiv \sum_{s_i = \pm 1} e^{\beta H_i s_i} \prod_{k \in \partial i} \left(\sum_{s_k = \pm 1} e^{\beta J_{ik} s_i s_k} Z_{k \rightarrow i}(s_k) \right), \quad (2.56)$$

where

$$Z_{i \rightarrow j}(s_i) \equiv e^{\beta H_i s_i} \prod_{k \in \partial i \setminus j} \left(\sum_{s_k = \pm 1} e^{\beta J_{ik} s_i s_k} Z_{k \rightarrow i}(s_k) \right). \quad (2.57)$$

These equations can be understood by starting from a site (for instance s_1 in Figure 2.4) and recursively writing all partial partition functions, to see that in the end all the terms contributing to the complete partition function are included.

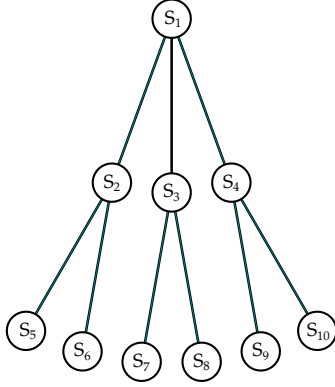


Figure 2.4. Example of a tree with $N = 10$ spins and constant connectivity $z = 3$, being s_5 to s_{10} the *boundary spins* (or *leaves*), with connectivity 1. The complete partition function can be written: $Z_N(\beta, \{H_i\}_i) = \sum_{s_1} Z_1(s_1)$. In this case $Z_1(s_1) = e^{\beta H_1 s_1} \prod_{k=2,3,4} \left(\sum_{s_k} e^{\beta J_{1k} s_1 s_k} Z_{k \rightarrow 1}(s_k) \right)$, where $Z_{k \rightarrow 1}(s_k) = e^{\beta H_k s_k} \prod_{j \in \partial k \setminus 1} \left(\sum_{s_j} e^{\beta J_{kj} s_k s_j} Z_{j \rightarrow k}(s_j) \right)$ and so on until the partial partition functions on the leaves s_j : $Z_{j \rightarrow k}(s_j) = e^{\beta H_j s_j}$.

On the other hand, to solve these equations, one starts from the partial partition functions at the leaves: $Z_{i \rightarrow j}(s_i) = e^{\beta H_i s_i}$, since for these nodes the connectivity is 1. Fixing H_i amounts to setting the boundary conditions.

Notice that normalizing these quantities it is possible to compute two kinds of marginal distribution functions: $\eta_i(s_i) \equiv \frac{Z_i(s_i)}{\sum_{s_i} Z_i(s_i)}$ and $\eta_{i \rightarrow j}(s_i) \equiv \frac{Z_{i \rightarrow j}(s_i)}{\sum_{s_i} Z_{i \rightarrow j}(s_i)}$, the marginal of spin s_i on the whole system and the one on the partial system where the edge between i and j is cut, respectively. Recursion relations can be written for these quantities, based on Eq. 2.56 and Eq. 2.57

$$\eta_{i \rightarrow j}(s_i) = \frac{1}{Z_{i \rightarrow j}} e^{\beta H_i s_i} \prod_{k \in \partial i \setminus j} \left(\sum_{s_k = \pm 1} e^{\beta J_{ik} s_i s_k} \eta_{k \rightarrow i}(s_k) \right) \quad (2.58)$$

and

$$\eta_i(s_i) = \frac{1}{Z_i} e^{\beta H_i s_i} \prod_{k \in \partial i} \left(\sum_{s_k = \pm 1} e^{\beta J_{ik} s_i s_k} \eta_{k \rightarrow i}(s_k) \right), \quad (2.59)$$

where the normalization constants are

$$Z_{i \rightarrow j} = \sum_{s_i = \pm 1} e^{\beta H_i s_i} \prod_{k \in \partial i \setminus j} \left(\sum_{s_k = \pm 1} e^{\beta J_{ik} s_i s_k} \eta_{k \rightarrow i}(s_k) \right) = \frac{\sum_{s_i} Z_{i \rightarrow j}(s_i)}{\prod_{k \in \partial i \setminus j} \sum_{s_k} Z_{k \rightarrow i}(s_k)}, \quad (2.60)$$

and

$$Z_i = \sum_{s_i = \pm 1} e^{\beta H_i s_i} \prod_{k \in \partial i} \left(\sum_{s_k = \pm 1} e^{\beta J_{ik} s_i s_k} \eta_{k \rightarrow i}(s_k) \right) = \frac{\sum_{s_i} Z_i(s_i)}{\prod_{k \in \partial i} \sum_{s_k} Z_{k \rightarrow i}(s_k)}. \quad (2.61)$$

Notice that the product

$$\prod_{k \in \partial i \setminus j} \eta_{k \rightarrow i}(s_k), \quad (2.62)$$

can be seen as the factorization of the joint probability distribution of the spins adjacent to i , excluding j . Without this factorization, the cavity equations, Eq. 2.58, cannot be written in such an easy closed form. Due to the absence of loops, this factorization is exact on a tree, while it is not in more complicated situations.

Defining the following quantity

$$z_{ij} \equiv \sum_{s_i, s_j = \pm 1} \eta_{i \rightarrow j}(s_i) e^{\beta J_{ij} s_i s_j} \eta_{j \rightarrow i}(s_j), \quad (2.63)$$

it is possible to express the free energy in a simple form [Zam10]⁴

$$F_N(\beta, \{H_i\}_i) = -\frac{1}{\beta} \ln Z_N(\beta, \{H_i\}_i) = -\frac{1}{\beta} \sum_{i=1}^N \ln z_i + \frac{1}{\beta} \sum_{(i,j)} \ln z_{ij}. \quad (2.64)$$

If, instead of a tree, the model of interest is defined on a generic graph, possibly with loops, how can the factorization displayed in Eq. 2.62, at the basis of the recursion relations, be justified? In the case of random graphs, as RRG or ER random graphs, loops are present but their length scales with $\mathcal{O}(\ln N)$ in the thermodynamic limit, thus the factorization is a good approximation. However, the Cavity method, in this form, can also be applied to a model on a graph with short loops, such as the FC lattice, as long as the configurations of nodes $k \in \partial i$ are almost independent from each other, once the configuration of node i is fixed to s_i . *This assumption is valid when the model is described by a single pure state.* In other words, the factorization can be used in the paramagnetic phase, where correlations are negligible, while it becomes problematic at the critical point.

Notice that, on a given generic graph, where the boundary condition is not set on some particular nodes, the starting point to solve the recursive equations can be exploited with a random initialization of the cavity marginals $\eta_{i \rightarrow j}(s_i)$, that is a valid procedure if a single pure state corresponds to a single stable solution of these recursive equations.

Interestingly, what is described above is the so-called RS Cavity method, that corresponds to the Bethe–Peierls approximation in Statistical Mechanics and to the Belief Propagation (BP) algorithm in Computer Science [Yed03].

Cavity field and bias in Ising-like models

If the microscopic degrees of freedom of the model under investigation are Ising variables, it is possible to simplify Eq. 2.58. Since in this case the variables s_i can only take two values, the marginal distribution functions are described by a single parameter, considering the normalization condition. It is usual to define the *cavity field* $h_{k \rightarrow i}$ and the *cavity bias* $u_{i \rightarrow j}$ as follows

$$\eta_{i \rightarrow j}(s_i) = \frac{e^{\beta h_{i \rightarrow j} s_i}}{2 \cosh(\beta h_{i \rightarrow j})}, \quad (2.65)$$

⁴Starting from the expression of the partition function in Eq. 2.56, using the second equality in Eq. 2.61 and then, recursively, the second equality in Eq. 2.60, the result shown for the free energy in Eq. 2.64 is obtained.

and

$$\sum_{s_k=\pm 1} e^{\beta J_{ik} s_i s_k} \eta_{k \rightarrow i}(s_k) = \frac{\cosh(\beta J_{ik})}{\cosh(\beta u_{k \rightarrow i})} e^{\beta u_{k \rightarrow i} s_i}. \quad (2.66)$$

Using these definitions, it turns out that solving Eq. 2.58 is equivalent to solving the following system of equations

$$\begin{cases} h_{i \rightarrow j} = H_i + \sum_{k \in \partial i \setminus j} u_{k \rightarrow i}, \\ u_{k \rightarrow i} = \hat{u}_\beta(J_{ik}, h_{k \rightarrow i}) \equiv \frac{1}{\beta} \operatorname{atanh}(\tanh(\beta J_{ik}) \tanh(\beta h_{k \rightarrow i})). \end{cases} \quad (2.67)$$

Notice that, in principle, a solution of these equations can be found for each edge (i, j) on a given graph. Such a solution is associated with a specific realization of the disorder and a specific graph.

On the other hand, in the thermodynamic limit $N \rightarrow \infty$ one can consider the field $h_{i \rightarrow j}$ and the bias $u_{i \rightarrow j}$ as random variables drawn from two distributions: $Q_B(h)$ and $P_B(u)$ respectively, where the subscript B stands for “Bethe”. In this way, it is possible to write recursion relations for the two distributions on a Bethe lattice with fixed connectivity z

$$\begin{cases} Q_B(h) = \overline{\int \prod_{i=1}^{z-1} P_B(u_i) du_i \delta\left(h - H - \sum_{i=1}^{z-1} u_i\right)}, \\ P_B(u) = \overline{\int dh Q_B(h) \delta(u - \hat{u}_\beta(J, h))}, \end{cases} \quad (2.68)$$

where $\overline{}$ denotes the average over all sources of disorder, which may include random Gaussian couplings and/or external fields, together with the graph. Similar equations hold for the ER random graph, where the product over i is up to k , being k the excess degree, drawn from a Poisson distribution, $q_z(r = k)$, with the same average value as the distribution for the connectivity. The distribution of the field, $Q_{ER}(h)$, must be averaged over the excess degree, thus

$$Q_{ER}(h) = \sum_{k=0}^{\infty} q_z(r = k) \overline{\int \prod_{i=1}^k P_B(u_i) du_i \delta\left(h - H - \sum_{i=1}^k u_i\right)}, \quad (2.69)$$

while $P_{ER}(u)$ has the same form as $P_B(u)$. In the following, the Bethe lattice case is considered unless explicitly stated. Moreover, the solution of Eq. 2.68 (or Eq. 2.69 for ER random graphs) is not valid on a single graph, instead it corresponds to the solution averaged over the disorder and over the ensemble of random graphs.

Eq. 2.68 are particularly suitable to be solved with a numerical algorithm, usually referred to as “Population Dynamics”. First of all the two equations can be condensed into one single equation for $P_B(u)$ (or equivalently $Q_B(h)$, here only the former is used)

$$P_B(u) = \overline{\int \prod_{i=1}^{z-1} P_B(u_i) du_i \delta\left(u - \hat{u}_\beta\left(J, H + \sum_{i=1}^{z-1} u_i\right)\right)}. \quad (2.70)$$

The starting point is to approximate the distribution $P_B(u)$ by a population of $N_p \gg 1$ biases u_i , for $i = 1, \dots, N_p$, i.e., by a sum of delta distributions $P_B(u) \propto \sum_{i=1}^{N_p} \delta(u - u_i)$. Then the population is plugged into the [Right-Hand Side \(RHS\)](#) of [Eq. 2.70](#) resulting in a new set of delta distributions. By iterating this process a sufficient number of times, the population could converge to a good enough approximation of the function $P_B(u)$.

2.2.2 The Spin Glass on the Bethe lattice

The standard procedure to make predictions on the phase diagram is to make use of the self-consistent equations, [Eq. 2.58](#) or [Eq. 2.68](#). However, these equations can be used to detect the spin glass transition only in the case of zero external field, $H = 0$ [[Par14](#)]. Using the Replica method, it has been shown that the critical point is related to the solution of certain integral equations [[Jan10b](#)], that has been confirmed in the Cavity method framework [[Jan10a](#)]. A simple derivation of the integral equation responsible for the location of the spin glass phase is presented below [[Par14](#)].

Location of the spin glass critical point

Here the [VB](#) model, defined in [Eq. 2.55](#), is considered on a Bethe lattice with fixed connectivity z . All the arguments shown below can be generalized to the case of the [ER](#) random graph, similarly to what is done above for $P_{ER}(u)$.

In order to identify the [dAT](#) line the computation of the spin glass susceptibility is needed:

$$\chi_{SG} = \frac{1}{N} \sum_{i,j} \overline{\langle s_i s_j \rangle_c^2} = \sum_{i=1}^N \overline{\langle s_0 s_i \rangle_c^2} = \frac{1}{\beta^2} \sum_{i=1}^N \overline{\left(\frac{d\langle s_0 \rangle}{dH_i} \right)^2}, \quad (2.71)$$

as already defined in [Eq. 1.5](#). In the second equality, due to homogeneity of the system, a reference spin, s_0 , can be considered. Moreover, using the Cavity method to compute the marginal on a given site, in [Eq. 2.59](#), the magnetization for a given realisation of the disorder reads

$$\langle s_i \rangle = \tanh \left(\beta \left(H_i + \sum_{j \in \partial i} u_{j \rightarrow i} \right) \right), \quad (2.72)$$

and its derivative

$$\frac{d\langle s_0 \rangle}{dH_i} = (1 - \langle s_0 \rangle^2) \frac{du_{j \rightarrow 0}}{dH_i}, \quad (2.73)$$

being j the node that connects i to 0. The average over the disorder gives

$$\overline{\langle s_i \rangle} = \int \prod_{i=1}^z dP_B(u_i) \tanh \left(\beta \left(H + \sum_{i=1}^z u_i \right) \right), \quad (2.74)$$

where, after the average, the cavity biases don't depend on the given edge (i, j) , their values are drawn from the probability distribution $P_B(u_i)$. Moreover, as already anticipated, the external magnetic field is considered constant on all sites at the end of a computation. Defining by $T_{i \rightarrow j}$ the set of sites connected to i by a path avoiding site j [[Per22a](#)]

$$\frac{1}{\beta^2} \sum_{i=1}^N \overline{\left(\frac{d\langle s_0 \rangle}{dH_i} \right)^2} = (1 - \langle s_0 \rangle^2)^2 \left[1 + \sum_{j \in \partial 0} \sum_{k \in T_{j \rightarrow 0}} \left(\frac{du_{j \rightarrow 0}}{dH_k} \right)^2 \right], \quad (2.75)$$

where the “1” in the square brackets accounts for the case $i = 0$. To write the result for χ_{SG} one can define the susceptibility on a path with a fixed value of the cavity bias u at one extremity

$$\chi(u) \equiv \overline{\delta(u - u_{i \rightarrow j}) \sum_{k \in T_{i \rightarrow j}} \left(\frac{du_{i \rightarrow j}}{dH_k} \right)^2}. \quad (2.76)$$

Indeed, using this definition in Eq. 2.75 and averaging over the disorder

$$\begin{aligned} \chi_{\text{SG}} = & \int \prod_{i=1}^z dP_B(u_i) \left[1 - \tanh^2 \left(\beta \left(H + \sum_{i=1}^z u_i \right) \right) \right]^2 + \\ & z \int \prod_{i=1}^{z-1} dP_B(u_i) du \chi(u) \left[1 - \tanh^2 \left(\beta \left(H + u + \sum_{i=1}^{z-1} u_i \right) \right) \right]^2. \end{aligned} \quad (2.77)$$

Interestingly, it is possible to compute $\chi(u)$ through a self-consistent relation. Two contributions are needed here, being the first one the case $k = i$ in Eq. 2.76

$$\begin{aligned} \overline{\delta(u - u_{i \rightarrow j}) \left(\frac{du_{i \rightarrow j}}{dH_i} \right)^2} = \\ \overline{\int \prod_{i=1}^{z-1} dP_B(u_i) \delta \left(u - \hat{u}_\beta \left(J, H + \sum_{i=1}^{z-1} u_i \right) \right) \left(\frac{d\hat{u}_\beta(J, H + \sum_{i=1}^{z-1} u_i)}{dH} \right)^2}. \end{aligned} \quad (2.78)$$

The second contribution is given by the $k \neq i$ terms. In this case, the derivative of the cavity bias can be computed by the chain rule, considering $k \in T_{q \rightarrow i}$ for some $q \in \partial i \setminus j$

$$\frac{du_{i \rightarrow j}}{dH_k} = \frac{du_{i \rightarrow j}}{du_{q \rightarrow i}} \frac{du_{q \rightarrow i}}{dH_k}, \quad (2.79)$$

where

$$\frac{du_{i \rightarrow j}}{du_{q \rightarrow i}} = \frac{du_{i \rightarrow j}}{dH_i}. \quad (2.80)$$

The second term in the RHS of Eq. 2.79 contributes to $\chi(u)$ while the first is simply $d\hat{u}_\beta/dH$:

$$\begin{aligned} \overline{\delta(u - u_{i \rightarrow j}) \sum_{k \in T_{i \rightarrow j} \setminus i} \left(\frac{du_{i \rightarrow j}}{dH_i} \right)^2} = (z-1) \times \\ \overline{\int \prod_{i=1}^{z-2} dP_B(u_i) du' \chi(u') \delta \left(u - \hat{u}_\beta \left(J, H + u' + \sum_{i=1}^{z-2} u_i \right) \right) \left(\frac{d\hat{u}_\beta(J, H + u' + \sum_{i=1}^{z-2} u_i)}{dH} \right)^2}. \end{aligned} \quad (2.81)$$

Putting these two contributions together

$$\chi(u) = \int \prod_{i=1}^{z-1} dP_B(u_i) \delta \left(u - \hat{u}_\beta \left(J, H + \sum_{i=1}^{z-1} u_i \right) \right) \left(\frac{d\hat{u}_\beta(J, H + \sum_{i=1}^{z-1} u_i)}{dH} \right)^2 + (z-1) \times$$

$$\int \prod_{i=1}^{z-2} dP_B(u_i) du' \chi(u') \delta \left(u - \hat{u}_\beta \left(J, H + u' + \sum_{i=1}^{z-2} u_i \right) \right) \left(\frac{d\hat{u}_\beta(J, H + u + \sum_{i=1}^{z-2} u_i)}{dH} \right)^2. \quad (2.82)$$

This last equation has the form

$$\chi(u) = F \chi(u) + \zeta(u), \quad (2.83)$$

that, if $(\mathbb{1} - F)$ is invertible, can be rewritten

$$\chi(u) = (\mathbb{1} - F)^{-1} \zeta(u). \quad (2.84)$$

At the critical point, corresponding to the [dAT](#) line in the plane $T - H$, the largest eigenvalue of F , λ , related to the eigenvector $g(u)$, tends to 1. The spin glass susceptibility diverges on this line. The onset of this divergence corresponds to the nontrivial solution of the homogeneous equation associated to [Eq. 2.82](#)

$$g(u) = (z - 1) \times \int \prod_{i=1}^{z-2} dP_B(u_i) du' g(u') \delta \left(u - \hat{u}_\beta \left(J, H + u' + \sum_{i=1}^{z-2} u_i \right) \right) \left(\frac{d\hat{u}_\beta(J, H + u' + \sum_{i=1}^{z-2} u_i)}{dH} \right)^2 \quad (2.85)$$

RSB phase insights

Beyond the [RS](#) framework, the Cavity method extends to [RSB](#) scenarios, allowing predictions across diverse disordered systems, including optimization and inference problems [[Mézo9](#)]. The result of this powerful tool for the zero-temperature spin glass phase of the [VB](#) model is peculiar [[Mézo3](#), [Per22a](#), [Per22b](#)]. At zero temperature, [RSB](#) leads to a thermodynamic behavior governed not by a unique global ground state, but by the existence of many *local ground states*, defined as configurations whose energy cannot be lowered by flipping any finite number of spins. These local ground states differ from the global ground state by an extensive number of spins, but the relative energy differences are $\mathcal{O}(1)$, subextensive. As a result, the cavity fields should be defined for each local ground state. At finite temperature, this structure is replaced by local minima of the free energy, and the transition across the [dAT](#) line is continuous: the distribution of local cavity fields broadens smoothly as the line is approached. Remarkably, this scenario changes at zero temperature. When a non-zero field is applied, only a finite fraction of sites (the so-called [RSB cluster](#)) exhibit [RSB](#), while the remaining sites have identical local fields across all local ground states. Thus, at zero temperature, only a subset of fields becomes nontrivial, in contrast to the $T > 0$ case where all fields “open” at the transition [[Per22b](#)]. This distinction allows one to differentiate between *closed* populations, which are supported on a single value (as in the [RS](#) phase), and *open* populations, which admit multiple values in the spin glass phase.

Chapter 3

Beyond Mean-Field: Finite-Dimensional Systems

This work focuses on finite-dimensional, $D < +\infty$, models. The preliminary approach to such problems, as discussed in the previous chapter, involves [MF](#) approximations. However, these approximations are not always valid: in high dimensions (i.e., in [FC](#) lattices, such as the [SK](#) model), order parameter fluctuations are negligible compared to its mean value, but they become increasingly significant as dimensionality decreases. Below the [Upper Critical Dimension \(UCD\)](#), where fluctuations must be taken into account, an alternative approach becomes necessary.

A generic model in the D -dimensional Euclidean space with order parameter $\phi(x)$, $x \in \mathbb{R}^D$, is considered. Two quantities need to be compared to find the [UCD](#). The first is the fluctuation around the mean value of the order parameter $\phi(x)$, which may represent the magnetization in the Ising model or the overlap in disordered systems. This is defined as

$$\sigma_\phi^2 \equiv \int_{\xi} d^D x (\phi(x) - \langle \phi(x) \rangle) (\phi(\mathbf{0}) - \langle \phi(\mathbf{0}) \rangle), \quad (3.1)$$

where $\phi(\mathbf{0})$ is the value of the order parameter at a reference position and the integral is performed up to the *correlation length* ξ , the scale beyond which correlations between $\phi(x)$ and $\phi(\mathbf{0})$ exponentially decay with $|x|$. This quantity is essentially proportional to the susceptibility χ_ϕ associated with the order parameter (see [Eq. 2.71](#) for the spin glass model, where the integration is replaced by a sum over sites). Although the integral should formally extend to infinity, the exponential decay of correlations for distances larger than ξ makes the truncation a good approximation. The second quantity is the square of the mean value of the order parameter,

$$\mu_\phi^2 \equiv \int_{\xi} d^D x \langle \phi(x) \rangle \langle \phi(\mathbf{0}) \rangle. \quad (3.2)$$

The *critical region*, defined as the regime where [MF](#) approximations break down, is identified by the condition $\sigma_\phi^2 / \mu_\phi^2 \gg 1$. This criterion can be expressed in terms of the distance from the critical point, denoted by t , by considering the scaling of the two quantities: $\sigma_\phi^2 \propto \chi_\phi \sim |t|^{-\gamma}$ and $\mu_\phi^2 \propto \xi^D \langle \phi \rangle^2$, where $\xi \sim R|t|^{-\nu}$ and $\langle \phi \rangle \sim |t|^\beta$, with R a constant representing the range of interactions. This leads to

$$R^{-D} \gg |t|^{\gamma - \nu D + 2\beta}, \quad (3.3)$$

and by substituting the MF critical exponents, the UCD is obtained as the value of D such that the RHS starts to diverge for $|t| \rightarrow 0$. This is one formulation of the well-known *Ginzburg criterion* [Ami93, Car96, Nis10, Gol18]. It is worth noting that if R is a microscopic length, the Left-Hand Side (LHS) is large and the critical region is correspondingly broader.

Within the critical region, fluctuations become significant and modify the values of the critical exponents. To account for these effects, the tools of Field Theory must be employed.

3.1 Field Theory and Renormalization Group

The Renormalization Group (RG) was developed to provide quantitative predictions for finite-dimensional models and to justify the scaling of observables at the critical point [Kad66, Wil71]. The fundamental idea is to exploit the *scale invariance* of a system at criticality by introducing specifically defined variables, known as *block variables*. Consider a spin system defined on an infinite D -dimensional lattice, with Hamiltonian $H_0 \equiv \beta \mathcal{H}_0(\{s_i\}_i)$. The lattice is partitioned into blocks k_n of size $L_n = 2^n$, where $n \in \mathbb{N}$ [Par98]. The block variable $\varphi_n(k_n)$ is given by

$$\varphi_n(k_n) = \sum_{i \in k_n} s_i, \quad (3.4)$$

and it describes the system at larger and larger length scales as n increases. When $L_n \gg \xi$, the correlations between different block variables become negligible. Consequently, the joint probability distribution of the block variables, denoted as $d\mathbb{P}(\{\varphi_n(k_n)\}_{k_n})$, factorizes for large n . In this regime, each $\varphi_n(k_n)$ is a sum of many uncorrelated variables, so the distribution becomes Gaussian by the central limit theorem:

$$d\mathbb{P}(\{\varphi_n(k_n)\}_{k_n}) = \prod_{k_n} d\varphi_n(k_n) \frac{1}{\sqrt{2\pi\sigma_\varphi^2}} \exp\left(-\frac{(\varphi_n(k_n) - L_n^D \langle s \rangle)^2}{2L_n^D \sigma_\varphi^2}\right), \quad (3.5)$$

where σ_φ^2 is the variance of the block variables and the average magnetization $\langle s \rangle$ may be non-zero, as in the Ising model in the presence of an external field. The procedure used to generate block variables is an example of *coarse-graining*, which constitutes only the first step of the RG approach. The next essential step is the rescaling of the variables. In general, rescaled variables $\psi_n(k_n)$ are introduced as

$$\psi_n(k_n) = \frac{\varphi_n(k_n) - L_n^D \langle s \rangle}{L_n^{\omega D}}, \quad (3.6)$$

where $1/2 < \omega < 1$ is a parameter that interpolates between two limiting regimes. As $n \rightarrow \infty$, the probability distribution for the rescaled variables is

$$d\mathbb{P}(\{\psi_n(k_n)\}_{k_n}) = \begin{cases} \prod_{k_n} d\psi_n(k_n) \delta(\psi_n(k_n)) & \text{if } \omega = 1, \\ \prod_{k_n} d\psi_n(k_n) \frac{1}{\sqrt{2\pi\sigma_\psi^2}} \exp\left(-\frac{\psi_n^2(k_n)}{2\sigma_\psi^2}\right) & \text{if } \omega = \frac{1}{2}. \end{cases} \quad (3.7)$$

The central assumption of the [RG](#) procedure is that, at the critical point, ω takes an intermediate value ω_c , so that the new variables are properly normalized and $\mathbb{P}(\{\psi_n(k_n)_{k_n}\})$ admits a nontrivial limiting distribution as $n \rightarrow \infty$. This is not always guaranteed, complications outside the scope of this work may arise [[Par98](#), [Jon75](#)]. The combination of coarse-graining and rescaling, as described above, constitutes the so-called *real-space RG*.

This procedure induces a recursion relation for the probability distribution, since $\mathbb{P}(\{\psi_{n+1}(k_{n+1})_{k_{n+1}}\})$ can be obtained from $\mathbb{P}(\{\psi_n(k_n)_{k_n}\})$. Define $b \equiv 2^n$ as the scaling factor of the coarse-graining. Recalling that $\mathbb{P} \propto \exp(-H)$ (β is included in the definition of H), the recursion for the probability distribution translates into a recursion relation for the Hamiltonian:

$$H' = R_b[H], \quad (3.8)$$

where H' is the Hamiltonian after rescaling by b , and R_b is the corresponding renormalization operation. A generic Hamiltonian can be expressed as a sum over operators O_a ,

$$H = \sum_a K_a O_a, \quad (3.9)$$

where K_a are the associated parameters, with $a = 1, \dots$ [[Car96](#), [Gol18](#)]. For the Ising model, for example, the operators $s_i s_j$ and s_i correspond to couplings βJ and βH , respectively. In this context, [Eq. 3.8](#) becomes

$$K' = R_b(K), \quad (3.10)$$

with $K = \{K_a\}_{a=1, \dots}$. Any physical quantity is rescaled by some power of b . By definition, lengths are reduced by a factor b , and in particular the correlation length:

$$\xi'[K'] = b^{-1} \xi[K]. \quad (3.11)$$

Iterating the coarse-graining and rescaling procedures infinitely many times, the correlation length tends to zero if it initially was finite, or diverges if it was infinite. This property characterizes the [Fixed Point \(FP\)](#) (possibly more than one) of the parameters recursion, as defined by [Eq. 3.10](#):

$$K^* = R_b(K^*). \quad (3.12)$$

A [FP](#) in parameters space is called *trivial* if $\xi[K^*] = 0$. Each [FP](#) is characterized by its basin of attraction: if all nearby points belong to its basin, it is attractive; if only the [FP](#) itself does, it is repulsive [[Car96](#), [Par98](#)].

To justify the scaling laws for the singular behavior of observables at the critical point, it is useful to study the properties of [Eq. 3.10](#) near a [FP](#):

$$K = K^* + \delta K, \quad (3.13)$$

so that [Eq. 3.10](#) can be linearized as

$$\delta K' = T_b(K^*) \cdot \delta K, \quad (3.14)$$

where

$$T_b(K^*)_{ij} = \left. \frac{\partial K'_i}{\partial K_j} \right|_{K^*}. \quad (3.15)$$

Notice that the matrix $T_b(\mathbf{K}^*)_{ij}$ is defined at a given FP: different FPs are associated with different matrices, eigenvalues, and eigenvectors. Considering the specific case (not always guaranteed) where $T_b(\mathbf{K}^*)_{ij}$ has real eigenvalues and using the property $T_{b_1} \cdot T_{b_2} = T_{b_1 b_2}$, the eigenvalues $\lambda_i(b)$ satisfy $\lambda_i(b_1)\lambda_i(b_2) = \lambda_i(b_1 b_2)$, leading to $\lambda_i(b) = b^{y_i}$, where y_i are known as *RG eigenvalues* [Car96]. Near the FP, it is convenient to project the parameters along the eigenvectors ϕ_i of $T_b(\mathbf{K}^*)_{ij}$:

$$\mathbf{K} = \mathbf{K}^* + \sum_i u_i \phi_i, \quad (3.16)$$

where u_i are the *scaling variables* (or *scaling fields*), which transform as

$$u'_i = b^{y_i} u_i. \quad (3.17)$$

Depending on the sign of the RG eigenvalue y_i near a given FP, three scenarios are possible:

- if $y_i < 0$ the scaling variable u_i is said to be *irrelevant*, in the sense that starting sufficiently near the FP and iterating the RG procedure, its value goes to zero and it doesn't contribute to the critical behavior of the system;
- if $y_i > 0$ u_i is said to be *relevant*, since its value is enhanced with the RG procedure and it brings the system far away from the FP;
- if $y_i = 0$ the corresponding scaling variable is said to be *marginal*, in this case the linearization of Eq. 3.10 is not sufficient to understand the behavior of u_i near the FP.

For a FP in a d -dimensional parameter space (usually d is infinite), with d' relevant scaling fields, the hypersurface generated near the FP by continuing the $(d - d')$ -dimensional hyperplane spanned by the $d - d'$ irrelevant eigenvalues is called the *critical manifold* (or *critical surface*). The critical behavior of all systems with parameters lying on this manifold is governed by the FP, and the correlation length diverges for all these systems, as implied by Eq. 3.11.

3.1.1 Scaling Laws

Within the framework described above, a direct connection between the RG procedure and scaling laws can be established [Par98]. Consider the logarithmic scaling parameter $l = \log_2 b$, which is the continuation of n , defined earlier, to real values. The linear recursion relation for the parameters, Eq. 3.14, can be rewritten near the FP as

$$\frac{d\delta H_l}{dl} = T_H \delta H_l, \quad (3.18)$$

where $H_l = \sum_a K_a^* O_a + \delta H_l$, and H_0 is related to the original Hamiltonian by $H_0 = \beta \mathcal{H}_0(\{s_i\})$. Suppose H_0 has two distinct basins of attraction corresponding to two different attractive FPs, separated by a one-dimensional critical surface. Depending on the value of β , H_0 is attracted to one of the two FPs. For a specific value β_c , it crosses the critical surface at H_c . In this scenario, the linear operator T_H at H_c has at least one positive and one negative (assumed real) eigenvalue. Denote the largest (in absolute

value) eigenvalues as $\omega_R(>0)$ and $\omega_A(<0)$, with corresponding eigenvectors H_R and H_A . Two possible asymptotic behaviors arise:

$$\begin{aligned} H_l &\sim H_c + H_A e^{\omega_A l} + (\beta - \beta_c) H_R e^{\omega_R l} \quad \text{if } l + \frac{\ln(|\beta - \beta_c|)}{\omega_R} \ll 0, \\ H_l &\sim H_0 \pm (\beta - \beta_c) H_R e^{\omega_R l} \quad \text{if } l + \frac{\ln(|\beta - \beta_c|)}{\omega_R} \gg 0, \end{aligned} \quad (3.19)$$

where the $\pm$ sign in the second case reflects the two possible trajectories for $\beta > \beta_c$ and $\beta < \beta_c$. For $\beta \approx \beta_c$, the trajectory remains near H_c for a time of order $\ln(1/|\beta - \beta_c|)/\omega_R$. In the first case of Eq. 3.19, the system behaves as at the critical point, with correlations at all length scales, whereas in the second case, correlations decay exponentially. The crossover value is $l_\xi \approx \ln(1/|\beta - \beta_c|)/\omega_R$, which, using $\xi = 2^{l_\xi}$, yields

$$\xi \sim |\beta - \beta_c|^{-\frac{1}{\omega_R}}. \quad (3.20)$$

This is the first derivation of a *scaling law*, typically expressed with the standard exponent $\nu = 1/\omega_R$.

The argument is still valid if the nontrivial FP is characterized by two relevant scaling fields, as in the Ising model with an external magnetic field, where the two unstable directions correspond to temperature and field, whose scaling variables are denoted u_t and u_h . With similar scaling arguments, it can be shown that the singular behavior of the free energy density, $f(\mathbf{K}) = \lim_{N \rightarrow \infty} \ln \int \text{Tr}_s e^{-H_0}$, scales as [Par98, Car96]

$$f_{\text{sing}}(\mathbf{K}) \sim b^{-D} f_{\text{sing}}(\mathbf{K}'). \quad (3.21)$$

For the Ising model, this can be rewritten in terms of the two relevant scaling variables u_t and u_h , with corresponding RG eigenvalues y_t and y_h :

$$f_{\text{sing}}(u_t, u_h) = b^{-D} f_{\text{sing}}(b^{y_t} u_t, b^{y_h} u_h). \quad (3.22)$$

The scaling law for the free energy density is obtained by choosing b such that $b^{y_t} = 1/u_t$:

$$f_{\text{sing}}(t, h) = t^{\frac{D}{y_t}} \Psi_f \left(\frac{h}{t^{y_h/y_t}} \right), \quad (3.23)$$

where the scaling fields are linearly related to the distances from the critical values, t and h , for temperature and external field: $u_t \propto t$ and $u_h \propto h$. The function $\Psi_f(x)$ is the *scaling function* for the free energy, defined such that $\Psi_f(0)$ is a non-vanishing constant and $\Psi_f(x) \sim x^{D/y_h}$ for $x \gg 1$. This relation is crucial for connecting different critical exponents. All observables can be derived from the free energy density via differentiation. The specific heat, $\frac{\partial^2 f(t,0)}{\partial t^2}$, has a critical exponent defined by $t^{-\alpha}$; the magnetization, $\frac{\partial f(t,h)}{\partial h} \Big|_{h=0}$, scales as t^β ; and the susceptibility, $\frac{\partial^2 f(t,h)}{\partial h^2} \Big|_{h=0}$, scales as $t^{-\gamma}$. These results allow the identification of relationships among critical exponents, so that only two are independent [Car96, Ami93].

Interestingly, the RG framework also yields a scaling law for generic correlation functions. The scaling exponent of correlation functions is determined by the dimension

of the block variables, which is directly related to the exponent ω_c at the critical point. From Eq. 3.6, one obtains [Par98]

$$\langle s_i s_j \rangle = \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{2D(1-\omega_c)}} \Psi_{C_2} \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{\xi} \right), \quad (3.24)$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the positions of spins s_i and s_j in a D -dimensional hypercubic lattice, and $\Psi_{C_2}(x) \sim e^{-x}$ for $x \gg 1$. This scaling relation generalizes to q -point correlation functions [Par98]

$$\langle s_{i_1} \dots s_{i_q} \rangle = \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{qD(1-\omega_c)}} \Psi_{C_q} \left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{\xi}, \frac{|\mathbf{x}_1 - \mathbf{x}_3|}{\xi}, \dots, \frac{|\mathbf{x}_1 - \mathbf{x}_q|}{\xi} \right). \quad (3.25)$$

The scaling of the 2-point correlation function is typically characterized by the critical exponent η , known as the *anomalous dimension*, with the relation

$$\eta = 2 + D(1 - \omega_c). \quad (3.26)$$

3.1.2 Critical Exponents at Fixed Dimension

The real-space RG provides a powerful framework for understanding scaling laws and provides a systematic approach for the computation of critical exponents. However, this is not the only possible method. In the context of continuum field theory, the perturbative expansion at fixed dimension can also be employed for this purpose [Par98, Par80b, Zin21]. The following section sets the stage for the application of this method, which will be used in the works presented below in Chapter 6 and Chapter 7.

For definiteness, the scalar ϕ^4 theory in generic dimension D is considered, defined by the action:

$$H[\phi(x)] = \int d^D x \left(\frac{\gamma}{2} (\nabla \phi(x))^2 + \frac{1}{2} r_0 \phi^2(x) + \frac{u_0}{4!} \phi^4(x) - h \phi(x) \right), \quad (3.27)$$

where γ is usually absorbed in the definition of the field, r_0 is a temperature-dependent parameter, proportional to the distance from the critical point ($r_0 \propto t$), and is referred to as the *bare mass* in Field Theory. The parameter u_0 is positive, its temperature dependence can be neglected, and it is called the *bare coupling constant*. Finally, h is the external magnetic field and $\phi(x)$ is the scalar field.

The action in Eq. 3.27 phenomenologically reproduces the critical behavior of the Ising model, it respects its $\mathbb{Z}_2$ symmetry, $\phi(x) \rightarrow -\phi(x)$, and it is usually referred to as the *Ginzburg-Landau* action. Interestingly, $H[\phi(x)]$ can be derived from Eq. 1.8 using the Hubbard-Stratonovich transformation [Kop10], neglecting higher-order terms such as $\phi^6(x)$ or $\phi^8(x)$, which can be shown to be irrelevant in the RG framework [Ami93, Gol18, Zin21, Par98]. For this reason, the action in Eq. 3.27 is considered the continuum Field Theory associated with the Ising model, as defined in Eq. 1.8 on a finite-dimensional lattice.

The first step in computing critical exponents is the evaluation of correlation functions, which can be obtained from the partition function. The following prescription is employed for the Fourier Transform in $\mathbb{R}^D$ for a generic function $f(x)$ ¹

$$\tilde{f}(\mathbf{k}) = \int f(\mathbf{x}) e^{i\mathbf{k}\mathbf{x}} d^D x, \quad f(\mathbf{x}) = \frac{1}{(2\pi)^D} \int \tilde{f}(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}} d^D k, \quad (3.28)$$

¹To ease the notation: i) the scalar product between $\mathbf{k}$ and $\mathbf{x}$ is denoted as $\mathbf{k}\mathbf{x}$ at the exponent; ii) the square of the norm of a generic vector $\mathbf{x}$ (or $\mathbf{k}$) is indicated with $|\mathbf{x}|^2 := \mathbf{x}^2$ (or $|\mathbf{k}|^2 := \mathbf{k}^2$).

that implies

$$(2\pi)^D \delta^D(\mathbf{k}) = \int e^{i\mathbf{k}\mathbf{x}} d^D x. \quad (3.29)$$

Thus, the partition function corresponding to Eq. 3.27 can be written in Fourier space as the following functional integral

$$Z = \int D[\tilde{\phi}(\mathbf{k})] e^{-H[\tilde{\phi}(\mathbf{k})]}, \quad (3.30)$$

with

$$\begin{aligned} H[\tilde{\phi}(\mathbf{k})] = & -h \tilde{\phi}(\mathbf{k}=0) + \frac{1}{2} \int_{\mathbf{k}} (k^2 + r_0) \tilde{\phi}(\mathbf{k}) \tilde{\phi}(-\mathbf{k}) \\ & + \frac{u_0}{4!} \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} \int_{\mathbf{k}_3} \int_{\mathbf{k}_4} (2\pi)^D \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i \right) \tilde{\phi}(\mathbf{k}_1) \tilde{\phi}(\mathbf{k}_2) \tilde{\phi}(\mathbf{k}_3) \tilde{\phi}(\mathbf{k}_4), \end{aligned} \quad (3.31)$$

where the following notation is employed,

$$\int_{\mathbf{k}} \equiv \int_{\Lambda} \frac{d^D k}{(2\pi)^D}, \quad (3.32)$$

with $|\mathbf{k}| < \Lambda$, where Λ is the **Ultraviolet (UV)** cutoff, arising from the finite lattice spacing a_l of the original lattice ($\Lambda \propto 1/a_l$).

Interpreting the exponential in Eq. 3.30 as proportional to the probability density for the order parameter to take the spatial configuration $\phi(\mathbf{x})$ (or $\tilde{\phi}(\mathbf{k})$ in momentum space), one recovers the results of the **MF** approximation by considering a uniform configuration:

$$\phi(\mathbf{x})_{\text{MF}} = \bar{\phi}, \quad (3.33)$$

and minimizing Eq. 3.27 by the saddle-point method,

$$\left. \frac{\partial H[\phi(\mathbf{x})]}{\partial \phi(\mathbf{x})} \right|_{\phi(\mathbf{x})=\bar{\phi}} = r_0 \bar{\phi} + \frac{u_0}{6} \bar{\phi}^3 - h = 0. \quad (3.34)$$

From here on, the case $h = 0$ is considered, thus the solution of the saddle-point equation is

$$\bar{\phi} = \begin{cases} 0 & \text{if } r_0 > 0 \ (T \geq T_c), \\ \sqrt{-6r_0/u_0} & \text{if } r_0 < 0 \ (T < T_c). \end{cases} \quad (3.35)$$

This is the so-called *Landau approximation*, equivalent to the **MF** result, and valid above the **UCD**, $D_{\text{uc}} = 4$ for the Ising model [Ami93, Par98].

Going further, to account for fluctuations in the underlying D -dimensional space, one must consider deviations around $\bar{\phi}$:

$$\tilde{\phi}(\mathbf{k}) = (2\pi)^D \delta^D(\mathbf{k}) \bar{\phi} + \delta \tilde{\phi}(\mathbf{k}). \quad (3.36)$$

By substituting Eq. 3.35 and Eq. 3.36 into the action (written in Fourier space in Eq. 3.30) and retaining only quadratic terms in $\delta \tilde{\phi}(\mathbf{k})$, it becomes [Kop10]

$$Z_0 \propto \int D[\tilde{\phi}(\mathbf{k})] \exp \left(-H_0[\tilde{\phi}(\mathbf{k})] \right), \quad (3.37)$$

where

$$H_0[\tilde{\phi}(\mathbf{k})] \equiv -\frac{1}{2} \int_{\mathbf{k}} (\xi_0^{-2} + \mathbf{k}^2) \delta\tilde{\phi}(\mathbf{k}) \delta\tilde{\phi}(-\mathbf{k}), \quad (3.38)$$

up to constant contributions (not shown here), and introducing the so-called “approximated” correlation length ξ_0 as

$$\xi_0^{-2} \equiv \begin{cases} r_0 & \text{for } T > T_c, \\ -2r_0 & \text{for } T < T_c. \end{cases} \quad (3.39)$$

This procedure is known as the *Gaussian approximation*. Again, by interpreting the exponential as the probability density for the order parameter configuration, it is clear from Eq. 3.37 that fluctuations at different wavelengths are independent, in contrast to what happens in the general partition function of Eq. 3.30.

Now it is possible to estimate the correlation functions. The 2-point connected correlation function is defined as

$$(2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \tilde{G}_2(\mathbf{k}_1) \equiv \langle \delta\tilde{\phi}(\mathbf{k}_1) \delta\tilde{\phi}(\mathbf{k}_2) \rangle, \quad (3.40)$$

where the average is taken over the probability density proportional to $\exp(-H[\tilde{\phi}(\mathbf{k})])$, corresponding to the full partition function Z . Using instead the distribution from the Gaussian approximation, Z_0 , the corresponding 2-point connected correlation function $\tilde{G}^{(0)}(\mathbf{k})$ reads

$$\tilde{G}^{(0)}(\mathbf{k}) = \frac{1}{\xi_0^{-2} + \mathbf{k}^2}, \quad (3.41)$$

which is the so-called *bare* (or *Gaussian*) *propagator*.

Diagrammatic Perturbative Expansion

To compute observables in the finite-dimensional system, one should average over the full distribution, $\propto \exp(-H[\tilde{\phi}(\mathbf{k})])$, as given in Eq. 3.31. However, since this cannot be done exactly, it is necessary to resort to a perturbative expansion as if u_0 were small. In fact, for perturbative expansions, the proper parameter is the dimensionless one. In this context, the only dimensionless parameter associated with the coupling u_0 is [Par98, Kop10]

$$u_0 \xi^{4-D}, \quad (3.42)$$

so the perturbative expansion loses validity for $D < 4$ near the critical point; how to solve this issue will be shown below.

For $T > T_c$, where $\delta\tilde{\phi}(\mathbf{k}) = \tilde{\phi}(\mathbf{k})$ and $\xi_0^{-2} = r_0$, the complete action can be written as

$$H[\tilde{\phi}(\mathbf{k})] = H_0[\tilde{\phi}(\mathbf{k})] + H_{\text{int}}[\tilde{\phi}(\mathbf{k})], \quad (3.43)$$

where $H_0[\tilde{\phi}(\mathbf{k})]$ is defined in Eq. 3.38 and

$$H_{\text{int}}[\tilde{\phi}(\mathbf{k})] \equiv \frac{u_0}{4!} \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} \int_{\mathbf{k}_3} \int_{\mathbf{k}_4} (2\pi)^D \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i \right) \tilde{\phi}(\mathbf{k}_1) \tilde{\phi}(\mathbf{k}_2) \tilde{\phi}(\mathbf{k}_3) \tilde{\phi}(\mathbf{k}_4). \quad (3.44)$$

The partition function becomes

$$Z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int D[\tilde{\phi}(\mathbf{k})] e^{-H_0[\tilde{\phi}(\mathbf{k})]} \left(-H_{\text{int}}[\tilde{\phi}(\mathbf{k})] \right)^n, \quad (3.45)$$

so that a generic q -point correlation function $\tilde{G}_q(\mathbf{k}_1, \dots, \mathbf{k}_q)$ can be computed as

$$(2\pi)^D \delta^D \left(\sum_{i=1}^q \mathbf{k}_i \right) \tilde{G}_q(\mathbf{k}_1, \dots, \mathbf{k}_{q-1}) = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \left(H_{\text{int}}[\tilde{\phi}(\mathbf{k})] \right)^n \tilde{\phi}(\mathbf{k}_1) \dots \tilde{\phi}(\mathbf{k}_q) \right\rangle_0}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \left(H_{\text{int}}[\tilde{\phi}(\mathbf{k})] \right)^n \right\rangle_0} \quad (3.46)$$

where the Gaussian average is denoted by $\langle \cdot \rangle_0$, such that, for a generic functional $\mathcal{F}[\tilde{\phi}(\mathbf{k})]$, this kind of average corresponds to

$$\langle \mathcal{F}[\tilde{\phi}(\mathbf{k})] \rangle_0 \equiv \frac{\int \mathcal{D}[\tilde{\phi}(\mathbf{k})] e^{-H_0[\tilde{\phi}(\mathbf{k})]} \mathcal{F}[\tilde{\phi}(\mathbf{k})]}{\int \mathcal{D}[\tilde{\phi}(\mathbf{k})] e^{-H_0[\tilde{\phi}(\mathbf{k})]}}. \quad (3.47)$$

In this way, the evaluation of a generic correlation function reduces to summing contributions that can be identified with the so-called *Feynman diagrams*, thanks to Wick's theorem [Zin21]. A schematic representation of each diagrammatic element is shown in Figure 3.1. Standard arguments show that for any connected correlation function, only *connected* diagrams, i.e., those represented by connected graphs, contribute [Ami93, Zin21]. It can be shown that the first two contributions to the 2-point correlation function, from Eq. 3.46 with $q = 2$, are

$$\tilde{G}_2(\mathbf{k}_1) = \tilde{G}^{(0)}(\mathbf{k}_1) - \frac{u_0}{2} \tilde{G}^{(0)}(\mathbf{k}_1)^2 \int_q \tilde{G}^{(0)}(q) + \mathcal{O}(u_0^2), \quad (3.48)$$

where the corresponding diagrams are shown in Figure 3.1. Similarly, the first three contributions (up to some permutations of the lines) to the 4-point function, represented diagrammatically in Figure 3.1, can be easily written as

$$\tilde{G}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \prod_{i=1}^4 \tilde{G}_2(\mathbf{k}_i) \left[-u_0 + \frac{1}{2} u_0^2 \left(I(\mathbf{k}_1 + \mathbf{k}_2) + I(\mathbf{k}_1 + \mathbf{k}_3) + I(\mathbf{k}_1 + \mathbf{k}_4) \right) \right] + \mathcal{O}(u_0^3), \quad (3.49)$$

where $-\mathbf{k}_4 = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$ due to momentum conservation and

$$I(q) \equiv \int_p \tilde{G}^{(0)}(p) \tilde{G}^{(0)}(p - q), \quad (3.50)$$

where in the products only terms up to $\mathcal{O}(u_0^2)$ are retained.

Note that the 2-point function can be generically written as

$$\begin{aligned} \tilde{G}_2(\mathbf{k}) &= \tilde{G}^{(0)}(\mathbf{k}) - \tilde{G}^{(0)}(\mathbf{k}) \Sigma(\mathbf{k}) \tilde{G}^{(0)}(\mathbf{k}) + \tilde{G}^{(0)}(\mathbf{k}) \Sigma(\mathbf{k}) \tilde{G}^{(0)}(\mathbf{k}) \Sigma(\mathbf{k}) \tilde{G}^{(0)}(\mathbf{k}) + \dots \\ &= \tilde{G}^{(0)}(\mathbf{k}) \sum_{n=0}^{\infty} (-\Sigma(\mathbf{k}) \tilde{G}^{(0)}(\mathbf{k}))^n = \frac{1}{(\tilde{G}^{(0)}(\mathbf{k}))^{-1} + \Sigma(\mathbf{k})}, \end{aligned} \quad (3.51)$$

where $\Sigma(\mathbf{k})$ is called the *self-energy*, and it is graphically represented as an *amputated, one-particle irreducible* diagram [Zin21]. For example, in Eq. 3.48:

$$\Sigma(\mathbf{k}) = \frac{u_0}{2} \int_q \tilde{G}^{(0)}(q), \quad (3.52)$$

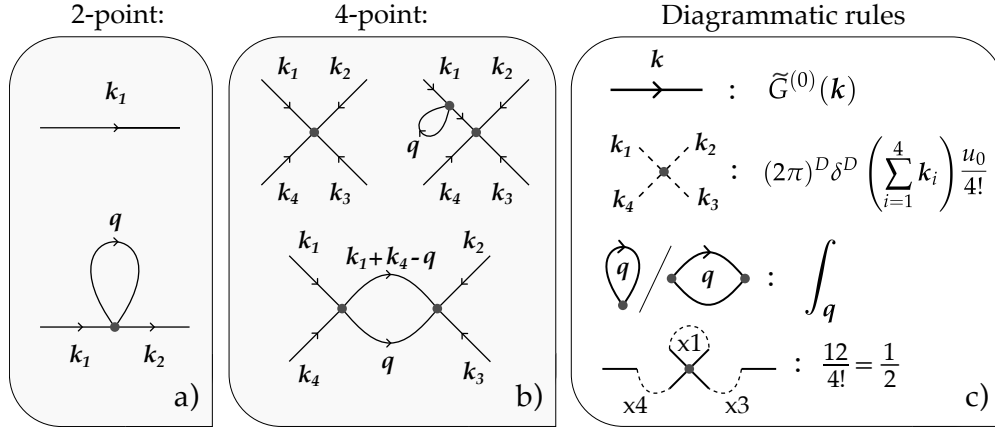


Figure 3.1. a) first two contributions to the 2-point correlation function: the bare line (Gaussian propagator) and the first one-loop correction; b) first three contributions to the 4-point correlation function (the 3-point correlation is zero by $\mathbb{Z}_2$ symmetry): bare vertex, the one-loop correction on external lines (four such diagrams contribute, each for external line) and the one-loop correction to the vertex itself (this contributes with three terms due to the permutation of the four external momenta); c) diagrammatic rules: a line contributes with a Gaussian propagator, a vertex contributes with the coupling u_0 while the delta function is absorbed in the definition of the correlation functions, Eq. 3.46, a loop contributes with a momentum integral; in the last line it is reported the example of the symmetry factor computation for the one-loop correction to the 2-point function: two external lines should be connected to two of the four lines of the vertex to reproduce the mentioned diagram, this can be done in 12 ways but the combinatorial factor from the vertex is $1/4!$, thus the associated factor is $1/2$.

and in this case, the self-energy does not depend on the external momentum k . In this way, it is possible to sum an infinite number of diagrams, all included in Eq. 3.51. Moreover, by expanding a generic $\Sigma(k)$ to second order in k^2 , in Eq. 3.51, one obtains

$$\tilde{G}_2(k) = \frac{z_1}{k^2 + \zeta^{-2}}, \quad (3.53)$$

defining z_1 as

$$z_1 \equiv \frac{1}{1 + \left. \frac{\partial \Sigma(k)}{\partial k^2} \right|_{k^2=0}}, \quad (3.54)$$

and the correlation length ζ as

$$\zeta^{-2} \equiv z_1 \Sigma(k=0), \quad (3.55)$$

related to the *mass* m , the distance from the physical critical point, as $\zeta = m^{-1}$.

Computation of Critical Exponents

As already mentioned, the perturbative expansion cannot be performed at the critical point near the UCD, due to the divergence of the associated dimensionless coupling constant. The solution is to find another dimensionless coupling constant that remains finite in the critical regime. The idea is to use scaling laws and define a specific ratio of

correlation functions such that the scaling with the correlation length yields a finite value [Par98]. For the Ising model, this can be identified as

$$\lambda \equiv \xi^{-D} \frac{\tilde{G}_4(0,0,0)}{\tilde{G}_2(0)^2}, \quad (3.56)$$

which remains finite at the critical point, as can be verified using the scaling laws of Eq. 3.25. Adopting standard jargon, λ is called the *renormalized coupling constant* [Par98, Bel92, Ami93]. Moreover,

$$\lambda = \begin{cases} u_0 m^{D-4} & \text{if } u_0 m^{D-4} \rightarrow 0 \\ \lambda_c & \text{if } u_0 m^{D-4} \rightarrow \infty \end{cases} \quad (3.57)$$

where $\lambda_c < +\infty$. Denoting $u = u_0 m^{D-4}$, the idea is to rewrite the perturbative expansion in powers of u as an expansion in powers of λ , which is no longer divergent at the critical point. As long as $\lambda_c \ll 1$, the new perturbative expansion is well-defined and reliable. Once λ_c is determined, critical exponents can be estimated using scaling laws:

$$z_1(\lambda) = \left(\frac{\partial \tilde{G}_2(k)^{-1}}{\partial k^2} \right)^{-1} \Big|_{k^2=0} \sim m^{2C_1} \quad (3.58)$$

$$m^2 \sim r_0^{2\nu} \longrightarrow z_2(\lambda) = \frac{dr_0}{dm^2} \sim m^{2C_2}, \quad (3.59)$$

where $C_1 = \frac{\eta}{2}$ and $C_2 = \frac{1}{2\nu} - 1$. Defining the following functions:

$$C_1(\lambda) \equiv m^2 \frac{\partial}{\partial m^2} \Big|_{u_0 \text{ fixed}} \ln[z_1(\lambda)], \quad (3.60)$$

$$C_2(\lambda) \equiv m^2 \frac{\partial}{\partial m^2} \Big|_{u_0 \text{ fixed}} \ln[z_2(\lambda)], \quad (3.61)$$

and evaluating them (using the perturbative expansion) at λ_c , the exponents are obtained as

$$C_1(\lambda_c) = C_1 \quad C_2(\lambda_c) = C_2. \quad (3.62)$$

Other scaling laws can be used, provided that the z 's and C 's functions are dimensionless and depend only on λ .

To compute λ_c , the so-called *β -function* for the coupling is defined as

$$\beta(\lambda) \equiv m^2 \frac{\partial}{\partial m^2} \Big|_{u_0 \text{ fixed}} \lambda = \frac{1}{2}(D-4)u \frac{\partial}{\partial u} \Big|_{m^2 \text{ fixed}} \lambda, \quad (3.63)$$

which vanishes at both $\lambda = 0$ and $\lambda = \lambda_c$, see Figure 3.2. Using the conditions on λ from Eq. 3.57, the following solution can be found:

$$u(\lambda) = \lambda \exp \left[\int_0^\lambda \left(\frac{D-4}{2\beta(\lambda')} - \frac{1}{\lambda'} \right) d\lambda' \right], \quad (3.64)$$

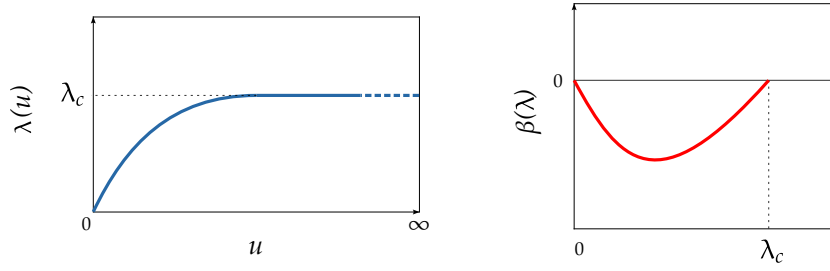


Figure 3.2. Left): behavior of the renormalized coupling constant λ as a function of the bare one, u . For $u \rightarrow \infty$, λ is expected to approach a finite and possibly small ($\ll 1$) value $\lambda = \lambda_c$; Right) the behavior of the corresponding β -function for $D < D_{uc}$. $\beta(\lambda)$ equals zero for the trivial value $\lambda = 0$ and for the nontrivial $\lambda = \lambda_c$.

which, for $\lambda \rightarrow \lambda_c$, can be rewritten by adding and subtracting $\frac{D-4}{2\beta'(\lambda_c - \lambda')}$ inside the exponential:

$$u = \alpha \left(\frac{\lambda_c - \lambda}{\lambda_c} \right)^{\frac{D-4}{2\beta'}}, \quad (3.65)$$

where

$$\beta' \equiv \left. \frac{d\beta(\lambda)}{d\lambda} \right|_{\lambda=\lambda_c}, \quad (3.66)$$

and α is a constant,

$$\alpha \equiv \lambda_c \exp \left[\int_0^{\lambda_c} \left(\frac{D-4}{2\beta(\lambda')} - \frac{1}{\lambda'} - \frac{4-D}{2\beta'(\lambda_c - \lambda')} \right) d\lambda' \right]. \quad (3.67)$$

This implies that at the critical point, the residual dependence of λ on the bare coupling u_0 vanishes as $m^{-2\beta'}$:

$$\lambda = \lambda_c \left[1 - \left(\frac{u}{\alpha} \right)^{-\frac{2\beta'}{4-D}} \right]. \quad (3.68)$$

This is in agreement with Eq. 3.19, where near the FP the residual dependence of the Hamiltonian on different trajectories vanishes as $m^{-\omega_A}$. Thus, here

$$\omega_A = -2\beta'. \quad (3.69)$$

For this reason, the value ω_A , a universal quantity, is called the *correction-to-scaling* critical exponent, usually (and in the following) denoted by ω .

The method described above allows for the estimation of critical exponents at fixed dimension D . The standard approach is to identify a ratio between observables that is finite at the critical point. Then, these quantities are evaluated perturbatively in powers of the bare dimensionless coupling constant (u above). The ratio (λ above) can then be computed, and its critical value used together with scaling laws to estimate critical exponents. Adding corrections, in terms of Feynman diagrams, to the computation of observables increases the accuracy of the exponents, but also increases the computational complexity due to the growing number of such diagrams that must be evaluated. For a detailed explanation of how to use this method to compute the critical exponents of the Ising model, see Chap. 8 of Ref. [Par98].

3.2 Field Theory of the Edwards–Anderson Model

Since the original formulation of the EA model in 1975, substantial effort has been dedicated to the analysis of its critical behavior in finite dimensions, using the methods described above and widely applied to previously well-studied systems [Har76, Lub75]. Nevertheless, it rapidly became apparent that extracting robust conclusions was highly nontrivial. This section partially summarizes the literature on the EA model that is relevant to the discussion of the results obtained in Chapter 7 via the M -layer construction. For a broader review of RG methods on spin glass models, see Chapter 4 of Ref. [Cha23].

3.2.1 Momentum Space RG Approach

The approach outlined in Section 3.1.2 can be applied to the EA model as defined in Eq. 1.12. Following the Replica method, the standard approach to get MF results in this kind of quenched disordered models, it is possible to construct an effective action that preserves the symmetries of the underlying Hamiltonian; in this case, permutation symmetry among replicas. Alternatively, the same result can be derived directly from the microscopic Hamiltonian by means of a Hubbard-Stratonovich transformation [Temo2], as described for the Ising model in Section 3.1.2. Both approaches yield the following generic replica symmetric Field Theory (truncated at cubic interactions):

$$H[\phi^{\alpha\beta}(\mathbf{p})] = \mathcal{L}^{(2)} + \mathcal{L}^{(3)}, \quad (3.70)$$

$$\begin{aligned} \mathcal{L}^{(2)} \equiv & \frac{1}{2} \int_{\mathbf{p}} \sum_{\alpha < \beta, \gamma < \delta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}) M_{\alpha\beta, \gamma\delta}(\mathbf{p}) \tilde{\phi}^{\gamma\delta}(-\mathbf{p}) = \\ & \frac{1}{2} \int_{\mathbf{p}} \left[\left(\frac{\mathbf{p}^2}{2} + m_1 \right) \sum_{\alpha\beta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}) \tilde{\phi}^{\alpha\beta}(-\mathbf{p}) + m_2 \sum_{\alpha\beta\gamma} \tilde{\phi}^{\alpha\gamma}(\mathbf{p}) \tilde{\phi}^{\beta\gamma}(-\mathbf{p}) + m_3 \sum_{\alpha\beta\gamma\delta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}) \tilde{\phi}^{\gamma\delta}(-\mathbf{p}) \right] \end{aligned} \quad (3.71)$$

$$\begin{aligned} \mathcal{L}^{(3)} = & -\frac{1}{6\sqrt{N}} \int_{\mathbf{p}_1} \int_{\mathbf{p}_2} \int_{\mathbf{p}_3} (2\pi)^D \delta^D(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \left[w_1 \sum_{\alpha\beta\gamma} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\beta\gamma}(\mathbf{p}_2) \tilde{\phi}^{\gamma\alpha}(\mathbf{p}_3) \right. \\ & + w_2 \sum_{\alpha\beta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\alpha\beta}(\mathbf{p}_2) \tilde{\phi}^{\alpha\beta}(\mathbf{p}_3) + w_3 \sum_{\alpha\beta\gamma} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\alpha\gamma}(\mathbf{p}_2) \tilde{\phi}^{\beta\gamma}(\mathbf{p}_3) \\ & + w_4 \sum_{\alpha\beta\gamma\delta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\alpha\beta}(\mathbf{p}_2) \tilde{\phi}^{\gamma\delta}(\mathbf{p}_3) + w_5 \sum_{\alpha\beta\gamma\delta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\alpha\gamma}(\mathbf{p}_2) \tilde{\phi}^{\beta\delta}(\mathbf{p}_3) \\ & + w_6 \sum_{\alpha\beta\gamma\delta} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\alpha\gamma}(\mathbf{p}_2) \tilde{\phi}^{\alpha\delta}(\mathbf{p}_3) + w_7 \sum_{\alpha\beta\gamma\delta\mu} \tilde{\phi}^{\alpha\gamma}(\mathbf{p}_1) \tilde{\phi}^{\beta\gamma}(\mathbf{p}_2) \tilde{\phi}^{\delta\mu}(\mathbf{p}_3) \\ & \left. + w_8 \sum_{\alpha\beta\gamma\delta\mu\nu} \tilde{\phi}^{\alpha\beta}(\mathbf{p}_1) \tilde{\phi}^{\gamma\delta}(\mathbf{p}_2) \tilde{\phi}^{\mu\nu}(\mathbf{p}_3) \right], \end{aligned} \quad (3.72)$$

The field, $\tilde{\phi}^{\alpha\beta}(\mathbf{p})$, describes fluctuations of the overlap:

$$\tilde{Q}^{\alpha\beta}(\mathbf{p}) = (2\pi)^D \delta^D(\mathbf{p}) Q^{\alpha\beta} + \tilde{\phi}^{\alpha\beta}(\mathbf{p}), \quad (3.73)$$

where $Q^{\alpha\beta}$ is determined by the MF equation of state, as detailed in Chapter 2. The eight cubic vertices above exhaust all possible interactions between the fields that respect replica permutation symmetry, up to cubic order. Neglect of higher-order terms is justified, as in the standard Ginzburg-Landau framework, by their RG irrelevance, as discussed in Section 3.1. Nevertheless, MF analysis and the stability of the RS solution indicate that the operator responsible for the vanishing of the replicon eigenvalue is quartic in the overlap. The Ginzburg-Landau action with quartic terms is considered in Ref. [Höl20] because of its essential role [Fis85].

The Hubbard-Stratonovich transformation enables explicit identification of the coupling constants and the three masses with spin correlation functions; the complete correspondence is given in Ref. [Temo2]. Specifically, the three masses are related to the stability analysis of the SK model, done in Section 2.1.2, through the parameters P , Q , and R (see Eq. 2.24) as follows:

$$m_1 = \frac{1}{2}(P - 2Q + R), \quad (3.74)$$

$$m_2 = Q - R, \quad (3.75)$$

$$m_3 = \frac{1}{4}R. \quad (3.76)$$

These relationships reflect that the quadratic operator $\mathcal{L}^{(2)}$ in Eq. 3.71 is not diagonal; its diagonalization exactly corresponds to the one described in Section 2.1.2 for the stability analysis of the RS solution. The result is the appearance of three new masses r_R , r_A , and r_L :

$$\sum_{\gamma < \delta} M_{\alpha\beta, \gamma\delta}(\mathbf{p}) \tilde{\phi}_i^{\gamma\delta}(\mathbf{p}) = r_i \phi_i^{\alpha\beta} \quad i = L, A, R, \quad (3.77)$$

corresponding, respectively, to the replicon, anomalous, and longitudinal sectors [Temo2]:

$$\tilde{\phi}^{\alpha\beta}(\mathbf{p}) = \tilde{\phi}_L^{\alpha\beta}(\mathbf{p}) + \tilde{\phi}_A^{\alpha\beta}(\mathbf{p}) + \tilde{\phi}_R^{\alpha\beta}(\mathbf{p}). \quad (3.78)$$

Coherently, r_L , r_A and r_R exactly correspond to the eigenvalues of the stability matrix in the MF framework; the relations with P , Q , and R are exactly those derived in Section 2.1.2 for a finite number of replicas n , and are reported here:

$$r_L = P + 2(n-2)Q + \frac{1}{2}(n-2)(n-3)R, \quad (3.79)$$

$$r_A = P + (n-4)Q - (n-3)R, \quad (3.80)$$

$$r_R = P - 2Q + R. \quad (3.81)$$

The diagonalization of the quadratic term also induces a nontrivial mapping for the cubic vertices, yielding the effective RS Field Theory for general n [Temo2]. In order to perform perturbative calculations with Feynman diagrams, one must introduce an explicit basis for each of the three invariant subspaces [Bra80, Temo2, De 97].

It is important to distinguish the zero field, $H = 0$, and the non-zero field, $H \neq 0$, cases:

- $H = 0$: In the RS phase without external field, Eqs. 3.79, 3.80, and 3.81 imply $r_L = r_A = r_R$ in the $n \rightarrow 0$ limit, since $Q = 0 = R$. In this case, only $w_1 \neq 0$, with $w_i = 0$ for $i = 2, \dots, 8$. The $n \rightarrow 0$ limit can be taken directly, and a nontrivial FP is found below the UCD $D_{uc} = 6$, with the corresponding ϵ -expansions for critical exponents [Har76, Che77, Pyt79].

- $H \neq 0$: For non-zero external field, both magnetization and overlap are different from zero, leading to all three masses and all eight couplings being present. In Ref. [Gre83], the behavior of the dAT line in finite dimensions is analyzed via Ginzburg–Landau actions as deviations from the $H = 0$ case; the MF prediction is found to fail below $D = 8$. An alternative approach, due to [Bra80], projects onto the replicon sector, yielding only the critical replicon mass and two non-zero cubic couplings, w_1 and w_2 . These works find that no nontrivial FP exists below the UCD $D_{uc} = 6$, and that the Gaussian FP above D_{uc} has a vanishing basin of attraction as $D \rightarrow 6$ from above [Moo11]. The absence of a FP is suggestive of the absence of an RSB transition, since the replicon is the mode responsible for such transitions in MF theory. Another possibility, supported by Ref. [Cha17], is that the FP is *nonperturbative*. In this view, *Functional RG* [Tar25] and real-space RG approaches (see below) are natural alternatives for extracting predictions in finite dimensions.

In both Refs. [Gre83] and [Bra80], the $n \rightarrow 0$ limit is taken, but, in contrast to the $H = 0$ case, this limit is nontrivial and plays an important role (see Chapter 4 of Ref. [Cha23]).

Notably, the simplified Field Theory proposed by Bray and Roberts [Bra80] has been confirmed by the numerical work in Ref. [Fer22]. In particular, the ratio w_2/w_1 is a universal quantity, unaffected by renormalization, and links equilibrium and nonequilibrium phenomenology [Riz13]. Such a ratio is therefore a valuable probe of the finite-dimensional critical behavior. Furthermore, Ref. [Ang18] matches the leading correction to the mass as computed via Feynman diagrams in the Bray–Roberts theory with results from the alternative Bethe M -layer construction, validating both the Field Theory and the new perturbative method.

3.2.2 Real-Space RG Approach

As outlined in Section 3.1, real-space RG is more than just an instructional tool. Analogously to momentum-space RG, it has been widely applied to disordered spin systems of spin glass type [You76, Sou77, Ben84, Ang13, Mon14, Ang15, Ang17]. However, on finite-dimensional lattices (other than the 1D chain), one must introduce approximations, and results remain inconclusive. Alternatively, special classes of lattices exist for which the coarse-graining procedure is exact [Kau84], allowing subsequent extrapolation to hypercubic lattices. Even in these situations, challenges comparable to those in momentum-space RG remain. For instance, Ref. [Gar84] shows that analytic continuation in the replica number is non-unique, and no RSB is observed, in analogy with the RS Field Theory for $H \neq 0$. More recently, a nontrivial zero-temperature FP for the EA model in an external field for $D > 8$ has been reported using both the Migdal–Kadanoff *real-space* method and the so-called *Ensemble Renormalization Group* [Ang15]. The latter methodology permits predictions for the critical phase of finite-dimensional systems, circumventing certain difficulties typical of quenched disordered systems. The basic principle of RG is to analyze a sample (with frozen disorder) at progressively larger scales. However, the presence of local heterogeneities complicates the application of standard RG approaches. In addition, sample-to-sample fluctuations may exceed thermal fluctuations in relevance, with the former typically neglected in conventional real-space RG methods but captured by the Ensemble Renormalization Group.

3.3 Exploring Possible Scenarios in Finite Dimensions

As the available results obtained via different RG schemes remain inconclusive, several possible scenarios are proposed for the critical behavior of the EA model in finite dimensions.

3.3.1 Droplet Theory

A central question is whether MF behavior survives in finite dimensions. If so, the phenomena observed could differ partially or substantially from those predicted by the RSB mechanism. An alternative, extensively investigated, scenario is the *Droplet Model*, wherein the ordered phase of the EA model for zero external field is governed by a single ground state (or two, by spin-flip symmetry) [Fis86, Hus87, Fis87, Fis88]. The contrast between the Droplet and RSB scenarios is most apparent in the statistics of low-energy excitations. Here, ground states are defined as configurations that cannot achieve a lower energy by flipping a finite number of spins. Interest then centers on the clusters of spins flipping in the first excited state and their associated energies (or free energies at finite temperature). More precisely, a *droplet* of scale L containing site i is defined as the minimal-energy connected cluster including i with size between L^D and L^{2D} . The Droplet theory predicts energy scaling as [Fis88]

$$F_L \sim L^\theta, \quad (3.82)$$

independently of the choice of site i . For instance, $\theta = D - 1$ in a ferromagnetic Ising model, while in spin glasses, θ is expected to be strictly less than $D - 1$, due to frustration and randomness. Most low-energy excitations involve a small number of spins and are irrelevant for large-scale critical behavior. Disorder and frustration imply that the energies of such droplets are broadly distributed. For large L , the predicted scaling for the distribution is [Fis88]

$$\rho(F_L) = \frac{1}{Y L^\theta} \tilde{\rho} \left[\frac{F_L}{Y L^\theta} \right], \quad (3.83)$$

where Y is the generalized stiffness modulus, and crucially, $\tilde{\rho}(0) > 0$, reflecting that configurations nearly degenerate in energy with the ground state exist down to $F_L = 0$. From Eq. 3.83 it also follows that the probability of finding a zero-energy droplet decays as $L^{-\theta}$. Moreover, arguments from the scaling law in Eq. 3.83 can be made to predict an upper bound for θ [Fis88]:

$$\theta \leq \frac{D - 1}{2}. \quad (3.84)$$

Surface Fractal Dimension: a Scaling Argument

The Droplet Model also predicts that the surface scaling of droplets, S_L , is [Fis87]

$$S_L \sim L^{D_s}, \quad (3.85)$$

with $D - 1 < D_s < D$. The lower bound $D - 1$ follows from the fact that minimal-energy surfaces avoid unfavorable local environments, leading to fractal (rough) geometries rather than smooth domain walls. Instead, the upper bound can be understood

with the following argument. If $D_s = D$, domain walls would be *space-filling*, i.e., separating ground states that differ by a positive fraction of the system's volume. In other words, this is what would happen if more than one single pure state existed. A scaling argument can rule out this possibility [Fis87]. The proof is made by contradiction: consider the presence of two ground states, α and β (for which the energy can be lowered only by an extensive spin flip, by definition), whose configurations differ by an extensive portion of spins (except for a global spin flip). In the RSB language, these are two distinct nontrivial pure states (at zero temperature). On a large length scale L these will differ by $\mathcal{O}(L^D)$ spins, thus $\mathcal{O}(L^D)$ bonds, or spin pairs. Assuming random independent spin pairs, the energy fluctuations on the length-scale L will be of $\mathcal{O}(L^{D/2})$. On the other hand, assume that α is lower (in energy) than β in the local region with linear size L . The energy cost to create a droplet scales with L^θ , but due to Eq. 3.84, the energy of β is lowered by flipping a finite number of spins, which is not possible, thus excluding the possibility of having multiple such states [Fis87].

To recap: the two central predictions of the Droplet Model are therefore: i) droplets are characterized by their excitation energy above the ground state with a nontrivial exponent $\theta > 0$, and ii) domain walls are fractal, i.e., $D_s < D$.

Most significantly, this picture predicts the absence of a phase transition in a field [Fis86] (i.e., no dAT line), following a generalization of the Imry–Ma argument [Imr75]. For a droplet excitation of linear size L , the magnetization scales as $L^{D/2}$, the excitation energy as L^θ , and the energy change from a field H as $L^\theta - HL^{D/2}$. Since $\theta < D/2$, the field term dominates for sufficiently large L , and long range spin glass order is destroyed.

The Droplet scenario is also supported by a real-space RG approach [Bra86]. Rather than following excitations, one considers the distribution $P^{(L)}(J)$ of effective couplings between blocks of size L , which for large L approaches a universal form with width L^θ as in (3.83). The properties of this distribution (particularly at low J) encode correlation functions at zero temperature [Bra86, De 06].

3.3.2 TNT and Chaotic Pairs

Thus, the Droplet and RSB scenarios are the principal competing possibilities. However, it is not guaranteed that either one or the other is fully consistent with the EA model's finite-dimensional behavior, and intermediate scenarios have been proposed. Notable among these are the so-called Trivial Non Trivial (TNT) [Krzoo, Paloo] and Chaotic Pairs [New96, New97] pictures.

The TNT hypothesis is most naturally discussed via the so-called *link overlap*:

$$q_{\text{lo}} \equiv \frac{1}{N_{\mathcal{E}}} \sum_{(i,j)} s_i s_j s'_i s'_j, \quad (3.86)$$

where $N_{\mathcal{E}}$ is the number of lattice edges and s'_i corresponds to the configuration in the droplet excited state. In the SK model, $q_{\text{lo}} = q^2$ (with q being the conventional overlap). Thus, a nontrivial link overlap distribution is expected whenever the overlap distribution is nontrivial, i.e., in the RSB scenario. The link overlap can also be written as $q_{\text{lo}} = 1 - L^{D_s - D}$, where D_s is a generic fractal dimension, as it reflects the fraction of links at the boundary of the droplet excitation. Thus, $D_s = D$ is required for a nontrivial link overlap distribution as in RSB. The TNT scenario allows $D_s < D$ (as in

the Droplet Model) together with typical droplet excitation energies of order $\mathcal{O}(1)$ (as in RSB), thereby interpolating between the two cases. The name thus refers to a *Trivial* link overlap distribution and a *Non Trivial* overlap distribution.

The Chaotic Pairs scenario of Newman and Stein is rooted in the formal concepts of Gibbs, pure, and mixed states. Without entering formalities, Ref. [New96] establishes that the thermodynamics of the EA (finite-dimensional) model cannot be described by a single mixed Gibbs state for arbitrary D and T ; mixed states necessarily decompose into infinitely many pure state pairs (in RSB) or a single spin-flip pair (in the Droplet scenario). The RSB phase thus contains many mixed states, each with uncountably many pure state pairs, while the Chaotic Pairs scenario predicts many states, each comprising only a trivial spin-flip pair, again, an intermediate between RSB and Droplet theories.

3.3.3 Comparison between Different Scenarios

TNT, Chaotic Pairs, RSB, and Droplet pictures constitute the four principal scenarios for the finite-dimensional spin glass phase. These may be classified via four central properties, as shown in Table 3.1 (adapted from [New22]): Whether a more refined

	$\theta = 0$	$\theta > 0$
$D_s = D$	RSB	Chaotic Pairs
$D_s < D$	TNT	Droplet

Table 3.1. Classification of scenarios for finite-dimensional spin glasses. Adapted from Ref. [New22].

classification involving additional features is required remains an open problem. Thus far, no definitive solution to the finite-dimensional case has been reached.

Part II

The *M*-layer Construction: General Framework

Chapter 4

The Bethe M -layer Construction

The importance of the Bethe–Peierls approximation relies on the fact that it is exact on locally tree-like graphs, as well as its relevance in computer science due to its equivalence with the [BP](#) algorithm. The absence of loops in these topologies allows the implementation of a powerful method to predict [MF](#) behavior on the Bethe lattice, as shown in [Section 2.2.1](#) through the [RS](#) formulation of the Cavity method. However, real systems in the world are defined on finite-dimensional lattices, where both finite connectivity (as in the Bethe lattice) and the presence of short loops play a crucial role in determining their critical behavior. Consequently, various methods have been developed to systematically compute corrections to the Bethe–Peierls approximation, taking into account the presence of short loops from different perspectives: improving results on Anderson localization and generalizing the approach for broader applicability [[Efeg0](#), [Sac07](#)]; obtaining a systematic method to compute corrections for lattice systems in Statistical Mechanics [[Mono5](#), [Par06](#)]; from a computer science perspective as well [[Cheo6](#), [Mooo07](#)], including variational approaches [[Waio8](#), [Ray17](#)] and other graphical models [[Bra19](#)]. One important issue is that many such problems lack a small parameter that could be used to construct a perturbative expansion, which would allow one to systematically account for corrections to the Bethe–Peierls approximation.

The general idea of the *Bethe M -layer construction* is to provide a method for computing a generic observable of a lattice system (in an arbitrary lattice, including finite-dimensional ones) in a perturbative series. The leading order of this series corresponds to the Bethe–Peierls approximation for the observables, while higher-order corrections are due to the effects of topological loops, which are “manually” introduced into the Bethe lattice. A crucial feature of this approach is that it explicitly identifies a small parameter for the perturbative computation. A similar idea has been used, independently, in computer science by Vontobel [[Von13](#)], though for a different purpose: in Ref. [[Von13](#)]. There, the idea is to use combinatorial methods (specifically, graph covering) to perform rigorous calculations. In contrast, this thesis aims to evaluate corrections to the Bethe–Peierls solution in order to make predictions for lattice models at the critical point (to estimate critical exponents). Although such problems are usually tackled using the methods described in [Chapter 3](#), it is interesting to develop this expansion around the Bethe solution for several reasons:

- As outlined in [Chapter 3](#), the machinery of the [RG](#) is applied to models defined by microscopic interactions through a Hamiltonian, whether in real or momentum space. Nevertheless, there exists an entire class of problems, characterized by

phase transitions, that are not defined directly in terms of interactions between microscopic components, such as all percolation-related models, making RG and field-theoretic tools ineffective. The coarse-grained Ginzburg-Landau action is not directly identified without the Hamiltonian. Then, the perturbative expansion for generic observables is not found, thus preventing the possibility of computing critical exponents. Through the Bethe M -layer construction, this perturbative expansion is possible for a generic model that can be defined on the Bethe lattice, even if the Hamiltonian is not directly defined [Riz19]. To be precise, it is possible to map bond percolation to a Potts model with q colors in the limit $q \rightarrow 1$ and thus make predictions on the critical behavior using RG, but it is generally not straightforward to implement this machinery.

- As already motivated by Efetov’s attempts [Efeg0], it would be valuable to construct a perturbative expansion around the Bethe–Peierls solution for Anderson localization. This problem requires the notion of the distance on the underlying lattice, a concept that is missing in FC networks, where a localization transition cannot be seen. It would be interesting to construct a perturbative expansion from the MF behavior on the Bethe lattice to understand what the finite-dimensional critical behavior is [Bar24].
- While the Ising model exhibits the same MF behavior whether defined on a FC graph or on a Bethe lattice, in general, there could exist problems whose behavior can be qualitatively different. This is the case for spin glass models in an external field at zero temperature: the SK model in a field predicts the divergence of the dAT line at $T = 0$, while on the Bethe lattice the VB model predicts a transition for a finite value of the external field even at zero temperature, as detailed in Chapter 2. If finite connectivity is assumed to be the relevant ingredient for studying finite-dimensional models, then an expansion around the Bethe–Peierls approximation on locally tree-like graphs, which incorporate finite connectivity, promises to yield better results than the techniques reviewed in Section 3.2 [Ang22].

A detailed explanation of the Bethe M -layer construction is provided here. The fundamental steps to build the perturbative expansion for the observables of a generic model are described, based on the seminal work of Ref. [Alt17], reviewed here as a background on which the original results of this thesis are built. In addition, some new concepts are presented, recently introduced in Ref. [Ang24], part of the original contributions of this work. Some recent applications of this construction will be discussed in the following chapters.

4.1 The Graph-Theoretical Loop Expansion

To show that the Bethe M -layer construction allows a systematic perturbative expansion for generic observables involving topological loops, one must first identify the perturbative parameter. Each term of the expansion then naturally takes the form of “diagrams”, resembling the standard Feynman diagram loop expansions introduced in Section 3.1. Within this framework, a set of “rules” can be established and applied to a wide range of models of interest.

4.1.1 The Perturbative Parameter

The goal of this section is to identify a parameter that can be tuned to take into account the presence of loops in the lattice for the computation of an observable in finite dimension D . More precisely, the leading order of the expansion should correspond to the observable computed using the Bethe–Peierls approximation. Corrections, which should become progressively smaller as the perturbative parameter decreases, arise from the presence of an increasing number of topological loops. These corrections should therefore be associated with higher powers of the perturbative parameter. An ideal perturbative parameter approaches zero when the lattice has no loops and increases as the probability of having loops in the lattice grows.

In order to be precise, consider as the starting lattice (in the following referred to as the *original lattice*) the hypercube in D dimensions, with N nodes, where $N \rightarrow \infty$. Here and in the following chapters $a_l \mathbb{Z}^D$ denotes the hypercube with *lattice spacing* a_l , which is the physical (dimensional) distance between nearest neighbors. The basic idea of the Bethe M -layer construction is to create M copies (called *layer*, from which the name of the method) of the original lattice and then create interlayer connections with the following random rewiring procedure. Each layer is identified by an index, $\alpha = 1, 2, \dots, M$, that is the subscript of the node, starting from the one corresponding to the original lattice, $\alpha = 1$. For instance, i_α refers to node i in layer α . Consider now one of the $2D$ neighbors of i_α in the same layer, say j_α . It is possible to identify two set of nodes, $\{i_\alpha\}_{\alpha=1,\dots,M}$ and $\{j_\alpha\}_{\alpha=1,\dots,M}$, connected by M edges (i_α, j_α) . The random rewiring procedure amounts to generate inter-layer connections creating a new set of M edges¹:

$$(i_\alpha, j_{\pi(\alpha)}) \quad \text{for } \alpha = 1, \dots, M \quad (4.1)$$

where $\pi(\alpha)$ is one of the possible $M!$ permutations of the set $(1, 2, \dots, M)$. Notice that, with this construction, the connectivity of each node is $2D$, unchanged. Repeating this random rewiring procedure for the remaining edges (each with the corresponding $M - 1$ copies) an instance of the so-called *M-layer lattice* is obtained. A pictorial representation of the construction is shown in [Figure 4.1](#). With simple combinatorial

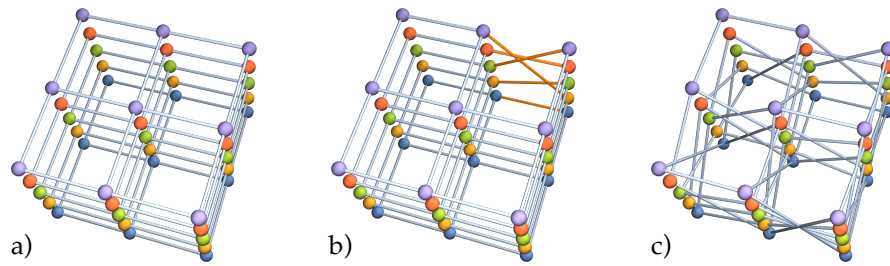


Figure 4.1. Pictorial representation of the three steps to obtain the M -layer construction in the case of a 3×3 square ($D = 2$) lattice as the original one and $M = 5$ layers: a) creation of M copies of the original lattice ; b) random rewiring of a set of M edges, colored in orange; c) rewirings of the remaining edges, the graph depicted is an instance of the M -layer graph. In general, N (here $N = 9$) and M should both be large, as explained below.

arguments, one can understand that the probability of closing a topological loop

¹Equivalently the new set of edges can be written as $(i_{\pi^{-1}(\alpha)}, j_\alpha)$, where $\pi^{-1}(\alpha)$ is the inverse of $\pi(\alpha)$, i.e. the choice of using α for node i and $\pi(\alpha)$ for node j is arbitrary.

decreases as M increases. Referring to Figure 4.1 for simplicity, consider an arbitrary node in an arbitrary layer, say the purple central one in the c) panel. The probability that a random rewiring creates a topological loop of length 4 including the selected node is $1/M$. Indeed, the first edge starting at the selected node can go to one of the 4 neighbors on an arbitrary layer, thus it is taken with probability 1. The same is true for the second and the third ones, starting from the second and from the third node of the cycle. Nevertheless, the last edge, starting from the fourth node, should pick exactly the selected first node, among the M possible layers, thus it is taken with probability $1/M$. The probability is independent of the length of the loop, as long as it is finite.

For this reason, it is clear that the perturbative parameter is exactly $1/M$: if $M \rightarrow \infty$ ($1/M \rightarrow 0$) a graph without loops is obtained (and possibly with disconnected components), while if $M = 1$ the original lattice is recovered. In the first limit, the Bethe–Peierls approximation is valid. In general $1/M$ interpolates between a loop-less graph and the original lattice, a hypercube in D dimensions in this context.

The next question is: how to use this construction to write the perturbative expansion for a generic observable on this M -layer lattice?

4.1.2 The Notion of “Diagram”

Since the whole rewiring procedure is random, the interesting quantity is the average value, over all possible rewiring, of the given observable. To be definite, consider a 2-point function. In all subsequent applications of the method only “connected” observables will be considered. If, in a given graph, 2 nodes are not connected in the graph sense (see Section 1.2.3), then the “connected” 2-point observable, computed on this graph gives zero. This notion can be generalized to a generic q -point observable, in this case, all q points must be connected in the graph sense².

Then, fix two nodes, i_α and j_β . The average value of the 2-point connected function between nodes i_α and j_β can be evaluated with the sum over all the rewirings instances of the value the 2-point function takes on a specific instance. Performing this sum is generically unfeasible. However, one can realise that, if M is large, the most probable instances are the ones that connect i_α and j_β with a single path, i.e., without loops (the instances in which the two nodes are not connected can be neglected since the observable will be zero). With a lower probability, in terms of $1/M$, the two nodes will be connected by a structure that includes one loop, two loops and so on. Three instances with an increasing number of loops are represented in Figure 4.2, where, in order to simply visually identify the path (or paths) connecting the two nodes, all the other edges of the M -layer lattice are removed. The set of paths connecting the q “external” nodes involved in the computation of the q -point function is called “diagrams”. In other words, a diagram is a portion of a Bethe lattice connecting the q nodes, in which, eventually, some loops of certain lengths are included. The diagrams is composed by different types of nodes, the *external vertices*, in Figure 4.2 the two orange ones; the *internal vertices* of a certain degree, in Figure 4.2 c) there are two internal vertices of degree 3 and there is also an internal vertex of degree 4; all the remaining nodes are “trivial” internal vertices of degree 2. One should imagine that all the nodes composing the given diagram, each with degree r , are attached to $2D - r$

²One can consider the 2-point connected correlation function for the Ising model as an example, but the following argument is generally valid for an arbitrary well-defined observable.

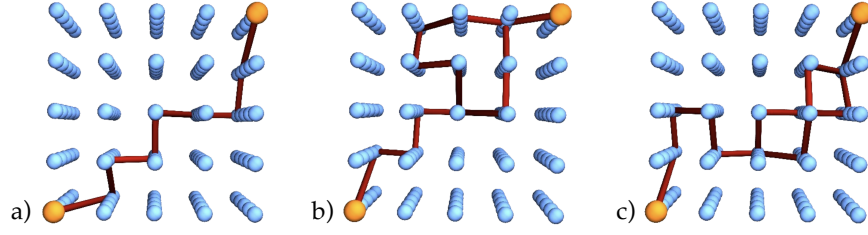


Figure 4.2. Instances of the random rewiring procedure in which all edges are pruned except for the ones belonging to the path that connects the two orange nodes: a) with probability $1/M$ the path is without loops; b) with less probability, $1/M^2$, the path includes a loop; c) with probability $1/M^3$ the path includes two loops. These powers of $1/M$ follow from the same combinatorial argument given to compute the probability to close a loop of length 4, explained in [Section 4.1.1](#).

infinite tree-like graphs, in addition to the r edges on the diagram itself. For this reason, diagrams are not represented with dangling edges (i.e., links that end in a degree-1 node that is not an external vertex), as it would be redundant, given that a tree is always considered to be attached. Some examples of such diagrams are reported in [Figure 4.3](#).

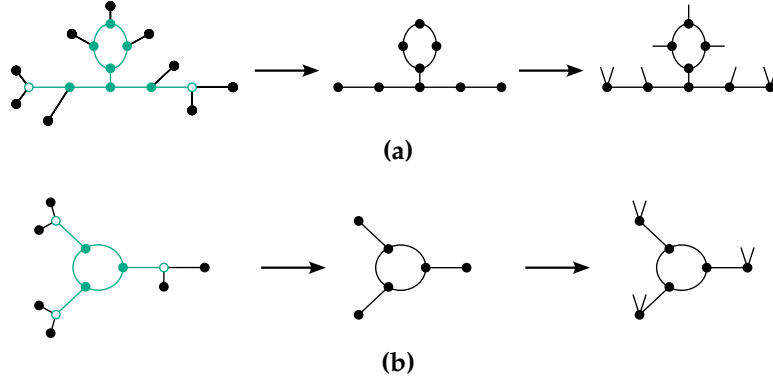


Figure 4.3. The figures in the center show two examples of diagrams, starting from a generic graph, on the left, while on the right there is the result of the complete construction to evaluate the corresponding observable, where the lines added represent infinite trees. White-filled points represent the external vertices, while the ones highlighted in green represent the portion of the graph, including loops, that connects them. All the graphs on the right have connectivity 3. Panel (a) displays the construction of a diagram for a 2-point observable. The result is a connected diagram, i.e., the two external vertices are connected by at least a linear path. In panel (b) there is an example of the procedure for a diagram that could contribute to a 3-point function.

It is also possible to identify *lines*, i.e. (loop-less) paths as defined in [Section 1.2.3](#), in the given diagram: in [Figure 4.2 a\)](#) a single line of length 8 is present; in b) the diagram is composed by 4 lines, two composing the loop, with lengths 5 and 3 and two connecting the loop to the external vertices, with lengths 4 and 1; similarly, in c), 4 lines of lengths 2 compose the loops and two lines, with lengths 5 and 1 connect them to the external nodes.

This notion of “diagram” is introduced in Ref. [\[Alt17\]](#), where it is called *fat diagram*, a name that first appeared in Ref. [\[Par12\]](#).

How to Average over Rewirings

With the identification of diagrams one realizes that the sum over all realizations of the rewirings, to compute the average value of the observable, is the sum over all possible diagrams connecting the external vertices. Moreover, noticing that the value the observable takes on the given diagram depends only on the lengths of the lines that compose it, the same value is taken for all the instances that share the same projection on the original lattice. This last statement is easier to understand (and to visualize) if the original lattice is a $2D$ square grid, but it is valid for generic D . Notice that a loop-less diagram that coils itself in the M -layer lattice can create a projection on the original lattice that is a diagram with a loop. This kind of diagram is called “boucle” [Ang24], but the discussion on them is postponed to Section 4.1.5, while for the moment they are ignored.

Thanks to this projection feature, the observable, averaged over all possible rewirings, is computed as the sum over all possible projected diagrams times the value the observable takes on that specific diagram, weighted by the fraction of the rewirings that realize that specific projection. For a generic q -point observable, $O_q(x_1, \dots, x_q)$, that depends on the position, on the original lattice, of the q external points: $x_1, x_2, \dots, x_q (\in a_l \mathbb{Z}^D$ if the D -dimensional hypercube is considered with lattice spacing a_l) one can write

$$\overline{O_q(x_1, \dots, x_q)}^{\text{rew}} = \sum_{\mathcal{G} \subseteq \mathcal{G}} \mathcal{P}(\mathcal{G}) \mathcal{O}_{q,\text{lc}}(\mathcal{G}), \quad (4.2)$$

where:

- $\mathcal{G}$ is each projected diagram;
- $\overline{\cdot}^{\text{rew}}$ denotes the average over the rewirings;
- $\mathcal{P}(\mathcal{G})$ is the fraction of rewirings that realize $\mathcal{G}$;
- $\mathcal{G}$ is the set of all projected diagrams;
- $\mathcal{O}_{q,\text{lc}}(\mathcal{G})$ is the value the observable O_q takes on the specific diagram $\mathcal{G}$ and the subscript “lc” stands for “line-connected”, a definition that is useful to avoid double counting of some of the contributions. For instance, the diagram in Figure 4.2 b) includes the contribution of the loop-less one in a) and to avoid this double counting the line-connected contribution is used.

4.1.3 Line-connected Observables

A *line-connected* observable is defined through the following implicit formula

$$\mathcal{O}(\mathcal{G}) = \sum_{\mathcal{G}' \subseteq \mathcal{G}} \mathcal{O}_{\text{lc}}(\mathcal{G}'), \quad (4.3)$$

where $\mathcal{G}'$ is a generic subdiagram of $\mathcal{G}$, that is a diagram where zero, one or more lines are removed from $\mathcal{G}$. The result of the computation of the observable on the diagram is $\mathcal{O}(\mathcal{G})$, thus the goal is to invert Eq. 4.3 to express $\mathcal{O}_{\text{lc}}(\mathcal{G})$ in terms of what one can compute: $\mathcal{O}(\mathcal{G})$. To do so, it is sufficient to write the same equation but considering subdiagrams of $\mathcal{G}$. Consider, for instance, the case

$$\mathcal{G} : \bullet \text{---} \bigcirc \text{---} \bullet \quad (4.4)$$

Eq. 4.3 for $\mathcal{O}(\mathcal{G})$ reads

$$\mathcal{O}(\text{diagram}) = \mathcal{O}_{\text{lc}}(\text{diagram}) + \mathcal{O}_{\text{lc}}(\text{diagram}) + \mathcal{O}_{\text{lc}}(\text{diagram}), \quad (4.5)$$

and all the other subdiagrams are neglected since, in this case, observables that are non-zero when computed on disconnected diagrams are not considered. The general definition, including disconnected diagrams, is extensively treated in Ref. [Alt17]. Then, Eq. 4.3 can be written for the subdiagrams, trivially:

$$\mathcal{O}(\text{diagram}) = \mathcal{O}_{\text{lc}}(\text{diagram}), \quad (4.6)$$

and

$$\mathcal{O}(\text{diagram}) = \mathcal{O}_{\text{lc}}(\text{diagram}). \quad (4.7)$$

Substituting these results in Eq. 4.5, and solving for $\mathcal{O}_{\text{lc}}(\mathcal{G})$:

$$\mathcal{O}_{\text{lc}}(\text{diagram}) = \mathcal{O}(\text{diagram}) - \mathcal{O}(\text{diagram}) - \mathcal{O}(\text{diagram}). \quad (4.8)$$

Similarly, the same computations can be done for the following one-loop 3-point diagram, widely used in the applications of the M -layer construction presented below

$$\mathcal{O}_{\text{lc}}(\text{diagram}) = \mathcal{O}(\text{diagram}) - \mathcal{O}(\text{diagram}) - \mathcal{O}(\text{diagram}) - \mathcal{O}(\text{diagram}). \quad (4.9)$$

The reason for introducing line-connected observables is now clear: Eq. 4.5 shows that the value $\mathcal{O}$ takes on the one-loop diagram is given also by tree-like contributions, which are already included in the tree-like term, so they must be subtracted from the one-loop term, as in Eq. 4.8.

4.1.4 Counting Paths

Even if the average over rewirings is, thanks to Eq. 4.2, more organised, it is still unfeasible. Nevertheless, it can be further simplified by grouping diagrams into topological classes. To be definite, consider diagrams with two external vertices. The first class of diagrams is the simple line, of length L , without loops, that is called $\mathcal{G}_1^3$, represented in Figure 4.4 a). To sum all the contributions due to this topology it is sufficient to sum over the length L :

$$\overline{\mathcal{O}_2(x_1, x_2)}^{\text{rew}} \Big|_{\mathcal{G}_1} = W(\mathcal{G}_1) \sum_{L=1}^{\infty} \mathcal{N}_L(x_1, x_2) \mathcal{O}_{2,\text{lc}}(\mathcal{G}_1; L), \quad (4.10)$$

where

- $W(\mathcal{G}_1)$ is the so-called *weight* associated to $\mathcal{G}_1$, that is the fraction of rewirings that realize the class of diagrams $\mathcal{G}_1$ on the original lattice⁴. $W(\mathcal{G}_1)$ only depends

³With a little abuse of notation, from now on, classes of diagrams are called and denoted with the same symbols as for single diagram

⁴The weight should not be confused with $\mathcal{P}(\mathcal{G})$ in Eq. 4.2, that is the fraction of rewirings that realize a specific diagram $\mathcal{G}$.

on M (not on the length) and for the simple line it results $W(\mathcal{G}_1) = 1/M$. Indeed, starting from x_1 the first $L - 1$ edges can be taken with probability 1 since every layer they are connected to is fine (they share the same projection). The last edge, instead, must be connected to the specific layer in which the external node, then projected on x_2 , is originally chosen. Thus, it contributes with $1/M$ and the overall result is $1/M$.

- $\mathcal{N}_L(x_1, x_2)$ is the number of all possible paths of length L between x_1 and x_2 on the original lattice. Due to the M -layer construction described above, $\mathcal{N}_L(x_1, x_2)$ corresponds to the number of **Non-Backtracking Paths (NBP)**, where “Non-Backtracking” means that three consecutive vertices must be pairwise different, that is simply due to the fact that the connectivity of each node is not altered by the random rewiring [Alt17]. Interestingly, one can compute the number of **NBP** with a given length between two points in $a_l \mathbb{Z}^D$ and the asymptotic behavior for large length is related to the number of **Random Walks (RW)** [Fit13].
- $\mathcal{O}_{2,\text{lc}}(\mathcal{G}_1; L)$ is again the line-connected observable computed on the specific topology $\mathcal{G}_1$, where the dependence of the length has been made explicit. To compute observables on given diagrams the Cavity method is employed.

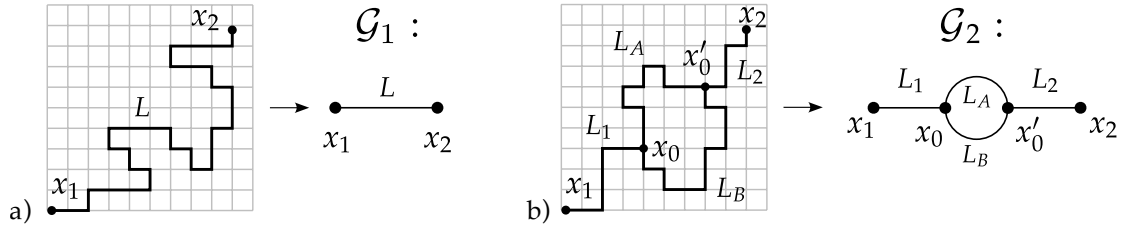


Figure 4.4. Example of **NBP** in $\mathbb{Z}^2$, with the associated representation of the corresponding diagrams: a) loop-less path; b) a one-loop diagram contributing to a 2-point observable.

Notice that, in principle, it is possible to compute the number of **NBP** of a certain length between two points fixing the starting and ending direction of the path [Fit13]. Notice also that $\mathcal{N}_L(x_1, x_2)$, as defined above, counts the number of all possible paths with all possible starting and ending directions. On the other hand, the goal of the Bethe M -layer construction is to compute perturbative expansions for observables to study the critical behavior of lattice systems. At the critical point, the most relevant contributions to the observables are the ones that come from diagrams with large-length lines. One expects that for large lengths the number of **NBP** is not dependent on the specific starting and ending directions of the path at the two points. Thus, to compute the number of **NBP** of (large) length L between two points x and y in $a_l \mathbb{Z}^D$ with fixed starting and ending directions one can use

$$\frac{\mathcal{N}_L(x, y)}{(2D)^2}, \quad (4.11)$$

one factor $2D$ accounts for all the possible directions on x and one for y .

Another possible contribution to the 2-point observable is given by diagrams that include one loop. For instance, consider the one depicted in Eq. 4.4 and in Figure 4.4 b), that is the same kind as the one in Figure 4.2 b) and will be referred to as $\mathcal{G}_2$.

Here the contributions can be taken into account summing over the four lengths that characterize the diagram, the set $\vec{L} = (L_1, L_2, L_A, L_B)$. Similarly

$$\overline{\mathcal{O}_2(x_1, x_2)}^{\text{rew}} \Big|_{\mathcal{G}_2} = \frac{W(\mathcal{G}_2)}{S(\mathcal{G}_2)} \sum_{\vec{L}} \mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) \mathcal{O}_{2,lc}(\mathcal{G}_2; \vec{L}), \quad (4.12)$$

where in this case

- $W(\mathcal{G}_2) = 1/M^2$, as can be computed similarly to $W(\mathcal{G}_1) = 1/M^1$, remembering that the internal and external vertices are originally fixed on a specific layer.
- $\mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2)$ is again the number of **NBP** that realizes $\mathcal{G}_2$ with given lengths $\vec{L}$. In this case, the number is given by the following combination of the four lines

$$\begin{aligned} \mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) &= \\ &= (2D)^2 \left(\frac{(2D)!}{(2D-3)!} \right)^2 \sum_{x_0, x'_0} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x'_0, x_2)}{(2D)^2}, \end{aligned} \quad (4.13)$$

where the fractions $\mathcal{N}_L(x, y)/(2D)^2$ are used to fix the starting and ending directions of each line. Moreover, each internal vertex of degree 3 contributes with a factor $(2D)!/(2D-3)!$ (while it is $(2D)!/(2D-k)!$ for a generic degree- k vertex). Each line can enter a vertex in different directions, $2D$ on a D -dimensional lattice. The factor arises from the assumption that in the large-lengths limit the number of **NBP** doesn't depend on the specific starting/ending directions at the vertices. $(2D)!/(2D-k)!$ is indeed the number of ordered k -plets out of the $2D$ possible directions [Alt17]. Since the contribution of each line is with fixed directions, a factor $2D$ should be multiplied for each external vertex, to count the correct number of paths with different directions.

Summing over all possible directions in each internal vertex, one is now overcounting the possible distinct realizations of a diagram. For this reason, the so-called *symmetry factor*, $S(\mathcal{G})$, should be included for a generic diagram $\mathcal{G}$. The overcounting for $\mathcal{G}_2$ is due to the equivalence between the two lines of lengths L_A and L_B , thus

$$S(\mathcal{G}_2) = 2. \quad (4.14)$$

Interestingly, these symmetry factors precisely correspond to the factors that in Field Theory arise due to Wick's Theorem [Alt17]. Finally, the position (projected) of the internal vertices, x_0 and x'_0 must be summed to get the number of all possible **NBP** from x_1 to x_2 with the topology of $\mathcal{G}_2$.

Notice that the symmetry factor has not been considered for the loop-less diagram $\mathcal{G}_1$, since in that case it is trivial: $S(\mathcal{G}_1) = 1$.

- $\mathcal{O}_{2,lc}(\mathcal{G}_2; \vec{L})$ is the line-connected observable computed on $\mathcal{G}_2$ with lengths $\vec{L}$.

Always keeping in mind that in the large-lengths limit, the directions of the lines don't affect the number of **NBP**, similar expressions can be found for the contribution to a generic q -point observable due to a generic class of diagrams. Indeed, the three

ingredients, $W(\mathcal{G})$, $\mathcal{N}(\mathcal{G}; \vec{L}; x_1, \dots, x_q)$ and $\mathcal{O}_{q,lc}(\mathcal{G}; \vec{L})$ can be computed case by case. The big advantage of this idea to compute the rewiring average is that the weights, $W(\mathcal{G})$, in powers of $1/M$, allow one to realise the initial goal: compute systematically the perturbative expansion. Indeed, while in Eq. 4.2, in principle, all contributions seem to be important, organising the sum in different topological diagrams allows one to identify the most important contributions, due to loop-less diagrams, and, if possible, add corrections that are due to the effect of topological loops. This last statement is valid in the large M regime, while if $M = 1$ all different topological diagrams contributions have the same weight and therefore should be equally considered.

Interestingly, if $M \neq 1$ and it is large but not infinite, universality arguments show that the critical quantities computed on the M -layer graph coincide with those of the original D -dimensional lattice. This crucial observation enables the use of the M -layer construction to make predictions about universal quantities in finite-dimensional lattice models.

4.1.5 Boucle-Type Diagrams

Possible issues may arise in performing the rewiring average projecting diagrams. As already mentioned in Section 4.1.2, one-loop projected diagrams may arise from loop-less lines in the M -layer lattice, the so-called “boucle”. An instance, associated to a one-loop 2-point diagram, is depicted in Figure 4.5. This *boucle* diagram is hereafter referred to as $\mathcal{G}'_3$, while the associated “real” loop diagram (not only in the projection but also in the M -layer graph) is $\mathcal{G}_3$. Performing the average over rewirings summing

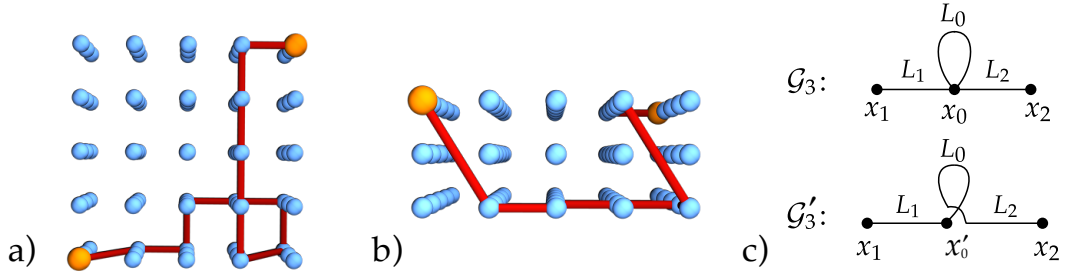


Figure 4.5. An example of *boucle* diagram: a) top view of the *boucle* diagram; b) different perspective for the same diagram: even if the projected diagram has a loop, the diagram in the M -layer graph is a loop-less line; c) representation of the two possible diagrams emerging from the one-loop projection of a).

contributions like the ones in Eq. 4.10 and Eq. 4.12, *boucle* diagrams can be taken into account, neglecting higher $1/M$ terms, as follows.

Consider $\mathcal{G}'_3$, this argument can then be generalized to a generic *boucle* diagram for a generic q -point function. Counting the number of NBP in the loop-less contribution, *boucle* diagrams such as $\mathcal{G}'_3$ are correctly considered, making higher order errors (at $\mathcal{O}(1/M^2)$ for the 2-point function). Indeed, the NBP are RW (with the constraint that three consecutive nodes must be pairwise distinct) and as such they can coil themselves creating loops in the projection. Then, in Eq. 4.10, the number of such paths, $\mathcal{N}_L(x_1, x_2)$, is multiplied by the value the observable takes on a diagram that is a loop-less line of length L , as it should be for a diagram like the one depicted in Figure 4.5. The problem is that the probability to realize the *boucle* projection is not

simply $1/M$, as in Eq. 4.10, but

$$\frac{1}{M} \left(1 - \frac{1}{M} \right), \quad (4.15)$$

where the factor $1 - 1/M$ is due to the fact that the edge that returns to the vertex of the intersection must avoid the layer previously chosen, thus can choose between $M - 1$ layers. This means that, neglecting $\mathcal{O}(1/M^2)$ the loop-less computation is correct. However, retaining only the leading order contributions corresponds to computing observables in the Gaussian approximation (see Section 3.1.2) as will be explained in the next chapters. Since the ultimate goal of the whole construction is to make predictions on the finite-dimensional critical behavior, corrections to the Gaussian approximation are needed.

While the $1/M$ contribution in Eq. 4.15 is already included in the loop-less contribution due to $\mathcal{G}_1$, the error made to $\mathcal{O}(1/M^2)$, due to *boucle* diagrams, can be identified by looking at the one-loop contribution due to $\mathcal{G}_3$

$$\overline{\mathcal{O}_2(x_1, x_2)}^{\text{rew}} \Big|_{\mathcal{G}_3} = \frac{W(\mathcal{G}_3)}{S(\mathcal{G}_3)} \sum_{\vec{L}'} \mathcal{N}(\mathcal{G}_3; \vec{L}'; x_1, x_2) \mathcal{O}_{2,\text{lc}}(\mathcal{G}_3; \vec{L}'), \quad (4.16)$$

where $\vec{L}' = (L_1, L_2, L_0)$, $W(\mathcal{G}_3) = 1/M^2$, then

$$\mathcal{N}(\mathcal{G}_3; \vec{L}'; x_1, x_2) = (2D)^2 \left(\frac{(2D)!}{(2D-4)!} \right) \sum_{x_0 \in a_l \mathbb{Z}^D} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_0}(x_0, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x_0, x_2)}{(2D)^2}, \quad (4.17)$$

and $S(\mathcal{G}_3) = 2$. So the number $\mathcal{N}(\mathcal{G}_3; \vec{L}', x_1, x_2)$ really counts all the **NBP** with the loop structure of the projection $\mathcal{G}_3$, that is the same as the one of $\mathcal{G}'_3$. The difference is that the observable $\mathcal{O}_{2,\text{lc}}(\mathcal{G}_3; \vec{L}')$ is computed on a diagram that has a real topological loop, while to consider $\mathcal{G}'_3$ the right observable should be

$$\mathcal{O}_{2,\text{lc}}(\mathcal{G}_1; L_1 + L_0 + L_2), \quad (4.18)$$

that is a line with length $L_1 + L_0 + L_2$, since the loop is not present in the M -layer graph. Thus, to correct the computation at $\mathcal{O}(1/M^2)$ due to the presence of $\mathcal{G}'_3$, it is sufficient to add the contribution to the *boucle* diagram

$$\overline{\mathcal{O}_2(x_1, x_2)}^{\text{rew}} \Big|_{\mathcal{G}'_3} = \frac{W(\mathcal{G}'_3)}{S(\mathcal{G}'_3)} \sum_{\vec{L}'} \mathcal{N}(\mathcal{G}_3; \vec{L}', x_1, x_2) \mathcal{O}_{2,\text{lc}}(\mathcal{G}_1; L_1 + L_0 + L_2), \quad (4.19)$$

where $W(\mathcal{G}'_3) = -1/M^2$ and $S(\mathcal{G}'_3) = 1$. The weight is simply the $\mathcal{O}(1/M^2)$ term in Eq. 4.15 (remember that the $\mathcal{O}(1/M)$ one is already included in the contribution of $\mathcal{G}_1$). The symmetry factor can be understood since the goal is to count all the possible paths that coil to realize a projection with (at least) one loop. Thus, exchanging the two directions of the lines of the loop entering the internal vertex gives two different possibilities one should take into account, at variance with the “real” loop case.

In general, to correctly compute the contributions to $\mathcal{O}(1/M^2)$ to the 2-point function, all relevant one-loop diagrams should be considered and corrections, with

weights $-1/M^2$ have to be added. The same is true for a generic q -point function, with higher orders of $1/M$. Notice here that such corrections are associated with even-degree internal vertices only. One important model that needs a careful treatment of these *boucle* diagrams is the Ising model, which is discussed in [Chapter 5](#) from [\[Ang24\]](#). Indeed, imagine to create a projection like $\mathcal{G}_2$ with a loop-less path in the M -layer graph. In this case, more lines than the four composing $\mathcal{G}_2$ are needed. In general, it can be argued that diagrams with more lines are less relevant to the critical behavior of the system, as is more precisely discussed in [Part III](#). Thus *boucle* diagrams can be neglected when associated with odd-degree internal vertices. Before the work presented in [Chapter 5](#), all applications of the M -layer method included the computations of diagrams with degree-3 internal vertices only.

Notice that, adding the *boucle* contributions as described here, errors to $\mathcal{O}(1/M^3)$ are made and should be considered if more corrections are needed. No such additional corrections are taken into account in all applications of the M -layer construction made so far, including those presented in [Part III](#).

4.2 Generic Rules to Compute Observables

As a recap: all the rules to systematically compute the perturbative expansion, up to one-loop order for a generic observable, are listed here [\[Ang24\]](#)

- Identify the possible diagrams (classes of projections on the original lattice) that contribute to the observable;
- For each diagram $\mathcal{G}$, compute its contribution as follows:
 - Compute the corresponding weight $W(\mathcal{G})$;
 - Multiply the preceding term by a factor $\mathcal{N}_L(x, y)/(2D)^2$ for each line, with length L , composing the diagram, starting at x and ending at y ; multiply by the combinatorial factors that emerge summing over the possible directions of the lines: $(2D)!/(2D - k)!$ for each internal vertex of degree k and $2D$ for each external vertex; multiply by the corresponding symmetry factor $S(\mathcal{G})$;
 - Multiply the precedent term by the value of the line-connected observable on a Bethe lattice in which the identified diagram has been manually injected, computed using the [RS](#) Cavity method;
 - Sum over the positions of the internal vertices and over the lengths of the lines;
- Sum the contributions coming from all diagrams, including *boucle*-type diagrams if even-degree internal vertices are involved, with the corresponding weight $-1/M^2$ at one-loop order.

To extend the computation to higher orders in the number of loops, one should carefully treat *boucle* diagrams, if present, but a complete treatment has not been formulated yet. No application of the M -layer construction has involved more than one-loop computations so far.

4.3 Final Remarks

Before moving to the applications, some remarks are necessary. Notice that the first two ingredients, $W(\mathcal{G})$ and $\mathcal{N}(\mathcal{G}; \vec{L}; x_1, \dots, x_q)$, are not model-dependent, they are the same independently on the characterization of the observable: given a class of diagram, the weight should be computed once for all different applications of the Bethe M -layer construction. The number of **NBP** depends on the geometry of the original lattice, where diagrams are projected and paths are counted. The computation of this number, in the large-lengths regime, is accessible for most of the interesting lattices.

The last ingredient, and the only model-dependent one, is the observable on the Bethe lattice, possibly with topological loop structures. The main difficulty of the M -layer construction lies in this computation: one must choose a specific model definable on the Bethe lattice, identify a well-defined observable, and compute it to make predictions for the finite-dimensional critical behavior, i.e., estimates of critical exponents. Once the perturbative calculations for the relevant observables are completed, they can be written in a form analogous to that of the 2- and 4-point functions of the Ising model in Eq. 3.48 and Eq. 3.49, as explained in Section 3.1.2. From these expressions, one can then apply the standard methods described in Section 3.1.2 to estimate critical exponents. This procedure will be illustrated in Chapter 5, Chapter 6, and Chapter 7, which contain the main results of this thesis.

One of the advantages of this construction is that once this whole procedure is done for a model in a given original lattice, similar results can be obtained readily for the same model on a different original lattice, simply changing the factors $\mathcal{N}_L(x, y)$ with the corresponding number of the lattice of interest. This is the case, for example, for the **EA** model defined on a one-dimensional **Long-Range (LR)** lattice, where the asymptotics of the number of **NBP** scales differently with respect to the case of the hypercubic lattice, as detailed in Section 7.5.

A last important remark: the random rewiring is a random procedure. One could think that the “disorder”, introduced by the construction, is not relevant. If one wants to analyse the critical behavior of the Ising model (a problem without any kind of disorder) using the Bethe M -layer construction, the hope is that the results are not affected by the randomness of the rewiring. This is what is concluded in [Ang24], where the construction is applied to the finite-dimensional Ising model, computing connected correlation functions up to one-loop (topological) order, obtaining the exact same results as standard Field Theory, in Eq. 3.48 and Eq. 3.49. It could be that adding more loop corrections affects the standard results. However, it is worth noting that the Bethe M -layer construction was originally conceived to address challenging quenched disordered problems, such as the **EA** model in a field or the **RFIM**, where the randomness of the rewiring is expected to play a minor role.

Part III

Applications of the *M*-layer Construction

Chapter 5

The Ising Model: A Benchmark Case

The first application of the M -layer construction, presented in this thesis, is on the Ising model in a hypercubic D -dimensional lattice. What is explained in the next sections is based on the generalities of the construction, presented in [Chapter 4](#). All the material of this chapter is based on [\[Ang24\]](#).

The main results of this application is to recover the results of the continuum quartic Field Theory presented in [Section 3.1.2](#) for the Ising model: the 2- and 4-point connected correlation functions are computed up to one-loop topological order and result in the same perturbative expansions expressed in [Eq. 3.48](#) and [Eq. 3.49](#). Moreover, the coupling constant and the other parameters of the theory are identified, in the M -layer framework, as functions of macroscopic and microscopic details of the model such as the lattice spacing, the dimension and the temperature.

After the definition of the model, in [Section 5.1](#), the basic tools to perform the following computations are presented through the Cavity method in [Section 5.2.1](#). These tools are used to locate the MF critical temperature for the case of zero external field. The application of the M -layer construction to evaluate the perturbative series for the 2- and 4-point connected correlation functions is detailed in [Section 5.2.2](#) and [Section 5.2.3](#). In [Section 5.3](#), field-theoretic parameters are defined as combinations of microscopic quantities to relate the results with the ϕ^4 theory. [Section 5.4](#) reports a review and recap of the main results of the chapter. Finally, as appendices, the soft spin variables case is treated in [Section 5.A.1](#), while a general discussion on the degree of the interaction in the M -layer construction is left for [Section 5.A.2](#).

5.1 Model's Definition

The model is defined on the D -dimensional cubic lattice, with $D > 1$, by the Hamiltonian of [Eq. 1.8](#) for $H = 0$, that is reported here:

$$\mathcal{H}_I(\{s_i\}_i) = -J \sum_{(i,j)} s_i s_j, \quad (5.1)$$

where $J > 0$ and the sum runs over the edges of the lattice. Here, consider N spin variables, $s_i = \pm 1$ $i = 1, \dots, N$. The sites are identified by the positions $\mathbf{x}_i \in a_l \mathbb{Z}^D$, where $a_l \mathbb{Z}^D$ denotes the D -dimensional hypercubic lattice with *lattice spacing* a_l , as defined in [Chapter 4](#). In order to make a connection with the Ising model Field Theory:

here the field corresponds to the spin values in the lattice $s : a_l \mathbb{Z}^D \mapsto \mathbb{R}$, in such a way that $s(\mathbf{x}_i)$ stands for s_i , being $\mathbf{x}_i$ the position on the lattice of the i -th spin.

5.2 Computation of Correlation Functions

Before going to the M -layer computations of the 2- and 4-point connected correlation functions, the basic building blocks are obtained through the Cavity method, using a “Transfer Matrix” formalism. The model is analysed in the [PM](#) phase, thus the [RS](#) version of the Cavity method is employed. Corrections to this [MF](#) result are considered below, in [Section 5.2.2](#) and in [Section 5.2.3](#), where the M -layer construction is applied.

5.2.1 Ising on the Bethe Lattice: Transfer Matrix through Cavity Method

The Cavity method can be applied to the Ising model on the Bethe lattice, as explained in [Section 2.2.1](#). It is possible to locate the transition point as a function of the temperature and, in general, it is possible to express the free energy density to get all the [MF](#) critical exponents. On the other hand, the goal here is to derive the expression for the (unnormalized) joint probability distribution of two spin variables for the Ising model, in the absence of an external field on a Bethe lattice. Consider N Ising variables $s = \pm 1$, so that on each spin the measure is

$$d\mu(s) = \delta(s^2 - 1)ds. \quad (5.2)$$

For definiteness, a [RRG](#) of connectivity $z > 2$ is considered. The cavity equation reads:

$$\hat{\eta}(s) = \frac{1}{Z_{\hat{\eta}}} \sum_{t=\pm 1} e^{\beta J s t} \hat{\eta}(t)^{z-1}, \quad (5.3)$$

with

$$Z_{\hat{\eta}} = \sum_{s,t=\pm 1} e^{\beta J s t} \hat{\eta}(t)^{z-1}. \quad (5.4)$$

The function $\hat{\eta}(s)$ is related to [Eq. 2.58](#) and [Eq. 2.60](#) considering the case of a homogeneous system without an external magnetic field, such that these equations are

$$\eta(s) = \frac{1}{z_{\eta}} \left(\sum_{t=\pm 1} e^{\beta J s t} \eta(t) \right)^{z-1}, \quad (5.5)$$

and

$$z_{\eta} = \sum_{s=\pm 1} \left(\sum_{t=\pm 1} e^{\beta J s t} \eta(t) \right)^{z-1}. \quad (5.6)$$

Then, the cavity marginal $\hat{\eta}(s)$ is defined as

$$\hat{\eta}(s) \equiv \frac{1}{\hat{Z}} \sum_{t=\pm 1} e^{\beta J s t} \eta(t), \quad (5.7)$$

such that [Eq. 5.3](#) is obtained with $Z_{\hat{\eta}} = \hat{Z}^{2-z} z_{\eta}$. The full marginal, denoted as $P_{\text{Bethe}}(s)$, is, up to a normalization constant:

$$P_{\text{Bethe}}(s) \propto \hat{\eta}(s)^z. \quad (5.8)$$

Since $\hat{\eta}(s)$ is connected with the probability of the configuration of the spin s , it is expected to be an even function for the zero external field case (in the [PM](#) phase), allowing easier computations:

$$\hat{\eta}(s) = \frac{e^{\beta J s} \hat{\eta}(1)^{z-1} + e^{-\beta J s} \hat{\eta}(-1)^{z-1}}{\sum_{s=\pm 1} (e^{\beta J s} \hat{\eta}(1)^{z-1} + e^{-\beta J s} \hat{\eta}(-1)^{z-1})} = \frac{\cosh(\beta J)}{2 \cosh(\beta J)} = \begin{cases} \frac{1}{2} & \text{if } s = 1 \\ \frac{1}{2} & \text{if } s = -1 \end{cases}, \quad (5.9)$$

where the second equality is justified by the fact that $\hat{\eta}(+1) = \hat{\eta}(-1)$.

The transfer matrix is defined as

$$T(s, t) = \sqrt{\hat{\eta}(s)^{z-2}} e^{\beta J s t} \sqrt{\hat{\eta}(t)^{z-2}} = \frac{1}{2^{z-2}} \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \quad (5.10)$$

with eigenvalues and corresponding normalized eigenvectors:

$$\lambda_{\max} = \frac{\cosh(\beta J)}{2^{z-3}} \longrightarrow \vec{e}_{\lambda_{\max}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad (5.11)$$

$$\lambda_- = \frac{\sinh(\beta J)}{2^{z-3}} \longrightarrow \vec{e}_{\lambda_-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (5.12)$$

Using the decomposition of symmetric matrices, the L -th power of $T(s, t)$, scaled by $\lambda_{\max}^L$, can be written as:

$$\frac{T^L(s, t)}{\lambda_{\max}^L} = \vec{e}_{\lambda_{\max}}(s) \vec{e}_{\lambda_{\max}}(t) + \lambda_-^L \vec{e}_{\lambda_-}(s) \vec{e}_{\lambda_-}(t), \quad (5.13)$$

where $\lambda \equiv \lambda_- / \lambda_{\max} = \tanh(\beta J)$. From [Eq. 5.3](#) and [Eq. 5.10](#) it follows that $\vec{e}_{\lambda_{\max}}(s) \propto \sqrt{\hat{\eta}(s)^z}$:

$$\hat{\eta}(s) \propto \frac{1}{\sqrt{\hat{\eta}(s)^{z-2}}} \sum_{t=\pm 1} T(s, t) \frac{1}{\sqrt{\hat{\eta}(t)^{z-2}}} \hat{\eta}(t)^{z-1}, \quad (5.14)$$

that is,

$$\sqrt{\hat{\eta}(s)^z} \propto \sum_{t=\pm 1} T(s, t) \sqrt{\hat{\eta}(t)^z}. \quad (5.15)$$

Moreover, since $\vec{e}_{\lambda_{\max}}$ has norm 1:

$$\vec{e}_{\lambda_{\max}} = \frac{\sqrt{\hat{\eta}(s)^z}}{\sqrt{\sum_{s'} \hat{\eta}(s')^z}}. \quad (5.16)$$

At this point, the (scaled) partition function $Z_L(s, t)$ can be defined. Consider an open chain with fixed boundary spins s and t , such that the connectivity of the central spins is z , while the connectivity of s and t is 1. The partition function, given the values of s and t is defined as:

$$Z_L(s, t) \equiv \frac{T^L(s, t) / \lambda_{\max}^L}{\sqrt{\hat{\eta}(s)^{z-2}} \sqrt{\hat{\eta}(t)^{z-2}}} = \frac{\vec{e}_{\lambda_{\max}}(s) \vec{e}_{\lambda_{\max}}(t)}{\sqrt{\hat{\eta}(s)^{z-2}} \sqrt{\hat{\eta}(t)^{z-2}}} + \lambda_-^L \frac{\vec{e}_{\lambda_-}(s) \vec{e}_{\lambda_-}(t)}{\sqrt{\hat{\eta}(s)^{z-2}} \sqrt{\hat{\eta}(t)^{z-2}}}, \quad (5.17)$$

that is,

$$Z_L(s, t) = \frac{\hat{\eta}(s)}{\sqrt{\sum_{s'} \hat{\eta}(s')^z}} \frac{\hat{\eta}(t)}{\sqrt{\sum_{t'} \hat{\eta}(t')^z}} + \lambda_-^L \frac{\vec{e}_{\lambda_-}(s) \vec{e}_{\lambda_-}(t)}{\sqrt{\hat{\eta}(s)^{z-2}} \sqrt{\hat{\eta}(t)^{z-2}}}. \quad (5.18)$$

Note that as $L \rightarrow \infty$ the marginal probability between the two spins factorizes because the second term on the [RHS](#) tends to zero, given $\lambda < 1$. In this case, $\hat{\eta}(s)$ must be the marginal probability distribution of the spin s . This is indeed the case (in the analyzed chain, the connectivity of the boundary spins is 1).

The previous quantity is compactly rewritten for the Ising model (without external field) as:

$$Z_L(s, t) = p_B(s)p_B(t) + \lambda^L g_\lambda(s) g_\lambda(t), \quad (5.19)$$

where $p_B(s)$ and $g_\lambda(s)$ are defined as:

$$p_B(s) \equiv \frac{\hat{\eta}(s)}{\sqrt{\sum_{s'} \hat{\eta}(s')^z}} \quad \text{and} \quad g_\lambda(s) \equiv \frac{\vec{e}_{\lambda_-}(s)}{\sqrt{\hat{\eta}(s)^{z-2}}}, \quad (5.20)$$

and from [Eq. 5.9](#):

$$p_B(s) = 2^{\frac{z-3}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad g_\lambda(s) = 2^{\frac{z-3}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (5.21)$$

As a remark, it is noticed that, while $p_B(s)$ is even with respect to its argument, $g_\lambda(s)$ is odd. This is an important consequence of the $\mathbb{Z}_2$ symmetry of the Ising model, which proves to be fundamentally important in the computation of the observable, on a specific topology, using the Cavity method.

The first important check is the *self-consistency* of $Z_L(s, t)$. More precisely: if two chains of length L_1 and L_2 are joined together, inserting $z - 2$ contributions $\hat{\eta}(s_0)$, once summed over the internal spin s_0 , the scaled partition function of the total chain of length $L_1 + L_2$ must have the same form

$$\sum_{s_0=\pm 1} Z_{L_1}(s_1, s_0) \hat{\eta}(s_0)^{z-2} Z_{L_2}(s_0, s_2) \stackrel{!}{=} Z_{L_1+L_2}(s_1, s_2). \quad (5.22)$$

This is tested by plugging in $Z_{L_1}(s_1, s_0)$ and $Z_{L_2}(s_0, s_2)$ with their definitions from [Eq. 5.17](#) and summing over s_0 , verifying that the definitions of $p_B(s)$ and $g_\lambda(s)$ are consistent:

$$\begin{aligned} \sum_{s_0=\pm 1} Z_{L_1}(s_1, s_0) \hat{\eta}(s_0)^{z-2} Z_{L_2}(s_0, s_2) &= \left(\sum_{s_0=\pm 1} p_B(s_0)^2 \hat{\eta}(s_0)^{z-2} \right) p_B(s_1) p_B(s_2) + \\ &+ \left(\sum_{s_0=\pm 1} g_\lambda(s_0)^2 \hat{\eta}(s_0)^{z-2} \right) \lambda^{L_1+L_2} g_\lambda(s_1) g_\lambda(s_2), \end{aligned} \quad (5.23)$$

where the last equality is justified by the fact that the function $g_\lambda(s)$ is odd with respect to its argument. This implies two identities:

$$\sum_{s=\pm 1} p_B(s)^2 \hat{\eta}(s)^{z-2} = 1 \quad \text{and} \quad \sum_{s=\pm 1} g_\lambda(s)^2 \hat{\eta}(s)^{z-2} = 1, \quad (5.24)$$

both can be easily verified using the definitions [Eq. 5.20](#), remembering that $\sum_s \hat{\eta}(s) = 1$ and $\sum_s e_{\lambda_-}(s)^2 = 1$.

Interestingly, from the expression of $Z_L(s, t)$, it is possible to locate the [MF](#) transition point, through the evaluation of the susceptibility. Considering a reference spin s_0 , in a homogeneous system

$$\chi = \beta \sum_{i=1}^N \langle s_0 s_i \rangle_c = \beta \sum_{L=1}^{\infty} z(z-1)^{L-1} \langle s_0 s_L \rangle_c, \quad (5.25)$$

where the last step is justified in the large N limit neglecting the presence of loops in the system; $z(z-1)^{L-1}$ is the number of nodes at distance L on a Bethe lattice with connectivity z and s_L is a generic spin at distance L from the reference one. The 2-point connected correlation function can be computed using $Z_L(s_0, s_L)$:

$$\langle s_0 s_L \rangle_c = \frac{\sum_{s_0, s_L = \pm 1} \hat{\eta}(s_0)^{z-1} s_0 Z_L(s_0, s_L) s_L \hat{\eta}(s_L)^{z-1}}{\sum_{s_0, s_L = \pm 1} \hat{\eta}(s_0)^{z-1} Z_L(s_0, s_L) \hat{\eta}(s_L)^{z-1}} = \lambda^L \frac{\left(\sum_{s=\pm 1} s g_\lambda(s) \hat{\eta}(s)^{z-1} \right)^2}{\left(\sum_{s=\pm 1} p_B(s) \hat{\eta}(s)^{z-1} \right)^2} = \lambda^L, \quad (5.26)$$

where the single magnetizations are zero and the simplification of the last step is due to the simple forms of $p_B(s)$, $g_\lambda(s)$ and $\hat{\eta}(s)$, given in Eq. 5.9 and Eq. 5.21. Interestingly, the 2-point connected correlation function on a Bethe lattice always decays exponentially with the distance. However, a phase transition can happen due to the exponential growth of the number of spins at distance L . Plugging the 2-point connected correlation in the expression for the susceptibility and using that $\lambda = \tanh(\beta J)$

$$\chi \propto \sum_{L=1}^{\infty} [(z-1) \tanh(\beta J)]^L = \begin{cases} +\infty & \text{if } \beta \leq \beta_c \equiv \frac{1}{J} \operatorname{atanh}\left(\frac{1}{z-1}\right) \\ < +\infty & \text{otherwise.} \end{cases} \quad (5.27)$$

The critical point is identified by the value

$$\lambda_c \equiv \frac{1}{z-1}. \quad (5.28)$$

Note that the expression for the joint probability distribution, given in Eq. 5.19, can be generalized for variables with a generic measure $d\mu(s)$, taking into account that the transfer matrix would have an infinite set of eigenvalues and eigenfunctions, so that

$$Z_L(s, t) = p_B(s) p_B(t) + \sum_{\substack{i=2, \dots \\ |\lambda_{i+1}| < |\lambda_i| < 1}} \lambda_i^L g_{\lambda_i}(s) g_{\lambda_i}(t), \quad (5.29)$$

where λ_2 corresponds to λ and is the one that assumes the critical value $\lambda_c = 1/(z-1)$ at the critical temperature $T_c = 1/\beta_c$. The case of the soft spin variables is explored in Section 5.A.1, yielding the same result as the Ising case.

5.2.2 2-point Connected Correlation Function

In this section, the computations for the 2-point function are presented, following the steps outlined below. Differently from the general rules described in Section 4.2, the calculation of the observable for the given diagram is postponed until after the “sum of the contributions” step. This choice is motivated by the fact that these computations, although straightforward, become lengthy if shown in detail. The same approach is applied to the 4-point function, as discussed in Section 5.2.3.

❶ Identification of relevant diagrams

Following the general rules, the starting point is to identify the possible diagrams that contribute to the chosen observable, that is the 2-point connected correlation

function. The simplest diagrams are obtained when two points are connected with a single loop-less line, of length L , in the M -layer graph. Given that the $1/M$ expansion turns out to be an expansion in the number of topological loops, the contribution of this type of diagram will give the leading order in the expansion¹. It is necessary to determine which type of projection on the original lattice is generated by a simple line in the M -layer graph. The simplest projection is a simple loop-less line also on the original lattice: this diagram is denoted as $\mathcal{G}_1$, and is shown in Figure 5.1. However, *boucle* diagrams have to be taken into account (see Section 4.1.5): the diagram with one loop on the original lattice is denoted as $\mathcal{G}_3$, it is displayed in Figure 5.1 in its two versions. Another possibility is to connect the two spins with a path with one loop already on the layered lattice, which thus will have (at least) one loop also on its projection on the original lattice: this is the case of $\mathcal{G}_2$ in Figure 5.1. $\mathcal{G}_2$ has a degree-4 vertex; in principle, one should also consider one-loop diagrams with degree-3 vertices. However, by introducing the RS Cavity method to compute the line-connected observable on a given diagram, it is shown that they give zero contribution for symmetry reasons in the case of the Ising model without an external field.

② Weights, number of projections and symmetry factors

Diagram $\mathcal{G}_1$:

- $W(\mathcal{G}_1) = \frac{1}{M}$;
- $\mathcal{N}(\mathcal{G}_1; L; \mathbf{x}_1, \mathbf{x}_2) = \mathcal{N}_L(\mathbf{x}_1, \mathbf{x}_2)$;
- $S(\mathcal{G}_1) = 1$.

Diagram $\mathcal{G}_2$:

- $W(\mathcal{G}_2) = \frac{1}{M^2}$;
- $\mathcal{N}(\mathcal{G}_2; \vec{L}; \mathbf{x}_1, \mathbf{x}_2) = (2D)^2 \frac{(2D)!}{(2D-4)!} \sum_{\mathbf{x}_0 \in a_l \mathbb{Z}^D} \frac{\mathcal{N}_{L_1}(\mathbf{x}_1, \mathbf{x}_0)}{(2D)^2} \frac{\mathcal{N}_{L_0}(\mathbf{x}_0, \mathbf{x}_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(\mathbf{x}_0, \mathbf{x}_2)}{(2D)^2}$;
- $S(\mathcal{G}_2) = 2$;

where $\vec{L} = (L_1, L_0, L_2)$.

Diagram $\mathcal{G}_3$:

- $W(\mathcal{G}_3) = -\frac{1}{M^2}$;
- $\mathcal{N}(\mathcal{G}_3; \vec{L}; \mathbf{x}_1, \mathbf{x}_2) = \mathcal{N}(\mathcal{G}_2; \vec{L}; \mathbf{x}_1, \mathbf{x}_2)$;
- $S(\mathcal{G}_3) = 1$.

③ Sum of the contributions

Summing all contributions yields, up to order $1/M^2$:

$$\overline{\langle s(\mathbf{x}_1) s(\mathbf{x}_2) \rangle}_c = \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}_L(\mathbf{x}_1, \mathbf{x}_2) \langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_1, \text{lc}} +$$

¹In general, additional diagrams in which the two points are disconnected must be considered as well. However, for connected observables such as the correlation functions, the contributions of these diagrams are null.

$$\begin{aligned}
& + \frac{1}{2M^2} \sum_{\vec{L}}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_2; \vec{L}; \mathbf{x}_1, \mathbf{x}_2) \langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_2, \text{lc}} \\
& - \frac{1}{M^2} \sum_{\vec{L}}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_2; \vec{L}; \mathbf{x}_1, \mathbf{x}_2) \langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_3, \text{lc}} + \mathcal{O}\left(\frac{1}{M^3}\right). \quad (5.30)
\end{aligned}$$

In the last equation, the contribution due to the *boucle* diagram $\mathcal{G}_3$ is given by the value the observable takes on a loop-less line of length $L_0 + L_1 + L_2$, as already discussed in Section 4.1.5. There, it is also noticed that the terms of order $1/M^2$ include *boucle* diagrams with two or more loops in the M -layer graph, which must be subtracted. However, these latter contributions are of order $1/M^3$, so they are neglected in the calculation of correlation functions at one-loop order. In the following, all computations regarding the 2-point function on the given diagrams are explicitly done.

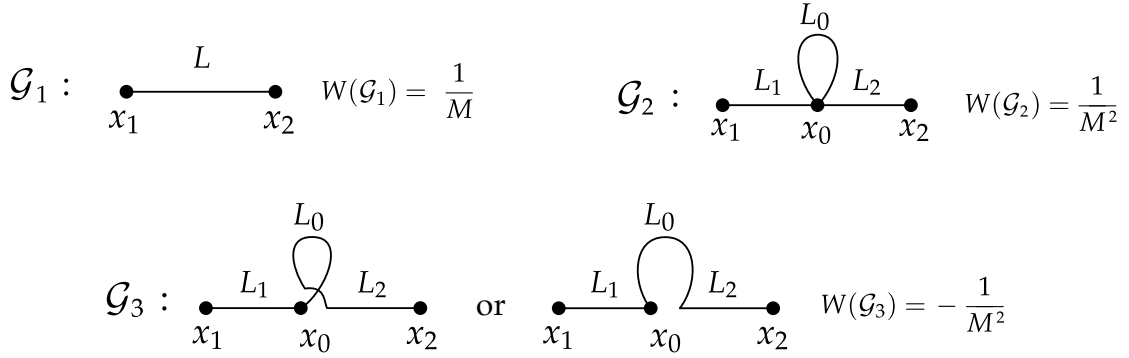


Figure 5.1. Topological diagram contributions to the 2-point connected correlation function. Top left: the order $\mathcal{O}(1/M)$, that is the simple line without loops in the M -layer graph nor in the original one. Top right: the diagram with one loop is of the order $\mathcal{O}(1/M^2)$. Bottom: two possible instances of the so-called "boucle" diagram, where the line does not close in a loop on the layered lattice but forms one in the projection.

Computation of the line-connected 2-point function through the RS Cavity Method:

The next step is to evaluate the observable on a diagram, merged into an infinite tree. As shown before, the (unnormalized) joint probability distribution of s and t , spins at the boundaries of an open chain of length L , merged in a tree-like graph, can be written as

$$Z_L(s, t) = p_B(s)p_B(t) + \lambda^L g_\lambda(s) g_\lambda(t). \quad (5.31)$$

Both $p_B(s)$ and $g_\lambda(s)$ are, respectively, even and odd functions of $s = \pm 1$, as a consequence of the $\mathbb{Z}_2$ symmetry of the Ising model. Once $Z_L(s, t)$ is computed, a generic observable on a generic diagram, including the presence of loops, can be evaluated. For a given diagram, a spin is assigned to each internal and external vertex, and for each line of length L (starting at s and ending at t), a factor $Z_L(s, t)$ is included. Since each node in the Bethe lattice has fixed connectivity z , a factor $\hat{\eta}(s_I)^{z-r}$ and a factor $\hat{\eta}(s_E)^{z-1}$ are multiplied, for each internal vertex of degree r on spin s_I and for each external vertex with spin s_E , respectively. To compute the 2-point correlation function, multiplication by the two external spin variables and integration over all spins with their measures is performed, then division by the partition function and

subtraction of all factorized contributions are used to compute the connected correlation. Finally, the "line-connected observable" is computed, following prescriptions from the M -layer construction in [Section 4.2](#).

The 2-point connected correlation function on the line $\mathcal{G}_1$ has already been computed for the MF susceptibility, at the end of [Section 5.2.1](#):

$$\langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_1, \text{lc}} = \frac{\sum_{s_1, s_2 = \pm 1} \hat{\eta}(s_1)^{z-1} s_1 Z_L(s_1, s_2) s_2 \hat{\eta}(s_2)^{z-1}}{\sum_{s_1, s_2 = \pm 1} \hat{\eta}(s_1)^{z-1} Z_L(s_1, s_2) \hat{\eta}(s_2)^{z-1}} = \lambda^L. \quad (5.32)$$

Proceeding to the one-loop diagram $\mathcal{G}_2^2$:

$$\begin{aligned} \langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_2} &= \frac{\sum_{s_1, s_0, s_2 = \pm 1} \hat{\eta}(s_1)^{z-1} s_1 Z_{L_1}(s_1, s_0) \hat{\eta}(s_0)^{z-4} Z_{L_0}(s_0, s_0) Z_{L_2}(s_0, s_2) s_2 \hat{\eta}(s_2)^{z-1}}{\sum_{s_1, s_0, s_2 = \pm 1} \hat{\eta}(s_1)^{z-1} Z_{L_1}(s_1, s_0) \hat{\eta}(s_0)^{z-4} Z_{L_0}(s_0, s_0) Z_{L_2}(s_0, s_2) \hat{\eta}(s_2)^{z-1}} \\ &= \lambda^{L_1+L_2}. \end{aligned} \quad (5.33)$$

For the "line-connected observable" computation the following is used:

$$\langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_2, \text{lc}} = \langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_2} - \langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_1} = 0, \quad (5.34)$$

where [Eq. 5.32](#) with $L = L_1 + L_2$ was used.

A remark: since the MF behavior of the Ising model is identical whether defined on the Bethe lattice or on the FC lattice, one should expect the M -layer construction to reproduce the same results as the ϕ_4 scalar Field Theory expansion for the 2-point correlation at one loop order. However, when considering only the diagram $\mathcal{G}_2$, this is not the case. This fact indicates that additional terms must be included.

In particular, the contribution arising from the *boucle* diagram. $\langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_3, \text{lc}}$ equals the value on $\mathcal{G}_1$ with L replaced by $L_1 + L_0 + L_2$:

$$\langle s_1 s_2 \rangle_c \Big|_{\mathcal{G}_3, \text{lc}} = \lambda^{L_1+L_0+L_2}. \quad (5.35)$$

By substituting [Eq. 5.32](#), [Eq. 5.34](#), [Eq. 5.35](#) into [Eq. 5.30](#) the result is

$$\begin{aligned} \overline{\langle s(\mathbf{x}_1) s(\mathbf{x}_2) \rangle_c} &= \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}_L(\mathbf{x}_1, \mathbf{x}_2) \lambda^L + \\ &- \frac{(2D)^{-4}}{M^2} \frac{(2D)!}{(2D-4)!} \sum_{\mathbf{x}_0 \in a_I \mathbb{Z}^D} \sum_{L_1=1}^{\infty} \mathcal{N}_{L_1}(\mathbf{x}_1, \mathbf{x}_0) \lambda^{L_1} \sum_{L_0=1}^{\infty} \mathcal{N}_{L_0}(\mathbf{x}_0, \mathbf{x}_0) \lambda^{L_0} \sum_{L_2=1}^{\infty} \mathcal{N}_{L_2}(\mathbf{x}_0, \mathbf{x}_2) \lambda^{L_2} \\ &+ \mathcal{O}\left(\frac{1}{M^3}\right). \end{aligned} \quad (5.36)$$

²Here and in the following computations, recurring combinations of $\hat{\eta}(s)$, $p_B(s)$ and $g_\lambda(s)$ appear in the numerator and denominator such that they simplify. Possible combinations are explicitly written in [Eq. 5.98](#) to [Eq. 5.99](#) for soft spin variables (for Ising ones they are the same with the sum over the spins instead of the integral). In order to simplify the computations these terms are not written for each observable, the interested reader can find them explicitly in Ref. [\[Ang24\]](#).

Introducing the generating function of the number of **NBP** as [Alt17, Fit13]:

$$B_\lambda(\mathbf{x}_f, \mathbf{x}_i) = \sum_{L=1}^{\infty} \mathcal{N}_L(\mathbf{x}_f, \mathbf{x}_i) \lambda^L, \quad (5.37)$$

Eq. 5.36 becomes:

$$\begin{aligned} \overline{\langle s(\mathbf{x}_1) s(\mathbf{x}_2) \rangle}_c &= \frac{1}{M} B_\lambda(\mathbf{x}_1, \mathbf{x}_2) + \\ &- \frac{(2D)^{-4}}{M^2} \frac{(2D)!}{(2D-4)!} \sum_{\mathbf{x}_0 \in a_l \mathbb{Z}^D} B_\lambda(\mathbf{x}_1, \mathbf{x}_0) B_\lambda(\mathbf{x}_0, \mathbf{x}_0) B_\lambda(\mathbf{x}_0, \mathbf{x}_2) + \mathcal{O}\left(\frac{1}{M^3}\right). \end{aligned} \quad (5.38)$$

The next step is to write this last expression in momentum space. The convention for the Fourier transform, from $\mathbf{x} \in a_l \mathbb{Z}^D$ to $\mathbf{k} \in [-\pi/a_l, \pi/a_l]^D$, of a generic function $f(\mathbf{x})$ is³

$$\widehat{f}(\mathbf{k}) = a_l^D \sum_{\mathbf{x} \in a_l \mathbb{Z}^D} f(\mathbf{x}) e^{i\mathbf{k}\mathbf{x}}, \quad f(\mathbf{x}) = \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \widehat{f}(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}}; \quad (5.39)$$

such that

$$\left(\frac{2\pi}{a_l}\right)^D \delta^D(\mathbf{k}) = \sum_{\mathbf{x} \in a_l \mathbb{Z}^D} e^{i\mathbf{k}\mathbf{x}}. \quad (5.40)$$

Since the number of **NBP** between $\mathbf{x}_1$ and $\mathbf{x}_2$ (and consequently the generating function $B_\lambda(\mathbf{x}_1, \mathbf{x}_2)$) is a function of only $|\mathbf{x}_1 - \mathbf{x}_2|$ one can write

$$\widehat{B}_\lambda(\mathbf{k}_1, \mathbf{k}_2) = (2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \widehat{B}_\lambda(\mathbf{k}_1). \quad (5.41)$$

Using this last result in Eq. 5.38, the 2-point function in momentum space reads

$$\begin{aligned} \overline{\langle s(\mathbf{p}) s(\mathbf{q}) \rangle}_c &= \\ &(2\pi)^D \delta^D(\mathbf{p} + \mathbf{q}) \frac{\widehat{B}_\lambda(\mathbf{p})}{M} \left(1 - \frac{(2D)^{-4}}{M} \frac{(2D)!}{(2D-4)!} \frac{1}{a_l^D} \widehat{B}_\lambda(\mathbf{q}) \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \widehat{B}_\lambda(\mathbf{k}) \right) \\ &+ \mathcal{O}\left(\frac{1}{M^3}\right). \end{aligned} \quad (5.42)$$

In the next section, similar computations are presented for the 4-point connected correlation function. In Section 5.3 both results, for the 2- and 4-point functions, are interpreted from the perspective of the ϕ^4 scalar Field Theory.

5.2.3 4-point Connected Correlation Function

❶ Identification of relevant diagrams

Regarding the 4-point connected correlation function, the leading term is $\mathcal{O}(1/M^3)$ and all contributions up to $\mathcal{O}(1/M^4)$ must be included to account for the diagrams up to one-loop level. The diagrams that are relevant to this computation

³Analogously to the convention employed in Chapter 3, to ease the notation: i) the scalar product between $\mathbf{k}$ and $\mathbf{x}$ is denoted as $\mathbf{k}\mathbf{x}$ at the exponent; ii) the square of the norm of a generic vector $\mathbf{x}$ (or $\mathbf{k}$) is indicated with $|\mathbf{x}|^2 := \mathbf{x}^2$ (or $|\mathbf{k}|^2 := \mathbf{k}^2$).

are reported in Figure 5.2. Due to the spin flip symmetry of the Ising model in the absence of an external field, some of the diagrams with degree-3 internal vertices give zero contribution. This can be seen performing the computations of the corresponding observables on such diagrams, as explained for the 2-point function in Section 5.2.2. However, similarly, one can realise that diagrams of the kind of $\mathcal{G}_{5_a}$ and $\mathcal{G}_{5_b}$ (see Figure 5.3) give non-zero contribution. The reason to neglect them is specified in a paragraph below, once the computations of the 4-point connected correlation functions are done on these topologies.

② *Weights, number of projections and symmetry factors*

Diagram $\mathcal{G}_4$:

- $W(\mathcal{G}_4) = \frac{1}{M^3}$;
- $\mathcal{N}(\mathcal{G}_4; \{L_i\}_{i=1,\dots,4}; \{\mathbf{x}_i\}_{i=1,\dots,4}) = (2D)^4 \frac{(2D)!}{(2D-4)!} \sum_{\mathbf{x}_0 \in a_l \mathbb{Z}^D} \prod_{i=1}^4 \frac{\mathcal{N}_{L_i}(\mathbf{x}_i, \mathbf{x}_0)}{(2D)^2}$;
- $S(\mathcal{G}_4) = 1$.

Diagram $\mathcal{G}_5$:

- $W(\mathcal{G}_5) = \frac{1}{M^4}$;
- $\mathcal{N}(\mathcal{G}_5; \vec{L}'; \{\mathbf{x}_i\}_{i=1,\dots,4}) = (2D)^4 \left(\frac{(2D)!}{(2D-4)!} \right)^2 \sum_{\mathbf{x}_0, \mathbf{x}'_0 \in a_l \mathbb{Z}^D} \prod_{i=1,3} \frac{\mathcal{N}_{L_i}(\mathbf{x}_i, \mathbf{x}_0)}{(2D)^2} \times$
 $\times \prod_{i=2,4} \frac{\mathcal{N}_{L_i}(\mathbf{x}_i, \mathbf{x}'_0)}{(2D)^2} \frac{\mathcal{N}_{L_A}(\mathbf{x}_0, \mathbf{x}'_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(\mathbf{x}_0, \mathbf{x}'_0)}{(2D)^2}$;
- $S(\mathcal{G}_5) = 2$;

where $\vec{L}' = (L_1, L_2, L_3, L_4, L_A, L_B)$.

Diagram $\mathcal{G}_6$:

- $W(\mathcal{G}_6) = -\frac{1}{M^4}$;
- $\mathcal{N}(\mathcal{G}_6; \vec{L}'; \{\mathbf{x}_i\}_{i=1,\dots,4}) = \mathcal{N}(\mathcal{G}_5; \vec{L}'; \{\mathbf{x}_i\}_{i=1,\dots,4})$;
- $S(\mathcal{G}_6) = 1$.

Diagram $\mathcal{G}_7$:

- $W(\mathcal{G}_7) = \frac{1}{M^4}$;
- $\mathcal{N}(\mathcal{G}_7; \vec{L}''; \{\mathbf{x}_i\}_{i=1,\dots,4}) = (2D)^4 \left(\frac{(2D)!}{(2D-4)!} \right)^2 \sum_{\mathbf{x}_0, \mathbf{x}'_0 \in a_l \mathbb{Z}^D} \prod_{i=1,3,4} \frac{\mathcal{N}_{L_i}(\mathbf{x}_i, \mathbf{x}_0)}{(2D)^2} \times$
 $\times \frac{\mathcal{N}_{L_{2A}}(\mathbf{x}'_0, \mathbf{x}_0)}{(2D)^2} \frac{\mathcal{N}_{L_0}(\mathbf{x}'_0, \mathbf{x}'_0)}{(2D)^2} \frac{\mathcal{N}_{L_{2B}}(\mathbf{x}_2, \mathbf{x}'_0)}{(2D)^2}$;
- $S(\mathcal{G}_7) = 2$;

where $\vec{L}'' = (L_1, L_{2A}, L_{2B}, L_3, L_4, L_0)$.

Diagram $\mathcal{G}_8$:

- $W(\mathcal{G}_8) = -\frac{1}{M^4}$;
- $\mathcal{N}(\mathcal{G}_8; \vec{L}''; \{\mathbf{x}_i\}_{i=1,\dots,4}) = \mathcal{N}(\mathcal{G}_7; \vec{L}''; \{\mathbf{x}_i\}_{i=1,\dots,4})$;
- $S(\mathcal{G}_8) = 1$.

③ *Sum of the contributions* The sum of all contributions to the 4-point connected correlation function is

$$\begin{aligned}
\overline{\langle s(\mathbf{x}_1)s(\mathbf{x}_2)s(\mathbf{x}_3)s(\mathbf{x}_4) \rangle}_c &= \frac{1}{M^3} \prod_{i=1}^4 \sum_{L_i=1}^{\infty} \mathcal{N}(\mathcal{G}_4; \{L_i\}_{i=1,\dots,4}; \{\mathbf{x}_i\}_{i=1,\dots,4}) \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_4, \text{lc}} \\
&+ \frac{1}{2M^4} \sum_{\vec{L}'}^{1,\dots,\infty} \mathcal{N}(\mathcal{G}_5; \vec{L}'; \{\mathbf{x}_i\}_{i=1,\dots,4}) \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_5, \text{lc}} + \\
&- 2 \frac{1}{M^4} \sum_{\vec{L}'}^{1,\dots,\infty} \mathcal{N}(\mathcal{G}_6; \vec{L}'; \{\mathbf{x}_i\}_{i=1,\dots,4}) \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_6, \text{lc}} + \\
&+ \frac{1}{2M^4} \sum_{\vec{L}''}^{1,\dots,\infty} \mathcal{N}(\mathcal{G}_7; \vec{L}''; \{\mathbf{x}_i\}_{i=1,\dots,4}) \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_7, \text{lc}} + \\
&- \frac{1}{M^4} \sum_{\vec{L}''}^{1,\dots,\infty} \mathcal{N}(\mathcal{G}_8; \vec{L}''; \{\mathbf{x}_i\}_{i=1,\dots,4}) \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_8, \text{lc}} \\
&+ \text{permutations of } \mathcal{G}_5, \mathcal{G}_6, \mathcal{G}_7 \text{ and } \mathcal{G}_8 + \mathcal{O}\left(\frac{1}{M^5}\right). \tag{5.43}
\end{aligned}$$

Some things to notice in Eq. 5.43: i) the factor 2 in the third term, corresponding to the *boucle* diagram $\mathcal{G}_6$ is due to the presence of two internal vertices, where line can, in principle, overlap to create the kind of *boucle* diagram; ii) similarly to the 2-point function, contributions $\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_6, \text{lc}}$ and $\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_8, \text{lc}}$ are the same as $\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_4, \text{lc}}$ where the lengths of the four lines are

$$L_1 \rightarrow L_1; \quad L_2 \rightarrow L_B + L_2; \quad L_3 \rightarrow L_3; \quad L_4 \rightarrow L_A + L_4, \tag{5.44}$$

and

$$L_1 \rightarrow L_1; \quad L_2 \rightarrow L_{2_A} + L_0 + L_{2_B}; \quad L_3 \rightarrow L_3; \quad L_4 \rightarrow L_4, \tag{5.45}$$

respectively.

Computation of the line-connected 4-point function through the RS Cavity Method:

Consider now the 4-point function, starting from diagram $\mathcal{G}_4$ in Figure 5.2. The following is obtained:

$$\langle s_1 s_2 s_3 s_4 \rangle \Big|_{\mathcal{G}_4} = \lambda^{\sum_{i=1}^4 L_i}, \tag{5.46}$$

using the definitions of $p_B(s)$, $g_\lambda(s)$, and $\hat{\eta}(s)$ from Eq. 5.21 and Eq. 5.9. The connected 4-point correlation function is:

$$\langle s_1 s_2 s_3 s_4 \rangle_c = \langle s_1 s_2 s_3 s_4 \rangle - \langle s_1 s_2 \rangle \langle s_3 s_4 \rangle - \langle s_1 s_3 \rangle \langle s_2 s_4 \rangle - \langle s_1 s_4 \rangle \langle s_2 s_3 \rangle, \tag{5.47}$$

where the 3-point and 1-point connected correlation functions are zero by symmetry. For the 2-point function between s_1 and s_2 (the others being similar):

$$\langle s_1 s_2 \rangle \Big|_{\mathcal{G}_4} = \lambda^{L_1 + L_2}. \tag{5.48}$$

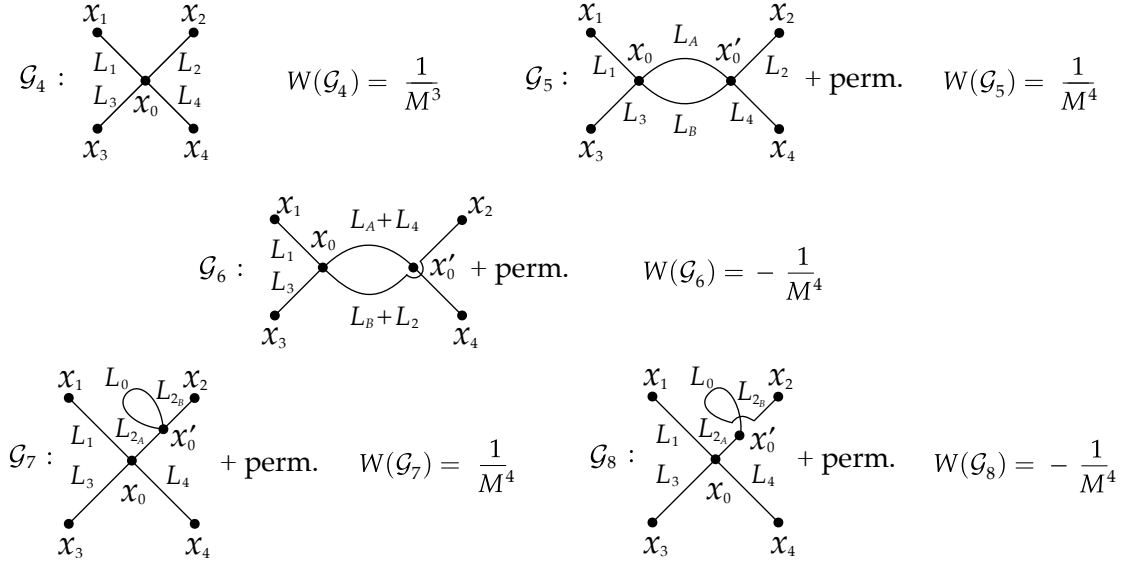


Figure 5.2. Topological diagram contributions to the 4-point connected correlation function. The order $\mathcal{O}(1/M^3)$ is the simple cross without loops connecting the four spins. At order $\mathcal{O}(1/M^4)$, that is at one-loop, for each topology there are two distinct contributions: $\mathcal{G}_5$ and $\mathcal{G}_7$. As in the case of the 2-point correlation, the *boucle* diagrams, $\mathcal{G}_6$ and $\mathcal{G}_8$, must also be considered, where the line does not close in a loop on the M -layer graph, but realizes it in the projection. A similar contribution for $\mathcal{G}_6$ involves the superimposition in the left internal vertex, not shown in the figure. Thus, a factor 2 is included for the $\mathcal{G}_6$ boucle contribution of the 4-point function.

Thus,

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_4, \text{lc}} = \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_4} = -2\lambda^{\sum_{i=1}^4 L_i}, \quad (5.49)$$

$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_4}$ coincides with the line-connected observable, since removing any of the lines of $\mathcal{G}_4$ disconnects the graph, yielding zero.

At one loop level, the 4-point correlation on a graph with the topology $\mathcal{G}_5$ (see Figure 5.2) is computed. For this level, the subtraction of the connected 4-point correlation functions computed on graphs $\mathcal{G}_{5_a}$ and $\mathcal{G}_{5_b}$ (see Figure 5.3) is required to accomplish the line-connected definition. The total 4-point correlation function on $\mathcal{G}_5$ is:

$$\langle s_1 s_2 s_3 s_4 \rangle \Big|_{\mathcal{G}_5} = \lambda^{\sum_{i=1}^4 L_i}; \quad (5.50)$$

while the 2-point functions can be of different kinds:

$$\langle s_1 s_2 \rangle \Big|_{\mathcal{G}_5} = \lambda^{\sum_{i=1}^2 L_i} \frac{\lambda^{L_A} + \lambda^{L_B}}{1 + \lambda^{L_A + L_B}}, \quad (5.51)$$

and

$$\langle s_1 s_3 \rangle \Big|_{\mathcal{G}_5} = \lambda^{L_1 + L_3}. \quad (5.52)$$

So:

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_5} = \langle s_1 s_2 s_3 s_4 \rangle \Big|_{\mathcal{G}_5} - \langle s_1 s_2 \rangle \langle s_3 s_4 \rangle \Big|_{\mathcal{G}_5} - \langle s_1 s_3 \rangle \langle s_2 s_4 \rangle \Big|_{\mathcal{G}_5} - \langle s_1 s_4 \rangle \langle s_2 s_3 \rangle \Big|_{\mathcal{G}_5} =$$

$$= -2 \frac{(\lambda^{L_A} + \lambda^{L_B})^2}{(1 + \lambda^{L_A+L_B})^2} \lambda^{\sum_{i=1}^4 L_i}. \quad (5.53)$$

Following the prescriptions of the M -layer construction, the line-connected 4-point con-

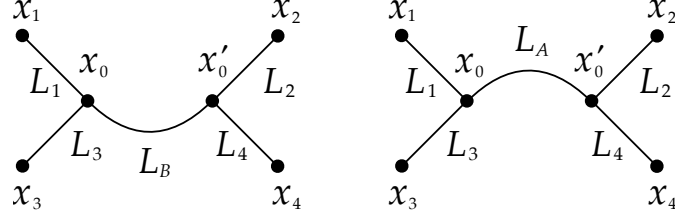


Figure 5.3. $\mathcal{G}_{5a}$ (left), $\mathcal{G}_{5b}$ (right).

nected correlation function is computed by subtracting the contributions of $\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5a}}$ and $\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5b}}$ (see Figure 5.3) from $\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_5}$. For $\mathcal{G}_{5a}$:

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5a}} = \lambda^{\sum_{i=1}^4 L_i} - 2\lambda^{2L_B + \sum_{i=1}^4 L_i} - \lambda^{\sum_{i=1}^4 L_i} = -2\lambda^{2L_B + \sum_{i=1}^4 L_i}, \quad (5.54)$$

analogously for $\mathcal{G}_{5b}$:

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5b}} = -2\lambda^{2L_A + \sum_{i=1}^4 L_i}. \quad (5.55)$$

Now the line-connected observable is:

$$\begin{aligned} \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5,lc}} &= \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_5} - \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5a}} - \langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5b}} \\ &= \lambda^{\sum_{i=1}^4 L_i} \left[-2 \frac{(\lambda^{L_A} + \lambda^{L_B})^2}{(1 + \lambda^{L_A+L_B})^2} + 2\lambda^{2L_A} + 2\lambda^{2L_B} \right], \end{aligned} \quad (5.56)$$

which can be Taylor-expanded up to $\mathcal{O}(\lambda^{6L})$ in the limit $L_A \sim L_B \sim L_i \sim L \rightarrow \infty$:

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{5,lc}} = -4\lambda^{L_A+L_B+\sum_{i=1}^4 L_i} + \mathcal{O}(\lambda^{8L}). \quad (5.57)$$

Before computing the contributions of the *boucle* diagrams for the 4-point function, it is noted that two more diagrams, with the same topology as $\mathcal{G}_5$ (as shown in Figure 5.2), contribute. One is obtained by exchanging x_1 and x_2 , the other by exchanging x_1 and x_4 . These are not included in the symmetry factor because they differ by the entrance directions of the internal vertices, so they must be added explicitly. Such contributions are present in the continuum Field Theory associated with the Ising model due to Wick's theorem. The same symmetry reasoning applies to the next *boucle* diagrams, $\mathcal{G}_6$. Writing $\mathcal{G}_5$ and $\mathcal{G}_6$ includes all external vertex permutations in what follows. The *boucle* diagrams for the 4-point function, reported in Figure 5.2, contribute:

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_{6,lc}} = \lambda^{L_A+L_B+\sum_{i=1}^4 L_i} - 3\lambda^{L_A+L_B+\sum_{i=1}^4 L_i} = -2\lambda^{L_A+L_B+\sum_{i=1}^4 L_i}. \quad (5.58)$$

Similarly, there is also the contribution due to the diagram where the overlap is on s_0 . Since in this case the value of the observable matches that defined on diagram $\mathcal{G}_6$, a

factor 2 is simply included (see Eq. 5.43). Now, the line-connected 4-point function on all four diagrams of type $\mathcal{G}_7$, that is where one external line includes a loop as in $\mathcal{G}_2$, is

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_7, \text{lc}} = 0. \quad (5.59)$$

The line-connected 4-point function on diagrams of type $\mathcal{G}_8$, that is where one external line includes a *boucle*, is

$$\langle s_1 s_2 s_3 s_4 \rangle_c \Big|_{\mathcal{G}_8, \text{lc}} = -2\lambda^{L_0 + \sum_{i=1}^4 L_i}. \quad (5.60)$$

For $\mathcal{G}_7$ and $\mathcal{G}_8$, all four permutations of the external vertices must be included to account for the loop positions on the external lines.

The Role of Cubic Vertices The roles of $\mathcal{G}_{5_a}$ and $\mathcal{G}_{5_b}$ are as follows. While their 4-point correlations were needed for the line-connected observable on $\mathcal{G}_5$ (see Eq. 5.56), they are not included as part of the o-loop order for the 4-point observable itself, as their potential contribution is $\mathcal{O}(\lambda^{6L})$ (for $L_A \sim L_B \sim L_i \sim L \rightarrow \infty$), while $\mathcal{G}_4$ gives $\mathcal{O}(\lambda^{4L})$. Thus, $\mathcal{G}_{5_a}$ and $\mathcal{G}_{5_b}$ can be neglected in the large-lengths limit compared to $\mathcal{G}_4$. Notably, $\mathcal{G}_{5_a}$ and $\mathcal{G}_{5_b}$ are diagrams with cubic vertices: the fact that they are subdominant compared to $\mathcal{G}_4$ is in agreement with expectations from standard Field Theory, which contains no cubic diagrams in this context.

The 4-point Function Now, all contributions can be combined, including weights in inverse powers of M and symmetry factors $S(\mathcal{G})$. In particular, Eq. 5.49, Eq. 5.57, Eq. 5.58, Eq. 5.59 and Eq. 5.60 can be plugged in Eq. 5.43, repeating the same steps done for the 2-point function, going to Fourier space:

$$\begin{aligned} \overline{\langle s(\mathbf{k}_1)s(\mathbf{k}_2)s(\mathbf{k}_3)s(\mathbf{k}_4) \rangle_c} = & -\frac{2(2D)^{-4}}{M^3} \frac{(2D)!}{(2D-4)!} \left(\frac{2\pi}{a_l}\right)^D \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i\right) \prod_{i=1}^4 \hat{B}_\lambda(\mathbf{k}_i) \left(1 + \right. \\ & \left. - \frac{(2D)^{-4}}{M} \frac{(2D)!}{(2D-4)!} \frac{1}{a_l^D} \sum_{j=2}^4 \int \frac{d^D q}{(2\pi)^D} \hat{B}_\lambda(\mathbf{q}) \hat{B}_\lambda(\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_j) \right) + \\ & + \sum_{j=1}^4 \frac{2(2D)^{-8}}{M^4} \left(\frac{(2D)!}{(2D-4)!}\right)^2 \frac{(2\pi)^D}{a_l^{2D}} \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i\right) \prod_{i=1}^4 \hat{B}(\mathbf{k}_i) \hat{B}(\mathbf{k}_j) \int \frac{d^D q}{(2\pi)^D} \hat{B}(\mathbf{q}) + \mathcal{O}\left(\frac{1}{M^5}\right), \end{aligned} \quad (5.61)$$

where all permutations of the external vertices on $\mathcal{G}_5$, $\mathcal{G}_6$, $\mathcal{G}_7$, and $\mathcal{G}_8$ are included in the sums over j . This is the result for the computation of the 4-point connected correlation function for the D -dimensional Ising model arising from the M -layer construction, up to order $\mathcal{O}(1/M^5)$ for $M \rightarrow \infty$.

The next section is devoted to the connection with field-theoretic results on the ϕ^4 scalar theory.

5.3 Connection with Field Theory

In this section, the way to cast the results of the M -layer construction into the Field Theory framework is presented. In particular, it is shown how the correlation functions

computed through the M -layer construction, in Eq. 5.42 and Eq. 5.61, can be expressed in the same form as the ones computed through the scalar ϕ^4 Field Theory, in Eq. 3.48 and Eq. 3.49.

The first step in this direction is to write the Fourier transform of the generating function of the number of NBP. Recalling Eq. 5.41, from Refs. [Alt17, Fit13]:

$$\widehat{B}_\lambda(\mathbf{k}) = a_l^D \sum_{\mathbf{x} \in a_l \mathbb{Z}^D} \sum_{L=1}^{\infty} \mathcal{N}_L(\mathbf{x}, 0) \lambda^L e^{i\mathbf{k}\mathbf{x}} = a_l^D \frac{2D\lambda (F(\mathbf{k}) - \lambda)}{1 + \lambda^2(2D - 1) - 2D\lambda F(\mathbf{k})}, \quad (5.62)$$

where:

$$F(\mathbf{k}) = \frac{1}{D} \sum_{\mu=1}^D \cos(k_\mu a_l). \quad (5.63)$$

The critical behavior is governed by the limit $k^2 \rightarrow 0$, for which:

$$\widehat{B}_\lambda^{-1} \left(|\mathbf{k}| \ll \frac{1}{a_l} \right) = \frac{1}{a_l^D} \frac{1 - \lambda(2D - 1)}{2D\lambda} + \frac{1}{a_l^D} \frac{\lambda + 1}{4D^2\lambda(1 - \lambda)} (ka_l)^2 + \mathcal{O}((ka_l)^4). \quad (5.64)$$

To make the connection with Field Theory expressions, the following quantities are defined

$$\rho \equiv M \frac{a_l^{2-D}(1 + \lambda)}{4D^2\lambda(1 - \lambda)}, \quad (5.65)$$

$$\tau \equiv M \frac{1}{2D a_l^D \lambda} \left(1 - \frac{\lambda}{\lambda_c} \right), \quad (5.66)$$

$$g \equiv 2M \frac{(2D)!}{(2D - 4)!} \frac{(2D)^{-4}}{a_l^D}, \quad (5.67)$$

recalling that $\lambda_c = 1/(2D - 1)$, from Eq. 5.28 since $z = 2D$ in the M -layer graph. In this way, it is possible to write the small momentum limit of $\widehat{B}_\lambda(\mathbf{k})$ in a simpler form:

$$\widehat{B}_\lambda^{-1}(\mathbf{k}) = \rho k^2 + \tau, \quad (5.68)$$

and the correlation functions, in Eq. 5.42 and Eq. 5.61, are

$$\overline{\langle s(\mathbf{p})s(\mathbf{q}) \rangle}_c = (2\pi)^D \delta^D(\mathbf{p} + \mathbf{q}) \frac{1}{\rho \mathbf{p}^2 + \tau} \left(1 - \frac{g}{2} \frac{1}{\rho \mathbf{q}^2 + \tau} \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \frac{1}{\rho \mathbf{k}^2 + \tau} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (5.69)$$

and

$$\overline{\langle s(\mathbf{k}_1)s(\mathbf{k}_2)s(\mathbf{k}_3)s(\mathbf{k}_4) \rangle}_c = (2\pi)^D \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i \right) \prod_{i=1}^4 \widehat{G}_2(\mathbf{k}_i) \left[-g + \frac{1}{2} g^2 \left(I(\mathbf{k}_1 + \mathbf{k}_2) + I(\mathbf{k}_1 + \mathbf{k}_3) + I(\mathbf{k}_1 + \mathbf{k}_4) \right) \right] + \mathcal{O}\left(\frac{1}{M^5}\right), \quad (5.70)$$

where

$$I(\mathbf{q}) \equiv \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D p}{(2\pi)^D} \frac{1}{\rho p^2 + \tau} \cdot \frac{1}{\rho(\mathbf{p} + \mathbf{q})^2 + \tau}, \quad (5.71)$$

$$\hat{G}_2(\mathbf{q}) \equiv \frac{1}{\rho q^2 + \tau} \left(1 - \frac{1}{2} g \frac{1}{\rho q^2 + \tau} \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \frac{1}{\rho k^2 + \tau} \right). \quad (5.72)$$

The dimensions of the parameters can be expressed in terms of the length and field units, respectively $[L]$ and $[s]$. Observables computed via the RS Cavity method simplify sums over $s = \pm 1$ involving powers of $p_B(s)$, $g_\lambda(s)$, and $\hat{\eta}(s)$ by using their definitions. In particular, the term $\sum_{s=\pm 1} s g_\lambda(s) \hat{\eta}(s)^{z-1}$, which has dimensions $[s]$, appears squared in the numerator of $\langle s_1 s_2 \rangle_c$ (cf. Eq. 5.32 and Eq. 5.33), giving the 2-point function the correct dimension $[s]^2$. Keeping track of these dimensional terms inside the defined parameters yields:

$$[\tau] = [L]^{-D} [s]^{-2}, \quad (5.73)$$

$$[\rho] = [L]^{2-D} [s]^{-2}, \quad (5.74)$$

$$[g] = [L]^{-D} [s]^{-4}. \quad (5.75)$$

The presence of these factors with the same dimension as the field $[s]$ is more explicit for soft spin variables, cf. Eq. 5.105 and Eq. 5.104 in Section 5.A.1.

To complete the mapping to the standard continuum theory, consider the ϕ^4 action in Eq. 3.27, reported here in the case of zero external field:

$$H[\phi(x)] = \int d^D x \left(\frac{\gamma}{2} (\nabla \phi(x))^2 + \frac{1}{2} r_0 \phi^2(x) + \frac{u_0}{4!} \phi^4(x) \right), \quad (5.76)$$

from which the Gaussian propagator is:

$$\tilde{G}^{(0)}(\mathbf{k}) = \frac{1}{\gamma \mathbf{k}^2 + r_0}, \quad (5.77)$$

where the parameters γ , r_0 , and u_0 depend on the temperature, and $\phi(x)$ denotes the field value at position x in the D -dimensional space. The dimensions of these quantities, expressed in terms of the unit of length $[L]$, are determined by requiring that the action H be dimensionless (being the exponent in the partition function):

$$1 \stackrel{!}{=} \left[\int d^D x \gamma (\nabla \phi)^2 \right] = [L]^{D-2} [\gamma] [\phi]^2 \Rightarrow [\gamma] = [L]^{2-D} [\phi]^{-2}, \quad (5.78)$$

$$1 \stackrel{!}{=} \left[\int d^D x r_0 \phi^2 \right] = [L]^D [r_0] [\phi]^2 \Rightarrow [r_0] = [L]^{-D} [\phi]^{-2}, \quad (5.79)$$

$$1 \stackrel{!}{=} \left[\int d^D x u_0 \phi^4 \right] = [L]^D [u_0] [\phi]^4 \Rightarrow [u_0] = [L]^{-D} [\phi]^{-4}. \quad (5.80)$$

Thus, the parameter ρ , defined in the M -layer framework, corresponds to the continuum parameter γ , τ corresponds to r_0 , and g corresponds to u_0 , once the spin field s is identified with the field ϕ . This completes the correspondence between the M -layer framework and the continuum ϕ^4 Field Theory.

The Ginzburg criterion Interestingly, the [UCD](#) of the Ising model can be identified using the results of the M -layer framework. Starting from g, ρ, τ , the only dimensionless combination of them (with one power of g) is: $g \rho^{-\frac{D}{2}} \tau^{\frac{D-4}{2}}$. Using [Eq. 5.65](#), [Eq. 5.66](#) and [Eq. 5.67](#) is simple to verify that

$$g \rho^{-\frac{D}{2}} \tau^{\frac{D-4}{2}} \propto \frac{1}{M}, \quad (5.81)$$

which turns out to be proportional to $1/M$: it is shown that the only existing dimensionless combination of parameters is indeed small in the limit $M \rightarrow \infty$, verifying the perturbative nature of the previous expansion. Indeed, while the bare coupling constant g , defined in [Eq. 5.67](#), is not small if M is large. However, the perturbative expansion is justified by the actual perturbative parameter, identified here in [Eq. 5.81](#), that is indeed small in the large- M limit. The corresponding dimensionless parameter in standard field-theoretic computations is only supposed to be small, while in the M -layer framework it is small by construction.

In this perspective, one can compute the shift of the critical temperature at $\mathcal{O}(1/M)$. Define the constants $\tau \equiv M\tau'$, $\rho \equiv M\rho'$ and $g \equiv Mg'$ and also define $\hat{G}_2(\mathbf{p})$ from the 2-point correlation function in [Eq. 5.69](#)

$$M \hat{G}_2(\mathbf{p}) \equiv \frac{1}{\rho' p^2 + \tau'} \left(1 - \frac{1}{2M} g' \frac{1}{\rho' p^2 + \tau'} \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \frac{1}{\rho' \mathbf{k}^2 + \tau'} \right). \quad (5.82)$$

Considering $M \hat{G}_2(\mathbf{p} = 0)$, the $1/M$ correction in the [RHS](#) is of order $1/\gamma'$ and therefore it diverges at the critical temperature of the Bethe lattice, i.e., when $\gamma' \rightarrow 0$. To overcome this problem, one can follow the standard procedure of considering the inverse of $M \hat{G}_2(\mathbf{p})$ [[Ang22](#)]:

$$(M \hat{G}_2(\mathbf{p}))^{-1} \equiv \rho' p^2 + \gamma' + \frac{1}{2M} g' \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \frac{1}{\rho' \mathbf{k}^2 + \gamma'}. \quad (5.83)$$

The condition $(M \hat{G}_2(0))^{-1} = 0$ determines the critical value τ'_c that is zero at leading order in $1/M$:

$$\tau'_c \equiv -\frac{1}{2M} g' \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \frac{1}{\rho' \mathbf{k}^2}. \quad (5.84)$$

Then consider the following expansion of $(M \hat{G}_2(\mathbf{p}))^{-1}$ for $\mathbf{p}$ and $\tau' - \tau'_c$ both small:

$$(M \hat{G}_2(\mathbf{p}))^{-1} \equiv A(\tau' - \tau'_c) + B p^2 + \mathcal{O}(p^4) + \mathcal{O}((\tau' - \tau'_c)^2). \quad (5.85)$$

Comparing with [Eq. 5.83](#) one obtains:

$$A = 1 - \frac{1}{2M} g' \int_{[-\frac{\pi}{a_l}, \frac{\pi}{a_l}]^D} \frac{d^D k}{(2\pi)^D} \frac{1}{(\rho' \mathbf{k}^2)^2} \quad (5.86)$$

$$B = \rho'. \quad (5.87)$$

The expression for τ'_c is divergent for $D \leq 2$ and the expression for A is divergent for $D \leq 4$, signaling that the [MF](#) expansion on [Eq. 5.85](#) is only valid above $D = 4$ that is indeed the [UCD](#) of the problem. This is a formulation of the Ginzburg criterion, already introduced at the beginning of [Chapter 3](#).

5.4 Application to Ising Model: Conclusions

In the present chapter, the M -layer construction is applied to the Ising model. Besides giving a methodological explanation on how to use this construction, it is shown that the 2- and 4-point connected correlation functions, up to one-loop topological order in this framework, have the same form as in the corresponding ϕ^4 Field Theory, up to one-loop order in terms of Feynman diagrams.

Some works [Ang18, Riz19, Ang20, Riz20, Ang22, Bar24] have already obtained results using the M -layer construction prior to this thesis. However, in these cases, the results did not need to take into account all constants in a systematic way as was done in this chapter. An example is the calculation of the UCD exploiting the Ginzburg criterion [Ang22]: for this calculation, it is sufficient to look at the dimension at which the one-loop correction to the 2-point correlation function diverges, thus the constants due to the vertices and the combinatorial factors are irrelevant. On the other hand, to obtain more accurate predictions on critical behavior, such as critical exponents, it is necessary to check all the various factors mentioned above before summing diagrams' contributions, in order to identify the right coupling constant or the mass.

In this chapter, the construction is applied to the case of the ferromagnetic Ising model, for which one expects that an expansion around the model on the Bethe lattice, or around the model on the FC lattice will give the same results because the ferromagnetic transitions have the same features on the two MF models. Here, this is indeed verified, validating the M -layer method. This is instructive and really important before applying the same method to more complex systems, such as the spin glass with an external field, in Chapter 7, for which the two expansions are expected to yield different results.

Moreover, in Eq. 5.65, Eq. 5.66 and Eq. 5.67, the field-theoretic constants of the model are written in terms of some physical observables such as the lattice spacing a_l , the physical dimension D , and the eigenvalues and eigenvectors of the transfer matrix $T(s, t)$, as defined in Section 5.2.1. Thus, it is possible to numerically control the present constants: this could be of crucial importance to understanding the finite-dimensional behavior of systems if the renormalization flux has more than one FP with a finite basin of attraction. This could be the case of the spin glass transition in a field. In the M -layer approach, once RG equations are written, one can run them for g , knowing exactly the initial value and thus understanding which will be the final reached FP. One could also think of manually changing the microscopic details of the model, changing $g_\lambda(s)$, $\hat{\eta}(s)$ and $p_B(s)$, trying in this way to enter different basins of attraction in the parameters space.

An important ingredient in this chapter is the presence of the so-called *boucle* diagrams. It is important to include these contributions if even-degree vertices are computed, as in the case of the Ising model. In all previous applications of the M -layer method, odd-degree vertices were considered, for which *boucle*-corrections to the vertices are not present in perturbative computations. For this reason, this kind of diagram is not considered in Chapter 6 and Chapter 7, where degree-3 internal vertices are the relevant ones.

Finally, a discussion on the role of the disorder in the M -layer approach is needed. The rewirings are extracted at random and thus one could expect that this randomness should reflect in an additional parameter, with respect to the ones (ρ , τ and g) that are present in the standard non-random Field Theory. However, it is shown that

at one-loop order, this is not the case: no additional parameter associated with the disorder is generated. This is, however, reasonable already thinking of the [RRG](#): in the large M limit, the M -layer graph tends to a Bethe lattice with fixed connectivity, that has a disordered topology. However, even if the Bethe lattice is disordered, for an Ising model the local cavity marginals, and the consequent “local critical temperature”, are the same for all the spins. It is thus natural to expect that the effect of the disorder will appear only at the next orders in the $1/M$ expansion, leading to a disordered local critical temperature and a different universality class for the critical exponents.

5.A Appendices to Chapter 5

5.A.1 Soft Spin Variables

In this section, the results derived for the Ising model with dichotomous variables are generalized to generic soft spin variables s , with measure $d\mu(s)$. The case of dichotomous variables is recovered simply with:

$$d\mu(s) = \delta(1 - s^2) ds. \quad (5.88)$$

In this more general case, the steps outlined in [Section 5.2.1](#) to compute the joint probability of two spins on an open chain, $Z_L(s, t)$, can be repeated. The cavity distribution, $\hat{\eta}(s)$, is introduced analogously:

$$\hat{\eta}(s) = \frac{1}{Z_{\hat{\eta}}} \int d\mu(t) e^{\beta J s t} \hat{\eta}(t)^{z-1} \quad (5.89)$$

$$Z_{\hat{\eta}} = \int d\mu(t) d\mu(s) e^{\beta J s t} \hat{\eta}(t)^{z-1}, \quad (5.90)$$

where z is the connectivity of the associated Bethe lattice. It is noted here that the transfer matrix

$$T(s, t) = \sqrt{\hat{\eta}(s)^{z-2}} e^{\beta J s t} \sqrt{\hat{\eta}(t)^{z-2}} \quad (5.91)$$

is no longer a 2×2 matrix, but an infinite-dimensional operator, which has an infinite set of eigenvalues and eigenvectors:

$$\frac{T(s, t)}{\lambda_{\max}} = \vec{e}_{\lambda_{\max}}(s) \vec{e}_{\lambda_{\max}}(t) + \sum_{\substack{i=2, \dots \\ |\lambda_{i+1}| < |\lambda_i| < 1}} \lambda_i \vec{e}_{\lambda_i}(s) \vec{e}_{\lambda_i}(t), \quad (5.92)$$

where λ_2 corresponds to λ of the Ising variables case, and is the one that assumes the critical value $\lambda_c = \frac{1}{2D-1}$ at the critical temperature T_c . The orthonormal eigenvector of $T(s, t)$, $\vec{e}_{\lambda_{\max}}(s)$, can be checked to be:

$$\vec{e}_{\lambda_{\max}}(s) = \frac{\sqrt{\hat{\eta}(s)^z}}{\sqrt{\int d\mu(s') \hat{\eta}(s')^z}}, \quad (5.93)$$

where $\lambda_{\max} = Z_{\hat{\eta}}$. On the other hand:

$$\int d\mu(t) T(s, t) \vec{e}_{\lambda_i}(t) = \lambda_i \vec{e}_{\lambda_i}(s) \quad \text{and} \quad \int d\mu(s) \vec{e}_{\lambda_i}(s)^2 = 1. \quad (5.94)$$

Given this general form of $T(s, t)$, the joint probability can be computed by repeating the same steps:

$$Z_L(s, t) = p_B(s) p_B(t) + \sum_{\substack{i=2, \dots \\ |\lambda_{i+1}| < |\lambda_i| < 1}} \lambda_i^L g_{\lambda_i}(s) g_{\lambda_i}(t), \quad (5.95)$$

where

$$p_B(s) \equiv \frac{\vec{e}_{\lambda_{\max}}(s)}{\sqrt{\hat{\eta}(s)^{z-2}}} \quad \text{and} \quad g_{\lambda_i}(s) \equiv \frac{\vec{e}_{\lambda_i}(s)}{\sqrt{\hat{\eta}(s)^{z-2}}}. \quad (5.96)$$

Given that $|\lambda_{i+1}| < |\lambda_i| < 1$ for all $i \geq 2$ and considering the aim to describe the critical limit in the regime of large lengths L , all terms with $i \geq 3$ can be neglected, yielding, for $Z_L(s, t)$, the same form as in the Ising variables case: $Z_L(s, t) = p_B(s)p_B(t) + \lambda^L g_\lambda(s) g_\lambda(t)$, with λ_2 denoted as λ . All computations of the observables on a specific diagram proceed analogously to the Ising case. A difference with respect to the Ising variables case is the absence of explicit expressions for $\hat{\eta}(s)$ and $\vec{e}_\lambda(s)$. This implies that, when computing observables, all constants involving the three functions $p_B(s)$, $g_\lambda(s)$, and $\hat{\eta}(s)$ must be retained. To this end, the following function is defined:

$$f(a, b, c, d) \equiv \int d\mu(s) s^a g_\lambda(s)^b p_B^c(s) \hat{\eta}(s)^{z-d}. \quad (5.97)$$

The relevant cases, useful in the following, over which f is computed, are:

$$\begin{aligned} f(0, 2, 2, 4) &= \int d\mu(s) g_\lambda(s)^2 p_B^2(s) \hat{\eta}(s)^{z-4} = \left(\int d\mu(s) \hat{\eta}(s)^z \right)^{-1} \\ &= \int d\mu(s) p_B^4(s) \hat{\eta}(s)^{z-4} = f(0, 0, 4, 4); \end{aligned} \quad (5.98)$$

$$\begin{aligned} f(0, 2, 1, 3) &= \int d\mu(s) g_\lambda(s)^2 p_B(s) \hat{\eta}(s)^{z-3} = \left(\int d\mu(s) \hat{\eta}(s)^z \right)^{-\frac{1}{2}} \\ &= \int d\mu(s) p_B^3(s) \hat{\eta}(s)^{z-3} = f(0, 0, 3, 3); \end{aligned} \quad (5.99)$$

$$f(0, 0, 1, 1) = \int d\mu(s) p_B(s) \hat{\eta}(s)^{z-1} = \left(\int d\mu(s) \hat{\eta}(s)^z \right)^{\frac{1}{2}}; \quad (5.100)$$

$$f(1, 1, 0, 1) = \int d\mu(s) s g_\lambda(s) \hat{\eta}(s)^{z-1} = \int d\mu(s) s \vec{e}_\lambda(s) \hat{\eta}(s)^{\frac{z}{2}}; \quad (5.101)$$

$$f(0, 4, 0, 4) = \int d\mu(s) g_\lambda(s)^4 \hat{\eta}(s)^{z-4} = \int d\mu(s) \frac{\vec{e}_\lambda(s)}{\hat{\eta}(s)^z}, \quad (5.102)$$

where Eq. 5.98 and Eq. 5.99 can be checked using the definitions of $p_B(s)$ and $g_\lambda(s)$ in Eq. 5.96 and the normalization of $\vec{e}_{\lambda_i}(s)$ in Eq. 5.94. With these ingredients, the results for the 2-point and 4-point connected correlation functions for the Ising model with generic soft spin variables can be written. It is remarked that these results are valid in the critical regime, where the lengths of the lines composing the diagrams are large. The final expressions are the same as Eq. 5.69 and Eq. 5.70; what changes are the definitions of the parameters⁴:

$$\rho \equiv M \frac{1}{a^{D-2}} \frac{1 + \lambda}{4D^2 \lambda (1 - \lambda)} \frac{f(0, 0, 1, 1)^2}{f(1, 1, 0, 1)^2}, \quad (5.103)$$

$$\tau \equiv M \frac{1 - \lambda(2D - 1)}{a_l^D 2D \lambda} \frac{f(0, 0, 1, 1)^2}{f(1, 1, 0, 1)^2}, \quad (5.104)$$

$$g \equiv - \left(\frac{f(0, 4, 0, 4)}{f(0, 0, 4, 4)} - 3 \right) \frac{M}{(2D)^4 a_l^D} \frac{(2D)!}{(2D - 4)!}. \quad (5.105)$$

⁴From these expressions and the definition of $f(a, b, c, d)$ in Eq. 5.97 it is possible to get the dimensions of the different parameters, as reported in Section 5.3

Using the definition of the function f , Eq. 5.97, and then the definitions of $p_B(s)$, $g_\lambda(s)$, and $\hat{\eta}(s)$ for the case of Ising variables, the results shown in the main text are recovered. It is important to emphasize that the definitions of the parameters in Eq. 5.105 and Eq. 5.104 are only valid at the critical point. This is a consequence of neglecting all the terms with λ_i for $i \geq 3$ in Eq. 5.95, which leads to the definitions of the functions $f(a, b, c, d)$ being valid only for large L . However, these additional neglected terms do not affect universal quantities, such as critical exponents.

5.A.2 Degree of the interaction in the M -layer framework

A simple question could come to mind: using the M -layer approach on the $a_l \mathbb{Z}^D$ lattice, how can one reproduce interactions with degree larger than the connectivity of the lattice, $2D$? Naively thinking, it seems impossible to map the M -layer construction results in every possible Field Theory, since the latter can always include all possible kinds of interactions that respect the symmetry of the problem. Nevertheless, in the M -layer construction, it is possible to find vertices with interaction degrees larger than $2D$. This can be realized by connecting, for instance in $D = 2$, two vertices of degree 4 by a line of length L_0 , as shown in Figure 5.4. When exploiting the large-length limit for all the lines, but keeping L_0 finite, the resulting contribution from this diagram is numerically the same as the one of a degree-6 vertex. Analogously, a similar procedure can be applied to realize a degree-8 vertex as depicted in Figure 5.4. A similar feature has already been found in some computations regarding the M -layer construction applied to the RFIM [Ang20]. It is easy to generalize this argument to generate a vertex with generic degree from the M -layer construction.

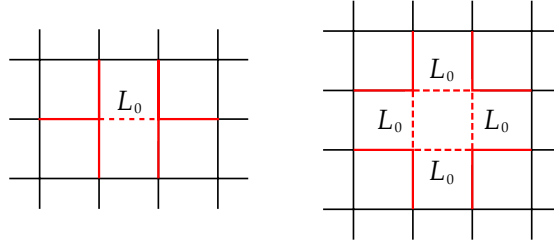


Figure 5.4. Left: In a 2-dimensional lattice, two vertices of degree 4 are connected by a line of length L_0 . When exploiting the large-length limit for all the lines, but keeping L_0 finite, the resulting contribution from this diagram is numerically the same as the one of a degree-6 vertex. Right: 4 vertices of degree 4 are connected by lines of length L_0 . In the large-length limit for all the lines, but keeping L_0 finite, the resulting contribution from this diagram is numerically the same as the one of a degree-8 vertex.

Chapter 6

Percolation: Extracting Critical Exponents

In this chapter, the application of the M -layer construction to the percolation problem in a finite- D hypercube is presented. What is explained in the next sections is based on the generalities of the construction, presented in [Chapter 4](#). All the material of this chapter is taken from [\[Ang25c\]](#).

The first result presented in this chapter is the confirmation of the validity of the M -layer construction as a tool to make predictions on the critical behavior of finite-dimensional lattice problems. In percolation theory, one usually maps the problem onto the q -state Potts model and then performs the analytic continuation $q \rightarrow 1$ in order to apply the field-theoretic methods described in [Chapter 3](#). The advantage of this construction is that it allows one to use the field-theoretic machinery without relying on such a mapping and analytic continuation. Moreover, with a good knowledge of the M -layer construction, as explained in [Chapter 4](#), computing correlation functions within this framework is not complicated. This method produces exactly the same expansion for the 2-point and 3-point correlation functions as derived from the Potts model Field Theory up to one-loop order. Furthermore, with this method, it is also possible to analytically verify (again, up to one loop) that the critical exponents of site percolation are equal to those of bond percolation, while the mapping to the Potts model is formally valid only for the bond percolation problem, thus such a universality statement cannot be done using the associated Field Theory. Before this work, the universality between these two problems had not been demonstrated analytically, but only supported by extensive numerical studies.

The additional value of this work is methodological: it is shown, for the first time, that from the $1/M$ expansion on the M -layer lattice one can obtain the ϵ -expansion, through the suitable introduction of a dimensionless β -function in analogy with what is usually done in standard Field Theory [\[Par98, Zin21\]](#), see [Section 3.1](#). This is a fundamental step that helped in applying the same techniques to more complicated systems, for which a finite-dimensional solution is still debated, such as the EA model [\[Ang18, Ang22\]](#) or Anderson localization [\[Bar24\]](#). Thanks to the work presented in this chapter, new results on the EA model in a field are obtained and presented in [Chapter 7](#).

In [Section 6.1](#), the problem of interest of this chapter is defined and its behavior on the Bethe lattice is described, serving as the starting point of the M -layer construction,

applied in [Section 6.2](#). The method to estimate critical exponents from the results of the construction is presented in [Section 6.3](#). In [Section 6.4](#), the relation with the results of the Potts model's Field Theory is described. Conclusions for the chapters are given in [Section 6.5](#). In the end, some appendices to the chapter are presented: the systematic identification of the non-universal constants for the M -layer construction is given in [Section 6.A.1](#); in [Section 6.A.2](#) additional contributions due to irrelevant diagrams are computed.

6.1 Model's Definition

In the standard site (respectively bond) percolation problem, each site (respectively bond) is present, or *active*, independently of the neighbors with probability p . In the site percolation problem, a cluster is defined as a subset of nearest-neighbor active sites, while in bond percolation a cluster is defined as a subset of sites connected by nearest-neighbor active bonds. At $p = p_c$ a giant cluster appears, containing a finite fraction of all the sites N . The analysis mainly applies to the non-percolating phase $p < p_c$ and from now on attention is restricted to this case. The critical behavior in the non-percolating phase is characterized by the average number $n(s, p)$ of clusters of size s in a system of size N . This distribution is cut off at a typical size s^* that diverges at the critical point. The generic q -point function $C_q(x_1, \dots, x_q)$, is defined as the probability that the sites at $x_1, \dots, x_q$ belong to the same cluster. The site positions are identified by vectors in the D -dimensional hypercube, denoted as $a_l \mathbb{Z}^D$, characterized by the lattice spacing a_l . According to scaling arguments [[Conoo](#), [Sta18](#)], the 2-point function is expected to obey the following scaling form for large $|x_1 - x_2|$ and for p close to p_c :

$$C_2(x_1, x_2) = \frac{1}{|x_1 - x_2|^{D-2+\eta}} f_{C_2} \left(\frac{|x_1 - x_2|}{\xi} \right), \quad (6.1)$$

where f_{C_2} is a proper scaling function, η is the anomalous dimension and ξ is the correlation length that diverges at the critical point with exponent ν :

$$\xi \sim \frac{1}{|p - p_c|^\nu}. \quad (6.2)$$

The typical size s^* scales with the correlation length as

$$s^* \sim \xi^{D_f}, \quad (6.3)$$

where D_f stands for the *fractal dimension* of the clusters. The distribution of cluster sizes also obeys a scaling law [[Sta18](#), [Conoo](#)]:

$$n(s, p) = s^{-\tau} f_n(|p - p_c| s^\sigma), \quad (6.4)$$

where f_n is another scaling function, σ is related to the scalings of the typical size and to the correlation length through

$$\sigma \equiv \frac{1}{\nu D_f}, \quad (6.5)$$

and τ is defined as

$$\tau \equiv 1 + \frac{D}{D_f}. \quad (6.6)$$

The space integrals of the $C_q(x_1, \dots, x_q)$, called susceptibilities, are also important:

$$\chi_q \equiv \sum_{x_2, \dots, x_q} C_q(x_1, \dots, x_q) \quad (6.7)$$

which are independent of x_1 in a homogeneous system (they only depend on the differences between the points). They are related to the moments of $n(s, p)$ by:

$$\chi_q = \sum_{s=0}^{\infty} s^q n(s, p). \quad (6.8)$$

By comparing Eq. 6.7 and Eq. 6.8 and using the scaling of $n(s, p)$, it follows that the susceptibilities diverge as

$$\chi_q \sim \xi^{-D+D_f q}. \quad (6.9)$$

6.1.1 Percolation on the Bethe Lattice

Considering the site percolation case, in this section, the equations for the critical behavior of $g(s, p)$ are derived, defined as the probability that a randomly chosen site belongs to a cluster of size s ($s = 0$ means that the site itself is not active). Given the definition of $n(s, p)$, it holds that

$$g(0, p) = 1 - p; \quad g(s, p) = s n(s, p), \quad \text{for } s > 0. \quad (6.10)$$

Here and in the following, the term “Bethe lattice” denotes a [RRG](#) with fixed connectivity z . Notice that $g(s, p)$ is normalized to 1, as the probability that a randomly chosen site belongs to a cluster is $\sum_{s=1}^{\infty} s n(s, p) = p$. The associated *cavity* probability, $g_{cav}(s, p)$, is defined as the probability that a randomly chosen site, for which one of its z edges is removed, belongs to a cluster of size s . This definition is useful since on a Bethe lattice two sites are connected by a unique sequence of adjacent edges in the large- N limit, so that, removing one edge, the two sites become completely independent and the resulting probability to belong to a cluster factorizes [[Méz87c](#)], see [Section 2.2.1](#). Thus, given that each site is active with probability p , an implicit equation for $g_{cav}(s, p)$ on the Bethe lattice with fixed connectivity z can be written:

$$g_{cav}(s, p) = (1 - p) \delta_{s,0} + p \sum_{s_1=0}^{\infty} \cdots \sum_{s_{z-1}=0}^{\infty} g_{cav}(s_1, p) \cdots g_{cav}(s_{z-1}, p) \delta_{s, 1+s_1+\cdots+s_{z-1}}, \quad (6.11)$$

where the first term describes the site not being present (resulting cluster size $s = 0$), and the second term is the probability that the site is present, p , times the product of the factorized probabilities that the $z - 1$ neighboring sites (one edge removed, see [Figure 6.1](#)) belong to clusters of sizes $s_1, s_2, \dots, s_{z-1}$. The resulting size is the sum of the sizes plus one (the given site). The probability $g(s, p)$ is then expressed in terms of the cavity probability as:

$$g(s, p) = (1 - p) \delta_{s,0} + p \sum_{s_1=0}^{\infty} \cdots \sum_{s_z=0}^{\infty} g_{cav}(s_1, p) \cdots g_{cav}(s_z, p) \delta_{s, 1+s_1+\cdots+s_z}, \quad (6.12)$$

the difference being that the product is over z terms g_{cav} , since all the z edges of the site are present.

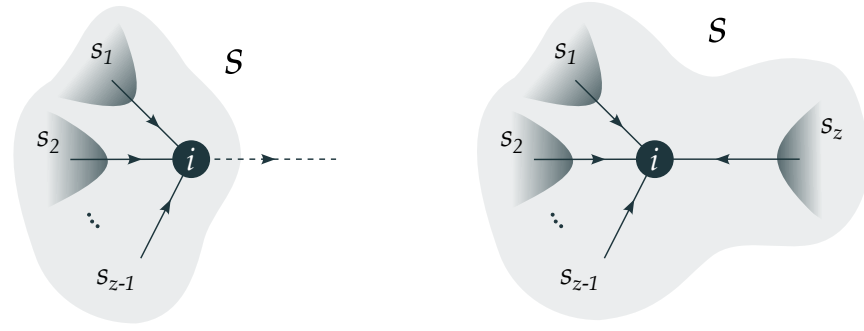


Figure 6.1. Graphic representation of Eq. 6.11 and Eq. 6.12. Left: on a Bethe lattice with connectivity z one of the edges of site i is removed, represented with a dashed line. The $z - 1$ remaining neighboring sites are connected by site i only, so the probabilities $g_{cav}(s_1, p), \dots, g_{cav}(s_{z-1}, p)$ factorize. The total resulting size of the cavity cluster containing i is $s = 1 + s_1 + \dots + s_{z-1}$. Right: none of the edges of site i is removed. Here, the cavity probabilities factorize as well, but the product includes $g_{cav}(s_z, p)$.

Next, the generating function $\tilde{g}(t, p) \equiv \sum_{s=0}^{\infty} g(s, p) e^{-ts}$ and its cavity counterpart $\tilde{g}_{cav}(t, p)$ are defined. Eq. 6.11 becomes:

$$\tilde{g}_{cav}(t, p) = 1 - p + p (\tilde{g}_{cav}(t, p))^{z-1} e^{-t}. \quad (6.13)$$

Differentiating the above equation with respect to t and setting $t = 0$ yields

$$\tilde{g}'_{cav}(0, p) = \frac{p}{p(z-1) - 1}. \quad (6.14)$$

The moments of $g(s, p)$ relate to derivatives of $\tilde{g}(t, p)$ at $t = 0$. In particular, recalling the definition in Eq. 6.8, it holds:

$$\chi_2 = -\tilde{g}'(0, p) = \frac{p(p+1)}{1 - p(z-1)}, \quad (6.15)$$

which diverges, as expected, at the critical point $p_c = 1/(z-1)$. The previous divergent behavior can also be obtained by considering the 2-point correlation previously defined. As explained in detail below, in Section 6.2, the correlation between sites at distance L on the Bethe lattice is $C_2(L) = p^{L+1}$. The associated susceptibility, χ_2 , is

$$\chi_2 = \sum_{L=0}^{\infty} C_2(L) n(L) = p + p \sum_{L=1}^{\infty} z(z-1)^{L-1} p^L = \frac{p(p+1)}{1 - p(z-1)}, \quad (6.16)$$

where $n(L)$ is the number of neighboring sites at distance L

$$n(L) = \begin{cases} 1 & \text{if } L = 0 \\ z(z-1)^{L-1} & \text{if } L \geq 1. \end{cases} \quad (6.17)$$

As noticed in Section 5.2.1, in the Bethe lattice the 2-point correlation always decays exponentially, even at the critical point $p = p_c$. The reason for the divergence of the susceptibility is the number of neighboring sites that grows exponentially with distance L . This means that the correlation length ξ_{BL} , implicitly defined by

$$C_2(L) \propto p^L \equiv e^{-\frac{L}{\xi_{BL}}} \quad (6.18)$$

is always finite and equal to $(-\log(p))^{-1}$. With this definition of ζ_{BL} the anomalous dimension η , associated to the power-law behavior of $C_2(L)$, and the exponent ν , associated to the power-law behavior of the correlation length, are not defined on the Bethe lattice¹. An interesting feature of the M -layer construction is that it allows the computation of η and ν also in the limit $M \rightarrow \infty$, as discussed later.

Scaling of $n(s, p)$ To compute the scaling of $n(s, p)$, attention is focused on $g(s, p)$ for p close to the critical point and large s , corresponding to small t in $\tilde{g}(t, p)$. Define

$$\delta\tilde{g}(t, p) \equiv \tilde{g}(t, p) - 1 = \sum_{s=0}^{\infty} g(s, p)(e^{-st} - 1) \quad (6.19)$$

and its cavity counterpart $\delta\tilde{g}_{\text{cav}}(t, p) \equiv \tilde{g}_{\text{cav}}(t, p) - 1$. Differentiating Eq. 6.13 with respect to t produces, for small t and p close to p_c :

$$\delta\tilde{g}'_{\text{cav}}(t, p)(1 - p/p_c - (c-2)\delta\tilde{g}_{\text{cav}}(t, p)) = -p_c, \quad (6.20)$$

which yields

$$\delta\tilde{g}_{\text{cav}}(t, p) = a(1 - (1 + t/t^*)^{1/2}), \quad (6.21)$$

where

$$\delta p \equiv p - p_c \quad a \equiv -\delta p \frac{z-1}{z-2}, \quad t^* \equiv \delta p^2 \frac{(z-1)^3}{2z-4}. \quad (6.22)$$

For small t and δp also

$$\delta\tilde{g}(t, p) = \frac{z}{z-1} \delta\tilde{g}_{\text{cav}}(t, p). \quad (6.23)$$

Replacing the sum with an integral (justified since small t corresponds to large s), computing the inverse Laplace transform of Eq. 6.21 and using Eq. 6.23 leads to

$$g(s, p) \sim \frac{1}{s^{3/2}} e^{-st^*} \rightarrow n(s, p) \sim \frac{1}{s^{5/2}} e^{-st^*}, \quad (6.24)$$

which obeys Eq. 6.4 with exponents

$$\sigma = \frac{1}{2} \quad \text{and} \quad \tau = \frac{5}{2}, \quad (6.25)$$

identified as the MF values. Percolation on the M -layer random lattice (associated with a finite- D hypercube) is tackled in the next sections. In the limit $M \rightarrow \infty$ the function $n(s, p)$ on the M -layer becomes identical to that of the Bethe lattice and thus $\tau = 5/2$ and $\sigma = 1/2$. Also, for $M \rightarrow \infty$ the 2-point function obeys the scaling form Eq. 6.1 with exponents

$$\nu = \frac{1}{2}, \quad \eta = 0, \quad (6.26)$$

in all dimensions $D \geq 2$, see the comment after Eq. 6.55. These relationships are consistent with Eq. 6.5 and Eq. 6.6 only for $D = D_{\text{uc}} = 6$. Indeed, $\tau = D/D_f + 1$ is

¹One can consider an alternative definition of ζ as $\zeta^2 \equiv \frac{\sum_L L^2 C_2(L)}{\sum_L C_2(L)}$ [Sta18, Chro5]. With this definition ζ diverges on the Bethe lattice. This discrepancy is a pathology related to the Bethe lattice topology, where volume grows exponentially with distance, while it grows as a power law in finite dimensions. This discrepancy is absent in finite dimensions and the second definition will be used.

a hyperscaling relation that is not generically valid [Con00], unlike the more general $\sigma^{-1} = \nu D_f$, which implies $D_f = 4$ for the $M \rightarrow \infty$ model in any dimension. Computing the $1/M$ corrections around the $M \rightarrow \infty$ limit shows that for finite M the critical exponents match those of the $M \rightarrow \infty$ limit for $D \geq D_{uc} = 6$ but differ for $D < D_{uc} = 6$. Conversely, for $D < 6$ both relationships in Eq. 6.5 and Eq. 6.6 hold. Note that the $M \rightarrow \infty$ model essentially plays the role of the Gaussian model in ferromagnetism, see [Par98], chapters 4 and 5.

6.2 Computation of Correlation Functions

Following the prescriptions of the M -layer construction (see Chapter 4), the results are reported here in the non-percolating phase, $p < p_c$. Two observables are considered: the 2- and 3-point correlation functions $\overline{C_2(x_1, x_2)}$ and $\overline{C_3(x_1, x_2, x_3)}$, where $\overline{\cdot}$ denotes the average over the rewirings of the M -layer procedure and over the realizations of the percolation lattice. According to the M -layer rules these correlation functions will be written as sums, over different diagrams, of $\mathcal{C}_{n,lc}(\mathcal{G}; \{\mathcal{L}\})$, the n -point line-connected correlation, averaged over the realizations of the percolation problem and computed on the diagram $\mathcal{G}$, embedded on a tree graph, where $\{\mathcal{L}\}$ indicates the set of lengths of the different lines of the diagram. For both site and bond percolation, the 2-point (3-point) correlation is defined as the probability that two (three) sites, at positions x_1 and x_2 (x_1, x_2 and x_3) are occupied and belong to the same cluster. At one loop level, pieces already considered in loop-free diagrams must be subtracted to compute the *line-connected*. The two observables are analyzed separately, following the steps listed in Section 4.2.

Observable: $\overline{C_2(x_1, x_2)}$

❶ Identify the relevant topological diagrams

The simplest diagram connecting two points is the bare line, called $\mathcal{G}_1$. Including the possibility of a loop to be present, the diagram composed of four lines with two vertices of degree 3, where the two internal lines compose a loop, is denoted $\mathcal{G}_2$.

Other possibilities include the tadpole-type diagrams, which connect two points with a loop generated by one degree-4 vertex or connect two points by two degree-3 vertices, respectively the diagrams $\mathcal{G}'$ and $\mathcal{G}''$ in Figure 6.2. These last two diagrams give no contributions to the *line-connected* 2-point observable for percolation, as will be seen in step 3 below, and are not considered in the following steps.

❷ Weights, number of projections and symmetry factors

Diagram $\mathcal{G}_1$:

- $W(\mathcal{G}_1) = \frac{1}{M}$;
- $\mathcal{N}(\mathcal{G}_1; L; x_1, x_2) = \mathcal{N}_L(x_1, x_2)$;
- $S(\mathcal{G}_1) = 1$.

Diagram $\mathcal{G}_2$:

- $W(\mathcal{G}_2) = \frac{1}{M^2}$;
- $\mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) = (2D)^2 \left(\frac{(2D)!}{(2D-3)!} \right)^2 \sum_{\substack{x_0, x'_0 \\ \in a_l \mathbb{Z}^D}} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x'_0, x_2)}{(2D)^2} \prod_{l=L_A, L_B} \frac{\mathcal{N}_l(x_0, x'_0)}{(2D)^2}$;
- $S(\mathcal{G}_2) = 2$.

where $\vec{L} = (L_1, L_A, L_B, L_2)$.

- ③ *Computation of $\mathcal{C}_{2,lc}(\mathcal{G}_1; L)$ and $\mathcal{C}_{2,lc}(\mathcal{G}_2; \vec{L})$* Given the definition of the line-connected 2-point correlation for both percolation problems, the contributions of diagrams $\mathcal{G}_1$ and $\mathcal{G}_2$ for the site percolation problem are first computed:

$$\mathcal{C}_{2,lc}(\mathcal{G}_1; L) = p p^L; \quad (6.27)$$

$$\mathcal{C}_{2,lc}(\mathcal{G}_2; \vec{L}) = -p^{L_1+L_2+L_A+L_B}. \quad (6.28)$$

The first result is immediate as, in the non-percolating phase, all the $L + 1$ sites connected by a line of length L must be active in order to connect the two sites at the extremities. The second result appears because, for the sites at the extremities to be connected, one or both lines of the loop must consist of active sites, in addition to the external lines, which also need to be composed of active sites. The associated probability for this to happen is $p^{L_1+1}(p^{L_A-1} + p^{L_B-1} - p^{L_A+L_B-2})p^{L_2+1}$. Eq. 6.28 is obtained by subtracting the straight line contributions, already taken into account with $\mathcal{G}_1$: $p^{L_1+1}p^{L_A-1}p^{L_2+1}$ and $p^{L_1+1}p^{L_B-1}p^{L_2+1}$. This last operation is the application of the *line-connected* definition, given in Section 4.1.3.

Performing the same computation for diagrams $\mathcal{G}'$ and $\mathcal{G}''$ yields zero, as anticipated. The reason is that the two tadpoles, which enter the site at x_0 , do not change the probability that sites x_1 and x_2 belong to the same cluster with respect to the case where the loop is not present. Indeed, independently of the lines of the tadpole, site x_0 must be active to connect the two sites; then, subtracting the contributions needed to define the line-connected observable, which are the simple lines without tadpoles, the net contribution is zero. These diagrams are instead relevant in the percolating phase to be studied in a subsequent work.

A similar computation can be performed for bond percolation. In this case, considering the contribution of $\mathcal{G}_1$, for the two sites to be in the same cluster, all the edges connecting the two must be active:

$$\mathcal{C}_{2,lc}^{bond}(\mathcal{G}_1; L) = p^L; \quad (6.29)$$

similarly, the line-connected contribution for $\mathcal{G}_2$ is

$$\mathcal{C}_{2,lc}^{bond}(\mathcal{G}_2; \vec{L}) = -p^{L_1+L_2+L_A+L_B}. \quad (6.30)$$

The same argument used for site percolation applies to the topological diagrams $\mathcal{G}'$ and $\mathcal{G}''$ in bond percolation, for which they also give zero contribution.

- ④ *Sum of the contributions*

The expression for $\overline{\mathcal{C}_2(x_1, x_2)}$ for site percolation is

$$\begin{aligned} \overline{C_2(x_1, x_2)} &= \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}(\mathcal{G}_1; L; x_1, x_2) \mathcal{C}_{2,lc}(\mathcal{G}_1; L) + \\ &\quad + \frac{1}{2M^2} \sum_{\vec{L}}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) \mathcal{C}_{2,lc}(\mathcal{G}_2; \vec{L}) + \mathcal{O}\left(\frac{1}{M^3}\right), \end{aligned} \quad (6.31)$$

while, for bond percolation:

$$\begin{aligned} \overline{C_2^{bond}(x_1, x_2)} &= \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}(\mathcal{G}_1; L; x_1, x_2) \mathcal{C}_{2,lc}^{bond}(\mathcal{G}_1; L) + \\ &\quad + \frac{1}{2M^2} \sum_{\vec{L}}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) \mathcal{C}_{2,lc}^{bond}(\mathcal{G}_2; \vec{L}) + \mathcal{O}\left(\frac{1}{M^3}\right). \end{aligned} \quad (6.32)$$

The only difference lies in the observable computed on a given diagram, here $\mathcal{G}_1$ and $\mathcal{G}_2$, which is the only model-dependent part of the M -layer computations.

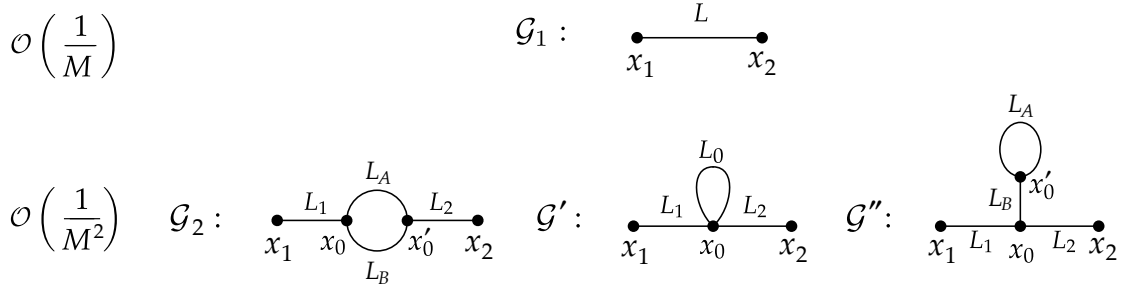


Figure 6.2. Diagrams that contribute to the 2-point correlation functions up to one loop. Even if, in principle, $\mathcal{G}'$ and $\mathcal{G}''$ should be considered, the 2-point correlation function is 0 when computed on them.

Observable: $\overline{C_3(x_1, x_2, x_3)}$

❶ *Identify the relevant topological diagrams*

The simplest diagram connecting three points is the bare degree-3 vertex, which is here denoted $\mathcal{G}_3$. Including the possibility of a loop, one should consider the diagram composed of six lines with three vertices of degree 3, which is here denoted $\mathcal{G}_4$. At one-loop level, three more diagrams connect three points with a single loop, which are the same as $\mathcal{G}_3$, but where one of the external legs is dressed with $\mathcal{G}_2$. Such diagrams, including all permutations, are denoted $\mathcal{G}_5$. All these diagrams are in [Figure 6.3](#).

❷ *Weights, number of projections and symmetry factors*

Diagram $\mathcal{G}_3$:

- $W(\mathcal{G}_3) = \frac{1}{M^2}$;
- $\mathcal{N}(\mathcal{G}_3; \vec{L}; x_1, x_2, x_3) = (2D)^3 \frac{(2D)!}{(2D-3)!} \sum_{x_0 \in a_l \mathbb{Z}^D} \prod_{i=1}^3 \frac{\mathcal{N}_{L_i}(x_i, x_0)}{(2D)^2}$;
- $S(\mathcal{G}_3) = 1$.

Diagram $\mathcal{G}_4$:

- $W(\mathcal{G}_4) = \frac{1}{M^3}$;
- $\mathcal{N}(\mathcal{G}_4; \vec{L}''; x_1, x_2, x_3) = (2D)^3 \left(\frac{(2D)!}{(2D-3)!} \right)^3 \sum_{\substack{x_0, x'_0, x''_0 \\ \in a_l \mathbb{Z}^D}} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x_2, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_3}(x_3, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x_0, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_C}(x'_0, x''_0)}{(2D)^2};$
- $S(\mathcal{G}_4) = 1$.

Diagram $\mathcal{G}_5$:

- $W(\mathcal{G}_5) = \frac{1}{M^3}$;
- $\mathcal{N}(\mathcal{G}_5; \vec{L}'''; x_1, x_2, x_3) = (2D)^3 \left(\frac{(2D)!}{(2D-3)!} \right)^3 \sum_{\substack{x_0, x'_0, x''_0 \\ \in a_l \mathbb{Z}^D}} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_{2A}}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_{2B}}(x_2, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_3}(x_3, x_0)}{(2D)^2} \prod_{l=L_A, L_B} \frac{\mathcal{N}_l(x'_0, x''_0)}{(2D)^2};$
- $S(\mathcal{G}_5) = 2$,

where $\vec{L}' = (L_1, L_2, L_3)$, $\vec{L}'' = (\vec{L}', L_A, L_B, L_C)$ and $\vec{L}''' = (L_1, L_{2A}, L_A, L_B, L_{2B}, L_3)$.

③ Computation of $\mathcal{C}_{3,lc}(\mathcal{G}_3; \vec{L}')$, $\mathcal{C}_{3,lc}(\mathcal{G}_4; \vec{L}'')$ and $\mathcal{C}_{3,lc}(\mathcal{G}_5; \vec{L}''')$

As for the 2-point function the contributions are computed, starting from the site percolation problem:

$$\mathcal{C}_{3,lc}(\mathcal{G}_3; \vec{L}') = p p^{L_1+L_2+L_3}; \quad (6.33)$$

$$\mathcal{C}_{3,lc}(\mathcal{G}_4; \vec{L}'') = -2p^{L_1+L_2+L_3+L_A+L_B+L_C}; \quad (6.34)$$

$$\mathcal{C}_{3,lc}(\mathcal{G}_5; \vec{L}''') = -p^{L_1+L_{2A}+L_{2B}+L_A+L_B+L_3}. \quad (6.35)$$

The result for $\mathcal{G}_3$ is easily derived, considering that all the sites of the topology must be active for the extremities to be connected. The result for $\mathcal{G}_5$ is obtained by multiplying the contribution for the bare vertex by the loop correction of the 2-point function, diagram $\mathcal{G}_2$, with the corresponding lengths. The contribution of $\mathcal{G}_4$ is a generalization of the computation for $\mathcal{G}_2$; to connect the three extremities, two of the three (or all three) lines of the loop must consist of all active sites. Moreover, in this case, one has to subtract three contributions due to the definition of line-connected observables. These terms correspond to cutting L_A , L_B , and L_C respectively, already included in the bare contribution $\mathcal{G}_3$.

Analogously to the 2-point function, the same arguments are used to compute the contributions for the bond percolation 3-point function:

$$\mathcal{C}_{3,lc}^{bond}(\mathcal{G}_3; \vec{L}') = p^{L_1+L_2+L_3}; \quad (6.36)$$

$$\mathcal{C}_{3,lc}^{bond}(\mathcal{G}_4; \vec{L}'') = -2p^{L_1+L_2+L_3+L_A+L_B+L_C}; \quad (6.37)$$

$$\mathcal{C}_{3,lc}^{bond}(\mathcal{G}_5; \vec{L}''') = -p^{L_1+L_{2A}+L_{2B}+L_A+L_B+L_3}. \quad (6.38)$$

④ Sum of the contributions

The expression for $\overline{C_3(x_1, x_2, x_3)}$ for site percolation is

$$\begin{aligned} \overline{C_3(x_1, x_2, x_3)} &= \frac{1}{M^2} \sum_{\vec{L}'}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_3; \vec{L}'; x_1, x_2, x_3) \mathcal{C}_{3,lc}(\mathcal{G}_3; \vec{L}') + \\ &\quad + \frac{1}{M^3} \sum_{\vec{L}''}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_4; \vec{L}''; x_1, x_2, x_3) \mathcal{C}_{3,lc}(\mathcal{G}_4; \vec{L}'') + \\ &\quad + \frac{1}{2M^3} \sum_{\vec{L}'''}^{1, \dots, \infty} \mathcal{N}(\mathcal{G}_5; \vec{L}'''; x_1, x_2, x_3) \mathcal{C}_{3,lc}(\mathcal{G}_5; \vec{L}''') + \text{perm.} + \mathcal{O}\left(\frac{1}{M^4}\right). \end{aligned} \quad (6.39)$$

As noticed for the 2-point function, the expression of $\overline{C_3^{bond}(x_1, x_2, x_3)}$ for bond percolation is the same as $\overline{C_3(x_1, x_2, x_3)}$ with corresponding observables $\mathcal{C}_{3,lc}^{bond}(\mathcal{G}_3; \vec{L}')$, $\mathcal{C}_{3,lc}^{bond}(\mathcal{G}_4; \vec{L}'')$ and $\mathcal{C}_{3,lc}^{bond}(\mathcal{G}_5; \vec{L}''')$. It is not written here for brevity.

In [Section 6.A.2](#), the reasons are discussed why other possible, but *irrelevant*, diagrams for the critical behavior of the percolation problem are not included, and in [Section 6.A.2](#), the explicit computation of the leading order critical behavior of the 4-point correlation function is presented.

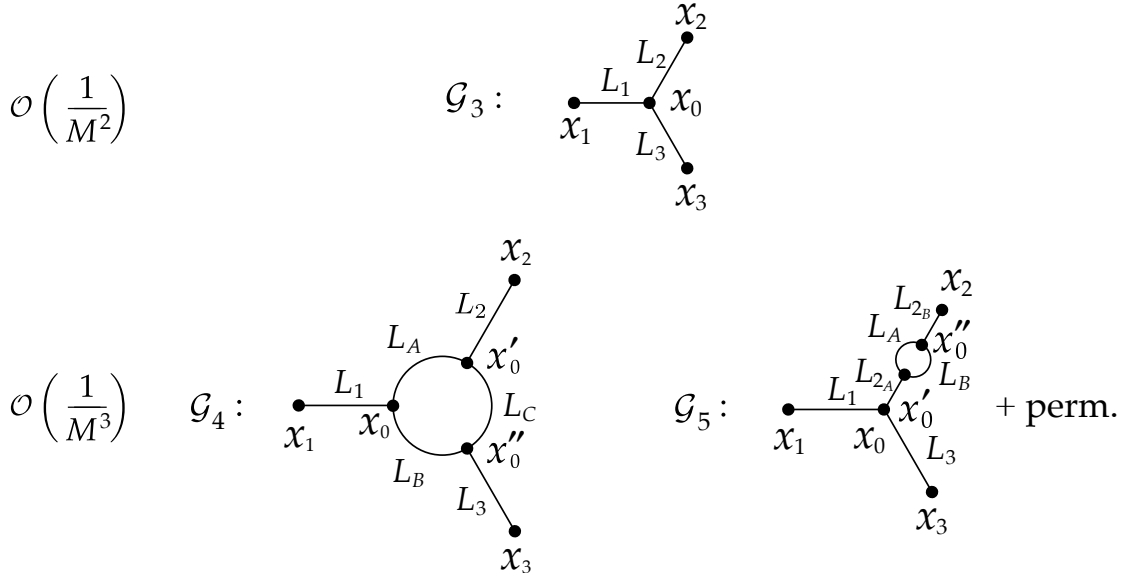


Figure 6.3. Diagrams that contribute to the 3-point correlation functions up to one loop.

6.2.1 Computation of the Moments of $n(s, p)$

To compute χ_2 and χ_3 the Fourier transform is performed for $\overline{C_2(x_1, x_2)}$ and $\overline{C_3(x_1, x_2, x_3)}$, given in [Eq. 6.31](#) and [Eq. 6.39](#), using the same convention as in [Chapter 5](#), see [Eq. 5.39](#) and [Eq. 5.40](#). It is also used that $\mathcal{N}_L(x_1, x_2)$ depends only on the difference between its start and end points, so that in Fourier space

$$\widehat{\mathcal{N}}_L(k_1, k_2) = (2\pi)^D \delta^D(k_1 + k_2) \widehat{\mathcal{N}}_L(k_1) \quad (6.40)$$

where, for small $|k|$ [Alt17, Riz19],

$$\widehat{\mathcal{N}}_L(k) \sim 2D(2D-1)^{L-1} a_l^D e^{-k^2 a_l^2 L/(2D-2)}. \quad (6.41)$$

Indeed, at variance with Chapter 5, one can directly write the asymptotics for the number of NBP, as shown in the previous equation.

Then, since in the critical region sums will be dominated by large L contributions, the sums over lengths can be written as integrals:

$$\sum_{L=1}^{\infty} \rightarrow \int_{1/\Lambda^2}^{\infty} dL, \quad (6.42)$$

with UV cutoff $\Lambda = 1$ to make contact with Field Theory. Unlike in Field Theory, where the UV cutoff is manually inserted, in the M -layer construction it arises naturally because of the lattice spacing (see Section 6.4 for more details). For site percolation, the resulting 2- and 3-point functions are respectively

$$\begin{aligned} \overline{\widehat{C}_2(k, k')} &= \frac{C B^2 a_l^D}{A\mu} \frac{1}{\widehat{k}^2 + 1} (2\pi)^D \delta^D(k + k') \times \\ &\left(1 - \frac{A\mu^{D/2-3}}{2(\widehat{k}^2 + 1)} \int \frac{d^D \widehat{q}}{(2\pi)^D} \int d\widehat{L}_A d\widehat{L}_B e^{-(1+(\widehat{k}-\widehat{q})^2)\widehat{L}_A} e^{-(1+\widehat{q}^2)\widehat{L}_B} \right) + \mathcal{O}\left(\frac{1}{M^3}\right) \end{aligned} \quad (6.43)$$

and

$$\begin{aligned} \overline{\widehat{C}_3(k_1, k_2, k_3)} &= \frac{C B^3 a_l^{2D}}{A\mu^3} \frac{(2\pi)^D \delta^D(k_1 + k_2 + k_3)}{(\widehat{k}_1^2 + 1)(\widehat{k}_2^2 + 1)(\widehat{k}_3^2 + 1)} \left(1 + \right. \\ &- 2A\mu^{D/2-3} \int \frac{d^D \widehat{q}}{(2\pi)^D} \int d\widehat{L}_A d\widehat{L}_B d\widehat{L}_C e^{-(1+(\widehat{k}_2+\widehat{k}_3+\widehat{q})^2)\widehat{L}_A} e^{-(1+(\widehat{k}_2+\widehat{q})^2)\widehat{L}_B} e^{-(1+\widehat{q}^2)\widehat{L}_C} + \\ &\left. - \frac{1}{2} \frac{A\mu^{D/2-3}}{\widehat{k}_2^2 + 1} \int \frac{d^D \widehat{q}}{(2\pi)^D} \int d\widehat{L}_A d\widehat{L}_B e^{-(1+(\widehat{k}_2+\widehat{q})^2)\widehat{L}_A} e^{-(1+\widehat{q}^2)\widehat{L}_B} + \text{perm.} \right) + \mathcal{O}\left(\frac{1}{M^4}\right), \end{aligned} \quad (6.44)$$

where μ is called *bare mass*. It is defined as the distance of the control parameter (in this case the probability p) to the critical value, near the critical point:

$$\mu \equiv -\ln\left(\frac{p}{p_c}\right) \quad \text{for } p \sim p_c, \quad (6.45)$$

where $p_c = 1/(2D-1)$ is the critical value for both site and bond percolation on a Bethe lattice with branching ratio $2D-1$. The following non-universal constants are defined:

$$A \equiv \frac{1}{M} \left(\frac{(2D)!}{(2D-3)!} \right)^2 p^{-1} (2D)^{-3} (2D-2)^{D/2} \left(\frac{1}{2D-1} \right)^3, \quad (6.46)$$

$$B \equiv \frac{1}{M} (2D)^{-1} \left(\frac{(2D)!}{(2D-3)!} \right) \left(\frac{1}{2D-1} \right)^2, \quad (6.47)$$

$$C \equiv (2D - 2)^{D/2}, \quad (6.48)$$

with momenta and lengths rescaled as:

$$\hat{k} \equiv k \frac{a_l}{\sqrt{\mu(2D - 2)}}, \quad \text{and} \quad \hat{L}_i \equiv L_i \mu. \quad (6.49)$$

Note that in Eq. 6.43 and Eq. 6.44 the extremes of integration $(\mu/\Lambda^2, \infty)$ of the integrals over $\hat{L}$ are omitted for the sake of simplicity. In Section 6.A.1, it is shown how to generalize this computation for a V_e -point function with $V_e \geq 2$, as well as the reasoning behind the identification of the constants A , B , and C . The same steps apply for bond percolation, with the only difference being the definition of the non-universal constant A :

$$A_{\text{bond}} \equiv \frac{1}{M} \left(\frac{(2D)!}{(2D-3)!} \right)^2 (2D)^{-3} (2D-2)^{D/2} \left(\frac{1}{2D-1} \right)^3, \quad (6.50)$$

where no factor p^{-1} appears, unlike Eq. 6.46. In the following, explicit computations for the site problem only will be performed; the reader can reproduce them for bond percolation by using Eq. 6.50 instead of Eq. 6.46.

From the above expressions, the functions χ_q are computed. Given a generic q -point function

$$\overline{\hat{C}_q(k_1, \dots, k_q)} \equiv (2\pi)^D \delta^D \left(\sum_{i=1}^q k_i \right) \hat{G}_q(k_1, \dots, k_{q-1}), \quad (6.51)$$

from which the q -point susceptibility is computed as a function of μ

$$\chi_q(\mu) = \frac{1}{a_l^{(q-1)D}} \hat{G}_q(0, \dots, 0). \quad (6.52)$$

This yields

$$\chi_2(\mu) = \frac{C B^2}{A \mu} \left(1 - \frac{A \mu^{D/2-3}}{2(4\pi)^{D/2}} \int \frac{d\hat{L}_A d\hat{L}_B}{(\hat{L}_A + \hat{L}_B)^{D/2}} e^{-\hat{L}_A - \hat{L}_B} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (6.53)$$

$$\begin{aligned} \chi_3(\mu) = \frac{C B^3}{A \mu^3} & \left(1 - \frac{2 A \mu^{D/2-3}}{(4\pi)^{D/2}} \int \frac{d\hat{L}_A d\hat{L}_B d\hat{L}_C}{(\hat{L}_A + \hat{L}_B + \hat{L}_C)^{D/2}} e^{-\hat{L}_A - \hat{L}_B - \hat{L}_C} + \right. \\ & \left. - \frac{3}{2} A \mu^{D/2-3} \int \frac{d\hat{L}_A d\hat{L}_B}{(\hat{L}_A + \hat{L}_B)^{D/2}} e^{-\hat{L}_A - \hat{L}_B} \right) + \mathcal{O}\left(\frac{1}{M^4}\right). \end{aligned} \quad (6.54)$$

Ginzburg criterion for D_{uc} Once the 2-point function is computed, here using the M -layer construction, the UCD of the problem, D_{uc} , can be deduced by applying the Ginzburg criterion in the non-critical phase [Ami93], as done in Section 5.3.

Using Eq. 6.51 with $q = 2$, together with Eq. 6.43, one has

$$\hat{G}_2(k) \propto \frac{1}{M} \frac{1}{\gamma k^2 + \mu} \left(1 + \right.$$

$$+ \frac{1}{M} \frac{c}{\gamma k^2 + \mu} \int \frac{d^D q}{(2\pi)^D} \sum_{L_A, L_B=1}^{\infty} e^{-(\gamma(k-q)^2 + \mu)L_A} e^{-(\gamma q^2 + \mu)L_B} \Big) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (6.55)$$

with $\gamma \equiv a_l^2/(2D-2)$ and $c \equiv MA/2$ defined to make the $1/M$ factors explicit. For $M \rightarrow \infty$ the correction can be neglected and the 2-point function assumes the MF expression, i.e. the Gaussian propagator, which leads to the MF values for the anomalous dimension, $\eta = 0$, and the exponent associated to the correlation length, $\nu = 1/2$. Moreover, in high dimensions, the 2-point function near the critical point and for $k^2 \rightarrow 0$ has the Gaussian form

$$\left(M\widehat{G}_2(k)\right)^{-1} \propto \mathcal{A}(\mu - \mu_c) + \mathcal{B}\gamma k^2 + \mathcal{O}(k^4), \quad (6.56)$$

from which one obtains the $1/M$ -order correction to the control parameter

$$\mu_c = -\frac{c}{M} \frac{1}{(4\pi\gamma)^{D/2}} \sum_{L_A, L_B=1}^{\infty} \frac{1}{(L_A + L_B)^{D/2}} \quad (6.57)$$

and the following two prefactors

$$\mathcal{A} = 1 - \frac{c}{M} \frac{1}{(4\pi\gamma)^{D/2}} \sum_{L_A, L_B=1}^{\infty} \frac{1}{(L_A + L_B)^{D/2-1}}, \quad (6.58)$$

$$\mathcal{B} = 1 - \frac{c}{M} \frac{1}{(4\pi\gamma)^{D/2}} \sum_{L_A, L_B=1}^{\infty} \frac{L_A L_B}{(L_A + L_B)^{D/2+1}}. \quad (6.59)$$

These three corrections, μ_c , $\mathcal{A}$ and $\mathcal{B}$, diverge respectively for $D \leq 4$, $D \leq 6$ and $D \leq 6$, revealing that the UCD for the site percolation problem, where MF behavior breaks down, is $D_{uc} = 6$. The analysis remains the same for bond percolation, where the only difference is the definition of the factor c , which does not affect these divergences.

To go below the UCD, one can rewrite the propagator including the cutoff in the integrals as prescribed by Eq. 6.42:

$$\begin{aligned} \left(M\widehat{G}_2(k)\right)^{-1} &\propto \mu(\widehat{k}^2 + 1) \times \\ &\left(1 + \frac{c}{M} \frac{\mu^{D/2-3}}{(\widehat{k}^2 + 1)} \frac{1}{(4\pi)^{D/2}} \int_{\mu/\Lambda^2}^{\infty} d\widehat{L}_A d\widehat{L}_B \frac{1}{(\widehat{L}_A + \widehat{L}_B)^{D/2}} e^{-\widehat{L}_A - \widehat{L}_B - \widehat{k}^2 \frac{\widehat{L}_A \widehat{L}_B}{\widehat{L}_A + \widehat{L}_B}}\right) + \mathcal{O}\left(\frac{1}{M^3}\right). \end{aligned} \quad (6.60)$$

The correction is not negligible for $D < 6$ in the limit $\mu \rightarrow 0$, due to the presence of the term $\mu^{D/2-3}$. Moreover, the integrals over $\widehat{L}_A$ and $\widehat{L}_B$ diverge in the UV regime, that is for $\mu/\Lambda^2 \rightarrow 0$ if $D \geq 4$ and especially near $D = 6$ from below. Ensuring finiteness of the integrals as $\mu/\Lambda^2 \rightarrow 0$ requires performing the standard mass renormalization, changing the variable from μ to $m^2 \equiv \xi^{-2}$, to be done explicitly in the next section.

6.3 Computation of Critical Exponents

In this section, the results of the M -layer construction for the 2- and 3-point observables are taken as starting point, and standard procedures are performed in order to compute

the ϵ -expansion for the critical exponents, where $\epsilon = 6 - D$. From the definition of $\widehat{G}_2(k)$, Eq. 6.51, the correlation length ξ can be defined as:

$$\xi^2 \equiv \widehat{G}_2(0) \frac{\partial \widehat{G}_2^{-1}(k)}{\partial k^2} \Big|_{k^2=0}, \quad (6.61)$$

where

$$\frac{\partial}{\partial k^2} = \frac{\partial \widehat{k}^2}{\partial k^2} \frac{\partial}{\partial \widehat{k}^2} = \frac{a_l^2}{\mu C^{2/D}} \frac{\partial}{\partial \widehat{k}^2}. \quad (6.62)$$

Then

$$\widehat{G}_2(0) = \frac{CB^2 a_l^D}{A\mu} \left(1 - \frac{A\mu^{D/2-3}}{2(4\pi)^{D/2}} \int \frac{d\widehat{L}_A d\widehat{L}_B}{(\widehat{L}_A + \widehat{L}_B)^{D/2}} e^{-\widehat{L}_A - \widehat{L}_B} \right), \quad (6.63)$$

and for small A (that is, for large M), neglecting corrections:

$$\widehat{G}_2^{-1}(k) = \frac{A\mu}{CB^2 a_l^D} \left(\widehat{k}^2 + 1 + \frac{A\mu^{D/2-3}}{2(4\pi)^{D/2}} \int \frac{d\widehat{L}_A d\widehat{L}_B}{(\widehat{L}_A + \widehat{L}_B)^{D/2}} e^{-\frac{\widehat{L}_A \widehat{L}_B}{\widehat{L}_A + \widehat{L}_B} \widehat{k}^2 - \widehat{L}_A - \widehat{L}_B} \right), \quad (6.64)$$

where in the RHS k has been replaced with $\widehat{k}$ according to the definition given in Eq. 6.49. One obtains:

$$\frac{\partial \widehat{G}_2^{-1}(k)}{\partial \widehat{k}^2} \Big|_{\widehat{k}^2=0} = \frac{A\mu}{CB^2 a_l^D} \left(1 + \frac{A\mu^{D/2-3}}{2(4\pi)^{D/2}} \int \frac{d\widehat{L}_A d\widehat{L}_B}{(\widehat{L}_A + \widehat{L}_B)^{D/2}} e^{-\widehat{L}_A - \widehat{L}_B} \frac{\partial}{\partial \widehat{k}^2} \left(e^{-\frac{\widehat{L}_A \widehat{L}_B}{\widehat{L}_A + \widehat{L}_B} \widehat{k}^2} \right) \Big|_{\widehat{k}^2=0} \right), \quad (6.65)$$

where

$$\int \frac{d\widehat{L}_A d\widehat{L}_B}{(\widehat{L}_A + \widehat{L}_B)^{D/2}} e^{-\widehat{L}_A - \widehat{L}_B} \frac{\partial}{\partial \widehat{k}^2} \left(e^{-\frac{\widehat{L}_A \widehat{L}_B}{\widehat{L}_A + \widehat{L}_B} \widehat{k}^2} \right) \Big|_{\widehat{k}^2=0} = - \int \frac{d\widehat{L}_A d\widehat{L}_B}{(\widehat{L}_A + \widehat{L}_B)^{D/2+1}} \widehat{L}_A \widehat{L}_B e^{-\widehat{L}_A - \widehat{L}_B}. \quad (6.66)$$

It is worth noting that in Eq. 6.63, Eq. 6.64 and Eq. 6.65, higher order terms, with respect to the one-loop corrections, in powers of $1/M$ have been neglected. For the rest of this chapter, these terms will be neglected unless explicitly specified. Defining

$$I_\alpha(\mu) \equiv \int_{\mu/\Lambda^2}^{\infty} d\widehat{L}_A d\widehat{L}_B \frac{e^{-\widehat{L}_A - \widehat{L}_B}}{(\widehat{L}_A + \widehat{L}_B)^{D/2}} \quad (6.67)$$

and

$$I_\beta(\mu) \equiv \int_{\mu/\Lambda^2}^{\infty} d\widehat{L}_A d\widehat{L}_B \frac{\widehat{L}_A \widehat{L}_B}{(\widehat{L}_A + \widehat{L}_B)^{D/2+1}} e^{-\widehat{L}_A - \widehat{L}_B}, \quad (6.68)$$

it follows

$$\xi^2(\mu) = \frac{1}{m^2(\mu)} = \frac{a_l^2}{C^{2/D} \mu} \left(1 - \frac{1}{2} \frac{A\mu^{D/2-3}}{(4\pi)^{D/2}} \left(I_\alpha(\mu) + I_\beta(\mu) \right) \right). \quad (6.69)$$

In the integrals in Eq. 6.67, Eq. 6.68, the extremes of integration have been written explicitly, which were previously omitted. Notice that the integral $I_\alpha(\mu)$ is UV divergent

in $D = 6$ for $\mu \rightarrow 0$ (i.e., $p \rightarrow p_c$). At this point the relation in Eq. 6.69 can be simply inverted to express μ as a function of m^2 :

$$\mu(m^2) = a_l^2 C^{-2/D} m^2 \left(1 - \frac{1}{2} \frac{A m^{D-6} C^{6/D-1} a_l^{D-6}}{(4\pi)^{D/2}} \left(I_\alpha(\mu(m^2)) + I_\beta(\mu(m^2)) \right) \right). \quad (6.70)$$

It should be noticed that the previous equations for χ_2 and χ_3 are written as functions of μ , which is not the "physical mass", thus they can be divergent for $\mu \rightarrow 0$, near the UCD, $D_{uc} = 6$. To avoid the divergences, the expression of μ as a function of m^2 is needed. To this aim, $\xi^2(\mu)$ (and so $m^2(\mu)$) is computed from its definition.

6.3.1 Susceptibilities

At this point, all the ingredients are available to write χ_2 and χ_3 as functions of the physical parameter m^2 . Inserting Eq. 6.70 into Eq. 6.53 and Eq. 6.54 gives:

$$\begin{aligned} \chi_2(m^2) &= \frac{B^2 C^{2/D+1}}{A a_l^2} m^{-2} \left(1 + \right. \\ &+ \left. \frac{1}{2} \frac{A(a_l m)^{D-6}}{(4\pi)^{D/2} C^{1-6/D}} \left(I_\alpha(\mu(m^2)) + I_\beta(\mu(m^2)) \right) \right) \left(1 - \frac{1}{2} \frac{A(a_l m)^{D-6}}{(4\pi)^{D/2} C^{1-6/D}} I_\alpha(\mu(m^2)) \right) \\ &= \frac{B^2 C^{2/D+1}}{A a_l^2} m^{-2} \left(1 + \frac{1}{2} \frac{A m^{D-6} C^{6/D-1} a_l^{D-6}}{(4\pi)^{D/2} C^{1-6/D}} I_\beta(\mu(m^2)) \right), \quad (6.71) \end{aligned}$$

$$\begin{aligned} \chi_3(m^2) &= \frac{B^3 C^{6/D+1}}{A a_l^6} m^{-6} \left(1 + \frac{3}{2} \frac{A m^{D-6} C^{6/D-1} a_l^{D-6}}{(4\pi)^{D/2}} \left(I_\alpha(\mu(m^2)) + I_\beta(\mu(m^2)) \right) \right) \times \\ &\times \left(1 - 2 \frac{A m^{D-6} C^{6/D-1} a_l^{D-6}}{(4\pi)^{D/2}} I_\gamma(\mu(m^2)) - \frac{3}{2} \frac{A m^{D-6} C^{6/D-1} a_l^{D-6}}{(4\pi)^{D/2}} I_\alpha(\mu(m^2)) \right) \\ &= \frac{B^3 C^{6/D+1}}{A a_l^6} m^{-6} \left(1 + \frac{A m^{D-6} C^{6/D-1} a_l^{D-6}}{(4\pi)^{D/2}} \left(\frac{3}{2} I_\beta(\mu(m^2)) - 2 I_\gamma(\mu(m^2)) \right) \right), \quad (6.72) \end{aligned}$$

where

$$I_\gamma(\mu) \equiv \int_{\mu/\Lambda^2}^{\infty} d\hat{L}_A d\hat{L}_B d\hat{L}_C \frac{e^{-\hat{L}_A - \hat{L}_B - \hat{L}_C}}{(\hat{L}_A + \hat{L}_B + \hat{L}_C)^{\frac{D}{2}}}. \quad (6.73)$$

Notice that $\chi_2(\mu)$ and $\chi_3(\mu)$ have UV divergences near 6 dimensions due to the presence of $I_\alpha(\mu)$, which disappears when they are written as functions of m^2 , i.e. $\chi_2(m^2)$ and $\chi_3(m^2)$ are free of UV divergences near 6 dimensions.

6.3.2 Critical Exponents at Fixed Dimension

Following Section 3.1.2, led by the scaling laws discussed in Section 6.1, the following dimensionless ratio is computed:

$$\lambda \equiv \left(\frac{a_l}{\xi} \right)^D \frac{\chi_3^2(m^2)}{\chi_2^3(m^2)}. \quad (6.74)$$

On the other hand, m^2 is connected to the bare distance from the critical point by

$$m^2 \sim |\mu - \mu_c|^{2\nu} \quad \text{and} \quad \xi \sim |\mu - \mu_c|^{-\nu}, \quad (6.75)$$

where ν is the critical exponent for the divergence of the correlation length. In the end, defining

$$u \equiv A C^{6/D-1} a_l^{D-6} m^{D-6} \equiv g m^{D-6}, \quad (6.76)$$

the ratio λ can be computed as

$$\lambda = u \left(1 - 2 \frac{u}{(4\pi)^{D/2}} \left(-\frac{3}{4} I_\beta(\mu(m^2)) + 2 I_\gamma(\mu(m^2)) \right) \right). \quad (6.77)$$

Note that λ depends on the microscopic parameters of the model only through the single parameter $u = \mathcal{O}(1/M)$. The integrals I_β and I_γ can now be computed in the limit $m^2 \rightarrow 0$, which are convergent near $D = 6$:

$$\lim_{m^2 \rightarrow 0} I_\beta(\mu(m^2)) = \frac{1}{6} \Gamma\left(3 - \frac{D}{2}\right), \quad (6.78)$$

$$\lim_{m^2 \rightarrow 0} I_\gamma(\mu(m^2)) = \frac{1}{2} \Gamma\left(3 - \frac{D}{2}\right). \quad (6.79)$$

Thus, in the limit $m^2 \rightarrow 0$

$$\lambda = u - \frac{7}{4} \frac{u^2}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right), \quad (6.80)$$

from which, neglecting higher-order corrections

$$u = \lambda + \frac{7}{4} \frac{\lambda^2}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right). \quad (6.81)$$

Now, following the procedure described in [Section 3.1.2](#), the function $\beta(\lambda)$ is defined as:

$$\begin{aligned} \beta(\lambda) &\equiv m^2 \frac{\partial}{\partial m^2} \Big|_{g \text{ fixed}} \quad \lambda = \frac{1}{2} (D-6) u \frac{\partial}{\partial u} \Big|_{m^2 \text{ fixed}} \quad \lambda = \\ &= \frac{1}{2} (D-6) \left(u - \frac{7}{2} \frac{u^2}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right) \right). \end{aligned} \quad (6.82)$$

From [Eq. 6.81](#) one obtains:

$$\beta(\lambda) = \frac{1}{2} (D-6) \left(\lambda - \frac{7}{4} \frac{\lambda^2}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right) \right). \quad (6.83)$$

λ has been constructed to be a dimensionless quantity that does not diverge at the critical point. For this reason, the “critical value” of λ can be identified as the point at

which the function $\beta(\lambda)$ is zero². While a trivial zero is always present at $\lambda = 0$, for $D < 6$ there also exists a nontrivial zero:

$$\lambda_c = \frac{4}{7} \frac{(4\pi)^{D/2}}{\Gamma(3 - \frac{D}{2})}. \quad (6.84)$$

The value of λ_c is universal, in the sense that it is no longer dependent on the specific value of g , and thus on the specific problem under consideration, bond or site percolation. From this point the computation is the same for the two cases, thus realizing universality between these two versions of the percolation problem.

Recalling that $m^2 \sim (\mu - \mu_c)^{2\nu}$, and from standard computations [Par98], the following is defined:

$$z(\lambda) \equiv \frac{\partial \mu}{\partial m^2} \sim m^{2D_1}, \quad (6.85)$$

where $D_1 = \frac{1}{2\nu} - 1$. Thus, it is possible to compute:

$$D_1(\lambda) \equiv m^2 \frac{\partial}{\partial m^2} \bigg|_{g \text{ fixed}} \ln(z(\lambda)). \quad (6.86)$$

In the same way, for the computation of η the following is needed:

$$D_2(\lambda) \equiv \frac{\partial \ln \chi_2}{\partial \ln m^2} \bigg|_{g \text{ fixed}}, \quad \chi_2 \sim m^{2\frac{\eta-2}{2}}, \quad D_2(\lambda_c) = -1 + \frac{\eta}{2}. \quad (6.87)$$

The computation of $z(\lambda)$ yields:

$$z(\lambda) = a_t^2 \hat{C}^{-2/D} \left(1 - \frac{1}{2} \frac{u}{(4\pi)^{D/2}} \frac{D-4}{2} I_\beta(\mu(m^2)) - \frac{1}{2} \frac{g}{(4\pi)^{D/2}} \frac{\partial}{\partial m^2} \left(m^{D-4} I_\alpha(\mu(m^2)) \right) \right) \quad (6.88)$$

where

$$\frac{\partial}{\partial m^2} \left(m^{D-4} I_\alpha(\mu(m^2)) \right) = -m^{D-6} \int_{\mu(m^2)/\Lambda^2}^{\infty} d\hat{L}_A d\hat{L}_B \frac{e^{-\hat{L}_A - \hat{L}_B}}{(\hat{L}_A + \hat{L}_B)^{\frac{D}{2}-1}} \equiv -m^{D-6} I'_\alpha(\mu(m^2)). \quad (6.89)$$

I'_α can be computed as:

$$\lim_{m^2 \rightarrow 0} I'_\alpha(\mu(m^2)) = \Gamma\left(3 - \frac{D}{2}\right), \quad (6.90)$$

giving

$$z(\lambda) \propto 1 - \frac{u}{2} \frac{1}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right) \frac{D-16}{12}, \quad (6.91)$$

²Note that nontrivial zeros of the β -function correspond to **FPs** in parameter space, as discussed in [Chapter 3](#). These **FPs** are identified as Hamiltonians (or equivalently, the full set of parameters defining the Hamiltonian) that remain invariant under a step of the **RG** procedure. However, since the M -layer construction framework does not necessarily involve a well-defined Hamiltonian, it is more appropriate to refer instead to zeros of the β -function. While Refs. [Ang25c, Ang25a] employ the term “fixed point” in a somewhat loose sense, this and the next chapter use the expression “zero of the β -function”.

and, from the definition of $D_1(\lambda)$, the critical exponent ν in D dimensions is obtained:

$$\nu_D = \frac{42}{84 + (6 - D)(D - 16)}. \quad (6.92)$$

The next exponent, η , requires the computation of $D_2(\lambda)$:

$$D_2(\lambda) \equiv \left. \frac{\partial \ln \chi_2}{\partial \ln m^2} \right|_{g \text{ fixed}} = \frac{m^2}{\chi_2} \left. \frac{\partial \chi_2}{\partial m^2} \right|_{g \text{ fixed}} = -1 + \frac{\lambda}{2} \frac{1}{(4\pi)^{D/2}} I_\beta(\mu(m^2)) \left(\frac{D}{2} - 3 \right), \quad (6.93)$$

which can be obtained using

$$\begin{aligned} \left. \frac{\partial \chi_2}{\partial m^2} \right|_{g \text{ fixed}} &\propto -m^{-4} + \frac{1}{2} \frac{u}{(4\pi)^{D/2}} m^{-4} I_\beta(\mu(m^2)) \frac{D-8}{2} \\ &= -m^{-4} \left(1 - \frac{1}{2} \frac{u}{(4\pi)^{D/2}} I_\beta(\mu(m^2)) \frac{D-8}{2} \right), \end{aligned} \quad (6.94)$$

$$\chi_2 \propto m^{-2} \left(1 + \frac{1}{2} \frac{u}{(4\pi)^{D/2}} I_\beta(\mu(m^2)) \right), \quad (6.95)$$

which gives

$$\eta_D = \frac{D-6}{21}. \quad (6.96)$$

ϵ -expansion Given the results of [Eq. 6.92](#) and [Eq. 6.96](#) in fixed dimension, it is possible to perform the computation for $D = 6 - \epsilon$:

$$\nu = \frac{1}{2} + \frac{5}{84} \epsilon + \mathcal{O}(\epsilon^2), \quad (6.97)$$

$$\eta = -\frac{1}{21} \epsilon + \mathcal{O}(\epsilon^2). \quad (6.98)$$

These results are, to first order in ϵ , equal to the expansion of the standard Field Theory associated with the percolation problem [[Ess80](#), [Har75](#), [Pri76](#), [dAlc81](#), [Gra15](#), [Bor21](#), [Sch25](#)].

6.4 Connection with Field Theory

The goal here is to express $\overline{\widehat{C}_{2,lc}(k, k')}$ and $\overline{\widehat{C}_{3,lc}(k_1, k_2, k_3)}$, in [Eq. 6.43](#) and [Eq. 6.44](#), in terms of scalar propagators, as in the corresponding Field Theory. Starting from these equations, the integrals over the lengths with lower and upper limits, μ/Λ^2 and ∞ respectively, are performed. Interest lies in the critical behavior, i.e., for $\mu \rightarrow 0$, so the lower limit is set to 0, neglecting higher orders in μ . The results are

$$\overline{\widehat{C}_2(k, k')} = \frac{C B^2 a_l^D}{A \mu} \frac{1}{\widehat{k}^2 + 1} (2\pi)^D \delta^D(k + k') \left(1 - \frac{A \mu^{\frac{D}{2}-3}}{2(\widehat{k}^2 + 1)} \int \frac{d^D \widehat{q}}{(2\pi)^D} \frac{1}{1 + (\widehat{k} - \widehat{q})^2} \frac{1}{1 + \widehat{q}^2} \right), \quad (6.99)$$

$$\begin{aligned} \overline{\widehat{C}_3(k_1, k_2, k_3)} &= \frac{C B^3 a_l^{2D}}{A \mu^3} \frac{(2\pi)^D \delta^D(k_1 + k_2 + k_3)}{(\widehat{k}_1^2 + 1)(\widehat{k}_2^2 + 1)(\widehat{k}_3^2 + 1)} \times \\ &\left(1 - 2A\mu^{\frac{D}{2}-3} \int \frac{d^D \widehat{q}}{(2\pi)^D} \frac{1}{1 + (\widehat{k}_2 + \widehat{k}_3 + \widehat{q})^2} \frac{1}{1 + (\widehat{k}_2 + \widehat{q})^2} \frac{1}{1 + \widehat{q}^2} + \right. \\ &\quad \left. - \frac{1}{2} \frac{A\mu^{\frac{D}{2}-3}}{\widehat{k}_2^2 + 1} \int \frac{d^D \widehat{q}}{(2\pi)^D} \frac{1}{1 + (\widehat{k}_2 + \widehat{q})^2} \frac{1}{1 + \widehat{q}^2} + \text{perm.} \right). \end{aligned} \quad (6.100)$$

Next, momenta are rescaled and the bare mass and coupling, respectively m_b and g_b , are defined as:

$$\tilde{k} \equiv \mu^{\frac{1}{2}} a_l^{\frac{2D}{D+2}} A^{\frac{1}{D+2}} B^{-\frac{2}{D+2}} \widehat{k} \quad (6.101)$$

$$m_b^2 \equiv \mu a_l^{-\frac{4D}{D+2}} A^{\frac{2}{D+2}} B^{-\frac{4}{D+2}} \quad (6.102)$$

$$g_b \equiv a_l^{D\frac{D-6}{D+2}} A^{\frac{4}{D+2}} B^{\frac{D-6}{D+2}} C^{-2+\frac{3}{D}+\frac{D}{4}} \quad (6.103)$$

producing

$$\begin{aligned} \overline{\widehat{C}_2(\tilde{k}, \tilde{k}')} &= (2\pi)^D \delta^D(k + k') \times \\ &\left(\frac{1}{\tilde{k}^2 + m_b^2} - \frac{1}{2} g_b^2 \frac{1}{(\tilde{k}^2 + m_b^2)^2} \int \frac{d^D \tilde{q}}{(2\pi)^D} \frac{1}{(\tilde{k} - \tilde{q})^2 + m_b^2} \frac{1}{\tilde{q}^2 + m_b^2} \right), \end{aligned} \quad (6.104)$$

$$\begin{aligned} \overline{\widehat{C}_3(k_1, k_2, k_3)} &= \frac{1}{(\tilde{k}_1^2 + m_b^2)(\tilde{k}_2^2 + m_b^2)(\tilde{k}_3^2 + m_b^2)} (2\pi)^D \delta^D(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3) \times \\ &\left(g_b - 2g_b^3 \int \frac{d^D \tilde{q}}{(2\pi)^D} \frac{1}{(\tilde{k}_2 + \tilde{k}_3 + \tilde{q})^2 + m_b^2} \frac{1}{(\tilde{k}_2 + \tilde{q})^2 + m_b^2} \frac{1}{\tilde{q}^2 + m_b^2} + \right. \\ &\quad \left. - \frac{1}{2} g_b^3 \frac{1}{\tilde{k}_2^2 + m_b^2} \int \frac{d^D \tilde{q}}{(2\pi)^D} \frac{1}{(\tilde{k}_2 + \tilde{q})^2 + m_b^2} \frac{1}{\tilde{q}^2 + m_b^2} + \text{perm.} \right), \end{aligned} \quad (6.105)$$

which correspond to the results of the Field Theory associated with the percolation problem [Ami76, dAlc81].

As a last remark, it is noted that it is not always possible to write the results of the M -layer construction in terms of scalar propagators. For the percolation problem, the observables computed on a given topology, such as Eq. 6.27 or Eq. 6.28, are powers of the control parameter (here the probability p) to some combination of the lengths of the lines, so the integrals over the lengths provide scalar propagator factors. The same is true for the Ising model, as seen in Chapter 5. For a generic problem, the observables can be more complicated functions of the lengths and the integrals do not produce the simple scalar propagator structure. This is the case of the EA model, see Chapter 7.

It is also interesting that the integrals in Field Theories are *actually computed* through formulas like:

$$\frac{1}{k^2 + m^2} = \int_0^\infty e^{-l(k^2 + m^2)} dl \quad (6.106)$$

(see e.g. appendix to Chap. 5 in [Par98]). This amounts to going from Eq. 6.99 and Eq. 6.100 back to Eq. 6.43 and Eq. 6.44. Thus, the M -layer approach directly yields expressions in the treatable form. Furthermore, the integration variable l , which seems artificial in Field Theory, has the natural meaning of the length of internal lines of the diagrams in the M -layer approach.

6.5 Application to Percolation: Conclusions

In this chapter, it is shown how to recover, at one-loop level of approximation, the results of the ϵ -expansion for the critical exponents of the percolation problem on a D -dimensional regular lattice, by means of a new method, the M -layer construction. To do so, the observables of interest for site and bond percolation in the non-percolating phase are computed on properly chosen graphs at the leading orders. More precisely, the observables are the 2- and 3-point correlation functions, i.e., the probability that two or three sites belong to the same cluster, respectively. From these results, it is shown how to obtain the ϵ -expansion for the critical exponents, recovering, at first order, the same values already obtained for bond percolation using the $q \rightarrow 1$ continuation of the Field Theory applied to the Potts model with q states. Moreover, it is shown that, within the M -layer construction, the bond percolation problem differs from site percolation only for non-universal constants, which directly implies the universality between these two problems in any dimension D . The analysis presented here clearly illustrates that the M -layer construction effectively allows one to extract quantitative information on the critical behavior even for problems that are not defined by a Hamiltonian, such as percolation.

As a methodological result, it is shown how this method can be applied to a known problem in order to obtain the ϵ -expansion of the critical exponents. This investigation is beneficial in guiding similar computations, done in Chapter 7 for the EA model.

Regarding the specific problem of percolation, it would be interesting to extend the calculations made in this work to the percolating phase $p > p_c$. In this sense, the preliminary calculation of the Ginzburg criterion at zero loop order has already been done using the M -layer construction, obtaining the known UCD, $D_{uc} = 6$ [Per22b]. To proceed further and obtain the values of the critical exponents in the percolating phase, it is necessary to calculate the same observables as in Ref. [Per22b] with the corrections due to the one-loop structures. This analysis is left for future work.

6.A Appendices to Chapter 6

6.A.1 Identification of the Constants in the M -layer expansion

In this section, the computation of the previous part of the chapter is generalized for the 2- and 3-point function, aiming to identify the minimal number of constants describing the loop expansion in the M -layer framework. In particular, computations are performed for both site and bond percolation problems, highlighting their differences. Starting with site percolation, all contributions in Fourier space of a generic V_e -point correlation computed on a generic topology, $\mathcal{G}$, with I lines, V_e external points, V_i internal vertices, and N_{loop} loops are written as:

$$\begin{aligned} \overline{\widehat{C}_{V_e}(\{\mathbf{k}_j\})} \Big|_{\mathcal{G}} &= \frac{(2D)^{V_e-2I}}{S(\mathcal{G}) M^{N_{\text{loop}}+V_e-1}} \left(\frac{(2D)!}{(2D-3)!} \right)^{V_i} \left(\prod_{i=1}^I \int dL_i \right) (2\pi)^D \delta^D \left(\sum_{j=1}^{V_e} \mathbf{k}_j \right) \times \\ &a_l^{-DV_i} \left(\int \prod_{l=1}^{N_{\text{loop}}} \frac{d^D q_l}{(2\pi)^D} \right) \left(\prod_{i=1}^I \widehat{\mathcal{N}}_{L_i}(\{\mathbf{q}_l\}, \{\mathbf{k}_j\}) \right) p^{I-2V_i} f(\mathcal{C}_{V_e,lc}) p^{\sum_{i=1}^I L_i}. \quad (6.107) \end{aligned}$$

Note that $\widehat{\mathcal{N}}_L$ are functions of linear combinations $g_i(\{\mathbf{q}_l\}, \{\mathbf{k}_j\})$ of internal ($\{\mathbf{q}_l\}$, $l = 1, \dots, N_{\text{loop}}$) and external momenta ($\{\mathbf{k}_j\}$, $j = 1, \dots, V_e$), ensuring momentum conservation at each vertex. The factor p^{I-2V_i} and the function $f(\mathcal{C}_{V_e,lc})$ originate from [Eq. 6.27](#), [Eq. 6.28](#), [Eq. 6.33](#), [Eq. 6.34](#), and [Eq. 6.35](#). The first is the extra factor p , present only for $\mathcal{C}_{2,lc}(\mathcal{G}_1; L)$ and $\mathcal{C}_{3,lc}(\mathcal{G}_3; \vec{L}')$, as verified by substituting the corresponding values for I and V_i ³. The function $f(\mathcal{C}_{V_e,lc})$ assumes the following values:

$$f(\mathcal{C}_{2,lc}(\mathcal{G}_1; L)) = 1, \quad (6.108)$$

$$f(\mathcal{C}_{2,lc}(\mathcal{G}_2; \vec{L})) = -1, \quad (6.109)$$

$$f(\mathcal{C}_{3,lc}(\mathcal{G}_3; \vec{L}')) = 1, \quad (6.110)$$

$$f(\mathcal{C}_{3,lc}(\mathcal{G}_4; \vec{L}'')) = -2, \quad (6.111)$$

$$f(\mathcal{C}_{3,lc}(\mathcal{G}_5; \vec{L}''')) = -1. \quad (6.112)$$

The observables on diagrams computed in this work take the form of [Eq. 6.107](#). It is believed that higher-order diagrams (with only vertices of degree 3) for generic V_e -point functions obey it as well, though this hypothesis is not required for the results discussed here. In principle the steps from [Eq. 6.107](#) to [Eq. 6.112](#) should be repeated for bond percolation. Generalizing similar arguments, the only difference in bond percolation is the factor p^{I-V_i} in [Eq. 6.107](#), which is absent for bond percolation.

Continuing with site percolation, employing the asymptotic expression of the NBP in Fourier space, [Eq. 6.41](#), along with the rescaling of momenta and lengths in [Eq. 6.49](#), leads to

$$\overline{\widehat{C}_{V_e}(\{\mathbf{k}_j\})} \Big|_{\mathcal{G}} = \frac{(2D)^{V_e-2I}}{S(\mathcal{G}) M^{N_{\text{loop}}-1+V_e}} \left(\frac{(2D)!}{(2D-3)!} \right)^{V_i} \mu^{-I} \left(\prod_{i=1}^I \int d\widehat{L}_i \right) \times$$

³Note that the expression p^{I-2V_i} is valid only for degree-3 vertices; for V_i degree- d vertices it generalizes as $p^{I-(d-1)V_i}$ and can be further generalized if vertices with different degrees are present. A similar generalization applies for the factor $(2D-3)!$, which for a degree- d vertex is $(2D-d)!$.

$$\begin{aligned}
& (2\pi)^D \delta^D \left(\sum_{j=1}^{V_e} \mathbf{k}_j \right) a_l^{-DV_i} \left(\int \prod_{l=1}^{N_{\text{loop}}} \frac{d^D \hat{q}_l}{(2\pi)^D} \right) p^{I-2V_i} f(\mathcal{C}_{V_e, lc}) \times \\
& \left(\frac{\mu(2D-2)}{a_l^2} \right)^{\frac{D}{2}(N_{\text{loop}}-1)} \left(\frac{\mu(2D-2)}{a_l^2} \right)^{\frac{D}{2}} \left(\frac{2D}{2D-1} \right)^I \prod_{i=1}^I e^{-g_i(\{\hat{q}_l\}, \{\hat{k}_j\})^2 \hat{L}_i - \hat{L}_i} a_l^{ID},
\end{aligned} \tag{6.113}$$

where $\hat{k}$ depends on k via Eq. 6.49. Note that, as in previous sections, the external momenta are not rescaled inside the delta function.

Using well-known relations for V_i , V_e , I and N_{loop} in a generic diagram with internal vertices of degree 3 the following relations hold:

$$V_i = V_e + 2(N_{\text{loop}} - 1) \quad \text{and} \quad I = 2V_e + 3(N_{\text{loop}} - 1). \tag{6.114}$$

Thus, Eq. 6.113 identifies the following topology-dependent term:

$$\frac{1}{s(\mathcal{G})} \left(\prod_{i=1}^I \int d\hat{L}_i \right) \left(\int \prod_{l=1}^{N_{\text{loop}}} \frac{d^D \hat{q}_l}{(2\pi)^D} \right) f(\mathcal{C}_{V_e, lc}) \prod_{i=1}^I e^{-g_i(\{\hat{q}_l\}, \{\hat{k}_j\})^2 \hat{L}_i - \hat{L}_i} \tag{6.115}$$

and three factors:

- a constant raised to $(N_{\text{loop}} - 1)$:

$$\frac{1}{M} \left(\frac{(2D)!}{(2D-3)!} \right)^2 p^{-1} (2D-2)^{\frac{D}{2}} (2D)^{-3} \left(\frac{1}{2D-1} \right)^3 \mu^{\frac{D}{2}-3} \equiv A \mu^{\frac{D}{2}-3}, \tag{6.116}$$

- a constant raised to V_e :

$$\frac{1}{M} (2D)^{-1} \left(\frac{(2D)!}{(2D-3)!} \right) \left(\frac{1}{2D-1} \right)^2 a_l^D \mu^{-2} \equiv B a_l^D \mu^{-2}, \tag{6.117}$$

- an overall factor:

$$(2\pi)^D \delta^D \left(\sum_{j=1}^{V_e} \mathbf{k}_j \right) \left(\frac{\mu(2D-2)}{a_l^2} \right)^{\frac{D}{2}} \equiv (2\pi)^D \delta^D \left(\sum_{j=1}^{V_e} \mathbf{k}_j \right) \mu^{D/2} C a_l^{-D}, \tag{6.118}$$

as defined in Eq. 6.46, Eq. 6.47 and Eq. 6.48. Again, the bond percolation case yields the same expressions except for A , which is defined as in Eq. 6.50.

Equation Eq. 6.113 facilitates identification of the relevant constants to perform the $1/M$ expansion. These quantities are all non-universal, while the critical exponents are independent of them.

6.A.2 Other (Irrelevant) Diagrams

Less Divergent Diagrams

In this appendix, other possible diagrams that may contribute to the 2- and 3-point correlations are considered. As discussed in Section 4.1.5, the computation of the line without loops should be corrected to $\mathcal{O}(1/M^2)$ by diagram $\mathcal{G}'_1$ in Figure 6.4, with

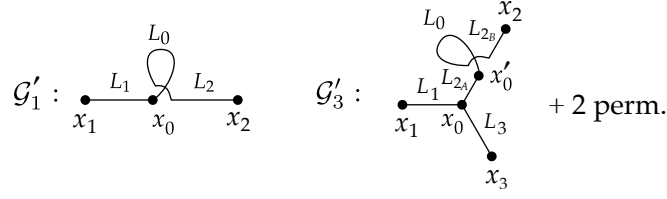


Figure 6.4. Less divergent diagrams that contribute to the 2-point and 3-point functions. In the figure, the vertex near x_0 in diagram $\mathcal{G}'_1$ (and analogously the vertex x'_0 in diagram $\mathcal{G}'_3$) represents a site of the M -layer lattice which projects onto the node x_0 of the original lattice. The two lines of length L_0 coil themselves in the M -layer lattice, in such a way that the projection on the original lattice is a loop.

the corresponding weight: $W(\mathcal{G}'_1) = -1/M^2$. While the contribution of $\mathcal{G}'_1$ at order $\mathcal{O}(1/M)$ is already included in Eq. 6.31, its contribution at order $\mathcal{O}(1/M^2)$ is not included there because $\mathcal{G}'_1$ diverges with a lower power of μ with respect to $\mathcal{G}_2$, which also contributes at order $\mathcal{O}(1/M^2)$.

The contribution of $\mathcal{G}'_1$ at order $\mathcal{O}(1/M^2)$ is

$$-\frac{(2D)^2}{M^2} \frac{(2D)!}{(2D-4)!} \sum_{L_1, L_0, L_2} \sum_{x_0 \in a_l \mathbb{Z}^D} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_0}(x_0, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x_0, x_2)}{(2D)^2} \mathcal{C}_{2,lc}(\mathcal{G}'_1; L_1, L_0, L_2), \quad (6.119)$$

where

$$\mathcal{C}_{2,lc}(\mathcal{G}'_1; L_1, L_0, L_2) = \mathcal{C}_{2,lc}(\mathcal{G}_1; L_1 + L_0 + L_2) = p p^{L_1+L_0+L_2}. \quad (6.120)$$

In Fourier space, using Eq. 6.40 and Eq. 6.41, it becomes:

$$-(2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \frac{(2D)^{-4}}{M^2} \frac{(2D)!}{(2D-4)!} \left(\frac{2D}{2D-1} \right)^3 a_l^{2D} p \frac{1}{\frac{a_l^2}{2D-2} \mathbf{k}_1^2 + \mu} \frac{1}{\frac{a_l^2}{2D-2} \mathbf{k}_2^2 + \mu} \times \\ \times \int \frac{d^D k_0}{(2\pi)^D} \int_{\mu/\Lambda^2}^{\infty} d\hat{L}_0 e^{-\left(\frac{a_l^2}{\mu(2D-2)} \mathbf{k}_0^2 + 1 \right) \hat{L}_0}, \quad (6.121)$$

which can be rewritten by scaling all the momenta, $\hat{\mathbf{k}} \equiv \mathbf{k} \frac{a_l}{\sqrt{\mu(2D-2)}}$, apart from the ones in the delta function, as:

$$-(2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \frac{(2D)^{-4}}{M^2} \frac{(2D)!}{(2D-4)!} \left(\frac{2D}{2D-1} \right)^3 \mu^{\frac{D}{2}-3} \frac{a_l^D}{(\hat{\mathbf{k}}_1^2 + 1)(\hat{\mathbf{k}}_2^2 + 1)} p (2D-2)^{\frac{D}{2}} \int \frac{d^D \hat{\mathbf{k}}_0}{(2\pi)^D} \frac{1}{\hat{\mathbf{k}}_0^2 + 1} \\ \propto \frac{\mu^{\frac{D}{2}-3}}{M^2}, \quad (6.122)$$

where, as usual, higher orders in μ are neglected by setting the lower limit of the length integration to 0. The other contribution to order $\mathcal{O}(1/M^2)$ comes from diagram $\mathcal{G}_2$, giving (following the same steps):

$$-(2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \frac{(2D)^{-4}}{2M^2} \left(\frac{(2D)!}{(2D-3)!} \right)^2 \left(\frac{2D}{2D-1} \right)^4 \mu^{\frac{D}{2}-4} \frac{a_l^D (2D-2)^{\frac{D}{2}}}{(\hat{\mathbf{k}}_1^2 + 1)(\hat{\mathbf{k}}_2^2 + 1)} \times$$

$$\times \int \frac{d^D \hat{k}}{(2\pi)^D} \frac{1}{\hat{k}^2 + 1} \frac{1}{(\hat{k}_1 - \hat{k})^2 + 1} \propto \frac{\mu^{\frac{D}{2}-4}}{M^2}, \quad (6.123)$$

from which it is clear that near the critical point, $\mu \approx 0$, the contribution of $\mathcal{G}'_1$ can be neglected with respect to that of $\mathcal{G}_2$. Analogously, diagram $\mathcal{G}'_3$ in Figure 6.4, is negligible with respect to $\mathcal{G}_4$ and $\mathcal{G}_5$. Thus, the expression for the 3-point correlation function in Eq. 6.39 yields the correct critical behavior up to one-loop order.

This argument can also be generalized, at least for the percolation problem. Since for each line of the diagram a factor proportional to $\mu^{-1}(\hat{k}^2 + 1)^{-1}$ appears, it follows that, at a given order in $\mathcal{O}(1/M)$, the most divergent diagrams in the limit $\mu \rightarrow 0$ are those with the largest number of lines. For this reason, the diagram $\mathcal{G}'_3$ in Figure 6.4 is less divergent with respect to the loop corrections to the 3-point function considered to compute critical exponents, diagrams $\mathcal{G}_4$ and $\mathcal{G}_5$. This argument is not valid generally for any problem or model. Indeed, the computation of the observables on a given diagram is the only model-dependent part of the M -layer procedure and, in general, the result can be a nontrivial function of the lengths, as stated at the end of Section 6.4.

4-point Correlation Function

Here, the computation for the most divergent contributions to the 4-point correlation function in the site percolation problem is presented; the same result can be obtained for bond percolation with the arguments given in Section 6.2. All the possible topologies, with only degree-3 and degree-4 vertices, are shown in Figure 6.5. Along the lines of

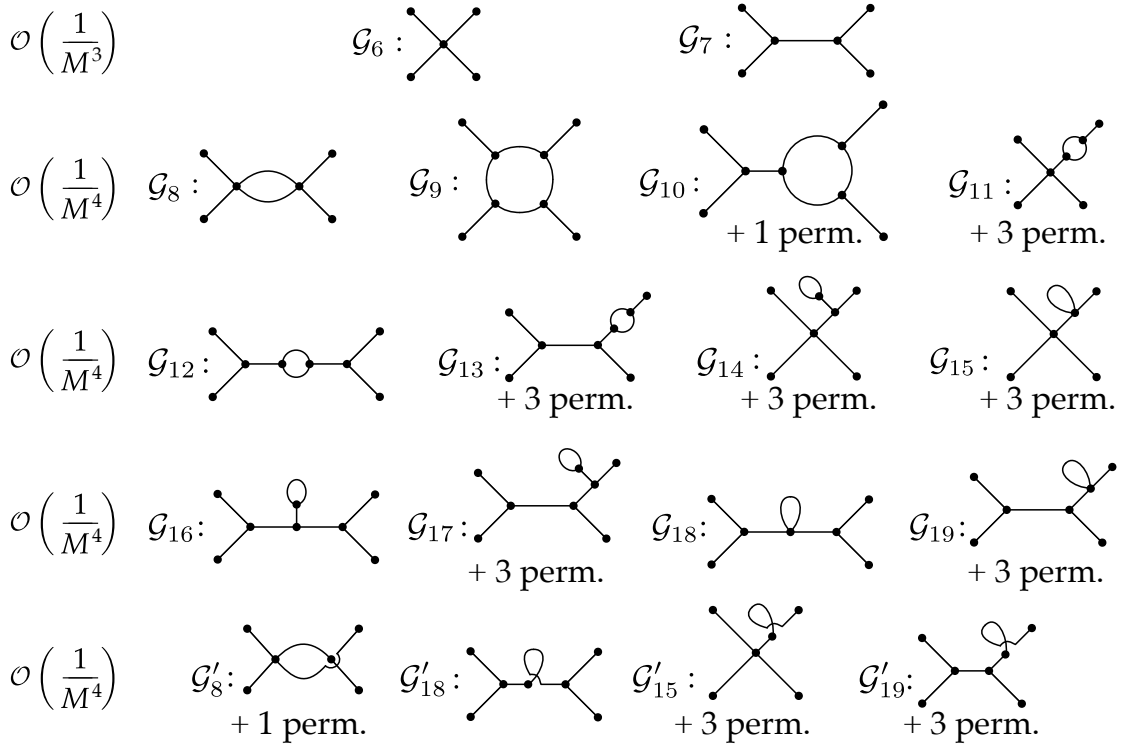


Figure 6.5. Diagrams contributing to the 4-point correlation function up to one loop.

the reasoning given for neglecting $\mathcal{G}'_1$ with respect to $\mathcal{G}_2$, it is possible to identify the

most divergent diagrams to each $\mathcal{O}(1/M)$ order for the 4-point correlation function simply by considering the diagrams with the largest number of lines. It turns out that the relevant diagrams, for the 4-point function, are the ones shown in Figure 6.6: $\mathcal{G}_7$ to order $\mathcal{O}(1/M^3)$, $\mathcal{G}_9$, $\mathcal{G}_{12}$ and $\mathcal{G}_{13}$ to order $\mathcal{O}(1/M^4)$. Notice that, in principle, one should also consider diagrams with vertices of degree larger than four, but they all have, at one loop order, fewer lines than the ones included in Figure 6.6, thus they are less divergent near the critical point $\mu \approx 0$. Now the contributions of the identified

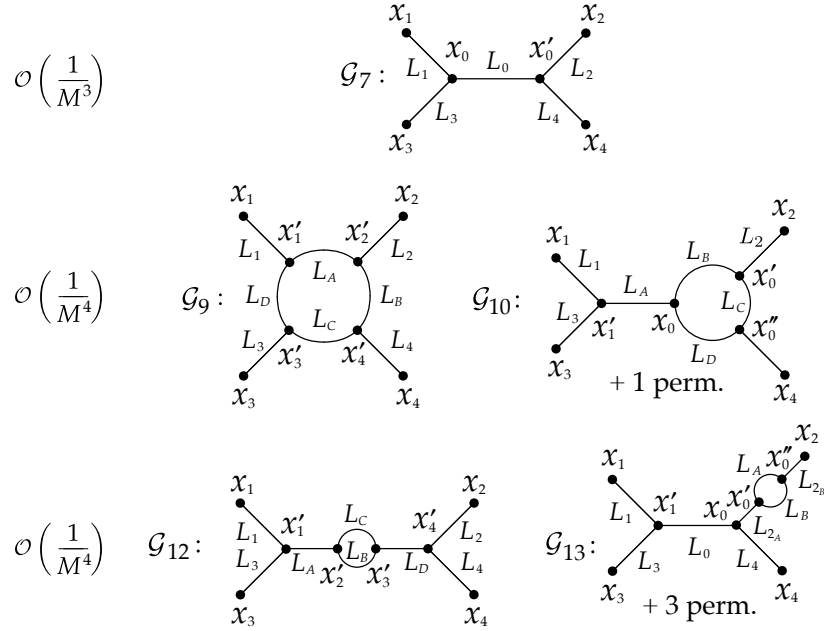


Figure 6.6. Most divergent diagrams contributing to the 4-point correlation functions up to one loop near the critical point.

diagrams can be written as:

$$\begin{aligned}
 \overline{C_4(x_1, x_2, x_3, x_4)} = & \frac{1}{M^3} \sum_{\vec{L}} \sum_{x_0, x'_0} \mathcal{N}(\mathcal{G}_7; \vec{L}; x_1, x_2, x_3, x_4, x_0, x'_0) \mathcal{C}_{4,lc}(\mathcal{G}_7; \vec{L}) + \\
 & + \frac{1}{M^4} \sum_{\vec{L}'} \sum_{\{x'_i\}_{i=1,\dots,4}} \mathcal{N}(\mathcal{G}_9; \vec{L}'; x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4) \mathcal{C}_{4,lc}(\mathcal{G}_9; \vec{L}') + \\
 & + \frac{1}{M^4} \sum_{\vec{L}'} \sum_{x'_1, x_0, x'_0, x''_0} \mathcal{N}(\mathcal{G}_{10}; \vec{L}'; x_1, x_2, x_3, x_4, x'_1, x_0, x'_0, x''_0) \mathcal{C}_{4,lc}(\mathcal{G}_{10}; \vec{L}') + \\
 & + \frac{1}{2M^4} \sum_{\vec{L}'} \sum_{\{x'_i\}_{i=1,\dots,4}} \mathcal{N}(\mathcal{G}_{12}; \vec{L}'; x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4) \mathcal{C}_{4,lc}(\mathcal{G}_{12}; \vec{L}') + \\
 & + \frac{1}{2M^4} \sum_{\vec{L}''} \sum_{x'_1, x_0, x'_0, x''_0} \mathcal{N}(\mathcal{G}_{13}; \vec{L}''; x_1, x_2, x_3, x_4, x'_1, x_0, x'_0, x''_0) \mathcal{C}_{4,lc}(\mathcal{G}_{13}; \vec{L}'') + \mathcal{O}\left(\frac{1}{M^5}\right),
 \end{aligned} \tag{6.124}$$

where the lengths are defined as $\vec{L} = (L_0, L_1, L_2, L_3, L_4)$, $\vec{L}' = (L_1, L_2, L_3, L_4, L_A, L_B, L_C, L_D)$, $\vec{L}'' = (L_1, L_3, L_4, L_{2A}, L_{2B}, L_A, L_B, L_C, L_D)$, and the number of **NBP**:

$$\mathcal{N}(\mathcal{G}_7; \vec{L}; x_1, x_2, x_3, x_4, x_0, x'_0) = (2D)^4 \left(\frac{(2D)!}{(2D-3)!} \right)^2 \prod_{i=1,3} \frac{\mathcal{N}_{L_i}(x_i, x_0)}{(2D)^2} \prod_{i=2,4} \frac{\mathcal{N}_{L_i}(x_i, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_0}(x_0, x'_0)}{(2D)^2}, \quad (6.125)$$

$$\mathcal{N}(\mathcal{G}_9; \vec{L}'; x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4) = (2D)^4 \left(\frac{(2D)!}{(2D-3)!} \right)^4 \prod_{i=1}^4 \frac{\mathcal{N}_{L_i}(x_i, x'_i)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x'_1, x'_2)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x'_2, x'_4)}{(2D)^2} \frac{\mathcal{N}_{L_C}(x'_3, x'_4)}{(2D)^2} \frac{\mathcal{N}_{L_D}(x'_3, x'_1)}{(2D)^2}, \quad (6.126)$$

$$\mathcal{N}(\mathcal{G}_{10}; \vec{L}'; x_1, x_2, x_3, x_4, x'_1, x_0, x'_0, x''_0) = (2D)^4 \left(\frac{(2D)!}{(2D-3)!} \right)^4 \prod_{i=1}^4 \frac{\mathcal{N}_{L_i}(x_i, x'_i)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x'_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_C}(x'_0, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_D}(x_0, x''_0)}{(2D)^2}, \quad (6.127)$$

$$\mathcal{N}(\mathcal{G}_{12}; \vec{L}'; x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4) = (2D)^4 \left(\frac{(2D)!}{(2D-3)!} \right)^4 \frac{\mathcal{N}_{L_1}(x_1, x'_1)}{(2D)^2} \times \frac{\mathcal{N}_{L_2}(x_2, x'_4)}{(2D)^2} \frac{\mathcal{N}_{L_3}(x_3, x'_1)}{(2D)^2} \frac{\mathcal{N}_{L_4}(x_4, x'_4)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x'_1, x'_2)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x'_2, x'_3)}{(2D)^2} \frac{\mathcal{N}_{L_C}(x'_2, x'_3)}{(2D)^2} \frac{\mathcal{N}_{L_D}(x'_4, x'_3)}{(2D)^2}, \quad (6.128)$$

$$\mathcal{N}(\mathcal{G}_{13}; \vec{L}''; x_1, x_2, x_3, x_4, x'_1, x_0, x'_0, x''_0) = (2D)^4 \left(\frac{(2D)!}{(2D-3)!} \right)^4 \frac{\mathcal{N}_{L_1}(x_1, x'_1)}{(2D)^2} \times \frac{\mathcal{N}_{L_3}(x_3, x'_1)}{(2D)^2} \frac{\mathcal{N}_{L_4}(x_4, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_0}(x'_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_{2B}}(x_2, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_{2A}}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x'_0, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x'_0, x''_0)}{(2D)^2}, \quad (6.129)$$

and finally the observables

$$\mathcal{C}_{4,lc}(\mathcal{G}_7; \vec{L}) = p p^{L_1+L_2+L_3+L_4+L_0}; \quad (6.130)$$

$$\mathcal{C}_{4,lc}(\mathcal{G}_9; \vec{L}') = -3p^{L_1+L_2+L_3+L_4+L_A+L_B+L_C+L_D}; \quad (6.131)$$

$$\mathcal{C}_{4,lc}(\mathcal{G}_{10}; \vec{L}') = -2p^{L_1+L_2+L_3+L_4+L_A+L_B+L_C+L_D}; \quad (6.132)$$

$$\mathcal{C}_{4,lc}(\mathcal{G}_{12}; \vec{L}') = -p^{L_1+L_2+L_3+L_4+L_A+L_B+L_C+L_D}; \quad (6.133)$$

$$\mathcal{C}_{4,lc}(\mathcal{G}_{13}; \vec{L}'') = -p^{L_1+L_3+L_4+L_0+L_{2A}+L_{2B}+L_A+L_B}. \quad (6.134)$$

Since the identified diagrams, $\mathcal{G}_7$, $\mathcal{G}_9$, $\mathcal{G}_{10}$, $\mathcal{G}_{12}$ and $\mathcal{G}_{13}$, contain only degree-3 vertices, the generic equation derived in [Section 6.A.1](#) for this kind of vertices, [Eq. 6.113](#), can be used where

$$f(\mathcal{C}_{4,lc}(\mathcal{G}_7; \vec{L}', L_4, L_0)) = 1, \quad (6.135)$$

$$f(\mathcal{C}_{4,lc}(\mathcal{G}_9; \vec{L})) = -3, \quad (6.136)$$

$$f(\mathcal{C}_{4,lc}(\mathcal{G}_{10}; \vec{L})) = -2, \quad (6.137)$$

$$f(\mathcal{C}_{4,lc}(\mathcal{G}_{12}; \vec{L}'')) = -1 = f(\mathcal{C}_{4,lc}(\mathcal{G}_{13}; \vec{L}''')) \quad (6.138)$$

and $S(\mathcal{G}_7) = S(\mathcal{G}_9) = S(\mathcal{G}_{10}) = 1$, $S(\mathcal{G}_{12}) = 2 = S(\mathcal{G}_{13})$:

$$\begin{aligned} \overline{\widehat{C}_4(\{\mathbf{k}_i\}_{i=1,\dots,4})} &= (2\pi)^D \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i \right) \frac{B^4 a_l^{3D} C}{A \mu^5} \prod_{i=1}^4 \frac{1}{\widehat{\mathbf{k}}_i^2 + 1} \left(\int d\widehat{L}_0 e^{-\widehat{L}_0 - (\widehat{\mathbf{k}}_1 + \widehat{\mathbf{k}}_3)^2 \widehat{L}_0} + \right. \\ &\quad - 3 A \mu^{\frac{D}{2}-3} \prod_{i=A,B,C,D} \int d\widehat{L}_i e^{-\widehat{L}_i} \int \frac{d^D \widehat{q}}{(2\pi)^D} e^{-\widehat{q}^2 L_A - (\widehat{q} + \widehat{\mathbf{k}}_2)^2 L_B - (\widehat{q} + \widehat{\mathbf{k}}_2 + \widehat{\mathbf{k}}_3)^2 L_C - (\widehat{\mathbf{k}}_1 - \widehat{q})^2 L_D} \\ &\quad - 2 A \mu^{\frac{D}{2}-3} \prod_{i=A,B,C,D} \int d\widehat{L}_i e^{-\widehat{L}_i} e^{-(\widehat{\mathbf{k}}_1 + \widehat{\mathbf{k}}_3)^2 L_A} \int \frac{d^D \widehat{q}}{(2\pi)^D} e^{-\widehat{q}^2 L_B - (\widehat{q} + \widehat{\mathbf{k}}_2)^2 L_C - (\widehat{q} + \widehat{\mathbf{k}}_2 + \widehat{\mathbf{k}}_4)^2 L_D} \\ &\quad - 2 A \mu^{\frac{D}{2}-3} \prod_{i=A,B,C,D} \int d\widehat{L}_i e^{-\widehat{L}_i} e^{-(\widehat{\mathbf{k}}_2 + \widehat{\mathbf{k}}_4)^2 L_A} \int \frac{d^D \widehat{q}}{(2\pi)^D} e^{-\widehat{q}^2 L_B - (\widehat{q} - \widehat{\mathbf{k}}_1)^2 L_C - (\widehat{q} - \widehat{\mathbf{k}}_1 - \widehat{\mathbf{k}}_3)^2 L_D} \\ &\quad - \frac{A \mu^{\frac{D}{2}-3}}{2} \prod_{i=A,B,C,D} \int d\widehat{L}_i e^{-\widehat{L}_i} e^{-(\widehat{\mathbf{k}}_1 + \widehat{\mathbf{k}}_3)^2 (L_A + L_D)} \int \frac{d^D \widehat{q}}{(2\pi)^D} e^{-\widehat{q}^2 L_B - (\widehat{q} - \widehat{\mathbf{k}}_1 - \widehat{\mathbf{k}}_3)^2 L_C} \\ &\quad \left. - \frac{A \mu^{\frac{D}{2}-3}}{2} \sum_{j=1}^4 \prod_{i=0,A,B,j_A,j_B} \int d\widehat{L}_i e^{-\widehat{L}_i} e^{-(\widehat{\mathbf{k}}_1 + \widehat{\mathbf{k}}_3)^2 \widehat{L}_0 - \widehat{\mathbf{k}}_2^2 (L_{j_A} + L_{j_B})} \int \frac{d^D \widehat{q}}{(2\pi)^D} e^{-\widehat{q}^2 L_A - (\widehat{q} + \widehat{\mathbf{k}}_j)^2 L_B} \right). \end{aligned} \quad (6.139)$$

As for the 2- and 3-point correlation functions, $\chi_4(\mu)$ can be defined as the 4-point correlation function at zero external momenta, divided by a_l^{3D} and without the factor $(2\pi)^D \delta^D \left(\sum_{i=1}^4 \mathbf{k}_i \right)$:

$$\chi_4(\mu) = \frac{B^4 C}{A \mu^5} \left(1 - \frac{5}{2} \frac{A \mu^{\frac{D}{2}-3}}{(4\pi)^{D/2}} I_\alpha(\mu) - 4 \frac{A \mu^{\frac{D}{2}-3}}{(4\pi)^{D/2}} I_\gamma(\mu) - 3 \frac{A \mu^{\frac{D}{2}-3}}{(4\pi)^{D/2}} I_\delta(\mu) \right) \quad (6.140)$$

where I_α and I_γ are defined in Eq. 6.67 and Eq. 6.73 respectively, while

$$I_\delta(\mu) \equiv \int_{\frac{\mu}{\Lambda^2}}^{\infty} d\widehat{L}_A d\widehat{L}_B d\widehat{L}_C d\widehat{L}_D \frac{e^{-\widehat{L}_A - \widehat{L}_B - \widehat{L}_C - \widehat{L}_D}}{\left(\widehat{L}_A + \widehat{L}_B + \widehat{L}_C + \widehat{L}_D \right)^{\frac{D}{2}}}, \quad (6.141)$$

and consequently

$$\lim_{\mu \rightarrow 0} I_\delta(\mu) = \frac{6-D}{12} \Gamma \left(3 - \frac{D}{2} \right). \quad (6.142)$$

Using the relation between μ and m^2 , Eq. 6.70, χ_4 can be written as a function of m^2 :

$$\chi_4(\mu(m^2)) = \frac{B^4 C^{10/D+1}}{A a_l^{10}} m^{-10} \left(1 + \frac{5}{2} \frac{u}{(4\pi)^{D/2}} I_\beta(\mu(m^2)) + \right.$$

$$\left. -4 \frac{u}{(4\pi)^{D/2}} I_\gamma(\mu(m^2)) - 3 \frac{u}{(4\pi)^{D/2}} I_\delta(\mu(m^2)) \right), \quad (6.143)$$

where u is the bare coupling constant, defined in [Eq. 6.76](#). Now the scaling, near the critical point $m^2 \rightarrow 0$, of the 4-point function can be considered:

$$D_4(\lambda) \equiv \left. \frac{\partial \ln \chi_4(\mu(m^2 \approx 0))}{\partial \ln m^2} \right|_{g \text{ fixed}}, \quad \chi_4(\mu(m^2 \approx 0)) \sim m^{2D_4(\lambda_c)}. \quad (6.144)$$

Using the expression of $\chi_4(m^2)$ it follows

$$D_4(\lambda_c) = \frac{1}{42} (D(3D - 55) + 12). \quad (6.145)$$

One can now compare with the scaling of the 4-point function:

$$\chi_q(\mu(m^2 \approx 0)) \sim m^{2D_q(\lambda_c)}, \quad D_q(\lambda_c) = \frac{q}{4}\eta - \frac{q}{2} + \frac{D}{2} \left(1 - \frac{q}{2}\right), \quad (6.146)$$

with $q = 4$, which gives the expected result of [Eq. 6.96](#)

$$\eta_D = \frac{D - 6}{21}. \quad (6.147)$$

Chapter 7

Spin Glass in a Field at Zero Temperature

In this chapter, the M -layer construction is applied to the EA model in an external field and in the zero-temperature regime, both on the hypercube in D dimensions and on the diluted one-dimensional LR lattice. Again, what is explained in the next sections is based on the generalities of the construction, presented in Chapter 4. All the material of this chapter is based on [Ang25a, Ang25b].

This construction has already been successfully applied to quenched disordered systems, such as the RFIM or the EA model in finite dimensions [Ang18, Ang20, Ang22]. Referring to the EA model, investigated here, authors in Ref. [Ang18] show that the results of the Bethe M -layer method for the EA model on external field in the limit of large connectivity reproduces the first loop correction to the mass, as prescribed by the Bray and Roberts simplified Field Theory for finite temperature [Bra80]. What is more important is that the large connectivity and the zero-temperature limits do not commute, being finite connectivity a fundamental ingredient to observe a zero-temperature spin glass transition in a field [Par14]. Then, in Ref. [Ang22], it was possible to identify the “unexpected” UCD, $D_{uc} = 8$, for the zero-temperature transition of the EA model in a field, again employing the M -layer method. In that work, only the 2-point connected (spin glass) correlation function was computed up to one-loop, from which the Ginzburg criterion establishes D_{uc} . In this chapter, those results are expanded to the 3-point connected correlation function and the 2-point disconnected one, which turns out to be crucial in the critical behavior at zero temperature.

The systematic computation of these observables, guided by the methodological value of Refs. [Ang24] and [Ang25c] (corresponding to Chapter 5 and Chapter 6), allows one to identify a new nontrivial zero of the coupling constant β -function that is associated to a spin glass transition. The main achievement of this work is indeed to reveal, for the first time since 50 years, the presence, in finite dimensions, of a phase transition in a field (at zero temperature). In addition, critical exponents, associated to the singular behavior of the above-mentioned observables, are estimated up to first order in the ϵ -expansion, where $\epsilon = 8 - D$.

Notably, the following analysis does not rely on the explicit identification of RG FPs, as described in the first part of Section 3.1. The strategy adopted here is instead, as in Chapter 6 based on the procedure described in Section 3.1.2. Following scaling hypotheses: the parameter λ is defined such that it remains finite at the critical point,

and the associated β -function is computed explicitly. The existence of a stable zero of the β -function below the UCD is then demonstrated. Importantly, no explicit Ginzburg-Landau action is introduced, and no real-space or momentum-space RG construction is carried out in this framework. Nevertheless, it is possible to connect the present results with the renormalization framework described in Section 3.2. In particular, Ref. [Ang15] implements an approximate real-space RG scheme and identifies an explicit FP at zero temperature in finite dimensions. Building on this work, it is natural to assume that the stable zero of the β -function, obtained here, corresponds to the limit $D \rightarrow D_{uc}$ of the FP found in [Ang15] with a suitable RG transformation. Within this interpretation, the discussion in terms of FPs can be understood from the M -layer construction perspective.

In Section 7.1, the spin glass model is introduced alongside a detailed discussion of the relevant observables and their phenomenology. Next, the MF critical behavior is examined in Section 7.2, which also establishes the building blocks for computing M -layer observables. The methodology for these computations is presented in Section 7.3, providing estimates for the critical exponents subsequently analyzed in Section 7.4. The same machinery is then applied to the one-dimensional LR model in Section 7.5. Chapter conclusions are presented in Section 7.6. The appendices to this chapter provide: further definitions of clusters and avalanches in Section 7.A.1; a detailed derivation of scaling laws in Section 7.A.2; a numerical validation of the cavity-derived expressions for observables in Section 7.A.3; and a discussion of irrelevant diagrams, analogous to those considered for the percolation case, in Section 7.A.4.

7.1 Model and Observables Definitions

Consider the EA spin glass model at zero temperature, defined on a generic lattice, by the following Hamiltonian:

$$\mathcal{H}(\{\sigma_i\}_{i \in \mathcal{L}}) = - \sum_{(i,j) \in \mathcal{E}} J_{ij} \sigma_i \sigma_j - \sum_{i \in \mathcal{L}} H_i \sigma_i, \quad (7.1)$$

where $\mathcal{L}$ is the set of lattice sites, $\mathcal{E}$ is the set of edges and $\sigma_i = \pm 1 \forall i \in \mathcal{L}$ are Ising spins at the lattice sites¹. In the language of the M -layer construction, the original lattice to be considered here is the D -dimensional hypercube with lattice spacing a_l , denoted with $a_l \mathbb{Z}^D$, where each site's position is identified by a vector $x \in a_l \mathbb{Z}^D$. The construction is also applied to the same model but considering a one-dimensional diluted LR lattice as the original one (rigorously defined in Section 7.5). The J_{ij} 's and the H_i 's are quenched random variables. In particular, one considers the cases of i) Gaussian distributed J 's with $\mathbb{E}_J[J_{ij}] = 0$ and constant field $H_i = H \forall i \in \mathcal{L}$; ii) Bimodal distributed $J_{ij} = \pm 1$ with zero mean and Gaussian distributed random fields H_i 's with variance Δ^2 and zero mean. In the first case, the transition occurs at some critical value H_c of the field, while in the second case, it occurs at some critical value Δ_c of the variance. The average over disorder is denoted here with $\mathbb{E}[\cdot]$.

On the Bethe lattice, the above model has a zero-temperature transition and it is of interest to assess the fate of this transition in finite-dimensional models. To do so, the M -layer lattice is considered. In any dimension, when M goes to infinity,

¹Note that, in this chapter, spins are identified by σ to avoid confusion with the cluster's size, defined in the following

all observables of the model converge to the Bethe lattice result; therefore a phase transition is observed with the Bethe lattice critical exponents. At finite values of M there are $\mathcal{O}(1/M)$ corrections that are harmless above the $\text{UCD} > D_{\text{uc}} = 8$, but are expected to alter the critical behavior for $D < 8$ [Ang22]. In the following, the first $1/M$ corrections to various observables are considered, showing that for $D < 8$ they do not destroy the transition, leading instead to an expansion of the critical exponents in powers of $\epsilon \equiv 8 - D$. A similar conclusion is found for the corresponding LR model.

Given a realization of the disorder, for each spin σ_i the local field h_i is considered, defined as usual such that i) σ_i is oriented along h_i in the ground state and ii) $\Delta E_i = 2|h_i|$ is the *excitation energy*, i.e., the energy difference between the ground state and the first excited state such that σ_i is constrained in the direction opposite to h_i . It is stressed that the local field h_i is a complicated function of all J_{ij} 's and of the external fields H_i 's in the system. The single-site probability density over the disorder that h_i is equal to h is given by $P_1(x_1, h)$, where x_1 is the position of σ_i in the hypercubic lattice. Due to the translational invariance of the disorder, the distribution is the same for all spins i.e., $P_1(x_1, h) = P_1(h)$.

Since either the J_{ij} 's or the H_i 's obey continuous distributions, $P_1(h)$ is also continuous, therefore it could be expected that the probability of finding two spins at a finite distance with the exact *same* local field is zero. It turns out that this is not the case and that for each value of ΔE there is a finite probability density of finding a cluster of spins with $|h_i| = \Delta E/2$. Therefore, for a given realization of the disorder, the whole system is partitioned into *clusters of spins with the same excitation energy*. In the paramagnetic phase, each cluster contains a finite number of sites and extends over a finite region of the lattice. The clusters are related to the concept of avalanches; indeed, if a spin is forced in the opposite direction of its local field, all the spins in the same cluster are also flipped in the same excited state. For the sake of readability, the detailed discussion of clusters and avalanches is postponed to Section 7.A.1. The clusters with nearly zero excitation energy $\Delta E = 0^+$ are of particular importance at the critical point: these are called *soft* clusters as they can be flipped with nearly zero-energy cost. Approaching the dAT line [dAlm78], the typical size of the soft clusters diverges, much as in percolation when the critical occupation probability is approached from the non-percolating phase [Sta18, Conoo].

Observables The notion of clusters allows for a straightforward definition of correlation functions. A generic q -point correlation function $\mathcal{P}_q(x_1, \dots, x_q; \Delta E)$ is defined as the probability density over the disorder that the q spins at positions $x_1, \dots, x_q$ are in the same cluster with excitation energy ΔE . Again, as the disorder distribution is translationally invariant, the correlation functions are translationally invariant as well. In particular, the probability $\mathcal{P}_1(x_1; \Delta E)$ that the spin at position x_1 belongs to a cluster with excitation energy ΔE does not depend on x_1 and is simply related to the local field distribution: $\mathcal{P}_1(x_1; \Delta E) = P_1(\Delta E/2)/2 + P_1(-\Delta E/2)/2$.

Guided by the percolation problem, the statistics of cluster sizes are also considered. The cluster density $n(s, \Delta E)$ is defined such that, in a system of size N , the number of clusters of size s and excitation energy between ΔE and $\Delta E + dE$ is given by $n(s, \Delta E)NdE$. In principle, $n(s, \Delta E)$ depends on the given realization of the disorder, but on general grounds it is expected to be self-averaging at large N . Much as in percolation, the correlation functions are related to the moments of $n(s, \Delta E)$, in

particular,

$$\sum_s s n(s, \Delta E) = \mathcal{P}_1(x_1; \Delta E). \quad (7.2)$$

Furthermore, the sum of $\mathcal{P}_2(x_1, x_2; \Delta E)$ over x_2 is related to the second moment of the cluster distribution:

$$\sum_s s^2 n(s, \Delta E) = \sum_{x_2} \mathcal{P}_2(x_1, x_2; \Delta E) \quad (7.3)$$

and in full generality:

$$\sum_s s^q n(s, \Delta E) = \sum_{x_2, \dots, x_q} \mathcal{P}_q(x_1, \dots, x_q; \Delta E). \quad (7.4)$$

Note that the [RHS](#) of the last three formulas does not depend on x_1 due to translational invariance.

The notion of local fields h_i can be generalized to any number of spins. In particular, given two spins σ_i and σ_j and a specific instance of the disorder, the effective energy function $E_{ij}(\sigma_i, \sigma_j) = -u_i\sigma_i - u_j\sigma_j - J_{ij}^{\text{eff}}\sigma_i\sigma_j$ is considered, yielding, apart from a constant, the ground state energy of the system when the two spins are constrained in the four possible configurations $\sigma_i = \pm 1, \sigma_j = \pm 1$. Note, again, that the effective fields u_i, u_j and J_{ij}^{eff} are complicated functions of the actual couplings J 's and fields H 's of the whole system, as given in [Eq. 7.1](#). One thus introduces the probability density over the disorder that the triplet $(u_i, u_j, J_{ij}^{\text{eff}})$ is equal to (u, u', J) as $P_2(x_1, x_2; u, u', J)$ where x_1 and x_2 denote respectively the position on the lattice of the sites i and j . Again, this distribution is translationally invariant with respect to a shift of x_1 and x_2 , and is symmetric with respect to the exchange $u \leftrightarrow u'$. At large distances $|x_1 - x_2|$, this distribution is expected to converge to the factorized form $P_1(u)P_1(u')\delta(J)$ where $P_1(u)$ is the probability distribution of the local field on a given site; indeed if $J = J_{ij}^{\text{eff}} = 0$, then $u_i = u$ ($u_j = u'$) coincides with the local field h_i (h_j). An important property of the triplets distribution is that, at any distance $|x_1 - x_2|$, there is a *finite* probability of having $J = 0$, see for instance [Table 7.1](#). Therefore, two additional distributions $P^{\text{dis}}(x_1, x_2; u, u')$ and $P^{\text{con}}(x_1, x_2; u, u', J)$ can be defined, that take into account triplets with respectively $J = 0$ and $|J| > 0$:

$$P_2(x_1, x_2; u, u', J) - P_1(u)P_1(u')\delta(J) = P^{\text{dis}}(x_1, x_2; u, u')\delta(J) + P^{\text{con}}(x_1, x_2; u, u', J), \quad (7.5)$$

where $P^{\text{con}}(x_1, x_2; u, u', J)$ is regular at $J = 0$. Note that triplets with strictly vanishing J are also present in the [RFIM](#), whose critical behavior is described by a zero-temperature [FP](#).

In the following, the probability densities P_1, P_2 , and $\mathcal{P}_q$ are studied on the M -layer random lattice. In this case, the average over the disorder includes the average over the rewirings, i.e., over all possible realizations of the M -layer. On the M -layer lattice, a point is specified by its position x projected on the original lattice and by its layer. However, after averaging over all possible realizations of the M -layer, the probabilities P_1, P_2 and $\mathcal{P}_q$ do not depend on the actual layers of the spins but only on their position on the original lattice, therefore the dependence on the layers is dropped.

Scaling Laws The $T = 0$ critical behavior is characterized by the presence of an additional independent critical exponent. This can be seen from a scaling perspective,

as done in Ref. [Bra85] for the RFIM, but the same argument applies for the EA model. This new exponent is related to the new observable, that is the 2-point disconnected correlation function. Indeed, an explicit computation, done below in Section 7.A.2, shows that *at leading order* in the $1/M$ expansion Eq. 7.5 takes the following form:

$$P_2(x_1, x_2; u, u', J) - P_1(u)P_1(u')\delta(J) \propto -\frac{1}{r^{D-4+\bar{\eta}}}f_{\text{dis}}\left(\frac{r}{\xi_{\text{dis}}}\right)\delta(J) + \frac{1}{r^{D-2+\eta}}f_{\text{con}}\left(\frac{r}{\xi_{\text{con}}}, \frac{|J|}{t^{\nu(\eta-\bar{\eta}+2)}}\right), \quad (7.6)$$

where t vanishes linearly at the critical point i.e., $t \propto H - H_c$ or $t \propto \Delta - \Delta_c$, $r \equiv |x_1 - x_2|$ and the new exponent is $\bar{\eta}$, associated to the scaling of the disconnected function. Two correlation lengths ξ_{dis} and ξ_{con} are identified in Eq. 7.6, respectively associated with the disconnected and connected functions. ξ_{dis} diverges as

$$\xi_{\text{dis}} \propto t^{-\nu}, \quad (7.7)$$

while ξ_{con} depends on both t and J (but not on u and u') and diverges if and only if *both* t and J vanish

$$\xi_{\text{con}} = t^{-\nu}f_{\xi}(|J|/t^{\nu(\eta-\bar{\eta}+2)}). \quad (7.8)$$

The scaling function f_{ξ} is such that ξ_{con} is finite unless *both* t and J vanish, $\xi_{\text{con}} \propto t^{-\nu}$ for $J = 0$ and $\xi_{\text{con}} \propto |J|^{-1/(\eta-\bar{\eta}+2)}$ for $t = 0$.

For $D < D_{\text{uc}}$ the critical exponents (ν , η , $\bar{\eta}$) and the scaling functions ($f_{\text{dis}}(\varrho)$, $f_{\text{con}}(\varrho, j)$ and $f_{\xi}(j)$) depend on the dimension but not on the microscopic details of the model, i.e., they are universal. The two scaling functions $f_{\text{dis}}(\varrho)$ and $f_{\text{con}}(\varrho, j)$ are finite for $\varrho = 0$ and decrease exponentially for $\varrho \rightarrow \infty$. The scaling variable j in $f_{\text{con}}(\varrho, j)$ and $f_{\xi}(j)$ defines the lines of approach of the point ($t = 0, J = 0$) and goes from $j = 0$, corresponding to $J = 0$, to $j = \infty$ corresponding to the line $t = 0$.

Interestingly, one can compute the 2-point connected and disconnected correlation functions from $P_2(x_1, x_2; u, u', J)$. The disconnected function is directly defined as the nontrivial $J = 0$ part of $P_2(x_1, x_2; u, u', J)$, that is P^{dis} , in the case $u = u' = 0$. Thus, the scaling with r is already displayed in Eq. 7.6. Instead, the 2-point connected function is defined as the probability density that the two spins, at positions x_1 and x_2 , belong to the same cluster with excitation energy ΔE . This probability can thus be computed from $P_2(x_1, x_2; u, u', J)$ imposing the following conditions on the triplet (u, u', J) :

- $|J| > |u|$ and $|J| > |u'|$ to ensure that flipping one spin the other flips too and vice versa;
- $|u + u' \text{sign}(J)| = \Delta E/2$ so as to fix the value of the local field to $\Delta E/2$.

More detailed computations, in Section 7.A.2, show that integrating $P_2(x_1, x_2; u, u', J)$ under the previous conditions the 2-point connected correlation scales as

$$\mathcal{P}_2(x_1, x_2; \Delta E) \propto \frac{1}{r^{D-4+\bar{\eta}+\theta}}f_2\left(\frac{r}{\xi}, \frac{\Delta E}{t^{\nu\theta}}\right), \quad (7.9)$$

where the exponent θ is defined from $\mathcal{P}_2(x_1, x_2, \Delta E) \propto 1/r^{D-2+\eta}$, thus

$$\eta = \bar{\eta} - 2 + \theta, \quad (7.10)$$

for $D \geq 8$ the values $\theta = 2$, $\bar{\eta} = 0$ imply the usual result $\eta = 0$. Moreover, $f_2(r/\xi, \Delta E/t^{\nu_0})$ is a scaling function, related to f_{con} and f_ξ as shown in [Section 7.A.2](#), that decreases exponentially for $r/\xi \rightarrow \infty$ and is finite for $r/\xi \rightarrow 0$. The correlation length ξ depends on both t and ΔE ; similarly to ξ_{con} , it obeys a scaling form $\xi = t^{-\nu} \tilde{f}_\xi(\Delta E/t^{\nu_0})$ and diverges as $\xi \propto t^{-\nu}$ for $\Delta E = 0$ and as $\xi \propto \Delta E^{-1/\theta}$ for $t = 0$ ².

Relation with Percolation Theory The above result, together with [Eq. 7.3](#) suggests that $n(s, \Delta E)$ obeys a scaling law as well. Following the analogous treatment of percolation [[Conoo](#), [Ang25c](#)] it is *assumed* that, changing the lengths by a factor b by means of a real-space [RG](#) transformation, the size s of large clusters, the value of the typical excitation energy ΔE and t change according to:

$$s' = s/b^{D_f}, \Delta E' = \Delta E/b^{-\theta}, t' = t/b^{-1/\nu}, \xi' = \xi/b. \quad (7.11)$$

$D_f \leq D$ is by definition the *fractal dimension* of the clusters: it relates the linear size l of a large cluster with its size s using $s \propto l^{D_f}$. Now it is assumed that *under the RG transformation the cluster number at large s is conserved*, i.e.

$$n(s, \Delta E, t) \Delta s \Delta(\Delta E) L^D \sim n(s', \Delta E', t') \Delta s' \Delta(\Delta E') (L')^D \quad (7.12)$$

where L is the linear size of the systems, $L^D = N$. Setting $b = (s/S)^{1/D_f}$ for some fixed large reference size S one then obtains :

$$n(s, \Delta E, t) = s^{\frac{\theta-D-D_f}{D_f}} n\left(S, \frac{\Delta E}{(s/S)^{-\theta/D_f}}, \frac{t}{(s/S)^{-1/(\nu D_f)}}\right),$$

that can be rewritten in terms of $\xi(t, \Delta E)$ (the correlation length of $\mathcal{P}_2(x_1, x_2; \Delta E)$) as

$$n(s, \Delta E, t) \equiv s^{(\theta-D-D_f)/D_f} f_0(s/\xi^{D_f}, \Delta E/t^{\nu_0}). \quad (7.13)$$

$f_0(s, e)$ is a scaling expression that decreases exponentially at large values of s and has a finite limit for $s = 0$. Therefore the cluster density at fixed ΔE and t follows a power-law $s^{(\theta-D-D_f)/D_f}$ up to a large value s^* that diverges as ξ^{D_f} approaching the critical point $t = 0$ when $\Delta E \approx 0$. It is remarked that the clusters with a finite $\Delta E > 0$ are characterized by a *finite* correlation length including at $t = 0$. Indeed $s^* \sim t^{-\nu D_f}$ at $\Delta E = 0$ and $s^* \sim \Delta E^{-D_f/\theta}$ at $t = 0$. From [Eq. 7.4](#) and [Eq. 7.13](#) it follows:

$$\sum_{x_2, \dots, x_q} \mathcal{P}_q(x_1, \dots, x_q; \Delta E) = \sum_s s^q n(s, \Delta E) \equiv \xi^{D_f q - D + \theta} \tilde{f}_q(\Delta E/t^{\nu_0}). \quad (7.14)$$

In the following, the susceptibilities are defined as the above integrals for $\Delta E = 0$. They diverge at the critical point as:

$$\chi_q \equiv \sum_{x_2, \dots, x_q} \mathcal{P}_q(x_1, \dots, x_q; 0) \propto \xi^{D_f q - D + \theta} \propto t^{-\nu(D_f q - D + \theta)}. \quad (7.15)$$

The comparison of [Eq. 7.14](#) for $q = 2$ with [Eq. 7.9](#) and [Eq. 7.10](#) leads to:

$$D_f = \frac{D - \theta + 2 - \eta}{2}, \quad \chi_q \propto \xi^{(\frac{q}{2}-1)(D-\theta) + \frac{q}{2}(2-\eta)}. \quad (7.16)$$

²Note that the universal scaling function of $\xi_{\text{con}}(t, J)$ is not the same as that of $\xi(t, \Delta E)$.

Note that this leads to $D_f = 4$ for $D = D_{uc}$. Analogously to the connected susceptibility χ_2 , the disconnected susceptibility is introduced as:

$$\chi_2^{\text{dis}} \equiv \sum_{x_2} P^{\text{dis}}(x_1, x_2; 0, 0) \propto - \int_0^\infty \frac{1}{r^{D-4+\bar{\eta}}} f_{\text{dis}}\left(\frac{r}{\xi_{\text{dis}}}\right) r^{D-1} dr \propto -\xi_{\text{dis}}^{4-\bar{\eta}} \propto -t^{4\nu-\bar{\eta}\nu}. \quad (7.17)$$

An explicit computation, done in [Section 7.3.3](#), shows that at leading order in $1/M$ the susceptibilities diverge as $\chi_q \propto \xi^{4q-6}$, for $D \geq 8$ this is the correct result since higher order corrections in $1/M$ are irrelevant. This implies that $\xi^{D_f q - D + \theta}$ as given in [Eq. 7.15](#) is wrong and this can be traced back to the failure of [Eq. 7.12](#)³. Indeed, in analogy with percolation [[Con00](#), [Ang25c](#)] it is expected that for all $D > 8$ the (hyper-scaling) cluster number expression, in [Eq. 7.13](#), must be replaced with the $D = 8$ result ($\eta = \bar{\eta} = 0$, $\nu = 1/2$, $\theta = 2$, $D_f = 4$):

$$n(s, \Delta E) = s^{-5/2} f_0(s/\xi^4, \Delta E/t). \quad (7.18)$$

For $\Delta E = 0$, it is verified that the above expression gives $n(s, 0)$ at leading order in $1/M$ leading to the aforementioned result $\chi_q \propto \xi^{4q-6}$. Note that the resulting expression is precisely the same as the cluster number $n(s)$ in percolation above the [UCD](#) $D = 6$, which has a $s^{-5/2}$ tail with a cut-off that diverges as t^{-2} and fractal dimension $D_f = 4$ [[Con00](#), [Ang25c](#)] (see [Chapter 6](#)).

7.2 Mean-Field Behavior on the Bethe Lattice

The qualitative [MF](#) behavior of the [EA](#) model on the Bethe Lattice is explained in [Section 2.2](#). In the presence of an external field and at zero temperature it is known that a phase transition occurs [[Par14](#)]. Throughout the following, it is assumed that all computations are performed in the $T = 0$ regime. Here, a useful approximation of the joint probability distribution of the two effective fields (or biases, in the Cavity method language of [Section 2.2.1](#)) and the effective coupling between two generic spins at large distance in the Bethe lattice is computed. This computation can be performed by exploiting the locally tree-like structure of this topology. Using this approximated distribution, it is possible to estimate the scaling functions cited in [Section 7.A.2](#) and the [MF](#) expressions of the observables of interest together with the corresponding corrections in the M -layer framework.

7.2.1 Semi-analytical Ansatz

A Bethe lattice with connectivity $2D$ is now considered, where D is the dimension of the associated M -layer graph. Consider the [EA](#) Hamiltonian and, in particular, the case of Gaussian couplings and fixed positive magnetic field, H . On a Bethe lattice a cavity bias $u_{j \rightarrow i}$ can be associated to any edge between nodes i and j , as done for [Eq. 2.66](#) in

³This can be rationalized employing percolation theory. Within percolation, a box of size b after a real-space [RG](#) transformation is considered occupied if there is a spanning cluster inside it [[Sta18](#)]. Below the [UCD](#), there is at most one spanning cluster in such a large box therefore the cluster number is conserved under the [RG](#) transformation. Instead, above the [UCD](#) there can be more than one spanning cluster in a large box of size b [[Con00](#)] and therefore [Eq. 7.12](#) is incorrect.

Section 2.2.1. In a homogeneous system, at zero temperature, the distribution of the cavity biases $P_B(u_i)$ over the disorder obeys Eq. 2.70, reported here

$$P_B(u) = \mathbb{E}_J \int \prod_{i=1}^{2D-1} P_B(u_i) du_i \delta\left(u - \text{sign}\left(J\left(H + \sum_{i=1}^{2D-1} u_i\right)\right) \min\left(|J|, \left|H + \sum_{i=1}^{2D-1} u_i\right|\right)\right), \quad (7.19)$$

with some differences: the average over the disorder is written as $\mathbb{E}_J$, the connectivity is $z = 2D$ and the function $\hat{u}_\beta$, defined in Eq. 2.67, is written in its $T = 0$ ($\beta = \infty$) form inside the Dirac delta function. As already stated, it is expected that, due to universality, the same results hold for the case of bimodal couplings and Gaussian external fields H_i : in the following, fixed H and Gaussian couplings will be considered.

To begin, the focus is put on computing the correlation between two spins, σ_1 and σ_2 , at a given distance. These spins are connected by a unique *path* (see Section 1.2.3, also referred to as a *chain*). This path can be effectively characterized by three parameters: two effective fields, u_1 and u_2 , acting on σ_1 and σ_2 , respectively, and an effective coupling J . A convenient way to describe the path connecting σ_1 and σ_2 is to consider only the sites and edges directly linking them, while imagining that each internal site of the chain is connected to its two neighboring sites along the chain and to $2D - 2$ infinite trees extending outside the chain. This construction preserves the original connectivity of the Bethe lattice, $2D$, on the internal spins. Notably, this approach implies that the two external spins have connectivity equal to 1, a feature useful in the following computation, as discussed a few lines below. Each of these infinite trees contributes a cavity bias u distributed according to $P_B(u)$. Summing over all internal spin configurations, the entire chain can then be effectively described by the three parameters mentioned above. As done in Ref. [Ang22], the probability density of the triplet (u_1, u_2, J) for two sites at distance L can be approximated by the semi-analytical *Ansatz*, valid at large L :

$$P_L(u_1, u_2, J) \equiv \delta(J)P_B(u_1)P_B(u_2) - 2aL\lambda^L\delta(J)g(u_1)g(u_2) + aL^2\lambda^L\rho e^{-\rho|J|L}g(u_1)g(u_2), \quad (7.20)$$

where a and ρ are parameters that can be computed, while λ and $g(u)$ are respectively the largest eigenvalue and the associated eigenfunction of the integral equation Eq. 2.85, in Section 2.2.2. The same equation is rewritten here with the notations used in this chapter and considering the zero temperature regime:

$$\lambda g(u) = \mathbb{E}_J \int \prod_{i=1}^{2D-2} dP_B(u_i) g(u') du' \mathbb{I}\left[\left|H + u' + \sum_{i=1}^{2D-2} u_i\right| < |J|\right] \delta\left(u - \text{sign}(J)\left|H + u' + \sum_{i=1}^{2D-2} u_i\right|\right), \quad (7.21)$$

where $\mathbb{I}[\cdot]$ is the indicator function. The function $g(u)$ satisfies $g(u) = g(-u)$, $\int du g(u) = 1$ and $g(u) \geq 0$. In this context, λ is the control parameter, and at the critical point, $H = H_c$, it assumes the critical value $\lambda_c = 1/(2D - 1)$. Notice that three terms contribute to the distribution in Eq. 7.20: the first is the trivial case in which the effective coupling is zero and the cavity biases are distributed according to $P_B(u)$, and it is the only term surviving in the limit $L \rightarrow \infty$; the second and third terms give corrections to it. In particular, the third term is an exponential distribution expected for large lengths, while the second term represents the finite weight for $J = 0$.

Notice that this approach to compute correlation functions on the Bethe lattice is very similar to what is done in [Chapter 5](#) for the Ising model: the idea is always to concentrate on the joint probability distributions of two spins at a certain distance and then use them to compose more complicated topologies. The additional complication of the EA model is the presence of disorder and the joint probability density can only be computed approximately. Indeed, it is important to remark that this distribution is valid in the large-length regime $L \gg 1$.

The expression of $P_L(u_1, u_2, J)$ is obtained by writing a quite generic form up to order $L \lambda^L$

$$P_L(u_1, u_2, J) = \delta(J) \left[P_B(u_1)P_B(u_2) - b L \lambda^L g(u_1)g(u_2) - c_1 L \lambda^L g'(u_1)g'(u_2) + \right. \\ \left. - c_2 L \lambda^L g''(u_1)g''(u_2) \right] + a L^2 \lambda^L \rho e^{-\rho|J|L} g(u_1)g(u_2), \quad (7.22)$$

and then, joining two lines and imposing self-consistency. The free parameters can be fixed [[Ang22](#)], leading to the expression given in [Eq. 7.20](#). Here, self-consistency refers to the requirement that the distribution obtained by joining two lines of lengths L_1 and L_2 must have the same form as the distributions of the original lines, but with the length parameter equal to $L_1 + L_2$. Now, consider two paths, one starting at spin σ_1 and ending at spin σ and the other starting at σ and ending at σ_2 . These can be seen as two chains joined at σ . The joining procedure of the two chains is the result of the marginalization over σ . At $T = 0$ the only configuration that contributes is the one that minimizes the energy, thus it depends on the field h acting on σ and the two couplings J_1 and J_2 linking σ with neighboring spins σ_1 and σ_2 , respectively. As output of this operation, two effective fields acting on spins σ_1 and σ_2 , an effective coupling J_{12} between them and an irrelevant constant are obtained:

$$-u_1 \sigma_1 - u_2 \sigma_2 - \sigma_1 J_{12} \sigma_2 + \text{const.} \equiv \min_{\sigma} (-\sigma_1 J_1 \sigma - \sigma_2 J_2 \sigma - h \sigma). \quad (7.23)$$

In [Table 7.1](#), the rules for this marginalization are reported depending on the input parameters h, J_1, J_2 and resulting in the output ones: u_1, u_2, J_{12} .

	u_1	u_2	J_{12}
$ h > J_2 + J_1 $	$\text{sign}(h)J_1$	$\text{sign}(h)J_2$	0
$ J_2 - J_1 < h \wedge h < J_2 + J_1 $	$\text{sign}(J_1)h_-(J_1, J_2)$	$\text{sign}(J_2)h_+(J_1, J_2)$	$\text{sign}(J_1 J_2) \tilde{J}$
$ h < J_2 - J_1 $	0	$\text{sign}(J_2)h$	$\text{sign}(J_2)J_1$

Table 7.1. Rules for the joining procedure in the case $|J_2| \geq |J_1|$. The complementary case $|J_1| \geq |J_2|$ can be obtained simply by exchanging index 1 with index 2, i.e., u_1 with u_2 and J_1 with J_2 .

with $\tilde{J} = \frac{|J_1| + |J_2| - |h|}{2}$ and $h_{\pm}(J_1, J_2) = \text{sign}(h) \frac{|h| \pm (|J_2| - |J_1|)}{2}$.

It is remarked that the *Ansatz* distribution on the effective triplet characterizes the effective parameters acting on two spins connected uniquely by $L - 1$ spins with connectivity $2D$, while the two external spins themselves have connectivity equal to 1. This choice ensures that lines can be joined by adding an arbitrary number of cavity

biases on the joining site, allowing to fix its connectivity to the chosen value $2D$. For instance, when connecting q lines on a central spin σ_0 to form a degree- q vertex, to retrieve connectivity equal to $2D$ for spin σ_0 , only the contributions of $2D - q$ cavity biases u_i for $i = 1, \dots, 2D - q$ have to be added.

To obtain the triplet distribution on the Bethe lattice, $2D - 1$ cavity biases distributed as $P_B(u)$ and the external field H can be attached to the external spins. From Eq. 7.20 one obtains the triplet distribution denoted with Q_L :

$$Q_L(u_1, u_2, J) - \delta(J)P_1(u_1)P_1(u_2) \equiv \hat{P}_1(u_1)\hat{P}_1(u_2) a \left(-2\delta(J)L\lambda^L + \rho L^2 \lambda^L e^{-\rho|J|L} \right), \quad (7.24)$$

where $P_1(u)$ is the total local field distribution on the Bethe lattice and $\hat{P}_1(u)$ is defined from:

$$\hat{P}_n(h) \equiv \int \prod_{i=1}^n g(u_i) du_i \prod_{j=n+1}^{2D} P_B(u_j) du_j \delta \left(h - \left(H + \sum_{i=1}^n u_i + \sum_{j=n+1}^{2D} u_j \right) \right), \quad (7.25)$$

in the specific case $n = 1$. It is noticed that $\hat{P}_2(h)$ satisfies the following relationship:

$$\frac{4a\hat{P}_2(0)}{\rho} = 1, \quad (7.26)$$

which follows from the self-consistency of the *Ansatz* distribution [Ang22]. Note that, according to the previous definition, $\hat{P}_0(h)$ can be identified with $P_1(h)$ on the Bethe lattice.

As a final comment: the connection between the expression for the distribution of the triplet in Eq. 7.5 and the expression on the Bethe lattice reported here in Eq. 7.24 is clarified. The former describes the effective triplet on the finite-dimensional *M*-layer lattice, hence additional information regarding the spatial positions is needed, while on the Bethe lattice no notion of Euclidean space exists. In the *M*-layer construction perspective the distribution in Eq. 7.24 is the observable on the diagram represented by a loop-less line, while Eq. 7.5 is the observable on the layered lattice (averaged over the rewirings) and it is obtained from Eq. 7.24 including the number of NBP and the weight $W = 1/M$ and summing over L . This last statement is made explicit in Eq. 7.174, in the appendix Section 7.A.2. Moreover, when working on tree-like graphs, it is useful to consider cavity distributions such as $P_B(u)$, which differ from the distribution $P_1(h)$ in Section 7.1, where h is the total local field acting on a single site, not a cavity quantity like the bias u . In Section 7.3, it will be shown how to use this distribution to compute observables on more complicated topologies, including those with loop structures.

7.3 *M*-layer: Computation of Observables

Following the prescriptions of the *M*-layer construction as exposed in Section 4.2, the expressions for the three observables defined above, $\mathcal{P}_2(x_1, x_2; 0)$, $P^{\text{dis}}(x_1, x_2; 0, 0)$ and $\mathcal{P}_3(x_1, x_2, x_3; 0)$, are derived here, taking into account the leading order and the one-loop corrections. To avoid all the zero inputs of the three observables, the 2-point connected and disconnected functions, together with the 3-point connected function, are redefined, respectively, as

$$\mathcal{P}_2(x_1, x_2; 0) \equiv C_2(x_1, x_2), \quad P^{\text{dis}}(x_1, x_2; 0, 0) \equiv D_2(x_1, x_2), \quad (7.27)$$

and

$$\mathcal{P}_3(x_1, x_2, x_3; 0) \equiv C_3(x_1, x_2, x_3), \quad (7.28)$$

averaged over the disorder and over the rewirings on the *M*-layer lattice. For each observable, the prescribed steps are followed: i) identify the relevant diagrams; ii) compute weights, number of projections and symmetry factors; iii) sum the contributions; iv) compute the observable on the identified diagrams. The last step, the only model-dependent one, will be done separately in the next sections. The connected and disconnected 2-point functions, $C_2(x_1, x_2)$ and $D_2(x_1, x_2)$, are treated together because the contributing diagrams are the same.

Moreover, the diagrams that are relevant for the critical behavior for this model are the same as for the percolation problem, the reader is referred to [Figure 6.2](#) and [Figure 6.3](#).

Observables: $C_2(x_1, x_2)$ and $D_2(x_1, x_2)$

❶ *Identify the possible topological diagrams*

The simplest diagram connecting two points is the bare line, denoted as $\mathcal{G}_1$. When including the possibility of a loop, the diagram composed of four lines and two degree-3 vertices, where the two internal lines form a loop, is considered. This diagram is referred to as $\mathcal{G}_2$ (see [Figure 6.2](#)).

Two additional diagrams could, in principle, contribute to 2-point observables: the *tadpole-type* diagrams $\mathcal{G}'$ and $\mathcal{G}''$, depicted in [Figure 6.2](#). When computing observables on given diagrams, in [Section 7.3](#), it is shown that these contributions are negligible in the large-length limit considered here. The argument leads to a more general conclusion: the insertion of a tadpole-type diagram on any generic line is negligible for the computations presented here. This observation allows disregarding such diagrams even when they are attached to a generic *k*-point function, in particular, for the 3-point function, as illustrated below.

Focus is exclusively on $\mathcal{G}_1$ and $\mathcal{G}_2$ henceforth.

❷ *Weights, number of projections and symmetry factors*

As explained in [Chapter 4](#), given a generic diagram $\mathcal{G}$ with lengths $\vec{L}$ and external vertices at $x_1, \dots, x_q$, to compute $\mathcal{N}(\mathcal{G}; \vec{L}; x_1, \dots, x_q)$ one has a factor $\mathcal{N}_L(x_1, x_2) / (2D)^2$ for each line between x_i and x_j . Then, to take into account the sum over allowed directions at each vertex, one multiplies by a factor $2D$ for each external vertex and a factor $(2D)! / (2D - 3)!$ for each internal cubic vertex. The factors W , $\mathcal{N}$ and S result to be the following for the 2-point diagrams:

Diagram $\mathcal{G}_1$

- $W(\mathcal{G}_1) = \frac{1}{M}$;
- $\mathcal{N}(\mathcal{G}_1; L; x_1, x_2) = \mathcal{N}_L(x_1, x_2)$;
- $S(\mathcal{G}_1) = 1$.

Diagram $\mathcal{G}_2$

- $W(\mathcal{G}_2) = \frac{1}{M^2}$;

- $\mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) = (2D)^2 \left(\frac{(2D)!}{(2D-3)!} \right)^2 \sum_{x_0, x'_0} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x'_0, x_2)}{(2D)^2};$
- $S(\mathcal{G}_2) = 2,$

where $\vec{L} = (L_1, L_A, L_B, L_2)$.

③ *Sum of the contributions*

The perturbative expression of the 2-point connected function on the M-layer lattice, averaged over the rewirings, $C_2(x_1, x_2)$, is the sum over the two relevant diagrams, up to one-loop contributions:

$$C_2(x_1, x_2) = \frac{1}{M} \sum_L \mathcal{N}_L(x_1, x_2) \mathcal{C}_2^{\text{lc}}(\mathcal{G}_1; L) + \frac{1}{2M^2} \sum_{\vec{L}} \mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) \mathcal{C}_2^{\text{lc}}(\mathcal{G}_2; \{\vec{L}\}) + \mathcal{O}\left(\frac{1}{M^3}\right). \quad (7.29)$$

Analogously, the same expansion can be written for the disconnected function

$$D_2(x_1, x_2) = \frac{1}{M} \sum_L \mathcal{N}_L(x_1, x_2) \mathcal{D}_2^{\text{lc}}(\mathcal{G}_1; L) + \frac{1}{2M^2} \sum_{\vec{L}} \mathcal{N}(\mathcal{G}_2; \vec{L}; x_1, x_2) \mathcal{D}_2^{\text{lc}}(\mathcal{G}_2; \{\vec{L}\}) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (7.30)$$

where again the only difference is the observable computed on a given diagram.

Notice also that the notation for the observables on the Bethe lattice is, including the superscript “lc” for “line-connected”:

$$\mathcal{C}_2^{\text{lc}}(\mathcal{G}; \{\vec{L}\}) \quad \text{and} \quad \mathcal{D}_2^{\text{lc}}(\mathcal{G}; \{\vec{L}\}), \quad (7.31)$$

where $\mathcal{G}$ denotes a generic diagram with generic set of lengths $\{\vec{L}\}$. The same notation is used in the following for the 3-point function: $\mathcal{C}_3^{\text{lc}}(\mathcal{G}; \{\vec{L}\})$.

Observable: $C_3(x_1, x_2, x_3)$

① *Identify the possible topological diagrams*

The simplest diagram connecting three points is the bare degree-3 vertex, denoted as $\mathcal{G}_3$. To account for the possibility of a loop, the diagram composed of six lines and three degree-3 vertices, where the three internal lines form a loop, is considered. This diagram is referred to as $\mathcal{G}_4$. At the one-loop level, three additional diagrams connect three points with a single loop. These are similar to $\mathcal{G}_3$, but with one of the external legs dressed by $\mathcal{G}_2$. Such a diagram is referred to as $\mathcal{G}_5$. All these diagrams are shown in [Figure 6.3](#).

Other possible diagrams are obtained by setting the length of one line to zero (e.g., the diagram $\mathcal{G}'''$ in [Figure 7.1](#)). They would, in principle, contribute as additional diagrams that are not included in the previous terms. This is because setting one length to zero increases the number of lines incident to a vertex, thereby altering the combinatorial factors associated with the number of [NBP](#). However, in [Section 7.A.4](#), an argument is provided explaining why such “sub-diagrams” can be neglected in this context.

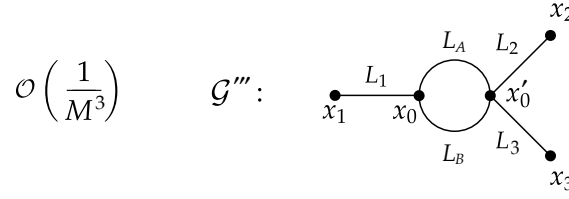


Figure 7.1. Graphic representation of the diagram $\mathcal{G}'''$. This diagram is obtained setting $L_C = 0$ in $\mathcal{G}_4$ or $L_{2_A} = 0$ in $\mathcal{G}_5$, both depicted in Figure 6.3. The contributions of such diagrams where a line is “contracted” can be shown to be less relevant than the ones corresponding to diagrams with the same weight, see Section 7.A.4.

② Weights, number of projections and symmetry factors

Diagram $\mathcal{G}_3$

- $W(\mathcal{G}_3) = \frac{1}{M^2}$;
- $\mathcal{N}(\mathcal{G}_3; L_1, L_2, L_3; x_1, x_2, x_3) = (2D)^3 \frac{(2D)!}{(2D-3)!} \sum_{x_0} \prod_{i=1}^3 \frac{\mathcal{N}_{L_i}(x_i, x_0)}{(2D)^2}$;
- $S(\mathcal{G}_3) = 1$.

Diagram $\mathcal{G}_4$

- $W(\mathcal{G}_4) = \frac{1}{M^3}$;
- $\mathcal{N}(\mathcal{G}_4; \vec{L}'; x_1, x_2, x_3) = (2D)^3 \left(\frac{(2D)!}{(2D-3)!} \right)^3 \sum_{x_0, x'_0, x''_0} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_2}(x_2, x'_0)}{(2D)^2} \times$
 $\times \frac{\mathcal{N}_{L_3}(x_3, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_A}(x_0, x'_0)}{(2D)^2} \frac{\mathcal{N}_{L_B}(x_0, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_C}(x'_0, x''_0)}{(2D)^2}$;
- $S(\mathcal{G}_4) = 1$.

Diagram $\mathcal{G}_5$

- $W(\mathcal{G}_5) = \frac{1}{M^3}$;
- $\mathcal{N}(\mathcal{G}_5; \vec{L}''; x_1, x_2, x_3) = (2D)^3 \left(\frac{(2D)!}{(2D-3)!} \right)^3 \sum_{x_0, x'_0, x''_0} \frac{\mathcal{N}_{L_1}(x_1, x_0)}{(2D)^2} \frac{\mathcal{N}_{L_{2_A}}(x_0, x'_0)}{(2D)^2} \times$
 $\times \frac{\mathcal{N}_{L_{2_B}}(x_2, x''_0)}{(2D)^2} \frac{\mathcal{N}_{L_3}(x_3, x_0)}{(2D)^2} \prod_{l=L_A, L_B} \frac{\mathcal{N}_l(x'_0, x''_0)}{(2D)^2}$;
- $S(\mathcal{G}_5) = 2$.

③ Sum of the contributions

As for the 2-point function, the perturbative expression of the 3-point connected function on the M -layer lattice, averaged over the rewirings, can be written as:

$$\begin{aligned}
 C_3(x_1, x_2, x_3) = & \frac{1}{M^2} \sum_{L_1, L_2, L_3} \mathcal{N}(\mathcal{G}_3; L_1, L_2, L_3; x_1, x_2, x_3) \mathcal{C}_3^{\text{lc}}(\mathcal{G}_3; \{L_1, L_2, L_3\}) + \\
 & + \frac{1}{M^3} \sum_{\vec{L}'} \mathcal{N}(\mathcal{G}_4; \vec{L}'; x_1, x_2, x_3) \mathcal{C}_3^{\text{lc}}(\mathcal{G}_4; \{\vec{L}'\}) + \\
 & + \frac{1}{2M^3} \sum_{\vec{L}''} \mathcal{N}(\mathcal{G}_5; \vec{L}''; x_1, x_2, x_3) \mathcal{C}_3^{\text{lc}}(\mathcal{G}_5; \{\vec{L}''\}) + \mathcal{O}\left(\frac{1}{M^4}\right), \quad (7.32)
 \end{aligned}$$

where $\vec{L}' = (L_1, L_2, L_3, L_A, L_B, L_C)$ and $\vec{L}'' = (L_1, L_{2_A}, L_A, L_B, L_{2_B}, L_3)$.

The last step of the *M*-layer construction is the actual computation of the observables on the given diagrams, to be done in the next section.

7.3.1 2-point Correlations

On the Bethe Lattice

At this point, it is possible to make use of the *Ansatz* distribution to compute the observables of interest on the Bethe lattice, that is the starting point of the *M*-layer expansion in powers of $1/M$. On the Bethe lattice, neglecting long loops, two sites can only be connected by a unique sequence of adjacent edges; thus the 2-point functions are computed on a line and the 3-point function on the loop-less topology connecting three sites, respectively represented by $\mathcal{G}_1$ and $\mathcal{G}_3$ depicted in Figure 6.2 and Figure 6.3.

In order to compute the 2-point functions, $\mathcal{C}_2$ and $\mathcal{D}_2$, their operational definitions are recalled. The disconnected function, on a line of length L , is simply defined as the “nontrivial” $J = 0$ part of the distribution $Q_L(u_1, u_2, J)$, with the external effective fields set to zero. On the other hand, the connected function can be obtained from the *Ansatz*, integrating with the conditions that the absolute values of the two effective fields at the extremities are lower than the absolute value of the effective coupling, together with the fact that the local field on the two sites equals the fixed value $\Delta E/2$, where $\Delta E = 0$ is set to take into account soft clusters. Notice that the first two conditions render the two $J = 0$ terms of the *Ansatz* irrelevant for the computation of the connected function since their contribution is null. The function $Q_L(h_1, h_2, J)$ defined in Eq. 7.24 is used:

$$Q_L(h_1, h_2, J) = \delta(J)P_1(h_1)P_1(h_2) - P_L^{J=0}\delta(J)\hat{P}_1(h_1)\hat{P}_1(h_2) + P_L^{J\neq 0}(J)\hat{P}_1(h_1)\hat{P}_1(h_2), \quad (7.33)$$

where the following definitions are introduced:

$$P_L^{J=0} \equiv 2aL\lambda^L \quad (7.34)$$

and

$$P_L^{J\neq 0}(J) \equiv aL^2\lambda^L \rho e^{-\rho|J|L}. \quad (7.35)$$

Now, the disconnected function is the second term of Eq. 7.33 with $h_1 = 0 = h_2$:

$$\mathcal{D}_2^{\text{lc}}(\mathcal{G}_1; L) = -2aL\lambda^L \left(\hat{P}_1(0) \right)^2. \quad (7.36)$$

Conversely, to compute the connected function, the last term, the only one giving a non-zero contribution, is focused on. Integrating over the three parameters h_1 , h_2 , and J , with the conditions given by its definition, yields the leading contribution

$$\begin{aligned} \mathcal{C}_2^{\text{lc}}(\mathcal{G}_1; L) &= \int dJ \int dh_1 \int dh_2 \prod_{i=1,2} \Theta(|J| - |h_i|) \delta(|h_1 + h_2 \text{sign}(J)|) P_L^{J\neq 0}(J) \hat{P}_1(h_1) \hat{P}_1(h_2) \\ &= \frac{4a}{\rho} \left(\hat{P}_1(0) \right)^2 \lambda^L + \mathcal{O}\left(\frac{1}{L}\lambda^L\right), \end{aligned} \quad (7.37)$$

where, to compute the integrals, $\mathcal{O}(\lambda^L/L)$ terms are neglected, since they cannot be controlled within the approximated *Ansatz* distribution. In the following, these corrections will not be explicitly written, so that one has

$$\mathcal{C}_2^{\text{lc}}(\mathcal{G}_1; L) = \frac{4a}{\rho} \left(\hat{P}_1(0) \right)^2 \lambda^L. \quad (7.38)$$

One-Loop Corrections

Now, the first corrections to $\mathcal{C}_2^{\text{lc}}$ and $\mathcal{D}_2^{\text{lc}}$ are computed. The corresponding diagram is $\mathcal{G}_2$, depicted in [Figure 6.2](#). The idea is to repeat the same steps done for $\mathcal{G}_1$: this requires computing the distribution of the effective triplet on the new topology given by the loop diagram.

In the following, $\mathcal{G}^{\text{amp}}$ will denote the *amputated* version of the generic diagram $\mathcal{G}$, obtained by pruning all the external lines (i.e., lines that connect an internal to an external vertex). In the case of a generic amputated diagram for a connected function, the contribution of the corresponding non-amputated diagram can be obtained simply by multiplying by the contribution of a loop-less line, given in [Eq. 7.38](#), for each external line (also called “leg”) with the corresponding length. This can be rationalized by considering that for the connected function to be different from zero, the spins at the external and internal vertices, connected by the external line, must follow the conditions on fields and coupling, thus contributing with the factor in [Eq. 7.38](#). Notice also that, given a generic diagram, when amputating an external line, the degree, say n , of the vertex to which the leg is connected is decreased by 1. This implies that the factor $\hat{P}_n(0)$ of the non-amputated diagram becomes $\hat{P}_{n-1}$ for the amputated one, since in the amputated case $2D - (n - 1)$ cavity biases, $u \sim P_B(u)$, are inserted in the vertex, while in the non-amputated case one of these biases is drawn from $g(u)$, coming from the external line. To simplify computations in the following sections, the amputated diagrams will be considered as if one field is drawn from $g(u)$, coming from the (amputated) external leg. In this way, the factor $\hat{P}_n$ is present for both the non-amputated and the amputated version of the same diagram.

Now, the distribution of the effective triplet on $\mathcal{G}_2$ is derived. The first step is to add together two lines to compose the amputated loop, $\mathcal{G}_2^{\text{amp}}$. In this case, a simple distribution results from the two lines of lengths L_A and L_B :

$$P_{L_A, L_B}^{\text{amp}}(v_1, v_2, J) = \int dv_1^A dv_2^A dv_1^B dv_2^B dJ_A dJ_B \delta(v_1^A + v_1^B - v_1) \times \\ \times \delta(v_2^A + v_2^B - v_2) \delta(J_A + J_B - J) P_{L_A}(v_1^A, v_2^A, J_A) P_{L_B}(v_1^B, v_2^B, J_B). \quad (7.39)$$

The result is the distribution of the effective triplet of the amputated $\mathcal{G}_2$ diagram, in the large-length limit: the fields on the extremities v_i are simply the sum of the of the two fields v_i^A and v_i^B of the lines and similarly for the coupling. At this point, the two external legs should be joined, using the rules of [Table 7.1](#). The resulting distribution does not take into account the line-connected definition, which in this case requires subtracting the two loop-less “sub-diagrams” given by $\mathcal{G}_2$ removing the upper or lower line composing the loop. To automatically subtract these two contributions, one can simply remove the first term from the complete *Ansatz* of the two lines of the loop

$$P_{L_i}(v_1^i, v_2^i, J_i) \rightarrow P_{L_i}(v_1^i, v_2^i, J_i) - \delta(J) P_B(v_1^i) P_B(v_2^i) \quad i = A, B, \quad (7.40)$$

while keeping the complete expression for the external lines. In this way, it is possible to figure out that the subtractions due to the line-connected definition are done; the only difference would be the subtraction of a contribution of a diagram in which the two lines of the loop are both removed, but for the observables computed in this paper this contribution is zero. The expression of the distribution of the effective triplet, once the external legs are added, is

$$\begin{aligned}
P_{\{\vec{L}\}}^{\text{lc}}(u_1, u_2, J) = & \\
& g(u_1)g(u_2) \left[64 \frac{a^4 \left(\widehat{P}_3(0) \right)^2}{\rho^2} \frac{\lambda^{L_1+L_2+L_A+L_B}}{L_A+L_B} L_A L_B (L_1+L_2+L_A+L_B) \delta(J) \right. \\
& - 32 \frac{a^4 \left(\widehat{P}_3(0) \right)^2}{\rho} \frac{\lambda^{L_1+L_2+L_A+L_B}}{L_A^2 - L_B^2} L_A L_B e^{-|J|(L_1+L_2)\rho} \left(e^{-|J|L_A\rho} L_A (L_1+L_2+L_A)^2 + \right. \\
& \left. \left. - e^{-|J|L_B\rho} L_B (L_1+L_2+L_B)^2 \right) \right], \quad (7.41)
\end{aligned}$$

where again $\vec{L} = (L_1, L_A, L_B, L_2)$. Before calculating the observables it is noticed that the integral over the effective coupling gives zero, as for the *Ansatz* distribution on $\mathcal{G}_1$. This property is expected to be general to every loop order for 2-point diagrams.

The same steps done for $\mathcal{G}_1$ should be performed in order to compute the loop contributions for the 2-point connected and disconnected functions. Once $2D - 1$ cavity biases are added to each end of the diagram, the disconnected function is simply the $J = 0$ term, with $u = u' = 0$:

$$\mathcal{D}_2^{\text{lc}}(\mathcal{G}_2; \{\vec{L}\}) = 64 \frac{a^4 \left(\widehat{P}_3(0) \right)^2 \left(\widehat{P}_1(0) \right)^2}{\rho^2} \frac{\lambda^{L_1+L_2+L_A+L_B}}{L_A+L_B} L_A L_B (L_1+L_2+L_A+L_B). \quad (7.42)$$

To obtain the result for the connected function, integration over the effective fields and coupling with the conditions that the absolute value of the coupling is greater than that of the fields and the local field is zero is also required. Finally

$$\mathcal{C}_2^{\text{lc}}(\mathcal{G}_2; \{\vec{L}\}) = -\frac{128a^4}{\rho^3} \left(\widehat{P}_1(0) \right)^2 \left(\widehat{P}_3(0) \right)^2 \frac{L_A L_B}{L_A+L_B} \lambda^{L_1+L_A+L_B+L_2}. \quad (7.43)$$

Notice that the dependence on the lengths in this loop correction of the connected function is a nontrivial function of the internal line lengths L_A and L_B , whereas the dependence on the external lines L_1 and L_2 factorizes in two exponentials λ^{L_1} and λ^{L_2} .

As a last remark it is noted that a generalization of the computation of $\mathcal{C}_2^{\text{lc}}(\mathcal{G}_2; \{\vec{L}\})$ is necessary to compute the 3-point connected function on the loop diagram $\mathcal{G}_4$, which is done in the next section.

7.3.2 3-point Correlation

On the Bethe Lattice

The leading contribution to the 3-point connected function is computed here, which comes from the diagram $\mathcal{G}_3$ in [Figure 6.3](#). In this case, 7 effective parameters can be identified, which constitute the following effective Hamiltonian:

$$\mathcal{H}_{\mathcal{G}_3}(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = -h_1\sigma_1 - J_1\sigma_1\sigma_0 - h_2\sigma_2 - J_2\sigma_2\sigma_0 - h_3\sigma_3 - J_3\sigma_3\sigma_0 - h_0\sigma_0. \quad (7.44)$$

The observable of interest is the probability that σ_1 , σ_2 and σ_3 belong to the same soft cluster. In other words, the goal is to compute the probability that the configuration of the three spins in the first excited state is flipped with respect to that of the ground

state and the energy difference is zero. This situation can be achieved only if the central spin, σ_0 , is also flipped in the first excited state with respect to its configuration in the ground state. The condition for this to happen is

$$\Theta(|J_1| - |h_1|)\Theta(|J_2| - |h_2|)\Theta(|J_3| - |h_{\rightarrow 0}|)\delta(h_{\rightarrow 3}), \quad (7.45)$$

where

$$h_{\rightarrow 0} \equiv h_0 + \text{sign}(J_1)h_1 + \text{sign}(J_2)h_2 \quad (7.46)$$

and

$$h_{\rightarrow 3} \equiv h_3 + \text{sign}(h_0 + \text{sign}(J_1)h_1 + \text{sign}(J_2)h_2)J_3. \quad (7.47)$$

Here, the first two Heaviside step functions ensure that σ_1 and σ_2 are susceptible to a perturbation of σ_0 , the third step function ensures that σ_0 is susceptible to a perturbation of σ_3 , and the Dirac delta function ensures that the energy difference with respect to the ground state is zero.

At this point, the integral over the distributions of the triplets of the three lines composing the diagram needs to be evaluated. It can be simplified by considering that the $J = 0$ parts of the distributions do not contribute to the integral, so that the following expression can be written, after insertions of $2D - 3$ cavity biases on σ_0 and $2D - 1$ on σ_1, σ_2 and σ_3 ,

$$\begin{aligned} \mathcal{C}_3(\mathcal{G}_3; L_1, L_2, L_3) = & \int \Theta(|J_1| - |h_1|)\Theta(|J_2| - |h_2|)\Theta(|J_3| - |h_{\rightarrow 0}|)\delta(h_{\rightarrow 3}) \times \\ & \times \hat{P}_3(h_0)dh_0 \prod_{i=1,2,3} P_{L_i}^{J \neq 0}(J_i)\hat{P}_1(h_i)dJ_i dh_i, \end{aligned} \quad (7.48)$$

which, neglecting $\mathcal{O}(\lambda^{3L}/L)$ terms (in the limit $L_1 \sim L_2 \sim L_3 \rightarrow L \gg 1$), leads to

$$\mathcal{C}_3(\mathcal{G}_3; L_1, L_2, L_3) = \frac{64a^3}{\rho^3} \left(\hat{P}_1(0) \right)^3 \hat{P}_3(0) \lambda^{L_1+L_2+L_3}. \quad (7.49)$$

Interestingly, apart from the factor $\hat{P}_3(0)$ coming from the internal vertex, the entire contribution factorizes into the three contributions of the (external) lines:

$$\frac{4a}{\rho} \hat{P}_1(0) \lambda^{L_i} \quad \text{for } i = 1, 2, 3. \quad (7.50)$$

This is the same feature of connected functions described by introducing amputated contributions for the 2-point function. To emphasize, consider $\mathcal{G}_3$ in [Figure 6.3](#). One of the external spins, say σ_1 , belongs to the same soft cluster as the other external spins if and only if it is susceptible to the flip of the internal spin σ_0 , i.e., σ_1 must flip from its ground state configuration along with σ_0 . This occurs when the absolute value of the effective coupling between these two spins exceeds the local field acting on σ_1 . Imposing this condition with the integration of the *Ansatz* for the corresponding external line of length L_1 yields the factor in [Eq. 7.50](#).

One-Loop Corrections

The two one-loop contributions to the 3-point connected function are now discussed. The diagram $\mathcal{G}_5$ is considered first, which is easily done by repeating the argument

given above for the external lines: the contribution of the “dressed” external leg, that is $\mathcal{C}_2^{\text{lc}}(\mathcal{G}_2; \{\vec{L}\})$, can be multiplied to the contributions of the remaining two external lines, resulting in

$$\mathcal{C}_3^{\text{lc}}(\mathcal{G}_5; \vec{L}') = -\frac{2048a^6}{\rho^5} \left(\hat{P}_1(0)\right)^3 \left(\hat{P}_3(0)\right)^3 \frac{L_A L_B}{L_A + L_B} \lambda^{L_1+L_{2A}+L_{2B}+L_A+L_B+L_3}. \quad (7.51)$$

The last diagram to be computed is $\mathcal{G}_4$. The procedure applied to the connected function on the 2-point diagram $\mathcal{G}_2$ is generalized. The computation is performed such that the contribution of a generic n -point connected function on an amputated loop diagram (i.e., a ring with n vertices and n lines) can be computed; then, as argued above, the contributions of the external lines can be simply multiplied. The goal is to

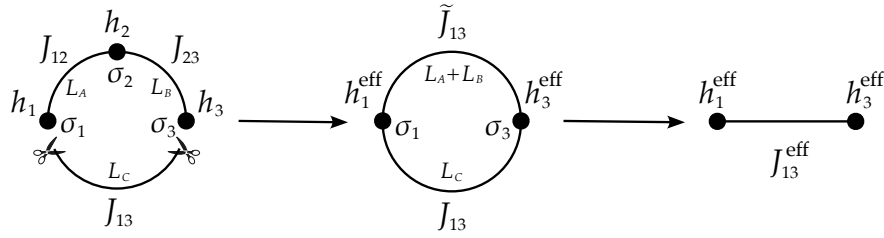


Figure 7.2. Graphic representation of the procedure to compute $\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4; \vec{L})$. Once the probability density that σ_2 is in the same soft cluster as σ_1 and σ_3 , the loop system can be treated as an effective 2-spin system, from which it is simple to compute the connected correlation.

compute the 3-point connected function of an effective system given by the following Hamiltonian

$$\mathcal{H}_{\mathcal{G}_4^{\text{amp}}}(\sigma_1, \sigma_2, \sigma_3) = -h_1\sigma_1 - J_{12}\sigma_1\sigma_2 - h_2\sigma_2 - J_{23}\sigma_2\sigma_3 - h_3\sigma_3 - J_{13}\sigma_1\sigma_3, \quad (7.52)$$

with three effective fields on the three spins σ_1 , σ_2 and σ_3 and three effective couplings. Notice that the amputated 3-point loop diagram is denoted by $\mathcal{G}_4^{\text{amp}}$. The amputated diagram is obtained by removing, as usual, the external legs and the requirement is that the three spins belong to the same soft cluster. As already explained, the amputated diagram is defined assuming that the distribution of the biases coming from the external legs is given by $g(u)$ instead of $P_B(u)$.

In practice for each line of length L there is a factor $g(u)g(u')(-\delta(J)P_L^{J=0} + P_L^{J \neq 0}(J))$ and for each internal vertex there are $2D - 3$ additional incoming biases distributed as $P_B(u)$ and one incoming field distributed as $g(u)$ (according to the definition of amputated diagrams). Then this pseudo-measure is integrated over h_1, h_2, h_3 and J_{12}, J_{23}, J_{13} with the condition that the three spins belong to the same soft cluster. The term “pseudo-measure” is used because the resulting product is not positive definite.

First, the case in which all three couplings are non-zero is considered, while the case in which one of them vanishes is analyzed at the end. If two or more couplings are zero, the three spins cannot belong to the same soft cluster. In the first case the factor is:

$$\hat{P}_3(h_1)\hat{P}_3(h_2)\hat{P}_3(h_3) a^3 \rho^3 e^{-|J_{12}|L_A - |J_{23}|L_B - |J_{13}|L_C} L_A^2 L_B^2 L_C^2 \lambda^{L_A+L_B+L_C}. \quad (7.53)$$

To study the problem, it is convenient to remove one line, say L_C , and then consider the effect of putting it back again in the system as depicted in Figure 7.2, which amounts

to assuming that $J_{13} = 0$. In this way, the 3-spin loop is reduced to a simple line for which the correlation function is known. The effective coupling between σ_1 and σ_3 , called $\tilde{J}_{13}$, and the two effective fields h_1^{eff} and h_3^{eff} can be computed. Since J_{12} and J_{23} are small, the effect of integrating out σ_2 produces a negligible shift on the fields, $h_1^{\text{eff}} \approx h_1$ and $h_3^{\text{eff}} \approx h_3$. Therefore it is correct to consider that integrating out h_2 , J_{12} and J_{23} only leads to an effective coupling $\tilde{J}_{13}$ while the fields distributions remain $\hat{P}_3(h_1)\hat{P}_3(h_3)$. Without L_C , the condition for the three spins to be in the same cluster is equivalent to the condition that spins σ_1 and σ_3 are in a soft cluster. Indeed, by flipping σ_3 , σ_1 flips if and only if σ_2 flips. The condition that spins σ_1 and σ_3 are in a soft cluster is simply that the absolute value of $\tilde{J}_{13}$ is larger than the absolute values of the effective fields h_1 and h_3 and that $h_1 = -h_3$. According to Table 7.1 the case $\tilde{J}_{13} = 0$ can occur only if spin σ_2 is blocked by a strong field h_2 or if either J_{12} or J_{23} vanish. Since the case where the three couplings J_{12}, J_{23}, J_{13} are different from zero is considered and σ_2 is in the soft cluster, it follows that $\tilde{J}_{13} \neq 0$.

To re-introduce L_C , the argument used for the loop of diagram $\mathcal{G}_2$ is followed. In particular the effective coupling is $J_{13}^{\text{eff}} = \tilde{J}_{13} + J_{13}$. Two possibilities are considered: i) in the first case the total effective coupling, $J_{13}^{\text{eff}} = \tilde{J}_{13} + J_{13}$, between σ_1 and σ_3 has the same sign as $\tilde{J}_{13}$, ii) in the second case it has the opposite sign.

First case - In the first case, adding L_C does not change the behavior of the three spins: if the fields on σ_1 and σ_3 , namely h_1 and h_3 , are smaller (in absolute value) than J_{13}^{eff} and $h_1 + \text{sign}(J_{13}^{\text{eff}})h_3 = 0$, all three spins belong to the same cluster. From now on, the case $\tilde{J}_{13} > 0$ is considered; the case $\tilde{J}_{13} < 0$ can be accounted for by multiplying the final result by 2, for symmetry reasons. The contribution to the 3-point connected function of the first case can already be computed: L_A and L_B are joined as described above for the self-consistency of the *Ansatz* and then the contribution of L_C is added with the constraint $J_{13} > -\tilde{J}_{13}$ that ensures $J_{13}^{\text{eff}} > 0$. Note however that, according to the definition of amputated diagrams, when L_A and L_B are joined the field on σ_2 coming from the external line is not distributed as $P_B(u)$ but as $g(u)$; therefore, the distribution of $\tilde{J}_{13}$ is *not* given by $(L_A + L_B)^2 a \lambda^{L_A+L_B} \rho \exp(-\rho(L_A + L_B)\tilde{J}_{13})$ but there is an additional factor $4a\hat{P}_3(0)/\rho^4$. Therefore, the effective coupling is distributed as

$$\tilde{P}_{L_A+L_B}^{J \neq 0}(\tilde{J}_{13}) \equiv P_{L_A+L_B}^{J \neq 0}(\tilde{J}_{13}) \frac{4a\hat{P}_3(0)}{\rho}. \quad (7.54)$$

The contribution of the first case is then:

$$\begin{aligned} & 2 \int_0^\infty dJ_{13}^{\text{eff}} \int_0^\infty d\tilde{J}_{13} \int_{-\tilde{J}_{13}}^\infty dJ_{13} \int_{-\infty}^\infty dh_1 \int_{-\infty}^\infty dh_3 \tilde{P}_{L_A+L_B}^{J \neq 0}(\tilde{J}_{13}) P_{L_C}^{J \neq 0}(J_{13}) \hat{P}_3(h_1) \hat{P}_3(h_3) \times \\ & \quad \delta(\tilde{J}_{13} + J_{13} - J_{13}^{\text{eff}}) \prod_{i=1,3} \Theta(|J_{13}^{\text{eff}}| - |h_i|) \delta(h_1 + h_3 \text{sign}(J_{13}^{\text{eff}})) = \\ & = \frac{16a^3 \left(\hat{P}_3(0)\right)^3}{\rho^2} \frac{L_A^2 + L_B^2 + 2L_C^2 + 2L_A L_B + 2L_B L_C + 2L_A L_C}{L_A + L_B + L_C} \lambda^{L_A+L_B+L_C}, \quad (7.55) \end{aligned}$$

where the factor 2 is included to take into account the case $\tilde{J}_{13} < 0$.

⁴If the fields were all distributed with $P_B(u)$ this factor, from the central spin σ_2 , would trivially be $4a\hat{P}_2(0)/\rho = 1$, from Eq. 7.26

Second case - In the second case, after the addition of J_{13} spin σ_1 and σ_3 are oriented differently in the soft cluster and *it may now happen that the spin σ_2 is no longer in the soft cluster*. Indeed, spin σ_2 always flips when σ_1 and σ_3 switch between the two configurations satisfying $\sigma_1\sigma_3 = \tilde{J}_{13}$, but not necessarily when they switch between the two configurations satisfying $\sigma_1\sigma_3 = -\tilde{J}_{13}$. The probability density of having $\tilde{J}_{13}$ constrained to this additional condition is defined as $P_{L_A, L_B}^{\text{NB}}(\tilde{J}_{13})$. The suffix (NB) stands for *Non-Blocking*, meaning that the central spin σ_2 is susceptible to the flip (i.e., it is not blocked) $(\sigma_1, \sigma_3) \rightarrow (-\sigma_1, -\sigma_3)$, independently of $\tilde{J}_{13}$. It is safely assumed that since the couplings are small the fields h_1 and h_3 are not changed. Again, the case $\tilde{J}_{13} > 0$ is considered; the case $\tilde{J}_{13} < 0$ can be accounted for by multiplying the result by 2 for symmetry reasons. In this case those instances of the disorder must be selected such that *without* the line L_C , spin σ_2 flips not only when σ_1 and σ_3 switch between $(1, 1)$ and $(-1, -1)$ but *also* when they switch between $(1, -1)$ and $(-1, 1)$. Consider only positive couplings $J_{12} > 0, J_{23} > 0$, the negative case can be argued to be the same. Also, the case $J_{12} > J_{23}$ is considered (the other case gives similar results). In this case, the condition for σ_2 to be susceptible to the flip of σ_1 or σ_3 is $(J_{12} - J_{23}) > |h_2|$, because in this second case the line L_C will change the sign of the coupling between σ_1 and σ_3 and they will be antiparallel in the ground state. When this condition is met, the effective coupling $\tilde{J}_{13}$ is equal to the *minimum* between J_{12} and J_{23} , as can be read from [Table 7.1](#). The *Ansatz* for the line of length $L_A + L_B$ joining the three spins is

$$\hat{P}_3(h_2) \rho^2 a^2 L_A^2 L_B^2 \lambda^{L_A+L_B} e^{-\rho L_A J_{12}} e^{-\rho L_B J_{23}}. \quad (7.56)$$

To obtain $P_{L_A, L_B}^{\text{NB}}(\tilde{J}_{13} > 0)$, the previous expression is integrated with the condition $|h_2| < (J_{12} - J_{23})$, which also implies that the effective coupling equals the minimum coupling $\tilde{J}_{13} = J_{23}$ (with $J_{12} > J_{23}$). The effect of the whole procedure on the distribution of h_1 and h_3 is neglected, consistent with [Table 7.1](#) since the field on σ_1 will not be changed after integrating out h_2, J_{12} , and J_{23} while the field on σ_3 will receive a contribution proportional to J_{23} which can be neglected due to $L_B \gg 1$ implying $J_{23} \ll 1$. The resulting contribution for $J_{12} > 0, J_{13} > 0$, and $J_{12} > J_{23}$ is then:

$$\begin{aligned} \rho^2 a^2 L_A^2 L_B^2 \lambda^{L_A+L_B} \hat{P}_3(0) e^{-\rho L_B \tilde{J}_{13}} \int_{\tilde{J}_{13}}^{\infty} 2(J_{12} - \tilde{J}_{13}) e^{-\rho L_A J_{12}} dJ_{12} \\ = 2a^2 L_B^2 \lambda^{L_A+L_B} \hat{P}_3(0) e^{-\rho(L_B+L_A)\tilde{J}_{13}}, \end{aligned} \quad (7.57)$$

where the integral over h_2 contributes a factor $2(J_{12} - \tilde{J}_{13})\hat{P}_3(0)$ because $|h_2| < (J_{12} - \tilde{J}_{13}) \ll 1$. The case $J_{12} < J_{23}$ leads to the same expression with $L_A \leftrightarrow L_B$, while the case $J_{12} < 0$ and $J_{23} < 0$ can be reduced to $J_{12} > 0$ and $J_{23} > 0$ by changing $\sigma_2 \rightarrow -\sigma_2$ and leads to an additional factor two. The final expression for $P_{L_A, L_B}^{\text{NB}}(\tilde{J}_{13})$ is then:

$$P_{L_A, L_B}^{\text{NB}}(\tilde{J}_{13}) = 4a^2 \hat{P}_3(0) (L_A^2 + L_B^2) \lambda^{L_A+L_B} e^{-\rho|\tilde{J}_{13}|(L_A+L_B)}. \quad (7.58)$$

Absolute value $|\tilde{J}_{13}|$ is written, as the above expression is valid also for $\tilde{J}_{13} < 0$ due to symmetry reasons.

Now it is possible to re-introduce the line with coupling J_{13} , contributing the factor

$$P_{L_C}^{J \neq 0}(J_{13}) = a \rho \lambda^{L_C} L_C^2 e^{-\rho|J_{13}|L_C}, \quad (7.59)$$

with the constraint $J_{13} < -\tilde{J}_{13}$. Adding a factor 2 to include the case $\tilde{J}_{13} < 0$, the following contribution is obtained:

$$2 \int_{-\infty}^0 dJ_{13}^{\text{eff}} \int_0^{\infty} d\tilde{J}_{13} \int_{-\infty}^{-\tilde{J}_{13}} dJ_{13} \int_{-\infty}^{\infty} dh_1 \int_{-\infty}^{\infty} dh_3 P_{L_A, L_B}^{\text{NB}}(\tilde{J}_{13}) P_{L_C}^{J \neq 0}(J_{13}) \hat{P}_3(h_1) \hat{P}_3(h_3) \times \\ \delta(\tilde{J}_{13} + J_{13} - J_{13}^{\text{eff}}) \prod_{i=1,3} \Theta(|J_{13}^{\text{eff}}| - |h_i|) \delta(h_1 + h_3 \text{sign}(J_{13}^{\text{eff}})) \\ = \frac{16a^3 \left(\hat{P}_3(0)\right)^3}{\rho^2} \frac{L_A^2 + L_B^2}{L_A + L_B + L_C} \lambda^{L_A + L_B + L_C}, \quad (7.60)$$

and adding it to the first contribution, Eq. 7.55, gives:

$$\frac{32a^3 \left(\hat{P}_3(0)\right)^2}{\rho^2} \frac{L_A^2 + L_B^2 + L_C^2 + L_A L_B + L_A L_C + L_B L_C}{L_A + L_B + L_C} \lambda^{L_A + L_B + L_C}. \quad (7.61)$$

The case in which one of the lines has zero coupling remains to be considered. Consider the case $J_{13} = 0$, the other cases are the same up to re-labeling of the lines. The pseudo-measure contribution is

$$-2\tilde{P}_3(h_1)\tilde{P}_3(h_2)\tilde{P}_3(h_3)a^3\rho^2\delta(J_{13})e^{-|J_{12}|L_A-|J_{23}|L_B}L_A^2L_B^2L_C\lambda^{L_A+L_B+L_C}. \quad (7.62)$$

Integration over h_2 , J_{12} , and J_{23} can be performed first, leading to a factor $\tilde{P}_{L_A+L_B}^{J \neq 0}(J_{13}^{\text{eff}})$. Then

$$-2aL_C\lambda^{L_C} \int_{-\infty}^{\infty} dJ_{13}^{\text{eff}} \int_{-\infty}^{\infty} dh_1 \int_{-\infty}^{\infty} dh_3 \tilde{P}_{L_A+L_B}^{J \neq 0}(J_{13}^{\text{eff}}) \hat{P}_3(h_1) \hat{P}_3(h_3) \\ \times \prod_{i=1,3} \Theta(|J_{13}^{\text{eff}}| - |h_i|) \delta(h_1 + h_3 \text{sign}(J_{13}^{\text{eff}})) = -\frac{32a^3 \left(\hat{P}_3(0)\right)^3}{\rho^2} L_C \lambda^{L_A + L_B + L_C}. \quad (7.63)$$

Note again the presence of $\tilde{P}_{L_A+L_B}^{J \neq 0}(J_{13}^{\text{eff}})$ instead of $P_{L_A+L_B}^{J \neq 0}(J_{13}^{\text{eff}})$ due to the definition of amputated diagrams, as in Eq. 7.54. The other two contributions are the same, exchanging L_C with L_A and L_B . Adding this result to Eq. 7.61, one finally obtains:

$$\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4^{\text{amp}}; L_A, L_B, L_C) = -\frac{32a^3}{\rho^2} \left(\hat{P}_3(0)\right)^3 \frac{L_A L_B + L_C L_B + L_A L_C}{L_A + L_B + L_C} \lambda^{L_A + L_B + L_C}. \quad (7.64)$$

It is now easy to compute the full diagram with the three external legs: multiply by

$$\frac{4a\hat{P}_1(0)}{\rho} \lambda^L \quad (7.65)$$

for each line of length L . The result is:

$$\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4; \vec{L}) = -\frac{2048a^6}{\rho^5} \left(\hat{P}_1(0)\right)^3 \left(\hat{P}_3(0)\right)^3 \frac{L_A L_B + L_C L_B + L_A L_C}{L_A + L_B + L_C} \lambda^{L_1 + L_2 + L_3 + L_A + L_B + L_C}. \quad (7.66)$$

Now all ingredients are available to compute the susceptibilities. Additional and irrelevant diagrams are discussed in Section 7.A.4.

7.3.3 Susceptibilities

In order to estimate critical exponents, the susceptibilities should be computed. The first step is to write the Fourier transform of Eq. 7.29, Eq. 7.30 and Eq. 7.32, using the convention of Eq. 5.39 and Eq. 5.40. To do so, the following functions are defined:

$$\widehat{C}_2(\mathbf{k}_1, \mathbf{k}_2) \equiv a_l^{2D} \sum_{\mathbf{x}_1, \mathbf{x}_2} e^{ik_1 x_1 + ik_2 x_2} C_2(\mathbf{x}_1, \mathbf{x}_2) \equiv (2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \widehat{C}_2(\mathbf{k}_1), \quad (7.67)$$

$$\widehat{D}_2(\mathbf{k}_1, \mathbf{k}_2) \equiv a_l^{2D} \sum_{\mathbf{x}_1, \mathbf{x}_2} e^{ik_1 x_1 + ik_2 x_2} D_2(\mathbf{x}_1, \mathbf{x}_2) \equiv (2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \widehat{D}_2(\mathbf{k}_1), \quad (7.68)$$

$$\widehat{C}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \equiv a_l^{3D} \sum_{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3} e^{ik_1 x_1 + ik_2 x_2 + ik_3 x_3} C_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \equiv (2\pi)^D \delta^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \widehat{C}_3(\mathbf{k}_1, \mathbf{k}_2). \quad (7.69)$$

Moreover, the distance from the MF critical point is defined as

$$\tau \equiv -\ln((2D-1)\lambda). \quad (7.70)$$

Consider the Fourier transform of Eq. 7.29, Eq. 7.30 and Eq. 7.32. Plugging the asymptotic expression of the number of NBP for small momenta (already found in Eq. 6.41) and the results of the observables on the given diagrams yields (up to negligible $1/M$ corrections):

$$\begin{aligned} \widehat{C}_2(\mathbf{k}) &= 4\rho a_l^D \frac{C B^2}{A} \left(\sum_{L=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} k^2 + \tau\right)L} \right) \left(1 + \right. \\ &\quad \left. - \frac{16 A}{a_l^{-D} C} \sum_{L=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} k^2 + \tau\right)L} \sum_{L_A, L_B}^{1, \dots, \infty} \frac{L_A L_B}{L_A + L_B} \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{a_l^2}{2D-2} q^2 + \tau\right)L_A - \left(\frac{a_l^2}{2D-2} (q-k)^2 + \tau\right)L_B} \right), \end{aligned} \quad (7.71)$$

$$\begin{aligned} \widehat{D}_2(\mathbf{k}) &= -2\rho^2 a_l^D \frac{C B^2}{A} \left(\sum_{L=1}^{\infty} L e^{-\left(\frac{a_l^2}{2D-2} k^2 + \tau\right)L} \right) \left(1 + \right. \\ &\quad \left. - \frac{32 A}{a_l^{-D} C} \left(\sum_{L=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} k^2 + \tau\right)L} \right) \sum_{L_A, L_B}^{1, \dots, \infty} \frac{L_A L_B}{L_A + L_B} \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{a_l^2}{2D-2} q^2 + \tau\right)L_A - \left(\frac{a_l^2}{2D-2} (q-k)^2 + \tau\right)L_B} \right. \\ &\quad \left. - \frac{16 A}{a_l^{-D} C} \frac{\left(\sum_{L=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} k^2 + \tau\right)L} \right)^2}{\sum_{L=1}^{\infty} L e^{-\left(\frac{a_l^2}{2D-2} k^2 + \tau\right)L}} \sum_{L_A, L_B}^{1, \dots, \infty} L_A L_B \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{a_l^2}{2D-2} q^2 + \tau\right)L_A - \left(\frac{a_l^2}{2D-2} (q-k)^2 + \tau\right)L_B} \right), \end{aligned} \quad (7.72)$$

$$\begin{aligned} \widehat{C}_3(\mathbf{k}_1, \mathbf{k}_2) &= 64\rho a_l^{2D} \frac{C B^3}{A} \prod_{i=1,2} \left(\sum_{L_i=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} k_i^2 + \tau\right)L_i} \right) \left(\sum_{L_3=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} (\mathbf{k}_1 + \mathbf{k}_2)^2 + \tau\right)L_3} \right) \left(1 + \right. \\ &\quad \left. - \frac{16 A}{a_l^{-D} C} \sum_{L_A, L_B, L_C}^{1, \dots, \infty} \frac{L_A L_B + L_A L_C + L_B L_C}{L_A + L_B + L_C} \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{a_l^2}{2D-2} q^2 + \tau\right)L_A - \left(\frac{a_l^2}{2D-2} (q-\mathbf{k}_1)^2 + \tau\right)L_B - \left(\frac{a_l^2}{2D-2} (q+\mathbf{k}_2)^2 + \tau\right)L_C} \right) \end{aligned}$$

$$- \frac{16 A}{a_l^{-D} C} \left(\sum_{L=1}^{\infty} e^{-\left(\frac{a_l^2}{2D-2} k_2^2 + \tau\right)L} \right) \sum_{L_A, L_B}^{1, \dots, \infty} \frac{L_A L_B}{L_A + L_B} \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{a_l^2}{2D-2} q^2 + \tau\right)L_A - \left(\frac{a_l^2}{2D-2} (q-k)^2 + \tau\right)L_B} \right), \quad (7.73)$$

where the following definitions are made:

$$A \equiv \frac{1}{M} \left(\frac{(2D)!}{(2D-3)!} \frac{\hat{P}_3(0)}{\rho} \right)^2 \left(\frac{a}{(2D-1)2D} \right)^3 (2D-2)^{\frac{D}{2}}, \quad (7.74)$$

$$B \equiv \frac{1}{M} \frac{(2D)!}{(2D-3)!} \frac{\hat{P}_3(0)}{\rho} \frac{2D \hat{P}_1(0)}{\rho} \left(\frac{a}{(2D-1)2D} \right)^2, \quad (7.75)$$

$$C \equiv (2D-2)^{\frac{D}{2}}. \quad (7.76)$$

Note that, while ρ and a_l are dimensional quantities, with dimension of energy and length respectively, τ is dimensionless and it quantifies the distance from the Bethe lattice (MF) critical point; for this reason, it can be thought of as the *bare* mass in Field Theory. Also note that the lengths of the paths are dimensionless, they are only parameters for the number of NBP. Similarly to the percolation case, in Section 6.A.1, the constants A , B and C are dimensionless as well, and the justification of their definitions is that, for a diagram with N_{loop} loops and V_e external vertices, a factor $A^{N_{\text{loop}}-1}$ and a factor B^{V_e} respectively appear. The constant C is defined only for brevity.

Another comment concerns the perturbative nature of the expressions for $\hat{C}_2$, $\hat{D}_2$ and $\hat{C}_3$. The *M*-layer construction provides a method to compute perturbative expansions for observables in the large M limit so that the expansion parameter $1/M$ is reasonably small. For simplicity, in Eq. 7.71, Eq. 7.72 and Eq. 7.73 corrections are dropped. Similarly, the following computations are done neglecting higher orders in $1/M$ (or analogously A or B , given that they both are $\mathcal{O}(1/M)$) if not explicitly written.

The quantities of interest are the susceptibilities

$$\sum_x C_2(x, 0) = \frac{1}{a_l^D} \hat{C}_2(0) = \chi_2(\tau), \quad (7.77)$$

$$\sum_{x,y} C_3(x, y, 0) = \frac{1}{a_l^{2D}} \hat{C}_3(0, 0) = \chi_3(\tau), \quad (7.78)$$

$$\sum_x D_2(x, 0) = \frac{1}{a_l^D} \hat{D}_2(0) = \chi_2^{\text{dis}}(\tau). \quad (7.79)$$

All ingredients are now available to compute the susceptibilities to one-loop order. To do so, the integrals over the “loop momenta”, denoted by q in Eq. 7.71, Eq. 7.72 and Eq. 7.73, are performed, then the sums over the lengths are written as integrals

$$\prod_{i=1}^I \sum_{L_i=1}^{\infty} \rightarrow \prod_{i=1}^I \int_{1/\Lambda^2}^{\infty} dL_i, \quad (7.80)$$

where, for later convenience, the small-length cut-off $\Lambda = 1$ is explicitly written. Then the lengths are scaled

$$\tilde{L}_i \equiv L_i \tau \quad (7.81)$$

and plugging Eq. 7.71, Eq. 7.72, Eq. 7.73 inside the definitions of the susceptibilities leads to

$$\chi_2(\tau) = 4 \frac{\rho}{\tau} C \frac{B^2}{A} \left(1 - \frac{16 A \tau^{D/2-4}}{(4\pi)^{D/2}} I_1(\tau/\Lambda^2) \right), \quad (7.82)$$

$$\chi_3(\tau) = 64 \frac{\rho}{\tau^3} C \frac{B^3}{A} \left(1 - \frac{32 A \tau^{D/2-4}}{(4\pi)^{D/2}} I_3(\tau/\Lambda^2) - \frac{48 A \tau^{D/2-4}}{(4\pi)^{D/2}} I_1(\tau/\Lambda^2) \right), \quad (7.83)$$

$$\chi_2^{\text{dis}}(\tau) = -2 \frac{\rho^2}{\tau^2} C \frac{B^2}{A} \left(1 - \frac{32 A \tau^{D/2-4}}{(4\pi)^{D/2}} (I_1(\tau/\Lambda^2) + I_2(\tau/\Lambda^2)) \right), \quad (7.84)$$

where the following integrals are defined

$$I_1(\tau/\Lambda^2) \equiv \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a \tilde{L}_b e^{-(\tilde{L}_a + \tilde{L}_b)}}{(\tilde{L}_a + \tilde{L}_b)^{D/2+1}}, \quad (7.85)$$

$$I_2(\tau/\Lambda^2) \equiv \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a^2 \tilde{L}_b e^{-(\tilde{L}_a + \tilde{L}_b)}}{(\tilde{L}_a + \tilde{L}_b)^{D/2+1}} = \frac{1}{2} \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a \tilde{L}_b e^{-(\tilde{L}_a + \tilde{L}_b)}}{(\tilde{L}_a + \tilde{L}_b)^{D/2}}, \quad (7.86)$$

and

$$I_3(\tau/\Lambda^2) \equiv \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b d\tilde{L}_c \frac{\tilde{L}_a \tilde{L}_b + \tilde{L}_c \tilde{L}_b + \tilde{L}_a \tilde{L}_c}{(\tilde{L}_a + \tilde{L}_b + \tilde{L}_c)^{D/2+1}} e^{-(\tilde{L}_a + \tilde{L}_b + \tilde{L}_c)}. \quad (7.87)$$

Note that $I_1(x)$ diverges for $x \rightarrow 0$ (UV regime) in $D = 8$, thus further complicating the expansion at small values of τ , while $I_2(x)$ and $I_3(x)$ are regular. These typical divergences can be “cured” by writing the susceptibilities as functions of the physical mass m , that is defined as the reciprocal of the correlation length, as follows. The correlation length is defined from the connected correlation through the following formula:

$$\zeta^2 = \left(\frac{d}{dk^2} \hat{C}_2^{-1}(k) \right) \Big|_{k^2=0} \hat{C}_2(0), \quad (7.88)$$

where, from Eq. 7.71, scaling the momenta by

$$\tilde{k} \equiv k \frac{a_l}{\sqrt{\tau(2D-2)}}, \quad (7.89)$$

one can write

$$\begin{aligned} \hat{C}_2(k) = & a_l^D \frac{\rho}{\tau} C \frac{B^2}{A} \left(4 \frac{1}{\tilde{k}^2 + 1} + \right. \\ & \left. - \frac{128}{2} A \tau^{D/2-4} \left(\frac{1}{\tilde{k}^2 + 1} \right)^2 \int \frac{d^D \tilde{q}}{(2\pi)^D} \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a \tilde{L}_b}{\tilde{L}_a + \tilde{L}_b} e^{-\tilde{q}^2 \tilde{L}_a - (\tilde{q} + \tilde{k})^2 \tilde{L}_b - (\tilde{L}_a + \tilde{L}_b)} \right). \end{aligned} \quad (7.90)$$

Then:

$$\hat{C}_2(0) = a_l^D \frac{\rho}{\tau} C \frac{B^2}{A} \left(4 - \frac{128}{2} A \tau^{D/2-4} \frac{1}{(4\pi)^{D/2}} \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a \tilde{L}_b}{(\tilde{L}_a + \tilde{L}_b)^{D/2+1}} e^{-(\tilde{L}_a + \tilde{L}_b)} \right), \quad (7.91)$$

$$\frac{1}{\widehat{C}_2(\mathbf{k})} = a_l^{-D} \frac{\tau}{\rho} \frac{A}{C B^2} \frac{1}{4} \left(\tilde{k}^2 + 1 + \frac{16 A \tau^{D/2-4}}{(4\pi)^{D/2}} \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a \tilde{L}_b}{(\tilde{L}_a + \tilde{L}_b)^{D/2+1}} e^{-\tilde{k}^2 \frac{\tilde{L}_a \tilde{L}_b}{\tilde{L}_a + \tilde{L}_b} - (\tilde{L}_a + \tilde{L}_b)} \right), \quad (7.92)$$

and

$$\left. \frac{\partial}{\partial \tilde{k}^2} \widehat{C}_2(\mathbf{k})^{-1} \right|_{\tilde{k}^2=0} = a_l^{-D} \frac{\tau}{\rho} \frac{A}{C B^2} \frac{1}{4} \left(1 - \frac{16 A \tau^{D/2-4}}{(4\pi)^{D/2}} \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a^2 \tilde{L}_b^2 e^{-(\tilde{L}_a + \tilde{L}_b)}}{(\tilde{L}_a + \tilde{L}_b)^{D/2+2}} \right), \quad (7.93)$$

to compute

$$\xi^2 = \left(\frac{\partial}{\partial \tilde{k}^2} \widehat{C}_2^{-1}(\mathbf{k}) \right) \Big|_{\tilde{k}^2=0} \widehat{C}_2(0) = a_l^2 C^{-2/D} \frac{1}{\tau} \left(1 - \frac{16 A \tau^{D/2-4}}{(4\pi)^{D/2}} (I_4(\tau/\Lambda^2) + I_1(\tau/\Lambda^2)) \right), \quad (7.94)$$

where

$$I_4(\tau/\Lambda^2) \equiv \int_{\tau/\Lambda^2}^{\infty} d\tilde{L}_a d\tilde{L}_b \frac{\tilde{L}_a^2 \tilde{L}_b^2 e^{-(\tilde{L}_a + \tilde{L}_b)}}{(\tilde{L}_a + \tilde{L}_b)^{D/2+2}}. \quad (7.95)$$

Note that $I_4(x)$ is finite for $x \rightarrow 0$ in $D = 8$. Analogously, the last equation can be written in terms of the physical distance from the critical point, called m

$$m^2 = \frac{1}{\xi^2} = \frac{C^{2/D}}{a_l^2} \tau \left(1 + \frac{16 A \tau^{D/2-4}}{(4\pi)^{D/2}} (I_4(\tau/\Lambda^2) + I_1(\tau/\Lambda^2)) \right). \quad (7.96)$$

From the last equation τ can be written as a function of the mass m (neglecting higher orders in A):

$$\tau = a_l^2 C^{-2/D} m^2 \left(1 - \frac{1}{(4\pi)^{D/2}} \frac{u}{2} (I_4(m^2/\Lambda^2) + I_1(m^2/\Lambda^2)) \right), \quad (7.97)$$

where the dimensionless coupling constant u is defined

$$u \equiv g m^{D-8}, \quad (7.98)$$

and

$$g \equiv 32A (a_l C^{-1/D})^{D-8}. \quad (7.99)$$

Given the last definition and using Eq. 7.97, the three observables can be rewritten as functions of m^2 :

$$\chi_2(m^2) = 4\rho C^{1+\frac{2}{D}} a_l^{-2} m^{-2} \frac{B^2}{A} \left(1 + \frac{1}{(4\pi)^{D/2}} \frac{u}{2} I_4(m^2/\Lambda^2) \right), \quad (7.100)$$

$$\chi_3(m^2) = 64\rho C^{1+\frac{6}{D}} a_l^{-6} m^{-6} \frac{B^3}{A} \left(1 + \frac{u}{(4\pi)^{D/2}} \left(\frac{3}{2} I_4(m^2/\Lambda^2) - I_3(m^2/\Lambda^2) \right) \right) \quad (7.101)$$

and

$$\chi_2^{\text{dis}}(m^2) = -2\rho^2 C^{1+\frac{4}{D}} a_l^{-4} m^{-4} \frac{B^2}{A} \left(1 + \frac{u}{(4\pi)^{D/2}} (I_4(m^2/\Lambda^2) - I_2(m^2/\Lambda^2)) \right). \quad (7.102)$$

Note that upon switching from the bare parameter τ to the physical correlation length $1/m$ the integral $I_1(\tau/\Lambda^2)$, that is UV divergent in $D = 8$, is no longer present and the expressions are free of UV divergences, providing the possibility to safely take the limit $m \rightarrow 0$.

7.4 Computation of Critical Exponents

The susceptibilities have been computed up to one loop order in powers of $1/M$, using for χ_2 and χ_2^{dis} the diagrams in Figure 6.2 and for χ_3 the diagrams in Figure 6.3. Diagrams involving tadpoles can be shown to be irrelevant with the help of the condition given by Eq. 7.184, see Section 7.A.4.

The results, expressed in terms of $m \equiv \zeta^{-1}$, are:

$$\chi_2 \propto m^{-2} \left(1 + \frac{1}{(4\pi)^{D/2}} \frac{u}{2} I_4 + \mathcal{O}(u^2) \right), \quad (7.103)$$

$$\chi_3 \propto m^{-6} \left(1 + \frac{u}{(4\pi)^{D/2}} \left(\frac{3}{2} I_4 - I_3 \right) + \mathcal{O}(u^2) \right), \quad (7.104)$$

$$\chi_2^{\text{dis}} \propto m^{-4} \left(1 + \frac{u}{(4\pi)^{D/2}} (I_4 - I_2) + \mathcal{O}(u^2) \right), \quad (7.105)$$

where $u \equiv g m^{D-8}$ and g is a $\mathcal{O}(1/M)$ quantity that depends on the microscopic details of the model. As will be shown, its actual value is irrelevant for the computation of the critical exponents leading to *universal* behavior. Moreover:

$$I_2 \equiv \int_0^\infty \int_0^\infty dL_A dL_B \frac{L_A^2 L_B e^{-(L_A+L_B)}}{(L_A + L_B)^{D/2+1}} = \frac{1}{12} \Gamma\left(\frac{\epsilon}{2}\right), \quad (7.106)$$

$$I_3 \equiv \int_0^\infty \int_0^\infty \int_0^\infty dL_A dL_B dL_C \frac{L_A L_B + L_C L_B + L_A L_C}{(L_A + L_B + L_C)^{D/2+1}} e^{-(L_A+L_B+L_C)} = \frac{1}{8} \Gamma\left(\frac{\epsilon}{2}\right), \quad (7.107)$$

$$I_4 \equiv \int_0^\infty \int_0^\infty dL_A dL_B \frac{L_A^2 L_B^2 e^{-(L_A+L_B)}}{(L_A + L_B)^{D/2+2}} = \frac{1}{30} \Gamma\left(\frac{\epsilon}{2}\right), \quad (7.108)$$

with $\epsilon \equiv 8 - D$.

Following the standard field-theoretic procedure described in Section 3.1.2 to estimate critical exponents, the quantity λ^5 is defined⁶

$$\lambda \equiv -\zeta^{-D} \frac{\chi_2^{\text{dis}} \chi_3^2}{\chi_2^4}, \quad (7.109)$$

that should remain *finite* at the critical point, due to the scaling laws for the susceptibilities, in Eq. 7.15 and Eq. 7.17. According to standard jargon, the ratio λ is also referred to as the *renormalized coupling constant*.

Inserting Eq. 7.103, Eq. 7.104, Eq. 7.105 into the definition of the renormalized coupling constant λ , yields at second order in u :

$$\lambda = u - \frac{4}{15} \frac{u^2}{(4\pi)^{D/2}} \Gamma\left(4 - \frac{D}{2}\right) + \mathcal{O}(u^3). \quad (7.110)$$

According to the scaling laws, λ has a finite value λ_c at the critical point for $D < 8$, but that value cannot be easily read from the above series because u instead diverges at the

⁵Not to be confused with the control parameter λ in Eq. 7.21

⁶Note that a minus sign is put in the definition of λ to have a positive value at the critical point since the leading order of χ_2^{dis} is negative.

critical point $m = 0$, according to its definition $u \equiv g m^{D-8}$. Therefore, it is convenient to transform the series in powers of u into a series in powers of λ . The β -function is introduced as

$$\beta \equiv m^2 \frac{\partial}{\partial m^2} \Big|_{g \text{ fixed}} \quad \lambda = \frac{1}{2}(D-8)u \frac{\partial}{\partial u} \Big|_{m^2 \text{ fixed}} \quad \lambda. \quad (7.111)$$

From Eq. 7.110 and Eq. 7.111 the β -function in powers of u is obtained, and then, computing u as a function of λ , inverting Eq. 7.110:

$$u = \lambda + \frac{4}{15} \frac{\lambda^2}{(4\pi)^{D/2}} \Gamma\left(4 - \frac{D}{2}\right) + \mathcal{O}(\lambda^3), \quad (7.112)$$

the following form for $\beta(\lambda)$ at second order in λ follows:

$$\beta(\lambda) = -\frac{\epsilon \lambda}{2} + \frac{2\epsilon}{15(4\pi)^{D/2}} \Gamma\left(\frac{\epsilon}{2}\right) \lambda^2 + \mathcal{O}(\lambda^3). \quad (7.113)$$

Since, at the critical point, λ converges to some λ_c , the requirement $\beta(\lambda_c) = 0$ holds. More precisely, close to the critical point, where m is small, $\lambda = \lambda_c + c_1 m^{-\omega}$ with the universal negative exponent ω controlling the leading finite-size corrections to scaling (see Section 3.1.2). This implies:

$$\beta(\lambda_c) = 0, \quad \omega = -2\beta'(\lambda_c). \quad (7.114)$$

From Eq. 7.113 the following scenario emerges for the zeros of the β -function: for $\epsilon \leq 0$ ($D \geq 8$) only the solution $\lambda_c = 0$ exists, meaning that λ tends to zero at the critical point with $\omega = \epsilon + \mathcal{O}(\epsilon^2)$ (in agreement with the discussion after Eq. 7.18), while for $\epsilon > 0$ a new solution $\lambda_c > 0$ appears while the solution $\lambda_c = 0$ becomes unstable as $\omega = \epsilon + \mathcal{O}(\epsilon^2)$ would be positive, thus $\lambda_c = 0$ is unphysical. Expanding at small ϵ , one obtains by means of $\lim_{\epsilon \rightarrow 0} \epsilon \Gamma(\epsilon) = 1$

$$\lambda_c = \frac{15}{8} (4\pi)^4 \epsilon + \mathcal{O}(\epsilon^2) \quad (7.115)$$

and $\omega = -\epsilon + \mathcal{O}(\epsilon^2)$. The critical exponents η and $\bar{\eta}$ can be evaluated considering the following effective exponent functions:

$$Q(\lambda) \equiv \frac{\partial \ln \chi_2}{\partial \ln m^2} \Big|_{g \text{ fixed}}, \quad Q^{\text{dis}}(\lambda) \equiv \frac{\partial \ln \chi_2^{\text{dis}}}{\partial \ln m^2} \Big|_{g \text{ fixed}}. \quad (7.116)$$

They can be obtained from Eq. 7.103, Eq. 7.105 and expressed in powers of λ from Eq. 7.112. The scaling laws $\chi_2 \propto m^{\eta-2}$, $\chi_2^{\text{dis}} \propto m^{\bar{\eta}-4}$ for $m \rightarrow 0$ imply that η and $\bar{\eta}$ are related to the effective exponent functions evaluated at the critical point $m = 0$ (i.e., $\lambda = \lambda_c$):

$$Q(\lambda_c) = -1 + \frac{\eta}{2}, \quad Q^{\text{dis}}(\lambda_c) = -2 + \frac{\bar{\eta}}{2}. \quad (7.117)$$

The exponent ν can be computed considering $\partial \tau / \partial m^2$. Since $m \propto \tau^\nu$ it follows that $\partial \tau / \partial m^2 \propto m^{1/\nu-2}$ for $m \rightarrow 0$ or equivalently $\lambda \rightarrow \lambda_c$. The M -layer computation leads to:

$$\frac{\partial \tau}{\partial m^2} \propto 1 - \frac{1}{(4\pi)^{D/2}} \frac{u}{2} \left(\frac{D-16}{60} \right) \Gamma\left(\frac{8-D}{2}\right) + \mathcal{O}(u^2). \quad (7.118)$$

To be combined with the following expressions:

$$c_2(\lambda) \equiv \left. \frac{\partial \ln(\partial \tau / \partial m^2)}{\partial \ln m^2} \right|_{g \text{ fixed}}, \quad c_2(\lambda_c) = \frac{1}{2\nu} - 1. \quad (7.119)$$

Finally, from these last expressions, as done for the percolation problem in [Chapter 6](#), the ϵ -expansion for the independent critical exponents can be carried out, with the following result:

$$\nu = \frac{1}{2} + \frac{1}{16}\epsilon + \mathcal{O}(\epsilon^2), \quad \eta = -\frac{1}{16}\epsilon + \mathcal{O}(\epsilon^2), \quad \bar{\eta} = \frac{3}{16}\epsilon + \mathcal{O}(\epsilon^2), \quad \omega = -\epsilon + \mathcal{O}(\epsilon^2). \quad (7.120)$$

7.5 One-Dimensional Long Range EA Model

One big advantage of the M -layer construction is that once it is applied to a model on a given lattice, similar results can be easily obtained for the same model but defined on a different topology, the only difference being the number of paths between two points in the original lattice. Exploiting this property, estimates for critical exponents are obtained in this section for the EA model defined on a one-dimensional LR lattice.

In [Section 7.5.1](#), the lattice is defined, together with the interesting critical exponents from corresponding scaling laws. Then, [Section 7.5.2](#) is devoted to the explanation of the M -layer procedure for the specific lattice considered. There, the asymptotic expression of the number of paths between two points in this new topology is derived, a fundamental ingredient to compute critical exponents, done in [Section 7.5.3](#). In this last section, the usual steps are done to present the main result: critical exponents are estimated for the EA model on the one-dimensional LR lattice.

7.5.1 The Model

The model of interest here is defined through the Hamiltonian of [Eq. 7.1](#), where $\mathcal{L}$ and $\mathcal{E}$ denote the set of nodes and edges, respectively, of a one-dimensional LR lattice, defined as follows.

Consider an infinite one-dimensional lattice with lattice spacing a_l . Each node i is located at a point $x_i \in a_l \mathbb{Z}^1$, so the distance between any two nodes i and j is given by $r_{ij} = |x_i - x_j|$. Initially, all edges between nodes are removed. They are then reintroduced according to the following probabilistic rule: the edge connecting two nodes at distance $r_{ij} \geq 1$ is restored with probability $\mathbb{P}_{r_{ij}} = \mathcal{A}/r_{ij}^\rho$, with $1 < \rho < 3$, not to be confused with the parameter ρ , relative to the *Ansatz* distribution for the triplet. The constant $\mathcal{A}$ is fixed by requiring that the average connectivity per node equals z :

$$z \stackrel{!}{=} 2 \sum_{r_{ij}=1}^{\infty} \mathbb{P}_{r_{ij}}, \quad (7.121)$$

which implies that $\mathbb{P}_{r_{ij}}$ is normalized to z . Notably, while this procedure is practically implemented in simulations with a finite number of nodes [[Leuo8](#)], the following calculations are directly done considering the thermodynamic limit. This construction generates a one-dimensional lattice where the degree of each node is Poisson

distributed with mean $z < \infty$ - in this sense “diluted” - and the connections may span arbitrarily large distances, in this sense “long-range”.

Since the goal is to study the EA model, once the network structure is established, to each existing edge a coupling constant is assigned, drawn from either a Gaussian or bimodal distribution, a choice that does not affect the critical behavior. Moreover, as discussed in Ref. [Leuo8], this diluted formulation is expected to belong to the same universality class as the corresponding FC one, where Gaussian couplings have zero mean and a variance that decays with the inter-site distance [Kot83].

One can identify a correspondence between the LR model and the associated finite-dimensional one. In particular it can be shown [Kot83, Bañ12b, Leuo8] that for $\rho_{lc} < \rho$ no transition is observed, for $\rho < \rho_{uc}$ the transition is characterized by the MF critical exponents while, for $\rho_{uc} < \rho < \rho_{lc}$, corrections to the MF values should be considered, for some values of ρ_{lc} and ρ_{uc} ($< \rho_{lc}$). This scenario, with different regions and values, is the same obtained varying the dimension, D , of the associated finite-dimensional (also called *short range*) model between the lower and upper critical dimensions, D_{lc} and D_{uc} respectively ($> D_{lc}$). Moreover, given a qualitative scaling argument [Bañ12b] it is possible to directly relate ρ to the dimension D through

$$D = \frac{2}{\rho - 1}, \quad (7.122)$$

valid for $D \geq D_{uc}$. With this relation, and the result of Refs. [Ang22, Ang25a], $D_{uc} = 8$, one finds the threshold value $\rho_{uc} = 5/4$. The same cannot be done for ρ_{lc} , both because the relation in Eq. 7.122 needs to be corrected for $D \leq D_{uc}$ and because D_{lc} is still debated for spin glass systems. While the features of the zero-temperature critical behavior of the system near the critical point is the same as described in Section 7.1, the only two differences in the application of the construction are: i) the number of paths between two points in the LR lattice; ii) the Poisson distribution of the connectivities (that must be considered when computing observables on the diagrams, as shown below). Furthermore, changing the underlying lattice structure leads to corresponding changes in the scaling laws. The next section presents these modified scaling laws before detailing the computation of the number of paths in the LR lattice.

Scaling Laws

In this section, definitions of the critical exponents are provided by stating the modified scaling laws. In particular, on the M -layer lattice defined above, one can get the following scaling laws, directly considering soft clusters. Starting from definitions similar to Eq. 7.67 and Eq. 7.68, it is possible to write

$$C_2^{\text{lr}}(r) = \int \frac{dk}{2\pi} \hat{C}_2^{\text{lr}}(k) e^{ikr} \propto \int dk \frac{1}{|k|^{\rho-1} + \xi_{\text{lr}}^{-\rho+1}} e^{ikr} = \frac{1}{r^{2-\rho+\eta_{\text{lr}}}} f_{\text{con}}^{\text{lr}} \left((r/\xi_{\text{lr}})^{\rho-1} \right), \quad (7.123)$$

$$D_2^{\text{lr}}(r) = \int \frac{dk}{2\pi} \hat{D}_2^{\text{lr}}(k) e^{ikr} \propto \int dk \frac{1}{(|k|^{\rho-1} + \xi_{\text{lr}}^{-\rho+1})^2} e^{ikr} = \frac{1}{r^{1-2(\rho-1)+\bar{\eta}_{\text{lr}}}} f_{\text{dis}}^{\text{lr}} \left((r/\xi_{\text{lr}})^{\rho-1} \right), \quad (7.124)$$

and then

$$\chi_2 \sim \xi^{\rho-1-\eta_{\text{lr}}}, \quad \chi_2^{\text{dis}} \sim \xi^{2(\rho-1)-\bar{\eta}_{\text{lr}}}, \quad \chi_3 \sim \xi^{\theta_{\text{lr}}-1+\frac{3}{2}(\rho-\theta_{\text{lr}})}, \quad (7.125)$$

where ξ_{lr} is the correlation length and new scaling functions, $f_{\text{con}}^{\text{lr}}$ and $f_{\text{dis}}^{\text{lr}}$, are identified. Following percolation theory, the exponent θ_{lr} can be expressed in terms of the anomalous dimensions (see Chapter 6)

$$\theta_{\text{lr}} = \eta_{\text{lr}} - \bar{\eta}_{\text{lr}} + \rho - 1. \quad (7.126)$$

Interestingly, η_{lr} doesn't get renormalized, i.e., it sticks to the MF value $\eta_{\text{lr}}^{\text{MF}} = 0$. This result is consistent with field-theoretic predictions, where the *non-locality* of the corresponding Gaussian interaction, due to its "long-range", is the reason for this non-renormalization in LR-type models. [Fis72, Pau16]. As explained below, the M -layer construction correctly reproduces this result, even if not *a priori* guaranteed.

The last exponent to be defined is ν_{lr} , from the relation between the correlation length ξ_{lr} and the distance from the critical point, in this context defined as $\tau \equiv -\ln(z\lambda)$

$$\xi_{\text{lr}}^{1-\rho} \sim \tau^{\nu_{\text{lr}}(\rho-1)}. \quad (7.127)$$

Remarkably, there is a connection between the LR model and the corresponding finite-dimensional version, see the argument in Refs. [Lar10, Bañ12b]. The main concepts are summarized here, including also the additional exponent θ_{lr} . The idea is to make use of the scaling form of the singular part of the ground state energy density of a generic system in D dimensions

$$E_{\text{sing}}(\tau, H/J, u) = \frac{1}{L^{D-y_J}} J f_{E_{\text{sing}}}(L^{y_\tau} \tau, L^{y_H-y_J} H/J, L^{y_u} u), \quad (7.128)$$

where τ is the distance from the critical point, H is the external field, J is the standard deviation of the random coupling, u is the operator which gives the leading correction to scaling and $f_{E_{\text{sing}}}$ is a scaling function. The exponents y_τ, y_H, y_J, y_u , can be written in term of the more commonly used ν, η, θ and ω

$$y_\tau = \frac{1}{\nu}, \quad y_H = \frac{D+2-\eta+\theta}{2}, \quad y_J = \theta, \quad y_u = \omega. \quad (7.129)$$

Notice that y_H , according to the scaling laws previously defined, has a different form for the LR model

$$y_H^{\text{lr}} = \frac{1 + (\rho - 1) + \theta_{\text{lr}}}{2}, \quad (7.130)$$

where the fact that $\eta_{\text{lr}} = 0$ is also used. Next, the idea is to compare the ground state energy densities of the finite-dimensional model with the one of the corresponding LR model, with the same number of spins $L^D = L_{\text{lr}}$ and with $J = J_{\text{lr}}$:

$$\frac{1}{L^{D-y_J}} J f(L^{y_\tau} \tau, L^{y_H-y_J} H/J, L^{y_u} u) = \frac{1}{L_{\text{lr}}^{1-y_J^{\text{lr}}}} J_{\text{lr}} f_{\text{lr}}(L_{\text{lr}}^{y_\tau^{\text{lr}}} \tau, L_{\text{lr}}^{y_H^{\text{lr}}-y_J^{\text{lr}}} H/J, L_{\text{lr}}^{y_u^{\text{lr}}} u). \quad (7.131)$$

Using $L^D = L_{\text{lr}}$ and $J = J_{\text{lr}}$ and imposing the validity of Eq. 7.131 one obtains:

$$\frac{y_i}{D} = y_i^{\text{lr}}, \quad \text{for } i = \tau, J, H, u. \quad (7.132)$$

Using Eq. 7.130 in the previous relation, it is possible to generalize the relation between ρ and the physical dimension D given in Eq. 7.122, for $D \leq D_{\text{uc}}$

$$D = \frac{2-\eta}{\rho-1}, \quad (7.133)$$

where it is remarked that η is the exponent of the finite-dimensional model.

As a last comment: the correspondence between the finite-dimensional and the associated LR version is only approximatively valid. Indeed, Eq. 7.131 is a conjecture that, for these models, can or not be a good approximation. It is not *a priori* expected that the relation in Eq. 7.133 holds at all orders in the ϵ -expansion. In the following, it is shown that at first order in ϵ it is satisfied for exponents η , $\bar{\eta}$ and θ .

7.5.2 M-layer Construction for the 1D lattice

The M -layer construction, as presented in Chapter 4, is generalized here. Indeed, while there the “original” lattice is the finite-dimensional one, here the corresponding LR lattice is considered as the starting point.

The construction described in Section 7.5.1 is generalized to the M -layer graph. Consider M copies of the infinite one-dimensional lattice, where all edges are removed. Then, consider a node pair (i^1, j^1) at distance r_{ij} in layer 1 (reference layer) and their copies (i^α, j^α) across all layers $\alpha = 1, \dots, M$. For each of the M^2 pairs (i^α, j^β) , with $\alpha, \beta = 1, \dots, M$ an edge is restored with probability $\mathbb{P}_{r_{ij}}/M$, such that, on average, $M\mathbb{P}_{r_{ij}}$ edges are added. This procedure is repeated for all other pairs of sites of the M -layer graph. Recalling that $\mathbb{P}_{r_{ij}}$ is normalized to z , each node in this resulting graph has average connectivity z . This yields inter-layer connections in which each edge links some i^α to some j^β , with α and β potentially equal (intra-layer) or different (inter-layer). Once the edges are placed, a random Gaussian coupling is associated to each of them. In this way, a sort of average over disorder is automatically done over different copies of the original lattice (see also Section 3 in [Alt17]).

It is possible to understand some properties of the resulting layered graph. First of all, the original lattice is recovered for $M = 1$. On the other hand, in the limit $M \rightarrow \infty$ the probability to close a topological loop goes to zero. This can be intuitively understood from the fact that, as the number of layers increases, the probability of connecting one node to another specific node on a generic layer decreases after the rewiring. In this sense the parameter $1/M$ can be varied to tune the probability of having topological loops in the system, whose presence affects the critical behavior.

A relevant difference is that, in this case, the paths projected on the original lattice are not necessarily non-backtracking, as in finite-dimensional lattices. This can happen since each site has, on average $z(> 1)$ nearest neighbors, meaning that, projecting on the original lattice, more than one line can start from the given site. For this reason, some subtleties on the construction can arise, as the *boucle*-type diagrams discussed in Chapter 4. It is expected that all these subtleties will be relevant at two-loop order, being irrelevant for the present work.

As anticipated in Section 7.5.1, it is important to remark that the degree distribution of the nodes is a Poisson distribution with mean z , as can be understood from the details of the construction of the one-dimensional LR lattice. In previous works [Ang20, Ang25c, Ang22], the same construction is applied to fixed-degree lattices. For this reason, the cavity equations should be rewritten for ER graphs, as already explained in Section 2.2.1. In particular, the definition of the cavity bias distribution changes into

$$P_{\text{ER}}(u) =$$

$$\mathbb{E}_J \sum_{k=0}^{\infty} q_z(r=k) \int \prod_{i=1}^k P_{\text{ER}}(u_i) du_i \delta \left(u - \text{sign} \left(J \left(H + \sum_{i=1}^k u_i \right) \right) \min \left(|J|, \left| H + \sum_{i=1}^k u_i \right| \right) \right), \quad (7.134)$$

where $P_{\text{ER}}(u)$ is defined in [Section 2.2.1](#) and $q_z(r=k) = z^k e^{-z}/k!$ is the excess degree distribution. Similarly, the definition of the eigenfunction $g(u)$ changes from [Eq. 7.21](#) into

$$\lambda g(u) = \mathbb{E}_J \sum_{k=0}^{\infty} q_z(r=k) \int \prod_{i=1}^k dP_{\text{ER}}(u_i) dg(u') \mathbb{I} \left[\left| H + u' + \sum_{i=1}^k u_i \right| < |J| \right] \delta \left(u - \text{sign}(J) \left| H + u' + \sum_{i=1}^k u_i \right| \right). \quad (7.135)$$

Finally, also the definition of the function $\hat{P}_n(h)$ changes from [Eq. 7.25](#) into

$$\hat{P}_n(h) = \sum_{k=0}^{\infty} q_z(r=k) \int \prod_{i=1}^n g(u_i) du_i \prod_{j=1}^k P_{\text{ER}}(v_j) dv_j \delta \left(h - \left(H + \sum_{i=1}^n u_i + \sum_{j=1}^k v_j \right) \right). \quad (7.136)$$

Counting Random Walks The goal of this paragraph is to evaluate the large L asymptotic expression for the number of [RW](#) between two points at distance L in the [LR](#) graph.

In the M -layer framework, one is interested in computing the number of [RW](#) with L steps between two nodes at x_i and x_j , that are denoted as $\mathcal{N}_L^{\text{lr}}(x_i, x_j)$, in a given lattice. To this aim one writes the recursive formula for $\mathcal{N}_L^{\text{lr}}(x_1, x_3)$:

$$\mathcal{N}_L^{\text{lr}}(x_1, x_3) = \sum_{x_2 \in \mathbb{Z}^1} \mathcal{N}_{L-1}^{\text{lr}}(x_1, x_2) \mathcal{W}(x_2, x_3), \quad (7.137)$$

where $\mathcal{W}(x_i, x_j) = \mathcal{W}(|x_i - x_j|)$ is the probability to go from x_i to x_j in one step, which is related to $\mathbb{P}_{r_{ij}}$, defined in [Section 7.5.1](#), through

$$\mathcal{W}(x_i, x_j) = \frac{\mathbb{P}_{r_{ij}}}{z}. \quad (7.138)$$

The Fourier transform of [Eq. 7.137](#), using the convention in [Eq. 5.39](#) and [Eq. 5.40](#) with $D = 1$ reads

$$\widehat{\mathcal{N}}_L^{\text{lr}}(k) = \widehat{\mathcal{N}}_{L-1}^{\text{lr}}(k) \widehat{\mathcal{W}}(k) / a_l, \quad (7.139)$$

where $\widehat{\mathcal{W}}(k)$ is defined from the Fourier transform of $\mathcal{W}(x_i, x_j)$

$$\widehat{\mathcal{W}}(k_i, k_j) = 2\pi \delta(k_i + k_j) \widehat{\mathcal{W}}(k_i), \quad (7.140)$$

as for $\widehat{\mathcal{N}}_L^{\text{lr}}(k)$ in [Eq. 6.40](#), again in the case $D = 1$. Now, since the interesting regime is where L is large, one can consider $k \approx 0$ and, using [Eq. 7.138](#) together with the expression of $\mathbb{P}_{r_{ij}}$ in [Section 7.5.1](#), one obtains

$$\widehat{\mathcal{W}}(k) - \widehat{\mathcal{W}}(0) = -\frac{2\mathcal{A}}{z} (a_l k)^{\rho-1} \sum_{r=1}^{\infty} \frac{1}{r^{\rho}} (1 - \cos(r)) \equiv -\alpha_{\rho} (a_l k)^{\rho-1}, \quad (7.141)$$

where α_ρ is a ρ -dependent constant, well-defined for $\rho < 3$, which belongs to the interval of definition of the exponent ρ . Plugging the asymptotic expression for $\widehat{\mathcal{W}}(k)$ in Eq. 7.139 it is possible to solve for $\widehat{\mathcal{N}}_L^{\text{lr}}(k)$

$$\widehat{\mathcal{N}}_L^{\text{lr}}(k) = z^L a_l \exp(-\alpha_\rho (a_l k)^{\rho-1} L), \quad (7.142)$$

which is the Fourier transform of the number of RW of length L , with momentum k , averaged over the connectivity of the first L nodes of the path, average that gives the factor z^L . Here and in the following, the momentum at the exponent, $k^{\rho-1}$, is intended to be the modulus to the power $\rho - 1$.

7.5.3 Critical Exponents

The computations to get critical exponents estimates are exactly the same as done in the finite-dimensional case, the only difference being the expression of the number of paths, in this case RW, given by Eq. 7.142, at variance with Eq. 6.41. The results of the computations for the susceptibilities read

$$\chi_2^{\text{lr}}(\tau) \propto \frac{1}{\tau} \left(1 - 16 A_{\text{lr}} \tau^{\frac{1}{\rho-1}-4} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} I_1^{\text{lr}}(\tau/\Lambda^2) \right), \quad (7.143)$$

$$\begin{aligned} \chi_3^{\text{lr}}(\tau) \propto \frac{1}{\tau^3} \left(1 + \right. \\ \left. - 32 A_{\text{lr}} \tau^{\frac{1}{\rho-1}-4} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} I_3^{\text{lr}}(\tau/\Lambda^2) - 48 A_{\text{lr}} \tau^{\frac{1}{\rho-1}-4} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} I_1^{\text{lr}}(\tau/\Lambda^2) \right), \end{aligned} \quad (7.144)$$

$$\chi_2^{\text{dis,lr}}(\tau) = -\frac{1}{\tau^2} \left(1 - 32 A_{\text{lr}} \tau^{\frac{1}{\rho-1}-4} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} \left(I_1^{\text{lr}}(\tau/\Lambda^2) + I_2^{\text{lr}}(\tau/\Lambda^2) \right) \right), \quad (7.145)$$

where the integrals $I_1^{\text{lr}}(x)$, $I_2^{\text{lr}}(x)$ and $I_3^{\text{lr}}(x)$ are defined as follows

$$I_1^{\text{lr}}(x) = \int_x^\infty dL_A dL_B \frac{L_A L_B e^{-(L_A+L_B)}}{(L_A + L_B)^{\frac{1}{\rho-1}+1}}, \quad (7.146)$$

$$I_2^{\text{lr}}(x) = \frac{1}{2} \int_x^\infty dL_A dL_B \frac{L_A L_B e^{-(L_A+L_B)}}{(L_A + L_B)^{\frac{1}{\rho-1}}} \quad (7.147)$$

and, finally

$$I_3^{\text{lr}}(x) = \int_x^\infty dL_A dL_B dL_C \frac{L_A L_B + L_C L_B + L_A L_C}{(L_A + L_B + L_C)^{\frac{1}{\rho-1}+1}} e^{-(L_A+L_B+L_C)}. \quad (7.148)$$

Notice that A_{lr} , B_{lr} and C_{lr} can be identified as done for the percolation (in Section 6.A.1) and for the EA model in the hypercube (in Section 7.3.3). The difference between A_{lr} ,

B_{lr} and C_{lr} and the corresponding ones defined in Eq. 7.74, Eq. 7.75 and Eq. 7.76 are constant factors that are irrelevant for the computation of the critical exponents.

As done for the finite-dimensional lattice, to regularize the divergence of I_1^{lr} one should rewrite the susceptibilities in terms of the *physical mass*, $m_{\text{lr}} = \xi_{\text{lr}}^{-1}$. To this aim, the definition of the correlation length is used

$$\xi_{\text{lr}}^{\rho-1} \equiv \left(\frac{d}{dk^{\rho-1}} (\hat{C}_2^{\text{lr}}(k))^{-1} \right) \bigg|_{k^{\rho-1}=0} \hat{C}_2^{\text{lr}}(0), \quad (7.149)$$

where $(\hat{C}_2^{\text{lr}}(k))^{-1}$ at one-loop order reads

$$(\hat{C}_2^{\text{lr}}(k))^{-1} \propto \left((a_l k)^{\rho-1} \frac{\alpha_\rho}{\tau} + 1 \right) \left(1 + \frac{16 A_{\text{lr}} \tau^{\frac{1}{\rho-1}-4}}{\frac{\alpha_\rho}{\tau} (a_l k)^{\rho-1} + 1} \int \frac{dq}{2\pi} \int dL_A dL_B \frac{L_A L_B}{L_A + L_B} e^{-q^{\rho-1} L_A - (q+k)^{\rho-1} L_B - (L_A + L_B)} \right), \quad (7.150)$$

neglecting higher order corrections in $A_{\text{lr}} \propto 1/M$. Here one realizes that the anomalous dimension η_{lr} doesn't renormalize for this kind of LR models. Indeed, when performing the derivative with respect to $k^{\rho-1}$ of the one-loop correction and then setting $k = 0$ the result is simply zero, as soon as $1 < \rho < 3$. Thus, the relation between τ and the mass is

$$\tau = \alpha_\rho (a_l m_{\text{lr}})^{\rho-1} \left(1 - 16 A_{\text{lr}} \alpha_\rho^{\frac{1}{\rho-1}-4} (a_l m_{\text{lr}})^{5-4\rho} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} I_1^{\text{lr}}(\tau/\Lambda^2) \right). \quad (7.151)$$

The 2-point susceptibility as a function of the physical mass is, at this order in perturbation theory

$$\chi_2^{\text{lr}}(m_{\text{lr}}^{\rho-1}) \propto m_{\text{lr}}^{-(\rho-1)}, \quad (7.152)$$

that is the MF relation, while the diverging correction disappeared. The same can be done for the other susceptibilities

$$\chi_3^{\text{lr}}(m_{\text{lr}}^{\rho-1}) \propto m_{\text{lr}}^{-3(\rho-1)} \left(1 - 32 A_{\text{lr}} \alpha_\rho^{\frac{1}{\rho-1}-4} (a_l m_{\text{lr}})^{5-4\rho} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} I_3^{\text{lr}} \right), \quad (7.153)$$

$$\chi_2^{\text{dis,lr}}(m_{\text{lr}}^{\rho-1}) \propto -m_{\text{lr}}^{-2(\rho-1)} \left(1 - 32 A_{\text{lr}} \alpha_\rho^{\frac{1}{\rho-1}-4} (a_l m_{\text{lr}})^{5-4\rho} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} I_2^{\text{lr}} \right). \quad (7.154)$$

As usual, a dimensionless coupling constant can be identified

$$u_{\text{lr}} \equiv g_{\text{lr}} m_{\text{lr}}^{5-4\rho} \equiv 32 A_{\text{lr}} \alpha_\rho^{\frac{1}{\rho-1}-4} (a_l m_{\text{lr}})^{5-4\rho}, \quad (7.155)$$

where $A_{\text{lr}} \propto 1/M$ and thus g_{lr} is small in the limit $M \rightarrow \infty$. On the other hand, u_{lr} diverges as $\rho > 5/4$ at the critical point $m_{\text{lr}} \rightarrow 0$. The standard procedure to tackle this

divergence is to rewrite the series expansions in terms of the following dimensionless ratio

$$\lambda_{\text{lr}} \equiv -m_{\text{lr}} \frac{\chi_2^{\text{dis,lr}} (\chi_3^{\text{lr}})^2}{(\chi_2^{\text{lr}})^4}. \quad (7.156)$$

In order to express the critical exponents in the ϵ -expansion, for $\epsilon_{\text{lr}} \equiv \rho - 5/4 \ll 1$ the integrals $I_2^{\text{lr}}(m_{\text{lr}}^{\rho-1}/\Lambda^{\rho-1})$ and $I_3^{\text{lr}}(m_{\text{lr}}^{\rho-1}/\Lambda^{\rho-1})$ need to be evaluated at the critical point $m_{\text{lr}} \rightarrow 0$:

$$\lim_{m_{\text{lr}} \rightarrow 0} I_2^{\text{lr}}(m_{\text{lr}}^{\rho-1}/\Lambda^{\rho-1}) = \frac{1}{12} \Gamma\left(\frac{4\rho-5}{\rho-1}\right), \quad \lim_{m_{\text{lr}} \rightarrow 0} I_3^{\text{lr}}(m_{\text{lr}}^{\rho-1}/\Lambda^{\rho-1}) = \frac{1}{8} \Gamma\left(\frac{4\rho-5}{\rho-1}\right). \quad (7.157)$$

Using these results and the expressions of the susceptibilities, it is possible to write the relation between the diverging dimensionless coupling constant and the renormalized one

$$\lambda^{\text{lr}} \propto -u_{\text{lr}} \left(1 - u_{\text{lr}} \frac{\Gamma\left(\frac{\rho}{\rho-1}\right)}{\pi} \frac{1}{3} \Gamma\left(\frac{4\rho-5}{\rho-1}\right) \right), \quad (7.158)$$

finite at the critical point. The following β -function is defined to compute the “critical value” λ_c^{lr}

$$\beta(\lambda^{\text{lr}}) \equiv m_{\text{lr}}^{\rho-1} \frac{\partial}{\partial m_{\text{lr}}^{\rho-1}} \bigg|_{g_{\text{lr}} \text{ fixed}} \lambda^{\text{lr}} = \frac{5-4\rho}{\rho-1} u_{\text{lr}} \frac{\partial}{\partial u_{\text{lr}}} \bigg|_{m_{\text{lr}}^{\rho-1} \text{ fixed}} \lambda^{\text{lr}}. \quad (7.159)$$

Imposing $\beta(\lambda_c^{\text{lr}}) = 0$ and from Eq. 7.158 one gets

$$\lambda_c^{\text{lr}} = 3\pi \frac{\Gamma\left(\frac{4\rho-5}{\rho-1}\right)}{\Gamma\left(\frac{\rho}{\rho-1}\right)}. \quad (7.160)$$

At this point, given the relation between τ and m_{lr} given in Eq. 7.151, the exponent ν_{lr} can be evaluated. The computation of critical exponents involves the definitions of the following functions

$$z_2(\lambda^{\text{lr}}) \equiv \frac{\partial \tau}{\partial m_{\text{lr}}^{\rho-1}} \quad (7.161)$$

and

$$c_2(\lambda^{\text{lr}}) \equiv m_{\text{lr}}^{\rho-1} \frac{\partial}{\partial m_{\text{lr}}^{\rho-1}} \bigg|_{g_{\text{lr}} \text{ fixed}} \ln z_2(\lambda^{\text{lr}}), \quad (7.162)$$

such that

$$c_2(\lambda_c^{\text{lr}}) = \frac{1}{\nu_{\text{lr}}(\rho-1)} - 1. \quad (7.163)$$

Given Eq. 7.151 and the value λ_c^{lr} one computes $c_2(\lambda_c^{\text{lr}})$

$$c_2(\lambda_c^{\text{lr}}) = \frac{5-4\rho}{4(\rho-1)}. \quad (7.164)$$

Notice that even if for $\epsilon_{\text{lr}} = \rho - 5/4 \ll 1$ the integral $I_1^{\text{lr}}(m_{\text{lr}}^{\rho-1}/\Lambda^{\rho-1})$ is diverging in the $m_{\text{lr}} \rightarrow 0$ limit, taking the derivative, as prescribed by the definition of $z_2(\lambda^{\text{lr}})$, the

result is no more diverging and can be expressed as a function of ρ . From Eq. 7.163 the expression for ν_{lr} is obtained to first order in ϵ_{lr}

$$\nu_{\text{lr}} = 4 + \mathcal{O}(\epsilon_{\text{lr}}^2). \quad (7.165)$$

The last two critical exponents to be computed are the anomalous dimensions, η_{lr} and $\bar{\eta}_{\text{lr}}$. Indeed, a nontrivial result from the M -layer construction, applied to the LR version of the EA, is the ϵ -expansion for these two exponents, which reveals that η_{lr} doesn't get renormalized, and it sticks to the MF value $\eta_{\text{lr}}^{\text{MF}} = 0$. To evaluate η_{lr} start from the 2-point susceptibility in Eq. 7.152 and define the following quantity from the scaling law $\chi_2^{\text{lr}} \sim m_{\text{lr}}^{\eta_{\text{lr}} - \rho + 1}$

$$Q_2(\lambda_c^{\text{lr}}) \equiv \left. \frac{\partial \ln \chi_2^{\text{lr}}}{\partial \ln m_{\text{lr}}^{\rho-1}} \right|_{\lambda^{\text{lr}} = \lambda_c^{\text{lr}}} = \frac{\eta_{\text{lr}}}{\rho - 1} - 1. \quad (7.166)$$

Now, given $Q_2(\lambda_c^{\text{lr}})$ and the expression of $\chi_2^{\text{lr}}(m_{\text{lr}})$ in Eq. 7.152, together with λ_c^{lr} in Eq. 7.160, from the perturbative computation it follows

$$\eta_{\text{lr}} = \mathcal{O}(\epsilon_{\text{lr}}^2). \quad (7.167)$$

Notice that this result is highly dependent on the values the relevant diagrams assume, which need to be included following the M -layer computation rules. Moreover, notice that the MF value $\eta_{\text{lr}}^{\text{MF}} = 0$ is due to the definition of the exponent in Eq. 7.123, other papers on the topic use a different definition [Bañ12b, Fis72].

A similar computation can be done for the exponent $\bar{\eta}_{\text{lr}}$, defining

$$Q_2^{\text{dis}}(\lambda_c^{\text{lr}}) \equiv \left. \frac{\partial \ln (-\chi_2^{\text{dis,lr}})}{\partial \ln m_{\text{lr}}^{\rho-1}} \right|_{\lambda_c} = \frac{\bar{\eta}_{\text{lr}}}{\rho - 1} - 2. \quad (7.168)$$

Given the expression of $\chi_2^{\text{dis,lr}}(m_{\text{lr}})$ in Eq. 7.154 and λ_c^{lr} one can compute $\bar{\eta}_{\text{lr}}$

$$\bar{\eta}_{\text{lr}} = \rho - \frac{5}{4}, \quad (7.169)$$

from which, using $\epsilon_{\text{lr}} = \rho - 5/4$, it reads

$$\bar{\eta}_{\text{lr}} = \epsilon_{\text{lr}} + \mathcal{O}(\epsilon_{\text{lr}}^2). \quad (7.170)$$

The result, again expected from field-theoretic results [Fis72, Pau16], is that while η_{lr} does not get renormalized, $\bar{\eta}_{\text{lr}}$ does.

Notice also that the correspondence between the finite-dimensional and the corresponding LR model, described in Section 7.5.1, is obtained at one-loop order for ν_{lr} , η_{lr} and $\bar{\eta}_{\text{lr}}$. Indeed, expressing ϵ_{lr} in terms of $\epsilon (= 8 - D)$, using Eq. 7.133, it is possible to show that

$$D\nu = \nu_{\text{lr}}, \quad (7.171)$$

$$\frac{D+2-\eta}{D} = \rho - \eta_{\text{lr}}, \quad (7.172)$$

and

$$\frac{\eta - \bar{\eta} + 2}{D} = \eta_{\text{lr}} - \bar{\eta}_{\text{lr}} + \rho - 1. \quad (7.173)$$

The same correspondence, at one-loop order, is identified for the EA model without external field at non-zero temperature [Har76, Kot83, Bañ12b].

7.6 Application to The Edwards-Anderson Model: Conclusions

Until now, no perturbative stable [FP](#) was found for the [EA](#) model in a field below the [UCD](#) $D_{uc}^{FC} = 6$. Several authors then look at this absence as an indication of the disappearance of the spin glass phase below the [UCD](#). In this work, a perturbative approach through a loop-expansion around the Bethe lattice solution is followed, and it reveals the presence of a phase transition. In particular, a nontrivial zero of the coupling constant β -function is found to be stable below $D_{uc} = 8$. As stated at the beginning of this chapter, an [RG FP](#) is not directly revealed through the M -layer construction. However, the natural expectation is that the nontrivial zero of the β -function, found here, corresponds to the limit $D \rightarrow D_{uc}$ of the [FP](#) found in [\[Ang15\]](#) with a suitable [RG](#) transformation.

Given the $T = 0$ nature of the [FP](#), the associated independent critical exponents are three and not just two. Here, the exponents ν , η and $\bar{\eta}$ are estimated using the ϵ -expansion around $D = 8$. These computations are done directly at $T = 0$. This would be impossible using standard field-theoretic methods, both because the [MF FC](#) spin glass model around which one expands has no transition in field exactly at $T = 0$, and because the Lagrangian is not well defined at $T = 0$. The M -layer expansion, that is used instead, is well-defined even at $T = 0$, and, moreover, the expansion is performed around the Bethe lattice solution that has a $T = 0$ transition at a critical value of the field. One could wonder if the results obtained here are also valid for $T > 0$. Following common folklore, if the exponent θ is positive, the temperature should be an irrelevant parameter and the critical line at $T > 0$ should be controlled by the $T = 0$ [FP](#), associated to the nontrivial zero of the coupling constant β -function. The exponents measured at finite temperature should thus be the ones associated with the $T = 0$ [FP](#), as first conjectured for the [RFIM](#) [\[Bra85\]](#) and then numerically verified.

It is now crucial to perform more precise numerical simulations, both on finite-dimensional and [LR](#) models, to numerically check these estimates for the critical exponents. In order to compute numerically the functions $P^{con}(x_1, x_2; u, u', J)$ and $P^{dis}(x_1, x_2; u, u')$ one should evaluate the triplet (u, u', J) between spins at positions x_1 and x_2 on a given realization of the disorder and then average over many realizations. To do so, one has to compute the ground state configuration of the whole system and the three excited states, obtained by forcing the two spins in the three possible configurations other than the ground state. Inverting equation $E_{ij}(\sigma_i, \sigma_j) = u\sigma_i + u'\sigma_j + J\sigma_i\sigma_j$ one obtains the triplet (u, u', J) . This procedure is computationally demanding, alternatively one could consider standard finite-temperature simulations. Indeed, while a detailed analysis of the implications of the present work for finite-temperature observables is left for future work, one could expect that the exponents ν and η , measured at $T \neq 0$ from the correlation length and from the 2-point susceptibilities correspond to the ones computed here at $T = 0$. This is what happens in the [RFIM](#). Besides, notice that, according to the results obtained here, η is negative and ν is larger than $1/2$ below the [UCD](#), in qualitative agreement with the finite-temperature numerical estimates $\eta \approx -0.3$ and $\nu \approx 1.46$ in four dimensions ($\epsilon = 4$) [\[Bañ12a\]](#). Numerical data in dimension six could not be used to obtain estimates, however, they are compatible with these results, within the numerical error [\[Agu24\]](#). In order to extract the other exponents, notice that [Eq. 7.16](#) should remain valid for $T > 0$ as well. Thus, a finite-temperature cubic connected function such as those measured in [\[Fer22\]](#) should obey the scaling laws of χ_3 and therefore one could extract θ from the

ratio $\chi_3^2/\chi_2^3 \propto \xi^{D-\theta}$ or similar. Apparently there are no finite temperature correlations that tend to the disconnected zero-temperature correlations defined in the limit $T = 0$ (standard disconnected correlation functions as defined at $T > 0$ go to zero in the $T \rightarrow 0$ limit); however, one can extract the exponent $\bar{\eta}$ from numerics determining η and θ , as suggested above. A detailed analysis of the implications of the present work for finite-temperature observables is left for future work.

Finally, remember the fact that for $D > 6$, also the Gaussian FP associated with the FC transition is stable (although it has a finite basin of attraction). Thus, if temperature is an irrelevant parameter in the vicinity of $T = 0$, for $6 < D < 8$ there are two stable FPs that could attract the RG flow. It could be possible that the RG flow on the critical line goes to the FC FP or to the $T = 0$ FP depending on the value of the field. Numerical investigation is crucial to understand this point.

Regarding the one-dimensional LR model: the M -layer construction is applied to estimate the critical exponents. This application significantly extends the range of the method beyond its previous use on finite-dimensional hypercubic lattices in D dimensions [Ang20, Ang22, Ang24, Ang25c, Ang25a], demonstrating its effectiveness also on more unconventional topologies. Notably, the key insight is that, once the loop expansion has been computed for a given model (in this case, a spin glass in a field at zero temperature), changing the underlying lattice requires only the knowledge of the number of RW (or NBP) between two points of the lattice. This observation significantly simplifies the analysis, making the M -layer approach an extremely useful tool for studying the same model on different topologies with minimal additional effort. While some models exist where the critical behavior is directly affected by the underlying lattice, in these cases the application of this method is simple and could give quite interesting results. [Cir25]

Moreover, the consistency of the results strongly supports the validity of the method. In particular, the anomalous field dimension—encoded in the critical exponent $\eta = 0$ —is a nontrivial outcome that strongly depends on the topological loop expansion. Its emergence in these results supports the consistency and robustness of the formalism. Remarkably, the scaling behavior observed aligns with the known correspondence between finite-dimensional and LR exponents, as discussed in Section 7.5.1. This provides further evidence that the predictions of the M -layer construction are not only theoretically consistent but also physically meaningful.

Finally, the most impactful contribution concerns the critical exponents themselves. The model defined on this one-dimensional LR lattice is particularly well-suited for numerical simulations, also due to its relatively simple structure. At the same time, this analysis yields the first (albeit approximate) analytical estimates of the critical exponents for this system. These estimates can serve as valuable benchmarks for future finite-size scaling analyses from simulations. In fact, a natural next step would be performing a thorough numerical investigation aimed at confirming or challenging the predictions presented here. Confirmation would represent a significant advancement in the field, offering a solid foundation for further analytical or numerical explorations of related systems.

7.A Appendices to Chapter 7

7.A.1 Clusters and Avalanches

In order to both clarify some physical points that were mentioned earlier and to connect the present framework to previous work [Ang22, Ang20], the relation between clusters (as introduced in Section 7.1) and avalanches is discussed.

Avalanches are a natural and intuitive concept that is often used in the literature of disordered systems at zero temperature, see *e.g.* [Mon11, Tar13]. Indeed, an intuitive way of studying zero-temperature correlations between two spins σ_i and σ_j on the lattice is to consider the difference between the ground state of the system and the new ground state in which spin σ_i is constrained in the direction $-h_i$. If in the new ground state, σ_j is flipped, then it is said that the two spins are correlated and a *response function* R_{ji} is defined, equal to one in this case and zero otherwise; this is the definition used in [Ang22, Ang20]. The *avalanche* of spin i is the set of spins that flip when i is flipped, i.e., all the spins such that $R_{ji} = 1$. A problem with defining correlation in terms of R_{ji} is that there may be instances of the disorder in which spin σ_j flips if σ_i is flipped but not vice versa, i.e., it may happen that $R_{ij} \neq R_{ji}$, meaning that responses are not symmetric.

In order to have a symmetric definition of the correlation function, the effective energy function $E_{ij}(\sigma_i, \sigma_j)$ is recalled. This function specifies, for a given realization of the disorder, the ground state configuration and the three excited configurations with the corresponding excitation energies. Let C_{ij} be defined as a quantity equal to one if the configuration with the lowest excitation energy among the three possible ones is the one where *both* spins are flipped. Obviously, $C_{ij} = 1$ implies $R_{ij} = R_{ji} = 1$ and $\Delta E_i = \Delta E_j = \Delta E$, where ΔE is the excitation energy of the first excitation. This means that, if $C_{ij} = 1$, spins i and j are in the same cluster of spins with local fields $|h_i| = |h_j| = \Delta E/2$.

It is now shown that the opposite is true, i.e., that if $|h_i| = |h_j|$ then $C_{ij} = 1$. To be precise, this holds with probability one over the disorder realization. By definition, the modulus of the local field $|h_i|$ ($|h_j|$) is equal to half the energy difference ΔE_i (ΔE_j) of the lowest configuration c_i (c_j) in which spin i (j) is flipped with respect to the ground state. Also by definition, if $C_{ij} = 1$, these configurations coincide, $c_i = c_j$, while if $C_{ij} = 0$ they are distinct, $c_i \neq c_j$. Since either the J_{ij} 's or the H_i 's obey a continuous distribution, the three excitation energies have a continuous distribution as well, meaning that the probability that a given excitation energy lies in a narrow interval $(\Delta E - dE/2, \Delta E + dE/2)$ around ΔE is $\mathcal{O}(dE)$. Therefore, if $c_i \neq c_j$ the probability that *both* the corresponding excitation energies ΔE_i and ΔE_j lie in the *same* narrow interval around ΔE is $\mathcal{O}(dE^2)$ while if $C_{ij} = 1$ the probability that the excitation energy of the *single* configuration $c_i = c_j$ lies in a narrow interval around ΔE is $\mathcal{O}(dE)$. It follows that if ΔE_i and ΔE_j lie in the *same* narrow interval around ΔE , with probability one they actually coincide, i.e., $C_{ij} = 1$. The connection with excited configurations can be generalized: given a cluster of size s with excitation energy ΔE , the lowest of the $2^s - 1$ excitations of the restricted system of the s spins in the cluster is the one in which all the s spins are flipped with respect to the ground state.

Concerning clusters and avalanches, it is noted that $R_{ij} \neq R_{ji}$ may occur, therefore the avalanche of a given spin *contains* the cluster of the spin but may be larger. The following statements hold: i) all spins in a cluster have the same avalanche, i.e., for

each cluster there is one and only one avalanche. ii) the excitation energy of any spin j in the avalanche of a given cluster must be smaller or at most equal to the excitation energy of the cluster $\Delta E_j \leq \Delta E$. This last statement notably implies that clusters and avalanches coincide if $\Delta E = 0$, i.e., in the case of *soft* clusters, most relevant to this work.

Finally, because of the analogy with percolation, it is important to remark that it is possible to find clusters that are *disconnected* in the sense of percolation. By this, it is meant that there exist clusters such that there is at least one pair of spins in the set not connected by a linear path made of spins belonging to the set. This disturbing feature is not present if avalanches are considered, because a spin may flip only if at least one of its neighbors flips. On the other hand, since clusters *coincide* with avalanches for $\Delta E = 0$, it follows that soft clusters are also connected in the sense of percolation.

7.A.2 Explicit Expressions of Scaling Laws

Here, the explicit expressions of the scaling laws enunciated in [Section 7.1](#) are obtained within the M -layer framework.

In order to do so, following the prescriptions of the M -layer construction in [Section 4.2](#), it is necessary to include the number of **NBP**, $\mathcal{N}_L(x, x')$, between two points x and x' and sum over the length of the path, L , with the corresponding weight W

$$P_2(x, x'; u, u', J) - \delta(J)P_1(u)P_1(u') = \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}_L(x, x') (Q_L(u, u', J) - \delta(J)P_1(u)P_1(u')). \quad (7.174)$$

In the large- L limit, the number of **NBP** in a D -dimensional hypercube is:

$$\mathcal{N}_L(x, x') = \left(\frac{2D}{2D-1} \right) \frac{(2D-1)^L}{(4\pi L/c)^{D/2}} e^{-c \frac{(x-x')^2}{4L}} \quad (7.175)$$

where $c = \frac{2D-2}{a_l^2}$ and a_l is the lattice spacing; in the remainder of this section $c = 1$ is set for simplicity with an appropriate rescaling of the length. Note that the sum over the lattice points x of the above expression yields $2D(2D-1)^{L-1}$ which is the number of different **NBP** of length L starting from x' . Going to Fourier space:

$$\widehat{\mathcal{N}}_L(k, k') \propto \delta^D(k + k') (2D-1)^L e^{-k^2 L}, \quad (7.176)$$

from which the Fourier transform of the distribution of the triplet is computed:

$$\begin{aligned} P_2(k, k'; u, u', J) - \delta(J)P_1(u)P_1(u') &\propto \\ \delta^D(k + k') \frac{1}{M} \sum_{L=1}^{\infty} (2D-1)^L e^{-k^2 L} \widehat{P}_1(u) \widehat{P}_1(u') &\left(-2\delta(J)L\lambda^L + \rho L^2 \lambda^L e^{-\rho|J|L} \right) = \\ \delta^D(k + k') \frac{1}{M} \sum_{L=1}^{\infty} e^{-k^2 L - \tau L} \widehat{P}_1(u) \widehat{P}_1(u') &\left(-2\delta(J)L + \rho L^2 e^{-\rho|J|L} \right) = \\ \delta^D(k + k') \frac{1}{M} \widehat{P}_1(u) \widehat{P}_1(u') &\left(-\delta(J) \frac{2}{(k^2 + \tau)^2} + \rho \frac{2}{(k^2 + \tau + \rho|J|)^3} \right), \end{aligned} \quad (7.177)$$

where the distance from the **MF** critical point is defined $\tau \equiv -\ln(\lambda(2D-1))$ and it is assumed that the region near the critical point $\tau \approx 0^+$ and at large distances $k^2 \approx 0$ are

considered. Moreover, given a generic translationally invariant 2-point function $C(x, x')$, the associated correlation length is defined as $\xi^2 \equiv \int d^D x |x|^2 C(x, 0) / \int d^D x C(x, 0)$. Thus, from Eq. 7.177 the two correlation lengths, ξ_{con} and ξ_{dis} , are identified. At the MF level the correlation lengths diverge at the critical point on the M -layer lattice as $\xi_{\text{dis}} \propto \tau^{-1/2}$ and $\xi_{\text{con}} \propto (\tau + \rho|J|)^{-1/2}$, while if loop corrections are taken into account these relations become more complicated.

Note that Eq. 7.177 is a special case of Eq. 7.6 with $\theta = 2$, $\nu = 1/2$, $\bar{\eta} = 0$ and appropriate scaling functions $f_{\text{con}}(\varrho, J)$, $f_{\text{dis}}(\varrho)$ and $f_{\xi}(J)$. Indeed, at leading order in $1/M$, t is proportional to τ .

In real space, one obtains

$$P^{\text{dis}}(x_1, x_2; u_1, u_2) = \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}_L(x_1, x_2) P_L^{J=0} \hat{P}_1(u) \hat{P}_1(u') \\ \propto \int_1^{\infty} dL L^{1-D/2} e^{-\tau L - \frac{(x_1-x_2)^2}{4L}} \hat{P}_1(u) \hat{P}_1(u') \quad (7.178)$$

where the sum over the length has been replaced with the corresponding integral. Near the critical point and for large $r = |x_1 - x_2| \sim \mathcal{O}(\xi)$ the disconnected part can be written as

$$\frac{1}{r^{D-4+\bar{\eta}}} f_{\text{dis}}\left(\frac{r}{\xi}\right) \propto \int_1^{\infty} dL L^{1-D/2} e^{-\frac{L}{\xi^2} - \frac{r^2}{4L}}, \quad (7.179)$$

where $\bar{\eta} = 0$ in the MF approximation. Changing the integration variable from L to $L' = L/r^2$ gives

$$\frac{1}{r^{D-4}} f_{\text{dis}}\left(\frac{r}{\xi}\right) \propto \frac{1}{r^{D-4}} \int_{1/r^2}^{\infty} dL' L'^{1-D/2} e^{-\frac{r^2}{\xi^2} L' - \frac{1}{4L'}}, \quad (7.180)$$

so that, since this scaling is valid for large r , the lower integration limit is set to zero and

$$f_{\text{dis}}(x) \propto \int_0^{\infty} dL' L'^{1-D/2} e^{-x^2 L' - \frac{1}{4L'}} \propto x^{D/2-2} K_{D/2-2}(x), \quad (7.181)$$

where $K_{\alpha}(x)$ is the modified Bessel function of the second kind. Note that, as expected, $f_{\text{dis}}(0)$ is finite due to the divergence of the Bessel function at small argument $K_{\alpha}(x) \propto x^{-\alpha}$. Similarly, to express the scaling function f_{con} , the same computation is performed using $P_L^{J \neq 0}(J)$ instead of $P_L^{J=0}$. In this case, the scaling function will have a second argument,

$$\frac{1}{r^{D-4+\bar{\eta}-\theta}} f_{\text{con}}\left(\frac{r}{\xi_{\text{con}}}, \frac{|J|}{t^{-\nu\theta}}\right) \propto \int_1^{\infty} dL L^{2-D/2} e^{-\frac{L}{\xi_{\text{con}}^2} - \frac{r^2}{4L}}, \quad (7.182)$$

where $\theta = 2$ in the MF approximation. Note that at first order in $1/M$ $f_{\text{con}}(x, y)$ has no dependence on the second argument. Performing the same change of integration variable, the result is

$$f_{\text{con}}(x, y) \propto \int_0^{\infty} dL' L'^{2-D/2} e^{-x^2 L' - \frac{1}{4L'}} \propto x^{D/2-3} K_{D/2-3}(x). \quad (7.183)$$

As for $f_{\text{dis}}(0)$ it is seen that $f_{\text{con}}(0, y)$ is finite due to the small argument divergence of the Bessel function $K_{\alpha}(x) \propto x^{-\alpha}$. Thus, the scaling functions displayed in Eq. 7.6 are written explicitly in the MF case.

An important relation can be found between f_{con} and f_{dis} . Much as in Eq. 7.177 the function $\hat{P}_1(u)$ is positive definite and model-dependent (at variance with f_{dis} and f_{con} that are universal). Given that, by definition, i) $P_2(x_1, x_2; u, u', J)$ is normalized with respect to (u, u', J) and ii) $P_1(u)$ is normalized with respect to u , it follows that the integral over (u, u', J) of the LHS of Eq. 7.6 vanishes. On the other hand, since $\hat{P}_1(u)$ on the RHS is positive definite, the RHS must vanish due to the integration over J . This has the following implications: i) the same exponent $\bar{\eta}$ appears in the connected and disconnected part, ii) $f_{\text{dis}}(\varrho)$ is related to $f_{\text{con}}(\varrho, J)$ and $f_{\xi}(J)$ by the relationship:

$$f_{\text{dis}}\left(\frac{r}{\xi_{\text{dis}}(t)}\right) = \int_{-\infty}^{\infty} f_{\text{con}}\left(\frac{r}{\xi_{\text{con}}(t, J)}, \frac{|J|}{t^{\nu_0}}\right) dJ, \quad (7.184)$$

iii) the ratio $\xi_{\text{dis}}(t)/\xi_{\text{con}}(t, J)$ depends solely on the ratio $|J|/t^{\nu_0}$ and is universal.

An explicit expression can also be computed for the scaling function f_2 , related to the 2-point connected correlation function, as in Eq. 7.9. $\mathcal{P}_2(x_1, x_2; \Delta E)$ can be expressed in terms of $P_2(x_1, x_2; u_1, u_2, J)$. According to the definitions given in Section 7.1, the condition that the two spins are on the same cluster is that $R_{12} = R_{21} = 1$. This implies that $|u_1| < |J|$ and $|u_2| < |J|$. In this case, the local fields on the two spins are respectively $u_1 + \text{sign}(J)u_2$ and $u_2 + \text{sign}(J)u_1$ leading to $\Delta E = 2|u_1 + \text{sign}(J)u_2|$. This leads to the following exact expression:

$$\mathcal{P}_2(x_1, x_2; \Delta E) = \int_{-\infty}^{\infty} dJ \prod_{i=1,2} \int_{-|J|}^{|J|} du_i P_2(x_1, x_2; u_1, u_2, J) \delta\left(|u_1 + \text{sign}(J)u_2| - \frac{\Delta E}{2}\right). \quad (7.185)$$

Taking the Fourier transform and plugging the leading order of the expression of $P_2(x_1, x_2; u_1, u_2, J)$ (in Eq. 7.174) one obtains:

$$\hat{\mathcal{P}}_2(k_1, k_2; \Delta E) \propto \delta^D(k + k') \int_{-\infty}^{\infty} dJ \prod_{i=1,2} \int_{-|J|}^{|J|} du_i \frac{\hat{P}_1(u_1)\hat{P}_1(u_2)}{(k^2 + \tau + \rho|J|)^3} \delta\left(|u_1 + \text{sign}(J)u_2| - \frac{\Delta E}{2}\right). \quad (7.186)$$

Note that only the contribution coming from $J \neq 0$ is selected due to the constraints $|u_1| < |J|$ and $|u_2| < |J|$. In the large-distance limit $k \ll 1$ and in the critical region $\tau \ll 1$, the integral is dominated by $|J| \ll 1$, thus $|u_1| \ll 1$ and $|u_2| \ll 1$ follow from $|u_1| < |J|$ and $|u_2| < |J|$. Therefore both $\hat{P}_1(u_1)$ and $\hat{P}_1(u_2)$ can be replaced with $\hat{P}_1(0)$. The integrals over u_1 and u_2 with the conditions $|u_1| < |J|$, $|u_2| < |J|$ and $|u_1 + u_2 \text{sign } J| = \Delta E$ lead to a factor $2|J| - \Delta E/2$ and the following expression is obtained:

$$\begin{aligned} \hat{\mathcal{P}}_2(k_1, k_2; \Delta E) &\propto \delta^D(k + k') \left(\hat{P}_1(0)\right)^2 \int_{\Delta E/4}^{\infty} dJ \frac{|J| - \Delta E/4}{(k^2 + \tau + \rho|J|)^3} = \\ \delta^D(k + k') \left(\hat{P}_1(0)\right)^2 \int_0^{\infty} dJ \frac{J}{(k^2 + \tau + \rho\Delta E/4 + \rho J)^3} &\propto \delta^D(k + k') \frac{\left(\hat{P}_1(0)\right)^2}{k^2 + \tau + \rho\Delta E/4}. \end{aligned} \quad (7.187)$$

The last equation is obtained using the explicit specific leading order of the function $P_2(x_1, x_2; u_1, u_2, J)$. However, this result can be extended to the case of a generic

$f_{\text{con}}(x, y)$ in order to obtain the scaling function $f_2(q, e)$ defined through:

$$\mathcal{P}_2(x_1, x_2; \Delta E) \equiv \frac{(\hat{P}_1(0))^2}{r^{D-4+\bar{\eta}+\theta}} f_2\left(\frac{r}{\xi}, \frac{\Delta E}{t^{\nu\theta}}\right). \quad (7.188)$$

It is first observed that at leading order in $1/M$ it is $\bar{\eta} = 0$ and $\theta = 2$ and the scaling function $f_2(x, y)$ is

$$f_2(x, y) \propto \int_0^\infty dL \frac{e^{-Lx^2-1/(4L)}}{L^{D/2}} \propto x^{D/2-1} K_{D/2-1}(x). \quad (7.189)$$

The above result is the Fourier transform of [Eq. 7.187](#). Note that, much as $f_{\text{con}}(x, y)$, $f_2(x, y)$ also does not depend on y at this order. To obtain the general form of $f_2(x, y)$ it is convenient to switch from the scaling variables r/ξ_{con} and $|J|/t^{\nu\theta}$ to the scaling variables $r/t^{-\nu}$ and $J/r^{-\theta}$. Indeed r/ξ_{con} depends on J through $\xi_{\text{con}} = t^{-\nu} f_\xi(|J|/t^{\nu\theta})$ and due to the integral over J in expression [Eq. 7.185](#) it is simpler to have a single scaling variable that is proportional to J . Thus a new function $\tilde{f}_{\text{con}}\left(\frac{r}{t^{-\nu}}, \frac{|J|}{r^{-\theta}}\right)$ is introduced through

$$\tilde{f}_{\text{con}}\left(\frac{r}{t^{-\nu}}, \frac{|J|}{r^{-\theta}}\right) \equiv f_{\text{con}}\left(\frac{r}{t^{-\nu}}/f_\xi\left(\frac{|J|}{r^{-\theta}} \frac{r^{-\theta}}{t^{\nu\theta}}\right), \frac{|J|}{r^{-\theta}} \frac{r^{-\theta}}{t^{\nu\theta}}\right). \quad (7.190)$$

Performing similar steps as those of [Eq. 7.187](#) one can obtain:

$$\begin{aligned} \mathcal{P}_2(x_1, x_2; \Delta E) &\propto \frac{(\hat{P}_1(0))^2}{r^{D-4+\bar{\eta}+\theta}} \int_{\frac{\Delta E}{4r^{-\theta}}}^\infty d\hat{J} \left(\hat{J} - \frac{\Delta E}{4r^{-\theta}}\right) \tilde{f}_{\text{con}}\left(\frac{r}{t^{-\nu}}, \hat{J}\right) \propto \\ &\frac{(\hat{P}_1(0))^2}{r^{D-4+\bar{\eta}+\theta}} \int_0^\infty dy' y' \tilde{f}_{\text{con}}(r/t^{-\nu}, y' + \Delta E/(4r^{-\theta})) \equiv \frac{(\hat{P}_1(0))^2}{r^{D-4+\bar{\eta}+\theta}} \hat{f}_2(r/t^{-\nu}, \Delta E/r^{-\theta}) \end{aligned} \quad (7.191)$$

leading to:

$$\hat{f}_2(x, y) \propto \int_0^\infty dy' y' \tilde{f}_{\text{con}}(x, y + y'). \quad (7.192)$$

From the scaling function $\hat{f}_2(r/t^{-\nu}, \Delta E/r^{-\theta})$ the scaling function $f_2(r/\xi, \Delta E/t^{\nu\theta})$ can be obtained by means of a change of scaling variables analogous to the one performed in [Eq. 7.190](#).

It is possible to obtain a last scaling expression

$$\sum_{x_2} \mathcal{P}_2(x_1, x_2; \Delta E) \propto \xi^{2-\eta} \tilde{f}_2\left(\frac{\Delta E}{t^{\nu\theta}}\right) \quad (7.193)$$

where $\eta = 0$ in the [MF](#) approximation. From [Eq. 7.188](#) and $-\bar{\eta} + \eta = \theta - 2$, performing similar computations it follows:

$$\tilde{f}_2(e) \propto \int_0^\infty q^{1-\eta} f_2(q, e) dq. \quad (7.194)$$

7.A.3 Numerical Tests on the *Ansatz*

In this section, some numerical tests are reported that have been performed to confirm the perturbative expressions obtained for the observables on the given diagrams, in Eq. 7.38, Eq. 7.36, Eq. 7.42, Eq. 7.43, Eq. 7.49, Eq. 7.51, and Eq. 7.66. To this aim, the numerical technique described in the SI of [Ang22] has been used. The main points are summarized here. Notice that the following numerical results are obtained for Gaussian couplings $J \sim \mathcal{N}(0, 1/(z-1))$ and fixed external field H on each site, where z is the connectivity of the graph. Due to universality, it is believed that the bimodal distributed coupling and Gaussian external field case would give the same results.

Given the rules for the joining procedure in Table 7.1, the idea is to start from a population of $N_T = 10^7$ triplets of two biases and one coupling characterized by length $L = 1$, i.e., distributed as $P_{L=1}(u_0, u_1, J_1) = \delta(u_0)\delta(u_1)\mathcal{N}(0, 1/(z-1))$. Joining one triplet of length $L-1$ with one of the kind $(0, 0, J)$, one iteratively constructs a new triplet of length L . It can be easily seen, from the evolution rules in Table 7.1, that at each step a fraction of the population that satisfies $|h| > |J_{L-1}| + |J|$, produces a new one with zero coupling, $J_L = 0$. Given that the effective coupling of two infinitely distant sites is zero, the size of the population with $J_L \neq 0$ shrinks to zero exponentially fast with L . Moreover, for the spin glass in a field the triplets with zero coupling, $(u_0, u_L, 0)$, can be of two different kinds. Indeed, when a triplet of the type $(u_0, u_L, 0)$, with u_0 and u_L correlated, is joined with one of the type $(0, 0, J)$, considering the case $|h| > |J|$ of Table 7.1, the correlation between the new biases u_0 and u_{L+1} is zero because of the random sign of J . In total, three different signals arise: one with non-zero coupling, one with zero coupling but correlated biases and one with zero coupling and uncorrelated biases. Thus, to amplify these three distinct signals, three populations of the same size N_T are evolved for each L with an enrichment procedure, alongside with their probabilities p_L and b_L that are the weights of $\mathcal{A}_L$ and $\mathcal{B}_L$ respectively: $\mathcal{A}_L$ stores the triplets with $J_L \neq 0$, $\mathcal{B}_L$ keeps correlated biases (u_0, u_L) but $J_L = 0$ and $\mathcal{C}_L$ keeps uncorrelated pairs (u_0, u_L) . Joining a triplet of length $L-1$, denoted as T_{L-1} , to one of length 1 of the type $(0, 0, J)$, to form a new longer triplet, T_L , different cases are realized:

- $T_{L-1} \in \mathcal{A}_{L-1}$ and $|h| < |J_{L-1}| + |J| \rightarrow T_L \in \mathcal{A}_L$,
- $T_{L-1} \in \mathcal{A}_{L-1}$ and $|h| > |J_{L-1}| + |J| \rightarrow T_L \in \mathcal{B}_L$,
- $T_{L-1} \in \mathcal{B}_{L-1}$ and $|h| < |J| \rightarrow T_L \in \mathcal{B}_L$,

in all other cases $T_L \in \mathcal{C}_L$. The associated initial probabilities are $p_1 = 1$, $b_1 = 0$. To compute the Bethe distribution of cavity biases $P_B(u)$ the corresponding cavity equations (at zero temperature) with 1000 iterations for a population of 10^6 biases have been used.

Once the population of the triplet for each line is obtained, the lines can be added by summing over the values of the internal spins in order to generate the distributions for the diagrams to be computed and to estimate the values of the observables. Particularly easy is the case of the disconnected function (line-connected), since it corresponds to the zero-coupling part of the distribution (once the trivial $P_B(u_1)P_B(u_2)\delta(J)$ part is subtracted). For the connected functions, some additional work is needed. Once the distribution of the effective parameters is obtained from the population of triplets, the connected function needs to be averaged. To do so, given the triplet, 1 is counted

every time the external spins belong to the same cluster. The result is averaged over the distribution of the triplets. This computation has then been repeated N_{data} times for each observable and each set of lengths of the given diagram, each time extracting independently a new distribution of the triplets and computing the mean value and its associated statistical error. The result corresponds to the connected functions computed analytically.

As anticipated for the *Ansatz* distribution of the triplet, each parameter can be computed or at least numerically estimated. Notice that here and for the following results the connectivity of the Bethe lattice is set to $z = 3$ and the associated critical field to $H = 0.358$. In Figure 7.4 the results for the estimate of the parameter a of the *Ansatz* in Eq. 7.20 are shown. To do so the two most numerically precise results have been used, that are for $\mathcal{C}_2^{\text{lc}}(\mathcal{G}_1; L)$ and $\mathcal{D}_2^{\text{lc}}(\mathcal{G}_1; L)$. With the former, the value of λ has been estimated as shown in Figure 7.3, fitting the data with free parameter α , which is the multiplicative factor of the term λ^L , see Eq. 7.38. Then it has been used to fit $\mathcal{D}_2^{\text{lc}}(\mathcal{G}_1; L)$ and extract the value of a . It is seen that the extrapolated values are

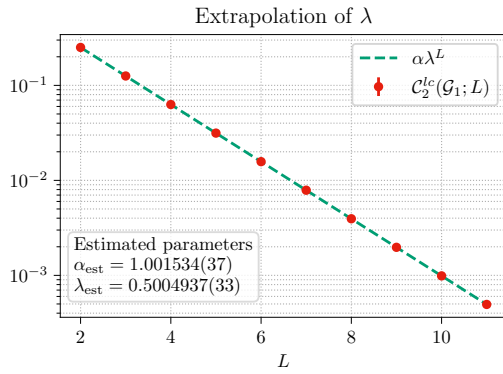


Figure 7.3. Numerical extrapolation of the control parameter λ . Points are the data while the dashed line is the fitting function with parameters λ_{est} and α_{est} .

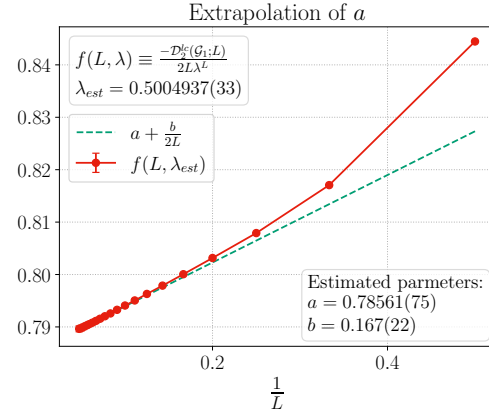


Figure 7.4. Numerical extrapolation of the *Ansatz* parameter a . Points come from the numerical evaluation of the bare 2-point disconnected correlation, scaled as $-\frac{\mathcal{D}_2^{\text{lc}}(\mathcal{G}_1; L)}{2L\lambda^L}$, with the value of λ obtained from Figure 7.3. The dashed line is the fitting function that allows for the extrapolation of the *Ansatz* parameter a .

compatible with the values $\lambda = 1/2$ and $\alpha = 1$. $\lambda = 1/2$ is expected exactly at the critical point and the observed deviation is due to a not fully precise value of the critical field $H = 0.358$. $\alpha = 1$ is expected because the parameter α can be expressed as the ratio between $g(u)$ and $P_B(u)$ and clear numerical evidence shows that the function $g(u) = P_B(u)$ in the Gaussian couplings case at the critical point. The deviation of the fitted value of α from 1 can be due to the finite size of the population used.

In Figure 7.5 the numerical results of Ref. [Ang22] for the 2-point connected function are reproduced. In particular, a joint fit of the parameters is performed, whose results are shown in the table on the right and, for large lengths, are in good agreement with the analytical computation. Notice that the amputated 2-point connected function

on $\mathcal{G}_2$ depends on two lengths, L_A and L_B . To check the result in a single plot the observable has been numerically computed for two different combinations of lengths: $L_A = L_B$ and $L_B = 2L_A$.

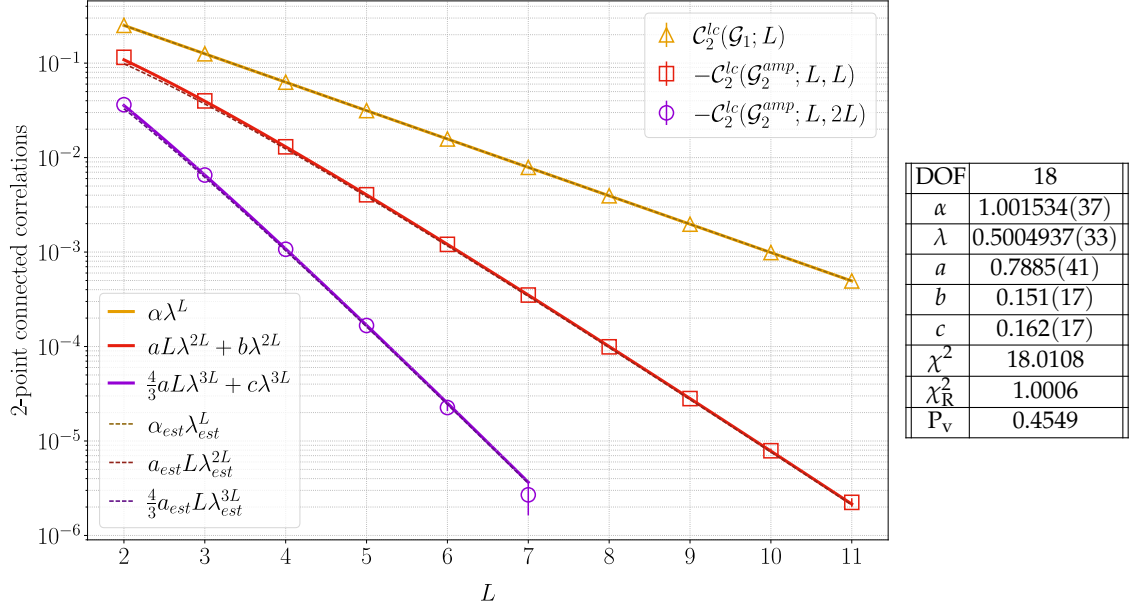


Figure 7.5. Left: Points correspond to the numerical evaluation of the 2-point connected correlation function on the different diagrams of Figure 6.2 as a function of the length L ; full lines correspond to the associated joint fit (that also includes next-to-leading order term with respect to L); dashed lines correspond to the analytical prediction, with the values α_{est} , λ_{est} and a_{est} extracted from Figure 7.4. Right: the evaluated corresponding fit parameters; χ_R^2 is the reduced χ^2 and P_v is the corresponding P -value.

The data in Figure 7.5 have been fitted with:

$$\mathcal{C}_2^{lc}(\mathcal{G}_1; L) = \alpha \lambda^L \quad \text{for } L \in [2, 11], \quad N_{data} = 200; \quad (7.195)$$

$$-\mathcal{C}_2^{lc}(\mathcal{G}_2^{amp}; L, L) = aL\lambda^{2L} + b\lambda^{2L} \quad \text{for } L \in [4, 11], \quad N_{data} = 200; \quad (7.196)$$

$$-\mathcal{C}_2^{lc}(\mathcal{G}_2^{amp}; L, 2L) = \frac{4}{3}aL\lambda^{3L} + c\lambda^{3L} \quad \text{for } L \in [3, 7], \quad N_{data} = 400. \quad (7.197)$$

In Figure 7.6 the results for the 3-point function are shown. The bare contribution is plotted together with four different combinations of lengths for the amputated diagram $\mathcal{G}_4^{amp}$. The results of the associated joint fit are collected in the table on the right. Again, there is good agreement between the numerical and the analytical results for the 3-point connected function on the given diagrams.

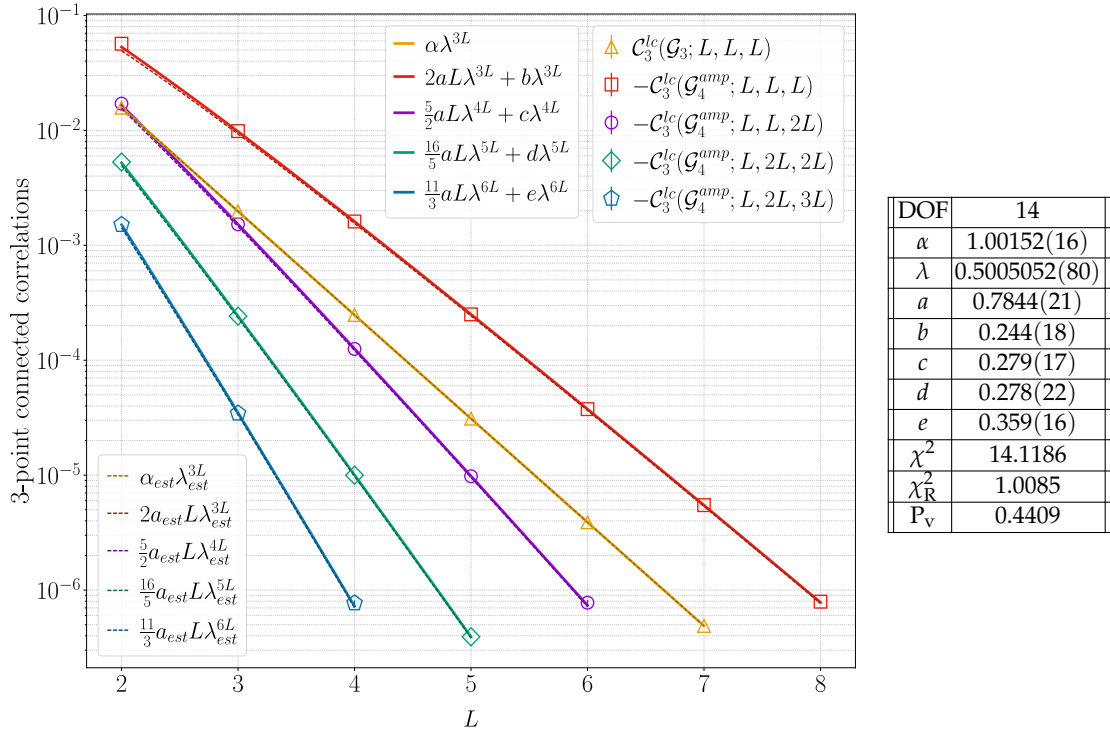


Figure 7.6. Left: Points correspond to the numerical evaluation of the 3-point connected correlation function on the different diagrams of Figure 6.3 as a function of the length L ; full lines correspond to the associated joint fit; dashed lines correspond to the analytical prediction, with the values $\alpha = 1$, $\lambda = 1/2$ and a extracted from Figure 7.4. Right: the evaluated corresponding fit parameters; χ^2_R is the *reduced* χ^2 and P_v is the corresponding P -value.

The fitting functions used are:

$$\mathcal{C}_3^{\text{lc}}(\mathcal{G}_3; L, L, L) = \alpha \lambda^{3L} \quad \text{for } L \in [2, 7], \quad N_{\text{data}} = 1800; \quad (7.198)$$

$$-\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4^{\text{amp}}; L, L, L) = 2aL\lambda^{3L} + b\lambda^{3L} \quad \text{for } L \in [4, 8], \quad N_{\text{data}} = 1800; \quad (7.199)$$

$$-\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4^{\text{amp}}; L, L, 2L) = \frac{5}{2}aL\lambda^{4L} + c\lambda^{4L} \quad \text{for } L \in [3, 6], \quad N_{\text{data}} = 1800; \quad (7.200)$$

$$-\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4^{\text{amp}}; L, 2L, 2L) = \frac{16}{5}aL\lambda^{5L} + d\lambda^{5L} \quad \text{for } L \in [3, 5], \quad N_{\text{data}} = 1800; \quad (7.201)$$

$$-\mathcal{C}_3^{\text{lc}}(\mathcal{G}_4^{\text{amp}}; L, 2L, 3L) = \frac{11}{3}aL\lambda^{6L} + e\lambda^{6L} \quad \text{for } L \in [2, 4], \quad N_{\text{data}} = 1800. \quad (7.202)$$

In Figure 7.7, numerical results for the disconnected function are shown. Results for both $\mathcal{G}_2^{\text{amp}}$ and $\mathcal{G}_2$ are plotted together with the bare case $\mathcal{G}_1$. In the table, the results of the joint fit are reported.

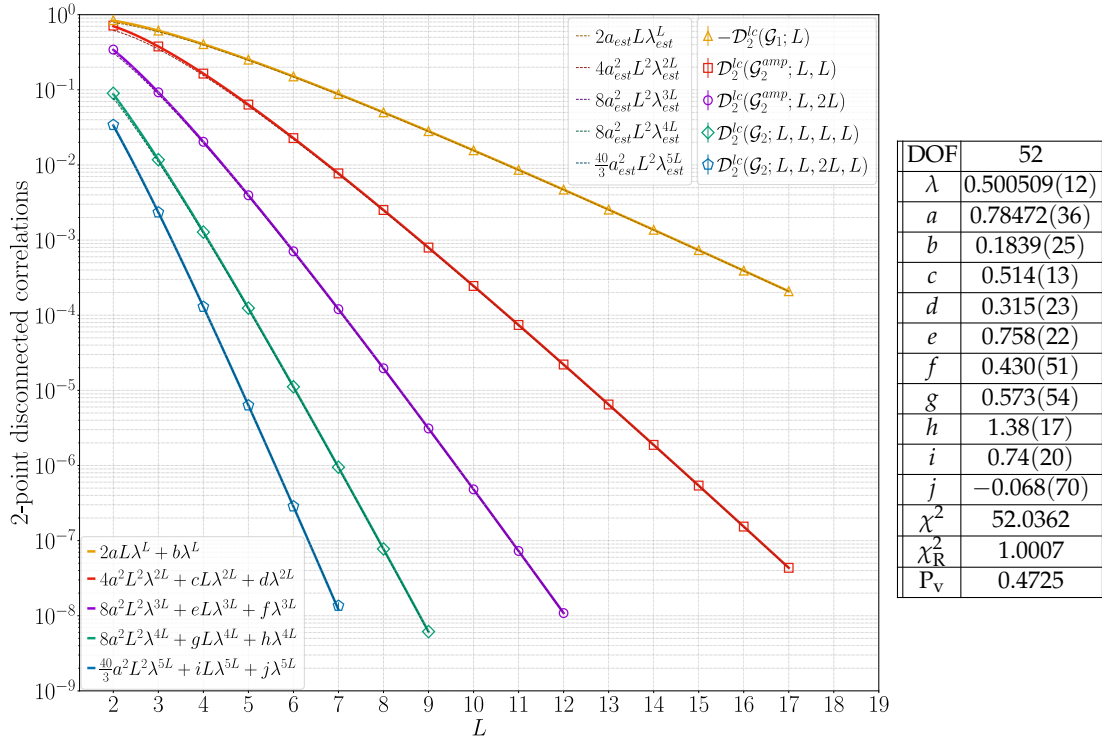


Figure 7.7. Left: Points correspond to the numerical evaluation of the 2-point disconnected correlation function on the different diagrams of Figure 6.2 as a function of the length L ; full lines correspond to the associated joint fit; dashed lines correspond to the analytical prediction, with the values $\alpha = 1$, $\lambda = 1/2$ and a extracted from Figure 7.4. Right: the evaluated corresponding fit parameters χ^2_R is the *reduced* χ^2 and P_v is the corresponding P -value.

The fit functions used here are:

$$-\mathcal{D}_2^{lc}(\mathcal{G}_1; L) = 2aL\lambda^L + b\lambda^L \quad \text{for } L \in [4, 19], \quad N_{data} = 100; \quad (7.203)$$

$$\mathcal{D}_2^{lc}(\mathcal{G}_2^{amp}; L, L) = 4a^2L^2\lambda^{2L} + cL\lambda^{2L} + d\lambda^{2L} \quad \text{for } L \in [3, 17], \quad N_{data} = 200; \quad (7.204)$$

$$\mathcal{D}_2^{lc}(\mathcal{G}_2^{amp}; L, 2L) = 8a^2L^2\lambda^{3L} + eL\lambda^{3L} + f\lambda^{3L} \quad \text{for } L \in [3, 15], \quad N_{data} = 150; \quad (7.205)$$

$$\mathcal{D}_2^{lc}(\mathcal{G}_2; L, L, L, L) = 8a^2L^2\lambda^{4L} + gL\lambda^{4L} + h\lambda^{4L} \quad \text{for } L \in [3, 14], \quad N_{data} = 100; \quad (7.206)$$

$$\mathcal{D}_2^{lc}(\mathcal{G}_2; L, L, 2L, L) = \frac{40}{3}a^2L^2\lambda^{5L} + iL\lambda^{5L} + j\lambda^{5L} \quad \text{for } L \in [3, 9], \quad N_{data} = 200. \quad (7.207)$$

7.A.4 Other (Irrelevant) Diagrams

In this section, some arguments are given about the irrelevance of certain diagrams that, in principle, should be considered to compute the observables of interest.

The first kind of diagrams neglected are the “tadpole-type” ones, depicted in Figure 6.2. In principle, such topologies should be considered not only for 2-point functions, but also to “dress” the external or internal lines of a generic diagram for a generic n -point function, for instance, the diagram $\mathcal{G}_3$ for the 3-point function computed here. The justification for neglecting this kind of topology is simple and makes use

of the property of the *Ansatz* distribution of the line, in Eq. 7.184. Consider two lines, of the kind of $\mathcal{G}_1$, joined by a central common spin σ_0 and with σ_1 and σ_2 at the extremities of the chain. Instead of simply joining them as done before to impose the self-consistency of the *Ansatz*, imagine adding a line starting and ending in σ_0 , thus forming a degree-4 vertex and a loop. In order to compute the distribution of the effective biases on σ_1 and σ_2 and the effective coupling between the two, a sum over the configurations of σ_0 , depending on the parameters of each of the three lines, is performed. In the end, these parameters are integrated following some rules similar to those in Table 7.1. However, the coupling of the loop line J_{00} is ineffective since its contribution to the effective Hamiltonian, $J_{00}\sigma_0^2$, is independent of σ_0 , given the fact that $\sigma_0^2 = 1$. This means that, integrating on J_{00} , without any constraint, the *Ansatz* distribution of the loop line, only the trivial part will survive:

$$\begin{aligned} \int dJ_{00} \left(\delta(J_{00}) P_B(u_1) P_B(u_2) - 2a L \lambda^L \delta(J_{00}) g(u_1) g(u_2) + P_L^{J \neq 0}(J_{00}) g(u_1) g(u_2) \right) \\ = P_B(u_1) P_B(u_2), \quad (7.208) \end{aligned}$$

since, due to the above-mentioned property, the integral over the coupling of the nontrivial zero-coupling part is equal and opposite in sign to the integral of the non-zero part. It is understood that, to this order in the large-length limit, closing a line on a site does not change the *Ansatz* distribution (a similar argument can be applied to the degree-3 vertex tadpole-type diagram). Since these tadpoles are one-loop diagrams it is necessary to be careful, in the M -layer framework, and apply the line-connected definition, which amounts to subtracting the simple line without the loop. The result is then zero, given that the distribution of the loop diagram is the same as the simple line. For this reason, every complicated diagram, dressed with such topologies, gives zero contribution to the observable for the spin glass model in a field at zero temperature.

Another kind of diagram completely neglected is $\mathcal{G}'''$ in Figure 7.1. This diagram can be seen as a special case of $\mathcal{G}_5$ where the internal line connecting the loop to the two other external lines has zero length (or, equivalently, by setting one of the loop lines in $\mathcal{G}_4$ to zero). However, the minimum length for a line is always 1, in order to correctly count all the combinatorial factors originating from each vertex. Indeed, $\mathcal{G}_5$ has three degree-3 vertices, while contracting the internal line two degree-3 vertices become a single degree-4 vertex. For this reason, in principle, the contributions of all diagrams originated from the contraction of a line of a generic diagram should be separately computed. Luckily, it can be argued that every time a line is contracted the resulting contribution misses a propagator $((k^2 + \tau)^{-1}$ for $k^2 \sim 0$), thus it will come with a multiplicative factor τ with respect to the “non-contracted” contribution. When the critical exponents are computed the critical limit $\tau \rightarrow 0$ (or equivalently $m^2 \rightarrow 0$) is taken, which is why these “contracted” diagrams are not relevant in the critical limit to this order in the large-length limit, because they give a subdominant contribution.

Generic n -point connected function at one loop Although all relevant diagrams have been computed, the idea of $P_{\{\bar{L}\}}^{\text{NB}}$ is extended to compute the contribution to a generic n -point connected function on diagrams of the kind depicted in Figure 7.8.

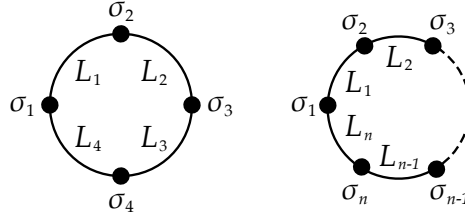


Figure 7.8. Left: 4-point diagram computed in this section: $\mathcal{G}_6^{\text{amp}}$. Right: relevant diagram for the amputated n -point function: $\mathcal{G}_n^{\text{amp}}$.

The case $n = 4$ is considered and a simple generalization leads to the generic formula for the n -point function. As for the 3-point function, the procedure starts by cutting one line, for instance, the one with length L_4 as in Figure 7.9, the probability that the four spins belong to the same cluster in this simple case is computed and then the line is reintroduced later.

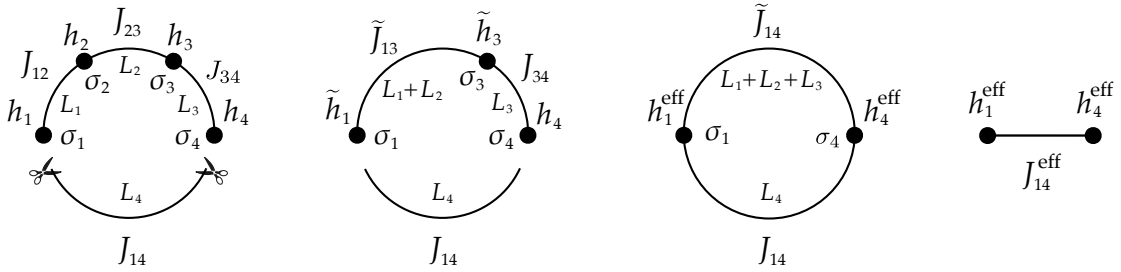


Figure 7.9. Representation of the steps of the procedure to compute the corresponding $P_{L_1, \dots, L_{n-1}}^{\text{NB}}$ distribution on the specific case $n = 4$.

Again, two possibilities have to be taken into account when re-introducing L_4 : i) the coupling J_{14} is such that the ground state without the line is not altered and ii) the coupling J_{14} is such that the ground state is altered. As for the 3-point function, the case where all the couplings are different from zero is considered first. The first case is simple: a sum over the configurations of σ_2 and σ_3 , putting one external field drawn from $g(u)$ in each vertex, is taken to obtain

$$\left(\frac{4a \hat{P}_3(0)}{\rho} \right)^2 P_{L_1+L_2+L_3}^{J \neq 0}(\tilde{J}_{14}) \hat{P}_3(h_1^{\text{eff}}) \hat{P}_3(h_4^{\text{eff}}) \quad (7.209)$$

and then the line of length L_4 is added with the usual condition for the resulting effective coupling and fields: $J_{14}^{\text{eff}} = \tilde{J}_{14} + J_{14}$, h_1^{eff} and h_4^{eff} . As for the 3-point function case it is noticed that summing over the configurations of σ_2 (before) and σ_3 (after), at this level of approximation in the large length limit, is ineffective on the values of the effective fields: $h_1^{\text{eff}} \approx h_1$, $h_3^{\text{eff}} \approx h_3$ and $h_4^{\text{eff}} \approx h_4$, see Figure 7.9. The result of this first contribution is:

$$2 \int_0^\infty dJ_{14}^{\text{eff}} \int_0^\infty d\tilde{J}_{14} \int_{-\tilde{J}_{14}}^\infty dJ_{14} \int_{-\infty}^\infty dh_1 \int_{-\infty}^\infty dh_4 P_{L_1+L_2+L_3}^{J \neq 0}(\tilde{J}_{14}) P_{L_4}^{J \neq 0}(J_{14}) \hat{P}_3(h_1) \times \\ \hat{P}_3(h_4) \delta(J_{14}^{\text{eff}} - (\tilde{J}_{14} + J_{14})) \left(\frac{4a \hat{P}_3(0)}{\rho} \right)^2 \prod_{i=1,4} \Theta(|J_{14}^{\text{eff}}| - |h_i|) \delta(|h_1 + h_4 \text{sign}(J_{14}^{\text{eff}})|)$$

$$= \frac{64 a^4 (\hat{P}_3(0))^4}{\rho^3} \lambda^{L_1+L_2+L_3+L_4} \frac{L_4^2 + (L_1 + L_2 + L_3 + L_4)^2}{L_1 + L_2 + L_3 + L_4}, \quad (7.210)$$

where the case of positive couplings is considered explicitly; the other case is the same by symmetry, giving the factor 2 in front. The same symmetry arguments are used for the next contribution.

The contribution of the second case is the nontrivial one. Again, the procedure starts by removing L_4 and then considering the scenario in which all four spins belong to the same cluster even if the coupling J_{14} , when re-introduced, changes the ground state configuration of σ_1 and σ_4 from the one where $J_{14} = 0$. To do so, the lines L_1 and L_2 are joined, resulting in the effective coupling $\tilde{J}_{13}$. Next L_3 is added with the condition that $|h_3| < |\tilde{J}_{13}| - |J_{34}|$ if $|\tilde{J}_{13}| > |J_{34}|$ or $|h_3| < |J_{34}| - |\tilde{J}_{13}|$ if $|\tilde{J}_{13}| < |J_{34}|$. The spin σ_3 follows σ_1 if $|\tilde{J}_{13}| > |J_{34}|$, otherwise it follows σ_4 . In the first situation, it suffices to consider $P_{L_1+L_2}^{J \neq 0}(\tilde{J}_{13})$, since the necessary condition for σ_3 to follow σ_1 is that σ_2 follows too, while in the second situation, since σ_3 follows σ_4 , it has to be ensured that σ_2 is “non-blocked”, thus $P_{L_1, L_2}^{\text{NB}}(\tilde{J}_{13})$ must be used. With this argument $P_{L_1, L_2, L_3}^{\text{NB}}(\tilde{J}_{14})$ can be computed, as the sum of the two contributions

$$\begin{aligned} & \frac{4a (\hat{P}_3(0))^2}{\rho} \int_{-\infty}^{\infty} dJ_{34} \int_{J_{34}}^{\infty} d\tilde{J}_{13} P_{L_1+L_2}^{J \neq 0}(\tilde{J}_{13}) P_{L_3}^{J \neq 0}(J_{34}) \delta(\tilde{J}_{14} - J_{34}) 2(\tilde{J}_{13} - J_{34}) + \\ & + \frac{4a (\hat{P}_3(0))^2}{\rho} \int_{-\infty}^{\infty} d\tilde{J}_{13} \int_{\tilde{J}_{13}}^{\infty} dJ_{34} P_{L_1, L_2}^{\text{NB}}(\tilde{J}_{13}) P_{L_3}^{J \neq 0}(J_{34}) \delta(\tilde{J}_{14} - \tilde{J}_{13}) 2(J_{34} - \tilde{J}_{13}), \end{aligned} \quad (7.211)$$

which generalizes Eq. 7.57. The result is

$$P_{L_1, L_2, L_3}^{\text{NB}}(\tilde{J}_{14}) = \frac{16 a^3 (\hat{P}_3(0))^2}{\rho} (L_1^2 + L_2^2 + L_3^2) \lambda^{L_1+L_2+L_3} e^{-\rho |\tilde{J}_{14}|(L_1+L_2+L_3)}. \quad (7.212)$$

The contribution of the second case follows:

$$\begin{aligned} & 2 \int_{-\infty}^0 dJ_{14}^{\text{eff}} \int_0^{\infty} d\tilde{J}_{14} \int_{-\infty}^{-\tilde{J}_{14}} dJ_{14} \int_{-\infty}^{\infty} dh_1 \int_{-\infty}^{\infty} dh_4 P_{L_1, L_2, L_3}^{\text{NB}}(\tilde{J}_{14}) P_{L_4}^{J \neq 0}(J_{14}) \hat{P}_3(h_1) \times \\ & \times \hat{P}_3(h_4) \prod_{i=1,4} \Theta(|J_{14}^{\text{eff}}| - |h_i|) \delta(|h_1 + h_4 \text{sign}(J_{14}^{\text{eff}})|) \\ & = \frac{64 a^4 (\hat{P}_3(0))^4}{\rho^3} \frac{L_1^2 + L_2^2 + L_3^2}{L_1 + L_2 + L_3 + L_4} \lambda^{L_1+L_2+L_3+L_4}. \end{aligned} \quad (7.213)$$

The last contribution is the case where one of the four lines has zero coupling:

$$- \frac{128 a^4 (\hat{P}_3(0))^4}{\rho^3} (L_1 + L_2 + L_3 + L_4). \quad (7.214)$$

All terms together give this contribution to the 4-point function, from $\mathcal{G}_6^{\text{amp}}$ in Figure 7.8:

$$\begin{aligned}
& \mathcal{C}_4^{\text{lc}}(\mathcal{G}_6^{\text{amp}}; L_1, L_2, L_3, L_4) = \\
& = -\frac{128 a^4}{\rho^3} \left(\widehat{P}_3(0) \right)^4 \frac{L_1 L_2 + L_3 L_2 + L_1 L_3 + L_1 L_4 + L_2 L_4 + L_3 L_4}{L_1 + L_2 + L_3 + L_4} \lambda^{L_1 + L_2 + L_3 + L_4}. \quad (7.215)
\end{aligned}$$

From this result the generic form of P^{NB} for a generic line of length $L_1 + L_2 + \dots + L_{n-1}$ is straightforwardly expressed generalizing the expression of [Eq. 7.212](#)

$$P_{L_1, \dots, L_{n-1}}^{\text{NB}}(\widetilde{J}_{1n}) = a \rho \left(\frac{4 a \widehat{P}_3(0)}{\rho} \right)^{n-2} \left(\sum_{i=1}^{n-1} L_i^2 \right) \prod_{i=1}^{n-1} \lambda^{L_i} e^{-\rho |\widetilde{J}_{1n}| L_i}, \quad (7.216)$$

from which the contribution of the diagram depicted in [Figure 7.8](#) for a generic n -point function is written:

$$\mathcal{C}_n^{\text{lc}}(\mathcal{G}_n^{\text{amp}}; \{L_i\}_{i=1, \dots, n}) = -\frac{\rho}{2} \left(\frac{4 a \widehat{P}_3(0)}{\rho} \right)^n \frac{\sum_{i < j}^{1, \dots, n} L_i L_j}{\sum_{i=1}^n L_i} \prod_{i=1}^n \lambda^{L_i}. \quad (7.217)$$

Part IV

Conclusions and Perspectives

Chapter 8

Conclusions

Throughout this thesis, the M -layer construction has been discussed and applied as a method to obtain predictions on finite-dimensional lattice models. Its application to the Ising model, detailed in [Chapter 5](#), provides both a validation of the method and a simple yet nontrivial working example. Well-known results are thereby recovered from a new perspective. Similarly, the application of the construction to the percolation problem, in [Chapter 6](#), shows how critical exponents can be extracted from this framework, making the method a valuable tool for obtaining quantitative information about lattice systems. The joint analysis of site and bond percolation within the same formalism concretely verifies their universality. An intriguing possible extension can be to perform the same analysis, but from the percolating phase: since observables can be computed on given diagrams even within this regime, whether analytically or numerically, the method may naturally be applied to that setting as well.

The central focus of this work remains the application of the method to the [EA](#) model in the presence of an external field. For systems with quenched disorder, the characterization of finite-dimensional critical behavior has long been debated due to the limitations of standard field-theoretic methods, often resulting in inconclusive outcomes. In particular, as discussed in [Chapter 3](#), in the presence of a field the classical questions remain unanswered: does a spin glass transition occur? If so, how is the critical phase characterized? And to what extent does the [MF](#) scenario persist in finite dimensions? The application of the Bethe M -layer construction to the [EA](#) model, in a field and at zero temperature, reveals the existence of a transition at a finite value of the external field. This transition is analyzed from above the critical value, in the non-critical phase; the collective behavior is detected by observables associated with the spin glass behavior. Scaling hypotheses for these observables are verified *a posteriori*, and the associated critical exponents are estimated within the ϵ -expansion framework. In order to facilitate numerical cross-checks, critical exponents were also computed in a one-dimensional model with [LR](#) interactions, which is more numerically tractable.

Unfortunately, the results presented in [Chapter 7](#) cannot be directly compared with the discussion of possible spin glass scenarios in finite dimensions of [Section 3.3](#). Indeed, the present analysis concerns the [EA](#) model in a strictly *non-zero field* (and at zero temperature), whereas those scenarios are usually discussed in the *zero-field* case. A connection with the finite-dimensional scenarios can nevertheless be established: according to the Droplet Theory, the external field suppresses the putative transition. In

contrast, the analysis developed in [Chapter 7](#) reveals the existence of such a transition, a result that would be in contradiction with the Droplet picture as a possible scenario, even in the zero-field case. On the other hand, given that the present work is based on a perturbative expansion around the [MF](#) (Bethe–Peierls) solution, it is natural to expect that the critical phase in finite dimensions retains the qualitative features of the [MF](#) critical behavior, already investigated for the spin glass in a field in the recent literature [[Per22b](#)]. The scenario that emerges from Ref. [[Per22b](#)] is, at least, qualitatively coherent with the [RSB](#) description of the critical phase.

This finding opens new directions for numerical investigation, guided by the quantitative predictions obtained through the Bethe M -layer construction. In particular, efficient algorithms for ground state computations in spin glass models could allow quantitative studies of soft cluster statistics, as defined in [Section 7.1](#). On the other hand, given that the zero-temperature [FP](#) may be attractive, as in the case of the [RFIM](#), Monte Carlo simulations at finite temperature also become highly relevant for testing the predictions for the critical exponents, especially in the [LR](#) interaction case.

Beyond the specific models examined in this thesis, the M -layer construction offers a systematic alternative to standard techniques, with particular interest in problems where the latter have faced intrinsic difficulties or yielded inconclusive results. Within the framework of lattice models, with or without quenched disorder, the construction allows for nontrivial predictions for any system that can be studied on a Bethe lattice, with or without a Hamiltonian. For instance, it would be valuable to test conjectures regarding dimensional reduction in the [RFIM](#) [[Tar13](#)]. More generally, questions remain open in percolation theory concerning the role of the underlying lattice in determining universality [[Cir25](#)]. Once the model is identified and the M -layer procedure applied, changing the lattice, on which the model is defined, amounts to a (often simple) modification of the counting of paths between two points. This significantly broadens the applicability of the method, positioning it as a systematic and effective alternative whenever existing approaches prove insufficient.

In conclusion, the analysis presented in this thesis shows that the Bethe M -layer construction provides a consistent and systematic framework for the study of finite-dimensional lattice models. Its application to different contexts, from the Ising model, to percolation, to the [EA](#) model in a field, demonstrates that the method is capable of reproducing known results, offering new perspectives on established problems, and yielding quantitative predictions where standard approaches remain inconclusive. These findings indicate that the M -layer construction constitutes a reliable and versatile tool for analytical investigations of systems defined on lattices, and that further applications may extend its range well beyond the cases explicitly considered here.

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List of Acronyms

ac.....	alternating current (pages 4 , 5)
BP.....	Belief Propagation (pages 29 , 55)
dAT.....	de Almeida–Thouless (pages 20 , 21 , 27 , 31 , 33 , 49 , 51 , 56 , 123)
EA.....	Edwards–Anderson (pages 9 , 10 , 15 , 26 , 47 , 49–52 , 67 , 93 , 111 , 112 , 121 , 122 , 125 , 127 , 129 , 148 , 149 , 153 , 156 , 157 , 175 , 176 , 204 , 205 , 207 , 208)
ER.....	Erdős–Rényi (pages 13 , 14 , 29–31 , 151 , 204 , 208)
FC.....	Fully Connected (pages 12 , 15 , 26 , 29 , 35 , 56 , 78 , 88 , 149 , 157 , 158)
fc.....	field cooled (pages 6 , 7 , 26 , 191)
FM.....	Ferromagnet (page 21)
FP.....	Fixed Point (pages 37–39 , 46 , 48 , 49 , 88 , 109 , 121 , 122 , 124 , 157 , 158 , 176)
fRSB.....	full Replica Symmetry Breaking (pages 22–25)
LHS.....	Left-Hand Side (pages 36 , 162)
LR.....	Long-Range (pages 67 , 121–123 , 148–152 , 154 , 156–158 , 175 , 176 , 201 , 205 , 207)
MF.....	Mean-Field (pages 6 , 7 , 10–12 , 15 , 25 , 35 , 36 , 41 , 47–50 , 55 , 56 , 71 , 72 , 74 , 78 , 87 , 88 , 97 , 105 , 122 , 127 , 142 , 143 , 149 , 150 , 154 , 156 , 157 , 160 , 161 , 163 , 175 , 176)
NBP.....	Non-Backtracking Paths (pages 62–65 , 67 , 79 , 85 , 103 , 113 , 118 , 130 , 132 , 142 , 143 , 158 , 160 , 193 , 201 , 203)
PM.....	Paramagnet (pages 21 , 72 , 73)

RFIM	Random Field Ising Model (pages 9 , 67 , 92 , 121 , 124 , 125 , 157 , 176)
RG.....	Renormalization Group (pages 36–40 , 47–51 , 55 , 56 , 88 , 109 , 121 , 122 , 126 , 127 , 157 , 158 , 205 , 207 , 208)
RHS	Right-Hand Side (pages 31 , 32 , 36 , 74 , 87 , 106 , 124 , 162)
RKKY	Ruderman–Kittel–Kasuya–Yosida (page 4)
RRG.....	Random Regular Graph (pages 13 , 14 , 26 , 29 , 72 , 89 , 95)
RS	Replica Symmetry (pages 17–23 , 25 , 27 , 29 , 33 , 48 , 49 , 55 , 66 , 72 , 76 , 77 , 81 , 86 , 204–207)
RSB.....	Replica Symmetry Breaking (pages 18 , 19 , 21–23 , 25 , 27 , 33 , 49–52 , 176 , 191 , 203 , 204 , 207)
RW	Random Walks (pages 62 , 64 , 152 , 153 , 158)
SG	Spin Glass (page 21)
SK	Sherrington–Kirkpatrick (pages 15–17 , 21 , 22 , 25–27 , 35 , 48 , 51 , 56 , 191 , 203–207)
TNT	Trivial Non Trivial (pages 51 , 52)
UCD.....	Upper Critical Dimension (pages 35 , 36 , 41 , 44 , 48 , 49 , 87 , 88 , 104 , 105 , 107 , 112 , 121–123 , 127 , 157)
UV.....	Ultraviolet (pages 41 , 103 , 105–107 , 144 , 145 , 207)
VB.....	Viana–Bray (pages 26 , 31 , 33 , 56)
zfc	zero field cooled (pages 6 , 7 , 26 , 191)

List of Symbols

$\langle \cdot \rangle$	Thermal average
$\langle \cdot \rangle_c$	Connected average
$\overline{\cdot}$	Average over the quenched disorder distribution
$\mathbb{E}_X$	Average over the distribution of the random variable X
$=$	Equality
$\neq$	Not equal to
$\equiv$	Defined as
$\sim$	Asymptotically equal up to a constant
$\approx$	Numerically similar
$\propto$	Proportional to
A, B, C	In Chapter 6 and in Chapter 7 : constants identified through the M -layer construction
$A_{\text{lr}}, B_{\text{lr}}, C_{\text{lr}}$	In Chapter 7 : constants identified through the M -layer construction for the LR model
A_n	In Chapter 2 : quenched free energy with n replicas
$\mathcal{A}, \mathcal{B}$	In Chapter 6 : prefactor in the Ginzburg criterion computations
a	In Chapter 7 : parameter in the <i>Ansatz</i> distribution for triplets
a_l	Lattice spacing of the system under investigation
B_λ	Generating function of the number of NBP
$\hat{B}_\lambda$	Generating function of the number of NBP in Fourier space

- b In [Chapter 3](#): length scale factor
- C_1, C_2 In [Chapter 3](#): auxiliary functions to compute critical exponents
- C_q In [Chapter 6](#) and in [Chapter 7](#): q -point connected correlation function
- $\hat{C}_q$ In [Chapter 6](#) and in [Chapter 7](#): q -point connected correlation function in Fourier space
- $\mathcal{C}_q$ q -point connected correlation function on a given diagram in the M -layer construction framework
- c In [Chapter 1](#): parameter fixing connectivity
- D $\triangleright$ Physical dimension of the system under investigation
 $\triangleright D_1, D_2$ and D_4 in [Section 6.3.2](#) auxiliary functions to compute critical exponents for the percolation problem
- Dx In [Chapter 2](#): Gaussian measure over the variable x
- D_f Critical exponent: fractal dimension
- D_s In [Chapter 3](#): surface fractal dimension in Droplet theory
- $\mathcal{D}_2$ In [Chapter 7](#): 2-point disconnected correlation function on a given diagram in the M -layer construction framework
- $\mathcal{E}$ Set of edges on a given graph
- f $\triangleright$ In [Chapter 3](#): intensive free energy
 $\triangleright$ In [Chapter 6](#) and [Chapter 7](#): generic scaling function
- G_q q -point correlation function
- $\tilde{G}_q, \hat{G}_q$ q -point correlation function in Fourier space
- $\mathcal{G}$ $\triangleright$ A generic graph
 $\triangleright$ In [Chapter 4](#), [Chapter 5](#), [Chapter 6](#) and [Chapter 7](#): a generic diagram in the M -layer construction framework
- g $\triangleright$ In [Chapter 5](#), [Chapter 6](#) and [Chapter 7](#): bare coupling constant
 $\triangleright$ In [Chapter 6](#): $g(s, p)$ probability to belong to a cluster of size s in the percolation problem with control parameter p

$\mathcal{H}$	▷ Hamiltonians ▷ In Chapter 1: $\mathcal{H}_I$ and $\mathcal{H}_{EA}$ for Ising and Edwards–Anderson Hamiltonians ▷ In Chapter 2: $\mathcal{H}_{SK}$ and $\mathcal{H}_{VB}$ for Sherrington-Kirkpatrick and Viana-Bray Hamiltonians ▷ In Chapter 3: $\mathcal{H}_0$ for a generic spin Hamiltonian ▷ In Chapter 5: $\mathcal{H}_I$ for Ising Hamiltonian ▷ In Chapter 7: $\mathcal{H}$ for Edwards–Anderson Hamiltonian in a field, $\mathcal{H}_{\mathcal{G}_3}$ and $\mathcal{H}_{\mathcal{G}_4}$ for effective Hamiltonians on diagrams $\mathcal{G}_3$ and $\mathcal{G}_4$
H	▷ External magnetic field ▷ In Chapter 3: $\beta\mathcal{H}$ for rescaled Hamiltonian
h	▷ Local or cavity field ▷ In Chapter 3: external magnetic field
J_{ij}	Microscopic coupling between spin at node i and spin at node j
K_a	In Chapter 3 : rescaled Hamiltonian parameter
L	▷ In Section 3.3: linear droplet size ▷ In Chapter 4, Chapter 5, Chapter 6 and Chapter 7: length of a line that belongs to a diagram in the M-layer construction framework
l	In Chapter 3 : logarithmic-length scaling parameter
M	▷ In Chapter 1: global magnetization ▷ In Chapter 4, Chapter 5, Chapter 6, Chapter 7: number of layers in the M-layer construction framework
m	▷ In Chapter 1 and Chapter 2: magnetization per site ▷ m_i with $i = 1, 2, \dots$ in Chapter 2: RSB parameter of the SK model ▷ From Chapter 3 on: mass, i.e. distance from the critical point
N	Number of microscopic components of a system
n	Number of replicas in the Replica Method framework
$\mathcal{N}$	From Chapter 4 on: number of NBP
$\mathcal{O}$	Order of magnitude of: $f(x) = \mathcal{O}(g(x)) \Rightarrow f(x) \sim g(x)$

- P $\triangleright P_f$ in [Section 2.1](#): couplings probability density of the [SK](#) model
 $\triangleright$ In [Section 2.1.2](#): parameter in the stability analysis of the [RS](#) solution of the [SK](#) model
 $\triangleright$ In [Section 2.1.4](#): overlap probability density
 $\triangleright P_B$ and P_{ER} in [Section 2.2](#) and in [Chapter 7](#): cavity bias probability density in the Bethe lattice and in the [ER](#) random graph respectively
 $\triangleright P_1$ in [Chapter 7](#): probability density of the local field
 $\triangleright P_2$ in [Chapter 7](#): probability density of the effective triplet (u, u', J) , Cavity method
 $\triangleright P^{\text{dis}}$ and P^{con} in [Chapter 7](#): disconnected and connected parts respectively of P_2
 $\triangleright \tilde{P}_n$ in [Chapter 7](#): auxiliary function, Cavity method
 $\triangleright P_L$ [Chapter 7](#): spin glass *Ansatz* distribution for the triplet (u, u', J) on a chain of length L at zero temperature, Cavity method
- $\mathcal{P}_q$ $\triangleright \mathcal{P}_q$ in [Chapter 7](#): q -point connected correlation function of the [EA](#) model in an external field at zero temperature in real space
 $\triangleright$ In [Chapter 7](#): q -point connected correlation function of the [EA](#) model in an external field at zero temperature in Fourier space
- $\mathbb{P}[\cdot]$ Probability of occurrence of the event “ $\cdot$ ”
- p $\triangleright p_B$ in [Chapter 5](#): auxiliary function proportional to the cavity marginal, Cavity method for the Ising model
 $\triangleright p$ in [Chapter 6](#): probability that a bond or node is active; p_c is the corresponding critical value
- Q $\triangleright$ In [Section 2.1.2](#): parameter in the stability analysis of the [RS](#) solution of the [SK](#) model
 $\triangleright Q_{\alpha\beta}$ in [Chapter 2](#) and in [Chapter 3](#): element of the overlap matrix of the [SK](#) model
 $\triangleright Q_B$ and Q_{ER} in [Section 2.2](#) and in [Chapter 7](#): cavity bias probability density in the Bethe lattice and in the [ER](#) random graph respectively
 $\triangleright Q_L$ in [Chapter 7](#): *Ansatz* distribution $P_L(u, u', J)$ including cavity biases at the extremities of the chain
- q $\triangleright$ In [Chapter 1](#): overlap in lattice spin systems
 $\triangleright$ In [Chapter 2](#): overlap value in the [RS](#) *Ansatz*
 $\triangleright q_{EA}$: Edwards–Anderson order parameter for the [EA](#) model
 $\triangleright q_c$ in [Section 1.2.3](#): excess degree distribution of a random graph in the graph theory framework
 $\triangleright q^{(k)}$ in [Section 2.1.4](#): k -th moment of the overlap distribution of the [SK](#) model
 $\triangleright q_i$ with $i = 1, 2, \dots$ in [Chapter 2](#): [RSB](#) parameter of the [SK](#) model

- R ▷ In [Section 2.1.2](#): parameter in the stability analysis of the [RS](#) solution of the [SK](#) model
 ▷ R_b in [Section 3.1](#): [RG](#) transformation of scaling factor b
- r ▷ Excess degree of a graph
 ▷ r_0 : bare mass of the ϕ^4 Field Theory
 ▷ r_L , r_A , and r_R in [Section 3.2](#): bare masses corresponding respectively to the *longitudinal*, *anomalous* and *replicon* sectors of the [RS](#) Field Theory
- s ▷ s_i or s'_i with $i = 1, 2, \dots$ from [Chapter 1](#) to [Chapter 5](#): Ising spin variable
 ▷ In [Chapter 6](#) and [Chapter 7](#): number of nodes in a given cluster in percolation framework
- T ▷ Temperature
 ▷ $T(s, t)$ in [Chapter 5](#): transfer matrix with Ising spins s and t , Cavity method
- t In [Chapter 5](#): Ising spin variable
- u ▷ u_i with $i = 1, 2, \dots, N$: cavity bias, Cavity method
 ▷ $u_{i \rightarrow j}$ in [Section 2.2](#): cavity bias message from node i to j , Cavity method
 ▷ $\hat{u}$ in [Section 2.2](#): filter function, Cavity method
 ▷ u_0 dimensional bare coupling of the ϕ^4 Field Theory
 u in [Chapter 6](#) and [Chapter 7](#): dimensionless bare coupling constant in the percolation and [EA](#) model frameworks
- v_i In [Chapter 7](#): with $i = 1, \dots, N$ cavity bias, Cavity method
- W Weight of the projected diagram in the M -layer construction framework
- w_i In [Chapter 3](#): with $i = 1, \dots, N$ coupling constants in the [RS](#) Field Theory
- $\mathcal{W}(x_i, x_j)$ Probability density to have an edge between nodes at x_i and x_j in the [LR](#) one-dimensional lattice
- Z ▷ Z_N : Partition function of a system with N microscopic variables
 ▷ $Z_{i \rightarrow j}(s_i)$ in [Section 2.2](#): partition function of a spin model on a tree where the configuration of s_i is fixed and the edge (i, j) is cut, Cavity method
 ▷ Z_i in [Section 2.2](#): partition function of a spin model on a tree where the configuration of s_i is fixed, Cavity method
 ▷ $Z_L(s_1, s_2)$ in [Chapter 5](#): joint probability distribution of spins s_1 and s_2 for the Ising model on the Bethe lattice

- z $\triangleright$ In [Section 2.1](#): Gaussian random variable involved in local field computations for the [SK](#) model
 $\triangleright z_{i \rightarrow j}$, z_i and z_{ij} in [Section 2.2](#): normalization factors, Cavity method
- $\mathbb{Z}^D$ Hypercubic lattice in D dimensions ($a_l \mathbb{Z}^D$ if lattice spacing is a_l)
- $\alpha, \beta, \gamma, \delta, \mu, \nu$ As indices, in [Chapter 2](#) and in [Chapter 3](#): replica indices
- β $\triangleright$ Reciprocal of temperature: $\beta = 1/T$
 $\triangleright$ As a function: “ β -function” in Field Theory framework
- β $\triangleright$ In [Chapter 1](#): overlap critical exponent in spin glass systems
 $\triangleright$ In [Chapter 3](#): magnetization critical exponent in the Ising model
- Γ Gamma function
- γ Parameter of ϕ^4 action in Field Theory
- γ $\triangleright$ In [Chapter 1](#): spin glass susceptibility critical exponent in spin glass systems
 $\triangleright$ In [Chapter 3](#): susceptibility critical exponent in the Ising model
- δ $\triangleright \delta(\cdot)$ (or $\delta^D(\cdot)$): Dirac delta function (or Dirac delta function in D dimensions)
 $\triangleright \delta f$: small variation of the quantity f
- ϵ Small perturbative parameter in the “ ϵ -expansion” in field theory framework
- ε In [Section 2.1.2](#): small perturbation to the magnetization in the stability analysis of the [RS Ansatz](#) of the [SK](#) model
- η $\triangleright$ In [Section 2.1.2](#): small perturbation to the overlap in the stability analysis of the [RS Ansatz](#) of the [SK](#) model
 $\triangleright \eta_i$ and $\eta_{i \rightarrow j}$ in [Section 2.2](#): marginal distribution of the spin variable s_i and cavity message from s_i to s_j , Cavity method
 $\triangleright$ In [Chapter 5](#): cavity marginal for a homogeneous system, Cavity method
- $\eta, \bar{\eta}$ Anomalous dimension critical exponents of the 2-point connected and disconnected correlation functions
- Θ Heaviside Theta function
- θ Free energy scaling exponent in the Droplet Theory framework

- θ Cluster energy critical exponent in the [EA](#) model in a field at zero temperature
- Λ [UV](#) cutoff in Field Theory framework
- λ $\triangleright \lambda^L, \lambda^A$ and λ^R in [Section 2.1.2](#): eigenvalues of the Hessian in the stability analysis of the [RS Ansatz](#) of the [SK](#) model
 $\triangleright$ In [Section 2.2](#) and in [Chapter 7](#): largest eigenvalue of the integral equation associated to the critical point of the [SK](#) model, Cavity method
 $\triangleright \lambda_i$ in [Section 3.1](#): eigenvalues of the infinitesimal [RG](#) transformation
 $\triangleright \lambda_{\max}, \lambda_-$ and λ in [Chapter 5](#): first two eigenvalues of the transfer matrix of the Ising model, Cavity method, $\lambda = \lambda_{\max}/\lambda_-$
- λ $\triangleright$ In [Section 3.1.2](#), [Section 6.3.2](#) and in [Section 7.4](#): renormalized coupling constant, with λ_c value at the critical point in Field Theory frameworks
 $\triangleright \lambda_{lr}$ in [Section 7.5.3](#): renormalized coupling constant for the [LR](#) version of the [EA](#) model, with λ_c^{lr} value at the critical point in Field Theory frameworks
- μ $\triangleright \mu_{\alpha\beta}^L, \mu_{\alpha\beta}^A$ and $\mu_{\alpha\beta}^R$ in [Section 2.1.2](#): eigenvectors of the Hessian in the stability analysis of the [RS Ansatz](#) of the [SK](#) model
 $\triangleright \mu_\phi^2$: square of the average of a generic order parameter $\triangleright$ In [Section 2.1.3](#) and in [Chapter 6](#): *bare mass* of the [SK](#) model and percolation theory respectively, in Field Theory framework
 $\triangleright d\mu(s)$ in [Chapter 5](#): measure over the spin variable s of a lattice spin model
- ν Correlation length critical exponent in Field Theory framework
- Ξ In [Section 2.1.3](#): random local effective field in the [RSB Ansatz](#) of the [SK](#) model
- ξ Equilibrium correlation length
- ρ $\triangleright \rho(\cdot)$ in [Section 3.3.1](#): probability density of the energy of droplets in Droplet Theory
 $\triangleright$ In [Chapter 5](#): auxiliary parameter defined of the Ising model in the M -layer construction framework
 $\triangleright$ In [Chapter 7](#): parameter in the definition of the *Ansatz* distribution for the triplet for the spin glass model, Cavity method
- ρ Exponent that defines the range of the microscopic interaction in the [LR](#) one-dimensional version of the [EA](#) model
- q, j In [Chapter 7](#): scaling functions variables in [RG](#) framework

Σ	<i>self-energy</i> in Field Theory framework
σ	▷ σ_ϕ^2 and σ_φ^2 in Chapter 3: variance of variables ϕ and φ ▷ In Chapter 6: critical exponent in percolation framework ▷ σ, σ_i in Chapter 7: Ising spin variable of the EA model
τ	In Chapter 5 and in Chapter 7 : <i>bare mass</i> of the Ising and EA models respectively, in Field Theory framework
τ	In Chapter 6 : critical exponent in percolation framework
Y	In Section 3.3.1 : generalized stiffness modulus in Droplet Theory
ϕ	Order parameter/field in Field Theory framework
φ	▷ In Section 1.2.3: probability of an edge to be present in an ER random graph in graph theory framework ▷ $\varphi_n(k_n)$ in Section 3.1: block variable corresponding to the block k_n in RG framework
χ	Different kinds of susceptibilities
$\Psi_f, \Psi_{C_2}, \Psi_{C_q}$	In Section 3.1 : scaling functions in RG framework
$\psi_n(k_n)$	In Section 3.1 : rescaled block variable corresponding to the block k_n in RG framework
ω	▷ In Section 3.1: scaling parameter for blocking variables (ω_c corresponding “critical” value) in RG framework ▷ ω_A, ω_R in Section 3.1: largest (in absolute value) eigenvalues of the infinitesimal RG transformation in RG framework
ω	Correction to scaling critical exponent