Sapienza Università di Roma

Recovering SU(2) Gauge Symmetry in Cosmological Implementations of Loop Quantum Gravity

Scuola di dottorato "Vito Volterra" in Scienze astronomiche, chimiche, fisiche, matematiche e della terra
Dottorato di Ricerca in Fisica (XXXVIII cycle)

Matteo Bruno
ID number 1807493

Advisor
Prof. Giovanni Montani

Academic Year 2024/2025

Thesis defended on 3 December 2025
in front of a Board of Examiners composed by:

Prof. Paolo Pani (chairman)
Prof. David Brizuela
Prof. Marc Mezard

Recovering SU(2) Gauge Symmetry in Cosmological Implementations of Loop Quantum Gravity

PhD thesis. Sapienza University of Rome

This thesis has been typeset by LaTeX and the Sapthesis class.

Author's email: matteo.bruno@uniroma1.it

Abstract

This thesis investigates the formal aspects of applying the quantisation techniques developed in Loop Quantum Gravity within a cosmological context. Its principal contribution is the proposal of a novel symmetry-reduction procedure that defines the cosmological sector of the Ashtekar-Barbero-Immirzi formulation of General Relativity for arbitrary cosmological models. Crucially, this method preserves the local $SU(2)$ gauge symmetry, thereby enabling a quantisation scheme closely aligned with Loop Quantum Gravity, in which spin-network states on suitably symmetric graphs arise naturally.

Loop Quantum Gravity is regarded as one of the most promising candidates for a theory of quantum gravity, primarily owing to its prediction that geometric quantum operators associated with area and volume have discrete spectra. The application of the loop quantisation scheme to cosmology leads to Loop Quantum Cosmology. Although Loop Quantum Cosmology has achieved significant success in predicting novel and intriguing phenomena, its reliability has been questioned, particularly concerning its consistency with the full Loop Quantum Gravity theory.

The original contributions of this thesis begin with an extension of the Loop Quantum Cosmology framework to more general cosmological models not thoroughly examined in the literature: the non-diagonal models, which are relevant to the Belinski-Khalatnikov-Lifshitz conjecture. We show that these models can be quantised through a minor modification of the standard approach, introducing angular variables that describe the directions of cosmic expansion. In this framework, the fate of the internal $SU(2)$ symmetry is analysed, revealing that the minisuperspace approximation effectively Abelianises the theory: the Gauß constraint reduces to three independent Abelian constraints, equivalent even at the quantum level.

To restore the local $SU(2)$ gauge symmetry, we need to move beyond the minisuperspace approach. To achieve this, we develop an appropriate geometric framework by employing principal bundle theory and emphasising the role of spin structures in the description of both the Ashtekar variables and the associated constraints. We apply the framework to cosmology, defining a precise notion of homogeneity for the Ashtekar connection that is consistent with Wang's theorem. This enables us to construct the classical cosmological sector while preserving the local $SU(2)$ gauge symmetry. Furthermore, we explore the properties of the moduli space of homogeneous Ashtekar connections, uncovering significant topological features that help address typical issues concerning the quantisation of gauge field theories.

Finally, we propose a novel interpretation of a class of theories often claimed to incorporate quantum-gravitational effects in an effective manner: the Generalised Uncertainty Principle theories. We introduce a semiclassical framework in which such models are described as symplectic manifolds endowed with a non-standard symplectic structure, conjecturing a link between modified symplectic geometry and the quantum-deformed Poisson algebra. Applying this formalism to constrained theories, with particular attention to cosmological scenarios, we demonstrate that the deformation is consistent with symplectic reduction and thus compatible with the imposition of constraints.

Acknowledgement

Many people outside academia have supported me throughout these doctoral years. However, as they are numerous and I would inevitably forget someone, I prefer to keep these acknowledgements strictly academic.
Beyond my supervisor, Giovanni Montani, and my close collaborators – Gabriele Peluso and Sebastiano Segreto, whose names already appear in this thesis – I wish to thank Eugenia Colafranceschi, Eleonora Giovannetti, and Fabio Mele for their support during the writing and publication of the works included in this manuscript. I am also grateful to David Brizuela and Francesca Vidotto for kindly agreeing to review this thesis.
I wish to thank Francesca Vidotto for her hospitality within the Quantum Gravity group at Western University, and Kristina Giesel for welcoming me to the Institute for Quantum Gravity at Friedrich-Alexander-Universität. I am likewise indebted to Carlo Rovelli and Thomas Thiemann for the valuable discussions held during my respective visits.
I am deeply grateful to the Mathematical Physics group at Sapienza University, in particular to Domenico Fiorenza and Gianluca Panati, for their interest and for the help they provided in developing the topics of this thesis. Finally, I wish to thank Simone Diverio, who above all has nurtured my love for Geometry.
These acknowledgements serve as a reminder that, although this thesis bears my name alone, it is the outcome of numerous shared efforts and contributions.

Contents

Introduction

General Relativity profoundly transformed our understanding of space and time, revolutionising both physics and our conception of the Universe. Since its formulation by A. Einstein [78], the theory has received continuous and striking experimental confirmation. Among the most notable recent achievements are the detection of gravitational waves [128] and the direct imaging of a black hole [162]. Mathematically, General Relativity intertwines physical intuition with geometric structure. Spacetime is modelled as a four-dimensional Lorentzian manifold, while gravity itself is understood as a manifestation of the curvature of this manifold, determined by the metric tensor. The dynamics are governed by the Einstein Field Equations, which link the geometry of spacetime to the distribution of mass and energy.

Despite its extraordinary successes, General Relativity is incomplete. The theory ceases to be predictive in the presence of *singularities*, where physical quantities diverge. Far from being mere mathematical artefacts, S. Hawking and R. Penrose demonstrated that singularities are robust predictions of the theory, formalised in the celebrated *singularity theorems* [141, 109]. These results establish that physically reasonable spacetimes inevitably contain singularities under certain conditions. As singularities appear precisely in regimes of extremely high curvature and energy density, it is natural to expect that quantum effects should play a fundamental role in resolving these pathologies.

The canonical programme of Quantum Gravity was outlined by P.A.M. Dirac [76, 77] and developed further by J.A. Wheeler and B.S. DeWitt [74], who formulated a functional equation governing the wave function of the Universe. Although conceptually pioneering, this approach faced severe technical and conceptual difficulties, most notably in defining an appropriate Hilbert space structure and well-defined quantum states.

A major advance came with the introduction of new variables by A. Ashtekar [12, 13]. The Ashtekar-Barbero-Immirzi formulation recasts General Relativity as an $SU(2)$ gauge theory, formally analogous to a Yang-Mills theory but dynamically equivalent to Einstein's gravity. This reformulation paved the way for the development of *Loop Quantum Gravity* (LQG), a non-perturbative and background-independent approach to quantisation. The quantisation is based on holonomies of the connection, non-local functionals of the gauge field similar to Wilson loops [62], and fluxes of the densitised triad. This construction leads naturally to a Hilbert space of L^2-functions over the space of connections, with a basis labelled by algebraic structures known as spin networks, which play a central role in the theory [145, 148].

The great success of LQG lies in the emergence of a *discrete quantum geometry* from an underlying continuum theory. The spectra of geometric operators such as area and volume are discrete, with the smallest non-zero eigenvalues of the order of the Planck scale [147, 21, 22]. This remarkable prediction provides the first consistent picture in which spacetime itself has an atomistic structure at the quantum level. While ambiguities remain in the precise definition of the Hamiltonian operator [146], the regularisation scheme proposed by T. Thiemann [163, 164, 165] provides a consistent framework for writing the quantum Einstein equations in the canonical setting. Complementary to the canonical programme, a covariant formulation via *spinfoam models* has also been developed [143, 31], offering a path-integral perspective on quantum spacetime dynamics (for a comprehensive review see [142, 150]). In this thesis, however, we shall restrict our attention to the canonical framework.

Cosmology offers a natural testing ground for candidate theories of Quantum Gravity. In its simplest form, cosmology implements the *cosmological principle* in General Relativity, assuming a spatial homogeneous and isotropic Universe: that the Universe looks the same at every point at a given cosmic time. This symmetry reduction drastically simplifies the Einstein Field Equations while still capturing the large-scale dynamics of the Universe [137, 138]. Yet, in the framework of the standard ΛCDM model, early-universe cosmology remains puzzling. Inflationary dynamics erase most information about the very early Universe [136], but a singularity persists at the beginning of time [109]. It is precisely in this regime that quantum gravitational effects are expected to dominate.

The application of LQG techniques to cosmology leads to *Loop Quantum Cosmology* (LQC) [27]. LQC provides a quantisation of symmetry-reduced cosmological models and offers one of the most striking results in the field: the classical Big Bang singularity is resolved and replaced by a *Big Bounce* [46, 24, 25, 26]. Beyond singularity resolution, LQC predicts corrections to inflationary dynamics and potential observational imprints in the Cosmic Microwave Background (for a review [14]). Furthermore, methods developed in LQC extend beyond cosmology, with applications to gravitational collapse and black hole physics [50, 151, 93, 84]. The formalism has also inspired related approaches such as Polymer Quantum Mechanics (PQM) [71], which can be framed within the broader context of Generalised Uncertainty Principle (GUP) theories [129, 130, 116, 117].

Nevertheless, the robustness of LQC has been questioned, particularly regarding its precise relation to the full LQG theory [63, 69, 54]. Concerns may be raised about whether the phenomena predicted in LQC are genuine consequences of LQG or artefacts of the minisuperspace reduction. A central issue is the loss of the non-Abelian $SU(2)$ gauge symmetry in the standard LQC framework. Despite the progress achieved in LQC over the years and the rich phenomenology that has emerged, its reliability in relation to LQG remains uncertain, representing one of the major open questions in the field. This thesis addresses this problem by proposing a formulation that restores the gauge symmetry and recovers spin-network states in a cosmological setting.

Before turning to the original contributions, the thesis also provides a review of the essential background material. In Chapter 1, we recall the Hamiltonian formulation of General Relativity, beginning with the ADM formalism and then recasting General Relativity as a theory of connections by introducing spin connections and Ashtekar variables. This sets the stage for the emergence of an $SU(2)$ gauge symmetry at the phase-space level. Chapter 2 discusses the canonical quantisation programme of Loop Quantum Gravity, emphasising its mathematical structure and making use of techniques from Lie theory commonly employed in constructive Quantum Field Theory [104] (for a brief overview of these concepts, see Appendix A). In particular, we review the discrete spectra of the geometric operators and the implementation of quantum dynamics. Chapter 3 introduces cosmology, clarifies the meaning of homogeneity and isotropy in this framework, and explores the implications of the loop quantisation scheme. The central results of Loop Quantum Cosmology are presented, together with their connections to GUP theories.

The original contributions of this thesis are organised into three main parts. First, we investigate the generality of the LQC approach when extended to more general cosmological models not yet systematically studied in the literature, namely the so-called *non-diagonal models*, in which the axes of cosmic expansion rotate during the time evolution. We show that a straightforward generalisation of the standard LQC construction is admissible: the fluxes can be quantised as in the diagonal case, yielding a comparable kinematical description, while additionally incorporating the angular variables that specify the directions of expansion. At the same time, we exclude alternative scenarios that have been proposed in the literature, thereby reinforcing the robustness of the Ashtekar school's approach in this broader setting. This analysis is presented in Chapter 4. Furthermore, in Chapter 5, we carry out a detailed investigation of the role of gauge symmetry in these models, showing how the minisuperspace reduction effectively "*Abelianises*" the theory. We establish that the quantum constraints associated with the gauge freedom form an Abelian algebra, and that their implementation yields the same Hilbert space as in the standard gauge-fixed theory.

The second part focuses on restoring the $SU(2)$ gauge symmetry, which is typically lost in the minisuperspace approach underlying LQC. We begin in Chapter 6 with a rigorous characterisation of the Ashtekar-Barbero-Immirzi formulation in the language of principal fibre bundles. This requires the introduction of advanced mathematical structures, most notably the *spin structure*, which proves crucial in providing a geometrically consistent interpretation of the theory. We extend this analysis to the phase space, clarifying the role of spin connections and highlighting the geometric subtleties often overlooked in the physical literature (cf. Appendix B for an introduction to principal bundles in physics). Building on this framework, in Chapter 7 we propose a new procedure for *symmetry reduction* in cosmology. This construction yields a reduced phase space that retains a local $SU(2)$ gauge symmetry, thereby recovering the non-Abelian character of the full theory within the cosmological context. Importantly, this reduction allows us to quantise the theory using the standard loop quantisation scheme. In doing so, we find that spin-network states naturally exhibit peculiar symmetries reflecting the classical notion of homogeneity, and at the same time they provide a direct link with the quantum states studied in LQC. In addition, we provide a detailed analysis of the *moduli space* of homogeneous connections modulo gauge transformations. This is carried out in Chapter 8, where we compute explicitly the moduli spaces associated with different cosmological models. Remarkably, these spaces exhibit particularly favourable topological properties: they are finite-dimensional topological manifolds (possibly with boundary), with contractible connected components, and their structure depends solely on the underlying symmetry of the cosmological models. These properties overcome several of the usual difficulties associated with moduli spaces in gauge field theory, such as singularities and the absence of regular measures, allow for a rigorous implementation of the quantum theory in cosmology.

Finally, in Chapter 9 we provide a complementary perspective by investigating the *classical formulation* of theories incorporating a Generalised Uncertainty Principle. Using the framework of symplectic geometry, we derive precise and general consistency relations for the phase-space functions controlling the deformation of the Poisson structure. This approach allows us to formulate the conjecture that a GUP theory is consistently implementable at the classical level if and only if the corresponding quantum commutators satisfy the Jacobi identities. We then apply this framework to analyse GUP theories with constraints, including two physically relevant cases: constraints interpretable as momentum maps, and Hamiltonian constraints. Particular attention is devoted to the cosmological setting, where we show that the induced deformations are well controlled and persist under the reduction of the phase space.

In summary, this thesis develops a rigorous mathematical framework for the cosmological sector of Loop Quantum Gravity, identifies the correct role of spin structures in the Ashtekar-Barbero-Immirzi formulation, and explores the interplay between Loop Quantum Cosmology and GUP classical theories within a geometric scheme.

The present thesis is mainly based on the following original research works

- M. Bruno, and G. Montani, "Loop quantum cosmology of nondiagonal Bianchi models", Phys. Rev. D **107**, 126013 (2023).
- M. Bruno, and G. Montani, "Is the diagonal case a general picture for loop quantum cosmology?", Phys. Rev. D **108**, 046003 (2023).
- M. Bruno, S. Segreto, and G. Montani, "Generalized uncertainty principle theories and their classical interpretation", Nucl. Phys. B **1009**, 116739 (2024).
- M. Bruno, "Fiber bundle structure in Ashtekar-Barbero-Immirzi formulation of General Relativity", J. Geom. Phys. **214**, 105537 (2025).
- M. Bruno, "Cosmology in loop quantum gravity: symmetry reduction preserving gauge degrees of freedom", Class. Quantum Grav. **42**, 185009 (2025).
- M. Bruno, and G. Peluso, "Moduli space of spin connections on three-dimensional homogeneous spaces". Preprint at arXiv.2507.06104 (2025).

Chapter 1

Hamiltonian formulations of General Relativity

General Relativity is the dynamical theory of the gravitational field. In its geometric interpretation, the gravitational field is described by a Lorentzian metric g on a four-dimensional manifold M called *spacetime*. The fundamental equations of the theory are the Einstein Field Equations, in the vacuum they read

$$\mathrm{Ric}_{\mathrm{g}} - \frac{1}{2}\mathrm{R}_{\mathrm{g}}\mathrm{g} + \Lambda \mathrm{g} = 0.$$

Where $\mathrm{Ric}_{\mathrm{g}}$ is the Ricci tensor of g, R_{g} its scalar curvature, and Λ a dimensional scalar constant known as *cosmological constant*. This equation can be derived by a variational principle, but generally does not admit a Hamiltonian formulation. The Hamiltonian formulation can be provided under some assumptions of regularity of the spacetime M and using the Arnowitt-Deser-Misner formalism. The Hamiltonian formulation is needed to construct a phase space and proceed with the Canonical Quantisation [75, 112] of the gravitational theory. For the rest of the text, we consider $\Lambda = 0$.

1.1 Geometrodynamics and canonical variables

To construct a Hamiltonian theory of General Relativity, it is necessary to introduce the ADM formalism [10]. This formulation requires a globally hyperbolic spacetime, specifically a time-orientable Lorentzian manifold that admits a Cauchy surface [110]. Let M be a globally hyperbolic spacetime, Geroch's theorem [92] ensures that M is homeomorphic to $\mathbb{R} \times \Sigma$, where Σ is a three-dimensional manifold. In particular, there exists a smooth surjective function $f : M \to \mathbb{R}$ such that $\Sigma_t \doteq f^{-1}(t)$ is a Cauchy surface homeomorphic to Σ for all $t \in \mathbb{R}$. The collection $\mathscr{F} \doteq \{\Sigma_t\}_{t\in\mathbb{R}}$ clearly defines a foliation for M, in fact, for each point $p \in M$, there exists a system of local coordinates $\{x^\mu\}$ such that each Σ_t is described locally by the equation $x^0 = const.$
Considering the splitting $M = \mathbb{R} \times \Sigma$, we can give a set of coordinates in a neighbourhood of $p = (t, x)$ as $x^0 = x^0(t)$, while $x^a = x^a(x)$ coordinates on Σ. Moreover, $\mathrm{T}M \cong \mathrm{T}\mathbb{R} \times \mathrm{T}\Sigma$, so, on every point, this choice induces a basis of tangent vectors

compatible with the splitting $\mathrm{T}_pM \cong \mathrm{T}_t\mathbb{R} \oplus \mathrm{T}_x\Sigma$, in particular $\partial_{x^0}|_t \in \mathrm{T}_t\mathbb{R}$ and $\partial_{x^a}|_x \in T_x\Sigma$.
In this case, the function of Geroch's theorem is the projector to the first factor $f \equiv \mathrm{pr}_1$. This function determines the vector field $n \doteq df^\sharp/||df||$, where $\sharp : \mathrm{T}^*M \to \mathrm{T}M$ is the musical isomorphism [124] and $||\cdot||$ is the norm induced by the metric g. This vector field is, for each fixed $t \in \mathbb{R}$, the unit normal vector of Σ_t. From this, we can write the following linear combination

$$\partial_0 = Nn + N^a\partial_a.$$

The functions $N(t,x)$ and $N^a(t,x)$ are called lapse function and shift vector respectively. These functions, together with the induced Riemannian metrics q on Σ_t (we omit the dependence of q on t), characterize the metric g at each point of M

$$\mathrm{g}_{ab} = q_{ab}, \quad \mathrm{g}_{0a} = N^b q_{ab}, \quad \mathrm{g}_{00} = -N^2 + N^aN^bq_{ab}.$$

The other geometric quantity that describes the geometry of the hypersurface Σ_t, together with q, is the extrinsic curvature K. Consider the Levi-Civita covariant derivative of the unit normal vector $\nabla n \in \Omega^1(M, \mathrm{T}M)$, we can define the Weingarten map $k : \mathrm{T}\Sigma_t \to \mathrm{T}\Sigma_t$; $X \mapsto \nabla_X n$ that is a linear endomorphism on the tangent space. In this way, we can define the extrinsic curvature as the quadratic form K on $\mathrm{T}_x\Sigma_t$ defined by $\mathrm{K}(X,Y) = q(k(X),Y)$, $\forall X,Y \in \mathrm{T}_x\Sigma_t$.
From the Einstein-Hilbert action [113], we can derive the ADM Lagrangian [11] using the Gauß equation for the scalar curvature in a Ricci flat spacetime, which links the scalar curvature of g and the one of q, namely

$$\mathrm{R_g} = \mathrm{R}_q + |k|_q^2 - (\mathrm{tr}_q\, k)^2, \tag{1.1.1}$$

where $|\cdot|_q$ and tr_q are the norm and trace induced by q. From which, we can derive the Hamiltonian $\mathbf{H}$ and the conjugate momenta $\Pi(x)$, $\Pi_a(x)$, and $P^{ab}(x)$ to $N(x)$, $N^a(x)$, and $q_{ab}(x)$, respectively, obtaining

$$\begin{aligned}
&\Pi = 0,\\
&\Pi_a = 0,\\
&P^{ab} = \frac{|N|}{\kappa N}\sqrt{\det(q)}\left(\mathrm{K}^{ab} - q^{ab}\mathrm{K}^c_c\right),\\
&\mathbf{H} = \tfrac{1}{\kappa}\int_\Sigma d^3x\,(\lambda\Pi + \lambda^a\Pi_a + N^aH_a + |N|H)\,,
\end{aligned} \tag{1.1.2}$$

where $\kappa = 16\pi G_N/c^3$ is a constant proportional to the Newtonian constant of gravitation G_N, λ, λ_a are Lagrangian multipliers, and

$$\begin{aligned}
&H_a = -2q_{ac}D_bP^{bc},\\
&H = \frac{\kappa}{\sqrt{\det(q)}}\left(q_{ac}q_{bd} - \tfrac{1}{2}q_{ab}q_{cd}\right)P^{ab}P^{cd} - \frac{1}{\kappa}\sqrt{\det(q)}\mathrm{R}_q.
\end{aligned} \tag{1.1.3}$$

The canonical Poisson brackets between configuration variables and their conjugate momenta are the following:

$$\begin{aligned}
&\{N(t,x), \Pi(t,y)\} = \delta^{(3)}(x,y)\\
&\{N^a(t,x), \Pi_b(t,y)\} = \delta^a_b\delta^{(3)}(x,y)\\
&\{q_{ab}(t,x), P^{cd}(t,y)\} = \delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x,y)
\end{aligned} \tag{1.1.4}$$

The lapse function and the shift vector have then trivial dynamics. Moreover, their conjugate momenta are subject to the *primary constraints*

$$\Pi = 0, \quad \Pi_a = 0.$$

The dynamical consistency of the primary constraints, namely their trivial evolution under dynamics, yields two secondary constraints called the *spatial Diffeomorphism constraint* and *Hamiltonian constraint*, respectively:

$$H_a = 0, \quad H = 0. \tag{1.1.5}$$

From here, we can see that General Relativity is a fully constrained Hamiltonian system. Indeed, $\mathbf{H} = 0$ on the solutions of the constraints.
Since the lapse function and the shift vector are just Lagrangian multipliers of the secondary constraints with trivial dynamics, we will consider the ADM phase space to consist of pairs (q_{ab}, P^{ab}), where q_{ab} is a 3-metric and P^{ab} is a symmetric tensor on Σ, together with the spacial Diffeomorphism and Hamiltonian constraints.
On this phase space the smeared constraints $\vec{H}(\vec{f}) = \int_\Sigma H_a f^a$ and $H(f) = \int_\Sigma Hf$ form the so-called *Dirac algebra* [75]

$$\begin{aligned}
&\{\vec{H}(\vec{f}), \vec{H}(\vec{f}')\} = \kappa \vec{H}(\mathcal{L}_{\vec{f}} \vec{f}'),\\
&\{\vec{H}(\vec{f}), H(f)\} = \kappa H(\mathcal{L}_{\vec{f}} f),\\
&\{H(f), H(f')\} = \kappa \vec{H}((f df' - f' df)^\sharp).
\end{aligned} \tag{1.1.6}$$

1.2 Gravity as a theory of connections

The metric formulation is not the only approach to General Relativity. Using the connection 1-forms as configuration variables, it can be expressed as a gauge theory. The main ingredient of a gauge theory is the gauge group. The vierbein formalism naturally encodes the correct gauge group of General Relativity. This gauge group is $SO(1,3)$, then, in other words, General Relativity has a local Lorentz symmetry, according to the equivalence principle. On a globally hyperbolic spacetime, one can perform a splitting into an $SO(3)$ gauge theory for any fixed time.
A vierbein is an object, with a coordinate index $\mu = 0,1,2,3$ and an internal index $\alpha = 0,1,2,3$, e^α_μ such that $\mathrm{g}_{\mu\nu} = \eta_{\alpha\beta} e^\alpha_\mu e^\beta_\nu$. The internal index is covariantly derived using the spin connection 1-form $\omega^{\alpha\beta}_\mu$, requiring

$$D_\mu e^\alpha_\nu = \partial_\mu e^\alpha_\nu - \Gamma^\sigma_{\mu\nu} e^\alpha_\sigma + {\omega_\mu}^\alpha{}_\beta e^\beta_\nu = 0. \tag{1.2.1}$$

Here, $\Gamma^\sigma_{\mu\nu}$ are the usual Christoffel symbol. General Relativity can be reformulated in terms of e^μ_α and its inverse e^α_μ. The internal index α can be raised and lowered using the Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$. The contraction $e_\alpha \doteq e^\mu_\alpha \partial_\mu$ is a orthonormal frame of tangent vector fields, while $e^\alpha \doteq e^\alpha_\mu dx^\mu$ is its dual frame. We can check that the covariant derivative induced by the spin connection is the usual Levi-Civita one

$$\begin{aligned}
\nabla X &= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\sigma} X^\sigma) dx^\mu \otimes \partial_\nu = (e^\nu_\beta \partial_\mu X^\beta + X^\beta \partial_\mu e^\nu_\beta + e^\sigma_\beta \Gamma^\nu_{\mu\sigma} X^\beta) e^\alpha_\nu dx^\mu \otimes e_\alpha\\
&= (\partial_\mu X^\alpha + (e^\alpha_\nu \partial_\mu e^\nu_\beta + \Gamma^\nu_{\mu\sigma} e^\sigma_\beta e^\alpha_\nu) X^\beta) dx^\mu \otimes e_\alpha = (\partial_\mu V^\alpha + \omega^\alpha_{\mu\beta} X^\beta) dx^\mu \otimes e_\alpha,
\end{aligned}$$

Indeed, the spin connection is metric-compatible and torsionless. This formulation can be adapted to the ADM splitting in the following form

$$e^{\alpha}{}_{\mu} = \begin{pmatrix} N & 0 \\ N^a e^i_a & e^i_a \end{pmatrix}, \; e^{\mu}{}_{\alpha} = \begin{pmatrix} \frac{1}{N} & 0 \\ -\frac{N^i}{N} & e^i_a \end{pmatrix} \tag{1.2.2}$$

where $q_{ab} = \delta_{ij} e^i_a e^j_b$. In this case, the e^i_a (called dreibein) emulates the role of the vierbein in the three-dimensional setting, and its inverse e^a_i is called triad. The internal index is i and the coordinate index is a. Moreover, a three-dimensional version of (1.2.1) holds. This choice for $e^{\alpha}{}_{\mu}$, commonly referred to as the *time gauge* in physics, is a specific choice and not the most general one. However, it reproduces a canonical construction in mathematics related to embedded hypersurfaces. In particular, its form can be deduced from the canonical embedding of the orthonormal frame bundle of the hypersurface into that of the entire manifold. A brief discussion of this geometric relationship between the vierbein and the dreibein is provided in Chapter 6, where we will introduce the proper mathematical language. The time gauge breaks the local Lorentz symmetry by fixing the temporal component of the vierbein to be normal to Σ_t and the spatial components to lie in its tangent bundle. For this reason, its role in the quantum theory has been discussed by many authors (e.g., see [7, 67, 68, 142] and references therein).
Using the dreibein, we can extend the ADM phase space. It is clear that the relation

$$q_{ab} = \delta_{ij} e^i_a e^j_b \tag{1.2.3}$$

is invariant under local $SO(3)$ rotations of the dreibein $e^i_a \mapsto O^i_j e^j_a$. We can encode this data in the dreibein by considering an $\mathfrak{su}(2)$-valued 1-form $e^i_a \tau_i$, called *solder form*. Here, τ_i are the generators of the Lie algebra $\mathfrak{su}(2)$, they are proportional to the Pauli matrices $\tau_i = -\frac{i}{2}\sigma_i$, and satisfy $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$. We recall that $SU(2)$ acts on its algebra via the adjoint representation $\mathrm{Ad} : SU(2) \to GL(\mathfrak{su}(2))$, and, in the basis of the generators, this action is given by a rotational matrix. Thus, this description introduces an $SU(2)$ gauge symmetry in the Hamiltonian formulation of the theory considering as phase space variables the pair (e^i_a, K^b_j), where $K^b_j = \mathrm{K}^b_a e^a_j$.

It is possible to introduce a new pair of conjugate variables (A^i_a, E^a_i), called Ashtekar-Sen connection and electric field (or densitised triad) [12, 13]. They can be written in terms of extended ADM variables as

$$A^i_a = \Gamma^i_a + \beta K^i_a, \tag{1.2.4}$$

$$E^a_i = |\det(e^j_b)|\, e^a_i. \tag{1.2.5}$$

where $\Gamma^i_a = \frac{1}{2}\epsilon^i{}_{kj}\omega^{jk}_a$, with ω^{jk}_a being the spin connection of the dreibein e^i_a. When β is a real parameter, the formulation exhibits an $SU(2)$ redundancy; namely, we find that, under local rotations, E^a_i transforms like the dreibein, while A^i_a transforms like a connection. In particular, we can interpret the Ashtekar-Sen connection as an $SU(2)$ gauge field (known as the Ashtekar-Barbero-Immirzi-Sen connection, in alphabetical order) [34]. In this case, β is called the Barbero-Immirzi parameter. Moreover, inverting the above relation, we can reconstruct the ADM variables, and this reconstruction is compatible with the gauge freedom. Indeed, there exists an $SU(2)$ gauge freedom in the choice of the pair (E^a_i, A^i_a) associated with the same $(q_{ab}, \mathrm{K}_{ab})$. Due to the $SU(2)$ redundancy, an additional constraint arises, characteristic of Yang-Mills theory: the Gauß constraint, which contains the structure constants of the gauge group. Thus, on this new phase space, the following constraints are present

$$G_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k, \tag{1.2.6}$$

$$V_a = E^b_i F^i_{ab}, \tag{1.2.7}$$

$$S = \left(F^i_{ab} - \tfrac{\beta^2+1}{\beta^2}\epsilon^i{}_{jk}(A^j_a - \Gamma^j_a)(A^k_b - \Gamma^k_b)\right)\frac{\epsilon_{imn}E^a_m E^b_n}{\sqrt{|\det(E^a_i)|}}, \tag{1.2.8}$$

called Gauß, Vector, and Scalar constraints, respectively. Here, $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ is the curvature of the Ashtekar-Sen connection, while Γ^i_a, as function of E^a_i, is a homogeneous rational function of degree zero of E^a_i and its derivatives (cf. Eq.4.2.18 in [166]), and β is the Barbero-Immirzi parameter that appears as a free parameter in the Holst action [115]. It is worth noting, indeed, that these variables and the constraints can be derived by the Holst action [114].
The fundamental Poisson brackets make the phase space $\mathcal{P}$, composed by the pairs (E^a_i, A^j_b), a symplectic manifold, rigorous constructed in Chapter 6

$$\{E^a_i(x), A^j_b(y)\} = \frac{\beta\kappa}{2}\delta^a_b\delta^j_i\delta(x,y). \tag{1.2.9}$$

The algebra of the smeared constraints on $\mathcal{P}$ can be obtained by considering an equivalent deformation of the Vector one $\mathrm{D}_a = V_a + A^i_a G_i$

$$\begin{aligned}
&\{G(\Lambda), G(\Lambda')\} = -\tfrac{\beta\kappa}{2}G([\Lambda,\Lambda']),\\
&\{G(\Lambda), \mathrm{D}(\mathrm{X})\} = \tfrac{\beta\kappa}{2}G(\mathcal{L}_X\Lambda),\\
&\{\mathrm{D}(X), \mathrm{D}(X')\} = -\tfrac{\beta\kappa}{2}\mathrm{D}(\mathcal{L}_X X'),\\
&\{\mathrm{D}(X), S(f)\} = -\tfrac{\beta\kappa}{2}S(\mathcal{L}_X f),\\
&\{S(f), S(f')\} = \tfrac{\beta\kappa}{2}\mathrm{D}((fdf' - f'df)^\sharp) + \tfrac{\beta\kappa}{2}G(A(fdf' - f'df)^\sharp).
\end{aligned} \tag{1.2.10}$$

The smearing functions have compact support to avoid issues with integrability on open manifolds. Namely, $\Lambda \in C^\infty_c(\Sigma, \mathfrak{su}(2))$, $X \in \mathfrak{X}_c(\Sigma)$, and $f \in C^\infty_c(\Sigma)$. On this phase space two gauge group acts: the $SU(2)$ gauge group $\mathcal{G} \doteq C^\infty(\Sigma, SU(2))$, and the diffeomorphism group $\mathrm{Diff}(\Sigma)$. Considering $g \in \mathcal{G}$, the action, as previously stated, is

$$A \mapsto \mathrm{Ad}_{g^{-1}} A + g^{-1}\,dg, \quad E \mapsto \mathrm{Ad}_{g^{-1}} E,$$

where $A = A^i_a \tau_i dx^a$, and $E = \frac{1}{2}\epsilon_{abc}E^a_i\tau_i dx^b \wedge dx^c$. While the diffeomorphisms act via pullback, let $\varphi \in \mathrm{Diff}(\Sigma)$,

$$A \mapsto \varphi^* A, \quad E \mapsto \varphi^* E.$$

The Gauß and Vector constraints are the infinitesimal generators of the gauge transformations on the phase space, indeed,

$$\begin{aligned}
\{G(\Lambda), A\} &= -d\Lambda - [A, \Lambda], \\
\{G(\Lambda), E\} &= [\Lambda, E], \\
\\
\{\mathrm{D}(X), A\} &= \mathcal{L}_X A, \\
\{\mathrm{D}(X), E\} &= \mathcal{L}_X E.
\end{aligned} \tag{1.2.11}$$

where $[\cdot, \cdot]$ are the Lie bracket of $\mathfrak{su}(2)$. Notice that, using the Cartan magic formula, we can compute $\mathcal{L}_X A = (X^b \partial_b A^i_a + A^i_b \partial_a X^b)\tau_i dx^a$, while $\mathcal{L}_X E = \frac{1}{2}\epsilon_{abc}(X^d \partial_d E^a_i - E^d_i \partial_d X^a + E^a_i \partial_d X^d)\tau_i dx^b \wedge dx^c$. As already stated, from $\mathcal{P}$ we can reconstruct the ADM phase space with the correct Poisson brackets, and so the proper constraints algebra. To be more precise, the following theorem holds

Theorem 1.2.1. *Consider the phase space $\mathcal{P}$ with Poisson brackets as in* (1.2.9) *and constraints algebra* (1.2.10)*. Then, solving only the Gauß constraint leads back to the ADM phase space together with the spatial Diffeomorphism and Hamiltonian constraints $H_a = V_a|_{G_i=0}$ and $H = S|_{G_i=0}$.*

This theorem has a clear mathematical interpretation in the setting of symplectic geometry, as explained in Chapter 6.

Chapter 2

Loop Quantum Gravity

Loop Quantum Gravity presents a successful way to implement Canonical Quantisation in the context of diffeomorphism-invariant theories of connections with local degrees of freedom [23]. The program started from the construction of a peculiar algebra on the phase space in Ashtekar variables: the holonomy-flux algebra. Then, an auxiliary Hilbert space of the theory is built as a representation of this algebra [19]. On this auxiliary Hilbert space, we implement the constraints as in Dirac's procedure, finding the physical Hilbert space as the common kernel of the quantum operators associated with the constraints [99]. The states in the auxiliary Hilbert space turn out to have remarkable properties; in particular, they decompose on a basis of states labelled by a continuous family of algebraic objects known as spin networks [145, 148]. This theory addresses one of the most significant aspects of quantum gravity: the quantisation of geometry, manifested in the discrete spectra of geometrical operators such as area and volume [147]. However, the implementation of the Scalar constraint poses some issues and ambiguities, leaving still open the quantisation program [165].

2.1 Holonomy-flux algebra

The holonomy-flux algebra $\mathfrak{P}$ is analogous to the Heisenberg algebra for Quantum Mechanics. The starting point is the definition of *holonomy* $h_c(A)$, which, given a connection A and a curve $c : [0, 1] \to \Sigma$, associates an element of $SU(2)$ satisfying the equation

$$\frac{d}{ds} h_{c_s}(A) = h_{c_s}(A) A(\dot{c}(s)), \tag{2.1.1}$$

where $c_s(t) \doteq c(st)$. This equation describes the inverse of the parallel transport. An explicit solution to this equation can be given by

$$h_c(A) = 1 + \sum_{k=1}^{\infty} \left(\int_0^1 ds_1 \int_{s_1}^1 ds_2 \cdots \int_{s_{k-1}}^1 ds_k \, A(s_1) A(s_2) \ldots A(s_k) \right).$$

The first factor of the holonomy-flux algebra $\mathfrak{P}$ is a peculiar class of functions on the space of smooth $SU(2)$ connections $\mathcal{A}$ which is canonically a Lagrangian submanifold of the Ashtekar phase space, the so-called *cylindrical functions.* Before that, we need to give a pair of definitions

Definition 2.1.1.
- A curve c is a continuous, piecewise semianalytic, embedding map $c : [0, 1] \to \Sigma$, such that $c([0, 1])$ is contained in a compact subset of Σ.
- An edge e is an equivalent class of a curve c semianalytic on $[0, 1]$. Two semianalytic curves c, c' are equivalent if $c(0) = c'(0)$, $c(1) = c'(1)$ and they are identical up to semianalytic reparametrization.
- A graph γ is a collection of edges that intersect at most at their endpoints, called vertices v. The set of vertices of a given γ is indicated by $V(\gamma)$, while the set of edges is $E(\gamma)$.

The collection of the all possible edges $\mathscr{C}$ is a groupoid with composition $[c] \circ [c'] = [c \circ c']$ if $c(1) = c'(0)$, and inverse $[c]^{-1} = [c^{-1}]$. The composition of curves $c \circ c'$ is defined by $c \circ c'(t) = c(2t)$ for $t \in [0, \frac{1}{2}]$ and $c \circ c'(t) = c'(2t - 1)$ for $t \in [\frac{1}{2}, 1]$, while the inverse is $c^{-1}(t) = c(1 - t)$. The holonomy has some remarkable properties with respect to these two operations

1. $h_{c_1 \circ c_2}(A) = h_{c_1}(A) h_{c_2}(A)$,
2. $h_{c^{-1}}(A) = h_c(A)^{-1}$

We can now define the cylindrical functions:

Definition 2.1.2. A function $f : \mathcal{A} \to \mathbb{C}$ is said to be cylindrical over a graph γ, if there exists a function $f_\gamma : SU(2)^{|E(\gamma)|} \to \mathbb{C}$ such that $f = f_\gamma \circ p_\gamma$, where

$$\begin{array}{rccl} p_\gamma : & \mathcal{A} & \to & SU(2)^{|E(\gamma)|}\,; \\ & A & \mapsto & \{h_e(A)\}_{e \in E(\gamma)}. \end{array}$$

We say f is r-times differentiable if f_γ is it with respect to the canonical smooth structure of $SU(2)^{|E(\gamma)|}$. The space of r-times differentiable cylindrical functions is indicated by $Cyl^r(\mathcal{A})$.

The space of smooth cylindrical functions $Cyl^\infty(\mathcal{A})$ is a subspace of the space $C^0_c(\mathcal{A})$ of continuous functions with compact support on $\mathcal{A}$, and it is an algebra over $\mathbb{C}$ with product the pointwise product of functions.
Thanks to the Poisson structure (1.2.9), we can construct vector fields on $\mathcal{P}$ with a non-trivial projection on $\mathcal{A}$ acting like a derivation on $C^1(\mathcal{A})$. These vector fields are generated by the *electric flux*. Given a piecewise analytic surface S in Σ, the electric flux of E on S, smeared by a function $\nu \in C^\infty(\Sigma, \mathfrak{su}(2))$ is

$$Y^\nu_S(E) = \int_S \epsilon_{abc} \nu^i E^a_i dx^b \wedge dx^c. \tag{2.1.2}$$

The action as a derivation of $Y^\nu_S : \mathcal{P} \to \mathbb{C}$ on $Cyl^\infty(\mathcal{A})$ is induced by the Poisson bracket

$$Y^\nu_S[f] \doteq \{Y^\nu_S, f\}. \tag{2.1.3}$$

The collection of fluxes varying S and ν forms a vector subspace of $\mathfrak{X}(\mathcal{A})$. The Poisson bracket with a cylindrical function can be computed explicitly knowing the Poisson bracket with the holonomies, indeed

$$\{Y^\nu_S, f\} = \sum_{e \in E(\gamma)} \frac{\partial f_\gamma(\{h_{e'}\}_{e' \in E(\gamma)})}{\partial (h_e)_{mn}} \{Y^\nu_S, (h_e)_{mn}\}|_{h_e(A)}.$$

For the sake of simplicity, we can consider the graphs over which the cylindrical functions are defined adapted to the surface S. Namely, the graph contains only edges of three kinds: completely outside the surface, completely lies in the surface, and intersects the surface only at its beginning point. An accurate analysis, involving a proper regularization for the δ-function, shows that the Poisson bracket with the holonomy reads as

$$\{Y_S^\nu, h_e\} = \tfrac{\beta\kappa}{4}\epsilon(e,S)\nu(e(0))h_e, \tag{2.1.4}$$

where $\epsilon(e,S)$ takes value 0 if the edge e lies inside or outside the surface, and ± 1 if the orientation of e intersecting S is according or not with the orientation of S. The action of the cylindrical functions can be written simply as

$$Y_S^\nu[f] = \frac{\beta\kappa}{4}\sum_{e\in E(\gamma)}\epsilon(e,S)\nu_i(e(0))R_e^i f_\gamma, \tag{2.1.5}$$

where we introduced the right and left-invariant vector fields on the copy of $SU(2)$ associated with the edge e, acting as a derivation on cylindrical functions via

$$(R_e^i f_\gamma)(\{h_{e'}\}_{e'\in E(\gamma)}) \doteq \frac{d}{dt}\Big|_{t=0} f_\gamma(\{h_{e'}\}_{e'\neq e}, e^{t\tau_i}h_e),$$
$$(L_e^i f_\gamma)(\{h_{e'}\}_{e'\in E(\gamma)}) \doteq \frac{d}{dt}\Big|_{t=0} f_\gamma(\{h_{e'}\}_{e'\neq e}, h_e e^{t\tau_i}).$$

These operators satisfy commutation relations $[R_e^i, R_e^j] = -\epsilon_{ijk}R_e^k$ and $[L_e^i, L_e^j] = \epsilon_{ijk}L_e^k$. We now have all the ingredients to properly define the holonomy-flux algebra.

Definition 2.1.3. The holonomy-flux algebra $\mathfrak{P}$ is a Poisson-Lie *-algebra, realized as the subspace of $C^\infty(\mathcal{A})\times\mathfrak{X}(\mathcal{A})$ given by the pairs (f, Y_S^ν), where $f\in Cyl^\infty(\mathcal{A})$ and Y_S^ν is an electric flux vector field. The involution is just the complex conjugation, and the Poisson-Lie brackets are

$$[(f, Y_S^\nu),(f', Y_{S'}^{\nu'})] = \left(Y_S^\nu[f'] - Y_{S'}^{\nu'}[f], [Y_S^\nu, Y_{S'}^{\nu'}]\right).$$

As the phase space $\mathcal{P}$, this algebra admits an action via automorphisms of the symmetry groups $\mathcal{G}$ and $\mathrm{Diff}(\Sigma)$. In particular, it is possible to combine them in a unique semidirect product group $\mathfrak{G} = \mathcal{G}\rtimes\mathrm{Diff}(\Sigma)$, in which the product of two elements is $(g,\varphi).(g',\varphi') = (g\cdot(g'\circ\varphi), \varphi\circ\varphi')$. The two groups have a left action on the holonomies given by

$$h_e \underset{g}{\to} g(e(0))h_e g(e(1))^{-1}, \quad h_e \underset{\varphi}{\to} h_{\varphi(e)}.$$

The $\mathfrak{G}$-action can be extended to the whole algebra by considering the following action α on the pairs

$$\begin{aligned}\alpha_g(f, Y_S^\nu) &= \left(f_\gamma(\{g(e(0))h_e g(e(1))^{-1}\}_{e\in E(\gamma)}), Y_S^{\mathrm{Ad}_g\nu}\right),\\ \alpha_\varphi(f, Y_S^\nu) &= \left((f_\gamma(\{h_{\varphi(e)}\}_{e\in E(\gamma)}), Y_{\varphi(S)}^{\nu\circ\varphi^{-1}}\right).\end{aligned} \tag{2.1.6}$$

To find the representations of such an algebra, analogously with Quantum Mechanics, we need to introduce the corresponding Weyl algebra, which has the huge advantage that it can be represented by bounded operators on some Hilbert space. We refer to this algebra as the quantum holonomy-flux algebra $\mathfrak{A}$.

Definition 2.1.4. The Weyl elements are defined for each fixed $t \in \mathbb{R}$ as

$$W_t^\nu(S) \doteq \exp(t\hbar Y_S^\nu).$$

The algebra $\mathfrak{A}$ is generated from all elements $f \in Cyl^\infty(\mathcal{A})$ and Weyl elements associated with all possible electric flux vector fields Y_S^ν. The involutive operation is

$$f^* \doteq \bar{f}, \text{ and } (W_t^\nu(S))^* \doteq W_{-t}^\nu(S) = W_t^{-\nu}(S) = (W_t^\nu(S))^{-1}.$$

The Weyl relations follow from the Lie bracket of $\mathfrak{P}$. Explicitly, they are

$$\begin{aligned}
&[f, f'] = 0,\\
&W_t^\nu(S) f (W_t^\nu(S))^{-1} \doteq W_t^\nu(S)[f] = f_\gamma(\{e^{t\frac{\beta\ell_P^2}{4}\nu(e(0))} h_e\}_{e\in E(\gamma)}),\\
&W_t^\nu(S) W_{t'}^{\nu'}(S')(W_t^\nu(S))^{-1} = \exp\left(t'\hbar \sum_{m=0}^{\infty} \frac{(t\hbar)^m}{m!}[Y_S^\nu, Y_{S'}^{\nu'}]_{(m)}\right).
\end{aligned}$$

Here the multiple commutator are defined by recursion $[Y_S^\nu, Y_{S'}^{\nu'}]_{(0)} = Y_{S'}^{\nu'}$ and $[Y_S^\nu, Y_{S'}^{\nu'}]_{(m+1)} = [Y_S^\nu, [Y_S^\nu, Y_{S'}^{\nu'}]_{(m)}]$.

We can now develop the representations of the algebra $\mathfrak{A}$. At the moment, only one representation has been found. It constitutes the auxiliary Hilbert space of Loop Quantum Gravity and is presented in the next section. We conclude this section with the main theorem about the representation of holonomy-flux algebra, the so-called LOST-F theorem [126, 85]

Theorem 2.1.1. *There is a unique, semi-weakly smooth, $\mathfrak{G}$-invariant state on $\mathfrak{A}$. Moreover, the corresponding cyclic GNS representation is irreducible.*

Without delving into the GNS construction (cf. [104], for a comprehensive treatment of these techniques in Quantum Field Theory), this theorem ensures that the action of $\mathfrak{G}$ is implemented unitary on the carrier space $\mathcal{H}_{aux}$ of the representation, and there exists a unique vacuum in $\mathcal{H}_{aux}$ which is invariant under the action of $\mathfrak{G}$. This vacuum is called the Ashtekar-Lewandowski vacuum.

2.2 Spin network Hilbert space

In this Section, we discuss the properties of the Hilbert space of the theory and implement the Dirac's procedure to find the physical states.
We need to introduce the quantum configuration space, a proper generalisation of $\mathcal{A}$, that can be thought of as a distributional extension. The quantum configuration space is the space of generalised connections $\overline{\mathcal{A}} \doteq \mathrm{Hom}(\mathscr{C}, SU(2))$, defined as the set of homomorphisms from the groupoid of the edges in $SU(2)$ [168]. The set of cylindrical functions $Cyl(\overline{\mathcal{A}})$, as in its straightforward generalisation, extends the holonomy-flux algebra on the space of generalised connections.
The space $\overline{\mathcal{A}}$ offers advantages from a functional analysis perspective. Using the projective limit technique [20], it is possible to equip $\overline{\mathcal{A}}$ with a natural topology, resulting in a compact Hausdorff space. On such a space, it is thus possible to define a regular Borel measure μ_0, while on $\mathcal{A}$, since it is a Fréchet space, a linear Borel measure cannot exist.

Definition 2.2.1. The Ashtekar-Lewandowski measure μ_0 on $\overline{\mathcal{A}}$ is the unique regular Borel measure induced via the Riesz-Markov representation theorem by the functional $\Lambda_0 : Cyl(\overline{\mathcal{A}}) \to \mathbb{C}$ defined by

$$\Lambda_0(f) = \int_{SU(2)^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e\in E(\gamma)}). \tag{2.2.1}$$

Where μ_H is the Haar measure on $SU(2)$.

For further details on the Haar measure and harmonic analysis on Lie groups, see Appendix A. Finally, we have a candidate for the Hilbert space of the quantum theory, namely $\mathcal{H}_{aux} = L^2(\overline{\mathcal{A}}, d\mu_0)$. The sets $Cyl^r(\overline{\mathcal{A}})$ are dense in this space.

Theorem 2.2.1. *Consider $\mathcal{H}_{aux} = L^2(\overline{\mathcal{A}}, d\mu_0)$ together with a map $\pi : \mathfrak{P} \to \mathcal{L}(Cyl^1(\overline{\mathcal{A}}))$ given by*

$$\begin{aligned}\pi(f).\psi &\doteq f\psi,\\ \pi(Y_S^\nu).\psi &\doteq i\hbar Y_S^\nu[\psi].\end{aligned}$$

Thus, $(\mathcal{H}_{aux}, \pi)$ is a representation for $\mathfrak{A}$.

We can thereby introduce the spin networks and the spin-network states:

Definition 2.2.2. A spin network is a quadruple

$$s \doteq (\gamma, \{j_e\}_{e\in E(\gamma)}, \{m_e\}_{e\in E(\gamma)}, \{n_e\}_{e\in E(\gamma)})$$

composed by a graph γ, with each edge e labelled with a triple of numbers (j_e, m_e, n_e) where j_e is an half-integer spin quantum number and m_e, n_e are magnetic quantum numbers. Given a representative π_j from each equivalence class of finite-dimensional irreducible unitary representation of $SU(2)$, a spin-network state of s is a function

$$\psi_s : \bar{\mathcal{A}} \to \mathbb{C}; \quad A \mapsto \prod_{e\in E(\gamma)} \sqrt{2j_e+1} D^{(j_e)}_{m_e n_e}(h_e).$$

Where $D^{(j_e)}_{m_e n_e}(h_e)$ are the elements of the D-Wigner matrix associated with the irreducible representation π_{j_e}.

By Peter-Weyl theorem (cf. Appendix A), the spin-network states form an orthonormal basis for $L^2(\overline{\mathcal{A}}, d\mu_0)$. Moreover, as anticipated, on this Hilbert space, the symmetry group $\mathfrak{G}$ acts via a unitary irreducible representation. In particular, we are interested in a slight modification of the original groups. Similarly to the quantum configuration space, we define the generalised gauge group $\overline{\mathcal{G}} \doteq \prod_{x\in\Sigma} SU(2) = SU(2)^\Sigma$, which is the collection of generic maps from Σ to $SU(2)$. This space is compact because of Tychonoff's theorem. While we need to restrict the diffeomorphisms to preserve the semianalytic structure, yielding the semianalytic diffeomorphism group $\mathrm{Diff}^\omega(\Sigma)$. The representation $\hat{U} : \mathfrak{G} \to \mathcal{B}(\mathcal{H}_{aux})$ on spin-network states is derived by the action (2.1.6)

$$\begin{aligned}&\hat{U}(g)\psi_s = \prod_{e\in E(\gamma)} \sqrt{2j_e+1} D^{(j_e)}_{m_e n_e}(g(e(0)) h_e g(e(1))^{-1}),\\ &\hat{U}(\varphi)\psi_s = \psi_{\varphi(s)},\ \varphi(s) \doteq (\varphi(\gamma), \{j_{\varphi(e)} \doteq j_e\}, \{m_{\varphi(e)} \doteq m_e\}, \{n_{\varphi(e)} \doteq n_e\}).\end{aligned} \tag{2.2.2}$$

Notably, the Ashtekar-Lewandowski measure is invariant under $\mathfrak{G}$-action since the functional Λ_0 is it. Indeed, since the Haar measure μ_H is bi-invariant on $SU(2)$, we obtain

$$\begin{aligned}\Lambda_0(\hat{U}(g)f) &= \int_{SU(2)^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(h_e) f_\gamma(\{g(e(0))h_e g(e(1))^{-1}\}_{e\in E(\gamma)})\\ &= \int_{SU(2)^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(g(e(0))^{-1}h_e g(e(1))) f_\gamma(\{h_e\}_{e\in E(\gamma)})\\ &= \int_{SU(2)^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e\in E(\gamma)}).\end{aligned}$$

Then, for the diffeomorphism, it is just a matter of relabelling

$$\begin{aligned}\Lambda_0(\hat{U}(\varphi)f) &= \int_{SU(2)^{|E(\varphi(\gamma))|}} \prod_{e\in E(\varphi(\gamma))} d\mu_H(h_e) f_{\varphi(\gamma)}(\{h_e\}_{e\in E(\varphi(\gamma))})\\ &= \int_{SU(2)^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e\in E(\gamma)}).\end{aligned}$$

This ensures the unitarity of the representation.

We construct now the quantum operators associated with the constraints. Dirac's procedure demands finding the states annihilated by the constraints to define the physical Hilbert space. This procedure has been revised in the operator algebra and functional analysis setting by the Refined Algebraic Quantisation (RAQ) [99, 98]. In RAQ, the constraints admit a common dense domain $\mathcal{D} \subset \mathcal{H}_{aux}$ on which they are self-adjoint. However, not always do the annihilated states live in $\mathcal{D}$; for this reason, we look for distributional solutions, analogously to the eigenvectors of position and momentum in Quantum Mechanics. The mathematical construction is built upon the "generalised" Gel'fand triple, a sequence of set-theoretical and topological inclusion

$$\mathcal{D} \subset \mathcal{H}_{aux} \subset \mathcal{D}'.$$

Where $\mathcal{D}'$ is the algebraic dual of $\mathcal{D}$, namely the space of linear (not necessary continuous) functionals on $\mathcal{D}$ equipped with the topology of pointwise convergence, we used the Riesz isomorphism to identify the Hilbert space $\mathcal{H}_{aux}$ with its continuous dual $\mathcal{H}^*_{aux}$. The solutions to the constraints are now elements in $\mathcal{D}'$. The hypothesis of RAQ is that there exists a "projector" on the solutions called *rigging map*:

Definition 2.2.3. A rigging map is an antilinear map $\eta : \mathcal{D} \to \mathcal{D}'$ such that:

- $\eta(\phi)[\psi] = \overline{\eta(\psi)[\phi]}$ for all $\psi, \phi \in \mathcal{D}$,
- $\eta(\psi)[\psi] \geq 0$ for all $\psi \in \mathcal{D}$,
- for any $\hat{O} \in \mathfrak{O}$, one has $\eta(\hat{O}\psi)[\phi] = \eta(\psi)[\hat{O}\phi]$ for all $\psi, \phi \in \mathcal{D}$, where $\mathfrak{O}$ denotes the algebra of observables (i.e. linear operators on $\mathcal{H}_{\text{aux}}$ such that $\mathcal{D} \subset \text{Dom}(\hat{O})$ and $\mathcal{D} \subset \text{Dom}(\hat{O}^\dagger)$) that commute with the constraints.

The rigging map allows us to define the *physical Hilbert space* of the theory as

$$\mathcal{H}_{\text{phys}} \doteq \overline{\mathcal{D}/\ker(\eta)}\,, \tag{2.2.3}$$

where the closure is taken with respect to the norm induced by the positive semidefinite Hermitian form $(\phi, \psi) = \eta(\phi)[\psi]$ on $\mathcal{D}$. Indeed, since the left and right radical of the Hermitian form are both equal to $\ker(\eta)$, the form passes to the quotient, defining an inner product

$$\langle \phi_{\text{phys}} | \psi_{\text{phys}} \rangle_{\mathcal{H}_{\text{phys}}} \doteq \eta(\phi)[\psi], \tag{2.2.4}$$

where $\phi_{\text{phys}}, \psi_{\text{phys}}$ are the equivalence classes of $\phi, \psi \in \mathcal{D}$, respectively. However, this program can be only partially applied in LQG since the rigging map can be constructed explicitly using the group averaging technique [97] only when the symmetry group of the theory is a locally compact Lie group. Thus, we will need an adaptation of this approach.

The quantum Gauß constraint is defined for any fixed $\Lambda \in C^\infty(\Sigma, \mathfrak{su}(2))$ by the Poisson bracket

$$\hat{G}(\Lambda) f = i\hbar\{G(\Lambda), f\}.$$

It is in fact possible to interpret the Gauß constraint as a vector field on $\mathcal{A}$, and then as a derivation on $Cyl^1(\mathcal{A})$ obtaining

$$\{G(\Lambda), f\} = \frac{\beta\kappa}{2} \sum_{e\in E(\gamma)} [\Lambda(e(0))h_e - h_e\Lambda(e(1))]_{mn} \frac{\partial f_\gamma}{\partial (h_e)_{mn}}.$$

Thus, the respective operator is densely defined, self-adjoint on $Cyl^1(\overline{\mathcal{A}})$

$$\hat{G}(\Lambda) f = \frac{i\beta\ell_P^2}{2} \sum_{v\in V(\gamma)} \Lambda_j(v) \left[\sum_{e\in E(\gamma);\, v=e(0)} R_e^j - \sum_{e\in E(\gamma);\, v=e(1)} L_e^j \right] f_\gamma. \tag{2.2.5}$$

It is easy to show that this family of operators satisfies the quantum version of the algebra (1.2.10)

$$[\hat{G}(\Lambda), \hat{G}(\Lambda')] = -\frac{i\beta\ell_P^2}{2} \hat{G}([\Lambda, \Lambda']). \tag{2.2.6}$$

Considering $\mathcal{D} = Cyl^1(\overline{\mathcal{A}})$, in this context, it is possible to apply the group averaging technique and provide a rigging map as

$$\eta : \mathcal{D} \to \mathcal{D}';\ \psi \mapsto \int_{\overline{\mathcal{G}}} d\mu_H(g) \langle \psi, \hat{U}(g)\cdot \rangle.$$

This map has range in $\mathcal{D}$, realizing gauge-invariant states as a finite linear combination of a special class of spin-network states. These spin-network states are the ones invariant under the action of $\overline{\mathcal{G}}$, which can be constructed explicitly.

Definition 2.2.4. We say a linear map is equivariant if it is a linear morphism between two representation spaces of the same group. E.g., given two irreducible representations $(\mathcal{V}^{j_1}, \pi_{j_1})$ and$(\mathcal{V}^{j_2}, \pi_{j_2})$ of $SU(2)$, a linear map $I : \mathcal{V}^{j_1} \to \mathcal{V}^{j_2}$ is equivariant if $I \circ \pi_{j_1} = \pi_{j_2} \circ I$.
An intertwiner is an equivariant map from some representation space $\mathcal{V}$ of a group G into the trivial representation. We indicate the collection of such maps as $\text{Inv}_G(\mathcal{V})$.

We recall that intertwiners do not always exist. For $SU(2)$, given two irreducible representations π_{j_1} and π_{j_2}, $\mathrm{Inv}_{SU(2)}(\mathcal{V}^{j_1} \otimes \mathcal{V}^{j_2}) \neq \emptyset$ if and only if $j_1 = j_2$, while $\mathrm{Inv}_{SU(2)}(\mathcal{V}^{j_1} \otimes \mathcal{V}^{j_2} \otimes \mathcal{V}^{j_3}) \neq \emptyset$ then j_1, j_2, j_3 must satisfy the triangle condition $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$, in this case the intertwiner is a tensor $I^{(j_1 j_2 j_3)}_{m_1 m_2 m_3}$ that can be written in terms of Wigner's $3j$-symbols, characterized by

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} I^{(j_1 j_2 j_3)}_{m_1 m_2 m_3} |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle = |00\rangle.$$

For an arbitrary number of irreducible representations, $I \in \mathrm{Inv}_{SU(2)}(\mathcal{V}^{j_1} \otimes \ldots \otimes \mathcal{V}^{j_k})$ is characterised by the following property with respect to the Wigner's D-matrices

$$\sum_{n_1,\ldots,n_k} I^{(j_1 \cdots j_k)}_{n_1 \ldots n_k} D^{(j_1)}_{n_1 m_1} \ldots D^{(j_k)}_{n_k m_k} = I^{(j_1 \cdots j_k)}_{m_1 \ldots m_k}. \tag{2.2.7}$$

Definition 2.2.5. The gauge-invariant spin network states are finite linear combinations of spin-network states, labelled by a triple $(\gamma, \vec{j}, \vec{I})$. Where γ is a graph, $\vec{j}$ is a collection of spin quantum numbers $\{j_e\}$, one for each edge $e \in E(\gamma)$, and $\vec{I}$ is a collection of intertwiners $\{I_v\}$, one for each vertex $v \in V(\gamma)$, such that $I_v \in \mathrm{Inv}_{SU(2)}(\mathcal{V}^{j_{e_1}} \otimes \ldots \otimes \mathcal{V}^{j_{e_k}})$, where $e_1, \ldots, e_k$ are all the edges with an endpoint in v.

The gauge-invariant spin network states span the subspace of solutions of the Gauß constraints in $\mathcal{H}_{aux}$, which is isomorphic to $L^2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, d\mu_0)$ [30]. In gauge-invariant spin network states, we can not consider 2-valent vertices; indeed, a 2-valent vertex is the endpoint of exactly two edges carrying representations j_1 and j_2, respectively. For the existence of the intertwiner, it must be $j_1 = j_2$, and the two edges can be considered as part of the same edge carrying the representation $j_1 = j_2$.

The Diffeomorphism constraint needs a different approach. Indeed, it is not possible to quantise directly the constraint since its action is not well-defined on cylindrical functions. The infinitesimal transformation of the holonomy under diffeomorphisms φ_t generated by a vector field $X \in \mathfrak{X}(\Sigma)$ is

$$\left.\frac{d}{dt}\right|_{t=0} h_c(\varphi_t^* A) = \int_0^1 ds\, h_{c([0,s])}(A) \mathcal{L}_X A\, h_{c([s,1])}(A). \tag{2.2.8}$$

This action does not provide a cylindrical function; moreover cannot be generalised to $A \in \overline{\mathcal{A}}$ since $\mathcal{L}_X A$ is not defined in that case. This is a consequence of the fact that the representation of $\mathrm{Diff}^\omega(\Sigma)$ in Eq.(2.2.2) is weakly discontinuous, hence does not admit infinitesimal generators. We then study the finite action of the group, noticing that, by construction, the diffeomorphism quantum constraint algebra is free of anomalies

$$\hat{U}(\varphi)\hat{U}(\varphi')\hat{U}(\varphi^{-1})\hat{U}(\varphi'^{-1}) = \hat{U}(\varphi \circ \varphi' \circ \varphi^{-1} \circ \varphi'^{-1})$$

In this case, defining a rigging map is still possible, but it requires a more subtle analysis. We are looking for the algebraic distribution $l \in \mathcal{D}'$ such that

$$l(\hat{U}(\varphi) f) = l(f), \quad \forall \varphi \in \mathrm{Diff}^\omega(\Sigma),\ f \in Cyl^\infty(\overline{\mathcal{A}}) \tag{2.2.9}$$

Since every element of $\mathcal{D}'$ can be written as a infinite sum $l = \sum_s c_s \langle \psi_s, \cdot \rangle$, it is clear that $c_\varphi(s) = c_s$. Introducing the orbits $[s] \doteq \{\varphi(s) \,|\, \varphi \in \mathrm{Diff}^\omega(\Sigma)\}$, we expect the rigging map satisfies, for some positive real number $c_{[s]}$,

$$\eta(\psi_s) = c_{[s]} l_{[s]}, \text{ where } l_{[s]} \doteq \sum_{s \in [s]} \langle \psi_s, \cdot \rangle.$$

Since $\mathrm{Diff}^\omega(\Sigma)$ is a Fréchet Lie group, it does not admit a Haar measure. However, we can consider it equipped with the discrete topology, so then the counting measure is trivially bi-invariant. Unfortunately, the map

$$\psi_s \mapsto \sum_{\varphi \in \mathrm{Diff}^\omega(\Sigma)} \langle \hat{U}(\varphi)\psi_s, \cdot \rangle$$

does not define an element of $\mathcal{D}'$ because the sum involves the same element uncountably many times. That is, there are uncountably many different diffeomorphisms that leave invariant the graph γ, and so the spin network s. Nevertheless, we notice that the action $\langle \hat{U}(\varphi)\psi_s, \psi_{s'} \rangle = 0$ for all $\varphi \in \mathrm{Diff}^\omega(\Sigma)$ if $[s] \neq [s']$. Thus, it is possible to decompose the space $\mathcal{H}_{aux}$ into sectors related to $[s]$. Specifically, studying the strongly diffeomorphism-invariant observables, namely operators $\hat{O}$ that satisfy $\hat{U}(\varphi)\hat{O}\hat{U}(\varphi^{-1}) = \hat{O}$ for all φ, the Hilbert space $\mathcal{H}_{aux}$ splits into mutually orthogonal superselection sectors labelled by knot classes of graphs $[\gamma]$ (i.e., graphs whose ranges are related by diffeomorphisms). So that, picking a representative $\tilde{\gamma}$ for each knot class $[\gamma]$ and choosing for each γ a diffeomorphism φ_γ such that $\varphi_\gamma(\tilde{\gamma}) = \gamma$, we define the rigging map applied on the spin-network states as

$$\eta(\psi_s) \doteq c_{[\gamma(s)]} \sum_{\gamma \in [\gamma(s)]} \sum_{\sigma \in \mathfrak{S}_{\tilde{\gamma}(s)}} \langle \hat{U}(\varphi_\gamma)\hat{U}(\varphi_\sigma)\hat{U}(\varphi^{-1}_{\gamma(s)})\psi_s, \cdot \rangle \,. \tag{2.2.10}$$

Here, $\mathfrak{S}_\gamma$ is the group of graph symmetries [23] defined as the quotient of the isotropy group $\mathrm{IDiff}^\omega_\gamma(\Sigma)$ of γ by the group of diffeomorphisms $\mathrm{TDiff}^\omega_\gamma(\Sigma)$ that act trivially on γ. Every element σ in such a group is associated to a class of diffeomorphisms $\varphi_\sigma \in \mathrm{IDiff}^\omega_\gamma(\Sigma)/\mathrm{TDiff}^\omega_\gamma(\Sigma)$, which have the same action on γ. The coefficients $c_{[\gamma]}$ are arbitrary positive real numbers. They may be constrained by further requests on graph-changing strongly diffeomorphism-invariant observable, yet yielding huge ambiguity in developing the RAQ program.

Setting aside the ambiguity in the rigging map, we can construct the Hilbert spaces of invariant states under gauge and diffeomorphism transformations. We usually refer to this space as the kinematical Hilbert space $\mathcal{H}_{kin}$, and to the related constraints as kinematical constraints. The basis of $\mathcal{H}_{kin}$ is labelled by equivalent classes of gauge-invariant spin network $[s]$ related by diffeomorphism, often called *s-knots*. The ambiguity is reflected in the possible inequivalent choices of a scalar product on this space.

2.3 Quantum geometry and quantum dynamics

The main result of Loop Quantum Gravity at the kinematical level is the quantisation of geometry. This is made clear by the geometric operators of area and volume, which can be well-defined in the theory and present discrete spectra [147, 21, 22]. For the sake of completeness, we briefly present the construction and the spectrum of the area operator, which represents the main result of the LQG program so far.

Let us consider a semianalytic oriented surface $S \subset \Sigma$ given by the embedding $\iota : \varsigma \hookrightarrow \Sigma$ of a compact oriented 2-dimensional manifold ς. Given a Riemannian metric q on Σ, we can easily compute the area as S as

$$\mathrm{Ar}(S) = \int_\varsigma dV_{\iota^* q} = \int_S n_S \lrcorner\, dV_q,$$

where dV_q is the volume form of the Riemannian metric q, n_S is the normal vector field to S, and $\lrcorner$ is the interior product [124]. Given a set of coordinates y^1, y^2 on ς, we can describe classically the area in terms of the electric field

$$\mathrm{Ar}(S) = \int_\varsigma d^2y \sqrt{\delta^{ij} n_a^S(\iota(y)) n_b^S(\iota(y)) E_i^a(\iota(y)) E_j^b(\iota(y))} \tag{2.3.1}$$

Consider now a partition $\mathscr{U}$ of ς by closed subsets U with open interior. The idea is to approximate the integral (2.3.1) by the Riemann sum

$$\mathrm{Ar}(S) = \sum_{U \in \mathscr{U}} \sqrt{\delta^{ij} E_i(\iota(U)) E_j(\iota(U))},$$

and then consider the limit through finer and finer partitions. Here, $E_i(\iota(U))$ is the electric flux on the surface $\iota(U)$ as defined in (2.1.2) with $\nu^j = \delta_i^j$. Since the electric flux is a well-defined self-adjoint operator on $Cyl^1(\overline{\mathcal{A}})$, the square root is meaningful in the sense of functional calculus. Let us now consider γ adapted to the surface S, and let $P_{\gamma,S}$ be the collection of isolated points of intersection between edges of γ and S. For a refined-enough partition, every $\iota(U)$ contains at most one of such points. Hence, for such partitions, the area operator depends only on the intersections, and the limit becomes trivial

$$\widehat{\mathrm{Ar}}(S) f = \frac{\beta \ell_P^2}{4} \sum_{x \in P_{\gamma,S}} \sqrt{-\sum_{i=1}^3 \left(\sum_{e \in E(\gamma),\, x \in \partial e} \epsilon(e,S) R_e^i \right)^2} f_\gamma \ . \tag{2.3.2}$$

The operator so constructed is a positive semidefinite, essential self-adjoint operator on $Cyl^2(\overline{\mathcal{A}})$. We end up with a well-defined geometric operator which encodes topological information of S (since it depends only on the number of intersection points with the graph of the quantum state). Furthermore, we can claim that we have successfully quantised the geometry (at least at the kinematical level) because the spectrum of this operator is purely discrete, and independent of S

$$\sigma\left(\widehat{\mathrm{Ar}}(S)\right) = \left\{ \begin{matrix} \frac{\beta \ell_P^2}{4} \sqrt{2j_1(j_1+1) + 2j_2(j_2+1) - j_3(j_3+1)} \\ \text{s.t. } j_1, j_2, j_3 \in \frac{1}{2}\mathbb{N},\ |j_1 - j_2| \le j_3 \le j_1 + j_2 \end{matrix} \right\}. \tag{2.3.3}$$

From the spectrum, we can clearly see that the theory yields an *area gap*. The smallest non-zero eigenvalue is obtained for $j_1 = j_3 = \frac{1}{2}$ and $j_2 = 0$

$$\lambda_0 = \frac{\sqrt{3} \beta \ell_P^2}{8}. \tag{2.3.4}$$

The area gap is order of Planck length square, as one expects from naive consideration about quantum effects in gravity, and explicitly depends on the Barbero-Immirzi parameter, making evident the reason to consider positive real values of the parameter. For the volume operator, one can follow a similar route. Starting from the classical expression of the volume for a compact 3-dimensional submanifold $R \subset \Sigma$

$$\mathrm{Vol}(R) = \int_R dV_q = \int_R d^3x \sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}E^a_i E^b_j E^c_k}. \tag{2.3.5}$$

One can proceed by quantising fluxes to arrive at the expression

$$\widehat{\mathrm{Vol}}(R) f = \ell_P^3 \sum_{v \in V(\gamma)\cap R} \sqrt{\left| \frac{i}{3!\cdot 8} \sum_{\substack{e,e',e'' \in E(\gamma),\\ e\cap e'\cap e'' = v}} \epsilon(e,e',e'')\epsilon_{ijk} R^i_e R^j_{e'} R^k_{e''} \right|} f_\gamma. \tag{2.3.6}$$

This is a densely defined, positive semidefinite, and essential self-adjoint operator on $Cyl^3(\overline{\mathcal{A}}/\overline{\mathcal{G}})$. It can not be diagonalised in a closed form, but it can be shown that its matrix elements depend explicitly on the Wigner's $3j$-symbols, namely on the intertwiners of the considered state. This operator will play a major role in the construction of the Hamiltonian operator.

The Hamiltonian operator is the quantisation of the Scalar constraint. Finding the kernel of the Hamiltonian operator is the main task of the Canonical Quantisation program of General Relativity, since it defines the physical Hilbert space of the theory. It is a particularly complicated task since the Hamiltonian constraint is tremendously non-linear (actually non-polynomial), and needs regularisation, which is a source of ambiguities. We are going to present an implementation of the Hamiltonian operator called Thiemann's regularised Hamiltonian constraint operator [163, 164], in which it is defined as a graph-changing operator on $\mathcal{H}_{aux}$ regularised via the introduction of a natural triangulation of the manifold Σ. This triangulation should not be regarded as a cut-off or a discretisation of the theory, since it depends on the quantum states themselves, which are constructed from a continuum theory, as previously shown. To support this interpretation, one can introduce a control parameter ε for the regularisation and demonstrate that, in the limit $\varepsilon \to 0$, Thiemann's regularised Hamiltonian constraint operator is well-defined. In fact, due to the diffeomorphism invariance of the states in $\mathcal{H}_{kin}$, the operator ultimately does not depend on ε.
The construction of the Hamiltonian operator starts from the classical expression of the volume V for the whole manifold Σ. In case Σ is open, we can define the integral via a limit over an exhaustion by compact sets $R_n \subset \Sigma$, namely $V \doteq \lim_{n\to\infty} \mathrm{Vol}(R_n)$. This starting point comes from noticing the following key identity

$$\frac{\beta\kappa}{2}\frac{\epsilon^{ijk}E^a_i(x)E^b_j(x)}{\sqrt{\det(q)}} = 2\epsilon^{abc}\{V, A^i_c(x)\}, \tag{2.3.7}$$

that allows us to hide the non-polynomiality introduced by the factor $1/\sqrt{\det(q)}$ into the Poisson bracket. Moreover, looking forward to the quantisation, the volume is a well-defined operator, while the Poisson bracket will be replaced by the commutator. Following [166], we quantise a multiple of the scalar constraint $H = \frac{1}{\beta k}S = \frac{\beta^2+1}{\beta^2}H_L -$

H_E, which is, for convenience, divided in two parts: an Euclidean part H_E, that is the Hamiltonian constraint for Euclidean General Relativity, and a Lorentzian part H_L. The Euclidean constraint can be written as

$$H_E \doteq -\frac{1}{\beta\kappa\sqrt{\det(q)}} F^i_{ab}\epsilon^{ijk}E^a_i E^b_j = \left(\frac{2}{\beta\kappa}\right)^2 \epsilon^{abc}F^i_{ab}\{A^i_c, V\}. \tag{2.3.8}$$

The remaining part of the Scalar constraint, namely the Lorentzian part, involves terms in the extrinsic curvature $\beta K^i_a = A^i_a - \Gamma^i_a$. To enable the definition of a proper quantum operator, we need to integrate its trace on Σ

$$K \doteq \int_\Sigma d^3x\, K^i_a E^a_i. \tag{2.3.9}$$

Noticing that $\frac{\beta\kappa}{2}K^i_a = \{K, A^i_a\}$, the Lorentzian part is written as

$$H_L \doteq -\epsilon_{ilm}K^l_a K^m_b \frac{\epsilon^{ijk}E^a_j E^b_k}{\beta\kappa\sqrt{\det(q)}} = \left(\frac{2}{\beta\kappa}\right)^4 \epsilon_{ijk}\epsilon^{abc}\{A^i_a, K\}\{A^j_b, K\}\{A^k_c, V\}. \tag{2.3.10}$$

In a more compact form, recalling that in the fundamental representation $\mathrm{Tr}(\tau_i\tau_j) = -\frac{1}{2}\delta_{ij}$ and $\mathrm{Tr}(\tau_i\tau_j\tau_k) = -\frac{1}{4}\epsilon_{ijk}$, we can write the smeared Hamiltonian constraint concisely as

$$\begin{aligned} H_L(N) &= -4\int_\Sigma \left(\frac{2}{\beta\kappa}\right)^4 N\,\mathrm{Tr}(\{A, K\}\wedge\{A, K\}\wedge\{A, V\}), \\ H_E(N) &= -2\int_\Sigma \left(\frac{2}{\beta\kappa}\right)^2 N\,\mathrm{Tr}(F\wedge\{A, V\}). \end{aligned} \tag{2.3.11}$$

The reason for the splitting also has a procedural sense. We can recognise in the integrated trace of the extrinsic curvature the time derivative of the total volume

$$K = -\{H_E(1), V\} = \{H(1), V\}.$$

Then, once one has a well-defined Euclidean Hamiltonian constraint operator $\hat{H}_E(N)$, the operator $\hat{K}$ reads as

$$\hat{K} \doteq \frac{i}{\hbar}[\hat{H}_E(1), \hat{V}], \tag{2.3.12}$$

and consequently define the Lorentzian part. Before proceeding to the quantisation, we must take care of the regularisation. Indeed, the connection A does not have a corresponding operator in the quantum theory, hence we need to regularise it in terms of holonomy. The idea is that the holonomy around an infinitesimal loop is the curvature F at the first order, while the connection 1-form approximates the parallel transport along an infinitesimal curve. The regularisation then takes the form of a "triangle with a nose". Consider a tetrahedron Δ embedded in Σ with three analytic edges $s_n(\Delta)$ meeting at the vertex $v(\Delta)$, the loop $\alpha_{nm}(\Delta)$ is constructed considering the edge $a_{nm}(\Delta)$ connecting the two other endpoints of $s_n(\Delta)$ and $s_m(\Delta)$, i.e. $\alpha_{nm}(\Delta) = s_n(\Delta)\circ a_{nm}(\Delta)\circ s^{-1}_m(\Delta)$, and we are going to consider the holonomy along $\alpha_{nm}\circ s_l\circ s^{-1}_l$. Then, our building block for the Hamiltonian operator will be

$$\begin{aligned} H^E_\Delta(N) &\doteq \frac{2}{3}\left(\frac{2}{\beta\kappa}\right)^2 N(v(\Delta))\epsilon^{lmn}\,\mathrm{Tr}(h_{\alpha_{lm}(\Delta)}h_{s_n(\Delta)}\{h^{-1}_{s_n(\Delta)}, V\}), \\ H^L_\Delta(N) &\doteq \frac{4}{3}\left(\frac{2}{\beta\kappa}\right)^4 N(v)\epsilon^{lmn}\,\mathrm{Tr}(h_{s_l}\{h^{-1}_{s_l}, K\}h_{s_m}\{h^{-1}_{s_m}, K\}h_{s_n}\{h^{-1}_{s_n}, V\}) \end{aligned} \tag{2.3.13}$$

which now can be implemented as an operator on $\mathcal{H}_{aux}$. Moreover, it has the correct limit for shrinking the loop, and it is intrinsically gauge invariant since the path $\alpha_{nm} \circ s_l \circ s_l^{-1}$ is closed. The idea for the quantisation is that, given a cylindrical function f_γ, we can construct such a Δ for each vertex $v \in V(\gamma)$. We can construct a triangulation for the manifold Σ, however, not every tetrahedron contributes to the dynamics, hence we limited ourselves to present only the relevant part. The tetrahedron is constructed starting from a vertex $v \in V(\gamma)$ and an unordered triple of distinguished edges (e_l, e_m, e_n) in the set of all the edges intersecting at v. Let us choose three segments s_l, s_m, s_n each along the corresponding edge e_l, e_m, e_n small enough, and choose three semianalytic path a_{lm}, a_{mn}, a_{nl} that connect the endpoints of the segments and do not intersect any edge of γ. Incredibly, this construction is invariant under semianalytic diffeomorphisms. Indeed, a change in the segments, namely choosing a different endpoint along the edges, is related to a diffeomorphism thanks to the following lemma

Lemma 2.3.1. *Let s_1, s_2 be two distinct analytic curves in a coordinate chart on a 3-dimensional manifold, which intersect each other only in their starting point v. There exist parametrisations of these curves, a number $\delta > 0$, and an analytic diffeomorphism such that in the corresponding frame the curves are given by*

- $s_1(t) = (t, 0, 0)$, $s_2(t) = (0, t, 0)$, $t \in [0, \delta]$ *if their tangents are linearly independent at v.*
- $s_1(t) = (t, 0, 0)$, $s_1(t) = (t, t^r, 0)$, $t \in [0, \delta]$ *for some $r \geq 2$ if their tangents are co-linear at v.*

Moreover, the choice of the arcs is labelled by the topology of the routing and three real numbers, both diffeomorphism-invariant quantities (for details [163]). Thus, for each vertex $v \in V(\gamma)$, we construct a number $E(v)$ of "relevant" tetrahedra, where $E(v)$ depends on the valence $n(v)$ of the vertex and it is just the number of all possible unordered triple $E(v) = \frac{n(v)!}{(n(v)-3)!3!}$. Finally, the action of the Euclidean Hamiltonian operator over a cylindrical function f reads as

$$\hat{H}_E(N)^\dagger f = \frac{32}{3i\kappa\beta^2\ell_P^2} \sum_{v\in V(\gamma)} \frac{N(v)}{E(v)} \sum_{\Delta,\, v(\Delta)=v} \epsilon^{lmn}\, \mathrm{Tr}(h_{\alpha_{lm}(\Delta)} h_{s_n(\Delta)} [h^{-1}_{s_n(\Delta)}, \hat{V}])\, f_\gamma. \tag{2.3.14}$$

This operator is well-defined on the finite span of spin-network states, so it is densely defined and is closable. Hence, the closure $\mathrm{cl}(\hat{H}_E(N)) = (\hat{H}_E(N)^\dagger)^\dagger$ is also densely defined, and its domain contains the spin-network states.
The Lorentzian part admits a similar treatment. Since both $\hat{H}_E(1)$ and $\hat{V}$ have domain and range in $Cyl^3(\overline{\mathcal{A}/\mathcal{G}})$, the operator $\hat{K}$ defined as in (2.3.12), has also dense domain and range in $Cyl^3(\overline{\mathcal{A}/\mathcal{G}})$. The Lorentzian Hamiltonian operator reads as

$$\begin{aligned}\hat{H}_L(N)^\dagger f =& \frac{64}{3\beta\kappa(i\beta\ell_P^2)^3} \sum_{v\in V(\gamma)} \frac{N(v)}{E(v)} \sum_{\Delta,\, v(\Delta)=v} \epsilon^{lmn}\, \mathrm{Tr}(h_{s_l(\Delta)} [h^{-1}_{s_l(\Delta)}, \hat{K}] h_{s_m(\Delta)} \\ &\times [h^{-1}_{s_m(\Delta)}, \hat{K}] h_{s_n(\Delta)} [h^{-1}_{s_n(\Delta)}, \hat{V}]) f_\gamma\ .\end{aligned} \tag{2.3.15}$$

Finally, we have the Hamiltonian operator $H(N)^\dagger = \frac{\beta^2+1}{\beta^2} H_L(N)^\dagger - H_E(N)^\dagger$. Accordingly to Refined Algebraic Quantisation program, the solutions of the Hamiltonian constraint are diffeomorphism-invariant distribution on $Cyl^\infty(\overline{\mathcal{A}}/\overline{\mathcal{G}})$, namely elements l of a dense subset $\mathcal{D}_{\text{diff}}$ of $\mathcal{H}_{kin}$ such that, for all smearing functions N

$$l(\hat{H}(N)^\dagger f) = 0, \ \forall f \in Cyl^\infty(\overline{\mathcal{A}}/\overline{\mathcal{G}}). \tag{2.3.16}$$

Noticing that we can rewrite the Hamiltonian constraint acting on a cylindrical function on a graph γ as $\hat{H}(N)^\dagger = \sum_v N_v \hat{H}_v^\dagger$. From which, we can find an equation equivalent to (2.3.16), that is, a pointwise implementation of the Hamiltonian constraint, namely

$$l(\hat{H}_v^\dagger f_\gamma) = 0, \ \forall v \in V(\gamma).$$

We can induce an appropriate topology on $\mathcal{H}_{aux}$ considering the weak* topology with respect to $\mathcal{D}_{\text{diff}}$. With this topology, it is possible to recover the constraints algebra. Despite the tetrahedra $\Delta_{\varphi(\gamma)}$ constructed by the transformed graph $\varphi(\gamma)$ is not, in general, equal to the transformed tetrahedra $\varphi(\Delta_\gamma)$, thanks to the diffeomorphism-invariant prescription to construct the arcs, the two are always linked by a semi-analytic diffeomorphism φ' such that $\varphi'(\varphi(\gamma)) = \varphi(\gamma)$ and $\varphi'(\Delta_{\varphi(\gamma)}) = \varphi(\Delta_\gamma)$. It follows that

$$\hat{U}(\varphi')\hat{H}(\varphi^* N)^\dagger \hat{U}(\varphi')^{-1} f_\gamma = \hat{U}(\varphi)\hat{H}(N)^\dagger \hat{U}(\varphi)^{-1} f_\gamma \ . \tag{2.3.17}$$

Thus, on the diffeomorphism-invariant distribution, we recover the correct covariance for the Hamiltonian operator

$$l(\hat{U}(\varphi)\hat{H}(N)^\dagger \hat{U}(\varphi)^{-1} \ f) = l(\hat{H}(\varphi^* N)^\dagger \ f), \ \forall f \in Cyl^\infty(\overline{\mathcal{A}}/\overline{\mathcal{G}}). \tag{2.3.18}$$

Hence, $\hat{U}(\varphi)\hat{H}(N)^\dagger \hat{U}(\varphi)^{-1} = \hat{H}(\varphi^* N)^\dagger$ is an exact operator identity in the topology described before. Furthermore, the complete Hamiltonian operator is anomaly-free in the given topology. Namely, $\hat{H}(N)^\dagger$ and $\hat{H}(M)^\dagger$ commute up to a diffeomorphism. From (2.3.14) and (2.3.15), we can write the Hamiltonian operator acting on a cylindrical function as

$$\hat{H}(N)^\dagger f_\gamma = \sum_{v \in V(\gamma)} N_v \sum_{\Delta,\, v(\Delta)=v} \hat{H}_\Delta^\dagger \, f_\gamma.$$

From which, the commutator reads

$$\begin{aligned}
&[\hat{H}(N)^\dagger, \hat{H}(M)^\dagger] f_\gamma \\
&= \sum_{v \in V(\gamma)} (M_v \hat{H}(N)^\dagger - N_v \hat{H}(M)^\dagger) \sum_{v(\Delta_\gamma)=v} \hat{H}_\Delta^\dagger \, f_\gamma \\
&= \sum_{v \in V(\gamma)} \sum_{v(\Delta_\gamma)=v} \sum_{v' \in V(\gamma \cup \Delta_\gamma)} (M_v N_{v'} - N_v M_{v'}) \sum_{v(\Delta'_{\gamma\cup\Delta_\gamma})=v'} \hat{H}^\dagger_{\Delta'_{\gamma\cup\Delta_\gamma}} \hat{H}^\dagger_{\Delta_\gamma} \, f_\gamma \\
&= \sum_{v \neq v' \in V(\gamma)} (N_{v'} M_v - N_v M_{v'}) \sum_{v(\Delta_\gamma)=v} \sum_{v(\Delta'_{\gamma\cup\Delta_\gamma})=v'} \hat{H}^\dagger_{\Delta'_{\gamma\cup\Delta_\gamma}} \hat{H}^\dagger_{\Delta_\gamma} \, f_\gamma \qquad (2.3.19) \\
&= \frac{1}{2} \sum_{v \neq v' \in V(\gamma)} (N_{v'} M_v - N_v M_{v'}) \sum_{\substack{v(\Delta_\gamma)=v,\, v(\Delta'_{\gamma\cup\Delta_\gamma})=v' \\ v(\Delta'_\gamma)=v',\, v(\Delta_{\gamma\cup\Delta'_\gamma})=v}} \left(\hat{H}^\dagger_{\Delta'_{\gamma\cup\Delta_\gamma}} \hat{H}^\dagger_{\Delta_\gamma} - \hat{H}^\dagger_{\Delta_{\gamma\cup\Delta'_\gamma}} \hat{H}^\dagger_{\Delta'_\gamma} \right) f_\gamma
\end{aligned}$$

Where we used the fact that the vertices $v \in V(\gamma \cup \Delta_\gamma)\backslash V(\gamma)$ are planar, then they are annihilated by the volume operator and do not contribute to the expression. The crucial point is that the prescription to attach the arcs near the vertex v and then v' is not the same as doing the opposite; then, the two terms in the reason of the summation are not the same. However, they are linked by a semianalytic diffeomorphism. In particular, there exists, for each vertex $v \in V(\gamma)$, a diffeomorphism $\varphi \in \mathrm{Diff}^\omega(\Sigma)$ that leaves invariant the graph γ in a neighbourhood of v but such that $\varphi(\Delta_{\gamma\cup\Delta'_\gamma}) = \Delta_\gamma$. Hence, $\hat{H}^\dagger_{\Delta_{\gamma\cup\Delta'_\gamma}}\hat{H}^\dagger_{\Delta'_\gamma} = \hat{U}(\varphi)\hat{H}^\dagger_{\Delta_\gamma}\hat{H}^\dagger_{\Delta'_\gamma}$, from which, on diffeomorphism-invariant distributions $l \in \mathcal{D}_{\text{diff}}$

$$\begin{aligned} & l\left((\hat{H}^\dagger_{\Delta'_{\gamma\cup\Delta_\gamma}}\hat{H}^\dagger_{\Delta_\gamma} - \hat{H}^\dagger_{\Delta_{\gamma\cup\Delta'_\gamma}}\hat{H}^\dagger_{\Delta'_\gamma})f_\gamma\right) \\ & = l\left((\hat{U}(\varphi')\hat{H}^\dagger_{\Delta'_\gamma}\hat{H}^\dagger_{\Delta_\gamma} - \hat{U}(\varphi)\hat{H}^\dagger_{\Delta_\gamma}\hat{H}^\dagger_{\Delta'_\gamma})f_\gamma\right) \\ & = l\left([\hat{H}^\dagger_{\Delta'_\gamma}, \hat{H}^\dagger_{\Delta_\gamma}]f_\gamma\right) \end{aligned} \tag{2.3.20}$$

Since $[\hat{H}^\dagger_{\Delta'_\gamma}, \hat{H}^\dagger_{\Delta_\gamma}] = 0$ when $v \neq v'$, then the commutator vanishes in the weak* topology with respect to $\mathcal{D}_{\text{diff}}$

$$[\hat{H}(N)^\dagger, \hat{H}(M)^\dagger] = 0. \tag{2.3.21}$$

Thus, it provides a quantum constraints algebra free of anomalies.
Looking for the solutions to the Hamiltonian constraint is not an easy task. However, we can recast the functional equation (2.3.16) in a finite system of linear equations for the coefficients of the states, enabling us to approach the problem with tools from linear algebra [58, 91]. The action of the Hamiltonian constraint, roughly, is the one of a fourth-order polynomial consisting of creation and annihilation operators, which spins associated to edge are created or annihilated modifying the previous spin by $\Delta j = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. When the spin becomes 0, the edge disappears; conversely, the Hamiltonian can generate new edges with spin $\pm\frac{1}{2}$. The new edges, related to the arcs, have a peculiar behaviour, being so-called *extraordinary edges*. Identifying the set $\mathcal{S}_0$ of spin networks with graphs without extraordinary edges, we can generate iteratively generic spin-network states starting from a given $s_0 \in \mathcal{S}_0$ and considering the action of the Hamiltonian operator. More precisely, fixing $\mathcal{S}_0(s_0) \doteq \{s_0\}$, $\mathcal{S}_{n+1}(s_0)$ is defined as the spin network that appear in the decomposition of $\hat{H}(N)^\dagger\psi_s$ for all lapse functions N and spin network $s \in \mathcal{S}_n(s_0)$; in such a way, for each spin network s there is a unique integer n and a unique $s_0 \in \mathcal{S}_0$ such that $s \in \mathcal{S}_n(s_0)$. Moreover, $\mathcal{S}_n(s_0)$ and $\mathcal{S}_{n'}(s'_0)$ are disjoint unless $s_0 = s'_0$ and $n = n'$
We recall that the diffeomorphism-invariant states decompose into distributions labelled by s-knot $l_{[s]} \in \mathcal{H}_{kin}$. Thus, giving a $s_0 \in \mathcal{S}_0$ and a finite collection of natural number n_k, the Ansatz for the solution is

$$\Psi_{[s_0],\vec{n}} = \sum_{k=1}^{N}\sum_{[s]\in\mathcal{S}_{n_k}(s_0)} c_{[s]}l_{[s]}, \tag{2.3.22}$$

where the complex coefficients $c_{[s]}$ must be determined by the constraint equation (2.3.16)

$$\Psi_{[s_0],\vec{n}}(\hat{H}(N)^\dagger\psi_s) = 0 \tag{2.3.23}$$

for all lapse function N and spin-network states ψ_s. Using the decomposition of the Hamiltonian operator in terms of the vertices of the graph of the spin network s, we can find a set of equivalent equations

$$\sum_{[s']\in\mathcal{S}_{n_k}(s_0)} c_{[s']} l_{[s']}(\hat{H}_v^\dagger \psi_s) = 0, \tag{2.3.24}$$

for each vertex $v \in V(\gamma(s))$. Even if we can deal with it via techniques of finite-dimensional linear algebra, it is not easy to find solutions to this equation. Recently, computational and numerical approaches seem to offer new possibilities to overcome this problem [153, 101].
The equation (2.3.16) represents the quantum analogue of the Einstein Field Equations, thereby providing rigorously defined solutions to Hamiltonian constraint of full, four-dimensional Lorentzian quantum General Relativity in the continuum. Although a complete characterisation of non-trivial solutions to the quantum Einstein equation remains unavailable, this result is nonetheless highly ambitious, particularly given that even in classical General Relativity the full space of solutions is not yet completely understood.

Chapter 3

Loop Quantum Cosmology

Cosmology concerns the evolution of the Universe and its large-scale structure. From the point of view of this thesis, Cosmology is the implementation of the *cosmological principle* in the framework of General Relativity.

Cosmological Principle*: At large scale, the Universe is spatially homogeneous and isotropic.*

Cosmology provides a natural testing ground for quantum gravity theories. The cosmological assumption of homogeneity simplifies the classical model by reducing the configuration space to a finite number of degrees of freedom, often referred to as the *minisuperspace*. Applying LQG techniques to cosmology has led to the development of Loop Quantum Cosmology [27]. In LQC, the quantum dynamics can be solved, leading to intriguing phenomena such as the Big Bounce [46, 24, 25, 26]. However, the reliability of LQC has been subject to debate [69, 54]. Indeed, the symmetry restriction imposed by homogeneity in LQC prevents the preservation of the $SU(2)$ gauge symmetry in both classical and quantum formulations. Therefore, the link between LQC and LQG remains an open problem, as does the broader issue of defining a cosmological sector within LQG [63]. The challenge of preserving the $SU(2)$ gauge symmetry in LQC is closely related to the symmetry-reduction problem in gauge theories [167]. Typically, symmetry reduction breaks the gauge symmetries of the field theory. While this approach can simplify computations in the classical theory, it does not yield an equivalent formulation of the theory at the quantum level. Analysing this connection is a crucial step toward demonstrating that the phenomena predicted by LQC are indeed robust predictions of LQG and comparing LQG with cosmological observations [14].

3.1 Symmetry reduction and minisuperspace

For the implementation of the homogeneity in General Relativity, we will follow the description made by L.D. Landau and E.M. Lifshitz [122]. We consider a globally hyperbolic spacetime $M \cong \mathbb{R} \times \Sigma$ and the corresponding ADM variables (q, k) at a given fixed time. Homogeneity implies identical metric properties at all points of the space. The space is homogeneous if it admits a group of transformations that can bring every point to any else, and leaves invariant the line element and the functional dependence of the metric on the point. Namely, for every transformation that sends $x \mapsto x'$, then

$$q_{ab}(x^1, x^2, x^3)dx^a dx^b = q_{ab}(x'^1, x'^2, x'^3)dx'^a dx'^b.$$

This definition coincides with the mathematical definition of Riemannian homogeneous space

Definition 3.1.1. A Riemannian homogeneous space is a pair (Σ, G) composed of a Riemannian manifold Σ and a Lie group G that acts transitively on Σ via isometries.

This means that there exists a left action of G on Σ such that, for each two points $x, y \in \Sigma$, there exists an element $g \in G$ such that $y = g.x$. Moreover, the image of the immersion of this group into $\mathrm{Diff}(\Sigma)$, called $L : G \to \mathrm{Diff}(\Sigma)$, is completely contained in the isometry group. Notice that the choice of g is not unique; this redundancy is encoded in the stabiliser of a point $H_x \doteq \{g \in G \mid g.x = x\}$. On a homogeneous space, the stabilisers of different points are all isomorphic to one another, and so the space is characterised by a unique stabiliser called the isotropy group H.
Generically, the group of transformations leaves invariant three independent differential 1-forms $\theta^I = \theta^I_a(x)dx^a$. Using them, we can rewrite the metric in a simple form

$$q_{ab}(x) = \eta_{IJ}\theta^I_a(x)\theta^J_b(x), \tag{3.1.1}$$

where η_{IJ} is a symmetric, positive definite, real matrix that does not depend on x. It is possible to define the dual frame of θ^I, that is, a frame of vector fields $\xi_I = \xi^a_I\partial_a$, such that $\theta^I_a\xi^a_J = \delta^I_J$. Using the properties of invariance under transformations, we can prove that this frame of vector fields satisfies

$$[\xi_I, \xi_J] = f^K_{IJ}\xi_K, \tag{3.1.2}$$

where $[\cdot,\cdot]$ are the usual bracket between vector fields, and f^K_{IJ} is a collection of real coefficients called *structure constants*. These structure constants characterise the cosmological models, providing a classification that identifies nine different models, known as *Bianchi classification* [42]. Furthermore, the frame of 1-forms satisfies an equation equivalent to (3.1.2), namely the Maurer-Cartan equation:

$$d\theta^I + \frac{1}{2}f^I_{JK}\theta^J \wedge \theta^K = 0. \tag{3.1.3}$$

It is quite clear that this formulation is demanding to deal with a homogeneous space that admits a Lie group structure, namely $\Sigma \cong G$. In this case, the action L is just the left multiplication. The fields ξ_I are left-invariant vector fields, i.e. $(L_g)_*\xi_I = \xi_I$ for all $g \in G$, where $(L_g)_*$ indicates the pushforward, and they are a basis of the Lie algebra $\mathfrak{g}$. Indeed, the Bianchi classification is the classification of the three-dimensional real Lie algebras. Correspondingly, θ^I are left-invariant 1-forms, i.e. $L^*_g\theta^I = \theta^I$, where L^*_g is the pullback. Every left-invariant tensor can be decomposed on these frames with constant coefficients. A homogeneous metric is then a left-invariant metric $q = \eta_{IJ}\theta^I \otimes \theta^J$, equivalently a scalar product on the Lie algebra $\mathfrak{g}$ (a review of these concepts can be found in Appendix A).

Additionally, we want to recall that the lapse function N and the shift vector N^a have a similar property in a spatially homogeneous Universe. In particular, the lapse function is a function of the time $N(t)$ and the shift vector can be factorised as $N^a(t,x) = \xi^a_I(x)N^I(t)$, where N^I are defined as $N^I(t) \doteq N^a(t,x)\theta^I_a(x)$ and can be showed that they depend on time only.
Notice that, in this case, the isometry group always contains G and can be bigger. Indeed, in this setting we are also able to define the *isotropic case*, also known as the Friedmann-Lemaître-Robertson-Walker (FLRW) model, demanding that η_{IJ} in Eq.(3.1.1) is a positive multiple of the identity. In the latter case, the isotropy group is the group of spatial rotations $H_x \cong SO(3)$. This analysis is further developed in Chapter 7.

The Einstein Field Equations for the cosmological setting simplify a lot. Cosmology is already formulated in the ADM formulation, and the only metric dynamical quantities are the matrix elements of η_{IJ}. This yields a finite-dimensional phase space composed of η_{IJ} and conjugate momenta P^{IJ}, known as minisuperspace. In the FLRW model [88, 89, 125], the matrix is $\eta_{IJ} = a^2\delta_{IJ}$, where a is the so-called *scale factor*, which is the only dynamical quantity and governs the expansion of the universe. Another possible homogeneous, but anisotropic, case considered in the literature is the diagonal case. In which the matrix $\eta_{IJ} = a^2_{(I)}\delta_{IJ}$ is diagonal, and the three configurational variables a_1, a_2, a_3 are three scale factors which control the expansion along the directions given by the vector fields ξ_I. In these cases, there is no Diffeomorphism constraint, then only the Hamiltonian constraint must be solved, usually put in the form of the Friedmann equation. More general non-diagonal models are usually not considered in phenomenological analyses. However, they are studied in the literature [152, 38] due to their connection with the Belinski-Khalatnikov-Lifshitz (BKL) conjecture [39, 40] (for a quantum analysis see [9, 60]) and the role they may play in introducing inhomogeneities into cosmological models.

In the cosmological setting, the Ashtekar variables admit a similar decomposition in terms of the 1-forms θ^I and the vector fields ξ_I [44]

$$A^i_a(x) = \phi^i_I\theta^I_a(x),\ E^a_i(t,x) = |\det(\theta^I_a)|\, p^I_i\xi^a_I(x). \tag{3.1.4}$$

The minisuperspace of the Ashtekar formulation of gravity is constituted by the pair (ϕ^i_I, p^J_j) with fundamental Poisson brackets

$$\{\phi^i_I, p^J_j\} = \frac{\beta\kappa}{2V_0}\delta^i_j\delta^J_I, \tag{3.1.5}$$

where β is the Barbero-Immirzi parameter, $\kappa = 16\pi G_N/c^3$, and V_0 is the *fiducial volume* $V_0 = \int |\det(\theta^I_a)|d^3x$, namely the volume with respect the fiducial metric $q_0 = \delta_{IJ}\theta^I \otimes \theta^J$. The reconstructed metric can be given in terms of inverse densitised dreibein as $q_{ab} = |\det(p^K_k)|\delta_{ij}p^i_Ip^j_J\theta^I_a\theta^J_b$. The metric is invariant under rotations $p^I_i \mapsto R^j_ip^I_j$, allowing for a global $SU(2)$ gauge transformation. For the peculiarity of the lapse function and the shift vector, we can explicitly integrate the smeared constraints, finding expressions in terms of phase space variables and independent of

x that, up to numerical constants, read as follows

$$
\begin{aligned}
G_i &= \epsilon_{ab}{}^c \phi_I^b p_c^I - f_{IJ}^J,\\
\mathrm{D}_I &= -f_{IJ}^K \phi_K^i p_i^J,\\
H &= -\frac{1}{\kappa\sqrt{|\det(p_c^K)|}}\Big(\epsilon_{ijk} f_{IJ}^K \phi_K^i p_j^I p_k^J - p_a^I \phi_I^a p_b^J \phi_J^b + p_a^I \phi_J^a p_b^J \phi_I^b \\
&\qquad + 2(1-\beta^{-2})(\phi_I^i - \Gamma_I^i)(\phi_J^j - \Gamma_J^j) p_i^{[I} p_j^{J]}\Big).
\end{aligned}
\tag{3.1.6}
$$

In the isotropic case [48], ϕ_I^i and p_i^I are multiple of the identity matrix via coefficient c and p with Poisson bracket $\{c, p\} = \frac{\beta\kappa}{6V_0}$. These variables have a direct connection with the scale factor via

$$
\begin{aligned}
p &= \operatorname{sgn}(a) a^2,\\
c &= \frac{\beta}{N}\frac{da}{dt}.
\end{aligned}
\tag{3.1.7}
$$

In this formulation, the Gauß constraint and the Diffeomorphism constraint identically vanish. Indeed, there are no residual gauge symmetries. The diagonal case [49] has a similar behaviour. For instance, we can study the diagonal Bianchi I model [132, 28]. In the Bianchi I model, the structure constants vanish $f_{IJ}^K = 0$, ensuring that the frame ξ_I defines a coordinate frame. The underlying simply connected topology is $\mathbb{R}^3$ equipped with a flat metric. It is also consistent with the topology of a torus $\mathbb{T}^3$. The full Lorentzian metric can be written as

$$
\mathrm{g} = -N^2 dt^2 + a_1^2 (dx^1)^2 + a_2^2 (dx^2)^2 + a_3^2 (dx^3)^2,
\tag{3.1.8}
$$

where a_I are the scale factors in each direction. In $\mathbb{R}^3$, the concept of *fiducial cell* plays a structural role. The fiducial cell is a parallelepiped with edges along the coordinate curves x^a with fiducial length L_I. The coordinate curves are the integral curves of the vector fields $\xi_I = \delta_I^a \partial_a$, and the fiducial length is the length with respect to the fiducial metric $q_0 = \delta_{IJ}\theta^I \otimes \theta^J = \delta_{ab} dx^a \otimes dx^b$. In this case, the fiducial volume is the volume of the fiducial cell $V_0 = \int \theta^1 \wedge \theta^2 \wedge \theta^3 = \int d^3x = L_1 L_2 L_3$ with respect to the fiducial metric.
Up to a rescaling, the Ashtekar variables have a similar expression as in the isotropic case

$$
A_a^i = \tilde{c}_i \delta_I^i (L_I)^{-1} \theta_a^I, \qquad E_i^a = \frac{\sqrt{\det(q_0)}}{V_0} L_I \tilde{p}^i \delta_i^I \xi_I^a.
\tag{3.1.9}
$$

These variables are fully characterised by the six variables $\tilde{c}_i$, $\tilde{p}^i$, which are canonically conjugate in that the following fundamental Poisson brackets hold

$$
\{\tilde{c}_i, \tilde{p}^j\} = \tfrac{\beta\kappa}{2}\delta_i^j.
\tag{3.1.10}
$$

The relation with the rescaled quantities is easy: $\tilde{p}_I = L_J L_K p_I$ with $\epsilon_{IJK} = 1$ and $\tilde{c}_I = c_I L_I$. They also have a direct link with the three scale factors by

$$
\begin{aligned}
\tilde{p}_I &= \operatorname{sgn}(a_I) a_J a_K L_J L_K \text{ with } \epsilon_{IJK} = 1,\\
\tilde{c}_I &= \frac{\beta L_I}{N}\frac{da_I}{dt}.
\end{aligned}
\tag{3.1.11}
$$

The wording "with $\epsilon_{IJK} = 1$" indicates that the indices I, J, K are not summed in the previous expression and satisfy the condition on the Levi-Civita symbol, so that, for given I, the other two are fixed uniquely. The Gauß and Diffeomorphism constraints are already satisfied, leaving only the scalar constraint, that is, the gravitational Hamiltonian, to be imposed. This takes the simple form

$$H = -\frac{2}{\kappa\beta^2}\frac{1}{\sqrt{|p_1p_2p_3|}}\left(p_1c_1p_2c_2 + p_2c_2p_3c_3 + p_3c_3p_1c_1\right). \tag{3.1.12}$$

This approach to Cosmology with homogeneous space is sometimes called mathematical cosmology. In this setting, the homogeneity is imposed at every scale, requiring a stronger condition than the usual physical principle. However, this approach is able to properly describe the evolution of the Universe from the inflationary phase to nowadays, comprehensive of the large-scale structure formation [137, 138]. Nevertheless, the meaning of homogeneity in a microscopic-sized Universe, especially in a quantum regime, is unclear to the author [1]. Notwithstanding, clarifying this point is beyond the scope of this work.[2]

3.2 Quantum states for Cosmology

The minisuperspace of the Ashtekar variables can be quantised using the technique of Loop Quantum Gravity. We first consider the isotropic case. The holonomy along an edge e_I of the fiducial cell has a decomposition in terms of trigonometric functions

$$h_{e_I}(A) = \exp\left(\lambda c\delta^i_I L_I \tau_i\right) = \cos\left(\frac{1}{2}\lambda\tilde{c}\right) + 2\tau_I \sin\left(\frac{1}{2}\lambda\tilde{c}\right). \tag{3.2.1}$$

Where λ is a real parameter that classically is linked with the parametrisation of the curve. Then, such holonomies belong to the space of *almost periodic functions*, namely, they are linear combinations of trigonometric polynomials. Analogously to periodic functions, which can be written as functions on $\mathbb{T}^1$, the almost periodic functions are equivalently described as functions on a peculiar Abelian topological group, that is the Bohr compactification of the real line $\overline{\mathbb{R}}_{\text{Bohr}}$, constructed as the Pontryagin dual of the discrete real line [161]. Since it is a compact group, it admits a Haar measure, and this pair is the quantum configuration space of Loop Quantum Cosmology [15].

The fluxes are constructed considering a constant smearing function ν and the face S_K of the fiducial cell "orthogonal" to the edge e_K. Since the space is isotropic, the fiducial cell is a cube with edges of fiducial length L and fiducial volume $V_0 = L^3$. The flux is

$$Y^{\nu}_{S_K} = \int_{S_K} p\,\theta^I \wedge \theta^J = p\,V_0^{\frac{2}{3}} = \tilde{p}. \tag{3.2.2}$$

[1] A possible interpretation arises from an effective quantum field theory perspective, in which the fiducial cell acts as an infrared regulator, scaling with the size of the Universe so that the modes of the inhomogeneous fields can be neglected and the minisuperspace approximation remains valid [53, 52].

[2] In covariant LQG, cosmological transition amplitudes are evaluated between semiclassical states peaked on homogeneous geometries, thereby overcoming the problem of imposing homogeneity at the quantum level [41, 111].

Clearly, this expression does not depend on the chosen face. It is quite immediate to compute also the classical expression of the volume (with the interpretation of the physical volume of the fiducial cell) $V = |p|^{\frac{3}{2}} V_0$.

The Hilbert space of the theory is then $L^2(\overline{\mathbb{R}}_{\text{Bohr}}, d\mu_H)$. This space is the completion of the space of almost periodic functions with the scalar product induced by the Haar measure. This space is not separable and has an orthonormal basis $|\lambda\rangle$, labelled by a real number λ, defined by the following functions of the variable $\tilde{c} \in \overline{\mathbb{R}}_{\text{Bohr}}$

$$|\lambda\rangle = e^{\frac{i\lambda\tilde{c}}{2}}, \ \lambda \in \mathbb{R}. \tag{3.2.3}$$

The inner product has an easy expression. Let f and g be complex-valued almost periodic functions on $\mathbb{R}$, then the inner product reads

$$\langle f, g\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(c)^* g(c)\, dc. \tag{3.2.4}$$

It is easy to check now that the basis $|\lambda\rangle$ is orthonormal with respect to this scalar product, i.e. $\langle\lambda'|\lambda\rangle = \delta_{\lambda\lambda'}$, where $\delta_{\lambda\lambda'}$ is the Kronecker delta. We can then derive the action of the fluxes on the almost periodic functions by the fundamental Poisson brackets. Let $f(\tilde{c}) = \sum_n f_{\lambda_n} e^{\frac{i\lambda_n\tilde{c}}{2}}$ for a finite collection of λ_n, then

$$\{f(\tilde{c}), \tilde{p}\} = \frac{\beta\kappa}{12} \sum_{n=1}^{N} i\lambda_n f_{\lambda_n} e^{\frac{i\lambda_n\tilde{c}}{2}}. \tag{3.2.5}$$

Hence, multiplicative and derivative operators are densely defined on $L^2(\overline{\mathbb{R}}_{\text{Bohr}}, d\mu_H)$: the holonomies act as multiplication on the space of almost periodic functions, while flux is a derivative operator. In details, let f, g be almost periodic functions, then

$$(\hat{f}g)(\tilde{c}) = f(\tilde{c})g(\tilde{c}), \quad (\hat{\tilde{p}}g) = -i\tfrac{\beta\ell_P^2}{6}\tfrac{dg}{d\tilde{c}}. \tag{3.2.6}$$

As in the LOST-F theorem in Loop Quantum Gravity [126, 85], there exist results establishing the uniqueness of the representation of the holonomy-flux algebra in the cosmological setting [16, 80]. Moreover, as in the full theory, the action of the holonomies is weakly discontinuous, thereby violating the assumptions of the Stone-von Neumann theorem and leading to an inequivalent representation of the Heisenberg algebra.

We now provide the action of the fundamental operators on the orthonormal basis, which is useful. Indeed, the operator $\widehat{e^{\frac{i\lambda\tilde{c}}{2}}} \doteq \hat{\lambda}$ is a shift operator, while $\hat{\tilde{p}}$ is diagonalised on the given basis

$$\hat{\lambda}|\lambda'\rangle = |\lambda + \lambda'\rangle, \quad \hat{\tilde{p}}|\lambda\rangle = \tfrac{\beta\ell_P^2}{12}\lambda|\lambda\rangle. \tag{3.2.7}$$

As a consequence, the states λ are eigenstates of the volume, which has a continuum spectrum

$$\hat{V}|\lambda\rangle = \left(\frac{\beta|\lambda|\ell_P^2}{12}\right)^{\frac{3}{2}} |\lambda\rangle \doteq V_\lambda|\lambda\rangle. \tag{3.2.8}$$

In the Hamiltonian constraint, we find the same issue of LQG: since the connection appears in the Hamiltonian, it has to be regularised. Taking inspiration from lattice gauge theory techniques, suitably adapted to take into account the features suggested by LQG, the idea is to compute the curvature of the Ashtekar connection on a plaquette with edges parallel to the edges of the fiducial cell. The edge of the plaquette has fiducial length $\mu_0 L$ and area $\mu_0^2 L^2$, and so shrinking the area to zero, we obtain

$$F^i_{ab}\tau_i = \lim_{\mu_0\to 0}\left(\frac{h_{\Box_{IJ}}-1}{\mu_0^2 V_0^{\frac{2}{3}}}\right)\theta^I_a\theta^J_b.$$

Here, $h_{\Box_{IJ}}$ is the holonomy along the plaquette

$$h_{\Box_{IJ}} = h_{e_I}h_{e_J}h^{-1}_{e_I}h^{-1}_{e_J}.$$

Then, using Thiemann's trick to reabsorb the non-polynomiality in the volume, one obtains the following Hamiltonian

$$H = -\frac{8}{\beta\kappa\mu_0^2}\sum_{IJK}\epsilon^{IJK}\,\mathrm{Tr}\left(h_{e_I}h_{e_J}h^{-1}_{e_I}h^{-1}_{e_J}h_{e_K}\{h^{-1}_{e_K},V\}\right). \tag{3.2.9}$$

Notice that, doing so, we have supposed that the edges along the directions ξ_I form a square. However, this condition is ensured only in the Bianchi I model; otherwise, we need to include in the Hamiltonian a factor that takes into account this defect, as well as terms from the Lorentzian part [45, 47]. We can now quantise, finding the Hamiltonian operator

$$\hat{H} = \frac{96i}{\beta\mu_0^3\ell_P^2}\cos(\tfrac{\mu_0\tilde{c}}{2})^2\sin(\tfrac{\mu_0\tilde{c}}{2})^2\left(\cos(\tfrac{\mu_0\tilde{c}}{2})\hat{V}\sin(\tfrac{\mu_0\tilde{c}}{2}) - \sin(\tfrac{\mu_0\tilde{c}}{2})\hat{V}\cos(\tfrac{\mu_0\tilde{c}}{2})\right), \tag{3.2.10}$$

where the operators associated with the trigonometric functions are $\cos(\frac{\mu_0\tilde{c}}{2}) = \frac{1}{2}(\widehat{e^{\frac{i\mu_0\tilde{c}}{2}}} + \widehat{e^{-\frac{i\mu_0\tilde{c}}{2}}})$ and $\sin(\frac{\mu_0\tilde{c}}{2}) = \frac{1}{2i}(\widehat{e^{\frac{i\mu_0\tilde{c}}{2}}} - \widehat{e^{-\frac{i\mu_0\tilde{c}}{2}}})$. Nevertheless, in the limit $\mu_0 \to 0$, this operator is not well-defined. Using input from LQG, we put a lower bound on the value of μ_0, fixing the area of the plaquette to be the area gap of LQG. However, from phenomenological computations, this choice of regularisation, with a constant μ_0 fixed once and for all, yields unphysical results, such as the occurrence of a quantum bounce even at low matter densities. To overcome this problem, a dynamical choice for the regularisation scheme is preferred [26]. The latter is known as the $\bar{\mu}$-scheme. The input is again the quantum geometry, this time the physical area of the plaquette, that is $\bar{\mu}|\tilde{p}|$, is subject to the lower bound

$$\bar{\mu}|\tilde{p}| = \frac{\sqrt{3}\beta\ell_P^2}{8}, \tag{3.2.11}$$

hence $\bar{\mu}$ is a function of $\tilde{p}$, and the new Hamiltonian is the one in (3.2.10), but with $\bar{\mu}$ instead of μ_0. The action of $\widehat{e^{\frac{i\bar{\mu}(\hat{p})\tilde{c}}{2}}}$ results extremely simple on the eigenstate of the volume. Consider the basis $|v\rangle$ such that $\hat{V}|v\rangle = \left(\frac{\beta|}{12}\right)^{\frac{3}{2}} v\ell_P^3|v\rangle$, then

$$\widehat{e^{\frac{i\bar{\mu}(\hat{p})\tilde{c}}{2}}}|v\rangle = |v+1\rangle. \tag{3.2.12}$$

Thus, the quantum Einstein equation in LQC has the form of a difference equation:

$$(v+2)\sqrt{v(v+4)}|v+4\rangle - 2v^2|v\rangle + (v-2)\sqrt{v(v-4)}|v-4\rangle = 0 \tag{3.2.13}$$

In the diagonal case, the picture is quite similar. The Hilbert space is the tensor product of three copies of $L^2(\overline{\mathbb{R}}_{\text{Bohr}}, d\mu_H)$, one for each direction. We can analogously regularise the Hamiltonian by introducing a fiducial cell with edges of different lengths and three plaquettes, one for each face. Then, introducing three regulators $\bar{\mu}_I$, the Hamiltonian operator has a similar form involving permutations on the index of the directions.
The LQC dynamics can be solved both at the quantum and the semiclassical level, even including matter. The characteristic phenomenon is the bounce, which predicts that the matter and the Universe itself have an expanding phase after a contraction phase reaches Planckian density [24, 25, 26]. Furthermore, LQC is an important source of phenomenology [55, 64]. Effective models based on LQC predict corrections to the Cosmic Microwave Background as well as to the inflationary era [4, 5, 132] (see also [14] for a brief review). There are also proposals to include inhomogeneity [149, 59], allowing quantum fluctuations near the classical singularity to be recovered while preserving the general features of LQC dynamics. Recently, these ideas have also found applications in the study of stellar collapse [56, 150, 93, 107, 84].

It is, however, important to keep in mind that Loop Quantum Cosmology *a priori* does not provide the cosmological sector of Loop Quantum Gravity. Despite the similarity, the LQC Hilbert space cannot be straightforwardly embedded into the LQG one. This is since, despite the homogeneous and isotropic connections being a subspace (in particular, an affine line) in the space of smooth Ashtekar connections, it is not possible to induce an immersion of $\overline{\mathbb{R}}_{\text{Bohr}}$ into $\overline{\mathcal{A}}$ [63]. The reason is that the holonomies are not almost periodic functions on generic curves. Furthermore, LQC exhibits an "Abelianisation", in which at the classical level the Gauß constraint identically vanishes, and at the quantum level no $SU(2)$ spin-network states emerge. This undermines the reliability of LQC in connection with the full theory, which remains one of the big open questions in the field.

3.3 Generalised Uncertainty Principle theories

The techniques developed in the Loop Quantum Cosmology program have a more general flavour. They can be contextualised in the framework of *Polymer Quantum Mechanics* [17, 87, 71]. Polymer Quantum Mechanics starts from the exponentiated version of the Canonical Commutation Relations (CCR), finding representations for Quantum Mechanics, inequivalent to the Shrödinger one. In detail, starting from the classical commutation relation $[\hat{q}, \hat{p}] = i\hbar\mathbb{1}$, we can construct the Weyl elements $\hat{U}(t) = e^{\frac{it\hat{q}}{\hbar}}$ and $\hat{V}(s) = e^{\frac{is\hat{p}}{\hbar}}$ satisfying

$$\hat{U}(t)\hat{V}(s) = e^{-\frac{ist}{\hbar}}\hat{V}(s)\hat{U}(t), \tag{3.3.1}$$

and look for discontinuous representations of one of the two generators. This breaks the Stone-von Neumann theorem, allowing inequivalent representations. We will call q-representation if the one-parameter unitary group $\hat{U}(t)$ is strongly continuous,

while $\hat{V}(s)$ is not. Hence, for the Stone's theorem, in the q-representation, the operator $\hat{q}$ is well defined, while $\hat{p}$ is not. The p-representation is defined the other way around.
Let us consider the q-representation. From the spectral theorem, we can construct the Hilbert space in the q-polarisation, i.e. the states are functions of q. It turns out to be $L^2(\mathbb{R}_\mathrm{d}, d\mu_H)$, where $\mathbb{R}_\mathrm{d}$ is the real line equipped with the discrete topology. The group $\mathbb{R}_\mathrm{d}$ is locally compact and Abelian, and the corresponding Haar measure is the counting measure. The orthonormal basis is provided by $\phi_\lambda(q) \doteq \delta_{\lambda,q}$, where $\delta_{\lambda,q}$ is the Kronecker delta. Then, the action of the Weyl algebra is

$$\hat{V}(s)\,\phi_\lambda(q) = \phi_\lambda(q+s) = \phi_{\lambda-s}(q), \quad \hat{q}\,\phi_\lambda(q) = \lambda\phi_\lambda(q). \tag{3.3.2}$$

For the p-polarisation, we can proceed by constructing the spectrum of the Abelian C*-algebra generated by $V(s)$, and then using the Gel'fand representation. In this case, we obtain the well-known Hilbert space $L^2(\overline{\mathbb{R}}_\mathrm{Bohr}, d\mu_H)$ with orthonormal basis $\psi_\lambda(p) \doteq e^{\frac{i\lambda p}{\hbar}}$. The action of the Weyl algebra is

$$\hat{V}(s)\,\psi_\lambda(p) = e^{\frac{i(\lambda+s)p}{\hbar}} = \psi_{\lambda+s}(p), \quad \hat{q}\,\psi_\lambda(p) = i\hbar\frac{\partial}{\partial p}e^{\frac{i\lambda p}{\hbar}} = -\lambda\psi_\lambda(p). \tag{3.3.3}$$

In the case of p-representation, the Hilbert spaces associated with the q- and p-polarisation switch. We summarise the polymer representations in Table 3.1

	q-representation	p-representation
q-pol	$L^2(\mathbb{R}_\mathrm{d}, d\mu_H)$ $\hat{q}\,\phi_\lambda(q) = \lambda\phi_\lambda(q)$ $\hat{V}(s)\,\phi_\lambda(q) = \phi_{\lambda-s}(q)$	$L^2(\overline{\mathbb{R}}_\mathrm{Bohr}, d\mu_H)$ $\hat{p}\,\psi_\lambda(q) = \lambda\psi_\lambda(q)$ $\hat{U}(t)\,\psi_\lambda(q) = \psi_{\lambda+t}(q)$
p-pol	$L^2(\overline{\mathbb{R}}_\mathrm{Bohr}, d\mu_H)$ $\hat{q}\,\psi_\lambda(p) = -\lambda\psi_\lambda(p)$ $\hat{V}(s)\,\psi_\lambda(p) = \psi_{\lambda+s}(p)$	$L^2(\mathbb{R}_\mathrm{d}, d\mu_H)$ $\hat{p}\,\phi_\lambda(p) = \lambda\phi_\lambda(p)$ $\hat{U}(t)\,\phi_\lambda(p) = \phi_{\lambda+t}(p)$

Table 3.1. Hilbert spaces together with the action of the Weyl group for the two different representations and polarisations.

Since $\overline{\mathbb{R}}_\mathrm{Bohr}$ is the Pontryagin dual of $\mathbb{R}_\mathrm{d}$ by construction, there exists an abstract version of the Fourier transform which map L^1 functions on $\overline{\mathbb{R}}_\mathrm{Bohr}$ in bounded continuous functions on $\mathbb{R}_\mathrm{d}$ which vanishes at infinity (a general overview of this topic can be found in Appendix A). It is possible to implement this transformation as a unitary operator from $L^2(\overline{\mathbb{R}}_\mathrm{Bohr}, d\mu_H)$ to $L^2(\mathbb{R}_\mathrm{d}, d\mu_H)$. Furthermore, both these spaces are isomorphic to a non-separable Hilbert space $\mathcal{H}_\mathrm{poly}$, with an orthonormal basis $|\lambda\rangle$ labelled by a real number λ, and with distinguished operators: a label operator $\hat{\varepsilon}$ and a one-parameter family of shift operators $\hat{s}(t)$ which act on the orthonormal basis as

$$\hat{\varepsilon}|\lambda\rangle = \lambda|\lambda\rangle, \quad \hat{s}(t)|\lambda\rangle = |\lambda + t\rangle. \tag{3.3.4}$$

The non-existence of one of the two fundamental bare operators implies an ambiguity in the quantisation of the polynomial functions. In general, the terms involving the ill-defined operators must be regularised. To do that, we choose a countable subset

of the orthonormal basis, usually provided fixing a lattice $\Gamma_{\mu_0} = \{n\mu_0 \in \mathbb{R} \mid n \in \mathbb{Z}\}$ on the index set $\mathbb{R}$, and then considering as regularised Hilbert space the linear span $\mathcal{H}_{\mu_0} \doteq Span\{|\lambda\rangle\}_{\lambda\in\Gamma_{\mu_0}} \subset \mathcal{H}_{\text{poly}}$.
Considering, for instance, the Hamiltonian $H = \frac{p^2}{2m} + V(q)$ in the q-representation, the quadratic term in the Hamiltonian must be treated appropriately. Choosing the lattice basis $\psi_{n\mu_0}(p)$ of the p-polarization, the operators defined in (3.3.3) induce operators on $\mathcal{H}_{\mu_0}$: the action of $\hat{q}$ is unchanged, while the one-parameter family becomes a cyclic group of operators generated by $\hat{V}(\mu_0)$. The idea is to approximate $\hat{p}$ by the sine function

$$p \approx \frac{\hbar}{\mu_0}\sin\left(\frac{\mu_0 p}{\hbar}\right) = \frac{\hbar}{2i\mu_0}\left(e^{\frac{i\mu_0 p}{\hbar}} - e^{\frac{-i\mu_0 p}{\hbar}}\right)$$

This trick provides a regularisation of the operator $\hat{p}$ on the regularised Hilbert space $\mathcal{H}_{\mu_0}$. The regularised momentum operator is defined by

$$\hat{p}_{\mu_0} \doteq \tfrac{\hbar}{2i\mu_0}\left(\hat{V}(\mu_0) - \hat{V}(-\mu_0)\right). \tag{3.3.5}$$

Hence, its action on the orthonormal basis reads

$$(\hat{p}_{\mu_0}\,\psi_{n\mu_0})(p) = \tfrac{\hbar}{2i\mu_0}\left(\psi_{(n+1)\mu_0}(p) - \psi_{(n-1)\mu_0}(p)\right).$$

Clearly, the classical approximation holds if $p \ll \hbar/\mu_0$, which in a quantum gravity context is reasonable since we expect that μ_0 is Planckian.

This regularisation introduces a modification of the usual Heisenberg algebra. Indeed, the fundamental Poisson brackets present a deformation

$$[\hat{q}, \hat{p}_{\mu_0}] = \frac{i\hbar}{2}\left(\hat{V}(\mu_0) + \hat{V}(-\mu_0)\right) = i\hbar\sqrt{1 - \tfrac{\mu_0^2}{\hbar^2}\hat{p}_{\mu_0}^2}. \tag{3.3.6}$$

This modification sets this kind of theory in the more general framework of Generalised Uncertainty Principle theories [116, 130]. These theories allow a deformation of the usual Heisenberg algebra, and a consequent modification of the Heisenberg Uncertainty Principle. In a 1-dimensional theory, the deformation is of the form

$$[\hat{q}, \hat{p}] = i\hbar f(\hat{p}), \tag{3.3.7}$$

where f is a positive real function usually dependent on a real parameter β that controls the deformation. This kind of theory appears at the effective level of different fundamental theories, i.e. Loop Quantum Gravity (as constructed above) and String Theory. In the context of quantum gravity, they encode at an effective level the spacetime structure supposed by the fundamental theories. Indeed, they include effects as minimal length, and, in the higher-dimensional case, non-commutativity of the space coordinates [117, 8].
In term of representation theory, the deformed commutation relation (3.3.7) is usually realised as an operator algebra on the Hilbert space $L^2(\mathbb{R}, \frac{d\mu}{f(p)})$, where $\frac{d\mu}{f(p)}$ is a deformation of the usual Lebesgue measure on $\mathbb{R}$. The action of the operator is

$$\hat{p}\,\psi(p) = p\psi(p), \quad \hat{q}\,\psi(p) = i\hbar f(p)\tfrac{d}{dp}\psi(p). \tag{3.3.8}$$

The deformed Lebesgue measure ensures that the operator $\hat{q}$ is symmetric. However, depending on the form of $f(p)$, it admits a unique, non-unique, or does not admit a self-adjoint extension. In case of a non-unique extension, a minimal length, as minimal non-zero variance of the position operator, appears. Thus, the latter are the most interesting cases from the physical viewpoint [116, 117].

GUP theories offer a unified framework within which to develop and explore the phenomenology of various quantum gravity models and regularisation proposals. The key differences among these approaches are encapsulated in the choice of the deformation function $f(p)$, which represents the sole deviation from standard Quantum Mechanics. This formulation allows the application of conventional tools from functional analysis to investigate the implications and physical consequences of the modified theory.

Chapter 4

Loop Quantum Cosmology of non-diagonal models

This chapter is devoted to extending the Loop Quantum Cosmology framework to non-diagonal Bianchi models. The significance of this study lies in the potential to implement the BKL conjecture [40, 18] within the quantum cosmological sector. This would correspond to applying a locally homogeneous, non-diagonal dynamics point by point in space. Indeed, in the context of a generic inhomogeneous cosmological model, the assumption of a diagonal representation must be abandoned. Naturally, one might assume that such a general scenario would fall within the unrestricted domain of Loop Quantum Gravity. However, the validity of the BKL conjecture, effectively freezing the dynamics of spatial gradients, could permit a pointwise description of the quantum dynamics using an extension of the current formulation, at least in regimes where spatial curvature contributions are negligible.

In non-diagonal models, the principal directions of cosmic expansion rotate during the evolution. This necessitates the inclusion of angular variables alongside the scale factors to adequately describe the dynamics. The central idea in addressing the non-diagonal case is to recover a diagonal kinematical description by fixing the directions of expansion. We construct the Ashtekar variables in this framework, and demonstrate that the fluxes can be consistently and coherently defined as in the diagonal case. As a result, the quantum states are labelled by the diagonal fluxes and the angles that determine the expansion directions, achieving the desired separation of variables. While alternative quantisation approaches have been proposed, they typically fail to yield a proper quantisation of the Hamiltonian or a well-posed physical interpretation. This study constitutes a foundational step towards assessing the generality and robustness of the LQC approach when extending beyond the standard minisuperspace framework to incorporate additional geometrical degrees of freedom.

4.1 Non-diagonal models in metric variables

In a non-diagonal Bianchi model, the hypersurface Σ has a metric that evolves in time as

$$q_{ab}(t,x) = \eta_{IJ}(t)\theta_a^I(x)\theta_b^J(x). \tag{4.1.1}$$

Following Belinski's approach to non-diagonal metric in [38], we want to factorise η_{IJ} in a diagonal matrix similar to it via a rotation. Since $\eta_{IJ}(t)$ is a symmetric matrix depending smoothly on the single parameter t, it always has three eigenvalues $\{a_1^2(t), a_2^2(t), a_3^2(t)\}$, that can be chosen to be continuously differentiable on the whole interval [144]. Moreover, if the continuous eigenvalues are such that, whenever two of them meet of infinite order at some $t_0 \in \mathbb{R}$, they are equal for all $t \in \mathbb{R}$, then all eigenvalues and all eigenvectors can be chosen C^∞ in t [6]. Two parametrised eigenvalues are said to meet of infinite order at $t_0 \in \mathbb{R}$ if all derivatives with respect to t coincide at t_0. Since two solutions of an ordinary differential equation that meet of infinite order coincide identically, this is a reasonable condition to impose on the eigenvalues. Thus, η_{IJ} can be decomposed as

$$\eta_{IJ} = d_{AB} R_I^A R_J^B \quad \text{with} \quad d_{AB} = \begin{pmatrix} a_1^2 & 0 & 0 \\ 0 & a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{pmatrix}, \tag{4.1.2}$$

and R is a rotation matrix that depends on the parameter t, and that is infinitely differentiable. The eigenvalues are positive because η_{IJ} are the components of the Riemannian metric q in basis θ^I, therefore, it is a positive definite, symmetric matrix. The rotation introduces three new variables: the angles. Using the Euler angles $\{\theta, \psi, \varphi\}$ the rotation reads

$$R = \exp(\theta T_3) \exp(\psi T_2) \exp(\varphi T_3), \tag{4.1.3}$$

where T_2 and T_3 are two of the three generators of $SO(3)$, as matrices:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These angles can be interpreted as a physical rotation of the left-invariant vector fields. Moreover, they are smooth functions of t.

4.1.1 Calculation of the ADM Lagrangian

To construct a Hamiltonian theory, one can choose as configuration variables the three scale factors and the three Euler angles $\{a_1, a_2, a_3, \theta, \psi, \varphi\}$. From now on, these variables are called metric variables. The Lagrangian can be calculated from the ADM formalism

$$\begin{aligned} L_{ADM} = N\sqrt{\det(q)}\Big(\mathrm{R}_q + &\frac{1}{4N^2} q^{ac}q^{bd}(\dot{q}_{ab} - N_{a;b} - N_{b;a})(\dot{q}_{cd} - N_{c;d} - N_{d;c}) + \\ &- \frac{1}{4N^2} q^{ab}q^{cd}(\dot{q}_{ab} - N_{a;b} - N_{b;a})(\dot{q}_{cd} - N_{c;d} - N_{d;c})\Big). \end{aligned}$$

A sketch of the calculation is presented below and passes through four terms. We rewrite the Lagrangian as the sum of four terms (excluding the scalar curvature R_q)

$$\begin{aligned} L_{ADM} = N\sqrt{\det(q)}\Bigg(\mathrm{R}_q + \frac{1}{4N^2}\Big(& q^{ac}q^{bd}\dot{q}_{ab}\dot{q}_{cd} + q^{ac}q^{bd}(N_{a;b} + N_{b;a})(N_{c;d} + N_{d;c}) \\ &- q^{ac}q^{bd}\dot{q}_{ab}(N_{c;d} + N_{d;c}) - q^{ac}q^{bd}\dot{q}_{cd}(N_{a;b} + N_{b;a}) \\ &- q^{ab}q^{cd}(\dot{q}_{ab} - N_{a;b} - N_{b;a})(\dot{q}_{cd} - N_{c;d} - N_{d;c})\Big)\Bigg). \end{aligned}$$

For the first term, we simply factorize the metric q_{ab} as in Eq.(4.1.1) and (4.1.2):

$$\begin{aligned}
&q^{ac}q^{bd}\dot{q}_{ab}\dot{q}_{cd} = \qquad (4.1.4)\\
=&d^{DC}d^{GH}(\dot{d}_{DH} + d_{A'H}\dot{R}^{A'}_{B}\Lambda^{B}_{D} + d_{DB'}\dot{R}^{B'}_{F}\Lambda^{F}_{H})(\dot{d}_{CG} + d_{G'G}\dot{R}^{G'}_{A}\Lambda^{A}_{C} + d_{CH'}\dot{R}^{H'}_{E}\Lambda^{E}_{G})\\
=&d^{AC}d^{BD}\dot{\Gamma}_{AB}\dot{d}_{CD} - 4d^{AC}\dot{d}_{AB}(R\dot{\Lambda})^{B}_{C} + 2d^{AB}d_{CD}(R\dot{\Lambda})^{D}_{A}(R\dot{\Lambda})^{C}_{B} + 2(R\dot{\Lambda})^{B}_{C}(R\dot{\Lambda})^{C}_{B},
\end{aligned}$$

where d^{AB} is the inverse of d_{AB} and Λ is the inverse matrix of R. Recalling that, since R is a rotation, $\Lambda = R^t$, therefore, the following properties hold

$$\begin{aligned}
&(a) \quad \dot{R}\Lambda + R\dot{\Lambda} = \Lambda\dot{R} + \dot{\Lambda}R = 0.\\
&(b) \quad (R\dot{\Lambda})^t = \dot{R}\Lambda = -R\dot{\Lambda}. \qquad (4.1.5)\\
&(c) \quad \mathrm{Tr}(R\dot{\Lambda}) = 0.
\end{aligned}$$

The property (4.1.5a) follows from the Leibniz rule. The (4.1.5b) can be proved using (4.1.5a) and it shows the skew-symmetry of $R\dot{\Lambda}$, so it implies (4.1.5c). Thus, in (4.1.4) one term vanishes and the first term reads

$$q^{ac}q^{bd}\dot{q}_{ab}\dot{q}_{cd} = d^{AC}d^{BD}\dot{d}_{AB}\dot{d}_{CD} + 2d^{AB}d_{CD}(R\dot{\Lambda})^{D}_{A}(R\dot{\Lambda})^{C}_{B} + 2(R\dot{\Lambda})^{B}_{C}(R\dot{\Lambda})^{C}_{B}.$$

For the other terms, it is useful to recall the Maurer-Cartan equation in local coordinates

$$\frac{\partial\theta^I_a}{\partial x^b} - \frac{\partial\theta^I_b}{\partial x^a} = f^I_{JK}\theta^J_a\theta^K_b. \qquad (4.1.6)$$

We are interested in computing a common factor to all the remaining terms $(N_{a;b} + N_{b;a})$.
Before this, we need to compute the Christoffel symbols

$$\begin{aligned}
\Gamma^c_{ab}N_c &= \frac{1}{2}q^{cd}(q_{ad,b} + q_{bd,a} - q_{ab,d})N_c\\
&= \frac{1}{2}(N_{a,b} + N_{b,a}) + \frac{1}{2}N^c\eta_{IJ}(\theta^I_a\theta^J_{c,b} + \theta^I_b\theta^J_{c,a} - (\theta^I_a\theta^J_b)_{,c}).
\end{aligned}$$

After some tedious calculations, the structure constants emerge using Eq.(3.1.3). In fact, the factor reads

$$\begin{aligned}
(N_{a;b} + N_{b;a}) &= (N_{a,b} + N_{b,a} - 2\Gamma^c_{ab}N_c)\\
&= -N^c\eta_{IJ}(\theta^I_a\theta^J_{c,b} + \theta^I_b\theta^J_{c,a} - (\theta^I_a\theta^J_b)_{,c})\\
&= -N^c\Big(\eta_{IJ}\theta^I_a(\theta^J_{c,b} - \theta^J_{b,c}) + \eta_{IJ}(\theta^I_b\theta^J_{c,a} - \theta^I_{a,c}\theta^J_b)\Big).
\end{aligned}$$

For symmetry of η_{IJ} and the Maurer-Cartan equation, we get

$$\begin{aligned}
(N_{a;b} + N_{b;a}) &= -N^c\Big(\eta_{IJ}\theta^I_a f^J_{KL}\theta^K_c\theta^L_b + 2\eta_{IJ}\theta^I_b\theta^J_{[c,a]}\Big)\\
&= 2N^K\eta_{IJ}f^J_{KL}\theta^I_{(a}\theta^L_{b)}.
\end{aligned}$$

We recall that the shift vector N^a on a homogeneous space can be factorised as $N^a(t,x) = N^I(t)\xi^a_I(x)$. For the second term, we find

$$\begin{aligned}
&q^{ac}q^{bd}(N_{a;b} + N_{b;a})(N_{c;d} + N_{d;c}) =\\
&=2N^A N^B(f^I_{AJ}f^J_{BI} + \eta^{IJ}\eta_{KL}f^K_{AI}f^L_{BJ}). \qquad (4.1.7)
\end{aligned}$$

The third term is the double product; it is trivial that

$$q^{ac}q^{bd}\dot{q}_{ab}(N_{c;d}+N_{d;c}) = q^{ac}q^{bd}(N_{a;b}+N_{b;a})\dot{q}_{cd}$$

since one can obtain one from another, via renaming $a \leftrightarrow c$, $b \leftrightarrow d$. This term can be easily calculated similarly to the second term

$$q^{ac}q^{bd}\dot{q}_{ab}(N_{c;d}+N_{d;c}) = -2N^K\eta^{IJ}\dot{\eta}_{JL}f^L_{KI}. \tag{4.1.8}$$

The fourth term appears quadratically in the Lagrangian; it is

$$q^{ab}(\dot{q}_{ab}-N_{a;b}-N_{b;b}) = \eta^{IJ}\dot{\eta}_{IJ}+2N^K f^J_{KJ}. \tag{4.1.9}$$

Recalling that for Bianchi A class models $f^J_{KJ}=0$, this term does not depend on the shift vector. To complete the decomposition, it remains to write η_{IJ} and its inverse η^{IJ} in terms of diagonal and rotation matrices. As matrix $\eta = \Lambda d R$, then,

$$\eta^{-1}\dot{\eta} = \Lambda d^{-1}R\dot{\Lambda}dR + \Lambda d^{-1}\dot{d}R + \Lambda\dot{R},$$

and, from properties (4.1.5), $\mathrm{Tr}(\eta^{-1}\dot{\eta}) = \mathrm{Tr}(d^{-1}\dot{d})$.
The ADM Lagrangian can be written in terms of scale factors and rotation matrices

$$\begin{aligned}L_{ADM} = {} & N|\det(\theta^I_a)|\sqrt{\det(d_{AB})}\Big(\mathrm{R}_q + \frac{1}{4N^2}\Big(d^{AC}d^{BD}\dot{d}_{AB}\dot{d}_{CD}\\ & + 2d^{AB}d_{CD}(R\dot{\Lambda})^D_A(R\dot{\Lambda})^C_B + 2(R\dot{\Lambda})^B_C(R\dot{\Lambda})^C_B + 2N^AN^B(f^I_{AJ}f^J_{BI}+\eta^{IJ}\eta_{KL}f^K_{AI}f^L_{BJ})\\ & + 4N^K\eta^{IJ}\dot{\eta}_{JL}f^L_{KI} - d^{IJ}\dot{d}_{IJ}d^{KL}\dot{d}_{KL}\Big)\Big).\end{aligned}$$

The Lagrangian can be further manipulated. First of all, the scalar curvature can be written in terms of the structure constants and η only [122]. Moreover, d can be written explicitly as a diagonal matrix $d_{AB} = a^2_{(A)}\delta_{AB}$, where a_1, a_2, a_3 are the scale factors, hence, $\sqrt{\det(d_{AB})} = |a_1a_2a_3|$.
Furthermore, we can bring out $|\det(\theta^I_i)|$ from the integration on the space-time to compute the Einstein-Hilbert action S_{EH}, because it is the only term that depends on the point of the hypersurface. It means

$$\begin{aligned}S_{\mathrm{EH}} &= -\frac{1}{\kappa}\int dt\, d^3x\, L_{ADM}(t,x) = -\frac{1}{\kappa}\int dt\,\mathcal{L}_{ADM}(t)\int_\Sigma d^3x\,|\det(\theta^I_i)|\\ &= -\frac{1}{\kappa}V_0\int dt\,\mathcal{L}_{ADM}(t).\end{aligned} \tag{4.1.10}$$

Hence, one can consider a Lagrangian that depends only on time $\mathcal{L}_{ADM}(t)$, which is defined as the homogeneous part of the Lagrangian defined in Eq.(4.1.1).
The Lagrangian is complex, and its writing in terms of elementary functions is quite difficult to read. An immediate simplification is to consider the Bianchi I model, in which the structure constants vanish. Notice that, in this case, all terms which contain the shift vector vanish.

4.1.2 Ashtekar variables

The loop approach requires computing the densitised dreibein and the Ashtekar variables. The dreibein vectors are pretty simple, and they follow from the metric

$$q_{ab} = \eta_{IJ}\theta^I_a\theta^J_b = \delta_{ij}e^i_a e^j_b. \tag{4.1.11}$$

Using the decomposition of η_{IJ} in Eq.(4.1.2), it easy to check

$$e^i_a = a_{(i)}R^i_I\theta^I_a \qquad i \text{ is not summed.} \tag{4.1.12}$$

While its dual reads

$$e^a_i = \tfrac{1}{a_{(i)}}\Lambda^I_i\xi^a_I \qquad i \text{ is not summed.} \tag{4.1.13}$$

The proof that one is the dual of the other is trivial

$$e^i_a e^a_j = \tfrac{a_{(i)}}{a_{(j)}}R^i_I\theta^I_a\Lambda^J_j\xi^a_J = \tfrac{a_{(i)}}{a_{(j)}}R^i_I\Lambda^I_j = \tfrac{a_{(i)}}{a_{(j)}}\delta^i_j = \delta^i_j.$$

From this, the densitised dreibein is defined as

$$\begin{aligned} E^a_i =& \sqrt{\det(q)}e^a_i = |\det(\theta^J_b)|\frac{|a_1a_2a_3|}{a_{(i)}}\Lambda^I_i\xi^a_I \\ =& \sqrt{\det(q_0)}\,\text{sgn}(a_{(i)})|a_ja_k|\Lambda^I_i\xi^a_I \qquad \text{with } \epsilon_{ijk}=1. \end{aligned} \tag{4.1.14}$$

Notice that Λ is not a pure gauge rotation, but it is a physical rotation, applied to the left-invariant vectors, necessary to have a non-diagonal metric.
To compute the connection, we use the definition $A^i_a = \Gamma^i_a + \beta K^i_a$. First of all, the extrinsic curvature is calculated

$$\begin{aligned} K^i_a &= K_{ab}e^{ib} = \frac{1}{2N}(\dot{q}_{ab} - N_{a;b} - N_{b;a})q^{bc}e^i_c \\ &= \frac{1}{2N}a_{(i)}R^i_L(\eta^{LJ}\dot{\eta}_{JI} + N^A\eta^{LK}\eta_{IJ}f^J_{AK} + N^Af^L_{AI})\theta^I_a. \end{aligned} \tag{4.1.15}$$

For the spin part $\Gamma^i_a = \frac{1}{2}\epsilon^i{}_{kj}\omega^{jk}_a$, the spin connection is evaluated

$$\omega^{ij}_a = \frac{1}{2}\tfrac{a_{(j)}}{a_{(i)}}\Lambda^I_iR^j_Jf^J_{IK}\theta^K_a - \frac{1}{2}\tfrac{a_{(i)}}{a_{(j)}}\Lambda^I_jR^i_Jf^J_{IK}\theta^K_a - \frac{1}{2}\tfrac{1}{a_{(i)}a_{(j)}}\eta_{LK}\Lambda^I_i\Lambda^J_jf^L_{JI}\theta^K_a. \tag{4.1.16}$$

One can check that this expression is skew-symmetric in a, b. Hence, for the spin term we obtain

$$\Gamma^i_a = \frac{1}{2}\epsilon^{ijk}\tfrac{a_k}{a_j}\Lambda^I_jR^k_Jf^J_{KI}\theta^K_a + \frac{1}{4}\epsilon^{ijk}\tfrac{1}{a_ja_k}\eta_{LK}\Lambda^I_j\Lambda^J_kf^L_{JI}\theta^K_a. \tag{4.1.17}$$

The connection A^i_a is now expressed in terms of scale factor, rotation matrices and structure constants

$$\begin{aligned} A^i_a = \Big(&\frac{1}{2}\epsilon^{ijk}\tfrac{a_k}{a_j}\Lambda^J_jR^k_Kf^K_{IJ} + \frac{1}{4}\epsilon^{ijk}\tfrac{1}{a_ja_k}\eta_{IJ}\Lambda^K_j\Lambda^L_kf^J_{LK} \\ &+ \frac{\beta}{2N}a_{(i)}R^i_L(\eta^{LJ}\dot{\eta}_{JI} + N^A\eta^{LK}\eta_{IJ}f^J_{AK} + N^Af^L_{AI})\Big)\theta^I_a. \end{aligned} \tag{4.1.18}$$

As the Lagrangian, the expression in terms of elementary functions is not easy to read. However, this calculation shows explicitly that the connection is linearly dependent on the left-invariant 1-form, as one expects.
Finally, one wants to find the dependence of p_i^I and ϕ_I^i on the scale factors and Euler angles. Since the dependence of A_a^i on the left-invariant 1-form is made explicit in Eq.(4.1.18), the components of the linear morphism ϕ can be easily written

$$\begin{aligned}\phi_I^i =& \frac{1}{2}\epsilon^{ijk}\tfrac{a_k}{a_j}\Lambda_j^J R_K^k f_{IJ}^K + \frac{1}{4}\epsilon^{ijk}\tfrac{1}{a_j a_k}\eta_{IJ}\Lambda_j^K\Lambda_k^L f_{LK}^J \\ &+ \frac{\beta}{2N}a_{(i)}R_L^i(\eta^{LJ}\dot{\eta}_{JI} + N^A\eta^{LK}\eta_{IJ}f_{AK}^J + N^A f_{AI}^L).\end{aligned} \tag{4.1.19}$$

While p_i^I can be computed from the metric

$$q_{ab} = d_{AB}R_I^A R_J^B\theta_a^I\theta_b^J = |\det(p_i^I)|p_I^k p_J^k\theta_a^I\theta_b^J,$$

from which one obtains

$$\sqrt{\det(q)} = |a_1a_2a_3||\det(\theta_a^J)| = \sqrt{|\det(p_i^I)|}|\det(\theta_a^J)|. \tag{4.1.20}$$

Recalling that the formula for the densitised dreibein is

$$E_i^a = \sqrt{\det(q)}e_i^a = \sqrt{\det(q_0)}|\frac{|a_1a_2a_3|}{a_{(i)}}\Lambda_i^I\xi_I^a.$$

Hence, p_i^I reads

$$p_i^I = \operatorname{sgn}(a_{(i)})|a_j a_k|\Lambda_i^I \quad \text{with } \epsilon_{ijk} = 1. \tag{4.1.21}$$

The classical theory presents eight disconnected cases, one for each choice of signs for a_1, a_2, a_3; in fact, the eigenvalues of the metric cannot vanish. Thus, without loss of generality, one can consider $a_1, a_2, a_3 > 0$, for all $t \in \mathbb{R}$; in the vierbein representation, this choice means that the densitised dreibein has the same orientation as the left-invariant vectors.

4.2 The Bianchi I case

In the non-diagonal Bianchi models, the Lagrangian and the connections have a long and difficult expression, so the study with respect to the scale factors and Euler angles is too complicated. The Bianchi I model provides a huge simplification of the formulae. In such a model, many terms in the Lagrangian and the connection vanish, and they come out very simplified. Considering the Bianchi I model with a metric as in (4.1.2), the ADM Lagrangian is derived by Eq.(4.1.1), and it reads

$$\begin{aligned}\mathcal{L}_{ADM} =& \frac{1}{4N}\sqrt{\det(d_{AB})}\Big(d^{AC}d^{BD}\dot{d}_{AB}\dot{d}_{CD} \\ &+ 2d^{AB}d_{CD}(R\dot{\Lambda})_A^D(R\dot{\Lambda})_B^C \\ &+ 2(R\dot{\Lambda})_C^B(R\dot{\Lambda})_B^C - d^{IJ}\dot{d}_{IJ}d^{KL}\dot{d}_{KL}\Big).\end{aligned} \tag{4.2.1}$$

Notice that the shift vector does not appear in the Bianchi I Lagrangian. In fact, as seen before, the shift vector is always paired with the structure constants; therefore, the gauge fixing $N^a = 0$ has no impact on the equations of motion. This is also valid for the connection, which we now calculate.
Since the spin connection Γ^i_a vanishes in the Bianchi I model, the connection has a really simple expression

$$\phi^i_I = \frac{\beta}{2N} a_{(i)} R^i_L \eta^{LJ} \dot{\eta}_{JI}. \tag{4.2.2}$$

However, its expression as a matrix function of scale factors and Euler angles remains difficult to read. Nevertheless, we can do some manipulation, considering the decomposition of η, we can rewrite the connection as

$$\phi^i_I = \frac{\beta}{2N a_{(i)}} \Lambda^J_i \, \dot{\eta}_{JI}. \tag{4.2.3}$$

Moreover, the diagonal case can be obtained by considering $R^i_I = \delta^i_I$. From the previous expression, we get

$$\phi^a_I = \frac{\beta}{N} \dot{a}_{(i)} \delta^i_I.$$

From which, in the isotropic case, given by $a_1 = a_2 = a_3 = a$, the connection reads

$$\phi^i_I = \frac{\beta}{N} \dot{a} \delta^i_I,$$

that is the same in [15].

Unlike the connection, we can write the Lagrangian explicitly in terms of the metric variables

$$\begin{aligned}
\mathcal{L}_{ADM} =& \frac{1}{2N a_1 a_2 a_3} \Big(-4 a_1 a_2 a_3 \left(a_1 \dot{a}_2 \dot{a}_3 + a_2 \dot{a}_1 \dot{a}_3 + a_3 \dot{a}_1 \dot{a}_2 \right) \\
& - 2 a_1^2 a_2^2 a_3^2 \left(2\dot{\theta}\dot{\varphi} \cos\psi + \dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2 \right) \\
& + a_1^4 \left(a_2^2 (\cos\theta \, \dot{\psi} + \sin\theta \sin\psi \, \dot{\varphi})^2 + a_3^2 (\dot{\theta} + \cos\psi \, \dot{\varphi})^2 \right) \\
& + a_2^4 \left(a_1^2 (\cos\theta \sin\psi \, \dot{\varphi} - \sin\theta \, \dot{\psi})^2 + a_3^2 (\dot{\theta} + \cos\psi \, \dot{\varphi})^2 \right) \\
& + a_3^4 \left(a_1^2 (\cos\theta \sin\psi \, \dot{\varphi} - \sin\theta \, \dot{\psi})^2 + a_2^2 (\cos\theta \, \dot{\psi} + \sin\theta \sin\psi \, \dot{\varphi})^2 \right) \Big).
\end{aligned} \tag{4.2.4}$$

The reduction to the diagonal case can be implemented considering the Euler angles constant and null; thus, only the first term remains

$$\mathcal{L}^{\text{diag}}_{ADM} = -\frac{2}{N} \left(a_1 \dot{a}_2 \dot{a}_3 + a_2 \dot{a}_1 \dot{a}_3 + a_3 \dot{a}_1 \dot{a}_2 \right).$$

Hence, for the isotropic case, the Lagrangian is proportional to the scalar curvature in the flat FLRW model, up to a total derivative

$$\mathcal{L}^{\text{iso}}_{ADM} = -\frac{6}{N} a \dot{a}^2.$$

From the Lagrangian $\mathcal{L}_{ADM}$, we can compute the conjugate momenta of the metric variables. Clearly, we have the momenta

$$\begin{aligned}
\frac{\partial \mathcal{L}_{ADM}}{\partial \dot{a}_1} &= -\frac{2}{N}(a_3\dot{a}_2 + a_2\dot{a}_3),\\
\frac{\partial \mathcal{L}_{ADM}}{\partial \dot{a}_2} &= -\frac{2}{N}(a_3\dot{a}_1 + a_1\dot{a}_3),\\
\frac{\partial \mathcal{L}_{ADM}}{\partial \dot{a}_3} &= -\frac{2}{N}(a_2\dot{a}_1 + a_1\dot{a}_2).
\end{aligned}$$

The conjugate momenta of the scale factors are the usual ones of the diagonal case. Instead, the conjugate momenta of the angles have a slightly difficult expression.

$$\begin{aligned}
\frac{\partial \mathcal{L}_{ADM}}{\partial \dot{\theta}} &= \frac{a_3}{Na_1a_2}(a_1^2 - a_2^2)^2(\dot{\theta} + \cos\psi\,\dot{\varphi}),\\
\frac{\partial \mathcal{L}_{ADM}}{\partial \dot{\psi}} &= \frac{1}{Na_1a_2a_3}\Big(a_1^4a_2^2(\cos^2\theta\,\dot{\psi} + \sin\theta\cos\theta\sin\psi\,\dot{\phi})\\
&\qquad + a_2^4a_1^2(\sin^2\theta\,\dot{\psi} - \sin\theta\cos\theta\sin\psi\,\dot{\phi})\\
&\qquad + a_3^4\Big(a_1^2(\sin^2\theta\,\dot{\psi} - \sin\theta\cos\theta\sin\psi\,\dot{\phi})\\
&\qquad\quad + a_2^2(\cos^2\theta\,\dot{\psi} + \sin\theta\cos\theta\sin\psi\,\dot{\phi})\Big)+\\
&\qquad - 2a_1^2a_2^2a_3^2\,\dot{\psi}\Big),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}_{ADM}}{\partial \dot{\varphi}} &= \frac{1}{Na_1a_2a_3}\Big(a_1^4\Big(a_2^2\sin\theta\sin\psi(\cos\theta\,\dot{\psi} + \sin\theta\sin\psi\,\dot{\varphi})\\
&\qquad\quad + a_3^2\cos\psi(\dot{\theta} + \cos\psi\,\dot{\varphi})\Big)\\
&\qquad + a_2^4\Big(a_1^2\cos\theta\sin\psi(\cos\theta\sin\psi\,\dot{\varphi} - \sin\theta\,\dot{\psi})\\
&\qquad\quad + a_3^2\cos\psi(\dot{\theta} + \cos\psi\,\dot{\varphi})\Big)\\
&\qquad + a_3^4\Big(a_1^2\cos\theta\sin\psi(\cos\theta\sin\psi\,\dot{\varphi} - \sin\theta\,\dot{\psi})\\
&\qquad\quad + a_2^2\sin\theta\sin\psi(\cos\theta\,\dot{\psi} + \sin\theta\sin\psi\,\dot{\varphi})\Big)+\\
&\qquad - 2a_1^2a_2^2a_3^2(\dot{\theta}\cos\psi + \dot{\varphi})\Big).
\end{aligned}$$

In addition, we now verify that the Lagrangian in Eq.(4.2.4) give us the correct Einstein equations for the Bianchi I universe. The details of this calculation are discussed in the following Section, in which we also compare the resulting equations with the Einstein equations presented in [38].

4.2.1 Classical dynamics for non-diagonal Bianchi I universe

In Eq.(4.2.4), we have a reduced Lagrangian which describes the Bianchi I universe. Currently, we want to get the Einstein equations from this Lagrangian and verify

that they are the correct ones. The final comparison will be with the equations (34-37) in [38]. We recall that Bianchi I is the kinetic part of Bianchi IX; hence, all terms in the equations must contain a time derivative. Consequently, there is no contribution in the equations from the homogeneous terms in scale factors.
First of all, we want to rewrite the Lagrangian similarly to Eq.(39) in [38]. We define the inertia momenta

$$I_1 = \frac{(a_2^2 - a_3^2)^2}{a_2^2 a_3^2}, \ I_2 = \frac{(a_1^2 - a_3^2)^2}{a_1^2 a_3^2}, \ I_3 = \frac{(a_1^2 - a_2^2)^2}{a_1^2 a_2^2},$$

and the angular velocities

$$\begin{aligned} \omega_1 &= \cos\theta \sin\psi\, \dot{\varphi} - \sin\theta\, \dot{\psi}, \\ \omega_2 &= \sin\theta \sin\psi\, \dot{\varphi} + \cos\theta\, \dot{\psi}, \\ \omega_3 &= \dot{\theta} + \cos\psi\, \dot{\varphi}. \end{aligned} \tag{4.2.5}$$

Moreover, we use the harmonic time gauge in which $N = a_1 a_2 a_3$. The Lagrangian reads

$$\mathcal{L}_{ADM} = -2\left(\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_1\dot{a}_3}{a_1a_3} + \frac{\dot{a}_2\dot{a}_3}{a_2a_3}\right) + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2,$$

now the dot indicates the derivative with respect to the harmonic time.
In this gauge, the conjugate momenta to the angles are

$$\begin{aligned} p_\theta &= I_3\omega_3, \\ p_\psi &= I_2\omega_2\cos\theta - I_1\omega_1\sin\theta, \\ p_\varphi &= I_1\omega_1\cos\theta\sin\psi + I_2\omega_2\sin\theta\sin\psi + I_3\omega_3\cos\psi. \end{aligned}$$

Then, we impose the Euler-Lagrange equation for the scale factors, obtaining three equations

$$\begin{aligned} &(a)\ \ \omega_2^2 b^2(a_1^4 - a_3^4) + \omega_3^2 a_3^2(a_2^4 - a_1^4) + 2a_1^2(a_2^2(\dot{a}_3^2 - a_3\ddot{a}_3) + a_3^2(\dot{a}_2^2 - a_2\ddot{a}_2)) = 0. \\ &(b)\ \ \omega_1^2 a_1^2(a_3^4 - a_2^4) + \omega_3^2 a_3^2(a_1^4 - a_2^4) + 2a_2^2(a^2(\dot{a}_3^2 - a_3\ddot{a}_3) + a_3^2(\dot{a}_1^2 - a_1\ddot{a}_1)) = 0. \\ &(c)\ \ \omega_1^2 a_1^2(a_2^4 - a_3^4) + \omega_2^2 a_2^2(a_1^4 - a_3^4) + 2a_3^2(a_1^2(\dot{a}_2^2 - a_2\ddot{a}_2) + a_2^2(\dot{a}_1^2 - a_1\ddot{a}_1)) = 0. \end{aligned} \tag{4.2.6}$$

Considering linear combinations, it is possible to show clearly that they are the right Einstein equations. From the calculation $(b) + (c) - (a)$ we obtain

$$\omega_3^2 a_3^2(a_1^4 - a_2^4) + \omega_2^2 a_2^2(a_1^4 - a_3^4) - 2a_1^2 a_2^2 a_3^2 \frac{d\log a_1}{d\tau} = 0,$$

where τ is the harmonic time. It can be rewritten as

$$2\frac{d\log a_1}{d\tau} - \frac{a_1^2a_2^2(a_1^2 + a_2^2)}{(a_1^2 - a_2^2)^3} I_3^2\omega_3^2 - \frac{a_1^2a_3^2(a_1^2 + a_3^2)}{(a_1^2 - a_3^2)^3} I_2^2\omega_2^2 = 0,$$

which is Eq.(35) from [38] up to homogeneous terms in scale factors. Clearly, permuting a_1, a_2, a_3 we find the Eq.(36) and (37) from [38].

We proceed in the same way for the equations of motion of the angles. The Euler-Lagrange equation for θ results

$$\frac{d}{d\tau}(I_3\omega_3) - (I_2 - I_1)\omega_1\omega_2 = 0.$$

Linear combinations of the Euler-Lagrange equations for ψ and φ give us

$$\begin{aligned}
&\frac{d}{d\tau}(I_1\omega_1) - (I_3 - I_2)\omega_2\omega_3 = 0,\\
&\frac{d}{d\tau}(I_2\omega_2) - (I_1 - I_3)\omega_1\omega_3 = 0.
\end{aligned}$$

These equations are equations for a free rotation as in Eq.(35) from [38].

4.2.2 Constraints and their algebra for the Bianchi I model

The Bianchi I model simplifies the expression of the connection with respect to the metric variables. Furthermore, the constraints referred to the Ashtekar variables, presented in Eq.(3.1.6) for a generic homogeneous model, have simple expressions in the Bianchi I model:

$$\begin{aligned}
&G_i = \epsilon_{ijk}\phi^j_I p^I_k,\\
&V_I = G_i\phi^i_I,\\
&S = -\frac{1}{\kappa\beta^2|\det(p^K_k)|}\left(p^I_i\phi^i_I p^J_j\phi^j_J - p^I_i\phi^i_J p^J_j\phi^j_I\right).
\end{aligned} \tag{4.2.7}$$

Notice that the Diffeomorphism constraint D_I does not play any role. It is a linear combination of the Gauß constraint and the Vector constraint, and it identically vanishes $\mathrm{D}_I = 0$. The Diffeomorphism constraint is not relevant for diagonal homogeneous models [51]. However, in a generic homogeneous model, it has a non-null expression, as shown in [44], but in the Bianchi I model. So, we will exclude the Diffeomorphism constraint from the computation of the classical constraint algebra.
We are interested in computing the Poisson brackets between the constraints to find their algebra. The algebra of the constraints is evaluated on the phase space (ϕ^I_i, p^j_J), with canonical fundamental Poisson brackets $\{\phi^I_i, p^j_J\} = \frac{\kappa\beta}{2V_0}\delta^j_i\delta^I_J$. It is convenient to introduce a rescaled scalar constraint $S' = \kappa\beta^2|\det(p^I_i)|S$, it is well defined because $|\det(p^I_i)|$ is always strictly greater than zero.
Furthermore, the Poisson brackets of the constraints with the phase space variables read

$$\begin{aligned}
&\{G_i, p^I_j\} = \tfrac{\kappa\beta'}{2}\,\epsilon_{ijk}p^I_k,\\
&\{G_i, \phi^j_I\} = \tfrac{\kappa\beta'}{2}\,\epsilon_{ijk}\phi^k_I,\\
&\{S', p^I_i\} = -\kappa\beta'\left(p^I_i p^J_j\phi^j_J - p^J_i p^I_j\phi^j_J\right),\\
&\{S', \phi^i_I\} = \kappa\beta'\left(\phi^i_I p^J_j\phi^j_J - \phi^i_J p^J_j\phi^j_I\right),
\end{aligned}$$

where $\beta' = \beta/V_0$ is the Immirzi parameter rescaled by the fiducial volume. From which we can easily figure out the Poisson brackets between the constraints. For the Gauß constraint, we get

$$\{G_i, G_j\} = \tfrac{\kappa\beta'}{2}\epsilon_{jkl}\left(\epsilon_{ikm}\phi_I^m p_l^I + \epsilon_{ilm}p_m^I\phi_I^k\right) = \tfrac{\kappa\beta'}{2}\epsilon_{ijm}\epsilon_{lkm}\phi_I^l p_k^I.$$

While for the Scalar constraint, we obtain

$$\{G_i, S'\} = \kappa\beta'\left(G_i p_l^J\phi_J^l - \epsilon_{ijk}p_k^I\phi_J^j p_l^J\phi_I^l - G_i p_l^J\phi_J^l + \epsilon_{ijk}\phi_I^j p_k^J p_l^I\phi_J^l\right) = 0.$$

Hence, the algebra generated by the Gauß constraint and the Scalar constraint reads

$$\begin{aligned}&\{G_i, G_j\} = \tfrac{\kappa\beta'}{2}\epsilon_{ijk}G_k,\\ &\{G_i, S'\} = 0.\end{aligned} \tag{4.2.8}$$

The algebra is closed, and all the constraints are first-class. It is also evident from the Poisson brackets that the Gauß constraint is the generator of the gauge transformation: it rotates the internal index with the $\mathfrak{su}(2)$ structure constant, while it leaves invariant the scalar quantities and the coordinate indices I.

Starting from the phase space (ϕ_i^I, p_J^j), we can perform a further reduction and use the metric variables. This reduction changes the degrees of freedom of the theory (in particular, we have six variables instead of nine), hence we expect some changes in the constraints. Using Eq.(4.2.2) and (4.1.21), the constraints can be written in terms of metric variables. For the Scalar constraint, we obtain

$$\begin{aligned}S^{\mathrm{ph}} = \frac{1}{2N^2 a_1 a_2 a_3}\Big(&-4a_1a_2a_3\left(a_1\dot{a}_2\dot{a}_3 + a_2\dot{a}_1\dot{a}_3 + a_3\dot{a}\dot{a}_2\right)\\ &-2a_1^2a_2^2a_3^2\left(2\dot{\theta}\dot{\varphi}\cos\psi + \dot{\theta}^2 + \dot{\psi}^2 + \dot{\varphi}^2\right)\\ &+a_1^4\left(a_2^2(\cos\theta\,\dot{\psi} + \sin\theta\sin\psi\,\dot{\varphi})^2 + a_3^2(\dot{\theta} + \cos\psi\,\dot{\varphi})^2\right)\\ &+a_2^4\left(a^2(\cos\theta\sin\psi\,\dot{\varphi} - \sin\theta\,\dot{\psi})^2 + a_3^2(\dot{\theta} + \cos\psi\,\dot{\varphi})^2\right)\\ &+a_3^4\left(a_1^2(\cos\theta\sin\psi\,\dot{\varphi} - \sin\theta\,\dot{\psi})^2 + a_2^2(\cos\theta\,\dot{\psi} + \sin\theta\sin\psi\,\dot{\varphi})^2\right)\Big).\end{aligned} \tag{4.2.9}$$

While the Gauß constraint vanishes identically

$$G_i^{\mathrm{ph}} = 0. \tag{4.2.10}$$

The vanishing of the Gauß constraint implies, due to Theorem 1.2.1, that the Scalar constraint is exactly the Hamiltonian constraint in ADM variables. One can check that S^{ph} is the Hamiltonian constraint obtained from the ADM Lagrangian in (4.2.4). Hence, the connection-flux description in the metric variables is equivalent to the ADM one.

The vanishing of the Gauß constraint is due to the set of variables chosen; the set $\{a_1, a_2, a_3, \theta, \psi, \varphi\}$ is a set of metric variables, while the Gauß constraint is associated with a gauge transformation. Thus, without the gauge freedom, as in this case, the Gauß constraint vanishes identically.

4.3 Quantisation of the non-diagonal Bianchi I model

The non-diagonal Bianchi I model is strongly linked to the diagonal one. In fact, we will show that the geometrical information is contained in some "diagonal" quantities and the quantisation of the non-diagonal model follows the one proposed in [28] for diagonal Bianchi I models. These considerations suggest that the angles do not play any role in the kinematical quantisation. We can find a justification for it in the classical description: a constant rotation of the left-invariant vectors does not affect the description; in fact, a linear combination of left-invariant vectors is a left-invariant vector, and it is always possible to lead back to a diagonal metric, i.e. the rotation can be absorbed in θ^I. Considering a time-dependent rotation, the linear combination remains in the Lie algebra of the homogeneous space, and for each fixed time, the previous consideration holds; hence, in each surface at a fixed time, it is possible to have a diagonal metric. At the quantum level, it means that the angles do not affect the kinematics, and they contribute only to the dynamics of the theory. Thus, it is possible to use the kinematical Hilbert space of the diagonal theory as the core of the Hilbert space that describes the kinematics of the non-diagonal theory.

4.3.1 Holonomy operator

The holonomy is the fundamental operator of the approach with loops. In LQC, the peculiar symmetries of the holonomy due to a simpler phase space are useful to quantise the Hamiltonian, and they enable us to define a suitable kinematical Hilbert space that renounces the $SU(2)$ construction to adopt an Abelian one. We want to verify if the properties of the homogeneous holonomies, studied in the diagonal case, hold in the non-diagonal models.
First, we want to find the usual expression of the holonomy in terms of trigonometric functions as in Eq.(3.1) in [28]. Considering the holonomy along an edge e_I with length λ_I

$$\exp\left(\int_{e_I} ds\, \dot{c}^a A_a^i \tau_i\right) = \exp\left(\lambda_I \phi_I^i \tau_i\right). \tag{4.3.1}$$

To do the expansion, the Taylor series of the exponential is considered

$$\begin{aligned}\exp\left(\lambda_I \phi_I^i \tau_i\right) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda_I \phi_I^i \tau_i)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\lambda_I \phi_I^i \tau_i)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\lambda_I \phi_I^i \tau_i)^{2n+1}.\end{aligned} \tag{4.3.2}$$

To proceed, we need to evaluate the square of the argument

$$\begin{aligned}(\lambda_I \phi_I^i \tau_i)^2 = \lambda_I^2 \phi_I^i \tau_i \phi_I^j \tau_j &= \lambda_I^2 \phi_I^i \phi_I^j \left(-\tfrac{1}{4}\delta_{ij} + \tfrac{1}{2}\epsilon_{ijk}\tau_k\right) \\ &= -\frac{\lambda_I^2}{4} \frac{\beta^2}{4N^2} \frac{\delta_{ij}}{c_i c_j} \Lambda_i^K \Lambda_j^J \dot{\eta}_{KI} \dot{\eta}_{JI} \\ &= -\frac{\lambda_I^2}{4} \frac{\beta^2}{4N^2} \eta^{KJ} \dot{\eta}_{KI} \dot{\eta}_{JI}.\end{aligned}$$

It is useful to define $c_{II} \doteq \frac{\beta}{2N}\sqrt{\eta^{KJ}\dot{\eta}_{KI}\dot{\eta}_{JI}}$. In such a way, the argument of the series reads

$$
\begin{aligned}
(\lambda_I \phi^i_I \tau_i)^{2n} &= \left(-\tfrac{1}{4}\lambda_I^2 c_{II}^2\right)^n = (-1)^n \left(\tfrac{1}{2}\lambda_I c_{II}\right)^{2n}, \\
(\lambda_I \phi^i_I \tau_i)^{2n+1} &= (-1)^n \left(\tfrac{1}{2}\lambda_I c_{II}\right)^{2n} \lambda_I \phi^i_I \tau_i = (-1)^n \left(\tfrac{1}{2}\lambda_I c_{II}\right)^{2n+1} \frac{2\phi^i_I \tau_i}{c_{II}}.
\end{aligned}
$$

Hence, the Taylor series reads

$$
\begin{aligned}
\exp\left(\lambda_I \phi^i_I \tau_i\right) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\lambda_I \phi^i_I \tau_i)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\lambda_I \phi^i_I \tau_i)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\tfrac{1}{2}\lambda_I c_{II}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\tfrac{1}{2}\lambda_I c_{II}\right)^{2n+1} \frac{(2\phi^i_I \tau_i)}{c_{II}} \qquad (4.3.3) \\
&= \cos\left(\tfrac{1}{2}\lambda_I c_{II}\right) + \frac{(2\phi^i_I \tau_i)}{c_{II}} \sin\left(\tfrac{1}{2}\lambda_I c_{II}\right).
\end{aligned}
$$

It is evident that the issue is the factor of the sine, which is not present in the isotropic and diagonal cases. However, we can check easily that, if $\phi^i_I = c_i \delta^i_I$, then $c_{II} = c_I$, where c_I are the diagonal variables. Thus, we find the usual expression $\exp\left(\lambda_I \phi^i_I \tau_i\right) = \cos\left(\frac{1}{2}\lambda_I c_I\right) + 2\tau_I \sin\left(\frac{1}{2}\lambda_I c_I\right)$ presented in Chapter 3 for the diagonal case.

With this expansion, we can derive some properties of the holonomies in the non-diagonal Bianchi I model. First of all, the commutation relations are enounced in the following lemma:

Lemma 4.3.1. *Considering the Lie group $SU(2)$ in its fundamental representation, i.e. as the group of 2×2 unitary matrices, associated with spin $j = \frac{1}{2}$. The commutator between holonomies reads as*

$$
\left[\exp\left(\lambda_I \phi^i_I \tau_i\right), \exp\left(\lambda_J \phi^j_J \tau_j\right)\right] = \frac{4}{c_{II} c_{JJ}} F^k_{IJ} \tau_k \sin\left(\tfrac{1}{2}\lambda_I c_{II}\right) \sin\left(\tfrac{1}{2}\lambda_J c_{JJ}\right).
$$

Proof. Fixed the representation $j = \frac{1}{2}$ means that the generators of the algebra are proportional to the Pauli matrices $\tau_i = -\frac{i}{2}\sigma_i$, then the matrix commutator defined on $M(2, \mathbb{C})$ still holds. The calculation follows from the expansion in (4.3.3), where the terms proportional to the identity are neglected because they do not contribute to the commutator

$$
\begin{aligned}
&[\exp\left(\lambda_I \phi^i_I \tau_i\right), \exp\left(\lambda_J \phi^j_J \tau_j\right)] = \\
&= \left[\frac{(2\phi^i_I \tau_i)}{c_{II}} \sin\left(\tfrac{1}{2}\lambda_I c_{II}\right), \frac{(2\phi^j_J \tau_j)}{c_{JJ}} \sin\left(\tfrac{1}{2}\lambda_J c_{JJ}\right)\right] \\
&= \frac{(2\phi^i_I)}{c_{II}} \sin\left(\tfrac{1}{2}\lambda_I c_{II}\right) \frac{(2\phi^j_J)}{c_{JJ}} \sin\left(\tfrac{1}{2}\lambda_J c_{JJ}\right) [\tau_i, \tau_j] \\
&= \frac{4}{c_{II} c_{JJ}} \epsilon_{ijk} \phi^i_I \phi^j_J \tau_k \sin\left(\tfrac{1}{2}\lambda_I c_{II}\right) \sin\left(\tfrac{1}{2}\lambda_J c_{JJ}\right).
\end{aligned}
$$

□

This is a completely generic formula valid for any homogeneous holonomy in Bianchi I. It is evident that, for $I = J$, the skew-symmetry of F^k_{IJ} assures the vanishing of the commutator of holonomy along the same edge. Moreover, it also holds in the diagonal case in a slightly simplified expression

$$[\exp(\lambda_I c_I \tau_I), \exp(\lambda_J c_J \tau_J)] = 4\,\epsilon_{IJk}\tau_k \sin\left(\tfrac{1}{2}\lambda_I c_I\right)\sin\left(\tfrac{1}{2}\lambda_J c_J\right).$$

In contrast, the isotropic case is fundamentally different. Owing to isotropy, no spatial direction can be distinguished, and therefore the holonomies along each edge must commute, with their commutator always vanishing. Consequently, the Abelian treatment of the isotropic case is justified: the group of isotropic holonomies is Abelian, and, in addition, the Gauß constraint identically vanishes.

Analogously, it is possible to derive another formula for the commutation property. It was originally derived by M. Bojowald in [44] for the diagonal case, but it is possible to generalise it to any homogeneous connection and to prove that it holds for $SU(2)$.

Lemma 4.3.2. *Let $g, h \in SU(2)$ in the fundamental representation. Then*

$$gh = hg + h^{-1}g + hg^{-1} - tr(hg^{-1}). \tag{4.3.4}$$

Proof. There are two possible approaches to proving the lemma: one based on (4.3.3), and the other using quaternions. To establish the result for the entire group $SU(2)$, the quaternionic formalism provides a more general framework.
Considering the isomorphism between $SU(2)$ and the unit quaternions

$$\begin{aligned} F :& SU(2) \to \mathbb{H} \\ & \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto q = \alpha + \beta j \qquad \alpha, \beta \in \mathbb{C} \end{aligned}$$

Recalling the quaternions algebra $ij = k, jk = i, ki = j$ and $i^2 = j^2 = k^2 = ijk = -1$. It is possible to define the complex conjugation as the change of the sign of i, j, k and for unit quaternions $q^{-1} = \bar{q}$. Notice that, using the fundamental representation of $SU(2)$, there is a link between trace and real part: $\mathrm{Tr}(g) = 2\mathrm{Re}(F(g)) = 2\mathrm{Re}(\alpha)$. Thus, it is enough to prove the formula for the unit quaternions, so the problem is reduced to doing some simple algebra. Let $q_1 = a+bi+cj+dk$ and $q_2 = w+xi+yj+zk$, the single terms are computed

$$\begin{aligned} q_1q_2 - q_2q_1 &= 2(cz-dy)i + 2(dx-bz)j + 2(by+cx)k, \\ q_2^{-1}q_1 &= (aw+bx+cy+dz) + (bw-ax+cz-dy)i \\ &\quad + (cw-ay+dx-bz)j + (dw-az+by-cx)k, \\ q_2q_1^{-1} &= (aw+bx+cy+dz) + (-bw+ax+cz-dy)i \\ &\quad + (ay-cw+dx-bz)j + (az-dw+by-cx)k. \end{aligned}$$

Where the real parts are evident. From which

$$\begin{aligned} & q_1q_2 - q_2q_1 - q_2^{-1}q_1 - q_2q_1^{-1} = \\ & = -2(aw+bx+cy+dz) = -2\mathrm{Re}(q_2q_1^{-1}). \end{aligned}$$

Where Re indicates the real part. □

Using the expansion (4.3.3), after some tedious calculations, we find that the formula by M. Bojowald is a subcase of this Lemma.

The curvature operator follows the usual definition of the canonical LQC, but the choice of the plaquette must be different. Recalling that

$$F^i_{ab}\tau_i = \lim_{\mathrm{Ar}\Box_{IJ}\to 0}\left(\frac{h_{\Box_{IJ}}-1}{Ar_{\Box_{IJ}}}\right)\theta^I_a\theta^J_b, \tag{4.3.5}$$

where $\Box_{IJ}$ is the plaquette, i.e. a rectangular closed path with edges along ξ_I and ξ_J. We want to emulate the procedure for the diagonal model and find a natural choice of the plaquette from which to compute the curvature.
In the non-diagonal case, p^I_i is not diagonal, but with a correct choice of the plaquette, we can bypass the problem and find three fluxes that are the same as the diagonal case. Considering the vectors

$$\zeta_i = \Lambda^I_i \xi_I, \tag{4.3.6}$$

they are elements of the Lie algebra of Σ and it is easy to check that $[\zeta_i, \zeta_j] = 0$. Moreover, along these vectors, the densitised dreibein vectors are diagonal

$$E^a_i = \sqrt{\det(q_0)}\frac{|a_1 a_2 a_3|}{a_{(i)}}\Lambda^I_i \xi^a_I = \sqrt{\det(q_0)}p_{(i)}\zeta^a_i, \tag{4.3.7}$$

where the expression of p_i can be derived by Eq.(4.1.21)

$$p_i = \mathrm{sgn}(a_i)|a_j a_k| \quad \text{with } \epsilon_{ijk} = 1. \tag{4.3.8}$$

Hence, since the choice of the plaquette for the regularisation of the curvature is completely arbitrary, we choose the plaquette $\Box_{ij}$ as a rectangular closed path with edges along ζ_i and ζ_j, that lies in the plane I-J rotated.
The holonomy along the edge ζ_i can be easily computed

$$\exp\left(\int_{\zeta_a} ds\, \dot{c}^a A^i_a \tau_i\right) = \exp\left(\lambda_i \zeta^a_i \phi^j_I \theta^I_a \tau_j\right) = \exp\left(\lambda_i \mathring{\phi}^j_i \tau_j\right), \tag{4.3.9}$$

where $\mathring{\phi}^j_i$ is defined as $\mathring{\phi}^j_i = L_i \Lambda^I_i \phi^j_I$. L_i is the fiducial length of the i-th edge of the rotated fiducial cell, which is equal to the canonical fiducial cell $L_i = L_I$ if $\delta_{iI} = 1$. In fact, the fiducial metric is invariant under rotations

$$q^{ab}_0 = \delta^{IJ}\xi^a_I \xi^b_J = \delta^{IJ} R^i_I \zeta^a_i R^j_J \zeta^b_j = \delta^{ij}\zeta^a_i \zeta^b_j. \tag{4.3.10}$$

Notice that $\mathring{\phi}^j_i$ has a explicit expression in terms of metric variables

$$\mathring{\phi}^j_i = \frac{\beta L_{(i)}}{2N a_{(j)}}\Lambda^J_j \Lambda^I_i\, \dot{\eta}_{JI}. \tag{4.3.11}$$

Writing down $\mathring{\phi}^j_i$, an interesting property becomes evident

$$\mathring{\phi}^j_i = \begin{pmatrix} L_1 c_1 & \frac{\beta L_2}{2N}\frac{a_2^2 - a_1^2}{a_1}\omega_3 & \frac{\beta L_3}{2N}\frac{a_1^2 - a_3^2}{a_1}\omega_2 \\ \frac{\beta L_1}{2N}\frac{a_2^2 - a_1^2}{a_2}\omega_3 & L_2 c_2 & \frac{\beta L_3}{2N}\frac{a_3^2 - a_2^2}{a_2}\omega_1 \\ \frac{\beta L_1}{2N}\frac{a_1^2 - a_3^2}{a_3}\omega_2 & \frac{\beta L_2}{2N}\frac{a_3^2 - a_2^2}{a_3}\omega_1 & L_3 c_3 \end{pmatrix} \tag{4.3.12}$$

where c_1, c_2, c_3 are the diagonal components of the connection as in the diagonal case

$$c_i = \frac{\beta}{N}\frac{da_i}{dt}, \tag{4.3.13}$$

and $\omega_1, \omega_2, \omega_3$ are defined in (4.2.5).
The connection is composed of a diagonal part that is exactly the connection of the diagonal case and out-of-diagonal terms, which depend on angles and are linear in their conjugate momenta. Notice that for a constant rotation, the connection is diagonal. Thus, since the holonomies are diagonal, one leads back to the diagonal case for both the kinematical and the dynamical theory.

4.3.2 Quantum geometry

In the previous section, with a suitable choice of the plaquette, the link with the diagonal case emerges. The role of the diagonal variables in the quantum geometry of the non-diagonal Bianchi I model will be examined in depth.
With the choice of the plaquette $\square_{ij}$ as above, one of the densitised dreibein vectors is orthogonal to the plaquette and its diagonal component can represent the area, so the argument for the diagonal case can also be emulated in the non-diagonal one. Moreover, the flux of the electric field over the face $\tilde{\sigma}_i$ of the rotated fiducial cell is proportional to p_i.
Recalling that the fiducial length is invariant under rotation of the edges, let us introduce new variables via rescaling $\tilde{p}_i = L_j L_k p_i$ with $\epsilon_{ijk} = 1$.

Lemma 4.3.3. *The area of $\tilde{\sigma}_i$ is $|\tilde{p}_i|$.*

Proof. Considering the face $\tilde{\sigma}_3$ on the rotated fiducial cell. Let $\{x^a\}$ local coordinates in which $\xi^a_I = \delta^a_I$. Since $[\zeta_i, \zeta_j] = 0$, then there exist local coordinates $\{\tilde{x}^a\}$ such that $\zeta^a_i = \delta^a_i$.
We can write the metric on Σ in these new coordinates $h = a_1^2(d\tilde{x}^1)^2 + a_2^2(d\tilde{x}^2)^2 + a_3^2(d\tilde{x}^3)^2$. Let $\iota : \tilde{\sigma}_3 \hookrightarrow \Sigma$ be the inclusion map, and $F : \Sigma \to \Sigma$ a rotation such that $F_*\xi_1 = \zeta_1$ and so on for 2 and 3.
For $\tilde{\sigma}_3$ we have:

1. $\tilde{\sigma}_3 = F(\sigma_3)$, where σ_3 is the face of the fiducial cell "transverse" to ξ_3.
2. The metric on the surface $\tilde{\sigma}_3$ is $\iota^* q = a_1^2(d\tilde{x}^1)^2 + a_2^2(d\tilde{x}^2)^2$. Hence, the volume form is $dV_{\iota^* q} = |a_1 a_2| d\tilde{x}^1 \wedge d\tilde{x}^2 = |p_3| d\tilde{x}^1 \wedge d\tilde{x}^2$.

Thus, the area of $\tilde{\sigma}_3$ reads

$$\begin{aligned}
\mathrm{Ar}(\tilde{\sigma}_3) &= \int_{\tilde{\sigma}_3} |p_3| d\tilde{x}^1 \wedge d\tilde{x}^2 = |p_3| \int_{F(\sigma_3)} d\tilde{x}^1 \wedge d\tilde{x}^2 \\
&= |p_3| \int_{\sigma_3} F^*(d\tilde{x}^1 \wedge d\tilde{x}^2) = |p_3| \int_{\sigma_3} \theta^1 \wedge \theta^2 \\
&= |p_3| L_1 L_2 = |\tilde{p}_3|.
\end{aligned}$$

The same procedure can be repeated for the other two faces. □

Considering also rescaled connection variables $\tilde{c}_i = L_i c_i$, the pair $(\tilde{c}_i, \tilde{p}_j)$ is exactly the one of the diagonal case. Hence, we can choose the phase space $\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, p_\theta, p_\psi, p_\varphi\}$ with (non-vanishing) Poisson brackets

$$\begin{aligned}
&\{\tilde{c}^i, \tilde{p}_j\} = \tfrac{\kappa\beta}{2}\delta^i_j,\\
&\{\theta, p_\theta\} = 1,\\
&\{\psi, p_\psi\} = 1,\\
&\{\varphi, p_\varphi\} = 1.
\end{aligned}$$

Considering the quantum states $|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle$, these states are eigenstates of the quantum geometry. For the Lemma 4.3.3, in such a state the face $\tilde{\sigma}_3$ in the plane ζ_1, ζ_2 of the fiducial cell rotated by (θ, ψ, φ) has area $|\tilde{p}_3|$. Furthermore, the classical volume has the same expression as the diagonal one, just like the volume operator. We obtain the classical expression

$$\begin{aligned}
V &= \int d^3x \sqrt{\left|\tfrac{1}{6}\epsilon^{ijk}\epsilon_{abc}E^a_i E^b_j E^c_k\right|}\\
&= V_0\sqrt{\left|\tfrac{1}{6}\epsilon^{ijk}\epsilon_{IJK}p^I_i p^J_j p^K_k\right|}\\
&= V_0\sqrt{\left|\tfrac{1}{6}\epsilon^{ijk}\epsilon_{IJK}\Lambda^I_i\Lambda^J_j\Lambda^K_k p_i p_j p_k\right|}\\
&= V_0\sqrt{|p_1p_2p_3|} = \sqrt{|\tilde{p}_1\tilde{p}_2\tilde{p}_3|}.
\end{aligned} \tag{4.3.14}$$

Thus, all the geometry information is contained in the "diagonal" variables $\tilde{p}_i$, which represent the fluxes, with their conjugate momenta $\tilde{c}_i$. Hence, as in the diagonal case, the pair $(\tilde{p}_i, \tilde{c}_i)$ can be quantised in the loop formalism, and their action on the states is

$$\begin{aligned}
&\hat{\tilde{p}}_1|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle = \tilde{p}_1|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle,\\
&\widehat{\exp(i\lambda\tilde{c}_1)}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle = |\tilde{p}_1 - \kappa\beta\hbar\lambda, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle,
\end{aligned} \tag{4.3.15}$$

and similarly for $\hat{\tilde{p}}_2$ and $\hat{\tilde{p}}_3$.

Since $\hat{\tilde{p}}_i$ is well-defined as an operator, the quantisation of the volume is trivial. On the previously defined states, the volume operator acts as

$$\hat{V}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle = \sqrt{|\tilde{p}_1\tilde{p}_2\tilde{p}_3|}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle.$$

The information about quantum geometry is wholly encoded by the fluxes $\tilde{p}_i$. The angles θ, ψ, φ represent how much non-diagonal the state is, i.e. greater the values more the dreibein is rotated with respect to the left-invariant vectors, which is important in the choice of a suitable plaquette on which to compute the curvature operator. As already seen, this choice is arbitrary and we can choose the plaquette whose area is proportional to $\tilde{p}_i$. Furthermore, without any ambiguity, the volume depends only on momenta. At the quantum level, the volume operator shares the same eigenstates as the flux operators, and its eigenvalues depend solely on those of the fluxes.

The kinematical states of the theory are linear combinations of $|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle$, in which the $\tilde{p}_i$ are quantised within the loop framework, while the angles can be quantised using the Weyl prescription. The fundamental operators read

$$e^{i\xi\hat{\theta}}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle = e^{i\xi\theta}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle, \tag{4.3.16}$$

$$e^{i\eta\hat{p}_\theta}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle = |\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta + \eta, \psi, \varphi\rangle, \tag{4.3.17}$$

and similarly for ψ and φ.
In this vector space, a norm can be defined such that it coincides with the one in the diagonal case. The diagonal states are the ones with null angles $|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0\rangle$. Thus, it is always possible to write any state as a linear transformation of a diagonal one

$$|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3\rangle = e^{i\varphi_1\hat{p}_\theta} e^{i\varphi_2\hat{p}_\psi} e^{i\varphi_3\hat{p}_\varphi}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0\rangle.$$

To define a kinematical Hilbert space, we need to equip it with a scalar product. From the quantisation properties, the scalar product must induce a norm that has to satisfy

$$\begin{aligned}
&||\,|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3\rangle||^2 = \\
&= \langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3\rangle \\
&= \langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0|e^{-i\varphi_3\hat{p}_\varphi} e^{-i\varphi_2\hat{p}_\psi} e^{-i\varphi_1\hat{p}_\theta} e^{i\varphi_1\hat{p}_\theta} e^{i\varphi_2\hat{p}_\psi} e^{i\varphi_3\hat{p}_\varphi}|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0\rangle \\
&= \langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0\rangle \\
&\doteq ||\,|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\rangle||^2_{diag} = 1,
\end{aligned}$$

in which $||\cdot||_{diag}$ is the norm of the state in the diagonal theory.
The property of leading back the scalar product to the diagonal one holds not only for the norm. In fact, consider a generic scalar product between two states

$$\begin{aligned}
&\langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3|\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3, \varphi'_1, \varphi'_2, \varphi'_3\rangle = \\
&= \langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0|e^{i(\varphi_1-\varphi'_1)\hat{p}_\theta} e^{i(\varphi_2-\varphi'_2)\hat{p}_\psi} e^{i(\varphi_3-\varphi'_3)\hat{p}_\varphi}|\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3, 0, 0, 0\rangle \\
&= \langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, 0, 0, 0|\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3, \varphi_1 - \varphi'_1, \varphi_2 - \varphi'_2, \varphi_3 - \varphi'_3\rangle.
\end{aligned}$$

If $\varphi_r = \varphi'_r$, we find the scalar product between diagonal states. So, the scalar product between equally rotated states is imposed to be

$$\langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3|\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3, \varphi_1, \varphi_2, \varphi_3\rangle \doteq \delta_{\tilde{p}_1\tilde{p}'_1}\delta_{\tilde{p}_2\tilde{p}'_2}\delta_{\tilde{p}_3\tilde{p}'_3}.$$

This is coherent with our interpretation of the quantum kinematics of the angles. There is no unique choice of a scalar product that induces the properties above. However, we require that $|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3\rangle$ are orthonormal states, then, in according with the quantisation procedure, we are allowed to choose the following scalar product

$$\begin{aligned}
&\langle\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3|\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3, \varphi'_1, \varphi'_2, \varphi'_3\rangle \\
&\doteq \delta_{\tilde{p}_1\tilde{p}'_1}\delta_{\tilde{p}_2\tilde{p}'_2}\delta_{\tilde{p}_3\tilde{p}'_3}\delta(\varphi_1 - \varphi'_1)\delta(\varphi_2 - \varphi'_2)\delta(\varphi_3 - \varphi'_3)
\end{aligned} \tag{4.3.18}$$

The kinematical Hilbert space of the theory is equipped with the scalar product (4.3.18) and it has an orthonormal basis $\{|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \theta, \psi, \varphi\rangle\}$. In the kinematical

Hilbert space, the angles do not play any role; hence, the kinematical theory reflects the properties of the geometric operators. The dynamics of the theory is governed by the quantisation of the following Hamiltonian

$$\begin{aligned} H = \frac{1}{\beta^2 \ell_P^2} \Bigg(&-(p_1 p_2 c_1 c_2 + p_1 p_3 c_1 c_3 + p_2 p_3 c_2 c_3) + \left(\tfrac{p_2}{p_1} - \tfrac{p_1}{p_2}\right)^{-2} p_\theta^2 \\ &+ \left(\tfrac{p_2}{p_3} - \tfrac{p_3}{p_2}\right)^{-2} (-\cos\theta \cot\psi\, p_\theta - \sin\theta\, p_\psi + \cos\theta \csc\psi\, p_\varphi)^2 \\ &+ \left(\tfrac{p_1}{p_3} - \tfrac{p_3}{p_1}\right)^{-2} (-\sin\theta \cot\psi\, p_\theta + \cos\theta\, p_\psi + \sin\theta \csc\psi\, p_\varphi)^2 \Bigg). \end{aligned}$$

This expression corresponds to the scalar constraint (4.2.9), written in the harmonic gauge and with respect to the diagonal fluxes and their conjugate momenta. It is noteworthy that the first term matches the Hamiltonian of the diagonal case and can therefore be quantised using the standard regularisation techniques developed in Loop Quantum Cosmology. The remaining terms are quadratic in the angular momenta, do not involve the connection components, and thus can be quantised unambiguously.

Another viable approach to defining the Hilbert space is to perform the quantisation at each fixed set of angles. In this setting, we obtain a family of Hilbert spaces $\mathcal{H}_a$ for each $a \in SO(3)$, all of which are isomorphic to the Hilbert space associated with the diagonal case $\mathcal{H}_{\text{LQC}}$. In analogy with the earlier construction, these spaces are spanned by basis elements of the form $|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3\rangle$, where the angular parameters $\varphi_1, \varphi_2, \varphi_3$ are held fixed. Formally, we can describe the total Hilbert space of the non-diagonal theory as a direct integral over $SO(3)$:

$$\int_{SO(3)}^{\oplus} \mathcal{H}_a \, d\mu_H(a) \cong L^2(SO(3), \mathcal{H}_{\text{LQC}}), \tag{4.3.19}$$

where μ_H denotes the Haar measure on the group $SO(3)$. In this construction, to be mathematically well-defined, each Hilbert space $\mathcal{H}_a$ must be separable. In this context, $\mathcal{H}_{\text{LQC}}$ may serve as the regularised Hilbert space. The resulting quantum states are square-integrable sections of the vector bundle $\bigsqcup_{a \in SO(3)} \mathcal{H}_a$. Notably, Kuiper's theorem ensures that this Hilbert bundle is topologically trivial, thereby justifying the identification with the Hilbert space $L^2(SO(3), \mathcal{H}_{\text{LQC}})$. The basis states $|\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3\rangle$ can then be understood as generalised eigenstates or distributions on this space, particularly as Dirac delta functions in the angular variables. This is analogous to the role of position eigenstates in standard Quantum Mechanics. Furthermore, this space is naturally equipped with an inner product induced by the Haar measure on $SO(3)$, which endows it with the necessary structure to serve as a well-defined and physically meaningful Hilbert space for the quantisation of the non-diagonal cosmological models.

4.4 Representation quantisation

The quantisation of homogeneous models is already implemented (see [51]). In that case, the same procedure of the full theory is used to find the kinematical Hilbert space and the fundamental operators: holonomies and the momentum operators. The theory is similar to the LQG, but it is described in terms of the homogeneous part of the connection ϕ^i_I instead of the full one. This similarity is due to the fact that there is a non-null Gauß constraint on the minisuperspace, and then, we must deal with an $SU(2)$ holonomy. In that approach, the states are vectors in a representation space of $SU(2)$. Nevertheless, in the 4.2.2, the cosmological theory of the non-diagonal Bianchi I model, described in metric variables, emerges naturally with a null Gauß constraint. Consequently, the non-diagonal models are close to the diagonal ones [15, 28], which present a null Gauß constraint and Abelian holonomies. In this common approach, the states are almost-periodic functions in the connection variables. Moreover, the holonomy has the same commutation relation as the diagonal case by Lemma 4.3.1. Hence, we want to express the theory as an Abelian theory, as in the usual approach to diagonal models. To do that, we implement the kinematical quantisation procedure shown in [51] in our case, imposing that the states are representations of $U(1)^3$. This formulation seems to be more general than the one proposed in 4.3.2, as it holds for any Bianchi models, but it presents some issues and problems with physical interpretation; in particular, in the regularisation of the Hamiltonian operator, and in the interpretation of such holonomies.

4.4.1 $U(1)^3$-holonomy

The choice of the Ashtekar variables c_1, c_2, c_3 for a cosmological setting seems to be the only reasonable one. However, in the non-diagonal case, there is an ambiguity. These variables do not emerge naturally from the geometry, but they are useful to show the relation with the diagonal case. We are interested in looking for a more natural set of variables. To do that, the expansion in (4.3.3) can be useful; we can choose the argument of the trigonometric functions as the configuration variables

$$c_{II} = \sqrt{\sum_i \phi^i_I \phi^i_I}. \tag{4.4.1}$$

The square root exists because the argument is a sum of squares (in this Section, we do not use the Einstein summation convention).
Moreover, it is possible to find the conjugate momenta in terms of the connection and dreibein. Considering a function F^J, imposing the invariance of the Poisson brackets

$$\frac{\kappa\beta'}{2}\delta^J_I = \{c_{II}, F^J\} = \frac{\kappa\beta'}{2}\frac{1}{c_{II}}\sum_{i,K}\delta^K_I \phi^i_K \frac{\partial F^J}{\partial p^K_i}, \tag{4.4.2}$$

we obtain that $\frac{\partial F^J}{\partial p^K_i} \propto (\phi^{-1})^J_i$. Hence, the momenta read

$$p^I = c_{II}\sum_i (\phi^{-1})^I_i p^I_i. \tag{4.4.3}$$

Notice that also $F^J = \frac{1}{c_{JJ}}\sum_i p^J_i \phi^i_J$ gives us the correct Poisson bracket, but the first one will be more useful in the next Section.

For the quantisation, the procedure proposed by M. Bojowald in [51] is implemented in the $U(1)^3$ case. In particular, to adapt the formalism to the original paper, rescaled variables are considered

$$\tilde{c}_{II} = L_I \sqrt{\sum_i \phi_I^i \phi_I^i}, \qquad \tilde{p}^I = L_J L_K p^I \quad \text{with } \epsilon_{IJK} = 1. \tag{4.4.4}$$

The holonomy-flux algebra is implemented considering representations of holonomies as kinematical states of the theory, where the quantum number n is the label of the representation. Thus, a state reads $\pi_{\lambda,n}(g_I) = (\exp(i\lambda\tilde{c}_{II}))^n$, with $\lambda \in \mathbb{Q}$ and $n \in \mathbb{N}$. The binary operations have really simple expressions; the multiplication reads

$$\pi_{\lambda_1,n_1}(g_I) \cdot \pi_{\lambda_2,n_2}(g_I) \doteq \pi_{z,n_1}(g_I)^{N_1} \pi_{z,n_2}(g_I)^{N_2} = \pi_{z,N_1n_1+N_2n_2}(g_I), \tag{4.4.5}$$

with z maximal rational such that $\lambda_1 = N_1 z$ and $\lambda_2 = N_2 z$. While multiplication between two elements with $I \neq J$ is the tensor product. The star operator is $\pi_{\lambda,n}(g_I)^* = \pi_{\lambda,-n}(g_I)$. For the inner product, the Haar measure on $U(1)$ is required. It can be defined by

$$\mu_H(S) = \frac{1}{2\pi} m(f^{-1}(S)),$$

for each subset $S \subset U(1)$, where f is the function $f : [0, 2\pi] \to U(1)$, $t \mapsto (\cos(t), \sin(t))$ and m is the usual Borel measure on the real line. Hence, the measure for the inner product is

$$\int_{U(1)} d\mu_H(\exp(z\lambda\tilde{c}_{II})) = \frac{1}{2\pi} \int_0^{2\pi/z} d(z\tilde{c}_{II}). \tag{4.4.6}$$

From this, we can define the inner product as

$$(\pi_{\lambda_1,n_1}(g_I), \pi_{\lambda_2,n_2}(g_J)) \doteq \frac{1}{(2\pi)^3} \prod_K \int_0^{2\pi/z} d(z\tilde{c}_{KK}) \pi_{\lambda_1,-n_1}(g_I) \cdot \pi_{\lambda_2,n_2}(g_J). \tag{4.4.7}$$

Notice that for $I \neq J$ the inner product vanishes unless $\lambda_1 n_1 = 0 = \lambda_2 n_2$. If $I = J$ one obtains

$$\begin{aligned} &(\pi_{\lambda_1,n_1}(g_I), \pi_{\lambda_2,n_2}(g_I)) \\ &= \frac{1}{2\pi} \int_0^{2\pi/z} d(z\tilde{c}_{II})\, \pi_{\lambda_1,-n_1}(g_I) \cdot \pi_{\lambda_2,n_2}(g_I) \\ &= \frac{1}{2\pi} \int_0^{2\pi} dx\, \pi_{N_2n_2 - N_1n_1}(e^{ix}) = \begin{cases} 0 & \text{if } \lambda_1 n_1 \neq \lambda_2 n_2 \\ 1 & \text{if } \lambda_1 n_1 = \lambda_2 n_2 \end{cases}. \end{aligned}$$

All the definitions above are coherent with the isotropic case studied in [51]. The momentum operator can be derived from the general case,

$$\hat{\tilde{p}}^I \pi_{\lambda,n}(g_J) = \beta \ell_P^2 \lambda n \delta_J^I \pi_{\lambda,n}(g_J). \tag{4.4.8}$$

With multiplication operator $\pi_{\lambda,n}(g_I)$ and momentum operator $\hat{\tilde{p}}^I$, one now can compute the commutator acting on a state $\psi(g_I) = \pi_{\lambda_2,n_2}(g_I)$ (for $J \neq I$ is the same since J-terms do not contribute). The calculation is pretty simple; we obtain

$$[\pi_{\lambda,n}(g_I), \hat{\tilde{p}}^I] = -\beta \ell_P^2 \lambda n \pi_{\lambda,n}(g_I). \tag{4.4.9}$$

There is no reordering operator $\hat{R}^I$, then the commutator is exactly the quantisation of the Poisson bracket. It is possible to show that the reordering operator vanishes in the Abelian theory

$$\begin{aligned}
&-i\hat{R}^I(\pi_{\lambda_1,n_1}(g_I)\cdots\pi_{\lambda_N,n_N}(g_I)) \\
&= z\sum_{l=1}^{N} N_l \pi_{\lambda_1,n_1}(g_I)\cdots\pi_{z,n_l}(g_I)^{N_l}\cdots\pi_{\lambda_N,n_N}(g_I) - \sum_{l=1}^{N}\lambda_l\,\pi_{\lambda_1,n_1}(g_I)\cdots\pi_{\lambda_N,n_N}(g_I) \\
&= 0,
\end{aligned}$$

with z maximal rational such that $\lambda_l = zN_l$. The reason is that refinement does not exist in the Abelian theory; in fact, the action of $\pi_{\lambda,0}(g_I)$ as a multiplication operator is trivial. In the $SU(2)$ theory, the reordering operator reads $R^I_i = -(\beta i\ell_P^2)^{-1}\tilde{p}^I_i(\pi_{z,0}-1)$, but in the Abelian theory the term in the parenthesis acts as a null projector on every state, then $\hat{R}^I = 0$.

This Hilbert space is isomorphic to the LQC's usual one: the space of functions on the Bohr compactification $\overline{\mathbb{R}}_{Bohr}$ of the real line (i.e. almost periodic functions). There exists a map between the Abelian states and almost periodic functions $B : \pi_{\lambda,n}(g_I) \mapsto \exp(i\lambda n\tilde{c}_{II})$. This map is a *-algebra morphism and commutes with the action of $\hat{\tilde{p}}^I$. The map is surjective: given any $\exp(ia\tilde{c}_{II})$ it is always possible to find $\pi_{\lambda,n}(g_I)$ such that $\lambda n = a$ and so, to satisfy $B(\pi_{\lambda,n}(g_I)) = \exp(ia\tilde{c}_{II})$. B is also injective [1] because of Lemma 4.4.1.

Lemma 4.4.1. *Two elements $\pi_{\lambda_1,n_1}(g_I)$ and $\pi_{\lambda_2,n_2}(g_I)$ such that $\lambda_1 n_1 = \lambda_2 n_2$ are the same point in the Hilbert space with the scalar product defined in* (4.4.7).

Proof. Let $\pi_{\lambda_1,n_1}(g_I)$ and $\pi_{\lambda_2,n_2}(g_I)$ such that $\lambda_1 n_1 = \lambda_2 n_2$. Recalling that

$$(\pi_{\lambda_1,n_1}(g_I), \pi_{\lambda_2,n_2}(g_I)) = 1 \text{ if } \lambda_1 n_1 = \lambda_2 n_2,$$

we want to compute the distance between these two elements

$$\begin{aligned}
&||\pi_{\lambda_1,n_1}(g_I) - \pi_{\lambda_2,n_2}(g_I)||^2 \\
&= ||\pi_{\lambda_1,n_1}(g_I)||^2 + ||\pi_{\lambda_2,n_2}(g_I)||^2 - (\pi_{\lambda_1,n_1}(g_I), \pi_{\lambda_2,n_2}(g_I)) - (\pi_{\lambda_2,n_2}(g_I), \pi_{\lambda_1,n_1}(g_I)) \\
&= 2 - 2\,\mathrm{Re}((\pi_{\lambda_1,n_1}(g_I), \pi_{\lambda_2,n_2}(g_I))) = 0.
\end{aligned}$$

□

Actually, the previous Lemma tells us that the preimage of an element under B consist of a single point.

The Hamiltonian can be implemented in the theory considering the expression shown in [47]. In the Bianchi I model, the Hamiltonian is given by the Euclidean term only, and it reads

$$\hat{H} = \frac{i}{\kappa\beta^2\ell_P^2}\sum_{I,J,K}\epsilon^{IJK}\,\mathrm{Tr}(h_I h_J h_I^{-1} h_J^{-1} h_K [h_K^{-1}, \hat{V}]). \tag{4.4.10}$$

[1] The injectivity of B descends also from the fact that, in the original paper [51], B is an isometry.

We can expand this expression in terms of trigonometric functions of $\tilde{c}_{II}$. The computation is quite tedious. First of all, one can separate the two terms of the commutator. The first term vanishes due to the symmetry of $\mathrm{Tr}(h_I h_J h_I^{-1} h_J^{-1})$. In fact, in the fundamental representation, the trace of any $SU(2)$ matrix is real, then

$$\mathrm{Tr}(h_I h_J h_I^{-1} h_J^{-1}) = \mathrm{Tr}((h_I h_J h_I^{-1} h_J^{-1})^\dagger) = \mathrm{Tr}(h_J h_I h_J^{-1} h_I^{-1}).$$

The second term can be computed using the expansion (4.3.3))and the Lemma 4.3.2 (the Einstein's summation convention holds for the indices i, j, k in the following formula)

$$\begin{aligned}
&\sum_{IJK} \epsilon^{IJK} tr(h_I h_J h_I^{-1} h_J^{-1} h_K \hat{V} h_K^{-1}) \\
&= \sum_{IJK} \epsilon^{IJK} \Bigg(4 \Bigg(\frac{\cos(\frac{1}{2}\tilde{c}_{II}) \sin(\frac{1}{2}\tilde{c}_{II}) \sin^2(\frac{1}{2}\tilde{c}_{JJ})}{\tilde{c}_{II}} (\tilde{\phi}^i_I - \frac{\tilde{c}^2_{IJ}}{\tilde{c}^2_{JJ}} \tilde{\phi}^i_J) - \text{terms } (I \leftrightarrow J) \Bigg) \\
&\times \Big(\cos(\tfrac{1}{2}\tilde{c}_{KK}) \hat{V} \frac{\tilde{\phi}^i_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) - \frac{\tilde{\phi}^i_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) \hat{V} \cos(\tfrac{1}{2}\tilde{c}_{KK}) \Big) \\
&+ 4 \big(\cos(\tfrac{1}{2}\tilde{c}_{II}) \cos(\tfrac{1}{2}\tilde{c}_{JJ}) - \frac{\tilde{c}^2_{IJ}}{\tilde{c}_{II}\tilde{c}_{JJ}} \sin(\tfrac{1}{2}\tilde{c}_{II}) \sin(\tfrac{1}{2}\tilde{c}_{JJ}) \big) \frac{\tilde{\phi}^i_I \tilde{\phi}^j_J}{\tilde{c}_{II}\tilde{c}_{JJ}} \sin(\tfrac{1}{2}\tilde{c}_{II}) \sin(\tfrac{1}{2}\tilde{c}_{JJ}) \times \\
&\times \epsilon_{ijk} \Big(\cos(\tfrac{1}{2}\tilde{c}_{KK}) \hat{V} \frac{\tilde{\phi}^k_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) - \frac{\tilde{\phi}^k_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) \hat{V} \cos(\tfrac{1}{2}\tilde{c}_{KK}) \Big) \\
&+ \Bigg(\frac{\cos(\frac{1}{2}\tilde{c}_{II}) \sin(\frac{1}{2}\tilde{c}_{II}) \sin^2(\frac{1}{2}\tilde{c}_{JJ})}{\tilde{c}_{II}} (\tilde{\phi}^i_I - \frac{\tilde{c}^2_{IJ}}{\tilde{c}^2_{JJ}} \tilde{\phi}^i_J) - \text{terms } (I \leftrightarrow J) \Bigg) \times \\
&\times \epsilon_{ijk} \frac{\tilde{\phi}^k_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) \hat{V} \frac{\tilde{\phi}^k_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) \\
&+ 4 \big(\cos(\tfrac{1}{2}\tilde{c}_{II}) \cos(\tfrac{1}{2}\tilde{c}_{JJ}) - \frac{\tilde{c}^2_{IJ}}{\tilde{c}_{II}\tilde{c}_{JJ}} \sin(\tfrac{1}{2}\tilde{c}_{II}) \sin(\tfrac{1}{2}\tilde{c}_{JJ}) \big) \frac{\tilde{\phi}^i_I \tilde{\phi}^j_J}{\tilde{c}_{II}\tilde{c}_{JJ}} \sin(\tfrac{1}{2}\tilde{c}_{II}) \sin(\tfrac{1}{2}\tilde{c}_{JJ}) \times \\
&\times \epsilon_{ijm} \epsilon_{mkl} \frac{\tilde{\phi}^k_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) \hat{V} \frac{\tilde{\phi}^l_K}{\tilde{c}_{KK}} \sin(\tfrac{1}{2}\tilde{c}_{KK}) \Bigg).
\end{aligned}$$

The problems of this Hamiltonian are evident. In addition to the complexity, linear terms in the connection appear. An operator for the connection does not exist, and it cannot be expressed in terms of $\tilde{c}_{II}$ and $\tilde{p}^I$. The same problem holds for the volume operator $\hat{V}$.
If one restricts to the diagonal case, the expression simplifies, obtaining the usual Hamiltonian of the diagonal case [49]

$$\begin{aligned}
&\sum_{IJK} \epsilon^{IJK} tr(h_I h_J h_I^{-1} h_J^{-1} h_K \hat{V} h_K^{-1}) = \\
&= \sum_{IJK} \epsilon^{IJK} \Big(4\, \epsilon_{IJK} \cos(\tfrac{1}{2}c_I) \cos(\tfrac{1}{2}c_J) \sin(\tfrac{1}{2}c_I) \sin(\tfrac{1}{2}c_J) \\
&\quad \times \Big(\cos(\tfrac{1}{2}c_K) \hat{V} \sin(\tfrac{1}{2}c_K) - \sin(\tfrac{1}{2}c_K) \hat{V} \cos(\tfrac{1}{2}c_K) \Big) \Big).
\end{aligned} \tag{4.4.11}$$

Despite the good properties of the Hilbert space and the holonomy-flux algebra, this approach is not useful. In particular, it fails in the implementation of angles in the kinematical states, which are dependent on only three variables $\tilde{c}_{II}$. It does not have a separable Hamiltonian, nor does it provide proof of the independence of the geometry from the angles. The same approach along the edges ζ_i can be considered. Still, it is not possible to find suitable conjugate momenta to the variables $\mathring{c}_{ii} = (\sum_j \mathring{\phi}_i^j \mathring{\phi}_i^j)^{\frac{1}{2}}$ because in this formulation ϕ_I^i and p_i^I are mixed.

4.4.2 $U(1)^6$-holonomy

The previous approach can be slightly modified, considering as a multiplication operator its "natural extension" in a symmetric matrix (still Einstein summation convention is not adopted)

$$c_{IJ} = \sqrt{\sum_i \phi_I^i \phi_J^i}. \tag{4.4.12}$$

Also, the momenta are the "generalisation" of the ones found before

$$p^{IJ} = c_{IJ} \sum_a p_a^I (\phi^{-1})_a^J. \tag{4.4.13}$$

It is not trivial to check that the conjugate momenta are symmetric. Considering the expression of the connection and of the dreibein as matrices

$$\begin{aligned} \phi &= \frac{\beta}{2N} C^{-1} R \dot{\eta}, \\ p &= \sqrt{\det(d_{IJ})} \Lambda C^{-1}, \end{aligned}$$

where $C = \mathrm{diag}(a_1, a_2, a_3)$. The conjugate momenta read

$$p^{IJ} \propto \phi^{-1} p^t \propto \dot{\eta}^{-1} \Lambda C C^{-1} R = \dot{\eta}^{-1}, \tag{4.4.14}$$

where η is symmetric, then p^{IJ} is symmetric too. In these variables, the classical scalar constraint S' can be written in a simple way

$$S' = \sum_{IJ} (p^{IJ} c_{IJ})^2 - \sum_{IJKL} \frac{p^{IL}}{c_{IL}} c_{LJ}^2 \frac{p^{JK}}{c_{JK}} c_{KI}^2. \tag{4.4.15}$$

If we implement Thiemann's trick for the Hamiltonian, we obtain the same Hamiltonian shown in the previous Section, since it comes from a general approach. This formalism seems to adapt better to the problem than the $U(1)^3$ one due to the natural emergence of six variables, but the quantisation program has several issues. To emulate the Hilbert space defined in Sec.4.4.1, we want to use as states the representations of $U(1)^6$ holonomies. In such a Hilbert space, the same properties of the $U(1)^3$ formulation hold. However, $U(1)^6$ holonomy has no physical meaning. In canonical LQC, the power of $U(1)$ is due to the different directions in the space, hence 6 holonomies can not have the usual interpretation as holonomy along an edge and do not have one. A possible way to find a "geometry+angles" interpretation from this approach is to diagonalise the matrix c_{IJ} and consider as variables the eigenvalues and the angles of the change-of-basis matrix. Unfortunately, the conjugate momenta are difficult to find, and the Hamiltonian does not seem to simplify further.

In conclusion, constructing a Hilbert space analogous to that of the general homogeneous case in [51] and isomorphic to the space of functions on the Bohr compactification of the real line does not provide a viable route forward. As demonstrated in the previous two Sections, several issues arise in the quantisation of the Hamiltonian and in the definition of the kinematical states. In particular, the linear term in the connection cannot be neglected, and it is not legitimate to consider almost-periodic functions derived from an expansion of the holonomy. Consequently, the quantisation scheme proposed in Sec. 4.3 is the only consistent framework. This approach implements a meaningful kinematical split between the diagonal fluxes, which depend solely on the scale factors, and the rotational angles, within a well-defined Hilbert space. Furthermore, the Hamiltonian can be regularised using the standard regularisation scheme of Loop Quantum Cosmology, yielding the Hamiltonian of the diagonal case augmented by quadratic terms in the angular momenta. Therefore, for the remainder of this work, we shall adopt this scheme as the LQC quantisation for non-diagonal cosmological models.

Chapter 5

Abelianisation of Gauß constraint in Cosmology

The quantisation scheme proposed in the previous chapter can be generalised to incorporate gauge transformations. Although the standard approach in Loop Quantum Cosmology typically does not preserve the full $SU(2)$ gauge symmetry, the symmetry reduction induced by homogeneity does not, in principle, preclude its preservation. In this chapter, we try to restore the $SU(2)$ gauge symmetry within a cosmological setting, alongside the associated Gauß constraint. The central idea is that spatial homogeneity still permits global, time-dependent rotations of the triad vectors, thereby reinstating a non-trivial Gauß constraint akin to that in the full theory.

This analysis explores in detail the relationship between the Ashtekar-Barbero-Immirzi-Sen connection and the electric field with the standard ADM Hamiltonian variables. It leads to the conclusion that the standard framework of LQC is sufficiently general to accommodate non-diagonal terms and gauge freedom. When the $SU(2)$ gauge connection is expressed in terms of metric variables (namely, the three scale factors, three Euler angles, and three gauge angles), a local expansion reveals a linear dependence of the Gauß constraint on the momenta conjugate to the gauge angles (which generate global rotations). This observation motivates an *ab initio* adoption of the Holst formulation, whereby the $SU(2)$ connection is parametrised directly by metric variables. This approach yields a crucial result: a linear correspondence between the components of the Gauß constraint and the momenta conjugate to the gauge angles, implying that the constraint is satisfied if and only if these momenta vanish. In particular, we demonstrate that the Gauß constraint can be reformulated as three Abelian constraints, reflecting the pure gauge character of the three angles that rotate the triads. An explicit expression for the matrix relating the original non-Abelian Gauß constraint to its Abelian form is derived.

The most significant consequence of this study is that gauge-invariant kinematical states must be independent of the gauge angles, since in the canonical formalism, they are annihilated by the vanishing momenta. As a result, the quantisation of the model reduces precisely to the framework discussed in the previous Chapter on non-diagonal Bianchi models. Even when starting from a fully general configuration with nine independent variables, three of these are merely gauge angles. The corresponding Gauß constraint reduces to three Abelian conditions on vanishing momenta, and the resulting quantisation matches that of a non-diagonal Bianchi universe, characterised by a diagonal representation of fluxes, in full agreement with the standard LQC approach. This supports the conclusion that the proper incorporation of the $SU(2)$ gauge group within the quantisation of the minisuperspace framework is not achievable.

5.1 Rotations as gauge transformations

In this Section, we aim to characterise the gauge transformation of the homogeneous Ashtekar variables (3.1.4). The gauge transformation for the densitised triads is discussed in Chapter 3, which allows a rotation $p_i^I \mapsto p_j^I O_i^j$, with $O \in SO(3)$. Due to the homogeneity hypothesis, p_i^I only depends on time, and this property must also hold after the gauge transformation. Hence, although O can be arbitrary and does not contribute in any physical sense, it must depend on time only too. Moreover, the gauge transformation can be seen as a rotation of the dreibein $e_i^a \mapsto O_i^j e_j^a$. This interpretation allows us to find the associated gauge transformation of the connection variables ϕ_I^i.
Consider the usual expression of the Ashtekar connection $A_a^i = \Gamma_a^i + \beta K_a^i$. We can treat the two terms separately. The second term $K_a^i = K_{ab} e^{ai}$ contains the external curvature K_{ab}, which is a geometrical quantity and is not affected by gauge transformations, while e_i^a is a dreibein vector, so it rotates under a gauge transformation. It is easy to check that the rotation matrix is the inverse of the transformation matrix that acts on e_a^i because $\delta_b^a = e_i^a e_b^i$ must be invariant. Then, $K_{ab} e^{ai} \mapsto K_{ab} (O^{-1})_j^i e^{aj}$. Moreover, the spin part transforms as $\Gamma_a^i \mapsto (O^{-1})_j^i \Gamma_a^j$. Thus, under a gauge transformation, a matrix rotation appears:

$$A_a^i = \phi_I^i \theta_a^I \mapsto (O^{-1})_j^i A_a^j = (O^{-1})_j^i \phi_I^j \theta_a^I. \tag{5.1.1}$$

Therefore, on the phase space (ϕ_I^i, p_j^J) the gauge transformation acts as

$$p_i^I \mapsto p_j^I O_i^j \ , \quad \phi_I^i \mapsto (O^t)_j^i \phi_I^j. \tag{5.1.2}$$

We can check that such a transformation leaves the hypersurface identified by the Gauß constraint invariant. In fact, the transformation of the Gauß constraint $G_i = \epsilon_{ijk} \phi_I^i p_k^I$ reads

$$G_i \mapsto \epsilon_{ijk} (O^t)_l^j O_k^m \phi_I^l p_m^I = \epsilon_{ljk} O_i^l \phi_I^j p_k^I = O_i^j G_j. \tag{5.1.3}$$

Now, we look for a description in metric variables like the ones in Chapter 4. The new phase space of the metric variables, composed of the three scale factors a_1, a_2, a_3 and the three Euler angles of the physics rotation θ, ψ, φ, needs to include variables of the gauge freedom. Since $O \in SO(3)$, it can be written in terms of Euler angles

$$O = \exp(\alpha T_3) \exp(\beta T_2) \exp(\gamma T_3), \tag{5.1.4}$$

Then, the three gauge variables are these three Euler angles (α, β, γ), seen as a chart on $SO(3)$, $\alpha, \gamma \in (0, 2\pi)$, $\beta \in (0, \pi)$. Hence, the new configuration coordinates are $\{a_1, a_2, a_3, \theta, \psi, \varphi, \alpha, \beta, \gamma\}$.

In order to construct a theory in which the Gauß constraint does not vanish identically and in which the role of the cosmological quantities is made explicit, the assumption of phase space with configuration variables $\{a_1, a_2, a_3, \theta, \psi, \varphi, \alpha, \beta, \gamma\}$ seems to be a reasonable solution. The idea is to impose a canonical transformation between the two phase spaces such that the conjugate momenta to the gauge variables are included in the expression of the Ashtekar variables. These momenta will play a role in the Gauß constraint, and they can be removed from the theory to recover the expressions in Sec. 4.1.2.

5.2 Examination of the Bianchi I model

For the non-diagonal Bianchi I model, we have a simple expression of Ashtekar variables in terms of metric variables, which allows us to do some computations. We want to use the connection and fluxes expression in Sec. 4.1.2 and Sec. 4.3.2 properly gauge rotated

$$\phi_I^i = \frac{\beta}{2Na_j}\Lambda_j^J\,\dot{\eta}_{JI}(O^t)_j^i\,, \tag{5.2.1}$$

$$p_i^I = a_j a_k \Lambda_l^I O_i^l \quad \text{with } \epsilon_{ijk} = 1\,, \tag{5.2.2}$$

where Λ is the physical rotation introduced in the previous Chapter. A direct computation shows that, despite the gauge freedom, the Gauß constraint identically vanishes (as well as in Ref. [49]). Thus, the gauge momenta play a fundamental role in a non-vanishing Gauß constraint description. Now, we want to analyse this aspect.

5.2.1 The Lie condition approach

Consider the two phase spaces: the connection-fluxes one (ϕ_I^i, p_i^I), and the metric one

$$(q_n, p_m) \doteq (a_1, a_2, a_3, \theta, \psi, \varphi, \alpha, \beta, \gamma, p_1, p_2, p_3, p_\theta, p_\psi, p_\varphi, p_\alpha, p_\beta, p_\gamma).$$

We want to impose a canonical transformation between them starting from the expression of p_i^I in Eq.(5.2.2) to find a formulation of the connection which includes a dependence on the gauge momenta. We expect that the gauge momenta can give a non-trivial contribution to the Gauß constraint.
For simplicity, we switch coordinates and momenta of (ϕ_I^i, p_i^I) and we implement a canonical transformation via the Lie condition with a null total differential

$$\phi_I^i dp_i^I - p_n dq_n = \left(\phi_I^i \frac{\partial p_i^I}{\partial q_n} - p_n\right) dq_n = 0. \tag{5.2.3}$$

Now, we want to find ϕ_I^i. It is useful to introduce the matrix M defined as $M \doteq \frac{\partial p_i^I}{\partial q_n}$, seen as a 9×9 matrix, such that the term $\phi_I^i dp_i^I$ can be written as a matrix-vector product

$$\phi^i_I \frac{\partial p^I_i}{\partial q_n} = \begin{pmatrix} \frac{\partial p^1_1}{\partial a_1} & \frac{\partial p^2_1}{\partial a_1} & \dots & \frac{\partial p^3_3}{\partial a_1} \\ \frac{\partial p^1_1}{\partial a_2} & \frac{\partial p^2_1}{\partial a_2} & \dots & \frac{\partial p^3_3}{\partial a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p^1_1}{\partial \gamma} & \frac{\partial p^2_1}{\partial \gamma} & \dots & \frac{\partial p^3_3}{\partial \gamma} \end{pmatrix} \begin{pmatrix} \phi^1_1 \\ \phi^1_2 \\ \vdots \\ \phi^3_3 \end{pmatrix}. \tag{5.2.4}$$

In such a way, the Lie condition can be written as $\vec{p} = M\vec{\phi}$, where $\vec{p} = (p_n)$ is the vector of the momenta, while $\vec{\phi}$ is the vector of the ϕ^i_I, ordered as in Eq.(5.2.4). Using the expression of p^I_i in terms of metric variables, M can be written explicitly, but the expression is too long to write here. However, the computation of the inverse is too difficult, and even computing programs such as Wolfram Mathematica do not give a solution. At least, the determinant does not vanish identically, and the matrix is invertible in an open subset of the configuration space.

The complexity of the matrix is due to the abundant presence of trigonometric functions. Consequently, to do an approximate analysis, we need to fix the values of the angles such that the matrix is invertible. These values are $\pi/2$ for all the angles. At this point, the matrix M is invertible, and ϕ^i_I can be written in terms of the gauge momenta. From which, a non-identically vanishing Gauß constraint is provided

$$G_i = (p_\beta, p_\alpha, p_\gamma). \tag{5.2.5}$$

We can check that, imposing the constraint (i.e. putting the gauge momenta to zero), we recover the same expression of ϕ^i_I as in Eq.(4.2.2). Since the non-vanishing condition of the determinant is an open condition, there exists an open neighbourhood of $\pi/2$ in which the matrix is invertible. So, now we expand M at the first order with respect to a small parameter ε, and, for ε small enough, we obtain an invertible matrix. We need a small parameter for each angle: the small parameter that affects the θ angle is defined as $\varepsilon_\theta = \theta - \pi/2$, and similarly for all the other angles. From this, we compute M at the first order. The inverse of the matrix can be found, as well as the connection. In the first-order approximation, the Gauß constraint has a simple expression

$$G_i \approx (p_\beta + \varepsilon_\gamma p_\alpha,\, p_\alpha + \varepsilon_\beta p_\gamma - \varepsilon_\gamma p_\beta,\, p_\gamma). \tag{5.2.6}$$

It is trivial to check that, for $\varepsilon \to 0$, one finds the quantities computed at $\pi/2$.

A non-vanishing Gauß constraint is found; this means that a formulation in terms of the gauge variables is the right way forward. However, this result is not satisfying. The expression for the Gauß constraint is an approximated form around an arbitrary point, in which the limit does not lead back to any well-known theory. Moreover, no explanation is given about the identically vanishing Gauß constraint that emerges in the previous Chapter. Hence, a deeper analysis is required.

5.2.2 The Lagrangian approach

We want to investigate what happens to the Hamiltonian formulation starting from the Holst action

$$S_{\text{Holst}} = \frac{1}{\beta\kappa} \int dt \left(p^I_i \dot{\phi}^i_I + \lambda^i G_i - N^I V_I - \frac{N}{2}\beta S \right), \tag{5.2.7}$$

and considering the connection and the dreibein written in terms of the metric variables as in (5.2.1) and (5.2.2). We want the Holst action to be explicit in terms of metric variables, the calculation of the single terms provides that the Gauß constraint vanishes $G_i = 0$, and so the Vector constraint $V_I = 0$, while the Scalar constraint has the same expression as in Eq.(4.2.9). As we expected, the gauge freedom does not appear in the Lagrangian, which is invariant under gauge transformation. Hence, the momenta can be computed and the gauge momenta are null (i.e., $p_\alpha = 0,\ p_\beta = 0,\ p_\gamma = 0$), while the others are the same as presented in Sec 4.2. We can now perform the Legendre transformation with Lagrangian multipliers λ_i to find the Hamiltonian

$$\mathbf{H} = \frac{1}{\beta\kappa}\left(\lambda_1 p_\alpha + \lambda_2 p_\beta + \lambda_3 p_\gamma - \frac{N}{2}S\right). \tag{5.2.8}$$

Thus, the theory of the non-diagonal Bianchi I model in metric variables is a constrained Hamiltonian theory with phase space

$$(a_1, a_2, a_3, \theta, \psi, \varphi, \alpha, \beta, \gamma, p_a, p_b, p_c, p_\theta, p_\psi, p_\varphi, p_\alpha, p_\beta, p_\gamma)$$

with four constraints

$$p_\alpha = 0\ ,\quad p_\beta = 0\ ,\quad p_\gamma = 0\ ,\quad S = 0, \tag{5.2.9}$$

and with a Hamiltonian which is a linear combination of these constraints.

Notice that the same theory written in terms of connection and dreibein (ϕ^i_I, p^J_j) has four constraints given by $G_i = 0$ and $S = 0$. The Scalar constraint is the same in both formulations in the sense that it is possible to switch from one to the other using the transformation (5.2.1). This property does not hold for the Gauß constraint, which also vanishes in gauge variables. However, it is replaced by the three constraints on the pure gauge momenta. We interpret this result as follows: the Gauß constraint after the canonical transformation becomes the gauge momenta constraint. In such a way, the dependence on the gauge momenta we introduce in the Ashtekar variables vanishes on the constraints' hypersurface, recovering the usual description.

5.3 Equivalence between Gauß constraint and pure gauge momenta

In this Section, we want to find an explicit expression for the Gauß constraint. Previously, we showed that there exists a relation between the Gauß constraint and the momenta constraint, which we interpreted as

$$G_i = 0 \qquad \Longleftrightarrow \qquad p_g = 0, \tag{5.3.1}$$

where $g \in \{\alpha, \beta, \gamma\}$.
Such a relation is satisfied if the Gauß constraint is linearly dependent on pure gauge momenta p_g only. For simplicity, it will be our ansatz:

Ansatz. The Gauß constraint depends on the gauge momenta via a 3×3 matrix L_{ig}:

$$G_i = L_{ig} p_g, \tag{5.3.2}$$

with i is $SU(2)$ internal index and $g \in \{\alpha, \beta, \gamma\}$.

Using this ansatz, we can explicitly compute the coefficients of the linear combination without using M, nor an explicit expression of ϕ^i_I. Let p^I_i be as in Eq.(5.2.2) and the gauge momenta p_g given by the transformation that satisfies the Lie condition (5.2.3). With these assumptions, the Gauß constraint reads

$$\begin{aligned} G_i &= \epsilon_{ijk} \phi^j_I p^I_k = \phi^j_I (p^{\mathrm{ph}})^I_l \epsilon_{ijk} O^l_k \\ &= L_{ig} p_g = L_{ig} \phi^j_I \frac{\partial p^I_j}{\partial q_g} = \phi^j_I (p^{\mathrm{ph}})^I_l L_{ig} \frac{\partial O^l_j}{\partial q_g}. \end{aligned}$$

Where $(p^{\mathrm{ph}})^I_l$ is the physical part of the dreibein (i.e. the one not gauge rotated). From this, we can derive the following equation

$$\epsilon_{ijk} = L_{ig} (O^t)^k_l \frac{\partial O^l_j}{\partial q_g}. \tag{5.3.3}$$

This equation is enough to fully characterise the matrix L_{ig}; in fact, the RHS has the same skew-symmetric property as the Levi-Civita symbol. Therefore, we obtain nine linearly independent equations. The equations' system has nine equations in nine variables, and the associated determinant is $\sin^3 \beta$, so it is non-degenerate. Hence, there exists a unique solution. The solution L_{ig} can be found easily and it reads

$$L_{ig} = \begin{pmatrix} -\csc\beta\cos\gamma & \sin\gamma & \cot\beta\cos\gamma \\ \csc\beta\sin\gamma & \cos\gamma & -\cot\beta\sin\gamma \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.3.4}$$

Finally, the Gauß constraint can be written explicitly in terms of gauge momenta

$$G_i = \begin{pmatrix} -\csc\beta\cos\gamma p_\alpha + \sin\gamma p_\beta + \cot\beta\cos\gamma p_\gamma \\ \csc\beta\sin\gamma p_\alpha + \cos\gamma p_\beta - \cot\beta\sin\gamma p_\gamma \\ p_\gamma \end{pmatrix}. \tag{5.3.5}$$

The ansatz enables us to find the explicit dependence of the Gauß constraint on the gauge momenta. The matrix of coefficients is invertible since its determinant is $\det(L_{ig}) = -\csc\beta$, then the equivalence condition (5.3.1) holds. This explains the vanishing Gauß constraint in our initial approach, and in general, in similar approaches of Loop Quantum Cosmology. In fact, the connection $A^i_a = \Gamma^i_a + \beta K^i_a$ is reduced and, when it is described in terms of metric variables, it results in a function defined on the constraints' hypersurface, as well as the dreibein, and so, the linear dependence (5.3.2) implies that the Gauß constraint computed from such a connection must vanish.

5.3.1 Relation with $\mathfrak{su}(2)$

The Gauß constraint is deeply linked with the gauge group $SU(2)$ and, in particular, with its Lie algebra $\mathfrak{su}(2)$. We want to explore this link in view of the previous results.

The expression in (5.3.5) has no evident $SU(2)$ symmetry. However, via a change of coordinates, a contribution of $\mathfrak{su}(2)$ appears in the expression. Consider the matrix O of the gauge transformation parametrized by three angles $\alpha_1, \alpha_2, \alpha_3$ such that $O = \exp(\alpha_i \tau_i) = \exp[\alpha(n_i \tau_i)]$, where τ_i are the generators of $\mathfrak{su}(2)$, $n_i = \alpha_i/\alpha$ and $\alpha = \sqrt{(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2}$. Imposing Eq.(5.3.3), we find a new set of equations. They admit a unique solution, and the matrix L_{ig} can be found. An expansion for small α of L_{ig} give us

$$L_{ig} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix} + \mathcal{O}(\alpha^2). \tag{5.3.6}$$

It is interesting to notice that the first-order term is a skew-symmetric 3×3 matrix, so it is an element of the Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$. There is another and more profound link between the Gauß constraint and the $SU(2)$ group that will be examined in the next Section.

5.4 Gauß constraint as the generator of gauge transformations

It is well known that the Gauß constraint G_i is the generator of the gauge transformations on the phase space (ϕ^i_I, p^J_j). This feature should also hold in the new variables. Therefore, we want to compute the canonical Poisson brackets with respect to $\{\alpha, \beta, \gamma, p_\alpha, p_\beta, p_\gamma\}$ of the Gauß constraint in (5.3.5). We obtain

$$\{G_i, G_j\} = -\epsilon_{ijk} G_k. \tag{5.4.1}$$

The sign is not relevant. We expect that this formulation comes out from the canonical transformation in (5.2.3) in which the connection and dreibein are switched, then a sign in the Poisson bracket appears.

Hence, the Gauß constraint respects the $\mathfrak{su}(2)$ Lie brackets and generates the $SU(2)$ gauge transformations. Furthermore, it is linear in gauge momenta, so the hypersurface defined by $G_i = 0$ is also described by $p_\alpha = p_\beta = p_\gamma = 0$. Thus, the Gauß constraint is equivalent to three constraints on the momenta. Consequently, the generators of the gauge transformation can be decomposed into three generators which commute with each other

$$\{p_\alpha, p_\beta\} = \{p_\alpha, p_\gamma\} = \{p_\beta, p_\gamma\} = 0. \tag{5.4.2}$$

This decomposition is particularly useful in simplifying the implementation of the Gauß constraint in a quantum theory.

5.4.1 Quantum Gauß constraint

In the previous Chapter is shown that the quantisation of the non-diagonal Bianchi I model can be provided in terms of diagonal fluxes and angle variables. It is reasonable that a similar quantisation can be provided for the other non-diagonal models, given a loop quantisation of homogeneous Universes. However, to complete the description in the loop framework, we need to include the gauge transformations and a non-vanishing Gauß constraint.
Supposing that we have a quantisation like in the non-diagonal Bianchi I case, it is enough to add the gauge variables to the phase space of the diagonal fluxes and angles. These gauge variables will be the Euler angles of the gauge rotation, and they will be quantised independently (likely the physical angles) via the Schrödinger picture. Thus, the wave functions are $\Psi(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3, \alpha, \beta, \gamma)$, where $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ are the diagonal fluxes, and $\varphi_1, \varphi_2, \varphi_3$ are the physical angles. Moreover, the Hamiltonian (such as the Lagrangian) is independent of the gauge variables, hence the wave function factorises $\Psi = \phi(\alpha, \beta, \gamma)\Phi(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \varphi_1, \varphi_2, \varphi_3)$. On these functions, the action of the Gauß constraint is essentially a first-order derivative, so the imposition of the weak constraint $\hat{G}_i\Psi = 0$ is equivalent to

$$-i\hbar\frac{\partial\Psi}{\partial\alpha} = 0, \ -i\hbar\frac{\partial\Psi}{\partial\beta} = 0, \ -i\hbar\frac{\partial\Psi}{\partial\gamma} = 0. \tag{5.4.3}$$

The solution of this set of equations is trivial: $\phi(\alpha, \beta, \gamma) = const$. Thus, in this Hilbert space, the Gauß constraint imposes that the wave function must be independent of the gauge angles. Consequently, the kinematical Hilbert space for the non-diagonal Bianchi I model, as presented in Section 4.3.2, remains unchanged even when gauge transformations are taken into account. This demonstrates that the proposed approach generalises consistently to include gauge freedom, and highlights that the Abelianisation of the theory is a fundamental feature of the loop quantisation scheme within the minisuperspace context. Furthermore, the gauge symmetry commutes with the quantisation procedure: the gauge-invariant states obtained here are precisely those that result from the conventional approach, in which the gauge is fixed before quantisation.

Chapter 6

Spin bundle for Ashtekar-Barbero-Immirzi formulation

The primary objective of this Chapter is to provide a framework within fibre bundle theory for giving a geometric interpretation to the Ashtekar variables. This framework incorporates the principal bundle structure, analogous to Yang-Mills theory, but with an additional component: a spin structure. Despite the existence of a rigorous mathematical literature on the subject [82, 83, 86], a deep analysis and comprehensive treatment of the Ashtekar-Barbero-Immirzi formulation within a proper geometric framework has been lacking. In particular, a unified description of the Ashtekar variables and constraints as geometric objects, rigorously defined within the context of principal bundles, together with a geometric characterisation of the reconstruction process, has not been available. In this Chapter we fill this gap: within our framework, the pair of Ashtekar variables is interpreted in terms of connections and sections in the adjoint bundle, while the spin structure is shown to be necessary for the geometric interpretation of the reconstruction. Furthermore, we can express the constraints of the theory in a coordinate-free manner and interpret them as sections of suitable vector bundles. From a purely mathematical perspective, we will precisely define and prove that, on a 3-dimensional spin Riemannian manifold (Σ, q), the following two sets of data are equivalent:

ω	K
connection in the principal Spin(3)-bundle	$(0,2)$-symmetric tensor on $\mathrm{T}\Sigma$
together with equations	together with equations
$\boldsymbol{G}(\omega) = 0$ Gauß constraint	
$\boldsymbol{V}(\omega) = 0$ Vector constraint	$\boldsymbol{D}(\mathrm{K}) = 0$ Codazzi equation
$S(\omega) = 0$ Scalar constraint	$H(\mathrm{K}) = 0$ Gauß equation

Table 6.1. The two datasets are shown. On the left, the data relate to the Ashtekar-Barbero-Immirzi formulation of General Relativity. On the right, along with the metric q, the data pertain to the ADM formulation.

The data set on the left originates from the Ashtekar-Barbero-Immirzi formulation of General Relativity. The connection ω will be associated with the Ashtekar-Barbero-Immirzi-Sen connection. We will show that the quantities $\boldsymbol{G}$, $\boldsymbol{V}$, and S correspond to the constraints in this formulation. Specifically, the Gauß constraint $\boldsymbol{G}$ will be a 3-form on Σ with values in the adjoint bundle of the spin bundle, analogous to the Yang-Mills theory [121]. The Vector constraint $\boldsymbol{V}$ will be a 3-form with values in the cotangent bundle of Σ, encoding the torsion of ω. Finally, the Scalar constraint S will be a 3-form related to the scalar curvature of ω.
On the right-hand side, the data correspond to the ADM formulation of General Relativity. The supermomentum $\boldsymbol{D}$ and the superhamiltonian H are a 1-form and a scalar function on Σ, respectively. These constraints are well-known equations in the context of Riemannian submanifolds; specifically, they are the contracted Codazzi equation and the Gauß equation for the scalar curvature. Using these constraints, we can interpret K as the extrinsic curvature of an isometric embedding of Σ into a Ricci-flat spacetime M [65]. Namely, a spacetime that satisfies the Einstein Field Equations for a vacuum Universe.

Afterwards, we will discuss the implementation of the symmetries of the physical theory and the formulation without a metric within the provided framework, thereby recovering background independence to better align with the principles of General Relativity. Furthermore, we will demonstrate that these symmetries are associated with the action of the automorphism group of the spin bundle. Finally, we can construct and analyse the phase space, examining its topological and symplectic properties. Using the tools of symplectic geometry, we will discuss the action of the automorphism group, providing a momentum map for this action constructed from the Gauß and Vector constraints, and demonstrating, within this framework, the well-known correspondence between the Ashtekar variables and the ADM phase space.

6.1 Gauge theory structure

In this section, we aim to understand how the $SU(2)$ symmetry emerges at the geometric level. Specifically, we will construct a principal $SU(2)$ bundle associated with the metric q and analyse its properties to implement the Ashtekar-Barbero-Immirzi-Sen connection as a connection (in the sense of a principal bundle) on it.
The bundle structure is inherently present on the spacetime M and becomes evident through the vierbein formalism. In this approach, the spacetime is equipped with a local frame at each point, which can be interpreted as a section of a principal bundle.
Suppose M is orientable, the orthonormal frame bundle $P^{SO}(M)$ of M is well-defined. For an oriented, n-dimensional Riemannian or Lorentzian manifold, the orthonormal frame bundle is defined as the collection of the fibre on a point $p \in M$

$$P_p^{SO}(M) = \{h : \mathbb{R}^n \to \mathrm{T}_pM \mid h \text{ orientation-preserving linear isometry}\}.$$

The orthonormal frame bundle $P^{SO}(M) \doteq \bigsqcup_{p\in\Sigma} P_p^{SO}(M)$ is a principal $SO(n)$-bundle over M with the right action of $SO(n)$ defined by

$$\begin{aligned} P_p^{SO}(M) \times SO(n) \to & P_p^{SO}(M) \\ (h, O) \mapsto & h \circ O. \end{aligned}$$

Because the metric g is Lorentzian and the spacetime M is four-dimensional, the orthonormal frame bundle is a principal $SO(1,3)$-bundle. An element of this bundle in the fibre over $p \in M$ is an orientation-preserving linear isometry $h : \mathbb{R}^{1,3} \to \mathrm{T}_pM$. Hence, the vierbein is the data of a section of $P^{SO}(M)$. It defines a set of orthonormal vectors $\{e_\alpha\}$, i.e. $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$, by $e_\alpha(p) = h_p(\mathfrak{e}_\alpha)$, where $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ and $\mathfrak{e}_\alpha$ is the canonical basis of $\mathbb{R}^{1,3}$.
We consider a spinnable spacetime, namely an oriented spacetime M that admits a spin structure. We use the definition of spin structure given in [32]:

Definition 6.1.1. A spin structure on M is a pair $(P^{Spin}(M), \bar{\rho})$ consisting of:

- a principal $\mathrm{Spin}(n)$-bundle $P^{Spin}(M)$ over M,
- a twofold covering $\bar{\rho} : P^{Spin}(M) \to P^{SO}(M)$ such that the following diagram commutes:

$$\begin{tikzcd} P^{Spin}(M) \times \mathrm{Spin}(n) \arrow[r] \arrow[dd, "\bar\rho\times\rho"] & P^{Spin}(M) \arrow[dd, "\bar\rho"] \arrow[dr] & \\ & & M \\ P^{SO}(M) \times SO(n) \arrow[r] & P^{SO}(M) \arrow[ur] & \end{tikzcd}$$

Here, $\rho : \mathrm{Spin}(n) \to SO(n)$ is the twofold covering homomorphism of $SO(n)$. The horizontal maps are the group operations (right multiplications) on the principal bundles.

In this setting, the spin connection $\omega_\mu^{\alpha\beta}$, defined in Eq.(1.2.1), is the connection 1-form of the Levi-Civita connection in the orthonormal basis:

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta.$$

Thus, Eq.(1.2.1) is interpreted as the covariant derivative of the Casimir element of the endomorphisms on the tangent space: $\nabla \operatorname{Id}_{TM} \equiv 0$, that is automatically satisfied in this framework. Hence, the condition (1.2.1) allows us to consider this fibre bundle construction.

The existence of spinor fields on spacetime is a physical fact, necessary for the description of the elementary particles. Hence, the existence of a spin structure on M is a proper hypothesis. Given a spacelike hypersurface in M, it can be equipped with a unique spin structure induced by the one on M. Let Σ be a spacelike hypersurface, there exists a canonical embedding $\iota : P^{SO}(\Sigma) \hookrightarrow P^{SO}(M)$. Let n be the normal vector field along Σ, this embedding map is given by $\iota : (h : \mathbb{R}^3 \to \mathrm{T}_p\Sigma) \mapsto (h' : \mathbb{R}^{1,3} \to \mathrm{T}_pM)$, where $h'(0, v^1, v^2, v^3) = h(v^1, v^2, v^3)$ and $h'(1, 0, 0, 0) = n(p)$. This embedding is compatible with the canonical embedding $SO(3) \hookrightarrow SO(1, 3)$, so the right action on the respective orthonormal frame bundles commutes with the embedding. Let $(P^{Spin}(M), \bar\rho)$ be the spin structure on M, the spin structure on Σ is $(P^{Spin}(\Sigma) \doteq \bar\rho^{-1}(P^{SO}(\Sigma)), \bar\rho)$ [32].

In the case of a globally hyperbolic spacetime $M \cong \mathbb{R} \times \Sigma$, every Cauchy surface $f^{-1}(t)$ admits a spin structure. Here, f is the surjective function of Geroch's theorem. Notice that the vierbein defined in (1.2.2) corresponds to a choice of an element h' of $P^{SO}(M)$, as defined above. Specifically, let $M = \mathbb{R} \times \Sigma$, with a chosen coordinate system as in Sec. 1.1, and h be a section of $P^{SO}(\Sigma_t)$ over the Cauchy surface Σ_t. The vierbein h' defined by the embedding $\iota \circ h$ has a matrix form that is exactly the inverse of the matrix in (1.2.2). Indeed, let us consider $\mathfrak{e}_\alpha$, with $\alpha = 0, 1, 2, 3$, as the canonical basis of $\mathbb{R}^{1,3}$. Hence, the vierbein satisfies $\iota(h(\mathfrak{e}_0)) = n = \frac{1}{N}\partial_0 - \frac{N_a}{N}\partial_a = e_0^\mu \partial_\mu$, and $\iota(e(\mathfrak{e}_i)) = e(\mathfrak{e}_i) = e_i^a \partial_a$.

We can also provide a topological algebraic argument to show that if $M = \mathbb{R} \times \Sigma$ admits a spin structure, then Σ admits one too. We will check this by computing the Stiefel-Whitney classes [123]. Notice that $\mathrm{T}M \cong \mathrm{T}\mathbb{R} \times \mathrm{T}\Sigma \cong \mathrm{pr}_1^*\mathrm{T}\mathbb{R} \oplus \mathrm{pr}_2^*\mathrm{T}\Sigma$. Here, $\mathrm{pr}_1 : M \to \mathbb{R}$ and $\mathrm{pr}_2 : M \to \Sigma$ are the projectors on the first and second factor, respectively. Using the naturality of Stiefel-Whitney classes and the Whitney sum formula, and recalling that $w_0 \equiv 1$, we get

$$\begin{aligned} w_1(\mathrm{T}M) &= w_1(\mathrm{pr}_1^*\mathrm{T}\mathbb{R} \oplus \mathrm{pr}_2^*\mathrm{T}\Sigma) = w_1(\mathrm{T}\Sigma), \\ w_2(\mathrm{T}M) &= w_2(\mathrm{pr}_1^*\mathrm{T}\mathbb{R} \oplus \mathrm{pr}_2^*\mathrm{T}\Sigma) = w_2(\mathrm{T}\Sigma). \end{aligned}$$

Recalling that M is spinnable if and only if $w_1(\mathrm{T}M) = 0$ and $w_2(\mathrm{T}M) = 0$, hence, if M is spinnable, then $w_1(\mathrm{T}\Sigma) = 0$ and $w_2(\mathrm{T}\Sigma) = 0$, meaning that Σ is also spinnable. Thus, the hypothesis of the existence of a spin structure on Σ is determined by the same condition on the globally hyperbolic spacetime M.

Let (Σ, q) be a 3-dimensional oriented Riemannian manifold. On such a manifold, the oriented orthonormal frame bundle $P^{SO}(\Sigma)$ is a principal $SO(3)$-bundle over Σ. Using the construction of the associated vector bundle, we can properly see the role

of the dreibein. The tangent bundle can be realized as an associated vector bundle $\mathrm{T}\Sigma \cong P^{SO}(\Sigma) \times_{\lambda_{st}} \mathbb{R}^3$, where $\lambda_{st} : SO(3) \to GL(3, \mathbb{R})$ is the defining representation of $SO(3)$. Hence, elements of the tangent bundle are equivalence classes of pairs $[h, \mathrm{v}]$, $h \in P^{SO}(\Sigma)$, $\mathrm{v} \in \mathbb{R}^3$ associated with the tangent vector $h(\mathrm{v})$. In this context, the index i serves purely as a counting index for the elements of the orthonormal basis. Therefore, it can be raised or lowered using the Kronecker delta.
In this setting, the dreibein is equivalent to the data of a (global) section $h : \Sigma \to P^{SO}(\Sigma)$. Indeed, given the canonical basis $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3\}$ of $\mathbb{R}^3$, we can associate a frame of orthonormal vector fields (the triad): $[h, \mathfrak{e}_i] = h(\mathfrak{e}_i) \doteq e_i = e^a_i \partial_a$. From this, the cotriad (or dreibein) is the 1-forms dual frame $\{e^i = e^i_a dx^a\} \subset \Omega^1(\Sigma)$. Notice that the vector $\mathrm{v} \in \mathbb{R}^3$ represents the components of $h(\mathrm{v})$ in the trivialization of the bundle induced by h, namely the components with respect to the frame e_i.
The dreibein is also related to a section in $\mathrm{End}(\mathrm{T}\Sigma) \cong \mathrm{T}^*\Sigma \otimes \mathrm{T}\Sigma$. Actually, the section that to each point associates the Casimir operator of the endomorphisms of the tangent space, namely the identity, can be written as $\mathrm{Id}_{\mathrm{T}\Sigma} = e^a_i e^i \otimes \partial_a = e^i_a dx^a \otimes e_i$. In this context, e^a_i and its inverse e^i_a play the role of change-of-basis matrices.

The $SU(2)$ structure emerges when spinors are introduced into the theory. The existence of a spin structure on Σ can be inferred from the natural requirement of a spin structure on M, as discussed above. With a spin structure, every associated vector bundle to $P^{SO}(\Sigma)$ is isomorphic to one to $P^{Spin}(\Sigma)$. Consider a representation $\bar{\lambda} : \mathrm{Spin}(3) \to GL(V)$ of the form $\bar{\lambda} = \lambda \circ \rho$, for some representation $\lambda : SO(3) \to GL(V)$, then $P^{Spin}(\Sigma) \times_{\bar{\lambda}} V \cong P^{SO}(\Sigma) \times_{\lambda} V$. In particular, $\mathrm{T}\Sigma \cong P^{SO}(\Sigma) \times_{\lambda_{st}} \mathbb{R}^3 \cong P^{Spin}(\Sigma) \times_{\lambda_{st} \circ \rho} \mathbb{R}^3$. Moreover, in dimension three, since the defining representation of Spin(3) is equivalent to the adjoint one, we get $\mathrm{T}\Sigma \cong \mathrm{ad}P^{Spin}(\Sigma) \doteq P^{Spin}(\Sigma) \times_{\mathrm{Ad}} \mathfrak{spin}(3)$. Thus, the dreibein defines (not uniquely) also a section $\bar{h} : \Sigma \to P^{Spin}(\Sigma)$ that satisfies $\bar{\rho} \circ \bar{h} = h$. In this way, it defines the same vector fields of $h : \Sigma \to P^{SO}(\Sigma)$, in the sense that $e_i = [\bar{h}, \mathfrak{e}_{\mathrm{i}}] \in P^{Spin}(\Sigma) \times_{\lambda_{st} \circ \rho} \mathbb{R}^3$.

Moreover, spinnable 3-manifolds are parallelizable. Since Spin(3) is simply connected and compact, then $P^{Spin}(\Sigma)$ is a trivial principal bundle over Σ. As associated bundles of a trivial principal bundle are trivial, $\mathrm{T}\Sigma$ is trivial. Hence, $P^{SO}(\Sigma)$ is trivial too. Thus, we can always consider global sections of such bundles.

From now on, the manifold Σ is always considered equipped with a spin structure $(P^{Spin}(\Sigma), \bar{\rho})$ and, to make the discussion closer to the usual physical notation, we identify $\mathrm{Spin}(3) \cong SU(2)$, and so $\mathfrak{spin}(3) \cong \mathfrak{su}(2)$.

6.1.1 Connection and gauge field

The connections on $P^{SO}(\Sigma)$ are the so-called metric-compatible connections. Among them, we can recognise a preferred one, which is the Levi-Civita connection ω^{LC}. The spin structure allows us to define a one-to-one correspondence between connections on $P^{SO}(\Sigma)$ and connections on $P^{Spin}(\Sigma)$. Given a connection ω on $P^{SO}(\Sigma)$, we can define a connection $\bar{\omega} = \rho_*^{-1} \bar{\rho}^* \omega$ on $P^{Spin}(\Sigma)$, where ρ_*^{-1} is the isomorphism $\mathfrak{so}(3) \xrightarrow{\sim} \mathfrak{su}(2)$ induced by ρ. The contrary is also true; every connection on $P^{Spin}(\Sigma)$ can be written in this way.

The Ashtekar connection is the local 1-form of a connection on $P^{Spin}(\Sigma)$, namely its pullback via a section

$$A = \bar{h}^* \bar{\omega}. \tag{6.1.1}$$

The Ashtekar connection A is a $\mathfrak{su}(2)$-valued 1-form $A = A^i_a dx^a \otimes \tau_i$, where τ_i is a basis of generators for $\mathfrak{su}(2)$. Under a change of section $\bar{h} \mapsto \bar{h}g$, with $g : \Sigma \to SU(2)$, it transforms as $A \mapsto \mathrm{Ad}_{g^{-1}} A + g^{-1}dg$. It is clear that, due to the properties of the spin structure, to deal with an Ashtekar connection is equivalent to dealing with a connection in $P^{SO}(\Sigma)$. Actually, since the bundle maps $\bar{h}^*$ and $\bar{\rho}^*$ commute with the Lie algebra isomorphism ρ_*, A is also the local 1-form of a metric-compatible connection

$$A = \bar{h}^* \bar{\omega} = \bar{h}^* \rho_*^{-1} \bar{\rho}^* \omega = \rho_*^{-1} (\bar{\rho} \circ \bar{h})^* \omega = \rho_*^{-1} h^* \omega.$$

In which the only difference is that τ_i is substituted by T_i, the generators of $\mathfrak{so}(3)$. In such a way, the local field A depends only on the section h and not on the specific lift $\bar{h}$, and in general, on the chosen spin structure.

A curvature form Θ is associated with a connection $\bar{\omega}$. It is a 2-form on $P^{Spin}(\Sigma)$ with values in $\mathfrak{su}(2)$ defined by:

$$\Theta = d\bar{\omega} + \frac{1}{2} \mathrm{ad}(\bar{\omega}) \wedge \bar{\omega}.$$

The pullback of Θ is the curvature (or field strength) $F = \bar{h}^* \Theta$. The curvature is written locally as

$$F = dA + \tfrac{1}{2} \mathrm{ad}(A) \wedge A. \tag{6.1.2}$$

The curvature transforms under a change of section $\bar{h} \mapsto \bar{h}g$ as $F \mapsto \mathrm{Ad}_{g^{-1}} F$ and in a coordinate chart reads:

$$\begin{aligned} F &= (\partial_a A^i_b \tau_i + \tfrac{1}{2} A^j_a A^k_b [\tau_j, \tau_k]) dx^a \wedge dx^b = (\partial_a A^i_b + \tfrac{1}{2} \epsilon^i{}_{jk} A^j_a A^k_b) \tau_i \otimes dx^a \wedge dx^b \\ &= \tfrac{1}{2} (\partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b) \tau_i \otimes dx^a \otimes dx^b = \tfrac{1}{2} F^i_{ab} \tau_i \otimes dx^a \otimes dx^b. \end{aligned}$$

We can associate with Θ an element of $\Omega^2(\Sigma, \mathrm{ad} P^{Spin}(\Sigma))$ uniquely. Explicitly, it is $\boldsymbol{F} = [\bar{h}, \tfrac{1}{2} F^i_{ab} \tau_i] \otimes dx^a \wedge dx^b$. Hence, F is the expression of $\boldsymbol{F}$ in the trivialization of the bundle $\mathrm{ad}\, P^{Spin}(\Sigma)$ induced by the section $\bar{h}$. For simplicity, by now we will omit the k-form part in this kind of object, so the previous reads $\boldsymbol{F} = [\bar{h}, F^i \tau_i] = [\bar{h}, F]$.

6.1.2 Electric field and solder form

In physics, the dreibein is collected in a $\mathfrak{su}(2)$-valued 1-form. Given the cotriad e^i, we are going to consider the contraction of the 1-forms with the basis of generators τ_i of $\mathfrak{su}(2)$

$$e = e^i_a dx^a \otimes \tau_i.$$

This construction, apparently unnatural, is linked with a solder form on $P^{Spin}(\Sigma)$.

On the orthonormal frame bundle, the tautological solder form θ is a 1-form on $P^{SO}(\Sigma)$ with values in $\mathbb{R}^3$ defined by

$$\theta_{(x,h)}(v) = h^{-1} \left(d_{(x,h)} \pi(v) \right),$$

where $x \in \Sigma, h \in P^{SO}_x(\Sigma), v \in \mathrm{T}_{(x,h)}P^{SO}(\Sigma)$ and π is the projection of the bundle. The solder form is right equivariant $R^*_g\theta = \lambda_{st}(g^{-1})\theta$. Let h be a section of $P^{SO}(M)$, then the cotriad is

$$(e^1, e^2, e^3) = h^*\theta. \tag{6.1.3}$$

Thus, point by point, for all $v \in \mathrm{T}_x M$

$$e^i_a(x)dx^a(v)\mathfrak{e}_i = h_x^{-1}\left(d_{(x,h_x)}\pi \circ d_x h(v)\right) = h_x^{-1}\left(d_x(\pi \circ h)(v)\right) = h_x^{-1}(v).$$

From the tautological solder form, we can induce a solder form on $P^{Spin}(\Sigma)$. Considering firstly the pullback of θ via $\bar{\rho}$ and then the composition with the $SU(2)$-equivariant isomorphism $\phi : \mathbb{R}^3 \xrightarrow{\sim} \mathfrak{su}(2)$, the solder form $\vartheta \in \Omega^1(P^{Spin}(\Sigma), \mathfrak{su}(2))$ is defined by:

$$\vartheta \doteq \phi \circ \bar{\rho}^*\theta. \tag{6.1.4}$$

It is horizontal because θ is horizontal. Moreover, it is $SU(2)$-equivariant:

$$R^*_g\vartheta = \phi \circ R^*_g\bar{\rho}^*\theta = \phi \circ \bar{\rho}^* R^*_{\rho(g)}\theta = \phi(\lambda_{st}(\rho(g^{-1}))\bar{\rho}^*\theta) = \mathrm{Ad}_{g^{-1}}\,\phi(\bar{\rho}^*\theta).$$

It is well-defined since the adjoint action of $SU(2)$ is equivalent to the adjoint action of $SO(3)$. With such a form, it is possible to associate uniquely a differential 1-form $\boldsymbol{e} \in \Omega^1(\Sigma, \mathrm{ad}P^{Spin}(\Sigma))$ that, in a trivialization $\bar{h}$, reads as

$$e = \bar{h}^*\vartheta. \tag{6.1.5}$$

From which, the transformation under change of trivialization is $e \mapsto \mathrm{Ad}_{g^{-1}}\,e$.

Now, we can introduce the electric field E. In a trivialization of $P^{Spin}(\Sigma)$, it can be defined as an element of $\Omega^2(\Sigma, \mathfrak{su}(2)^*)$ such that

$$\frac{1}{3}tr(E \wedge e) = dV_q, \tag{6.1.6}$$

where tr is the natural pairing between $\mathfrak{su}(2)^*$ and $\mathfrak{su}(2)$ (or, analogously, the multiple of Killing form such that the generators of $\mathfrak{su}(2)$ form an orthonormal basis. In particular, it is proportional to the standard trace Tr, $tr = -2\,\mathrm{Tr}$). It is clear that E is the contraction of the Hodge dual of the cotriad

$$\star e^i = \tfrac{1}{2}\sqrt{\det q}\,e^i_{a'}q^{a'a}\epsilon_{abc}dx^b \wedge dx^c = \tfrac{1}{2}\delta^{ij}E^a_j\epsilon_{abc}dx^b \wedge dx^c, \tag{6.1.7}$$

with the dual basis of $\mathfrak{su}(2)$ given by $\{\mathfrak{t}^i\} \subset \mathfrak{su}(2)^*$. Under a change of trivialization $\bar{h} \mapsto \bar{h}g$, the electric field must transform as $E \mapsto \mathrm{Ad}^*_{g^{-1}}\,E$, where Ad^* is the coadjoint representation, to maintain the invariance of Eq.(6.1.6).
Hence, $[\bar{h}, \frac{1}{2}E^a_i\epsilon_{abc}\mathfrak{t}^i] \otimes dx^b \wedge dx^c$ is a well-defined 2-form with values in $\mathrm{ad}^*P^{Spin}(\Sigma)$, namely $\boldsymbol{E} = [\bar{h}, E] \in \Omega^2(\Sigma, \mathrm{ad}^*P^{Spin}(\Sigma))$. Such a vector field is associated with a horizontal 2-form $\tau \in \Omega^2(P^{Spin}(\Sigma), \mathfrak{su}(2)^*)$ such that $\boldsymbol{E} = [\bar{h}, \bar{h}^*\tau]$.

6.1.3 Equivalence of the data

We want to show how to reconstruct a $(0,2)$-tensor field K starting from a connection ω on $P^{Spin}(\Sigma)$.
On $P^{Spin}(\Sigma)$, there exists a special connection that is the spin connection $\bar{\omega}^{LC}$ associated with the Levi-Civita connection $\bar{\omega}^{LC} = \rho_*^{-1}\bar{\rho}^*\omega^{LC}$. The difference $\omega - \bar{\omega}^{LC}$ is a horizontal and $SU(2)$-invariant 1-form on $P^{Spin}(\Sigma)$, so, it is associated with an element of $\Omega^1(\Sigma, \mathrm{ad}P^{Spin}(\Sigma))$. In a trivialization $\bar{h}$, it reads $\beta K = \bar{h}^*(\omega - \bar{\omega}^{LC})$, where $\beta \in \mathbb{R}^+$ is fixed. From the global trivialization, we can define the tensor

$$\mathrm{K} = tr(e \otimes K), \tag{6.1.8}$$

where $e = \bar{h}^*\vartheta$ is the solder form. K is invariant under change of trivialization since tr is proportional, pointwise, to the Killing form on $\mathfrak{su}(2)$, then $tr(\mathrm{Ad}_g\, e \otimes \mathrm{Ad}_g\, K) = tr(e \otimes K)$ for all $g : \Sigma \to SU(2)$. Therefore, $\mathrm{K} \in \Omega^1(\Sigma) \otimes \Omega^1(\Sigma)$ is a well-defined tensor.
For an *a priori* trivialization-free definition, we must use the bundle isomorphism $\Phi : \mathrm{T}\Sigma \xrightarrow{\sim} \mathrm{ad}P^{Spin}(\Sigma)$ induced by the solder form ϑ and defined by $\Phi : X \mapsto \boldsymbol{e}(X)$. Then, we can define $k = \Phi^{-1}([\bar{h}, \beta\bar{h}^*(\bar{\omega}^A - \bar{\omega}^{LC})])$ and conclude $\mathrm{K}(X,Y) = q(k(X), Y)\, \forall X, Y \in \mathrm{T}_x\Sigma$.
Finally, we must check that, in local coordinates, K is the map (1.2.4):

$$\begin{aligned} \beta\mathrm{K} &= tr(e \otimes \beta K) = e^j \otimes (A^i - \Gamma^i) tr(\tau_i\tau_j) = e^j \otimes (A^i - \Gamma^i)\delta_{ij} \\ &= e^i_a (A^j_b - \Gamma^j_b)\delta_{ij} dx^a \otimes dx^b. \end{aligned}$$

However, the symmetry of this tensor is not deduced from this construction. The Gauß constraint ensures this property.

6.2 Constraints

The constraints arise from the Hamiltonian formulation of General Relativity in Ashtekar variables, and they have a well-known interpretation as the generators of infinitesimal transformations of the phase space. However, they can also acquire a geometric interpretation and can be described within the framework of fibre bundle theory.

We recall that associated with a connection on a principal bundle, there exists a unique covariant derivative on the associated vector bundles. Hence, a connection ω on $P^{Spin}(\Sigma)$ induces a covariant derivative D on $\mathrm{T}\Sigma$ and its functorial constructions (for details see Appendix B). Its action on the triad and its dual frame read in terms of Ashtekar connection

$$\begin{aligned} De_i &= [\bar{h}, d\mathfrak{e}_i + \lambda_*(\bar{h}^*\omega)\mathfrak{e}_i] = [\bar{h}, A^j\lambda_*(\tau_j)\mathfrak{e}_i] = [\bar{h}, A^j\epsilon_{jik}\mathfrak{e}_k] = A^j\epsilon_{ji}{}^k e_k, \\ De^i &= [\bar{h}, d\mathfrak{e}_i - {}^t\lambda_*(\bar{h}^*\omega)\mathfrak{e}_i] = [\bar{h}, -A^j\, {}^t\lambda_*(\tau_j)\mathfrak{e}_i] = [\bar{h}, A^j\epsilon_{jik}\mathfrak{e}_k] = A^j\epsilon_{jik}e^k. \end{aligned}$$

Here, $-{}^t\lambda_*$ is the dual representation, that, in terms of matrices, is minus the transpose of the defining representation λ_* of the Lie algebra $\mathfrak{su}(2)$. The indices i, j, k are raised and lowered using the Kronecker delta. Moreover, let ∇ be the

Levi-Civita covariant derivative, we recall that $\nabla e_i = \Gamma^k \epsilon_{ki}{}^j e_j$.
A connection ω also induces the so-called exterior covariant derivative d_ω on k-forms with values in a vector bundle. The derivation rule is the following: consider η a k-form on Σ and s a section in a associated vector bundle

$$d_\omega(\eta \otimes s) = d\eta \otimes s + (-1)^k \eta \wedge Ds.$$

Notice that the $\mathfrak{su}(2)$-valued 1-form K, defined in Sec.6.1.3, plays a role in the induced covariant derivative. Indeed, the covariant derivative on TΣ can be written as $D = \nabla + \mathbf{K}$, where $\mathbf{K} \in \Omega^1(\Sigma, \mathrm{AntiSym}(\mathrm{T}\Sigma))$. Here, AntiSym(T$\Sigma$) is the bundle of skew-symmetric linear operators on the tangent space. In particular, its action on the dreibein is

$$\begin{aligned}\mathbf{K}e_i &= De_i - \nabla e_i = [\bar{h}, d\mathfrak{e}_i + \lambda_*(\bar{h}^*\omega)\mathfrak{e}_i] - [\bar{h}, d\mathfrak{e}_i + \lambda_*(\bar{h}^*\bar{\omega}^{LC})\mathfrak{e}_i] \\ &= [\bar{h}, \lambda_*(\bar{h}^*(\omega - \bar{\omega}^{LC}))\mathfrak{e}_i] = [\bar{h}, \lambda_*(\beta K)\mathfrak{e}_i] = [\bar{h}, \beta K^j \epsilon_{jik}\mathfrak{e}_k] = \beta K^j \epsilon_{ji}{}^k e_k.\end{aligned}$$

From which $q(\mathbf{K}(\partial_a)e_i, e_j) = \beta K^l_a \epsilon_{lij}$, showing the skew-symmetry property.

6.2.1 Gauß constraint

It is well-known that the Gauß constraint forces the tensor K to be symmetric. This fact follows from a simple computation; for completeness, we reproduce it here

$$\begin{aligned}G_i &= \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k = \sqrt{\det q}\left(\nabla_a e^a_i + \epsilon_{ij}{}^k A^j_a e^a_k\right) \\ &= \sqrt{\det q}\left(\Gamma^j_a \epsilon_{ji}{}^k e^a_k + \epsilon_{ij}{}^k A^j_a e^a_k\right) = \sqrt{\det q}\left(-\beta K^j_a \epsilon_{ji}{}^k e^a_k\right) = \beta\sqrt{\det q} K_{ab}\epsilon_i{}^{jk} e^a_k e^b_j.\end{aligned}$$

It is quite evident that

$$G_i = 0 \iff \epsilon^{ikj} K_{ab} e^a_k e^b_j = 0 \iff K_{ab} = K_{ba}.$$

We are now interested in a coordinate-free definition and a geometric interpretation of the Gauß constraint. It can be found quite straightforwardly via the exterior covariant derivative of the electric field:

$$\boldsymbol{G} = d_\omega \boldsymbol{E}. \tag{6.2.1}$$

This defines an element of $\Omega^3(\Sigma, \mathrm{ad}^* P^{Spin}(\Sigma))$. Considering a global trivialization, we can define the $\mathfrak{su}(2)^*$-valued 3-form

$$G = d_A E \doteq dE + \mathrm{ad}^*(A) \wedge E. \tag{6.2.2}$$

In such a way, $\boldsymbol{G} = [\bar{h}, G]$. The proof that this object is the usual Gauß constraint follows immediately:

$$\begin{aligned}d_\omega \boldsymbol{E} &= d(\tfrac{1}{2}E^a_i \epsilon_{abc} dx^b \wedge dx^c) \otimes [\bar{h}, \mathfrak{t}^i] + (\tfrac{1}{2}E^a_i \epsilon_{abc} dx^b \wedge dx^c) \wedge D[\bar{h}, \mathfrak{t}^i] \\ &= (\tfrac{1}{2}\partial_{a'} E^a_i \epsilon_{abc} dx^{a'} \wedge dx^b \wedge dx^c) \otimes [\bar{h}, \mathfrak{t}^i] + \tfrac{1}{2}E^a_i \epsilon_{abc} dx^b \wedge dx^c \wedge dx^{a'} \otimes [\bar{h}, A^j_{a'} \epsilon_{jik}\mathfrak{t}^k] \\ &= (\partial_a E^a_i + \epsilon_{ijk} A^j_a E^a_k) dx^1 \wedge dx^2 \wedge dx^3 \otimes [\bar{h}, \mathfrak{t}^i] = G_i dx^1 \wedge dx^2 \wedge dx^3 \otimes [\bar{h}, \mathfrak{t}^i].\end{aligned}$$

Thus, the constraint $\boldsymbol{G} = 0$ imposes the vanishing of the exterior covariant derivative for the electric field, and the equation $d_A E = 0$ is analogous to the equations of motion of a Yang-Mills theory.

6.2.2 Vector constraint

The Vector constraint admits an expression in terms of the exterior covariant derivative. Its expression in terms of the cotriad is clear and concise and gives us an element of $\Omega^3(\Sigma, \mathrm{ad}P^{Spin}(\Sigma))$

$$\boldsymbol{V} = d^2_\omega \boldsymbol{e}. \tag{6.2.3}$$

In a global trivialization, its components read as

$$\mathcal{V} = d^2_A e = \mathrm{ad}(F) \wedge e. \tag{6.2.4}$$

However, in physics, the Vector constraint is a 1-form. We can obtain this formulation by considering the bundle isomorphism induced by the solder form, analogously to Sec. 6.1.3. In this case, we can define $\mathrm{V} \in \Omega^3(\Sigma) \otimes \Omega^1(\Sigma)$ as

$$\mathrm{V} = tr(e \otimes \mathcal{V}). \tag{6.2.5}$$

Now, we can check that this expression of the Vector constraint is the usual one by performing a computation in coordinates

$$\mathcal{V} = \mathrm{ad}(F) \wedge e = F^i \wedge e^j \otimes [\tau_i, \tau_j] = \tfrac{1}{2}\epsilon_{ijk} F^i_{ab} e^j_c dx^a \wedge dx^b \wedge dx^c \otimes \tau_k,$$

and then

$$\begin{aligned}\mathrm{V} &= e^l \otimes \epsilon_{ijk} F^i \wedge e^j tr(\tau_l \tau_k) = \tfrac{1}{2}\epsilon^{abc}\epsilon_{ijk} F^i_{ab} e^j_c e^k_{a'} dx^1 \wedge dx^2 \wedge dx^2 \otimes dx^{a'} \\ &= \tfrac{1}{2}\epsilon^{abc} F^i_{ab} \epsilon_{a'b'c} E^{b'}_i dx^1 \wedge dx^2 \wedge dx^2 \otimes dx^{a'} = F^i_{ab} E^b_i dx^1 \wedge dx^2 \wedge dx^2 \otimes dx^a.\end{aligned}$$

Where we used the property $E^i_a = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc} e^b_j e^c_k$, obtained by the Hodge duality (6.1.7). Clearly, $\mathcal{V} = 0$ if and only if $\mathrm{V} = 0$.
The equation $d^2_A e = 0$ does not resemble one from Yang-Mills theory, even considering e as the "Hodge dual" of E (see Eq.(6.1.6)). In this context, a Yang-Mills-like equation is $d_A e = 0$, and if this equation holds, then the Vector constraint is necessarily satisfied. This Yang-Mills-like equation is the local version of the torsion-less condition for the connection ω because e is the local realisation of the solder form ϑ. Hence, it can be written as a pullback, with respect to a global section, of the equation on $P^{Spin}(\Sigma)$

$$d\vartheta + \mathrm{ad}(\omega) \wedge \vartheta = 0. \tag{6.2.6}$$

The left-hand side of this equation is the torsion form $\mathfrak{T}$ of ω. Thus, the Vector constraint requires a connection ω with parallel torsion form, namely a torsion form with vanishing exterior covariant derivative:

$$d_\omega \mathfrak{T} = \mathrm{ad}(\Theta) \wedge \vartheta = 0. \tag{6.2.7}$$

Now, we show that the Vector constraint yields the Codazzi equation when the Gauß constraint is satisfied. This is a standard result that we are going to prove in this bundle language.
First of all, we can rewrite the curvature F in a fixed trivialization splitting Γ and K:

$$F = R + \beta dK + \beta\, \mathrm{ad}(\Gamma) \wedge K + \tfrac{\beta^2}{2}\, \mathrm{ad}(K) \wedge K. \tag{6.2.8}$$

Here, R is the curvature of the Levi-Civita connection Γ. Therefore, the Vector constraint is composed by four terms. The first term $\mathrm{ad}(R) \wedge e$ vanishes because it is the Vector constraint for the connection Γ, but it is the Levi-Civita connection and so it is torsion-free.
The fourth term $\mathrm{ad}(\mathrm{ad}(K) \wedge K) \wedge e = 2\,\mathrm{ad}(K) \wedge \mathrm{ad}(K) \wedge e$ is proportional to the Gauß constraint. Thus, considering the Gauß constraint satisfied, the only non-trivial contribution to the Vector constraint comes from the middle terms of F, which are the exterior covariant derivative of K with respect to the Levi-Civita connection. Hence, the reduced Vector constraint reads

$$\mathrm{ad}(d_\Gamma K) \wedge e = 0. \tag{6.2.9}$$

Passing to a coordinate system, we obtain the following equation

$$\left(\partial_a K^i_b - \partial_b K^i_a + \epsilon^i{}_{jk} \Gamma^j_a K^k_b + \epsilon^i{}_{jk} K^j_a \Gamma^k_b\right) E^b_i = 0.$$

Using the relation with the usual Christoffel symbols $\Gamma^k_a \epsilon_{kj}{}^i = e^i_c \Gamma^c_{ab} e^b_j + e^i_b \partial_a e^b_j$ the previous equation reads

$$\sqrt{\det q}\left(\partial_a K^b_b - \partial_b K^b_a + \Gamma^b_{ac} K^c_b - \Gamma^c_{bc} K^b_a\right) = 0. \tag{6.2.10}$$

This equation is known as the contracted Codazzi equation [119] in a Ricci flat spacetime.

6.2.3 Scalar constraint

Even for the Scalar constraint, we can give a formulation in terms of the connection and the electric field. The Scalar constraint can be written as

$$S = tr\left(E \wedge \star(F - \tfrac{1}{2}(\beta^2 + 1)\,\mathrm{ad}(K) \wedge K)\right). \tag{6.2.11}$$

Hence, S is a top-form, namely an element of $\Omega^3(\Sigma)$. Here, K is a function of A and the dreibein as defined in Sec.6.1.3. The Scalar constraint is interpreted as a section of the trivial bundle $\Sigma \times \mathbb{R}$, which is an associated bundle of $P^{Spin}(\Sigma)$ with the vector space $\mathbb{R}$ under the trivial representation $\lambda_{\mathrm{tr}} : SU(2) \to GL(\mathbb{R})$, mapping every group element to the identity element on $\mathbb{R}$. Since sections of $\Sigma \times \mathbb{R}$ are canonically identified with real-valued functions on Σ, we have

$$\Omega^3(\Sigma, \Sigma \times \mathbb{R}) \cong \Omega^3(\Sigma) \otimes \Omega^0(\Sigma) \cong \Omega^3(\Sigma).$$

The computation that Eq.(6.2.11) yields Eq.(1.2.8) in a local coordinate system is quite simple, and we will show it only for the so-called Euclidean part:

$$\begin{aligned} tr(E \wedge \star F) &= \tfrac{1}{4}\epsilon_{ijk} F^i_{ab} e^a_l e^b_m \epsilon^{lm}{}_n e^j \wedge e^k \wedge e^n = \tfrac{1}{2}\epsilon_{ijk}\epsilon^{jkn} F^i_{ab} e^a_l e^b_m \epsilon^{lm}{}_n dV_q \\ &= \tfrac{1}{2}\epsilon^{lm}{}_i F^i_{ab} e^a_l e^b_m dV_q = \frac{1}{2}\epsilon_i{}^{jk} F^i_{ab} \frac{E^a_j E^b_k}{\sqrt{\det q}} dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

We can also notice that the definition of scalar curvature from Riemannian geometry provides a similar result

$$\begin{aligned}\mathrm{scal}_F &\doteq \sum_{i,j} q(D^2_{(e_i,e_j)}e_i, e_j) = \sum_{i,j,k,l} \frac{1}{2} F^k_{ab} e^a_i e^b_j \epsilon_{kil} q(e_l, e_j)\\ &= \sum_{i,j,k} \frac{1}{2} F^k_{ab} \epsilon_{kij} e^a_i e^b_j.\end{aligned}$$

Thus, $tr(E \wedge \star F) = \mathrm{scal}_F\, dV_q$.

From the Scalar constraint, we can extract the Gauß equation using the decomposition (6.2.8) and impose the Gauß constraint (i.e., considering a symmetric K). The β^2 terms cancel out while the other terms must be studied one by one. The first term $tr(E \wedge \star R) = \mathrm{scal}_R dV_q$ is the Ricci curvature. The second term $tr(E \wedge \star d_\Gamma K) = tr(e \wedge d_\Gamma K) = d\, tr(e \wedge K) + tr(d_\Gamma e \wedge K)$ vanishes because $tr(e \wedge K) = 0$ when K is symmetric and $d_\Gamma e = 0$ since Γ is torsion-less. The last term is $tr(E \wedge \star(\mathrm{ad}(K) \wedge K))$ that is the same of $(\sum_{i,j} \mathrm{K}(e_i, e_j)^2 - \sum_i \mathrm{K}(e_i, e_i)^2) dV_q$. Thus, the constraint $S = 0$ reads

$$\mathrm{scal}_R + \sum_{i,j} \mathrm{K}(e_i, e_j)^2 - \sum_i \mathrm{K}(e_i, e_i)^2 = 0. \tag{6.2.12}$$

This equation is the Gauß equation for the scalar curvature in a Ricci flat spacetime [119].

6.3 Elements from the physical theory

A crucial property of General Relativity is background independence. In our description, we fixed a metric to construct the spin structure and the related geometrical quantities. However, we want to describe the metric as a dynamical quantity reconstructed from the electric field. Indeed, the relevant geometric quantities are objects on $P^{Spin}(\Sigma)$ independent from the spin structure and from the metric, such as the physical variables. Therefore, we now aim to emancipate from the metric by encoding the metric data, and even the orientation, in the electric field, using the correspondence between q and $\boldsymbol{E}$, or equivalently τ (cf. Sec. 6.1.2), and prove the independence of the set of connections from the metric.
Thus, on a spinnable closed 3-manifold Σ, our data will consist of the pair (ω, τ), which allows us to reconstruct the pair of tensors (q, K). Here, ω is a connection on the unique principal $SU(2)$ bundle $P^{Spin}(\Sigma)$ over Σ, while τ is a non-degenerate element of $\Omega^2(P^{Spin}(\Sigma)) \otimes \mathfrak{su}(2)^*$ that is horizontal and $SU(2)$-equivariant.

6.3.1 Independence from the metric

In the previous Sections, we used the spin structure as a necessary tool to give a geometrical interpretation to the physical variables and to prove the equivalence of the data. To do that, we fixed a metric. However, we will prove that the set of Ashtekar connections is actually independent of the metric.
Fixing a spin structure on a given metric fixes the spin structure for every other metric.

Let us consider a metric q associated with an orthonormal frame bundle $P^{SO}(\Sigma, q)$ and with a spin structure $(P^{Spin}(\Sigma), \bar{\rho})$. A different metric q' on Σ is associated with a different orthonormal frame bundle $P^{SO}(\Sigma, q')$. Between these two principal bundles, there exists a vertical isomorphism $\psi : P^{SO}(\Sigma, q) \to P^{SO}(\Sigma, q')$, given pointwise by the matrix which maps an orthonormal basis of q in an orthonormal basis of q'. Moreover, there exists a vertical automorphism $\bar{\psi} : P^{Spin}(\Sigma) \to P^{Spin}(\Sigma)$. Defining $\bar{\rho}' : P^{Spin}(\Sigma) \to P^{SO}(\Sigma, q')$ such that the following diagram commutes:

$$\begin{tikzcd}
P^{Spin}(\Sigma) \arrow[rr, "\bar{\psi}"] \arrow[d, "\bar{\rho}"] & & P^{Spin}(\Sigma) \arrow[d, "\bar{\rho}'"'] \\
P^{SO}(\Sigma, q) \arrow[rr, "\psi"] \arrow[dr] & & P^{SO}(\Sigma, q') \arrow[dl] \\
& \Sigma &
\end{tikzcd}$$

The pair $(P^{Spin}(\Sigma), \bar{\rho}' = \psi \circ \bar{\rho} \circ \bar{\psi}^{-1})$ clearly defines a spin structure for $P^{SO}(\Sigma, q')$. Furthermore, the isomorphism ψ defines an isomorphism between the sets of metric-compatible connections, namely for any connection ω on $P^{SO}(\Sigma, q)$ there exists a unique connection ω' on $P^{SO}(\Sigma, q')$ such that $\omega = \psi^* \omega'$. For any pair of connections of this kind, the respective spin connections are linked by a gauge transformation $\bar{\rho}^* \omega = \bar{\psi}^* \bar{\rho}'^* \omega'$. Thus, the space of spin connections does not depend on the metric.

6.3.2 Reconstructing the metric

Fixed a global trivialization of $P^{Spin}(\Sigma)$, the non-degeneracy of τ means that the three 2-forms $E_i \in \Omega^2(\Sigma)$ that characterise $E = \bar{h}^* \tau = E_i \mathrm{t}^i$ are linearly independent. Hence, they define a global frame for $\Lambda^2 \mathrm{T}^* \Sigma$. In such a way, it is possible to induce a scalar product and an orientation on $\mathrm{T}\Sigma$.

To show that, let us see what happens in $\mathbb{R}^3$. Let us consider a basis $\{W_i\} \subset \Lambda^2 \mathbb{R}^3$. Each vector of $\Lambda^2 \mathbb{R}^3$ can be decomposed as $W = u \wedge v$ with $u, v \in \mathbb{R}^3$. We want to prove that the basis is decomposed uniquely on a basis of $\mathbb{R}^3$.
There exist $u_2, u_3 \in \mathbb{R}^3$ such that $W_1 = u_2 \wedge u_3$, we can complete $\{u_2, u_3\}$ to a basis for $\mathbb{R}^3$ adding the vector u_1, then W_2 is a linear combination $W_2 = \alpha u_1 \wedge u_2 + \beta u_2 \wedge u_3 + \gamma u_3 \wedge u_1$. We have two cases:

1. If $\alpha \neq 0$, we can substitute u_1 with the vector $u_1 + \frac{\beta}{\alpha} u_3$. And then u_2 with $u_2 + \frac{\gamma}{\alpha} u_3$, obtaining $W_2 = \alpha u_1 \wedge u_2$ and leaving W_1 invariant.

2. If $\gamma \neq 0$, we can substitute u_1 with the vector $u_1 + \frac{\beta}{\gamma} u_2$. And then u_3 with $u_3 + \frac{\alpha}{\gamma} u_2$, obtaining $W_2 = \gamma u_3 \wedge u_1$ and leaving W_1 invariant.

The two results are linked to a choice of orientation for the basis $\{W_1, W_2, W_3\}$, so, without loss of generality, we can consider $\gamma \neq 0$. The last vector is again a linear combination $W_3 = \alpha' u_1 \wedge u_2 + \beta' u_2 \wedge u_3 + \gamma' u_3 \wedge u_1$, where necessarily $\alpha' \neq 0$. We can substitute u_1 with the vector $u_1 + \frac{\beta'}{\alpha'} u_3$. And then u_2 with $u_2 + \frac{\gamma'}{\alpha'} u_3$, obtaining $W_3 = \alpha' u_1 \wedge u_2$ and leaving W_1 and W_2 invariant. The last step is the rescaling of u_1 by $\sqrt{\frac{1}{\alpha' \gamma}}$, u_2 by $\sqrt{\frac{\gamma}{\alpha'}}$, and u_3 by $\sqrt{\frac{\alpha'}{\gamma}}$, the radicand is positive under

a proper choice of the orientation, i.e. a relabelling of W_i. Thus we obtain that, given a basis $\{W_1, W_2, W_3\} \subset \Lambda^2\mathbb{R}^3$ there exists a basis $\{u_1, u_2, u_3\} \subset \mathbb{R}^3$ such that $W_1 = u_2 \wedge u_3$, $W_2 = u_3 \wedge u_1$, $W_3 = u_1 \wedge u_2$.
These bases are metric compatible in the sense that if $\{u_i\}$ is an orthonormal basis, then $\{W_i\}$ is an orthonormal basis with respect to the scalar product induced by the one on $\mathbb{R}^3$. The proof is straightforward

$$\begin{aligned}\langle W_i, W_j\rangle &= \frac{1}{4}\epsilon_{ilk}\epsilon_{jmn}\langle u_l \wedge u_k, u_m \wedge u_n\rangle = \frac{1}{4}\epsilon_{ilk}\epsilon_{jmn}(\delta_{lm}\delta_{kn} - \delta_{ln}\delta_{km}) \\ &= \frac{1}{2}\epsilon_{ilk}\epsilon_{ilk} = \delta_{ij}.\end{aligned}$$

Moreover, the decomposition is unique. Suppose there exists $u'_i = \alpha_{ik}u_k$ such that $W_i = \frac{1}{2}\epsilon_{ijk}u_j \wedge u_k = \frac{1}{2}\epsilon_{ijk}u'_j \wedge u'_k$, then the coefficients α_{ij} must satisfy

$$\det(\alpha)\epsilon_{ijk} = \epsilon_{ljk}\alpha_{li}.$$

Forcing the matrix $\alpha = \{\alpha_{nm}\}$ to be diagonal and $\det(\alpha)$ to be the diagonal terms. Hence, the only solution is the identity matrix.
Moreover, if two bases $\{W'_i\}, \{W_i\} \subset \Lambda^2\mathbb{R}^3$ are linked by an orthogonal transformation, the two bases of the decomposition are linked by the same transformation.

Hence, since the fibre of $\mathrm{T}^*\Sigma$ is isomorphic to $\mathbb{R}^3$, for each $E \in \Omega^2(\Sigma) \otimes \mathfrak{su}(2)$, there exists a unique coframe of 1-forms $\{e^i\} \subset \Omega^1(\Sigma)$ that decomposes the frame of 2-forms $\{E_i\} \subset \Omega^2(\Sigma)$ as $E_i = \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k$. The 1-forms frame e^i is the reconstructed dreibein or cotriad. Its dual frame $\{e_i\} \subset \mathfrak{X}(\Sigma)$ is the reconstructed triad and, declaring it to be orthonormal, is associated with a unique metric q. Moreover, it induces an orientation on $\mathrm{T}\Sigma$.
This procedure, starting from τ, defines a unique metric q and provides an orientation for Σ. Changing the global trivialization, E changes via the adjoint representation, and thus the frame $\{E_i\}$ is rotated by a special orthogonal matrix. As a result, the dreibein is also rotated by the same special orthogonal matrix, meaning that the associated metric and the orientation do not change. Even considering a different set of generators of $\mathfrak{su}(2)$, the metric remains invariant. Indeed, a change of generators corresponds to an automorphism of the Lie algebra, but $\mathrm{Aut}(\mathfrak{su}(2)) = SO(3)$, leading to the same conclusion.

6.3.3 Symmetries

An important aspect of the physical theory is its symmetries. We now aim to analyse the known symmetries of the Ashtekar-Barbero-Immirzi formulation of General Relativity within our framework. The physical theory tells us that two groups act on the local fields E and A, encoding the symmetries of the theory. These are the group of vertical automorphisms $\mathcal{G} = \mathrm{Gau}(P^{Spin}(\Sigma))$ and the diffeomorphism group $\mathrm{Diff}(\Sigma)$.

The geometrical description of the Ashtekar-Barbero-Immirzi formulation is given by a pair τ and ω of $\mathfrak{su}(2)$-valued differential forms on $P^{Spin}(\Sigma)$. However, in the physical description, we work with fields on Σ. Since $P^{Spin}(\Sigma)$ is trivial over Σ, it is possible to establish a natural correspondence between the geometric data and the local fields by fixing a global trivialization. Let s be the global section associated with the trivialization. In this case, τ is mapped to an electric field $E = s^*\tau \in \Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*$ and ω to an Ashtekar connection $A = s^*\omega \in \Omega^1(\Sigma) \otimes \mathfrak{su}(2)$. The group of gauge transformations has a convenient isomorphism $\mathcal{G} \cong C^\infty(\Sigma, SU(2))$, and the group action can be deduced by the pullback. Let $f \in \mathcal{G}$ be identified by $g : \Sigma \to SU(2)$, then $f^*\tau \mapsto \mathrm{Ad}^*_{g^{-1}} E$ and $f^*\omega \mapsto \mathrm{Ad}_{g^{-1}} A + g^{-1}dg$. While $\mathrm{Diff}(\Sigma)$ acts via pullback of the k-form part of the local fields.

This is the local version of saying that $\mathrm{Aut}(P^{Spin}(\Sigma))$ is the symmetry group of the theory. Indeed, it is well known that, since $P^{Spin}(\Sigma)$ is trivial, the following short sequence holds [1]

$$1 \to \mathcal{G} \to \mathrm{Aut}(P^{Spin}(\Sigma)) \to \mathrm{Diff}(\Sigma) \to 1.$$

Moreover, there exists an homomorphism from $\mathrm{Diff}(\Sigma)$ to $\mathrm{Aut}(P^{Spin}(\Sigma))$ which maps, in a fixed trivialization, $\varphi \mapsto \varphi \times \mathrm{id}_{SU(2)}$. That map is projected by the second homomorphism into the identity on $\mathrm{Diff}(\Sigma)$. This condition is necessary and sufficient for $\mathrm{Aut}(P^{Spin}(\Sigma))$ to be isomorphic to $\mathcal{G} \rtimes \mathrm{Diff}(\Sigma)$. Notice that $\mathrm{Aut}(P^{Spin}(\Sigma))$ is the group of isomorphisms in the category of principal bundles, providing a natural parallelism between physical symmetry and geometric equivalence.

A deeper analysis of these symmetry groups is provided in the context of symplectic geometry in the next Section.

6.4 A note on phase space

In this Section, we aim to characterise the phase space of the theory in terms of symplectic geometry. We are going to provide a symplectic form and show that the Gauß and Vector constraints can be interpreted as momentum maps on this symplectic manifold. The construction of the phase space emulates the procedure proposed in [166], namely we will consider as elements of the phase space the pairs of local fields (E, A), which in our framework means to fix a global trivialization (equivalently choose a section) for $P^{Spin}(\Sigma)$. However, the generalisation to vector-valued sections and connections is quite straightforward.

Let Σ be a spinnable closed 3-manifold. From what we have seen before, we can collect the data of the metric in an $\mathrm{ad}^*P^{Spin}(\Sigma)$-valued 2-form $\boldsymbol{E}$. In a fixed trivialization, it is represented by a 2-form $E = E^a_i \epsilon_{abc} dx^a \wedge dx^b \mathfrak{t}^i$ with values in $\mathfrak{su}(2)^*$. Such a 2-form has to be nondegenerate, namely $\det(E^a_i)$ is never zero. Thus, the configuration space is $\mathcal{E} \doteq \{E \in \Omega^2(\Sigma) \otimes \mathfrak{su}(2)^* \mid E \text{ is non degenerate}\}$. The space $\mathcal{E}$ is a Fréchet manifold. Indeed, it is an open subset of $\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*$, which is a Fréchet topological vector space. The Fréchet space structure is inherited, via pullback map of the chosen section, by the one of $\Omega^2(\Sigma, \mathrm{ad}^*P^{Spin}(\Sigma))$, which is a Fréchet space for a fixed choice of a triple composed by an auxiliary metric on Σ, a Riemannian product on $\mathrm{ad}^*P^{Spin}(\Sigma)$, and a auxiliary connection on $P^{Spin}(\Sigma)$. By compactness of Σ, the Fréchet space structure does not depend on the choice of the triple [33]. The family of seminorms on $\Omega^2(\Sigma, \mathrm{ad}^*P^{Spin}(\Sigma))$ is given by

$$||\boldsymbol{E}||_m = \max_{l \in \{0,\dots,m\}} \max_{x \in \Sigma} |\nabla^l \boldsymbol{E}(x)|, \; m \in \mathbb{N}_0, \tag{6.4.1}$$

where $|\cdot|$ is the norm induced on $\mathrm{T}^*\Sigma^{\otimes^l} \otimes \Lambda^2\mathrm{T}^*\Sigma \otimes \mathrm{ad}^* P^{Spin}(\Sigma)$ by the auxiliary metric and the Riemannian product, and ∇ is the covariant derivative induced on $\mathrm{T}^*\Sigma^{\otimes^l} \otimes \Lambda^2\mathrm{T}^*\Sigma \otimes \mathrm{ad}^* P^{Spin}(\Sigma)$ by the Levi-Civita connection and the chosen auxiliary connection. More concretely, we can consider a collection of seminorms on $\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*$ as given by an auxiliary metric, the scalar product on $\mathfrak{su}(2)$ defined in (6.1.6), and a flat connection, namely

$$
\begin{aligned}
||E||_0 &= \max_{x\in\Sigma} |E_a(x)\mathfrak{t}^a|, \\
||E||_1 &= \max\left\{||E||_0,\ \max_{x\in\Sigma}\left|\left(\nabla^{\Lambda^2}E_a(x)\right)\mathfrak{t}^a\right|\right\}, \\
||E||_2 &= \max\left\{||E||_1,\ \max_{x\in\Sigma}\left|\nabla^{\Lambda^1\otimes\Lambda^2}\left(\nabla^{\Lambda^2}E_a(x)\right)\mathfrak{t}^a\right|\right\}, \\
||E||_3 &= \max\left\{||E||_2,\ \max_{x\in\Sigma}\left|\left(\nabla^{\Lambda^1\otimes\Lambda^1\otimes\Lambda^2}\nabla^{\Lambda^1\otimes\Lambda^2}\nabla^{\Lambda^2}E_a(x)\right)\mathfrak{t}^a\right|\right\}, \\
&\vdots
\end{aligned}
$$

where ∇^{Λ^i} is the Levi-Civita connection of the auxiliary metric induced on the exterior power $\Lambda^i\mathrm{T}^*\Sigma$ and $|\cdot|$ is the norm induced by the auxiliary metric on $\mathrm{T}^*\Sigma^{\otimes^l} \otimes \Lambda^2\mathrm{T}^*\Sigma$ and the scalar product tr defined before. As a consequence, the tangent space on each point is isomorphic to $\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*$. In particular, the tangent bundle is trivial and $\mathcal{E}$ is parallelizable, i.e. $\mathrm{T}\mathcal{E} \cong \mathcal{E} \times (\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*)$. Notice that $\mathcal{E}$ has two connected components associated with the two possible orientations of Σ, namely the sign of $\det(E^a_i)$.

Each E is associated with a metric, constructed as in Sec.6.3.2. Given this metric and fixing a spin structure, we can define, as in Sec.6.1, the space of Ashtekar connections $\mathcal{A}_E$. These spaces do not actually depend on the metric, and they are canonically the space of $SU(2)$-connections $\mathcal{A}$.
The space of connections $\mathcal{A}$ is an affine space associated with the vector space $\Omega^1(\Sigma, \mathrm{ad}P^{Spin}(\Sigma))$, then it is a Fréchet manifold. Fixing a global trivialization, it is in one-to-one correspondence with $\Omega^1(\Sigma) \otimes \mathfrak{su}(2)$. From which we are going to consider as phase space $\mathcal{P}$ the fibre bundle with base manifold $\mathcal{E}$ and fibre $\mathcal{A}$ (the fibre over each point is $\mathcal{A}_E$). Such a bundle admits a global section, assigns to each E the corresponding spin connection $\Gamma(E)$, hence it is a trivial bundle $\mathcal{P} \cong \mathcal{E} \times \mathcal{A}$. The tangent bundle $\mathrm{T}\mathcal{P}$ can be easily characterized $\mathrm{T}\mathcal{P} \cong \mathrm{T}\mathcal{E} \times \mathrm{T}\mathcal{A} \cong \mathcal{E} \times (\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*) \times \mathcal{A} \times (\Omega^1(\Sigma) \otimes \mathfrak{su}(2))$. Hence, on each point, $\mathrm{T}_{(E,A)}\mathcal{P} \cong \mathrm{T}_E\mathcal{E} \oplus \mathrm{T}_A\mathcal{A} \cong (\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*) \oplus (\Omega^1(\Sigma) \otimes \mathfrak{su}(2))$. Notice that there is a weakly nondegenerate pairing between $\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*$ and $\Omega^1(\Sigma) \otimes \mathfrak{su}(2)$:

$$
\begin{aligned}
\left(\Omega^2(\Sigma) \otimes \mathfrak{su}(2)^*\right) \times \left(\Omega^1(\Sigma) \otimes \mathfrak{su}(2)\right) &\to \mathbb{R}; \\
(\nu, \alpha) &\mapsto \int_\Sigma tr(\nu \wedge \alpha).
\end{aligned}
$$

From which, we can define the canonical symplectic form ω [131] on the tangent space of a point $(E, A) \in \mathcal{P}$ by

$$\begin{aligned}\omega_{(E,A)} : \mathrm{T}_{(E,A)}\mathcal{P} \times \mathrm{T}_{(E,A)}\mathcal{P} &\to \mathbb{R} \\ ((\nu_1, \alpha_1), (\nu_2, \alpha_2)) &\mapsto \int_\Sigma tr(\nu_1 \wedge \alpha_2) - \int_\Sigma tr(\nu_2 \wedge \alpha_1).\end{aligned} \tag{6.4.2}$$

The symplectic form is defined modulo a constant real factor β, which represents the Barbero-Immirzi parameter. Actually, it can be associated with the symplectic potential

$$\begin{aligned}\theta_{(E,A)} : \mathrm{T}_{(E,A)}\mathcal{P} &\to \mathbb{R} \\ (\nu, \alpha) &\mapsto \int_\Sigma tr(E \wedge \alpha).\end{aligned}$$

Notice that it is the same symplectic structure of Yang-Mills theory [156, 159]. Introducing the exterior derivative δ on $\mathcal{P}$, those objects can be written, with a slight abuse of notation, as $\omega = \int tr(\delta E \,\bar{\wedge}\, \delta A)$ and $\theta = \int tr(E \wedge \delta A)$, where $\bar{\wedge}$ represents the wedge product between k-forms on Σ and between l-forms of $\mathcal{P}$.

The constraints are well-defined on this phase space. The Gauß constraint G and the Vector constraint V can be regarded as functions on the phase space with values in suitable infinite-dimensional Fréchet Lie algebras. These functions are globally defined due to the smoothness of the fields and the compactness of Σ. Moreover, they have an interpretation in terms of momentum maps, and we will implement them within the framework of symplectic geometry via symplectic reduction. Note that if we were to consider the space of L^2 connections instead, we would also need to analyse the domain of the constraints, likely involving some type of Sobolev space.

6.4.1 Gauge transformations and momentum maps

The infinite-dimensional group of gauge transformations $\mathcal{G} \cong C^\infty(\Sigma, SU(2))$ acts on this phase space. The group has a natural right action on $\mathcal{P}$:

$$\begin{aligned}\mathcal{G} \times \mathcal{P} &\to \mathcal{P}; \\ (g, (E, A)) &\mapsto g.(E, A) \doteq (\mathrm{Ad}^*_{g^{-1}} E, \mathrm{Ad}_{g^{-1}} A + g^{-1}dg).\end{aligned} \tag{6.4.3}$$

The action preserves the symplectic form ω, i.e. it acts via symplectomorphisms.

$$\begin{aligned}(g.^*\omega)_{(E,A)}((\nu_1, \alpha_1), (\nu_2, \alpha_2)) &= \omega_{g.(E,A)}(\delta g.(\nu_1, \alpha_1), \delta g.(\nu_2, \alpha_2)) \\ &= \omega_{g.(E,A)}((\mathrm{Ad}^*_{g^{-1}} \nu_1, \mathrm{Ad}_{g^{-1}} \alpha_1), (\mathrm{Ad}^*_{g^{-1}} \nu_2, \mathrm{Ad}_{g^{-1}} \alpha_2)) \\ = \int_\Sigma tr(\mathrm{Ad}^*_{g^{-1}} \nu_1 \wedge \mathrm{Ad}_{g^{-1}} \alpha_2) &- \int_\Sigma tr(\mathrm{Ad}^*_{g^{-1}} \nu_2 \wedge \mathrm{Ad}_{g^{-1}} \alpha_1) \\ &= \int_\Sigma tr(\nu_1 \wedge \alpha_2) - \int_\Sigma tr(\nu_2 \wedge \alpha_1)\end{aligned}$$

Moreover, it is possible to define the momentum map for this action as follows:

$$\begin{array}{rccl}\mu_G : & \mathcal{P} & \to & \Omega^3(\Sigma) \otimes \mathfrak{su}(2)^*; \\ & (E, A) & \mapsto & G(E, A) = d_A E.\end{array} \tag{6.4.4}$$

The map μ_G is equivariant with respect to the $\mathcal{G}$-action, indeed, $G(g.(E,A)) = \mathrm{Ad}^*_{g^{-1}} G(E,A)$. The coupling with $Lie(\mathcal{G}) \cong \Omega^0(\Sigma) \otimes \mathfrak{su}(2)$ is given by

$$\begin{aligned} \left(\Omega^3(\Sigma) \otimes \mathfrak{su}(2)^*\right) \times \left(\Omega^0(\Sigma) \otimes \mathfrak{su}(2)\right) &\to \mathbb{R}; \\ (\chi, \Lambda) \mapsto \langle \chi, \Lambda \rangle &\doteq \int_\Sigma tr(\chi\Lambda). \end{aligned}$$

We can now prove the momentum map property. Considering the vector field on $\mathcal{P}$ defined by

$$\varrho(\Lambda)_{(E,A)} \doteq \frac{d}{dt}\Big|_0 \exp(t\Lambda).(E,A) = \left(-\operatorname{ad}^*(\Lambda)E, d\Lambda + \operatorname{ad}(A)\Lambda\right),$$

we must show $\iota_{\varrho(\Lambda)}\omega = \delta\langle \mu_G, \Lambda\rangle$. Let us compute first the interior product with the symplectic form

$$\begin{aligned} (\iota_{\varrho(\Lambda)}\omega)(\nu,\alpha) &= \omega\left((-\operatorname{ad}^*(\Lambda)E, d\Lambda + \operatorname{ad}(A)\Lambda), (\nu,\alpha)\right) \\ &= -\int_\Sigma tr(\operatorname{ad}^*(\Lambda)E \wedge \alpha) - \int_\Sigma tr(\nu \wedge (d\Lambda + \operatorname{ad}(A)\Lambda)). \end{aligned}$$

Then, we compute the differential of the coupling with a slight abuse of notation

$$\begin{aligned} \delta\langle \mu_G, \Lambda\rangle &= \delta \int_\Sigma tr(G(E,A)\Lambda) = \int_\Sigma tr\left((d\delta E + \operatorname{ad}^*(\delta A) \wedge E + \operatorname{ad}^*(A) \wedge \delta E)\Lambda\right) \\ &= -\int_\Sigma tr(\delta E \wedge d\Lambda) - \int_\Sigma tr(\operatorname{ad}^*(\Lambda)E \wedge \delta A) - \int_\Sigma tr(\delta E \wedge \operatorname{ad}(A)\Lambda). \end{aligned}$$

Thus, the $\mathcal{G}$-action on $\mathcal{P}$ is Hamiltonian and we can consider the symplectic reduction $\mathcal{P} /\!\!/ \mathcal{G} = \mu_G^{-1}(0)/\mathcal{G}$. However, this action is not free because there exists a non-trivial centre. We can slightly modify the definition of gauge transformation considering $\mathcal{G} \doteq C^\infty(\Sigma, \operatorname{Ad} SU(2))$. This choice does not modify the Lie algebra and the previous considerations; moreover, the action is free on $\mathcal{E}$ and so on $\mathcal{P}$ even if $\mathcal{A}$ admits reducible connections [140].

6.4.2 Diffeomorphisms group and ADM phase space

A second group acts via symplectomorphism on $\mathcal{P}$, it is $\mathrm{Diff}(\Sigma)$.
The action is $\varphi.(E,A) = (\varphi^*E, \varphi^*A)$, it passes to the quotient, and the symplectic form is invariant under it:

$$\begin{aligned} (\varphi.^*\omega)_{(E,A)}((\nu_1,\alpha_1),(\nu_2,\alpha_2)) &= \omega_{\varphi.(E,A)}(\delta\varphi.(\nu_1,\alpha_1), \delta\varphi.(\nu_2,\alpha_2)) \\ &= \omega_{\varphi.(E,A)}((\varphi^*\nu_1, \varphi^*\alpha_1), (\varphi^*\nu_2, \varphi^*\alpha_2)) \\ &= \int_\Sigma tr(\varphi^*\nu_1 \wedge \varphi^*\alpha_2) - \int_\Sigma tr(\varphi^*\nu_2 \wedge \varphi^*\alpha_1) \\ &= \int_\Sigma \varphi^* tr(\nu_1 \wedge \alpha_2) - \int_\Sigma \varphi^* tr(\nu_2 \wedge \alpha_1) \\ &= \int_{\varphi(\Sigma)} tr(\nu_1 \wedge \alpha_2) - \int_{\varphi(\Sigma)} tr(\nu_2 \wedge \alpha_1) \\ &= \int_\Sigma tr(\nu_1 \wedge \alpha_2) - \int_\Sigma tr(\nu_2 \wedge \alpha_1). \end{aligned}$$

The map given by $(E, A) \mapsto \mathrm{V}(E, A) = tr\,(e(E) \otimes \mathrm{ad}^*(F(A)) \wedge e(E))$ is invariant under the action of $\mathcal{G}$, hence it defines a well-defined map on the quotient space. Moreover, its range is in $\Omega^3(\Sigma) \otimes \Omega^1(\Sigma)$ and that map is $\mathrm{Diff}(\Sigma)$-equivariant $\mathrm{V}(\varphi^* E, \varphi^* A) = \varphi^* \mathrm{V}(E, A)$. Its projection to the quotient is the momentum map associated with $\mathrm{Diff}(\Sigma)$, which is the supermomentum constraint.
Using the procedure explained in Sec.6.3.2, it is possible to associate a metric $q(E)$ to E, which is the same for the entire equivalent class. In the same proof, the cotriad $e(E)$ reconstructed from E is also defined. Moreover, we can define the function $\Gamma(E)$ that is the Levi-Civita connection of the metric $q(E)$ with respect to the triad reconstructed by E. Such a function is a homogeneous function of E and its derivative of degree zero [166]. Finally, we can associate a tensor $\mathrm{K}(E, A) = tr\,(e(E) \otimes (A - \Gamma(E)))$ that it constant along the $\mathcal{G}$-orbits in $\mathcal{P}$ and symmetric on $\mu_G^{-1}(0)$. Changing the orientation is equivalent to consider the transformation $E \mapsto -E$, we can notice that $q(E) = q(-E)$ and $\mathrm{K}(E, A) = -\mathrm{K}(-E, A)$. Hence, the equivalence classes $[E, A]$ in $\mathcal{P} /\!\!/ \mathcal{G}$ are identified by pairs $(q(E), \mathrm{K}(E, A))$. Therefore, the reduced symplectic manifold $\mathcal{P} /\!\!/ \mathcal{G}$ is a fibre bundle with base manifold the space of Riemannian metrics $Met(\Sigma)$ and as fibre the vector space $C^\infty(\Sigma, \mathrm{Sym}(\mathrm{T}\Sigma))$ of symmetric tensors. Since $Met(\Sigma)$ is contractible, the fibre bundle is trivial.
The action of the Diffeomorphism group is induced by the one on $\mathcal{P}$ and reads $(q, \mathrm{K}) \mapsto (\varphi^* q, \varphi^* \mathrm{K})$. It acts via symplectomorphisms and has a momentum map that is the supermomentum constraint. This is induced on the reduced manifold by the projection of the Vector constraint, while the Scalar constraint is projected into the Hamiltonian one. Hence, this phase space is isomorphic to the ADM one [12, 13].

6.4.3 Full group of symmetries

We can consider the action of the full group $\mathfrak{G} = \mathcal{G} \rtimes \mathrm{Diff}(\Sigma)$, where the group product is

$$(g_1, \varphi_1) \cdot (g_2, \varphi_2) = (g_1 \cdot (g_2 \circ \varphi_1^{-1}), \varphi_1 \circ \varphi_2).$$

The action of the pair (g, φ) on $\mathcal{P}$ is

$$(g, \varphi).(E, A) = (\mathrm{Ad}^*_{g^{-1}}\, \varphi^* E, \mathrm{Ad}_{g^{-1}}\, \varphi^* A + g^{-1} dg). \tag{6.4.5}$$

The Lie algebra of this group is isomorphic as a vector space to $Lie(\mathcal{G}) \oplus Lie(\mathrm{Diff}(\Sigma))$. Following some basic calculations about semidirect products of groups, we find the adjoint action on (Λ, X), where $\Lambda \in C^\infty(\Sigma, \mathfrak{su}(2)) \cong Lie(\mathcal{G})$ and $X \in \mathfrak{X}(\Sigma) \cong Lie(\mathrm{Diff}(\Sigma))$:

$$\mathrm{Ad}_{(g,\varphi)}(\Lambda, X) = (\mathrm{Ad}_g \Lambda \circ \varphi^{-1} - g d g^{-1}(\varphi_* X), \varphi_* X). \tag{6.4.6}$$

From which we can find the coadjoint representation by imposing

$$\begin{aligned}
&\langle \mathrm{Ad}^*_{(g,\varphi)}(\chi, \lambda), (\Lambda, X) \rangle = \langle (\chi, \lambda), \mathrm{Ad}_{(g,\varphi)^{-1}}(\Lambda, X) \rangle \\
&= \int_\Sigma tr\left(\chi \otimes (\mathrm{Ad}_{g^{-1}\circ\varphi} \Lambda \circ \varphi - (g^{-1} \circ \varphi) d(g \circ \varphi)(\varphi_*^{-1} X))\right) + \int_\Sigma \lambda(\varphi_*^{-1} X) \\
&= \int_\Sigma tr\left((\mathrm{Ad}^*_g (\varphi^{-1})^* \chi) \otimes \Lambda\right) - \int_\Sigma tr\left((\varphi^{-1})^* \chi \otimes g^{-1} dg(X)\right) + \int_\Sigma (\varphi^{-1})^* \lambda(X).
\end{aligned}$$

Then, the coadjoint action reads

$$\mathrm{Ad}^*_{(g,\varphi)}(\chi,\lambda) = (\mathrm{Ad}^*_g(\varphi^{-1})^*\chi, (\varphi^{-1})^*\lambda + tr((\varphi^{-1})^*\chi \otimes gdg^{-1})). \tag{6.4.7}$$

For completeness, we can compute the adjoint action of the algebra on itself, namely the Lie bracket:

$$\mathrm{ad}_{(\Lambda_1,X_1)}(\Lambda_2,X_2) = (\mathrm{ad}_{\Lambda_1}\Lambda_2 + d\Lambda_1(X_2) - d\Lambda_2(X_1), -\mathcal{L}_{X_1}X_2). \tag{6.4.8}$$

We can now discuss the momentum map of this full group, which acts via symplectomorphisms.
It is known that the correct constraint that implements diffeomorphisms is a deformation of (but equivalent to) the Vector constraint. In our case, the Diffeomorphism constraint reads

$$\mathrm{D}(E,A) = \mathrm{V}(E,A) + tr(G(E,A)\otimes A). \tag{6.4.9}$$

The momentum map is defined by:

$$\begin{aligned} \mu : \mathcal{P} \to &\left(\Omega^3(\Sigma)\otimes \mathfrak{su}(2)^*\right) \oplus \left(\Omega^3(\Sigma)\otimes\Omega^1(\Sigma)\right); \\ (E,A) \mapsto &\mu(E,A) = (G(E,A), \mathrm{D}(E,A)). \end{aligned} \tag{6.4.10}$$

This map is $\mathfrak{G}$-equivariant with respect to the right action

$$\begin{aligned} \mu\circ(g,\varphi).(E,A) &= (\mathrm{Ad}^*_{g^{-1}}\,\varphi^*G(E,A), \varphi^*\mathrm{D}(E,A) + tr(\varphi^*G(E,A)\otimes g^{-1}dg)) \\ &= \mathrm{Ad}^*_{(g^{-1},\varphi^{-1})}\,\mu(E,A). \end{aligned}$$

With the same slight abuse of notation as before, we can find that

$$\delta\langle\mu_D, X\rangle = \int_\Sigma tr(\mathcal{L}_X E\wedge\delta A) - \int_\Sigma tr(\delta E\wedge\mathcal{L}_X A).$$

Considering now the vector field $\varrho(X)_{(E,A)} \doteq (\mathcal{L}_X E, \mathcal{L}_X A)$ on $\mathcal{P}$ that generates the Diffeomorphism transformations, the interior product matches the differential of the pairing

$$\begin{aligned} (\iota_{\varrho(X)}\omega)(\nu,\alpha) &= \omega\left((\mathcal{L}_X E, \mathcal{L}_X A), (\nu,\alpha)\right) \\ &= \int_\Sigma tr(\mathcal{L}_X E\wedge\alpha) - \int_\Sigma tr(\nu\wedge\mathcal{L}_X A). \end{aligned}$$

The previous computations show that the Diffeomorphism constraint $\mathrm{D}(E,A)$ is the momentum map for the $\mathrm{Diff}(\Sigma)$-action and has the correct transformation behaviour under $\mathcal{G}$-action in order to μ be the momentum map for the $\mathfrak{G}$-action on $\mathcal{P}$

$$\delta\langle\mu,(\Lambda,X)\rangle = \iota_{\varrho(\Lambda,X)}\omega.$$

Concerning the usual physical notation, the Hamiltonian functions of the respective group actions, namely the natural pairing between momentum maps and Lie algebra elements, are the so-called smeared constraints.

Chapter 7

Cosmology in Loop Quantum Gravity preserving local gauge degrees of freedom

In this chapter, we aim to address the loss of internal $SU(2)$ gauge symmetry in the cosmological setting. As demonstrated in Chapter 5, to properly recover gauge symmetry at both the classical and quantum levels, it is necessary to go beyond the conventional minisuperspace framework. This is achieved by allowing the gauge transformations to possess local degrees of freedom. Importantly, this extension does not contradict the assumption of homogeneity. Since gauge transformations are not physical observables, it is not required that the gauge itself be homogeneous to yield homogeneous physical quantities. The common imposition of a homogeneous gauge in Loop Quantum Cosmology is precisely what leads to the effective disappearance of the $SU(2)$ gauge symmetry.

The approach we take here is to permit the Ashtekar variables to include local gauge degrees of freedom, while enforcing the reconstructed ADM variables to remain homogeneous. To facilitate this, we adopt a geometric perspective based on the tools introduced in Chapter 6, which enable a rigorous definition of homogeneous connections and densitised triads. More specifically, we introduce the mathematical concepts of homogeneous spaces and invariant principal bundles. We carry out a parallel analysis to that of the previous chapter, now within a cosmological context. We demonstrate that the notion of a homogeneous connection, as formalised by Wang's theorem, provides precisely the structure needed to complete the classical formulation and to identify the appropriate cosmological sector of the Ashtekar-Barbero-Immirzi formulation of General Relativity.

Following the classical construction, we proceed to discuss the properties of the corresponding cylindrical functions and examine the classical constraint algebra. Remarkably, we find that the resulting description closely resembles that of full Loop Quantum Gravity. This approach successfully restores the full $SU(2)$ gauge symmetry and, at the same time, introduces significant simplifications in the formulation, laying a robust foundation for the subsequent quantisation.

7.1 Classical gauge field theory

We want to find a picture of the Ashtekar variables in the cosmological setting using the prescriptions of the gauge theories and the language presented in the previous Chapter. We aim to analyse the classical description and identify the cosmological sector of General Relativity in Ashtekar variables. To achieve this, we must articulate the cosmological hypothesis and the Landau description in a pure group-theoretic form.

7.1.1 Homogeneous hypothesis on Cauchy hypersurface

The homogeneous hypothesis compelled us to regard Σ as a homogeneous space.

Definition 7.1.1. A homogeneous space is a pair $(\mathcal{X}, G)$ with $\mathcal{X}$ a topological space and G a group that acts transitively on $\mathcal{X}$. In this case, $\mathcal{X} \cong G/H$, where H is the stabiliser of a fixed point.

In cosmology, we require that (Σ, q) is a Riemannian homogeneous manifold, namely, that the isometry group $\mathrm{Isom}(\Sigma, q)$ acts transitively. However, the formalism presented in Sec.3.1 and proposed by L.Landau (c.f. [122]) describes the pair (Σ, q) as a 3-dimensional Lie group G equipped with a left-invariant metric η. In this case, we have a copy of G into the isometry group $G_L \leq \mathrm{Isom}(G, \eta)$. Moreover, considering a Lie group with a left-invariant metric instead of a generic Riemannian homogeneous manifold has a well-posed mathematical aspect. Every class A [1] simply connected Riemannian homogeneous manifold admits a Lie group structure, hence it is isometric to a suitable group G equipped with a left-invariant metric η [2]. Furthermore, from simply connected Lie groups we can generate a 3-dimensional Lie group with non-trivial topology taking the quotient by a normal subgroup (two interesting examples are $\mathbb{R}^3/\mathbb{Z}^3 = \mathbb{T}^3$ and $SU(2)/\mathbb{Z}^2 = SO(3)$). In addition, there exist Lie groups associated with class B [2] Riemannian homogeneous manifolds. Finally, the description through Lie groups is quite general and includes all the physically relevant models.

Once we deal with the pair (G, η), we need to discuss which group acts on this Riemannian manifold. In fact, on the same Riemannian manifold, different groups can act via isometries. In our case, on (G, η), there are two relevant groups: the group of orientation-preserving isometries $S = \mathrm{SIsom}(G, \eta)$ or the copy of G in it: G_L, that acts on G via left multiplication. We choose the smaller group of S that acts transitively on G, which is G_L. Then, our homogeneous space is the pair (G, G_L). This choice is interpreted as the weakest request for homogeneity and is equivalent to dealing with Bianchi models, without any further symmetry. To consider the stronger constraint given by isotropy, we need to enlarge the symmetry group. Another equivalent way to state the hypothesis is to consider the homogeneous space (Σ, G) with a transitive and free action of G [51]. Such a request provides $G \cong \Sigma$, and the action is always equivalent to the left multiplication. Namely, there exist a

[1] A homogeneous Riemannian space is class A if it admits a compact quotient. If the space is a Lie group, it is class A if it is unimodular [135].

[2] A homogeneous Riemannian space is class B if it is not class A.

diffeomorphism $\varphi : \Sigma \to G$ such that the action can be written as $\varphi^{-1} \circ L_g \circ \varphi$. To include isotropic models, we can ask for an effective action instead of a free one.

Recalling the Definition 7.1.1 for homogeneous space, we can use it to define the configuration space of a dynamical theory of geometry for a spatially homogeneous universe with spatial slice a homogeneous space (G, G_L). Namely, we consider the set of left-invariant metrics on G:

$$\mathcal{S}_G = \{\eta \in Met(G) \mid L_g^* \eta = \eta \quad \forall g \in G\}.$$

As a consequence, the conjugate momenta, as well as the extrinsic curvature, will be represented by left-invariant symmetric tensors. These are the conditions that the quantities reconstructed from the Ashtekar variables must satisfy. As already noted in [37], this choice breaks the full diffeomorphism gauge symmetry. However, a residual gauge symmetry remains, given by the action of the semidirect product $G \rtimes \mathrm{Aut}(G)$, where G acts by right multiplication and $\mathrm{Aut}(G)$ denotes the automorphism group. The structure and implications of this residual symmetry will be discussed in detail later.

7.1.2 Homogeneous spin bundle

We aim to analyse the properties required by the Ashtekar variables to understand a spatially homogeneous Universe. We aspire to utilise the Yang-Mills approach, where the connection is desired to be homogeneous in the sense of Wang's theorem. Thus, we need to begin studying the homogeneity of the corresponding principal bundle.

Considering the formulation in Chapter 6, and let G be a connected 3-dimensional Lie group, we must work with the orthonormal frame bundle $P^{SO}(G)$. It is defined as the disjoint union $P^{SO}(G) := \bigsqcup_{x \in G} P_x^{SO}(G)$, where

$$P_x^{SO}(G) = \{h : \mathbb{R}^3 \to \mathrm{T}_x G \,|\, h \text{ orientation-preserving linear isometry}\}.$$

The group $S = \mathrm{SIsom}(G, \eta)$ acts freely and transitively via automorphisms on $P^{SO}(G)$. In fact, let $f \in S$, the action

$$\begin{aligned} \tilde{f} : P_x^{SO}(G) &\to P_{f(x)}^{SO}(G); \\ h &\mapsto d_x f \circ h, \end{aligned}$$

preserves the principal bundle structure. Hence, the orthonormal frame bundle is homogeneous with respect to S (properly, it is S-invariant), and so, the desirable connections would be too. Remark that this defines also an action $\tilde{L}$ of G on $P^{SO}(G)$, such that $\tilde{L}_g$ is the lift of L_g for all $g \in G$, then $P^{SO}(G)$ is also G_L-invariant (or just G-invariant).

However, in the Ashtekar formulation, we are interested in the involvement of classical spinors. Hence, we must include in our analysis the spin bundle $P^{Spin}(G)$ (cf. [32, 123]). Moreover, since the homogeneous property will be verified on the reconstructed quantities (q, K) (on which we have a standard notion of homogeneity), we need to deal with a (possibly invariant) spin structure.
There exists a notion of homogeneous spin structure:

Definition 7.1.2. On a homogeneous space $(\mathcal{X}, G)$, a spin structure $(P^{Spin}(\mathcal{X}), \bar{\rho})$ is homogeneous (G-invariant) if $P^{Spin}(\mathcal{X})$ is equipped with a G-action covering the natural action of G on $P^{SO}(\mathcal{X})$.

On a homogeneous space (G, G_L), where G is a connected Lie group, there always exists a unique G-invariant spin structure. Namely, there exists an action $\bar{L}$ of G on $P^{Spin}(G)$ such that $\bar{\rho} \circ \bar{L}_g = \tilde{L}_g \circ \bar{\rho}$ for all $g \in G$.
The previous property does not hold for every homogeneous space, even with the same base manifold. The example of the 3-sphere $\mathbb{S}^3$ is enlightening [72, 3]. Considering the 3-sphere as a group $\mathbb{S}^3 = SU(2)$, in this case there exists a $SU(2)$-invariant spin structure, while, if $\mathbb{S}^3 \cong SO(4)/SO(3)$, a $SO(4)$-invariant spin structure does not exist. In our language, $\Sigma = \mathbb{S}^3 \cong SU(2) = G$ with the action of the groups $G_L = SU(2)$ and $S = SO(4)$, respectively.

Thus, we will examine the connected Lie group G, the orthonormal frame bundle $P^{SO}(G)$, and the spin structure $P^{Spin}(G)$ equipped and invariant under the action of G_L.

7.1.3 Homogeneous Ashtekar variables

Such as the homogeneous space (G, G_L), we require the Ashtekar connection A to be G_L-invariant. Motivated by the approach in Yang-Mills theory [61, 43] and in previous works in Loop Quantum Gravity [57], we demand that the connection 1-form ω on $P^{SO}(G)$ must be G_L-invariant, namely $\tilde{L}_g^* \omega = \omega$. The homogeneous spin structure helps us to define a proper notion of homogeneity for the connections on $P^{Spin}(G)$. We require that the homogeneous Ashtekar connections be the ones G-invariant on the unique invariant spin structure. These two requirements are equivalent. Indeed, considering a connection ω on $P^{SO}(G)$ and the associated connection $\bar{\omega}$ on $P^{Spin}(G)$, since the spin structure is homogeneous, the following equation holds:

$$\bar{L}_g^* \bar{\omega} = \bar{L}_g^* \rho_*^{-1} \bar{\rho}^* \omega = \rho_*^{-1} \bar{\rho}^* \tilde{L}_g \omega = \rho_*^{-1} \bar{\rho}^* \omega = \bar{\omega}. \tag{7.1.1}$$

Showing that, if ω is homogeneous, the $\bar{\omega}$ is homogeneous too. The contrary is also true. Proceeding along the same steps, we end up with $\rho_*^{-1} \bar{\rho}^* \tilde{L}_g \omega = \rho_*^{-1} \bar{\rho}^* \omega$, since ρ_*^{-1} is an isomorphism and $\bar{\rho}^*$ is injective, then it implies $\tilde{L}_g \omega = \omega$.

These properties ensure that Wang's theorem can be applied to both $P^{SO}(G)$ and $P^{Spin}(G)$, so that the homogeneous Ashtekar variables are precisely those classified by the theorem:

Theorem 7.1.1 (Wang's theorem [169])**.** *Let P be a G-invariant principal K-bundle over a manifold $\mathcal{X}$, where $\mathcal{X} \cong G/H$ is a homogeneous space and H is the stabiliser of a point $x_0 \in \mathcal{X}$. Let $p_0 \in P$ be a point in the fibre over x_0, and let $\lambda : H \to K$ denote the isotropy homomorphism associated with p_0.*
Then, the G-invariant connections ω on P are in one to one correspondence with linear maps $\phi : \mathfrak{g} \to \mathfrak{k}$ such that

1. $\phi \circ \mathrm{Ad}_h = \mathrm{Ad}_{\lambda(h)} \circ \phi$, *for all* $h \in H$,
2. $\phi|_{\mathfrak{h}} = d\lambda$.

The correspondence is given by

$$\phi(\xi) \doteq \omega_{p_0}(X_\xi)$$

where X_ξ is the vector field on P induced by $\xi \in \mathfrak{g}$.

In our framework, the theorem is simplified a lot. Our homogeneous space is a Lie group (G, G_L) and, so, the stabiliser contains only the identity element $H = \{\mathbb{1}\}$. The principal bundle we are considering is $P^{Spin}(G)$ as a principal $SU(2)$-bundle, and it is G_L-invariant, as the desirable connections. Hence, the theorem can be recast in a simplified version:

Corollary 7.1.2. *Let $P^{Spin}(G)$ be the unique G_L-invariant spin structure on G. Then, the G_L-invariant connections ω are in one-to-one correspondence with linear maps $\phi : \mathfrak{g} \to \mathfrak{su}(2)$.*

Notice that considering the orthonormal frame bundle $P^{SO}(G)$ as a principal $SO(3)$-bundle, due to the isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, we obtain the same classification for the G_L-invariant connections on $P^{SO}(G)$.
For such connections, there exists a preferred trivialization $s : G \to P^{Spin}(G)$ such that the corresponding local gauge field can be written as

$$A = s^*\omega = \phi \circ \theta_{MC}, \tag{7.1.2}$$

where θ_{MC} is the Maurer-Cartan form on $\mathfrak{su}(2)$ [108].

Once we have clarified the properties of the Ashtekar connection, we need to discuss the dreibein. The restriction we want to impose is that the reconstructed physical quantities, the Riemannian metric q and the extrinsic curvature K (i.e., the ADM variables), be homogeneous.
We interpret the dreibein as a section $h : G \to P^{SO}(G)$. Since $P^{SO}(G)$ is G_L-invariant, the associated metric q is therefore homogeneous by construction.
It is useful to notice that there exists a G_L-equivariant dreibein, namely a section such that the following diagram commutes

$$\begin{tikzcd} P^{SO}(G) \arrow[r, "\tilde{L}_g"] \arrow[d, "\pi"] & P^{SO}(G) \arrow[d, "\pi"'] \\ G \arrow[r, "L_g"] \arrow[u, bend left, "h"] & G \arrow[u, bend right, "h"'] \end{tikzcd}$$

Thus, the equation for the G_L-equivariant dreibein is

$$h_{L_g(x)} = d_x L_g \circ h_x, \ \forall x \in G, \, g \in G, \tag{7.1.3}$$

where h_x is the section h evaluated on the point $x \in G$, which is an element of $P^{SO}_x(G)$. Moreover, given a G_L-equivariant dreibein h, $h(\mathrm{v})$ (the image of a vector $\mathrm{v} \in \mathbb{R}^3$ via h) is a left-invariant vector field and, so, an element of the Lie algebra $\mathfrak{g}$.
It is easy to prove that every section is gauge equivalent to a G_L-equivariant section, and that a G_L-equivariant section is the preferred section in Eq.(7.1.2).
Invariant sections also exist for the spin bundle. Suppose $\bar{h} : G \to P^{Spin}(G)$ such that $\bar{\rho} \circ \bar{h} = h$ then $\bar{h}$ is G_L-equivariant if and only if h is it.

Proof. Suppose $\bar{h}$ is G_L-equivariant, i.e. it satisfies $\bar{h} \circ L_g = \bar{L}_g \circ \bar{h}$, then

$$\bar{\rho} \circ \bar{h} \circ L_g = \bar{\rho} \circ \bar{L}_g \circ \bar{h} \quad \Longrightarrow \quad h \circ L_g = \tilde{L}_g \circ \bar{\rho} \circ \bar{h} = \tilde{L}_g \circ h.$$

Thus, $\bar{\rho} \circ \bar{h}$ defines a invariant section of $P^{SO}(G)$.

Suppose h is G_L-equivariant, i.e. it satisfies $h \circ L_g = \tilde{L}_g \circ h$, then

$$\bar{\rho} \circ \bar{h} \circ L_g = h \circ L_g = \tilde{L}_g \circ h = \tilde{L}_g \circ \bar{\rho} \circ \bar{h} = \bar{\rho} \circ \bar{L}_g \circ \bar{h}.$$

Fixing a point $x \in G$ and an open neighbourhood $U \subset G$ small enough such that the image $U' = h \circ L_g(U) = \tilde{L}_g \circ h(U) \subset P^{SO}(G)$ is such that $\bar{\rho}^{-1}(U') \cong U' \times \mathbb{Z}^2$. For every point $y \in U$, we can fix $\bar{h} \circ L_g(y)$ to be in the $+1$ connected components while $\bar{L}_g \circ \bar{h}(y)$ will be in the $+1$ or -1 connected component appropriately. When g is the identity, the two terms $\bar{L}_g \circ \bar{h}(x)$ and $\bar{h} \circ L_g(x)$ are equal. Let us consider g in a small enough open neighbourhood $V \subset G$ of the identity such that $L_g x$ is in U, then $\bar{h} \circ L_g(x) = \bar{h}(y)$ for some $y \in U$, moreover $\bar{h} \circ L_g(x)$ remains in the same connected components as $\bar{h}(x)$ for all $g \in V$. Due to the continuity of the action, $\bar{L}_g \circ \bar{h}(x)$ must also remain in the same connected components of $\bar{h}(x)$. Hence, $\bar{L}_g \circ \bar{h}(x)$ and $\bar{h} \circ L_g(x)$ must define the same element.
However, since G is connected, V generates G (Prop.7.14 in [124]). Thus, every g can be written as a finite product of elements of V. Suppose $g = g_N g_{N-1} \dots g_2 g_1$, where $g_1, \dots, g_N \in V$ then

$$\begin{aligned}\bar{L}_g \circ \bar{h} &= \bar{L}_{g_N} \circ \dots \circ \bar{L}_{g_2} \circ \bar{L}_{g_1} \circ \bar{h} = \bar{L}_{g_N} \circ \dots \circ \bar{L}_{g_2} \circ \bar{h} \circ L_{g_1} \\ &= \bar{L}_{g_N} \circ \dots \circ \bar{h} \circ L_{g_2} \circ L_{g_1} = \bar{h} \circ L_{g_N} \circ \dots \circ \ L_{g_2} \circ L_{g_1} = \bar{h} \circ L_g.\end{aligned}$$

□

This means, it is always possible to find two sections, $h : \Sigma \to P^{SO}(G)$ and $\bar{h} : \Sigma \to P^{Spin}(G)$, such that the following diagram commutes:

$$\begin{tikzcd}
P^{Spin}(G) \arrow[r, "\bar{L}_g"] \arrow[d, "\bar{\rho}"] & P^{Spin}(G) \arrow[d, "\bar{\rho}"'] \\
P^{SO}(G) \arrow[r, "\tilde{L}_g"] \arrow[d, "\pi"] & P^{SO}(G) \arrow[d, "\pi"'] \\
G \arrow[r, "L_g"] \arrow[u, bend left, "h"] \arrow[uu, bend left=60, "\bar{h}"] & G \arrow[u, bend right, "h"'] \arrow[uu, bend right=60, "\bar{h}"']
\end{tikzcd}$$

These sections induce an automorphism between $P^{SO}(G)$ with G-action $\tilde{L}_g$ and the trivial bundle $G \times SO(3)$ equipped with the action $(L_g, \mathrm{id}_{SO(3)})$. In such a way, the homogeneous spin structure becomes $\bar{\rho} : P^{Spin}(G) \cong G \times SU(2) \to G \times SO(3)$; $\bar{\rho}(g, a) = (g, \rho(a))$, with G-action $(L_g, \mathrm{id}_{SU(2)})$. With this automorphism, a connection on $P^{Spin}(G)$ reads as

$$\omega_{(g,a)} = a^{-1} da + a^{-1} \hat{\omega}_g a,$$

where $\hat{\omega}$ is a $\mathfrak{su}(2)$-valued 1-form on G. When ω is G-invariant, $\hat{\omega}$ is a left-invariant form, identified with A in Eq. (7.1.2).

Reconstruction and reduced phase space

Recall that the extrinsic curvature is defined starting from the Weingarten map $k : \mathrm{T}G \to \mathrm{T}G$ as the symmetric bilinear form K on T_xG such that $\mathrm{K}(v,w) = q(k(v),w)$ for all $v,w \in \mathrm{T}_xG$. We can obtain the Weingarten map from the Ashtekar connection A and the local Levi-Civita connection Γ associated with h by

$$k = \Phi([\bar{h}, A - \Gamma]), \tag{7.1.4}$$

where $\Phi : \mathrm{ad}P^{Spin}(G) \xrightarrow{\sim} TG$ is the natural vector bundle isomorphism.
We now verify that the extrinsic curvature is homogeneous. Let ω be a homogeneous connections on $P^{Spin}(G)$ and ω^{LC} be the "lifted" Levi-Civita connection, then $\Omega = \omega - \omega^{LC}$ is a G_L-invariant and horizontal $\mathfrak{su}(2)$-valued 1-form on $P^{Spin}(G)$. Let $\bar{h}$ be a G_L-equivariant section and fix $x \in G$ and $v,w \in T_xG$

$$\begin{aligned}
(\bar{h}^*\Omega)_{L_gx}(d_xL_g(v)) &= \Omega_{\bar{h}_{L_gx}}(d_{L_gx}\bar{h} \circ d_xL_g(v)) = \Omega_{\bar{h}_{L_g(x)}}(d_x(\bar{h} \circ L_g)(v)) \\
&= \Omega_{\bar{L}_g(\bar{h}_x)}(d_x(\bar{L}_g \circ \bar{h})(v)) = \Omega_{\bar{L}_g(\bar{h}_x)}(d_{\bar{h}_x}\bar{L}_g \circ d_x\bar{h}(v)) = \Omega_{\bar{h}_x}(d_x\bar{h}(v)) \\
&= (\bar{h}^*\Omega)_x(v).
\end{aligned}$$

Since Φ is an isomorphism between homogeneous fibre bundles, it preserves the homogeneous structure. Let V be a $\mathbb{R}^3$-valued 1-form on G such that $\Phi([\bar{h}, \bar{h}^*\Omega]) = [\bar{h}, V]$, hence $L_g^*V = V$ when $\bar{h}$ is G_L-equivariant. Thus, the Weingarten map satisfies

$$k_{L_gx}(d_xL_g(v)) = [\bar{h}_{L_gx}, V_{L_gx}(d_xL_g(v))] = d_xL_g \circ h_x(V_x(v)) = d_xL_g(k_x(v)). \tag{7.1.5}$$

From which descend that

$$\begin{aligned}
(L_g^*\mathrm{K})_x(v,w) &= q_{L_gx}(k_{L_gx}(d_xL_g(v)), d_xL_g(w)) = q_{L_gx}(d_xL_g(k_x(v)), d_xL_g(w)) \\
&= q_x(k_x(v), w) = \mathrm{K}_x(v,w).
\end{aligned} \tag{7.1.6}$$

Since the last expression is gauge independent, it holds for every choice of the dreibein. Hence, from generality of $x \in G$, $v,w \in \mathrm{T}_xG$ and $L_g \in G_L$, the extrinsic curvature is homogeneous.

The data of the classical cosmological sector of General Relativity in Ashtekar variables consists of an invariant connection ω together with a section e in the homogeneous orthonormal frame bundle $P^{SO}(G)$. In particular, the reduced phase space is composed of the local fields $A = h^*\omega$, referred to as the homogeneous Ashtekar connections, and the densitised dreibein E of e as defined in Sec.6.1.2. As in Chapter 6, the set of connections will depend on the metric, and so on the dreibein $\mathcal{A}_E^G$, and a fixed choice of a spin structure (in this case, we have a canonical choice: the homogeneous one). However, the reason to maintain the orthonormal frame bundle in the description is that the procedure can be generalized in the case in which there is no homogeneous spin structure, and no lift of the action on the principal $SU(2)$-bundle, that unable us to provide a definition in terms of homogeneous $SU(2)$ connections. For instance, this happens in the closed FLRW model, namely the discussed case of the 3-sphere with $SO(4)$-action. In those cases, we can still define $\mathcal{A}_E^G$ to be the image by a fixed spin structure (not homogeneous) of G-invariant metric-compatible connections on $P^{SO}(\Sigma)$ (now G acts transitively and effectively). All these sets are isomorphic to the set $\mathcal{A}^G$ of G-invariant $SO(3)$ connections on the trivial bundle $\Sigma \times SO(3)$ pullbacked by the map $\bar{\rho} = (\mathrm{id}_\Sigma, \rho) : \Sigma \times SU(2) \to \Sigma \times SO(3)$. When a homogeneous spin structure exists, the two approaches are the same. The collection of homogeneous Ashtekar connections $\mathcal{A}^G$ will play the role of the configuration space, providing the classical cosmological sector of the Ashtekar-Barbero-Immirzi-Set connections.

Isotropic case

The isotropic case deserves a peculiar treatment. The isotropic hypothesis prevents finding a preferred direction even during the dynamics evolution of the metric and the extrinsic curvature [79].
In this case, the usual mathematical definition of an isotropic manifold does not catch the whole feature we desire.

Definition 7.1.3. A Riemannian manifold (Σ, q) is isotropic if, for any $x \in \Sigma$ and any unit vectors $v, w \in \mathrm{T}_x\Sigma$, there exists an isometry $f : \Sigma \to \Sigma$ with $f(x) = x$ such that $f_*(v) = w$.

This definition means that Σ is a homogeneous space and the stabiliser group is isomorphic to the group of rotations. Herewith, G admits an action of $H \cong SO(3) \subset S$ with a fixed point (the fixed point will be the identity element $1 \in G$). However, we can see in Table 1 from [103] that the full isometry group of $\mathbb{R}^3$ equipped with any left-invariant metric has an $SO(3)$ subgroup. Moreover, for every two left-invariant metric q, q' on $\mathbb{R}^3$ always exists an automorphism $F \in \mathrm{Aut}(\mathbb{R}^3)$ such that $q = F^*q'$ and their stabilizers are conjugate under it, $H_q = \mathrm{Ad}_F\, H_{q'}$ [102]. Roughly speaking, let $Q = \{q_{IJ}\}$ and $Q' = \{q'_{IJ}\}$ be the matrices associated with the two metrics in a given basis of generators of the Lie algebra $\mathbb{R}^3 \cong \mathrm{T}_{\mathbb{1}}\mathbb{R}^3$, then, there exists a matrix M such that $Q = MQ'M^t$. Hence, given a transformation Λ that leaves the metric Q invariant $\Lambda Q \Lambda^t = Q$, we get an invariant transformation for Q', namely $M\Lambda M^{-1}$. Thus, the stabiliser group H_q is conjugate to the stabiliser of the canonical scalar product of $\mathbb{R}^3$, i.e., $SO(3)$.
Thinking about the two different metrics as a time evolution, the stabiliser group evolves in time but is still isomorphic to $SO(3)$. Nevertheless, this can not be a description of an isotropic universe because of the generality of q and q'. Then, we have to restrict our definition to be the stabiliser group not isomorphic to $SO(3)$ but exactly $H = SO(3)$, corresponding to fixing Q as a positive multiple of the identity matrix.

Definition 7.1.4. An isotropic model is a Riemannian homogeneous space (G, S) together with a metric q such that the stabiliser is $H = SO(3) \subset S$.

In our description of class A simply connected Bianchi models, there are only two Lie groups available for the isotropic model, $\mathbb{R}^3$ and $\mathbb{S}^3$ [103]. So they are homogeneous spaces together with the action of $E^3_0 = \mathbb{R}^3 \rtimes SO(3)$ and $SO(4)$, respectively. However, $\mathbb{S}^3$ does not admit an $SO(4)$-invariant spin structure. We can avoid this problem considering connection on $P^{SO}(\mathbb{S}^3)$ because in the usual

formulation they are in one-to-one correspondence with the connection in the $SU(2)$-principal bundle.
Hence, we require that the connection be invariant under the action of the chosen group.

For consistency, we will show that there exists a gauge in which we can write the Ashtekar connection in the canonical form [48].
Consider $\mathbb{R}^3$ equipped with a positive multiple of the canonical scalar product $\alpha^2\langle\cdot,\cdot\rangle$ and $P^{SO}(\mathbb{R}^3)$ together with the natural action of E_0^3. The fibre on a point $x \in \mathbb{R}^3$ is canonically $SO(3)$ times the positive factor α

$$P_x^{SO}(\mathbb{R}^3) = \{U : \mathbb{R}^3_{\langle\cdot,\cdot\rangle} \to \mathrm{T}_x\mathbb{R}^3 \cong \mathbb{R}^3_{\alpha^2\langle\cdot,\cdot\rangle} \mid \text{orientation-preserving linear isometry}\}$$

The action of $g = tR \in E_0^3$ on $P^{SO}(\mathbb{R}^3) = \mathbb{R}^3 \times SO(3)$ (forgetting the α factor), with t translation and R rotation, can be written explicitly

$$\begin{aligned} \tilde{L}_g : P_x^{SO}(\mathbb{R}^3) &\to P^{SO}(\mathbb{R}^3); \\ U &\mapsto d_x g \circ U, \qquad \Longrightarrow \; L_g(x,U) = (Rx+t, RU). \end{aligned}$$

In this case, a section equivariant under the action of E_0^3 does not exist. But, invariant connections exist due to Wang's theorem.
A connection ω over $P^{SO}(\mathbb{R}^3)$ can be written as $\omega = a^{-1}da + a^{-1}\hat{\omega}a$, where $\hat{\omega}$ is a $\mathfrak{so}(3)$-valued 1-form over $\mathbb{R}^3$ and $a \in SO(3)$. Given a section $h : \mathbb{R}^3 \to P^{SO}(\mathbb{R}^3)$, we obtain the local connection 1-form $h^*\omega = h^{-1}dh + h^{-1}\hat{\omega}h$. Fixing $h_x = (x, \mathbb{1})$, $h^*\omega = \hat{\omega}$. The action of L_g on the connection is

$$\tilde{L}_g^*\omega = (Ra)^{-1}d(Ra) + (Ra)^{-1}g^*\hat{\omega}(Ra) = a^{-1}da + a^{-1}R^{-1}g^*\hat{\omega}Ra.$$

From which the condition of invariance is

$$(g^*\hat{\omega})_x(v) = \hat{\omega}_{Rx+t}(Rv) = R\hat{\omega}_x(v)R^{-1}.$$

That is, in the gauge fixing, $\hat{\omega}$ invariant under the adjoint action of $SO(3)$, hence,the unique solution is $\hat{\omega} = c\sum_a T_a \otimes \mathfrak{e}^a$, where $\{\mathfrak{e}^a\}$ is the canonical dual basis of $\mathbb{R}^3$, T_a are the generators of $\mathfrak{so}(3)$, and c is a real number. It corresponds on each $\mathrm{T}_x\mathbb{R}^3 \cong \mathbb{R}^3$ to the equivariant isomorphism $\mathbb{R}^3 \xrightarrow{\sim} \mathfrak{so}(3)$. In a matrix form, it is

$$\hat{\omega}_i^{IJ} = c\,\delta_i^a (T_a)^{IJ}. \tag{7.1.7}$$

Dropping the Lie algebra generators T_a, we notice that it is exactly the expression of the Ashtekar connection in a spatially homogeneous and isotropic Universe: $A_i^a = c\,\delta_i^a$.

Such a connection reconstructs an isotropic extrinsic curvature. Let ω^A be a E_0^3-invariant connection and ω^{LC} be the Levi-Civita one. Their difference is

$$\Omega = \omega^a - \omega^{LC} = a^{-1}\hat{\omega}^A a - a^{-1}\hat{\omega}^{LC}a = a^{-1}\hat{\Omega}a,$$

and it satisfies $g^*h^*(a^{-1}\hat{\Omega}a) = \mathrm{Ad}_R\,\hat{\Omega}$. Let v be a $\mathbb{R}^3$-valued 1-form such that $\Phi([h, h^*\Omega]) = [h, \mathrm{v}]$, if $h = (x, \mathbb{1})$, then $(g^*\mathrm{v})_x(v) = \mathrm{v}_{Rx+t}(Rv) = R\mathrm{v}_x(v)$. Hence, the Weingarten map rotates under this transformation

$$k_{Rx+t}(Rv) = [h_{Rx+t}, \mathrm{v}_{Rx+t}(Rv)] = [h_{Rx+t}, R\mathrm{v}_x(v)] = Rk_x(v). \tag{7.1.8}$$

While the extrinsic curvature keeps the E_0^3-invariant (i.e., isotropic and homogeneous) property

$$\begin{aligned}(g^*\mathrm{K})_x(v,w) = \mathrm{K}_{Rx+t}(Rv,Rw) &= \alpha\langle k_{Rx+t}(Rv),Rw\rangle = \alpha\langle Rk_x(v),Rw\rangle \\ =&\alpha\langle k_x(v),w\rangle = \mathrm{K}_x(v,w).\end{aligned} \tag{7.1.9}$$

Thus, in our formulation, we are able to recover the canonical form of the Ashtekar connection in an isotropic case and be consistent with the request of an isotropic metric and extrinsic curvature.

7.2 The fate of the gauge symmetries

We now discuss the role of the gauge symmetries, namely the invariance of this formulation under the action of the spatial diffeomorphism group $\mathrm{Diff}(G)$ and the gauge group (vertical automorphisms) of the bundle $\mathcal{G} \doteq \mathrm{Gau}(P^{Spin}(G))$. We are going to verify that these two groups have a well-defined action on the set of homogeneous Ashtekar connections $\mathcal{A}^G$.
Notice that the homogeneous connections on $P^{Spin}(G)$ form an affine space associated to the vector space $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g},\mathfrak{su}(2))$ as a consequence of Wang's theorem. On such a space, not the whole $\mathcal{G}$ acts but only a subgroup given by the automorphisms that commute with the action $\bar{L}$

$$\mathcal{G}^G = \{f \in \mathcal{G} \mid f \circ \bar{L}_g = \bar{L}_g \circ f,\, \forall g \in G\} \cong SU(2).$$

However, on $\mathcal{A}^G$ we can recover the action of the whole $\mathcal{G}$. Let us consider $A \in \mathcal{A}^G$, then there exists a section $s : G \to P^{Spin}(G)$ and a homogeneous connection ω such that $A = s^*\omega$. The action of $\mathcal{G}$ is given by the transformation of ω

$$\begin{aligned}&\mathcal{G} \times \mathcal{A}^G \to \mathcal{A}^G; \\ &(f, A) \mapsto A' = s^* f^* \omega.\end{aligned} \tag{7.2.1}$$

Despite $f^*\omega$ being no more homogeneous, A' is it. Indeed, $A' = s^*f^*\omega = (f \circ s)^*\omega$, and $f \circ s$ is a section of the spin bundle, then A' is still the local field of a homogeneous connection.

The behaviour of the $\mathrm{Diff}(G)$-action requires more attention since a diffeomorphism changes the spin structure and the orthonormal frame bundle as well. Let us first consider the automorphism group $\mathrm{Aut}(G) < \mathrm{Diff}(G)$; this group preserves the homogeneity condition for the metric. Indeed, let $\varphi \in \mathrm{Aut}(G), g \in G, x \in G$, then $\varphi(L_g x) = L_{\varphi(g)}\varphi(x)$. Hence, considering a metric η such that $L_g^*\eta = \eta, \forall g \in G$, we get that $\varphi^*\eta$ is still homogeneous:

$$L_g^*\varphi^*\eta = \varphi^* L_{\varphi(g)}^*\eta = \varphi^*\eta.$$

Together with the automorphism group, the right multiplication preserves the homogeneous structure because it commutes with the left multiplication. In particular, if ξ is a left-invariant vector field, $(R_g)_*\xi$ is left invariant. Thus, we can notice that the request of homogeneity for the metric breaks the diffeomorphism-invariance, reducing

the group to $G \rtimes \mathrm{Aut}(G)$ (for a group-theoretic focus on this group, see the next section). Analytically, this reflects the expression of the Vector constraint, which in cosmology is dependent on the structure constants of $\mathfrak{g}$ [44]. As a consequence, $\mathbf{f}^*\eta$, with $\mathbf{f} = (g, \varphi) \in G \rtimes \mathrm{Aut}(G)$ acting on G as $\mathbf{f} = R_{g^{-1}} \circ \varphi$, still defines a orthonormal frame bundle that is homogeneous under the natural left action $\tilde{L}$ of G, and there exists the associated unique G_L-invariant spin structure $(P^{Spin}(G), \bar{\rho}', \bar{L}'_g)$.
The map $\mathbf{f}$ intertwines two equivalent actions $\mathbf{f} \circ L_g = L_{\varphi(g)} \circ \mathbf{f}$. Now, considering the two actions L and L_φ on the same frame bundle $P^{SO}(G)$, an invariant section e with respect to one action will be invariant with respect to both. Namely, the two actions on the same trivialized bundle reads as $\tilde{L}_g = (L_g, \mathrm{id}_{SO(3)})$ and $\tilde{L}_{\varphi(g)} = (L_{\varphi(g)}, \mathrm{id}_{\mathrm{SO}(3)})$. Moreover, there exists an automorphism $\tilde{\varphi} = (\varphi, \mathrm{id}_{SO(3)})$ such that $\tilde{L}_{\varphi(g)} = \tilde{\varphi} \circ \tilde{L}_g \circ \tilde{\varphi}^{-1}$. Hence, the spin structure $(P^{Spin}(G), \bar{\rho})$ is the same for both the actions $\bar{L}_g = (L_g, \mathrm{id}_{\mathrm{SU}(2)})$ and $\bar{L}_{\varphi(g)} = (L_{\varphi(g)}, \mathrm{id}_{SU(2)})$, and a lift of the automorphism exists $\bar{\varphi} = (\varphi, \mathrm{id}_{SU(2)})$. Thus, a local field $A = s^*\omega$ transform under the group $\mathrm{Aut}(G)$ as $\varphi^*A = (s \circ \varphi)^*\omega = (\bar{\varphi} \circ s')^*\omega = s'^*\bar{\varphi}^*\omega$, where $s' = \bar{\varphi} \circ s \circ \varphi^{-1}$, and $\bar{\varphi}^*\omega$ is invariant under action of $\bar{L}^*_{\varphi(g)}$ for all $g \in G$, and so under $\bar{L}^*_g$. Analogous computations are performed when we include the term $R_{g^{-1}}$ because it does not modify the left action. Hence, $\mathcal{A}^G$ is invariant under the action of $G \rtimes \mathrm{Aut}(G)$.
It also has an interpretation in terms of spin structure on different orthonormal frame bundles. Let $\mathbf{f} \in G \rtimes \mathrm{Aut}_0(G)$, there exists a isomorphism of principal bundles between $P^{SO}(G, \eta)$ and $P^{SO}(G, \mathbf{f}^*\eta)$ given by

$$
\begin{aligned}
\tilde{\mathbf{f}} : P^{SO}_x(G, \varphi^*\eta) &\to P^{SO}_{\varphi(x)}(G, \eta);\\
h &\mapsto d_x\mathbf{f} \circ h,
\end{aligned}
$$

such that it is a lift of $\mathbf{f}$. Moreover $\tilde{\mathbf{f}} \circ \tilde{L}_g = \tilde{L}_{\varphi(g)} \circ \tilde{\mathbf{f}}$. Considering two G-equivariant sections: h for $P^{SO}(G, \eta)$, and h' for $P^{SO}(G, \mathbf{f}^*\eta)$. In a fixed point $x \in G$, $\tilde{\mathbf{f}}(h'_x) = R_u h_{\mathbf{f}(x)}$, for some $u \in SO(3)$. Here, R_u is the right action of a element in $SO(3)$ on $P^{SO}(G, \eta)$. Moving on another point via left multiplication, we obtain

$$
\tilde{\mathbf{f}}(h'_{gx}) = \tilde{\mathbf{f}}\tilde{L}_g(h'_x) = \tilde{L}_{\varphi(g)}\tilde{\mathbf{f}}(h'_x) = \tilde{L}_{\varphi(g)} R_u(h_{\tilde{\mathbf{f}}(x)}) = R_u \tilde{L}_{\varphi(g)}(h_{\tilde{\mathbf{f}}(x)}) = R_u(h_{\tilde{\mathbf{f}}(gx)}).
$$

This means that, with respect to the trivialization induced by the two sections, $\tilde{\mathbf{f}}$ acts as $(x, \rho(a)) \mapsto (\mathbf{f}(x), \rho(a)u)$. Hence, we can lift the action on the trivialised $P^{Spin}(G)$ by choosing a preimage by ρ of u in $SU(2)$. We can fix it by continuity, asking that $u = \mathbb{1}$ when $\mathbf{f} = \mathrm{id}_G$. To do that, $\mathbf{f}$ must lie in the identity component of $G \rtimes \mathrm{Aut}(G)$. Since G is connected, then the identity component is $G \rtimes \mathrm{Aut}_0(G)$.

Thus, we have found an automorphism $\bar{\mathbf{f}}$ such that the following diagram commutes:

$$\begin{tikzcd}
& P^{Spin}(G) \arrow[rr, "\bar{\mathbf{f}}"] \arrow[dd, "\bar{\rho}'"] & & P^{Spin}(G) \arrow[dd, "\bar{\rho}"] \\
P^{Spin}(G) \arrow[ur, "\bar{L}'_g"] \arrow[rr, "\bar{\mathbf{f}}"] \arrow[dd, "\bar{\rho}'"] & & P^{Spin}(G) \arrow[ur, "\bar{L}_{\varphi(g)}"] \arrow[dd, "\bar{\rho}"] & \\
& P^{SO}(G,\mathbf{f}^*\eta) \arrow[rr, "\tilde{\mathbf{f}}"] & & P^{SO}(G,\eta) \\
P^{SO}(G,\mathbf{f}^*\eta) \arrow[ur, "\tilde{L}_g"] \arrow[rr, "\tilde{\mathbf{f}}"] & & P^{SO}(G,\eta) \arrow[ur, "\tilde{L}_{\mathbf{f}(g)}"] &
\end{tikzcd}$$

Furthermore, $\tilde{\mathbf{f}}^*$ maps homogeneous connections on $P^{SO}(G,\eta)$ to homogeneous connections on $P^{SO}(G,\varphi^*\eta)$, and $\bar{\mathbf{f}}^*$ maps homogeneous connections in homogeneous connections on $P^{Spin}(G)$ with respect to the proper unique invariant spin structure (the spin structures are the same once the bundles are trivialized)

$$\bar{L}_g^*\omega = \omega \iff \bar{L}'_g\bar{\mathbf{f}}^*\omega = \bar{\mathbf{f}}^*\omega.$$

Hence, $A = s^*\bar{\mathbf{f}}^*\omega = (\bar{\mathbf{f}} \circ s)^*\omega = (s' \circ \mathbf{f})^*\omega = \mathbf{f}^*A'$, where $s' = \bar{\mathbf{f}} \circ s \circ \mathbf{f}^{-1}$ is a new section.

7.2.1 The broken diffeomorphism group $G \rtimes \mathrm{Aut}(G)$

We are going to discuss the group-theoretic and spectral properties of $G \rtimes \mathrm{Aut}(G)$, for a connected Lie group G. The semidirect Lie group $G \rtimes \mathrm{Aut}(G)$, which is topologically $G \times \mathrm{Aut}(G)$, is characterized by the following product

$$(g_1,\varphi_1).(g_2,\varphi_2) = (g_1\varphi_1(g_2), \varphi_1 \circ \varphi_2).$$

The inverse is $(g,\varphi)^{-1} = (\varphi^{-1}(g^{-1}), \varphi^{-1})$, and identity $(1, \mathrm{id}_G)$. This group has an effective left action on G, and so a smooth injective homomorphism

$$\begin{array}{rccc} \alpha: & G \rtimes \mathrm{Aut}(g) & \to & \mathrm{Diff}(G); \\ & (g,\varphi) & \mapsto & R_{g^{-1}} \circ \varphi. \end{array} \tag{7.2.2}$$

Notice that, under the usual choice of left action $(g,\varphi) \mapsto L_g \circ \varphi$, the image in $\mathrm{Diff}(G)$ is the same since $L_g \circ \varphi = R_{g^{-1}} \circ (\mathrm{Ad}_{g^{-1}} \circ \varphi)$. We want to describe the Lie algebra of $G \rtimes \mathrm{Aut}(G)$ in terms of vector fields on G. Namely, find the Lie subalgebra in $\mathfrak{X}(G)$ of the image of $G \rtimes \mathrm{Aut}(g)$ in $\mathrm{Diff}(G)$ via α.
Let us start from the normal subgroup G. Let $\xi \in \mathfrak{g}$, and $v = \xi_{\mathbb{1}} \in \mathrm{T}_{\mathbb{1}}G$ its value on the identity. Then, $\alpha(\exp(tv)) = R_{\exp(-tv)}$ is the flow of the left-invariant vector field $-\xi$ on G. Hence, $\alpha_*(\mathfrak{g}) = -\mathfrak{g}$. So, on the image $[\xi_1,\xi_2]_{\alpha_*\mathfrak{g}} := \mathrm{ad}_{\xi_1}\xi_2 = -[\xi_1,\xi_2]$, where $[\cdot,\cdot]$ are the usual Lie brackets between vector fields, i.e. $[X,Y] = \mathcal{L}_X Y$. We recall that the adjoint action of G on $\mathfrak{g}$ in terms of left-invariant vector fields is $\mathrm{Ad}_g \xi = (R_{g^{-1}})_*\xi$. Since $\mathrm{Aut}(G)$ is canonically a subgroup of $\mathrm{Diff}(G)$, it inherits the same adjoint representations. Let $X \in Lie(\mathrm{Aut}(G)) \subset \mathfrak{X}(G)$, then, for all $\varphi \in \mathrm{Aut}(G)$, $\mathrm{Ad}_\varphi X = \varphi^* X$, and, for all $Y \in \mathfrak{X}(G)$, $\mathrm{ad}_Y X = [X,Y]$. Moreover,

the action of $\mathrm{Aut}(G)$ on G lift to $\mathfrak{g}$ simply by $(\varphi, \xi) \mapsto \varphi_* \xi$. Now, we have all the ingredient to describe $\alpha_*(Lie(G \rtimes \mathrm{Aut}(G))) \subset \mathfrak{X}(G)$. As vector space, it is $\mathfrak{g} \oplus Lie(\mathrm{Aut}(G))$. The adjoint representation of the group is given by

$$\mathrm{Ad}_{(g,\varphi)}(\xi, X) = (\mathrm{Ad}_g(\varphi_*\xi) + \sigma_g(\varphi_* X), \varphi_* X),$$

where, $\sigma_g : Lie(\mathrm{Aut}(G)) \to \mathfrak{g}$ is given by identify $\mathfrak{g}$ with $\mathrm{T}_1 G$ and map X to $d_g L_{g^{-1}}(X_g)$. From which, we can derive the inner automorphism of the Lie algebra:

$$\mathrm{ad}_{(v,Y)}(\xi, X) = ([\xi, v] + [X, v] - [Y, \xi], [X, Y]).$$

Notice that, if X is the vector field associated to an automorphism φ, then $[X, \xi] \in \mathfrak{g}$ for all $\xi \in \mathfrak{g}$. So, this expression is well defined.

Since $G \rtimes \mathrm{Aut}(G)$ is a Lie group, it admits a Haar measure. We can construct it explicitly starting from left-invariant measures μ_G on G and μ_A on $\mathrm{Aut}(G)$. It cannot be just the product of the two due to the bad behaviour of μ_G under the action of $\mathrm{Aut}(G)$ on G. To deal with that, notice that $\mu_G \circ \varphi$ is still a left-invariant measure on G, for each $\varphi \in \mathrm{Aut}(G)$, then $\mu_G \circ \varphi = \lambda(\varphi)\mu_G$, where $\lambda(\varphi)$ is a real positive number. More properly, it is easy to show that $\lambda : \mathrm{Aut}(G) \to \mathbb{R}^\times$ is a continuous homomorphism. Thus, considering the functional

$$\begin{aligned} I : C^0_c(G \rtimes \mathrm{Aut}(G)) \qquad &\to \mathbb{C}; \\ f \qquad &\mapsto \int_G \int_{\mathrm{Aut}(G)} f(g, \varphi)\lambda(\varphi^{-1}) d\mu_A(\varphi) d\mu_G(g), \end{aligned}$$

thanks to the Riesz-Markov-Kakutani representation theorem, it defines a left-invariant measure on $G \rtimes \mathrm{Aut}(G)$.

7.3 Abelian artefact, gauge fixing, and point holonomies

In this framework, generically, the holonomy maintains its value in $SU(2)$ (cf. Theorem 4.1 in [119]). Therefore, it is capable of reintroducing the $SU(2)$ internal degree of freedom lost in the canonical approach in LQC [66, 70, 69, 54]. Moreover, this formulation does not require a minisuperspace, allowing gauge transformations to possess local degrees of freedom that do not adhere to the homogeneous property, as necessary to have a non-Abelian nor identically vanishing Gauß constraint. Furthermore, we can recover the classical formulation in minisuperspace. The description of the classical Ashtekar variables in cosmology by M. Bojowald [44, 51] coincides with the restriction of our framework to G-equivariant dreibeins, as can be seen from Eq.(7.1.2). Indeed, our pair of variables (A, E) can be described by the usual collection of functions $(A^i_a(x), E^a_i(x))$ such that, in a specific gauge, they can be written as $A^i_a(x) = \phi^i_I \theta^I_a(x)$ and $E^a_i(x) = |\det(\theta)| p^I_i \xi^a_I(x)$. Here, ξ_I is a set of generators for the Lie algebra $\mathfrak{g}$, and θ^I its dual basis. That specific gauge h is a G-equivariant one. As discussed in Sec. 7.1.3, A is left-invariant in such a gauge and, due to Eq.(7.1.2), $A^i_a(x) = \phi^i_I \theta^I_a(x)$, where ϕ^i_I is the matrix associated with the map $\phi : \mathfrak{g} \to \mathfrak{su}(2)$ with respect to the basis of generators ξ_I and τ_i (we recall that $\theta_{MC} = \xi_I \otimes \theta^I$). The frame of vector fields $h(\mathfrak{e}_i)$ is left-invariant, here $\mathfrak{e}_i$ is the canonical basis of $\mathbb{R}^3$, and so $e_i^a(x) = c_i^I \xi_I(x)$, where c_i^I is a change-of-basis matrix. The usual description of LQC [15, 48, 49] can be recovered by a complete gauge fixing, namely choosing a specific G-equivariant dreibein, as shown explicitly in the isotropic case. Furthermore, as a special case, we can rescue the point holonomies, which characterise LQC. Let us consider the parallel transport along the integral curve of a left-invariant vector field ξ. Considering $c : [0,1] \to G;\, t \mapsto c(t)$ an integral curve of a left-invariant vector field ξ associated with a vector $v = \xi_{\mathbb{1}} \in \mathrm{T}_{\mathbb{1}}G$. Hence,

$$\dot{c}(t) := d_t c(\partial_t) = \xi(c(t)).$$

Let $u : [0,1] \to SU(2)$ be the parallel transport along the curve c, fixing e be a G-equivariant dreibein, the parallel transport equation reads

$$\dot{u}(t) = -A(\dot{c}(t))u(t) = -\phi \circ \theta_{MC}(\xi(c(t))u(t) = -\phi(v)u(t). \tag{7.3.1}$$

The solution of this equation is $u(t) = \exp(-\phi(v)t)$. The holonomy is defined as the inverse of the parallel transport operator evaluated in $t = 1$

$$h_c(A) = u(t=1)^{-1} = \exp(\phi(v)).$$

However, a remembrance of the path-ordering product appears. If we consider a composition of that kind of integral curves associated with vectors $v_1, \ldots, v_n$, we can easily convince ourselves that the holonomy along this curve is

$$h_c(A) = \exp(\phi(v_1)) \ldots \exp(\phi(v_n)).$$

To connect the geometrical point of view on gauge fixing with an analytical one, we recall that, in Chapter 5, we found that, in the minisuperspace, the Gauß constraint loses the divergence term $\partial_a E_i^a$, and we showed how to recast it into three Abelian constraints. The equation $\partial_a E_i^a = 0$ is proper for the minisuperspace model, and it is analogous to some previous results [133, 134]. Actually, a G-equivariant section of $P^{Spin}(G)$ yields a dreibein e_i composed of left-invariant vector fields, which have constant covariant divergence $\mathrm{div}(e_i) = const$. For class A Bianchi models, the constant is zero. Thus, we can interpret $\partial_a E_i^a = 0$ as the minisuperspace constraint, which should enable us to recover Bojowald's description of reduced phase space.

7.4 Cylindrical functions and constraints algebra

In the phase space presented in Sec.7.1.3, the functional dependence of the constraints on the Ashtekar variables remains the same as that of Loop Quantum Gravity. Since no minisuperspace is required, not only does the Gauß constraint persist, but even the Diffeomorphism constraint does not vanish, thereby recovering the full set of constraints of LQG. Beginning with this observation, in this Section, we are going to analyse the classical constraints algebra and the cylindrical functions restricted to the classical cosmological sector $\mathcal{A}^G$.

7.4.1 Homogeneous cylindrical function

The holonomy-flux algebra can be properly defined on the reduced phase space, integrating the connections and the electric fields over all the possible suitable curves and surfaces. Hence, the space of cylindrical functions comes out naturally as functions on $\mathcal{A}^G$ supported on a graph. We recall the definition of graph:

Definition 7.4.1. A graph γ is a collection of piecewise analytic, continuous, oriented curves $c : [0,1] \to G$ (c is an embedding in G of an open subset of $\mathbb{R}$ that contains $[0,1]$) which intersect each other at their endpoints.

However, there is a distinguished collection of graphs with useful properties, which we will call homogeneous graphs.
We recall that the classical notion of an invariant subset cannot be properly implemented. Let $S \subset G$ be a subset, it is invariant if $L_g(S) \subset S$, $\forall g \in G$. Since G acts transitively, the orbit of a point is the whole space G, then no proper invariant subsets exist. However, a finer property can be found for curves on G. On a Lie group, the exponential map $\exp : \mathfrak{g} \to G$ is well-defined. Locally, this map is a diffeomorphism, specifically, there exists an open neighbourhood $U \subset G$ of the identity element and a ball of radius δ, $\mathbb{B}_\delta(0) \subset \mathfrak{g} \cong \mathrm{T}_{\mathbb{1}}G$ such that for all $g \in U$ there exists a unique $v \in \mathbb{B}_\delta(0)$ such that $g = \exp(v)$.
This property can be extended to all pairs of close enough points. Let g_1 and g_2 be two elements of G such that $g_1^{-1}g_2 \in U$, then $\exists! v \in \mathbb{B}_\delta(0)$ such that $g_1^{-1}g_2 = \exp(v)$, hence $g_2 = g_1 \exp(v)$.
The object $\exp(v)$ can be interpreted as the endpoint of the integral curve of the left-invariant vector field associated with v. Thus, between two close enough points g_1 and g_2, there always exists an analytic curve $\zeta : [0,1] \to G$ such that $\zeta(0) = g_1$ and $\zeta(1) = g_2$, this curve is $\zeta(t) = g_1 \exp(tv)$. Using such curves, we can approximate every continuous curve $c : [0,1] \to G$. Consider a partition of the $[0,1]$ interval $\mathscr{P}_n = \{0, t_1, \dots, t_n, 1\}$, if the partition is fine enough, the points $g_i = c(t_i)$ and $g_{i-1} = c(t_{i-1})$ will be close enough and, so, there exists an integral curve that connects them $\zeta_i(t) = g_{i-1}\exp(tv_i)$. Thus every curve $c : [0,1] \to G$ is approximated by a continuous, piecewise analytic curve $\zeta_{\mathscr{P}_n} : [0,1] \to G$ composed by integral curves:

$$\zeta_{\mathscr{P}_n}(t) = \begin{cases} \zeta_1(\frac{t}{t_1}) & \textit{if } 0 \le t < t_1, \\ \zeta_2(\frac{t-t_1}{t_2-t_1}) & \textit{if } t_1 < t < t_2, \\ \vdots & \\ \zeta_{n+1}(\frac{t-t_n}{1-t_n}) & \textit{if } t_n < t \le 1. \end{cases} \tag{7.4.1}$$

The finer the partition, the better the approximation. Thus, every curve c can be realised as the limit of some sequence composed of integral curves. In this sense, the set of curves composed of integral curves is dense in the set of piecewise analytic, continuous curves with the topology of pointwise convergence.

Definition 7.4.2. A homogeneous graph γ^G is a collection of integral curves of left-invariant vector fields and their compositions.

Such as the curves, the set of homogeneous graphs is dense in the set of graphs. Poorly speaking, a homogeneous graph is made starting from a graph and substituting at each curve its approximation in integral curves. (Notice that this definition differs from the usual mathematical notion of a homogeneous graph as used in [36].) We will refer to cylindrical functions supported on homogeneous graphs as *homogeneous cylindrical functions*, and denote their space by $Cyl_G^\infty \subset Cyl^\infty(\mathcal{A}^G)$. The computation in Sec. 7.3 concerning point holonomies suggests that, up to gauge transformations, homogeneous cylindrical functions effectively localise point holonomies along the edges.

7.4.2 Classical constraints algebra

As we stated before, since our pair of variables for the phase space are still $(A_a^i(x), E_i^a(x))$, the constraints have the same form as the usual LQG ones. They can also be derived geometrically from the description proposed in Chapter 6, considering A to be homogeneous in the sense given before. Thus,

$$\begin{aligned} G_i &= \partial_a E_i^a + \epsilon_{ijk} A_a^j E_k^a, \\ V_a &= F_{ab}^i E_i^b - A_a^i G_i, \\ H &= \left(F_{ab}^i - \tfrac{\beta^2+1}{\beta^2} \epsilon^i{}_{jk} (A_a^j - \Gamma_a^j)(A_b^k - \Gamma_b^k)\right) \frac{\epsilon_{imn} E_m^a E_n^b}{\sqrt{|\det(E_i^a)|}}. \end{aligned} \tag{7.4.2}$$

We observe that these are precisely the constraints of standard Loop Quantum Gravity. The imposition of homogeneity, as implemented in this work, does not yield any significant simplification in the form of the constraints. They retain the same functional dependence on the phase space variables $E_j^a(x), A_b^k(y)$ even though these variables now belong to the cosmological sector of the theory. Furthermore, since we adopt the standard symplectic structure $\{E_j^a(x), A_b^k(y)\} = \frac{\beta\kappa}{2}\delta_b^a\delta_j^k\delta(x-y)$, the resulting smeared constraint algebra coincides with that of LQG:

$$\begin{aligned} \{G(\Lambda), G(\Lambda')\} &= \tfrac{\beta\kappa}{2} G([\Lambda, \Lambda']), \\ \{\mathrm{D}(X), \mathrm{D}(X')\} &= \tfrac{\beta\kappa}{2} \mathrm{D}(\mathcal{L}_X X'), \\ \{\mathrm{D}(X), H(f)\} &= \tfrac{\beta\kappa}{2} H(\mathcal{L}_X f), \\ \{H(f), H(f')\} &= \tfrac{\beta\kappa}{2} \mathrm{D}(q(f df' - f' df)). \end{aligned}$$

Nevertheless, we must carefully consider the choice of smearing functions, as we are ultimately interested in their action on cylindrical functions and must ensure that they generate the correct gauge transformations. For the Gauß constraint, there are no such restrictions: since we have recovered the full gauge action on $\mathcal{A}^G$, we can take $\Lambda \in C_c^\infty(\Sigma, \mathfrak{su}(2))$. A more subtle analysis is required for the smearing functions X and f. In cosmological settings, these are typically associated with the shift vector and the lapse function, respectively, both of which are required to be left-invariant. From the action of the Hamiltonian constraint on the ADM metric q, it is evident that, to preserve homogeneity, f must be constant in our case as well. Regarding the vector field X, which is the generator of diffeomorphisms on the space, we must note that diffeomorphism symmetry is broken in our setup. Therefore, X must preserve the structure of left-invariant tensors. In other words, X must lie in the Lie algebra $Lie(G \rtimes \mathrm{Aut}(G))$, as constructed in Section 7.2.1. With this choice of smearing functions, the classical constraint algebra is significantly simplified:

$$\begin{aligned}
&\{G(\Lambda), G(\Lambda')\} = \tfrac{\beta\kappa}{2} G([\Lambda, \Lambda']),\\
&\{G(\Lambda), \mathrm{D}(\mathrm{X})\} = \tfrac{\beta\kappa}{2} G(\mathcal{L}_X \Lambda),\\
&\{\mathrm{D}(X), \mathrm{D}(X')\} = -\tfrac{\beta\kappa}{2} \mathrm{D}(\mathcal{L}_X X'),\\
&\{G(\Lambda), H(f)\} = 0,\\
&\{\mathrm{D}(X), H(f)\} = 0,\\
&\{H(f), H(f')\} = 0.
\end{aligned} \tag{7.4.3}$$

In this form, the constraint algebra is isomorphic to the Lie algebra

$$(C_c^\infty(\Sigma, \mathfrak{su}(2)) \hat{\oplus} Lie(G \rtimes \mathrm{Aut}(G))) \oplus \mathbb{R}.$$

Where the first sum is a semidirect sum, and the third term is the direct sum with the real 1-dimensional Abelian algebra $\mathbb{R}$. Thus, $C_c^\infty(\Sigma, \mathfrak{su}(2))$ is an ideal as in the full theory, while $H(f)$ generate the centre.
If the Lie group on which the theory is defined is not compact, we may encounter issues with the convergence of $\mathrm{D}(X)$ and $H(f)$. However, since we are ultimately interested in their action on cylindrical functionals, the support of the smeared constraint is irrelevant as long as it is large enough to contain the entire graph. Therefore, we can define the integral via a limit over an exhaustion by compact sets $\Sigma_n \subset G$, ensuring that it is well-defined, similarly to the procedure used in the definition of the volume in [163].

The action of the infinitesimal transformation of the cylindrical functions is the same as the full LQG. The Gauß constraint can be implemented in the same way; indeed, its action on the holonomies reads as

$$\{G(\Lambda), h_c(A)\} = \frac{\beta\kappa}{2} \left(\Lambda(c(0)) h_c(A) - h_c(A) \Lambda(c(1))\right),$$

providing the infinitesimal version of the gauge transformations implemented by the group $\mathcal{G} = C_c^\infty(\Sigma, SU(2))$

$$U(a) h_c(A) = h_c(\mathrm{Ad}_{a^{-1}} A + a^{-1} da) = a(c(0)) h_{\varphi(c)}(A) a^{-1}(c(1)). \tag{7.4.4}$$

We need to pay attention to the Diffeomorphism constraint. Indeed, despite $\mathcal{L}_X A$ being well-defined on our configuration space, the action on smooth holonomies

$$\{V(X), h_c(A)\} = \int_0^1 dt\, h_{c([o,t])}(A) (\mathcal{L}_X A)_{c(t)} h_{c([t,1])}(A)$$

does not produce a cylindrical function. Hence, we are forced to consider the finite action of the group $G \rtimes \mathrm{Aut}(G)$ as

$$U(\varphi) h_c(A) = h_c(\varphi^* A) = h_{\varphi(c)}(A). \tag{7.4.5}$$

Notice that the subspace Cyl_G^∞ is invariant under these actions. This lays the foundation for the quantisation of the theory and the implementation of the quantum constraint algebra. Thanks to the simplified structure of the algebra and the symmetric properties of the cylindrical functions, this framework appears more tractable using the established techniques of canonical LQG. It may also offer a concrete link with the structure of LQC, while potentially avoiding some of the ambiguities present in the full theory.

7.4.3 Quantum constraint operators

The natural candidate for the auxiliary Hilbert space is the completion of Cyl_G^∞ with respect to the scalar product

$$\Lambda_0(f) = \int_{SU(2)^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(h_e)\, f_\gamma(\{h_e\}_{e\in E(\gamma)}).$$

As discussed previously, this space is invariant under the action of gauge transformations and diffeomorphisms. Moreover, the Gauß and Diffeomorphism constraints can be implemented in the same way as in canonical LQG, as shown in Eqs. (7.4.4) and (7.4.5).

The auxiliary Hilbert space is not of the form $L^2(\mathcal{A}^G)$ because $\mathcal{A}^G$ is not locally compact, and therefore the Riesz-Markov theorem does not apply. However, this issue can be circumvented after imposing the Gauß constraint. Indeed, due to homogeneity, the moduli space $\mathcal{A}^G/\mathcal{G}$ is particularly simple: in fact, it is always a topological manifold (possibly with boundary), which allows one to define an induced measure via the linear functional Λ_0. The topological properties of the moduli space $\mathcal{A}^G/\mathcal{G}$ are studied in detail in the next section.

The operators defined in Eqs. (7.4.4) and (7.4.5) provide a representation of the group $\mathcal{G} \rtimes (G \rtimes \mathrm{Aut}(G))$. To obtain a quantum constraint algebra free of anomalies, it is necessary to define the Hamiltonian operator appropriately. We adopt a graph-preserving version of Thiemann's Hamiltonian operator, first introduced in [154] and later extensively studied in [94, 95, 96]. The form of the operator at a vertex, acting on a spin-network state, remains the same; however, the loops are not constructed by adding new edges, but instead by using edges already present in the graph. In particular, the construction involves a normalised sum over the so-called *minimal loops*: given a graph γ, a vertex $v \in V(\gamma)$, and two distinct edges $e, e' \in E(\gamma)$ starting at v, a loop within γ starting along e and ending along e' is said to be minimal if there is no other loop with the same properties that traverses fewer edges of γ. The Euclidean Hamiltonian constraint then reads

$$\hat{H}_v^E = \frac{1}{E(v)} \sum_{e_1\cap e_2\cap e_3=v} \frac{\epsilon(e_1,e_2,e_3)}{|L(v,e_1,e_2)|} \sum_{\alpha\in L(v,e_1,e_2)} \mathrm{Tr}\left((h_\alpha - h_\alpha^{-1})\, h_{e_3} [h_{e_3}^{-1}, \hat{V}] \right),$$

where $L(v, e_1, e_2)$ denotes the set of minimal loops associated with the pair of distinct edges e_1, e_2 starting at v.

While the Hamiltonian is manifestly gauge invariant, and the commutation relations between the Gauß and Diffeomorphism constraints are correctly reproduced at the quantum level, owing to our representation of the group $\mathcal{G} \rtimes (G \rtimes \mathrm{Aut}(G))$

via its finite action on holonomies, we must still verify the commutation between the Diffeomorphism transformations and the Hamiltonian constraint.

The Hamiltonian constraint is, in general, diffeomorphism invariant, since the notion of a minimal loop is purely topological. Considering the smeared constraint $\hat{H}(N) = \sum_v N\, \hat{H}_v^E$, where N is a constant lapse function (and thus acts as an overall multiplicative factor), it follows immediately that

$$U(\varphi)\, \hat{H}(N)\, U(\varphi^{-1}) = \hat{H}(N),$$

which is the finite version of the commutation relation between the Diffeomorphism and Hamiltonian constraints in the algebra (7.4.3). Moreover, the commutator of two Hamiltonian constraints vanishes:

$$[\hat{H}(N), \hat{H}(M)] = 0.$$

This follows from the fact that the commutator contains a factor $(M_v N_{v'} - N_v M_{v'})$, which vanishes identically when both N and M are constant. Indeed, $\hat{H}(N)$ and $\hat{H}(M)$ are multiples of a common Hamiltonian operator $\hat{H}(1)$; hence, there is a single generator of the dynamical evolution, rather than one for each point, in accordance with the homogeneity assumption. The Lorentzian Hamiltonian constraint exhibits the same behaviour. Thus, the quantum constraint algebra is anomaly-free and reproduces the classical algebra (7.4.3).

We now briefly discuss the solution of the quantum constraints. The implementation of the Gauß constraint straightforwardly mirrors that in the full theory, yielding as the space of solutions the invariant spin-network states, labelled by intertwiners at the vertices. This construction may be interpreted (though it remains to be rigorously demonstrated) as the quantisation of the moduli space of connections, a procedure that is typically both mathematically well-motivated and physically consistent [139]. A detailed analysis of the moduli space of homogeneous Ashtekar connections is provided in the following chapter

Since the Diffeomorphism and Hamiltonian constraints operators commute on the auxiliary Hilbert space, most of the issues typically related to the construction of physical states appear to be resolved. However, constructing a generic solution of the quantum Einstein equations remains non-trivial. One might expect that, since $G \rtimes \mathrm{Aut}(G)$ is a Lie group, it would be possible to construct a proper rigging map to solve the Diffeomorphism constraint. In practice, this is not the case. Owing to the weak discontinuity of the representation, the rigging map is trivial. Indeed, the only non-vanishing contribution to

$$\eta(\psi_s) = \int_{G \rtimes \mathrm{Aut}(G)} d\mu_H(\varphi)\, \langle \psi_s, \hat{U}(\varphi) \cdot \rangle$$

comes from those φ such that $U(\varphi)\psi_s \in \mathrm{Span}(\psi_s)$, namely when $\varphi(s) = s$. These φ form a subgroup of $G \rtimes \mathrm{Aut}(G)$, which cannot be an open subset, otherwise it would coincide with the whole group (which it cannot, as the group acts transitively on Σ). Any closed subgroup, however, has measure zero, implying $\eta(\psi_s) = 0$.

Since $\mathrm{Aut}(G)$ depends strongly on the chosen cosmological model, a detailed analysis of the solutions to the Diffeomorphism and Hamiltonian quantum constraints, as well as the characterisation of the physical Hilbert space, will require an investigation on a model-by-model basis.

Chapter 8

Moduli space of homogeneous Ashtekar connections

In this chapter, we analyse the homogeneous Ashtekar connections introduced in the previous chapter within a rigorous mathematical framework. This in-depth treatment of the previously outlined approach enables both a generalisation and a classification of homogeneous Ashtekar connections across all cosmological models. We demonstrate that, in this setting, a clear distinction emerges between homogeneous spin connections and homogeneous $SU(2)$ connections. Only the former exhibits the correct behaviour required by the physical theory, as anticipated from the geometric considerations discussed in earlier chapters. Furthermore, we explicitly compute the moduli spaces of the two types of connections associated with each three-dimensional homogeneous space, corresponding to the various cosmological models.

The moduli space plays a central role in the quantisation of gauge field theories. While such spaces typically exhibit singularities, we find that, in our case, the moduli spaces of both types of connections possess exceptionally favourable properties:

- The moduli space depends only on the isotropy group of the homogeneous space, which encodes the symmetry class of the corresponding cosmological model.
- The moduli space admits the structure of a finite-dimensional topological manifold (possibly with boundary).

While the first property aligns with expectations based on physical considerations, the second introduces a significant regularisation within the quantum theory of the homogeneous Ashtekar-Barbero-Immirzi formulation. This regularisation arises from the ability to employ standard, well-defined tools typically used in the quantisation of systems with finitely many degrees of freedom. As a result, this structure paves the way for a mathematically rigorous analysis of the associated quantum theory.

The explicit computation of the moduli spaces reveals a distinction between the two types of connections. This distinction allows us to conclude that the Ashtekar connection, when interpreted in the cosmological setting, should be understood as a homogeneous spin connection (up to a slight generalisation, as discussed below).

This interpretation not only aligns with the mathematical structure of the moduli space but also agrees with the physical requirements of the theory, reinforcing the necessity of incorporating spin structures in any rigorous approach to the Ashtekar-Barbero-Immirzi formulation.

8.1 Homogeneous spaces, spin structures and connections

We start from a review of the structure and objects that will be the main characters of our study, and some well-known facts about them. A *Riemannian homogeneous space* is a pair (M, G) where M is an n-dimensional orientable Riemannian manifold and G a connected Lie group which acts transitively from the left via orientation-preserving isometries. Fixing a point $o \in M$, the stabiliser of that point is the so-called *isotropy subgroup* $H \subset G$. Whenever H is connected and compact, $M \simeq G/H$ is reductive, namely $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie subalgebra of H and $\mathfrak{m}$ is a vector subspace invariant under the action of H via adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$.
We denote with $L_g : M \to M$ the group action of G on M. This action defines a representation called *linear isotropy representation* of H in the linear group of T_oM,

$$\lambda : H \to GL(\mathrm{T}_oM); \quad g \mapsto \lambda(g) = d_oL_g. \tag{8.1.1}$$

In general, the action L can be lifted to the orthonormal frame bundle $P^{SO}(M)$ of M

$$\begin{aligned} \tilde{L} : G \times P^{SO}_x(M) &\to P^{SO}_{L_gx}(M); \\ (g, u) &\mapsto d_xL_g \circ u, \end{aligned}$$

where an element u in the fibre over a point x is a orientation-preserving linear isometry $u : \mathbb{R}^n \to \mathrm{T}_xM$. With this action, G acts on $P^{SO}(M)$ via automorphisms, hence $P^{SO}(M)$ is a G-invariant principal $SO(n)$-bundle. Fixing a frame u_o in the fibre over $o \in M$, the linear isotropy representation can be identified with the homomorphism $\lambda : H \to SO(n)$ defined by

$$\lambda(g) = u_o^{-1} \circ d_oL_g \circ u_o.$$

When $M \simeq G/H$ is reductive, then λ is faithful and homogeneous metric-compatible connections exist. In our case, *homogeneous metric-compatible connections* are connections, interpreted as $\mathfrak{so}(n)$-valued 1-forms ω on the orthonormal frame bundle $P^{SO}(M)$, that are G-invariant, namely

$$\tilde{L}_g^*\omega = \omega, \ \forall g \in G.$$

The existence of such a connection is ensured by a specialised version of Wang's theorem for reductive homogeneous spaces [119]. Moreover, this Theorem also provides a classification of such connections.

Theorem 8.1.1 (Wang's theorem for reductive spaces)**.** *Let P be a G-invariant principal K-bundle on a reductive homogeneous space $M \simeq G/H$ with decomposition*

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. *Then there is a one-to-one correspondence between the set of G-invariant connections on P and the set of linear maps $\Lambda : \mathfrak{m} \to \mathfrak{k}$ such that*

$$\Lambda(\mathrm{Ad}_h(v)) = \mathrm{Ad}_{\lambda(h)}(\Lambda(v)), \quad \textit{for all } v \in \mathfrak{m},\, h \in H,$$

where λ denotes the isotropy homomorphism $H \to K$.

In case of $P^{SO}(M)$, we just need to set $K = SO(n)$. We would like to stress that the following vector spaces are isomorphic

$$\mathfrak{m} \xrightarrow{\sim} \mathrm{T}_o M \xrightarrow{\sim} \mathbb{R}^n,$$

where the first isomorphism is given by the tangent map of the projection $\pi : G \to M$ on the identity element, while the second, by the choice of a frame u_o. Through those isomorphisms, the adjoint representation $\mathrm{Ad} : H \subset G \to GL(\mathfrak{g})$ is mapped to the linear isotropy representation, and to the isotropy homomorphism, respectively.

We are interested in the cases in which M admits a spin structure, namely a pair $(P^{Spin}(M), \bar{\rho})$ where $P^{Spin}(M)$ is a principal $\mathrm{Spin}(n)$-bundle over M and $\bar{\rho} : P^{Spin}(M) \to P^{SO}(M)$ is a double-covering such that the following diagram commutes [32].

$$\begin{tikzcd}
P^{Spin}(M) \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\bar{\rho}\times\rho"] & P^{Spin}(M) \arrow[d, "\bar{\rho}"] \arrow[dr] & \\
P^{SO}(M) \times SO(n) \arrow[r] & P^{SO}(M) \arrow[r] & M
\end{tikzcd}$$

Here, $\rho : \mathrm{Spin}(n) \to SO(n)$ is the twofold covering homomorphism of $SO(n)$. Over a homogeneous manifold $M \simeq G/H$, we call *homogeneous spin structure* a spin structure $(P^{Spin}(M), \bar{\rho})$ equipped with an action of G on $P^{Spin}(M)$ covering the action of G on $P^{SO}(M)$ and that is invariant under it. Namely, for every $g \in G$ there exists an automorphism $\bar{L}_g : P^{Spin}(M) \to P^{Spin}(M)$ such that $\bar{\rho} \circ \bar{L} = \tilde{L} \circ \bar{\rho}$. In this case we can define a *homogeneous spin connection* $\bar{\omega}$ on $P^{Spin}(M)$ as the pullback of a homogeneous metric-compatible connection via a homogeneous spin structure $(P^{Spin}(M), \bar{\rho})$, i.e.

$$\bar{\omega} = \rho_*^{-1} \bar{\rho}^* \omega.$$

However, a homogeneous spin structure does not always exist. To remedy this problem, we can notice that the set of homogeneous spin connections is in one-to-one correspondence, due to the injectivity of the pullback, with the set of homogeneous metric-compatible connections when a homogeneous spin structure exists. And, in the physical theory, the local fields do not depend on the specific spin structure, so they reflect directly the properties of homogeneous metric-compatible connections. Thus, we will study the latter set as a "generalisation" of the first.

We recall that on the set of metric-compatible connections acts the group of vertical automorphisms, or gauge group, $\mathcal{G}$ of $P^{SO}(M)$, whose elements are automorphisms of $P^{SO}(M)$ that cover the identity on M. A subgroup of $\mathcal{G}$ acts on the set of homogeneous metric-compatible connections, and it is defined by

$$\mathcal{G}^G = \{f \in \mathcal{G} \mid f \circ \tilde{L}_g = \tilde{L}_g \circ f, \forall g \in G\}. \tag{8.1.2}$$

On a homogeneous space $M \simeq G/H$, principal Spin(n)-bundles have their own homogeneous connections for which homogeneous spin connections represent a subset. The collection of all possible homogeneous connections on all possible principal Spin(n)-bundles modulo gauge transformations has been studied in the past years due to its relation with Yang-Mills theory. This collection is a moduli space defined as follows (cf. [43] for details)

$$\mathcal{M} = \left\{ \begin{array}{c} (\mu, \Lambda) \in \mathrm{Hom}(H, \mathrm{Spin}(n)) \times \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h}, \mathfrak{spin}(n)) \\ \text{s.t. } \Lambda \circ \mathrm{Ad}_h = \mathrm{Ad}_{\mu(h)} \circ \Lambda \ \forall h \in H \end{array} \right\} / \mathrm{Spin}(n). \tag{8.1.3}$$

where Spin(n) acts via conjugation on the pair (μ, Λ), i.e. the action is $(\mu, \Lambda) \mapsto (g\mu g^{-1}, \mathrm{Ad}_g \circ \Lambda)$ for $g \in \mathrm{Spin}(n)$.

8.1.1 Speciality of dimension 3

In the peculiar and physically relevant case of dimension 3, we want to analyse the collection of homogeneous spin connections and compare it with the moduli space of homogeneous Spin(3)-connections.

We recall the speciality of dimension 3: the simply connected homogeneous Riemannian manifolds are classified. It is possible to prove that the isotropy group H is a Lie subgroup of $O(3)$, presenting three cases according to the physical symmetries. Since H is connected and compact, it must be isomorphic to $SO(3)$ or $U(1)$, or be trivial. These cases correspond, respectively, to a homogeneous and isotropic Universe, to a homogeneous Universe with axial symmetry, and, finally, to the so-called Bianchi Universes. We will provide our analysis, separating these three cases.

In dimension 3, we also have some useful isomorphisms. The spin bundle $P^{Spin}(M)$ is trivial, namely it is diffeomorphic to $M \times \mathrm{Spin}(3)$. Moreover, Spin(3) is isomorphic as Lie group to $SU(2)$, and the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic. Furthermore, $\mathfrak{su}(2)$ equipped with the adjoint representation of $SU(2)$, $\mathfrak{so}(3)$ equipped with the adjoint representation of $SO(3)$, and $\mathbb{R}^3$ equipped with the defining representation of $SO(3)$ are all isomorphic representations of $SU(2)$ considering the last two actions precomposed with $\rho : SU(2) \to SO(3)$. Namely, there exists an invertible equivariant map between each two of those vector spaces.

Using the previous properties, in the next Sections we are going to compute the moduli space of the two different types of connections in the three cosmological cases identified by the isotropy group H. The first moduli space is the moduli space of homogeneous metric-compatible connections. According to Wang's theorem, the set of homogeneous metric-compatible connections $\mathcal{A}^G$ is an affine space associated with the vector space

$$\{\Lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{so}(3)) \mid \Lambda \circ \lambda(h) = \mathrm{Ad}_{\lambda(h)} \circ \Lambda \ \forall h \in H\}. \tag{8.1.4}$$

The moduli space we are interested in is the space of the orbits $\mathcal{A}^G/\mathcal{G}^G$, where the dependence on the homogeneous space will be only encoded by H. The second moduli space is the moduli space of homogeneous $SU(2)$ connections, defined by

$$\mathcal{M} = \left\{ \begin{array}{c} (\mu, \Lambda) \in \mathrm{Hom}(H, SU(2)) \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{su}(2)) \\ \text{s.t. } \Lambda \circ \lambda(h) = \mathrm{Ad}_{\mu(h)} \circ \Lambda \quad \forall h \in H \end{array} \right\} / SU(2). \tag{8.1.5}$$

Since $\mathfrak{g}/\mathfrak{h} = \mathfrak{m} \cong \mathbb{R}^3$, even this moduli space depends only on the isotropy group H.

8.2 Homogeneous Connections on Bianchi groups

Let M be a 3-dimensional orientable connected Riemannian manifold equipped with a Lie group structure G and with a left-invariant metric. It is a particular case of a Riemannian homogeneous space in which the stabiliser H is trivial, $M \simeq G/\{e\}$. The simply connected spaces of this kind are known as Bianchi groups. On such spaces, the natural action we are going to consider is the left multiplication

$$\begin{aligned} L : G \times G &\to G; \\ (g, g') &\mapsto L_g g' = gg'. \end{aligned}$$

Remark ([72])**.** On a Lie group G, there exists a unique homogeneous spin structure $(P^{Spin}(G), \bar{\rho})$ equipped with an action $\bar{L}$ of G that covers $\tilde{L}$.

In this case, the space of homogeneous metric-compatible connections $\mathcal{A}^G$ is simply given by Wang's theorem as an affine space over the finite-dimensional vector space $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{so}(3))$. The gauge group $\mathcal{G}^G$, which acts on $\mathcal{A}^G$, is the group of G-equivariant vertical automorphisms defined in (8.1.2).

Proposition 8.2.1. *$\mathcal{G}^G$ is isomorphic to $SO(3)$.*

Proof. Let $p, p' \in P^{SO}(G)$ be such that $\pi(p) = x$ and $\pi(p') = y$, then there exists a unique $g \in G$ such that $L_g x = y$. Hence, there exists a unique $a_{p,p'} \in SO(3)$ such that $\tilde{L}_g p = p' a_{p,p'}$. Let f be in $\mathcal{G}^G$, the value of $f(p')$ is fixed by the value of $f(p)$. Recalling that there exists a unique $b_p \in SO(3)$ such that $f(p) = p\, b_p$, $\forall p \in P^{SO}(G)$:

$$\begin{aligned} f(p') = f(\tilde{L}_g p\, a^{-1}_{p,p'}) = f(\tilde{L}_g p) a^{-1}_{p,p'} &= \tilde{L}_g f(p) a^{-1}_{p,p'} = \\ = \tilde{L}_g (p b_p) a^{-1}_{p,p'} = \tilde{L}_g p\, b_p a^{-1}_{p,p'} &= \\ = p' a_{p,p'} b_p a^{-1}_{p,p'}. \end{aligned} \tag{8.2.1}$$

From which $b_{p'} = a_{p,p'} b_p a^{-1}_{p,p'}$. Thus, the isomorphism is given, fixed a point $p \in P^{SO}(G)$, by the map $f \mapsto b_p$. □

Thus, the space $\mathcal{A}^G/\mathcal{G}^G$, composed by equivalence classes of homogeneous metric-compatible connections, in reason of Wang's theorem and Prop.8.2.1, is in one-to-one correspondence with $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{so}(3))/SO(3)$ with the adjoint action of $SO(3)$ on its Lie algebra.

Considering just the spin bundle $P^{Spin}(G)$, in [43], we can find an expression of the moduli space $\mathcal{M}$ which can be interpreted as the disjoint union of the equivalence classes of homogeneous connections with respect to all the possible nonequivalent lifts of the G-action on the all possible spin bundles. In this peculiar case, the spin bundle can be only the trivial one, and all the lifts are equivalent. Indeed, the moduli space (8.1.5) is pretty simple $\mathcal{M} = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{su}(2))/SU(2)$ and the following proposition holds

Proposition 8.2.2. *$\mathcal{A}^G/\mathcal{G}^G$ and $\mathcal{M}$ are in one-to-one correspondence. Furthermore, there exists a bijective map from $\mathcal{M}$ to $M_3(\mathbb{R})/SO(3)$, with the action of $SO(3)$ on the space of the 3×3 real matrices $M_3(\mathbb{R})$ given by the left composition.*

Proof. The Lie algebra $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic. Let $\rho : SU(2) \to SO(3)$ be the natural double covering homomorphism. The isomorphism between the Lie algebras is given by the pushforward $\rho_* : \mathfrak{su}(2) \to \mathfrak{so}(3)$ [123]. Considering $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ as carrying spaces of the respective adjoint action, the isomorphism ρ_* is also an equivariant map, in the meaning that $\rho_*(\mathrm{Ad}_a v) = \mathrm{Ad}_{\rho(a)} \rho_*(v)$ for all $v \in \mathfrak{su}(2)$, $a \in SU(2)$.
Elements of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{su}(2))/SU(2)$ are equivalent classes of linear maps $[\Lambda]$ with $\Lambda \sim \mathrm{Ad}_a \circ\Lambda$ for any $a \in SU(2)$. We can define the map

$$\begin{aligned} \varrho : \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{su}(2))/SU(2) &\to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{so}(3))/SO(3); \\ [\Lambda] &\mapsto [\rho_* \circ \Lambda]. \end{aligned}$$

This map is well-defined thanks to the equivariant property

$$\varrho[\mathrm{Ad}_a \circ\Lambda] = [\rho_* \circ \mathrm{Ad}_a \circ\Lambda] = [\mathrm{Ad}_{\rho(a)} \circ\rho_* \circ \Lambda] = [\rho_* \circ \Lambda] = \varrho[\Lambda].$$

Clearly, it is invertible because ρ_* is it.

Moreover, $\mathfrak{so}(3)$ with the adjoint action of $SO(3)$ and $\mathbb{R}^3$ with the standard action of $SO(3)$ are isomorphic representation. The isomorphism is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

Hence, following a similar procedure we conclude that there exist a one-to-one correspondence between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{so}(3))/SO(3)$ and $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3)/SO(3)$. Since $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3) \cong M_3(\mathbb{R})$, we conclude

$$\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{su}(2))/SU(2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{so}(3))/SO(3) \xrightarrow{\sim} M_3(\mathbb{R})/SO(3), \tag{8.2.2}$$

with the action of $SO(3)$ on $M_3(\mathbb{R})$ to be the left composition. □

Notice that the vector spaces associated to homogeneous connections are isomorphic as vector spaces $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{su}(2)) \cong \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{so}(3))$. At the level of connections, this isomorphism can be implemented by the unique homogeneous spin structure, which maps homogeneous metric-compatible connections ω into homogeneous spin connections $\bar{\omega} = \rho_*^{-1}\bar{\rho}^*\omega$.

Our aim is characterize the moduli space $\mathcal{M}$ realized as $M_3(\mathbb{R})/SO(3)$. Before to do that, let us notice that the set of positive (negative) semi-definite symmetric 3×3

matrices $\mathcal{C}_+$ ($\mathcal{C}_-$) is a convex cone in the vector space of symmetric 3×3 matrices, and its boundary $\partial\mathcal{C}_+$ ($\partial\mathcal{C}_-$) is composed by matrices with a vanishing determinant. Hence, we can enounce the following Lemma:

Lemma 8.2.3. *The moduli space $M_3(\mathbb{R})/SO(3)$ is in one-to-one correspondence with $\mathcal{C}_+\cup\mathcal{C}_-/\sim$. Where the equivalence relation is:*

$$P\sim P' \quad \textit{iff} \quad P=-P' \textit{ and } P,P'\in\partial\mathcal{C}_+\cup\partial\mathcal{C}_-. \tag{8.2.3}$$

Proof. We consider $M_3(\mathbb{R})$ equipped with the following action of $SO(3)$:

$$\begin{aligned} SO(3)\times M_3(\mathbb{R}) &\to M_3(\mathbb{R});\\ (O,M)&\mapsto OM. \end{aligned}$$

This action is clearly not free.
Let $M\in M_3(\mathbb{R})$, in reason of the polar decomposition, there exists a unique positive semi-definite symmetric 3×3 matrix $P=\sqrt{M^tM}$ such that $M=UP$ for some $U\in O(3)$.
If $\det(M)\neq 0$, then the orthogonal matrix is unique and P is positive definite. Multiplying both for $\operatorname{sgn}(\det(M))$, we always obtain that M is given by the composition of a special orthogonal matrix and a positive-definite (if $\det(M)>0$) or negative-definite (if $\det(M)<0$) symmetric matrix. Hence, the map $M\mapsto\operatorname{sgn}(\det(M))\sqrt{M^tM}$ is constant on the orbit of the $SO(3)$-action and differs on different orbits. Thus, the maps passes to the quotient and defines a bijection between the orbit space of invertible matrices in $M_3(\mathbb{R})$ and the set of positive-definite and negative-definite symmetric matrices.
If $\det(M)=0$, we need to study the different ranks separately; in such cases, U is no longer unique. If $\operatorname{rk}(M)=2$, let $U,U'\in O(3)$ be such that $M=UP=U'P$, then $(U-U')P=0$, and so $U=U'$ on $\operatorname{Ran}(P)$. Furthermore, since $U,U'\in O(3)$ and $\ker(P)\perp\operatorname{Ran}(P)$, hence $U(\ker(P))\subset\ker(P)$ and $U'(\ker(P))\subset\ker(P)$, and then or $U=U'$, or $U=-U'$ on $\ker(P)$. Thus, exactly one of U and U' is in $SO(3)$. This means that the stabiliser of M is trivial. Therefore, for all M such that $\operatorname{rk}(M)=2$, there exists a unique pair $U_+,U_-\in SO(3)$, and a unique positive and a unique negative semi-definite symmetric matrices P_+ and P_- such that $M=U_+P_+=U_-P_-$. Where, U_+,P_+ is the pair constructed before, and $U_-=-\mathrm{Id}_{\operatorname{Ran}(P_+)}U_+$ and $P_-=-\mathrm{Id}_{\operatorname{Ran}(P_+)}P_+=-P_+$.
If $\operatorname{rk}(M)=1$, we can proceed in a similar analysis as before. In this case, there are infinite $U,U'\in O(3)$; in fact, the stabiliser of such an M is isomorphic to $SO(2)$. Moreover, there exist infinite pairs $U_+,U_-\in SO(3)$ as before, but the semi-definite symmetric matrices are unique: $P_+=\sqrt{M^tM}$ and $P_-=-P_+$.
If $\operatorname{rk}(M)=0$, the orbit contains only the null matrix, and so $P_+=P_-=0$.
This means that, when $\det(M)=0$, P_+ and P_- are in the same orbit. So, it is no longer true that different semi-definite symmetric matrices identify different orbits. Thus, we need to impose the following equivalence relation on semi-definite symmetric matrices with $\det(P)=0$: $P\sim P'$ if and only if $P=-P'$.

□

Let us consider $\mathcal{M}$ as in the above identification, and the space of traceless symmetric 3×3 real matrices $\operatorname{Sym}_3^0(\mathbb{R})$. Both spaces admit a filtration in terms of algebraic manifolds with edges. Consider $\mathcal{M}$ with collection of subspaces

$$M_n = \{P \in M_3(\mathbb{R})/SO(3) \mid \mathrm{rk}(P) \leq n\} \quad \text{for } n = 0, \dots, 3.$$

While, for $\mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R}$, we can consider the subspaces S_n given by

$$\begin{aligned} S_3 &= \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R}, \\ S_2 &= \Big\{(A, \lambda) \in \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R} \mid \lambda = 0\Big\}, \\ S_1 &= \left\{(A, \lambda) \in \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R} \;\middle|\; \begin{array}{c} 27(\det(A))^2 = -4(\min(A))^3, \\ \lambda = 0,\ \det(A) \geq 0 \end{array}\right\}, \\ S_0 &= \Big\{(A, \lambda) \in \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R} \mid \lambda = 0,\ A = 0\Big\}. \end{aligned}$$

Where $\min(A)$ denotes the second coefficient of the characteristic polynomial, namely:

$$\min(A) = \frac{1}{2}\Big(\mathrm{Tr}(A^2) - (\mathrm{Tr}\, A)^2\Big), \quad \text{for } A \in M_3(\mathbb{R}).$$

This filtration induces a stratification of the spaces, which on $\mathcal{M}$ coincides with the stratification naturally induced by the quotient. Moreover, the equation that describes S_1 is extremely singular; in fact, if there were a point on which the gradient of the equation is non-trivial, then it would be possible to describe S_1 locally as a four-dimensional manifold. This is impossible because the choice of an element in S_1 is equivalent to the choice of an eigenvalue $2\mu > 0$ and the relative eigenvector, and those conditions can be done with three degrees of freedom. So, as a topological manifold, S_1 has dimension 3 while S_2 has dimension 5. As a result, we can characterise the moduli space $\mathcal{M}$ in terms of traceless symmetric matrices:

Proposition 8.2.4. *There exists a weakly stratified isomorphism between $\mathcal{M}$ and $\mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R}$.*

Proof. To provide the topological manifold structure, we specify a bijective map $F : \mathcal{M} \to \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R}$, which, using the isomorphism $\mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R} \cong \mathbb{R}^6$, gives us a global chart for $\mathcal{M}$. Specifically, we are going to construct a global chart for $\mathcal{C}_+$ and $\mathcal{C}_-$, showing that the glueing of them gives us a global chart for the whole $\mathcal{M}$. We define the following maps

$$\begin{aligned} F_+ : \mathcal{C}_+ &\to \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R}_+; \\ P &\mapsto \Big(P - \tfrac{1}{3}\mathrm{Tr}(P) I_3, \lambda_P\Big), \end{aligned}$$

where λ_P is the eigenvalue of P with the smallest modulus $|\lambda_P|$, and I_3 is the identity matrix of order 3. This map is bijective with an inverse

$$\begin{aligned} F_+^{-1} : \mathrm{Sym}_3^0(\mathbb{R}) \times \mathbb{R}_+ &\to \mathcal{C}_+; \\ (A, \lambda) &\mapsto A + (\lambda - \mu_A) I_3, \end{aligned}$$

where A is a null-trace symmetric 3×3 matrix and μ_A is its smallest eigenvalue (surely non-positive). Hence, this map defines a global chart for $\mathcal{C}_+$.

With a change of sign, we obtain the global chart for $\mathcal{C}_-$

$$\begin{array}{cccc} F_- : & \mathcal{C}_- & \to & \mathrm{Sym}^0_3(\mathbb{R}) \times \mathbb{R}_-; \\ & P & \mapsto & \left(-P + \frac{1}{3}\operatorname{Tr}(P) I_3, \lambda_P\right). \end{array}$$

$$\begin{array}{cccc} F_-^{-1} : & \mathrm{Sym}^0_3(\mathbb{R}) \times \mathbb{R}_- & \to & \mathcal{C}_-; \\ & (A, \lambda) & \mapsto & -A + (\lambda + \mu_A) I_3. \end{array}$$

Remains to show that the glueing of these two maps is well-defined. Considering $P \in \partial\mathcal{C}_+$ and so $-P \in \partial\mathcal{C}_-$, hence $\lambda_P = 0 = \lambda_{-P}$, thus

$$F_-(-P) = \left(-(-P) + \tfrac{1}{3}\operatorname{Tr}(-P) I_3, 0\right) = \left(P - \tfrac{1}{3}\operatorname{Tr}(P) I_3, 0\right) = F_+(P).$$

We can check that the projection to the quotient $\pi : M_3(\mathbb{R}) \to \mathcal{M}$ is continuous with respect to the global chart $(\mathcal{M}, F)$. Let us consider the map $M \mapsto \operatorname{sgn}(\det(M))\sqrt{M^t M}$, where $\operatorname{sgn}(\det(M))$ can be ± 1 indifferently when $\det(M)$ vanishes, and noticing $\lambda_{-P} = -\lambda_P$ in general, we obtain

$$F \circ \pi(M) = \left(\sqrt{M^t M} - \tfrac{1}{3} tr(\sqrt{M^t M}) I_3,\ \operatorname{sgn}(\det(M))\lambda_{\sqrt{M^t M}}\right), \tag{8.2.4}$$

which is a continuous map.

We now prove that the strata are mapped appropriately. Evidently, $F(M_3) = S_3$ and $F(M_0) = S_0$. Moreover, it is immediate to see that $F(M_2) \subset S_2$. Conversely, $\operatorname{rk}(F^{-1}(A)) \leq 2$, indeed, the eigenvector of A with the smallest eigenvalues μ_A is in the kernel of $F^{-1}(A) = A - \mu_A I_3$, then $F(M_2) = S_2$.

Let us now consider $P \in M_1$, given a basis of eigenvectors v_0, v_1, v_2, with $v_1, v_2 \in \ker(P)$ and v_o with eigenvalues $\lambda \geq 0$ (without loss in generality), they are eigenvectors of $F(P) = P - \frac{1}{3}\operatorname{Tr}(P) I_3$ with eigenvalues $-\frac{\lambda}{3}$ and $\frac{2\lambda}{3}$, respectively. Hence, $\det(F(P)) = \frac{2}{27}\lambda^3 \geq 0$ and $\min(F(P)) = -\frac{\lambda^2}{3}$, from which $27(\det(A))^2 = -4(\min(A))^3$. Then, $F(M_1) \subset S_1$.
Let $A \in S_1$, its eigenvalues μ_1, μ_2, μ_3 satisfy

$$\mu_1 + \mu_2 + \mu_3 = 0, \ \text{ and } \ \tfrac{1}{4}(\mu_1\mu_2\mu_3)^2 = -\tfrac{1}{27}(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)^3.$$

Plugging the first into the second, we obtain an equation

$$4\mu_1^6 + 12\mu_1^5\mu_2 - 3\mu_1^4\mu_2^2 - 26\mu_1^3\mu_2^3 - 3\mu_1^2\mu_2^4 + 12\mu_1\mu_2^5 + 4\mu_2^6 = 0.$$

A possible solution is $\mu_2 = 0$, in this case $\det(A) = 0$, and so $\min(A) = 0$ and $\operatorname{Tr}(A) = 0$, imposing that $A = 0$. Considering $\mu_2 \neq 0$, we can introduce the variable $t = \mu_1/\mu_2$, and the previous equation reads

$$4t^6 + 12t^5 - 3t^4 - 26t^3 - 3t^2 + 12t + 4 = 0.$$

This polynomial equation has three roots, each with multiplicity 2: $t = 1$, $t = -2$, $t = -\frac{1}{2}$, providing three similar cases: $\mu_1 = \mu_2 = -\frac{1}{2}\mu_3$, $\mu_2 = \mu_3 = -\frac{1}{2}\mu_1$, and $\mu_3 = \mu_1 = -\frac{1}{2}\mu_2$. The condition $\det(A) > 0$ forces the two equal eigenvalues to be negative. Thus, the spectrum is

$$\sigma(A) = \{-\mu, -\mu, 2\mu\} \ \text{ with } \ \mu \geq 0.$$

Now, $F^{-1}(A)$ is given by $A + \mu I_3$ which has rank almost 1. Thus, $F(M_1) = S_1$. $\square$

As part of the proof, we get the following corollary

Corollary 8.2.5. *The moduli space $\mathcal{M}$ is a topological manifold homeomorphic to $\mathbb{R}^6$.*

8.3 Axial symmetry: a peculiar case

This case represents a peculiar symmetry in physics, characterised by the stabiliser being the abelian group $U(1)$. The peculiarity of this case is that there are two possible inequivalent choices for the isotropy homomorphism; however, they provide the same classification of homogeneous connections.
Since we do not fix the group G neither the action on M, we can figure out the isotropic homomorphism looking for injective homomorphisms in the set of inequivalent homomorphism $\mathrm{Hom}(U(1), SO(3))/SO(3)$, with $SO(3)$ acting via conjugation. This classification is provided by the maps into the maximal tori of $SO(3)$. Since the maximal tori are conjugate to one another, we can pick a representative as the map into the natural $SO(2)$ subgroup. The only two elements of the set represented by injective maps are

$$\begin{aligned} \lambda_\pm : & U(1) \to SO(3); \\ & e^{\pm i\theta} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{8.3.1}$$

Using the equivariant map between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, we obtain two vector spaces associated to homogeneous metric-compatible connections, one for each isotropy homomorphism

$$\vec{\mathcal{A}}^{ax}_\pm = \{\Lambda \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3) \mid \Lambda \circ \lambda_\pm(h) = \lambda_\pm(h) \circ \Lambda, \text{ for all } h \in H\}.$$

Both these vector spaces read as the space of linear maps on $\mathbb{R}^3$ that commute with the rotation about the x axis. Such matrices are the multiples of rotational matrices on the plane (y, z), thus

$$\mathcal{A}^{ax} \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & -c \\ 0 & c & b \end{pmatrix} \quad \text{s.t.} \quad a, b, c \in \mathbb{R} \right\} \cong \mathbb{R}^3. \tag{8.3.2}$$

As anticipated, the group $\mathcal{G}^G$ is independent of the group G and depends only on the stabiliser $H = U(1)$. Then, we call the group in this axial symmetric case $\mathcal{G}^{ax}$.

Proposition 8.3.1. *$\mathcal{G}^{ax}$ is independent of the choice of isotropy homomorphism, and it is isomorphic to $SO(2)$.*

Proof. The proof is similar to the proof of Prop.8.2.1. In this case, we have a preferred point $p_0^\pm \in P^{SO}(M)$ in the fibre over $o \in M$ for each choice of the isotropic homomorphism $\lambda_\pm$, such that $\tilde{L}_h p_0^\pm = p_0^\pm \lambda_\pm(h)$ for all $h \in H$. Considering the homomorphism $f \mapsto b_{p_0^\pm}$, where $f(p_0^\pm) = p_0^\pm b_{p_0^\pm}$, we have a constraint on the values of $b_{p_0^\pm}$:

$$p_0^\pm b_{p_0^\pm} \lambda_\pm(h) = f(\tilde{L}_h p_0^\pm) = \tilde{L}_h f(p_0^\pm) = p_0^\pm \lambda_\pm(h) b_{p_0^\pm},$$

namely $b_{p_0^\pm}$ must commute with $\lambda_\pm(h) \in SO(3)$ for any $h \in H$. Thus, $b_{p_0^\pm} \in \lambda_\pm(H) \cong SO(2)$ because $\lambda_\pm(H)$ is a maximal torus in $SO(3)$. □

As a consequence, $\mathcal{G}^{ax}$ acts via the adjoint representation of $\lambda(H) \cong SO(2) \subset SO(3)$ on the $\mathfrak{so}(3)$ factor, namely via left matrix multiplication on $\mathcal{A}^{ax}$ as expressed in (8.3.2). From which, we can compute the moduli space:

Proposition 8.3.2. *$\mathcal{A}^{ax}/\mathcal{G}^{ax}$ is a smooth manifold with boundary diffeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$.*

Proof. Using polar coordinates $b = r\cos\theta, c = r\sin\theta$, every element of $\mathcal{A}^{ax}$ can be written as

$$\begin{pmatrix} a & 0 & 0 \\ 0 & r\cos\theta & -r\sin\theta \\ 0 & r\sin\theta & r\cos\theta \end{pmatrix}, \quad a, \theta \in \mathbb{R},\ r \in \mathbb{R}_{\geq 0}.$$

Varying θ, we move along an orbit, and we can clearly see that every orbit has a different representative for $\theta = 0$, obtaining:

$$\mathcal{A}^{ax}/\mathcal{G}^{ax} \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix} \mid a \in \mathbb{R},\ r \in \mathbb{R}_{\geq 0} \right\} \cong \mathbb{R} \times \mathbb{R}_{\geq 0}.$$

□

Nevertheless, on such a homogeneous space, a homogeneous spin structure does not always exist (see [72] for the case $U(2)/U(1)$). Hence, a proper notion of homogeneous spin connection can not be found. However, we can look at the space of homogeneous connections on the spin bundle $P^{Spin}(M)$.

The formula 8.1.5 gives us a rapid way to compute the moduli space of homogeneous connections on a spin bundle. First of all, we need to compute the space $\mathrm{Hom}(U(1), SU(2))/SU(2)$, which counts the inequivalent lifts on $P^{Spin}(M)$ of the G-action on M. The classification is analogous to the $SO(3)$ homomorphism. Looking at a fixed maximal torus in $SU(2)$ to find the representatives

$$\begin{array}{rccc} \mu_n: & U(1) & \to & SU(2); \\ & e^{i\theta} & \mapsto & \begin{pmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{pmatrix}, \end{array} \tag{8.3.3}$$

we describe countable distinguishable classes, each one represented by μ_n with $n \in \mathbb{Z}$. We can now compute the moduli space of homogeneous $SU(2)$ connections in this axially symmetric case.

Proposition 8.3.3. *The moduli space $\mathcal{M}^{ax}$ has countable connected components, each diffeomorphic to $\mathbb{R}$.*

Proof. We recall that we have a priori two moduli spaces, each associated with a possible isotropy homomorphism

$$\mathcal{M}^{ax}_{\pm} = \left\{ \begin{matrix} (\mu, \Lambda) \in \mathrm{Hom}(U(1), SU(2)) \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{su}(2)) \\ \text{s.t.} \quad \Lambda \circ \lambda_{\pm}(h) = \mathrm{Ad}_{\mu(h)} \circ \Lambda \ \forall h \in H \end{matrix} \right\} / SU(2).$$

Using the isomorphism between $\mathfrak{su}(2)$ and $\mathbb{R}^3$, we can construct the action $\mu_n(U(1)) \subset SU(2)$ on $\mathbb{R}^3$:

$$\begin{pmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{pmatrix} \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} =$$
$$= \begin{pmatrix} ix & (-y+iz)e^{-2ni\theta} \\ (y+iz)e^{2ni\theta} & -ix \end{pmatrix}.$$

Hence, $\mu_n(U(1))$ acts on the vector (x, y, z) as a rotational matrix

$$R(2n\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2n\theta & -\sin 2n\theta \\ 0 & \sin 2n\theta & \cos 2n\theta \end{pmatrix}.$$

This means that, for a fixed μ_n and $\lambda_{\pm}$, we are looking for endomorphisms Λ of $\mathbb{R}^3$ such that $\Lambda R(\pm\theta) = R(2n\theta)\Lambda$. Those Λ must be like

$$\Lambda = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ c \in \mathbb{R}.$$

Clearly the description is the same for λ_+ or λ_-. This is enough to provide the classification. Considering an equivalent class $[\mu, \Lambda']$, there exists an $a \in SU(2)$ and $n \in \mathbb{Z}$ such that $[\mu, \Lambda'] = [\mu_n, \mathrm{Ad}_a \circ \Lambda']$, where $\mathrm{Ad}_a \circ \Lambda'$ is in the type of Λ found before. If there exists $a \in SU(2)$ such that $[\mu_n, \Lambda] = [\mu_n, \mathrm{Ad}_a \circ \Lambda]$, then a can be associated to a special orthogonal matrix O_a such that, when applied to Λ, gives a matrix $\Lambda' = O_a\Lambda$ with all entries zero but the first. However, considering the first column as a vector, it must conserve its modulus, hence the first entry must be the same as Λ, then $O_a\Lambda = \Lambda$.
In conclusion, every class of homomorphism defines a connected component diffeomorphic to $\mathbb{R}$. □

Here, we observe a clear distinction between the two moduli spaces. This significant difference from the purely homogeneous case arises from the non-existence of a homogeneous spin structure, which prevents the embedding of homogeneous metric-compatible connections into the $SU(2)$ framework. This obstruction is due to the failure of the G-action to lift from the orthonormal frame bundle to the spin bundle. In the next section, it will become evident which of the two moduli spaces corresponds to the classification of homogeneous Ashtekar-Barbero-Immirzi-Sen connections as required by the physical theory.

8.4 Isotropic spaces

This class of symmetry is one of the most studied in physics; it is characterised by a stabiliser $H = SO(3)$. The most relevant cases consist of the three simply connected homogeneous spaces $(\mathbb{R}^3, E_0(3))$, $(\mathbb{S}^3, SO(4))$ and $(\mathbb{H}^3, SO^+(1,3))$. However, as in the previous cases, we can provide a unique treatment dependent only on the stabiliser subgroup.

The classification of homogeneous metric-compatible connections starts by noticing that the isotropic homomorphism $\lambda : H \to SO(3)$ is just the identity. Indeed, since $SO(3)$ has no normal subgroup, a homomorphism $\lambda : SO(3) \to SO(3)$ is trivial or bijective. But the isotropic homomorphism must be injective for a reductive homogeneous space. Moreover, all the automorphisms of $SO(3)$ are inner automorphisms, then they are conjugate to one another. Hence, a suitable choice of p_0 in the orthonormal frame bundle give us $\lambda = \mathrm{id}_{SO(3)}$. Using the version of Wang's theorem for reductive homogeneous spaces, we can compute the set of homogeneous metric-compatible connections $\mathcal{A}^{iso}$, and for any isotropic homogeneous space we get

$$\mathcal{A}^{iso} \cong \Big\{ \Lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{so}(3)) \mid \Lambda \circ R = \mathrm{Ad}_R \circ \Lambda,\ \forall R \in SO(3) \Big\}. \tag{8.4.1}$$

The invariant gauge group $\mathcal{G}^{iso}$ is simplified a lot on this class of spaces

Lemma 8.4.1. *$\mathcal{G}^{iso}$ contains only the identity element.*

Proof. The proof follows from the computation done for Prop.8.3.1. Since b_{p_0} must commute with $\lambda(SO(3))$ and λ is bijective, b_{p_0} must leave in the centre of $SO(3)$, which is trivial. Hence, $b_{p_0} = \mathbb{1}$, and so $b_p = \mathbb{1}$ for all $p \in P^{SO}$. □

Proposition 8.4.2. *$\mathcal{A}^{iso}/\mathcal{G}^{iso}$ is diffeomorphic to $\mathbb{R}$.*

Proof. We already discussed that $\mathcal{G}^{iso}$ contains only the identity element. Hence, we just need to discuss $\mathcal{A}^{iso}$. Using the equivariant isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, we reduce to discussing the linear endomorphisms of $\mathbb{R}^3$ that commute with the rotational matrices, namely, we are looking for $\Lambda : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\Lambda \circ R = R \circ \Lambda$ for all $R \in SO(3)$. It is easy to show that they are only multiples of the identity. Thus, the vector space associated to $\mathcal{A}^{iso}$ is canonically $\mathbb{R}$. □

As in the axial-symmetric case, on such spaces does not always exist an invariant homogeneous structure (see [72] for the case $SO(4)/SO(3)$). Hence, we can not define a proper homogeneous spin structure. However, in this case, there is only one homogeneous connection on the spin bundle.

Proposition 8.4.3. *The moduli space of homogeneous $SU(2)$ connections for isotropic spaces is trivial.*

Proof. Let us start from the computation of $\mathrm{Hom}(SO(3), SU(2))$. Such a homomorphism cannot be injective, but the kernel can be only trivial or the whole $SO(3)$. Thus, there exists one homomorphism only, which is the trivial one.
The moduli space formula (8.1.5) says us, independently by the specific space

$$\mathcal{M}^{iso} = \Big\{ \Lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathfrak{su}(2)) \mid \Lambda \circ \lambda(h) = \Lambda\ \forall h \in SO(3) \Big\} / SU(2).$$

Using the equivariant isomorphism between $\mathfrak{su}(2)$ and $\mathbb{R}^3$, we reduce to discuss the linear endomorphisms of $\mathbb{R}^3$ such that $\Lambda \circ R = \Lambda$ for all $R \in SO(3)$. Clearly, the only possible solution is $\Lambda = 0$. Thus, the moduli space contains only the canonical connection. □

8.5 Relation with Loop Quantum Cosmology

The isotropic case provides an opportunity to compare the two spaces with the standard treatment of cosmology in the Ashtekar-Barbero-Immirzi formulation. In [15], it is stated that the homogeneous and isotropic Ashtekar-Barbero-Immirzi-Sen connections are classified by a gauge-invariant parameter, which is simply a real number. Then, the moduli space of homogeneous and isotropic Ashtekar-Barbero-Immirzi-Sen connections is just the real line [63]. This classification aligns with the moduli space of homogeneous metric-compatible connections, which become spin connections upon fixing a spin structure, while it automatically excludes the $SU(2)$ connection interpretation, as the latter is represented by a single point.

The topological features of the moduli space of homogeneous spin connections have direct consequences for the mathematical foundations of the quantum theory. Most notably, they guarantee the existence of a regular Borel measure and allow the use of the Riesz-Markov-Kakutani representation theorem, thereby facilitating a rigorous construction of gauge-invariant quantum states. Additionally, the contractibility of the moduli space implies the triviality of the fibre bundle $\mathcal{A}^G \to \mathcal{A}^G/\mathcal{G}^G$, enabling a global treatment of the gauge-fixing procedure without encountering topological obstructions.

Chapter 9

Classical interpretation of Generalised Uncertainty Principle theories

It is well understood that the natural formulation of Generalised Uncertainty Principle theories lies within the quantum regime. Nevertheless, the literature includes numerous works in which the GUP framework is implemented at the classical level. This is typically achieved by transferring the deformation in the commutator relations to the Poisson brackets, which in turn govern the Hamiltonian dynamics. From a physical perspective, these studies offer a semiclassical formulation, whereby classical trajectories receive quantum-inspired corrections –presumed to be significant at certain energy scales– originating, in principle, from the underlying quantum theory.

Despite the physical plausibility of this approach, from a formal standpoint, it is not immediately obvious whether a given set of deformed commutators can be consistently translated into a classical Hamiltonian framework. That is, compatibility with the structure of classical mechanics –particularly the Poisson bracket formalism– is not automatically guaranteed.

In this work, we investigate whether it is possible to implement a GUP theory consistently at the classical level using the formalism of symplectic geometry. Given an even-dimensional smooth manifold $\mathcal{P}$, we interpret the variables (q, p) as local coordinates on $\mathcal{P}$, and we define a 2-form that reproduces the desired deformed Poisson brackets. We require this 2-form to be symplectic, i.e. both closed and non-degenerate, which allows us to fully characterise the symplectic structure necessary for a consistent formulation of the theory. As a result, we obtain explicit and fully general conditions on the phase space functions that control the deformation of the Poisson structure. This comprehensive understanding of the classical symplectic structure enables us to formulate the following conjecture:

> *A GUP theory can be consistently implemented at the classical level if and only if the corresponding quantum commutators satisfy the Jacobi identities.*

Building upon this general framework, we also investigate the role of symmetries in such deformed theories. We begin by considering rotational invariance, in which the deformed algebra is preserved under the action of the rotation group $SO(3)$, interpreted as a group of symplectomorphisms acting on the symplectic manifold. This analysis follows a line of reasoning similar to that in [129, 81], but the classical setting allows for greater freedom in the construction of the models.

The study of symmetries naturally leads us to examine constrained Hamiltonian systems. In particular, we are interested in how the deformation of the algebra is affected by the presence of first-class constraints. We analyse two representative cases: in the first, the constraints can be interpreted as arising from a momentum map associated with a gauge symmetry group; in the second, we consider a single Hamiltonian constraint. In both cases, we find that the deformation is well controlled and passes consistently to the reduced phase space, preserving the essential structure of the theory.

9.1 Classical interpretation and Jacobi identity

The class of GUP theories, of which we aim to discuss the classical interpretation, is the one described by the following commutation relations:

$$\begin{aligned} [\hat{p}_i, \hat{p}_j] &= 0, \\ [\hat{q}_i, \hat{q}_j] &= i\hbar L_{ij}(\hat{q}, \hat{p}), \\ [\hat{q}_i, \hat{p}_j] &= i\hbar \delta_{ij} f(\hat{q}, \hat{p}). \end{aligned} \tag{9.1.1}$$

By *classical interpretation* of a n-dimensional GUP theory, we can give the following definition:

Definition 9.1.1. The classical interpretation of a GUP theory in the class given above consists of interpreting the set $(q_1, \dots, q_n, p_1, \dots, p_n)$ as coordinates on a $2n$-dimensional smooth manifold $\mathcal{P}$ equipped with a symplectic form ω such that it induces the following fundamental Poisson brackets

$$\begin{aligned} \{p_i, p_j\} &= 0, \\ \{q_i, q_j\} &= L_{ij}(q, p), \\ \{q_i, p_j\} &= f(q, p)\delta_{ij}. \end{aligned} \tag{9.1.2}$$

We refer to the symplectic manifold $(\mathcal{P}, \omega)$ as phase space.

Remark. A symplectic form ω induces a Lie algebra structure on $\mathcal{C}^\infty(\mathcal{P})$ via the Poisson brackets. Let $f, g \in \mathcal{C}^\infty(\mathcal{P})$, and let X_f be Hamiltonian vector field of f defined by $df = X_f \lrcorner \omega$, the Poisson brackets of two functions are $\{f, g\} = -\omega(X_f, X_g)$. The Poisson brackets satisfy the Jacobi identity and define a non-degenerate Poisson structure on $\mathcal{P}$.

From Definition 9.1.1 we can compute the matrix associated with the symplectic form that induces the Poisson brackets (9.1.2) in the coordinate system $x^a = (q_1, \dots, q_n, p_1, \dots, p_n)$. Indeed, the inverse of the symplectic form matrix is easily computable:

$$\omega^{ab} = \{x^a, x^b\} = \begin{pmatrix} L & fI_n \\ -fI_n & 0 \end{pmatrix}, \tag{9.1.3}$$

The inverse of this matrix is immediate, and so we obtain the matrix of the symplectic form

$$\omega_{ab} = \begin{pmatrix} 0 & -\frac{1}{f} I_n \\ \frac{1}{f} I_n & \frac{1}{f^2} L \end{pmatrix}, \tag{9.1.4}$$

where clearly $L \doteq \{L_{ij}\}$ is a skew-symmetric submatrix and I_n is the identity matrix in n dimensions.

From (9.1.4) it is straightforward to realise that, if we want to deal with a properly defined smooth 2-form, we have to require $f(x) \neq 0$, $\forall x \in \mathcal{M}$. Hence, f is a function with a definite sign all over $\mathcal{P}$. Specifically, to obtain ordinary classical mechanics in a proper limit, these functions have to be strictly positive. Usually, this limit is managed by introducing some suitable parameter that controls the deformation of the Poisson structure. As this parameter approaches zero, ordinary Poisson brackets are fully recovered.

However, the 2-form in (9.1.4) is not a symplectic form *a priori*. This classical structure is a relevant feature for the study of quantum physics theory. Indeed, although in the literature it is debated if the semiclassical dynamics of a GUP is implemented by its classical interpretation, we can, anyway, relate the algebraic properties of the quantum commutation relations to the symplectic structure. When L_{ij} and f can be interpreted as smooth functions of the coordinates on an open subset $\mathcal{U} \subset \mathbb{R}^{2n}$, and f is strictly positive on $\mathcal{U}$, we can formulate the following Conjecture:

Conjecture 9.1.1. *A* n*-dimensional GUP theory as in 9.1.1 admits a classical interpretation defined in Definition 9.1.1 if and only if the commutators satisfy the Jacobi identity.*

In Section 9.3, we provide a series of cases in which the Conjecture is verified. As we will see, we can postulate that the Conjecture provides us with a correct prescription for the quantisation of some relevant phase-space functions, selecting suitable operator orderings among all the possible ones. In the context of deformation quantisation, this conjecture is a corollary of a known theorem by M. Kontsevich [120].
The following Section shows how to construct the symplectic form ω just imposing the *non-degeneracy* and the *closure* of ω, imposing some constraints for the expressions of L and f. The considerations expressed in Section 9.2 will constitute the main part of the construction of the Conjecture 9.1.1.

The non-degeneracy condition is quite trivial since $\det(\omega_{ab}) = 1/f^{2n}$. Therefore, to fulfil the non-degeneracy request, it suffices to deal with a smooth strictly positive function f on $\mathcal{M}$, as already required.
The closed condition, on the other hand, implies that we have to verify that $d\omega = 0$. As we are going to see, this condition will give us a complete set of equations for L and f, and it will establish a set of relations between these two objects.

9.2 Building the function $f(q,p)$

In this section, we are going to explicitly compute the equations coming from $d\omega = 0$, for the *1*, *2*, and finally the n-dimensional case. Furthermore, we will provide some explicit examples and we will discuss, within this symplectic framework, some well-known GUP models scattered in the literature.

9.2.1 1-dimensional GUP theories

The 1-dimensional case is highly trivial. Being, in this case, the phase space a 2-dimensional manifold, every 2-form is closed. This means that the closed condition is automatically satisfied. From this, it immediately follows that every 1-dimensional GUP theory, with a suitable f respecting the minimal conditions imposed above, has a symplectic structure and hence a classical interpretation.

9.2.2 2-dimensional GUP theories

In the 2-dimensional case, the matrix L_{ij} has a unique degree of freedom $L_{12}(q,p) \doteq l(q,p)$. Let us call $h = \frac{1}{f}$. The symplectic form matrix now reads:

$$\omega_{ab} = \begin{pmatrix} 0 & 0 & -h & 0 \\ 0 & 0 & 0 & -h \\ h & 0 & 0 & h^2 l \\ 0 & h & -h^2 l & 0 \end{pmatrix}. \tag{9.2.1}$$

The closure condition, namely $\partial_c \omega_{ab} dx^c \wedge dx^a \wedge dx^b = 0$, gives us $\binom{4}{3} = 4$ scalar equations, i.e. the components of the 3-form $\partial_c \omega_{ab} dx^c \wedge dx^a \wedge dx^b = \frac{2}{3}(\partial_c \omega_{ab} + \partial_a \omega_{bc} + \partial_b \omega_{ca}) dx^c \otimes dx^a \otimes dx^b = 0$. Explicitly, we obtain the following set of equations:

$$\begin{cases} (123) & \frac{\partial h}{\partial q_2} = 0, \\ (124) & -\frac{\partial h}{\partial q_1} = 0, \\ (134) & \frac{\partial h^2 l}{\partial q_1} - \frac{\partial h}{\partial p_2} = 0, \\ (234) & \frac{\partial h^2 l}{\partial q_2} + \frac{\partial h}{\partial p_1} = 0. \end{cases} \tag{9.2.2}$$

The first two say that h, hence f, does not depend on q_1, q_2, but only on the momentum variables p_1, p_2. Therefore, the last two equations of (9.2.2) can be rewritten as:

$$\frac{\partial l}{\partial q_1} + \frac{\partial f}{\partial p_2} = 0, \quad \frac{\partial l}{\partial q_2} - \frac{\partial f}{\partial p_1} = 0. \tag{9.2.3}$$

These relations can be read as a gradient equation for f in an open set $U \subset \mathbb{R}^2$ in terms of p_1, p_2. Together with the independence of f from q_1, q_2, from (9.2.3) we can deduce a general form for the l function:

$$l(q,p) = s(p) + a(p)q_1 + b(p)q_2, \tag{9.2.4}$$

where s, a, b are smooth functions of p_1, p_2. If U is simply connected, the system (9.2.3) admits solution if and only if

$$\frac{\partial a(p)}{\partial p_1} = -\frac{\partial b(p)}{\partial p_2}. \tag{9.2.5}$$

In this case, the solution for f is given by the integral along a curve $\gamma \subset U$ with final point (p_1, p_2) modulo an integration constant:

$$f(p_1,p_2) = \int_\gamma \left(\frac{\partial l}{\partial q_2}, -\frac{\partial l}{\partial q_1}\right) \cdot \dot{\gamma}(t) dt. \tag{9.2.6}$$

Note that, in this setting, the system (9.2.3) can be equivalently read as a gradient equation for l.

2D classical mechanics and 2D non-commutative classical mechanics

Fixing $l = c$, where c is an arbitrary constant, we find a trivial solution for f. The two equations (9.2.3) impose indeed $f = const$. With a proper choice of the constants involved, here we can recover both ordinary 2D classical mechanics (c=0 and f=1) and one of the most studied formulations of 2D non-commutative classical mechanics (c=1 and f=1), which does not involve a modification of the Poisson brackets between conjugate variables (see e.g. [90]).

Interesting example: Kempf-Mangano-Mann 2D-GUP

Consider $l(q,p)$ proportional to the usual angular momentum $l = c(q_1p_2 - q_2p_1)$. This function is clearly compatible with the general form (9.2.4) and satisfies the integrability condition (9.2.5).
In this case the gradient equation (9.2.3) for f assumes a simple form:

$$cp_2 + \frac{\partial f}{\partial p_2} = 0, \quad cp_1 + \frac{\partial f}{\partial p_1} = 0.$$

The solution is $f(p_1,p_2) = -\frac{c}{2}(p_1^2 + p_2^2) + const$, which essentially represents, modulo the factor c, a kinetic energy term.
The integration constant can be fixed by introducing a control parameter and requiring that, in a certain limit, one recovers the ordinary Poisson brackets. In the case taken into the analysis, this can be done by evaluating the limit for $c \to 0$, which fixes the integration constant to be 1.
Clearly we are not considering a possible dependence of the integration constant on the control parameter c. The idea is indeed to introduce the deformation parameter naturally only once and let it properly propagate along the equations.
If we now fix $c = -2\beta$, where β is a positive parameter, what we obtain is the well-known GUP model first proposed by Kempf, Mangano, and Mann in [117], at the quantum level.
Note that to deal with a strictly positive f, for the reason discussed above, either $c < 0$ or, for $c > 0$, our phase space is a ball in the space of momenta $\frac{c}{2}(p_1^2 + p_2^2) < 1$.

It is worth noticing that the symplectic form naturally induces a volume form in the phase space, which in the 2-dimensional case reads as $\mathrm{Vol}_\omega \doteq \omega \wedge \omega = \frac{2!}{f(p)^2}\, dq^1 \wedge dq^2 \wedge dp^1 \wedge dp^2$. This volume form is clearly well-defined if and only if the ω is non-degenerate, that is, if $f(p) \neq 0$ everywhere. The resulting condition is very close to the one obtained in the quantum framework from the request of symmetry of the fundamental operators, in particular, the position operator. In this context, to recover the symmetry of the operators, a necessary but not sufficient condition for the self-adjointness, we need to deform the Lebesgue measure in a very similar way, introducing a factor $1/f$. As a consequence, both measures result in being well-defined if and only if $f \neq 0$, which in turn is a condition related, in the classical context, to the non-degeneracy of ω.

Explicit computation: Polynomial functions

In this Section, we aim to derive an explicit and general form for f from a wide class of functions for l. We consider a polynomial function l:

$$l(q,p) = \sum \alpha_{rsmn} q_1^r q_2^s p_1^m p_2^n. \tag{9.2.7}$$

We already know that $l(q,p)$ can be a polynomial with maximal degree one with respect to q variables. Given that, we can write:

$$l(q,p) = \sum \alpha_{mn} p_1^m p_2^n + q_1 \sum \beta_{mn} p_1^m p_2^n + q_2 \sum \gamma_{mn} p_1^m p_2^n, \tag{9.2.8}$$

The necessary and sufficient condition (9.2.5) now reads as:

$$m\beta_{m,n-1} + n\gamma_{m-1,n} = 0, \ \forall m,n \in \mathbb{N}. \tag{9.2.9}$$

From the constraint (9.2.9), the polynomial l has the following form:

$$\begin{aligned} l(q,p) = &\sum_{m,n\geq 0} \alpha_{mn} p_1^m p_2^n + \sum_{m\geq 1,\, n\geq 0} \beta_{mn} p_1^m p_2^n q_1 + \sum_{n\geq 0} \beta_n p_2^n q_1 + \\ &- \sum_{m\geq 0,\, n\geq 1} \tfrac{m+1}{n} \beta_{m+1,n-1} p_1^m p_2^n q_2 - \sum_{m\geq 0} \gamma_m p_1^m q_2, \end{aligned} \tag{9.2.10}$$

with $\beta_n \equiv \beta_{0n}$ and $\gamma_m \equiv -\gamma_{m0}$.
The f function, by reason of (9.2.6), can be computed on a simple curve

$$\gamma(t) = \begin{cases} (t,0) & \text{if } 0 \leq t \leq p_1, \\ (p_1, t-p_1) & \text{if } p_1 \leq t \leq p_2 + p_1, \end{cases}$$

resulting in

$$f(p_1,p_2) = \int_0^{p_1} \frac{\partial l}{\partial q_2}(p_1',0)dp_1' - \int_0^{p_2} \frac{\partial l}{\partial q_1}(p_1,p_2')dp_2' + c.$$

From this solution, we can give an explicit form of $f(p)$ in the polynomial case.

$$f(p_1,p_2) = -\sum_{m\geq 0} \tfrac{\gamma_m}{m+1} p_1^{m+1} - \sum_{n\geq 0} \tfrac{\beta_n}{n+1} p_2^{n+1} - \sum_{m,\,n\geq 1} \tfrac{1}{n} \beta_{m,n-1} p_1^m p_2^n + c. \tag{9.2.11}$$

We can check *a posteriori* that f is smooth and, if $c > 0$, there exists $\varepsilon > 0$ such that f is strictly positive in $\mathbb{B}^2_\varepsilon \subset \mathbb{R}^2$. Thus, our phase space is $\mathcal{M} = \mathbb{R}^2 \times \mathbb{B}^2_\varepsilon$ with a symplectic form given in coordinates by (9.2.1), where the functions l and f are defined in (9.2.10) and (9.2.11), respectively.
Note that the "angular momentum case" is included in this description. It can be recovered by setting $\beta_1 = c = \gamma_1$ and all other coefficients to null.

9.2.3 n-dimensional GUP theories

We can extend this analysis to GUP theories in every dimension n. Analogously to the 2-dimensional case, from the closed condition of the 2-form in (9.1.4), we find a set of differential equations for L and f.
Notice that we can rewrite the 2-form as $\omega = h\tilde{\omega}$, where $h = \frac{1}{f}$ and

$$\tilde{\omega}_{ab} = \begin{pmatrix} 0 & -I_n \\ I_n & hL \end{pmatrix}.$$

In coordinates $x^a = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, the closure condition $d\omega = 0$ reads

$$\tilde{\omega}_{ab}\partial_c f + \tilde{\omega}_{bc}\partial_a f + \tilde{\omega}_{ca}\partial_b f - f\partial_c\tilde{\omega}_{ab} - f\partial_a\tilde{\omega}_{bc} - f\partial_b\tilde{\omega}_{ca} = 0. \tag{9.2.12}$$

The equations can be collected in four sets, depending on the values of the indices. Clearly, the equations are symmetric under permutations of the indices, obtaining at most $\binom{2n}{3}$ independent equations.

For $a, b, c \leq n$, we have a set of trivial identities. So, that choice of indices is not relevant.

For $c > n$ and $a, b \leq n$, defining $k = c - n$ and $i, j \in \{1, \ldots, n\}$, we get:

$$-\frac{\partial f}{\partial q_i}\delta_{jk} + \frac{\partial f}{\partial q_j}\delta_{ik} = 0. \tag{9.2.13}$$

That is the independence of f on qs. We can show it by noticing that the equations are antisymmetric in i, j and the only non-trivial equations are for $i = k$ (equivalently $j = k$), providing a set of n independent equations

$$\frac{\partial f}{\partial q_k} = 0.$$

For $a, b > n$ and $c \leq n$, let $i = a - n, j = b - n, k \in \{1, \ldots, n\}$, we get the following equations:

$$\frac{1}{f^2}\frac{\partial f}{\partial q_k}L_{ij} + \frac{\partial f}{\partial p_i}\delta_{jk} - \frac{\partial f}{\partial p_j}\delta_{ik} + \frac{1}{f}\frac{\partial f}{\partial q_k}L_{ij} - \frac{1}{f}\frac{\partial L_{ij}}{\partial q_k} = 0.$$

Using the independence of f on qs, we obtain

$$\frac{\partial f}{\partial p_i}\delta_{jk} - \frac{\partial f}{\partial p_j}\delta_{ik} - \frac{\partial L_{ij}}{\partial q_k} = 0. \tag{9.2.14}$$

That contains the gradient equation. There are n^3 non-independent equations. If $k \neq i, j$ we obtain $\frac{1}{2}n(n-1)(n-2)$ independent equations

$$\frac{\partial L_{ij}}{\partial q_k} = 0.$$

Due to antisymmetry of (9.2.14), $k = i$ gives us the same equations as $k = j$. So, considering $k = j$ and $i \neq j$, we have $n(n-1)$ independent equations

$$\frac{\partial f}{\partial p_i} = \frac{\partial L_{ij}}{\partial q_j},$$

which give us constraints on $L_{ij}(q,p)$ and a gradient equation for $f(p)$. The constraints read

$$\frac{\partial L_{ij}}{\partial q_j} = \frac{\partial L_{ik}}{\partial q_k}, \ \forall i,j,k \le n,$$

and give us the general form of L_{ij}

$$L_{ij}(q,p) = S_{ij}(p) - g_j(p)q_i + g_i(p)q_j, \tag{9.2.15}$$

where $S_{ij} = -S_{ji}$ and g_i are a set of smooth functions on p only.
Thus, the remaining equations are simply recast as the gradient equation

$$\frac{\partial f}{\partial p_i} = g_i(p). \tag{9.2.16}$$

Let us consider this differential equation defined on a simply connected subset $U \subset \mathbb{R}^n$ and define the 1-form $g = \sum_{i=1}^{d} g_i dp_i$. The equation is equivalent to $df = g$. Hence, a necessary and sufficient condition for integrability is $dg = 0$.
On the other hand, considering a given $f(p)$, the Eq. (9.2.16) fully characterises L_{ij} modulo a function on the momenta without any further request.

For $a, b, c > n$, a new equation appears, which is not present in the 2-dimensional case:

$$2\left(\frac{\partial f}{\partial p_k}L_{ij} + \frac{\partial f}{\partial p_i}L_{jk} + \frac{\partial f}{\partial p_j}L_{ki}\right) - f\left(\frac{\partial L_{ij}}{\partial p_k} + \frac{\partial L_{jk}}{\partial p_i} + \frac{\partial L_{ki}}{\partial p_j}\right) = 0. \tag{9.2.17}$$

The implication of this "strange equation" is not clear immediately.
We can study it by considering the general form of L_{ij} given before, with $S_{ij} \equiv 0$. The first term vanishes

$$\begin{aligned}
&\frac{\partial f}{\partial p_k}L_{ij} + \frac{\partial f}{\partial p_i}L_{jk} + \frac{\partial f}{\partial p_j}L_{ki} \\
&= g_k(-g_j q_i + g_i q_j) + g_i(-g_k q_j + g_j q_k) + g_j(-g_i q_k + g_k q_i) \\
&= 0,
\end{aligned}$$

while the second term contains the integrability conditions

$$\begin{aligned}
&\frac{\partial L_{ij}}{\partial p_k} + \frac{\partial L_{jk}}{\partial p_i} + \frac{\partial L_{ki}}{\partial p_j} = \\
&= -\frac{\partial g_j}{\partial p_k}q_i + \frac{\partial g_i}{\partial p_k}q_j - \frac{\partial g_k}{\partial p_i}q_j + \frac{\partial g_j}{\partial p_i}q_k - \frac{\partial g_i}{\partial p_j}q_k + \frac{\partial g_k}{\partial p_j}q_i \\
&= \left(\frac{\partial g_k}{\partial p_j} - \frac{\partial g_j}{\partial p_k}\right)q_i + \left(\frac{\partial g_i}{\partial p_k} - \frac{\partial g_k}{\partial p_i}\right)q_j + \left(\frac{\partial g_j}{\partial p_i} - \frac{\partial g_i}{\partial p_j}\right)q_k.
\end{aligned}$$

Hence, considering the gradient equation satisfied, the strange equation becomes a constraint on the S_{ij} functions, which are not fixed by the form of $f(p)$:

$$2\left(\frac{\partial f}{\partial p_k}S_{ij} + \frac{\partial f}{\partial p_i}S_{jk} + \frac{\partial f}{\partial p_j}S_{ki}\right) - f\left(\frac{\partial S_{ij}}{\partial p_k} + \frac{\partial S_{jk}}{\partial p_i} + \frac{\partial S_{ki}}{\partial p_j}\right) = 0. \tag{9.2.18}$$

In reason of the previous discussion, $S_{ij} \equiv 0$ is a solution of the strange equation.

The 3-dimensional case

The simplest case in which the strange equation appears is the three-dimensional one. Following the results of the previous Section, in dimension 3, the closure condition implies $\binom{6}{3} = 20$ equations.
Ten of them are trivial identities if we consider $f(p)$ depending only on momentum variables. We then proceed to examine the remaining 9+1 equations. Let us consider the first nine of these that correspond to (9.2.14):

$$\begin{cases} (145) & \frac{\partial f}{\partial p_2} + \frac{\partial L_{12}}{\partial q_1} = 0, \\ (146) & \frac{\partial f}{\partial p_3} + \frac{\partial L_{13}}{\partial q_1} = 0, \\ (156) & \frac{\partial L_{23}}{\partial q_1} = 0, \\ (245) & \frac{\partial f}{\partial p_1} - \frac{\partial L_{12}}{\partial q_2} = 0, \\ (246) & \frac{\partial L_{13}}{\partial q_2} = 0, \\ (256) & \frac{\partial f}{\partial p_3} + \frac{\partial L_{23}}{\partial q_2} = 0, \\ (345) & \frac{\partial L_{12}}{\partial q_3} = 0, \\ (346) & \frac{\partial f}{\partial p_1} - \frac{\partial L_{13}}{\partial q_3} = 0, \\ (356) & \frac{\partial f}{\partial p_2} - \frac{\partial L_{23}}{\partial q_3} = 0. \end{cases} \tag{9.2.19}$$

As already discussed, that system brings out the following form for L_{ij}

$$\begin{aligned} L_{12}(q,p) &= S_{12}(p) - P(p)q_1 + Q(p)q_2, \\ L_{23}(q,p) &= S_{23}(p) - R(p)q_2 + P(p)q_3, \\ L_{13}(q,p) &= S_{13}(p) - R(p)q_1 + Q(p)q_3. \end{aligned}$$

Here, S_{ij}, P, Q, R are smooth functions of p_1, p_2, p_3. In such a way, the system (9.2.19) is recast as three independent equations

$$\begin{cases} \frac{\partial f}{\partial p_1} = Q(p_1,p_2,p_3), \\ \frac{\partial f}{\partial p_2} = P(p_1,p_2,p_3), \\ \frac{\partial f}{\partial p_3} = R(p_1,p_2,p_3). \end{cases} \tag{9.2.20}$$

Again, we obtain a gradient equation in an open subset $U \subset \mathbb{R}^3$. If U is simply connected, the system is solvable if and only if

$$\frac{\partial P}{\partial p_1} - \frac{\partial Q}{\partial p_2} = 0, \quad \frac{\partial R}{\partial p_1} - \frac{\partial Q}{\partial p_3} = 0, \quad \frac{\partial P}{\partial p_3} - \frac{\partial R}{\partial p_2} = 0,$$

that is the irrotational condition for the vector field $\vec{V} = (Q, P, R)$. In this case, the solution for $f(p)$ is given by fixing a curve $\gamma \subset U$ with endpoint (p_1, p_2, p_3)

$$f(p_1,p_2,p_3) = \int_\gamma \vec{V} \cdot d\gamma. \tag{9.2.21}$$

In dimension three, the strange equation has only one component, which reads

$$2\left(\frac{\partial f}{\partial p_1}L_{23} + \frac{\partial f}{\partial p_2}L_{31} + \frac{\partial f}{\partial p_3}L_{12}\right) - f\left(\frac{\partial L_{12}}{\partial p_3} + \frac{\partial L_{23}}{\partial p_1} + \frac{\partial L_{31}}{\partial p_2}\right) = 0.$$

Considering the gradient equation (9.2.20) satisfied, we obtain a constraint on S_{ij}:

$$2\left(Q\, S_{23} + P\, S_{31} + R\, S_{12}\right) - f\left(\frac{\partial S_{12}}{\partial p_3} + \frac{\partial S_{23}}{\partial p_1} + \frac{\partial S_{31}}{\partial p_2}\right) = 0.$$

We note that solutions of the following system are solutions of the above equation as well:

$$\begin{cases} 2R\, S_{12} - f\frac{\partial S_{12}}{\partial p_3} = 0, \\ 2Q\, S_{23} - f\frac{\partial S_{23}}{\partial p_1} = 0, \\ 2P\, S_{31} - f\frac{\partial S_{31}}{\partial p_2} = 0. \end{cases}$$

In the subset of $\mathbb{R}^3$ where $S_{ij}(p) \neq 0$ and $f(p) \neq 0$, the system can be written compactly as

$$\frac{1}{S_{ij}}\frac{\partial S_{ij}}{\partial p_k} = 2\frac{1}{f}\frac{\partial f}{\partial p_k} \quad \text{with } i \neq j \neq k. \tag{9.2.22}$$

Hence, a possible solution is $S_{ij}(p) = \pm f(p)^2 e^{c_{ij}(p_i,p_j)}$, where $c_{ij} = c_{ji}$ are arbitrary functions on p_i, p_j only.

Interesting example: Kempf-Mangano-Mann 3D-GUP

We now want to analyse the "angular momentum case" in three dimensions, but from the perspective of a fixed $f(p)$.
Let $f(p) = \beta(p_1^2 + p_2^2 + p_3^2) + c$, with β strictly positive. This choice forces $Q = 2\beta p_1$, $P = 2\beta p_2$, $R = 2\beta p_3$ because of (9.2.20). Hence,

$$L_{ij} = S_{ij}(p) + 2\beta(q_j p_i - q_i p_j).$$

The Kempf-Mangano-Mann (KMM) GUP theory is recovered for $c = 1$ and $S_{ij} \equiv 0$. However, as we already said, this is not the unique choice for S_{ij}. Nevertheless, considering β the control parameter and requiring that the usual Heisenberg algebra is recovered for $\beta \to 0$, it is possible to exclude some S_{ij} functions. For instance, $S_{ij} = f^2 e^{c_{ij}(p_i,p_j)}$ is an admissible solution of 9.2.18 but it does not have the correct limit because $f \xrightarrow[\beta\to 0]{} c$, forcing $c = 1$, and so $S_{ij} \xrightarrow[\beta\to 0]{} e^{c_{ij}(p_i,p_j)}$, which cannot be zero. For the same reason discussed in Section 9.2.2, here we are not considering a possible dependence of the c_{ij} functions, nor of the constant c, on the control parameter β.

9.3 Verifying the Conjecture

In this Section, we are going to discuss the validity of the Conjecture 9.1.1, providing some cases that satisfy it. Let us start from the following Proposition (Prop.8.11 in [127]):

Proposition 9.3.1. *A symplectic form on $\mathcal{P}$ is equivalent to a non-degenerate Poisson structure.*

The equivalence is given identifying the Poisson bivector field π [170] with the inverse of the symplectic form ω, i.e., in coordinates, $\pi^{ab} = \omega^{ab}$.

Consider the fundamental Poisson brackets expressed in local coordinates $x^a = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ as in (9.1.2) , where L_{ij} and f are smooth functions of the coordinates, and f is strictly positive on $\mathcal{P}$. If these Poisson brackets satisfy the Jacobi identities, then they define a non-degenerate Poisson structure. The non-degeneracy can be easily checked, considering the bivector field in coordinates

$$\pi^{ab} \doteq \{x^a, x^b\} = \begin{pmatrix} L & fI_n \\ -fI_n & 0 \end{pmatrix}, \tag{9.3.1}$$

for which $\det(\pi^{ab}) = f^{2n} > 0$. From this and Prop. 9.3.1, the following Lemma holds:

Lemma 9.3.2. *The 2-form ω defined in* (9.1.4) *is a symplectic form if and only if the Poisson brackets in* (9.1.2) *satisfy the Jacobi identity.*

Since a rigorous and consistent quantisation procedure does not exist for GUP theories, we just check that the quantum Jacobi identity is satisfied if we consider f and L as in Section 9.2.3 for two possible “natural” operator ordering in the quantum theory, namely all the momenta to the left and all the momenta to the right.

Let us consider a GUP theory in the class (9.1.1), with L and f that, as functions on the phase space, satisfy (9.2.13),(9.2.15),(9.2.16). The main problem is the choice of the operator ordering in $L_{ij}(\hat{q}, \hat{p})$, since, in the classical functions, monomials of the kind $p_i q_i$ appear. The other functions, which depend only on p, do not have any ambiguity in the ordering.
Let us first check the Jacobi identity for the fundamental Poisson brackets that do not depend on L_{ij}. On the momenta, the commutation is trivially null $[\hat{p}_i, [\hat{p}_j, \hat{p}_k]] + [\hat{p}_j, [\hat{p}_k, \hat{p}_i]] + [\hat{p}_k, [\hat{p}_i, \hat{p}_j]] = 0$.
Moreover, we have

$$\begin{aligned} &[\hat{p}_i, [\hat{p}_j, \hat{q}_k]] + [\hat{p}_j, [\hat{q}_k, \hat{p}_i]] + [\hat{q}_k, [\hat{p}_i, \hat{p}_j]] \\ &= -i\hbar\delta_{jk}[\hat{p}_i, f(\hat{p})] + i\hbar\delta_{ik}[\hat{p}_j, f(\hat{p})] = 0. \end{aligned}$$

Notice that, independently from the chosen ordering for $L_{ij}(\hat{q}, \hat{p})$, we get

$$[\hat{p}_k, L_{ij}(\hat{q}, \hat{p})] = i\hbar \left(\delta_{ik} g_j(\hat{p}) f(\hat{p}) - \delta_{jk} g_i(\hat{p}) f(\hat{p})\right). \tag{9.3.2}$$

Furthermore, since there are no ambiguities in $f(p)$, its derivative is well-defined $\frac{\partial f}{\partial p_i}(\hat{p})$ in its clear meaning, and so it is not difficult to prove that:

$$[\hat{q}_k, f(\hat{p})] = i\hbar f(\hat{p}) \frac{\partial f}{\partial p_k}(\hat{p}) \doteq i\hbar f(\hat{p}) g_k(\hat{p}). \tag{9.3.3}$$

This leads us to conclude that another commutator is free from ambiguities:

$$\begin{aligned} &[\hat{q}_i, [\hat{q}_j, \hat{p}_k]] + [\hat{q}_j, [\hat{p}_k, \hat{q}_i]] + [\hat{p}_k, [\hat{q}_i, \hat{q}_j]] \\ &= i\hbar\delta_{jk}[\hat{q}_i, f(\hat{p})] - i\hbar\delta_{ik}[\hat{q}_j, f(\hat{p})] + i\hbar[\hat{p}_k, L_{ij}(\hat{q}, \hat{p})] \\ &= -\hbar^2 \left(\delta_{jk} f(\hat{p}) g_i(\hat{p}) - \delta_{ik} f(\hat{p}) g_j(\hat{p}) + \delta_{ik} g_j(\hat{p}) f(\hat{p}) - \delta_{jk} g_i(\hat{p}) f(\hat{p})\right) \\ &= 0. \end{aligned}$$

We now need to fix an operator ordering in the theory to proceed. Let us first consider all the momenta on the left, so the commutation between configuration coordinates reads

$$[\hat{q}_i, \hat{q}_j] = i\hbar\left(S_{ij}(\hat{p}) - g_j(\hat{p})\hat{q}_i + g_i(\hat{p})\hat{q}_j\right), \tag{9.3.4}$$

from which we can write the double commutator:

$$\begin{aligned}
&[\hat{q}_k, [\hat{q}_i, \hat{q}_j]] \\
&= i\hbar\Big([\hat{q}_k, S_{ij}(\hat{p})] - [\hat{q}_k, g_j(\hat{p})]\hat{q}_i - g_j(\hat{p})[\hat{q}_k, \hat{q}_i] + [\hat{q}_k, g_i(\hat{p})]\hat{q}_j + g_i(\hat{p})[\hat{q}_k, \hat{q}_j]\Big) \\
&= -\hbar^2\left(f(\hat{p})\frac{\partial S_{ij}}{\partial p_k}(\hat{p}) - f(\hat{p})\frac{\partial g_j}{\partial p_k}(\hat{p})\hat{q}_i - g_j(\hat{p})L_{ki} + f(\hat{p})\frac{\partial g_i}{\partial p_k}(\hat{p})\hat{q}_j + g_i(\hat{p})L_{kj}\right).
\end{aligned}$$

Fixing $S_{ij} \equiv 0$, the last Jacobi identity is satisfied

$$\begin{aligned}
&\frac{1}{\hbar^2}\left([\hat{q}_i, [\hat{q}_j, \hat{q}_k]] + [\hat{q}_j, [\hat{q}_k, \hat{q}_i]] + [\hat{q}_k, [\hat{q}_i, \hat{q}_j]]\right) \\
&= -f(\hat{p})\frac{\partial g_k}{\partial p_i}(\hat{p})\hat{q}_j - g_k(\hat{p})L_{ij} + f(\hat{p})\frac{\partial g_j}{\partial p_i}(\hat{p})\hat{q}_k + g_j(\hat{p})L_{ik} \\
&\quad - f(\hat{p})\frac{\partial g_i}{\partial p_j}(\hat{p})\hat{q}_k - g_i(\hat{p})L_{jk} + f(\hat{p})\frac{\partial g_k}{\partial p_j}(\hat{p})\hat{q}_i + g_k(\hat{p})L_{ji} \\
&\quad - f(\hat{p})\frac{\partial g_j}{\partial p_k}(\hat{p})\hat{q}_i - g_j(\hat{p})L_{ki} + f(\hat{p})\frac{\partial g_i}{\partial p_k}(\hat{p})\hat{q}_j + g_i(\hat{p})L_{kj} \\
&= f(\hat{p})\left(\frac{\partial g_k}{\partial p_j}(\hat{p}) - \frac{\partial g_j}{\partial p_k}(\hat{p})\right)\hat{q}_i + f(\hat{p})\left(\frac{\partial g_i}{\partial p_k}(\hat{p}) - \frac{\partial g_k}{\partial p_i}(\hat{p})\right)\hat{q}_j \\
&\quad + f(\hat{p})\left(\frac{\partial g_j}{\partial p_i}(\hat{p}) - \frac{\partial g_i}{\partial p_j}(\hat{p})\right)\hat{q}_k - 2g_k(\hat{p})\left(-g_j(\hat{p})\hat{q}_i + g_i(\hat{p})\hat{q}_j\right) \\
&\quad - 2g_i(\hat{p})\left(-g_k(\hat{p})\hat{q}_j + g_j(\hat{p})\hat{q}_k\right) - 2g_j(\hat{p})\left(-g_i(\hat{p})\hat{q}_k + g_k(\hat{p})\hat{q}_i\right) \\
&= 0.
\end{aligned}$$

For a not identically null S_{ij}, the Jacobi identity imposes

$$\begin{aligned}
0 &= \frac{1}{\hbar^2}\left([\hat{q}_i, [\hat{q}_j, \hat{q}_k]] + [\hat{q}_j, [\hat{q}_k, \hat{q}_i]] + [\hat{q}_k, [\hat{q}_i, \hat{q}_j]]\right) \\
&= f(\hat{p})\frac{\partial S_{jk}}{\partial p_i}(\hat{p}) - g_k(\hat{p})S_{ij}(\hat{p}) + g_j(\hat{p})S_{ik}(\hat{p}) + f(\hat{p})\frac{\partial S_{ki}}{\partial p_j}(\hat{p}) \\
&\quad - g_i(\hat{p})S_{jk}(\hat{p}) + g_k(\hat{p})S_{ji}(\hat{p}) + f(\hat{p})\frac{\partial S_{ij}}{\partial p_k}(\hat{p}) - g_j(\hat{p})S_{ki}(\hat{p}) + g_i(\hat{p})S_{kj}(\hat{p}) \\
&= f(\hat{p})\left(\frac{\partial S_{ij}}{\partial p_k}(\hat{p}) + \frac{\partial S_{jk}}{\partial p_i}(\hat{p}) + \frac{\partial S_{ki}}{\partial p_j}(\hat{p})\right) \\
&\quad - 2\left(g_k(\hat{p})S_{ij}(\hat{p}) + g_i(\hat{p})S_{jk}(\hat{p}) + g_j(\hat{p})S_{ki}(\hat{p})\right),
\end{aligned}$$

which is the strange equation (9.2.18). If S_{ij} fulfil the classic equation, the Jacobi identity is verified and no ambiguities in the ordering arise in the final expression,

where everything is just a function of the momentum operator.
Nevertheless, the choice of the ordering of the L_{ij} operators is relevant for this last Jacobi identity, even after S_{ij} is fixed by solving the strange equation. That any ordering might lead to the same result and thus verify the validity of the latter Jacobi identity is neither clear nor straightforward to confirm in the general case. However, we can notice that in the other possible common choice with all the momenta on the right, the Jacobi identity is still achieved.

The converse, namely if the quantum commutators of a GUP in the class (9.1.1) satisfy the Jacobi identities then the Poisson brackets of its classical interpretation satisfy them too, is trivial.

9.4 Maggiore GUP as an application

The mathematical relations we have derived in Section 9.2 are completely general and are based only on the request that the 2-form ω (9.1.4) is symplectic, without further assumptions on the f and L_{ij} functions.
Nevertheless, usually, in constructing a GUP theory, other constraints are imposed on the general algebra (9.1.1). The most common additional demands include the rotational invariance of the algebra and the condition that the L_{ij} functions satisfy an angular momentum algebra.
This specific subclass of GUP theories is extensively discussed in [130, 81] within the quantum framework, where formal and physical arguments supporting this choice are presented, showing in particular how it still maintains a certain level of generality. In these works, the authors consider the following structure for the GUP algebra:

$$\begin{aligned} [\hat{p}_i, \hat{p}_j] &= 0, \\ [\hat{q}_i, \hat{q}_j] &= \frac{\hbar^2}{\kappa^2 c^2} a(\hat{p}) i\epsilon_{ijk} \hat{J}_k, \\ [\hat{q}_i, \hat{p}_j] &= i\hbar f(\hat{p})\delta_{ij}. \end{aligned} \tag{9.4.1}$$

where J_k satisfies

$$[\hat{J}_i, \hat{q}_j] = i\epsilon_{ijk}\hat{q}_k, \quad [\hat{J}_i, \hat{p}_j] = i\epsilon_{ijk}\hat{p}_k, \quad [\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k,$$

a and f depend on p via its modulus $\rho = \sqrt{p_1^2 + p_2^2 + p_3^2}$ only, while κ is a deformation parameter with the dimension of a mass, and c is the speed of light in vacuum.

By means of Jacobi identities, they can find two equations relating f and a:

$$\frac{da(\hat{\rho})}{d\rho}\,\hat{\vec{p}}\cdot\hat{\vec{J}} = 0, \tag{9.4.2}$$

$$f(\hat{\rho})\frac{df(\hat{\rho})}{d\rho} = -\frac{a(\hat{\rho})\hat{\rho}}{\kappa^2 c^2}. \tag{9.4.3}$$

Two cases can be distinguished:

- $\hat{\vec{p}}\cdot\hat{\vec{J}} = 0$. This case arises when $\hat{\vec{J}}$ is regarded only as orbital angular momentum. Due to the zero scalar product, the constraint (9.4.2) is removed, leaving the choice of a arbitrary. Fixed a suitable a, it is possible to determine correctly f or vice versa.

- $\hat{\vec{p}}\cdot\hat{\vec{J}} \neq 0$. This is the most general situation, happening when we are considering $\hat{\vec{J}}$ as a total angular momentum, including spin.
  In this case (9.4.2) sets $a(\hat{p}) = \pm k$, where k is a constant. By rescaling, we can fix $a(\hat{p}) = \pm 1$ and from (9.4.3) we obtain, modulo a sign, a unique solution for f, namely $f(\hat{p}) = \sqrt{1 \mp \frac{\hat{\rho}^2}{\kappa^2 c^2}}$.
  Once again, we have set the integration constant equal to one for the reason discussed above.

The final algebra then reads:

$$\begin{aligned}
[\hat{q}_i, \hat{q}_j] &= \mp\frac{\hbar^2}{\kappa^2 c^2} i\epsilon_{ijk}\hat{J}_k, \\
[\hat{q}_i, \hat{p}_j] &= i\hbar\delta_{ij}\sqrt{1 \pm \tfrac{\hat{\rho}^2}{\kappa^2 c^2}}.
\end{aligned}$$

As it should be clear from the derivation, these specific GUP models acquire relevance since they are the most general models, defined only by the imposition of Jacobi identities and the inclusion of the spin operator. We stress that the generality of the model has to be intended in the sense specified above.

In light of this, in this section, we aim to discuss this particular scheme at the classical level, further constrain the general relations for f and L_{ij} we have obtained only asking for the ω to be symplectic. Exactly as in [129, 81], we limit our discussion to the 3-dimensional case.

9.4.1 Closure condition and Maggiore algebra

Consider the corresponding classical interpretation of the algebra (9.4.1):

$$\begin{aligned}
\{p_i, p_j\} &= 0, \\
\{q_i, q_j\} &= a(p)\epsilon_{ijk}J_k, \\
\{q_i, p_j\} &= f(p)\delta_{ij}.
\end{aligned} \tag{9.4.4}$$

Note that now J_i is a dimension-full quantity, and we have absorbed the term $(\kappa c)^{-2}$ in the definition of a. We demand that f and a depend on the momenta modulus ρ only, and that the J_is satisfy the algebra of an angular momentum, that is

$$\{J_i, q_j\} = \epsilon_{ijk}q_k, \quad \{J_i, p_j\} = \epsilon_{ijk}p_k, \quad \{J_i, J_j\} = \epsilon_{ijk}J_k.$$

With specific reference to the Poisson brackets above, the previously derived system of equations (9.2.19) coming from the closure condition, yields the following relations:

$$\begin{cases}
(145) & \frac{\partial f}{\partial p_2} + a(p)\frac{\partial J_3}{\partial q_1} = 0, \\
(146) & \frac{\partial f}{\partial p_3} - a(p)\frac{\partial J_2}{\partial q_1} = 0, \\
(156) & \frac{\partial J_1}{\partial q_1} = 0, \\
(245) & \frac{\partial f}{\partial p_1} - a(p)\frac{\partial J_3}{\partial q_2} = 0, \\
(246) & \frac{\partial J_2}{\partial q_2} = 0, \\
(256) & \frac{\partial f}{\partial p_3} + a(p)\frac{\partial J_1}{\partial q_2} = 0, \\
(345) & \frac{\partial J_3}{\partial q_3} = 0, \\
(346) & \frac{\partial f}{\partial p_1} + a(p)\frac{\partial J_2}{\partial q_3} = 0, \\
(356) & \frac{\partial f}{\partial p_2} - a(p)\frac{\partial J_1}{\partial q_3} = 0.
\end{cases} \tag{9.4.5}$$

By the same considerations developed for the system (9.2.19), we can obtain the following general structure for the Js functions:

$$\begin{aligned} J_1(q,p) &= s_1(p) - R(p)q_2 + P(p)q_3, \\ J_2(q,p) &= s_2(p) + R(p)q_1 - Q(p)q_3, \\ J_3(q,p) &= s_3(p) - P(p)q_1 + Q(p)q_2. \end{aligned} \tag{9.4.6}$$

By imposing an additional structure, namely the angular momentum algebra on the Js functions, we expect to be able to further constrain the previous expressions and relations.
As a first thing, we are going to explicitly evaluate the Poisson brackets $\{J_m, p_l\}$. This procedure leads to the following result:

$$\epsilon_{mlk}p_k = \{J_m, p_l\} = \frac{\partial J_m}{\partial q_l} f(p). \tag{9.4.7}$$

From here, using the Js formulae (9.4.6), we consistently find:

$$R(p) = -\frac{p_3}{f(p)}, \quad P(p) = -\frac{p_2}{f(p)}, \quad Q(p) = -\frac{p_1}{f(p)}. \tag{9.4.8}$$

This allows us to rewrite the functions (9.4.6) as:

$$\begin{aligned} J_1(q,p) &= s_1(p) + \tfrac{1}{f(p)}(p_3q_2 - p_2q_3), \\ J_2(q,p) &= s_2(p) + \tfrac{1}{f(p)}(p_1q_3 - p_3q_1), \\ J_3(q,p) &= s_3(p) + \tfrac{1}{f(p)}(p_2q_1 - p_1q_2). \end{aligned} \tag{9.4.9}$$

Accordingly, the gradient equation for f that can be extracted from the system (9.4.5) becomes:

$$\begin{cases} \frac{\partial f}{\partial p_1} + \frac{a(p)}{f(p)}p_1 = 0, \\ \frac{\partial f}{\partial p_2} + \frac{a(p)}{f(p)}p_2 = 0, \\ \frac{\partial f}{\partial p_3} + \frac{a(p)}{f(p)}p_3 = 0. \end{cases} \tag{9.4.10}$$

We now proceed in calculating explicitly the set of Poisson brackets $\{J_m, q_l\}$, which leads to the following relation:

$$\epsilon_{mlk}q_k = \{J_m, q_l\} = -\frac{\partial J_m}{\partial p_l} f(p) + a(p)\frac{\partial J_m}{\partial q_i}\epsilon_{ilk}J_k. \tag{9.4.11}$$

By means of the expressions (9.4.9) for the Js functions, we obtain a system of equations, which we can compactly write as:

$$-\frac{\partial s_m}{\partial p_l} f + \frac{a}{f}(s_m p_l - s_i p_j \delta_{ij}\delta_{ml}) + \epsilon_{ijm}q_i p_j \left(\frac{1}{f}\frac{\partial f}{\partial p_l} + \frac{a}{f^2}p_l\right) = 0.$$

This set of equations can be simplified by considering the system (9.4.10), leading to the following partial differential equations system:

$$\begin{cases} f^2(p)\frac{\partial s_1}{\partial p_1} + a(p)(s_2(p)p_2 + s_3(p)p_3) = 0, \\ f^2(p)\frac{\partial s_2}{\partial p_2} + a(p)(s_1(p)p_1 + s_3(p)p_3) = 0, \\ f^2(p)\frac{\partial s_3}{\partial p_3} + a(p)(s_1(p)p_1 + s_2(p)p_2) = 0, \\ f^2(p)\frac{\partial s_1}{\partial p_2} - a(p)p_2 s_1(p) = 0, \\ f^2(p)\frac{\partial s_1}{\partial p_3} - a(p)p_3 s_1(p) = 0, \\ f^2(p)\frac{\partial s_2}{\partial p_1} - a(p)p_1 s_2(p) = 0, \\ f^2(p)\frac{\partial s_2}{\partial p_3} - a(p)p_3 s_2(p) = 0, \\ f^2(p)\frac{\partial s_3}{\partial p_1} - a(p)p_1 s_3(p) = 0, \\ f^2(p)\frac{\partial s_3}{\partial p_2} - a(p)p_2 s_3(p) = 0. \end{cases} \tag{9.4.12}$$

Finally, we carry out the calculations regarding the last set of Poisson brackets for the angular momentum algebra:

$$\epsilon_{mnl}J_l = \{J_m, J_n\} = \left(\frac{\partial J_m}{\partial q_i}\frac{\partial J_n}{\partial p_j} - \frac{\partial J_m}{\partial p_i}\frac{\partial J_n}{\partial q_j}\right)\delta_{ij}f + a\frac{\partial J_m}{\partial q_i}\frac{\partial J_n}{\partial q_j}\epsilon_{ijk}J_k. \tag{9.4.13}$$

Due to the antisymmetry of the brackets, we need to evaluate just three of them:

$$\begin{aligned} J_3 = \{J_1, J_2\} &= \frac{a(p)}{f(p)^2}p_3\,\vec{p}\cdot\vec{J} + J_3 - s_3(p) + p_3\left(\frac{\partial s_1}{\partial p_1} + \frac{\partial s_2}{\partial p_2}\right) - p_2\frac{\partial s_2}{\partial p_3} - p_1\frac{\partial s_1}{\partial p_3}, \\ J_1 = \{J_2, J_3\} &= \frac{a(p)}{f(p)^2}p_1\,\vec{p}\cdot\vec{J} + J_1 - s_1(p) + p_1\left(\frac{\partial s_2}{\partial p_2} + \frac{\partial s_3}{\partial p_3}\right) - p_3\frac{\partial s_3}{\partial p_1} - p_2\frac{\partial s_2}{\partial p_3 1}, \\ J_2 = \{J_3, J_1\} &= \frac{a(p)}{f(p)^2}p_2\,\vec{p}\cdot\vec{J} + J_2 - s_2(p) + p_2\left(\frac{\partial s_1}{\partial p_1} + \frac{\partial s_3}{\partial p_3}\right) - p_3\frac{\partial s_3}{\partial p_2} - p_1\frac{\partial s_1}{\partial p_2}. \end{aligned}$$

Taking now into account the relations established by the system (9.4.12) and the derived form (9.4.9) of the Js functions, we can rewrite these conditions on the Poisson brackets as a simple algebraic system for the s_i:

$$\begin{cases} \frac{a}{f^2}p_1\,\vec{p}\cdot\vec{s} + s_1 = 0, \\ \frac{a}{f^2}p_2\,\vec{p}\cdot\vec{s} + s_2 = 0, \\ \frac{a}{f^2}p_3\,\vec{p}\cdot\vec{s} + s_3 = 0. \end{cases} \tag{9.4.14}$$

As a homogeneous linear system with respect to the vector $\vec{s}$, its determinant is $1 + \frac{a(p)}{f(p)^2}(p_1^2 + p_2^2 + p_3^2)$. The singular points are represented by the equation:

$$a(p)(p_1^2 + p_2^2 + p_3^2) = -f^2(p). \tag{9.4.15}$$

Clearly, this equation admits a solution only for $a(p) < 0$. If it is satisfied in an open subset of $\mathbb{R}^3$, then we can put the $a(p)$ derived from the above equation in the gradient equation (9.2.20). As a result, we obtain a unique solution for f, which is $f(p) = c\sqrt{p_1^2 + p_2^2 + p_3^2}$. This solution is incompatible with the undeformed limit, since there is no possible dependence of c on the deformation parameter β which results in $f \xrightarrow[\beta\to 0]{} 1$.

As a consequence, the equation (9.4.15) can be satisfied only on a submanifold with at least codimension one. Where in $\mathbb{R}^3$ the determinant is not singular, the unique solution of (9.4.14) is $\vec{s} = 0$. Hence, since s_i is a smooth function everywhere null but on a closed surface, for continuity it must be zero everywhere in $\mathbb{R}^3$.
It is trivial to verify that this solution is compatible with the system (9.4.12).

Now that we have further specified the form of the Js functions, we analyse the remaining condition to be satisfied to have the closure of the 2-form ω. This is exactly the Eq. (9.2.17), the specific form of which in this context is:

$$2\frac{a^2}{f}\left(J_1p_1 + J_2p_2 + J_3p_3\right) + f\left(\frac{\partial(aJ_1)}{\partial p_1} + \frac{\partial(aJ_2)}{\partial p_2} + \frac{\partial(aJ_3)}{\partial p_3}\right) = 0. \tag{9.4.16}$$

By employing the last obtained form of the Js we get:

$$J_1\frac{\partial a}{\partial p_1} + J_2\frac{\partial a}{\partial p_2} + J_3\frac{\partial a}{\partial p_3} = 0. \tag{9.4.17}$$

If we now recall our assumption of a depending only on the modulus $\rho = |\vec{p}|$, we finally obtain the equation:

$$\vec{p}\cdot\vec{J}\,\frac{\partial a}{\partial \rho} = 0, \tag{9.4.18}$$

which is the same of (9.4.2).
Nevertheless, due to the explicit form of the Js and specifically to the fact that the s_i functions have to be zero, it is immediate to show that $\vec{p}\cdot\vec{J}$ is always equal to zero in this context. This fact leads us to conclude that (9.4.18) is always satisfied for any smooth, rotational invariant $a(\rho)$ and rules out the case in which $\vec{p}\cdot\vec{J} \neq 0$. From here, we can also deduce the potential role played by the functions s_i, as the only part of the Js able to ensure the condition $\vec{p}\cdot\vec{J} \neq 0$. As a consequence, we conclude that the generality of the solution $f(p) = \sqrt{1 \mp \rho^2}$ in the quantum framework seems to be lost at the classical level, being just one of the possible GUP models.

Finally, we can write down the general solution for f by examining the gradient equation (9.4.10). Being $a(\rho)$ rotational invariant and considering that ρ is the length $\rho = \sqrt{p_1^2 + p_2^2 + p_3^2}$, the system (9.4.10) can be recast in a gradient equation for $g = f^2$ in $\vec{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$, completely equivalent to (9.4.3):

$$\nabla_{\vec{p}}\, g = -2a(\rho)\vec{p}. \tag{9.4.19}$$

Clearly, the right-hand side is irrotational

$$\nabla_{\vec{p}} \times (a(\rho)\vec{p}) = a(\rho)(\nabla_{\vec{p}} \times \vec{p}) + (\nabla_{\vec{p}}\, a) \times \vec{p} = \frac{1}{\rho}\frac{\partial a}{\partial \rho}\vec{p} \times \vec{p} = 0.$$

Then, if $a(\rho)$ is defined on a simply connected domain, the system admits a solution:

$$f(p) = \left(-2\int_\gamma a(p')\vec{p}' \cdot d\vec{p}'\right)^{\frac{1}{2}} = \left(-2\int_0^{\sqrt{p_1^2+p_2^2+p_3^2}} a(\rho)\rho d\rho + c\right)^{\frac{1}{2}}. \tag{9.4.20}$$

9.4.2 Kempf-Mangano-Mann case in Maggiore's scheme

One of the most studied GUP algebras is the one first presented in [117].
In Section 9.2.2 we have already shown how this particular algebra can be recovered by choosing the L_{ij} functions as proportional to the ordinary angular momentum, and in Section 9.2.3 we were able to derive its algebra by fixing the f function and discussing the limit $\beta \to 0$. Precisely, the full agreement is obtained by setting $S_{ij} \equiv 0$ and the integration constant c equal to one.

Here we briefly show how the KMM algebra can be reproduced in Maggiore's scheme as well, unveiling the presence of the rotation generators in the modified Poisson brackets of the configuration variables.
Starting from the gradient equation (9.4.10) in the form:

$$\nabla_{\vec{p}} f + \frac{a}{f}\vec{p} = 0. \tag{9.4.21}$$

We can fix the desired f and solve algebraically the system for a. Hence, setting $f(\rho) = 1 + \beta\rho^2$, we easily obtain $a(\rho) = -2\beta(1 + \beta\rho^2)$. According to the algebra (9.4.4), we should have:

$$\{q_i, q_j\} = a(\rho)\epsilon_{ijk}J_k = -2\beta(1 + \beta\rho^2)\epsilon_{ijk}J_k. \tag{9.4.22}$$

At the same time, we know the exact form of the J_k functions from (9.4.9):

$$J_k = \frac{1}{2f(p)}\epsilon_{ijk}(q_i p_j - q_j p_i) = \frac{1}{1 + \beta\rho^2}\epsilon_{ijk} q_i p_j. \tag{9.4.23}$$

Putting everything together, we finally obtain:

$$\{q_i, q_j\} = -2\beta(q_i p_j - q_j p_i), \tag{9.4.24}$$

which is the correct expression of the KMM algebra at the classical level and the same formula obtained in Section 9.2.3, with the identification previously mentioned, i.e. $S_{ij} = 0$ and $c = 1$.

9.5 Deformed algebra with a single constraint

We aim to analyse the consistency of the deformation of the Heisenberg algebra in the setting of constrained Hamiltonian systems, providing a procedure to induce the deformation on the Poisson algebra after symplectic reduction. We investigate this within the classical interpretation of Generalised Uncertainty Principle theories, treating two cases separately.

In the first case, we consider a group action on the phase space together with a set of first-class constraints that can be interpreted as a momentum map. We provide an explicit example involving rotationally invariant deformed algebras. In the second case, we consider a single constraint given by the Hamiltonian, which is a common situation in General Relativity, with direct applications in Cosmology.

The framework for this investigation is provided by the construction of symplectic reduction. Recall that, given a symplectic action of a Lie group G on a symplectic manifold $\mathcal{P}$ with symplectic form ω, a momentum map for the action may exist. That is, a map $\mu : \mathcal{P} \to \mathfrak{g}^*$ such that:

1. $d\mu(\xi) = \varrho(\xi) \lrcorner\, \omega$ for all $\xi \in \mathfrak{g}$;

2. $\mu \circ g = \mathrm{Ad}^*_g \mu$ for all $g \in G$.

That is, the action is Hamiltonian and the momentum map is G-equivariant. Here, ϱ denotes the Lie algebra homomorphism $\varrho : \mathfrak{g} \to \mathfrak{X}(\mathcal{P})$ induced by the group action.

If G acts freely and properly on $\mu^{-1}(0)$, then 0 is a regular value of μ, and the quotient $\mu^{-1}(0)/G$ is a smooth manifold. This quotient inherits a symplectic form from $\mathcal{P}$; specifically, there exists a unique symplectic form on the quotient whose pullback to $\mu^{-1}(0)$ equals the restriction of ω to $\mu^{-1}(0)$. The resulting symplectic manifold is called the *symplectic reduction* of $\mathcal{P}$ by G and is denoted $\mathcal{P} /\!\!/ G$.

In the classical interpretation of GUP theories, the deformation of the algebra is encoded in the fact that the physical position q and momentum p are interpreted as coordinates on a symplectic manifold that do not form a Darboux frame. However, since Darboux coordinates always exist locally on a symplectic manifold, it becomes necessary to characterise the coordinate system in which the symplectic form is evaluated on the reduced manifold, starting from the original coordinate frame.

Definition 9.5.1. A coordinate system $(x_1, \dots, x_m)$ on $\mathcal{P} /\!\!/ G$ is said to be *natural* with respect to the classical interpretation of a GUP theory if it is possible to induce coordinates on $\mu^{-1}(0)$ of the form $(\theta_1, \dots, \theta_l)$ such that

$$(x_1, \dots, x_m) = \pi(\theta_1, \dots, \theta_l),$$

where π is the projection $\mu^{-1}(0) \to \mu^{-1}(0)/G = \mathcal{P} /\!\!/ G$, and where the parametrisation $(q_1(\theta), \dots, q_n(\theta), p_1(\theta), \dots, p_n(\theta))$ of $\mu^{-1}(0)$ in $\mathcal{P}$ is provided. Furthermore, if (q_i, p_i) form a Darboux frame when $f(p) = 1$ and $L_{ij} = 0$, then $(x_1, \dots, x_m)$ is required to be a Darboux frame for the induced symplectic form.

This definition generalises straightforwardly to the case in which there is no group action and the reduced manifold is simply a symplectic submanifold of $\mathcal{P}$.

9.5.1 Rotational-invariant deformed algebra in two dimensions

To illustrate the procedure, we begin with a simple case: a GUP particle in two dimensions with preserved angular momentum. Specifically, we consider a two-dimensional version of the Maggiore algebra with the constraint $J = 0$:

$$\begin{aligned} \{p_i, p_j\} &= 0, \\ \{q_i, q_j\} &= a(p)J, \\ \{q_i, p_j\} &= f(p)\delta_{ij}, \end{aligned} \tag{9.5.1}$$

where J is the generator of rotations, satisfying

$$\{J, p_i\} = \epsilon_{ij} p_j, \qquad \{J, q_i\} = \epsilon_{ij} q_j.$$

It can be expressed explicitly as

$$J = \frac{1}{f(p)}(q_1 p_2 - q_2 p_1). \tag{9.5.2}$$

To ensure rotational invariance, it is known that the functions $a(p)$ and $f(p)$ must depend only on the modulus $|p|$.

We consider as phase space the manifold $\mathcal{P} = \mathrm{T}^*(\mathbb{R}^2 \setminus \{0\}) \cong (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$, equipped with a symplectic form in the coordinate system (q_1, q_2, p_1, p_2) given by

$$\omega_{ab} = \frac{1}{f}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \frac{a}{f}J \\ 0 & 1 & -\frac{a}{f}J & 0 \end{pmatrix}. \tag{9.5.3}$$

Where the functions a and f must satisfy the following differential equation

$$f\frac{\partial f}{\partial p_i} = -ap_i. \tag{9.5.4}$$

The action of $SO(2)$ on this space is the usual one, and it is represented by the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

The generator of rotations is the candidate to be the momentum map for this action: the coadjoint action is trivial because $SO(2)$ is Abelian, indeed J is invariant under rotations. We can check that this action is Hamiltonian. The Lie algebra $\mathfrak{so}(2)$ is a real 1-dimensional Abelian Lie algebra, hence, it is isomorphic to $\mathbb{R}$. The generator of this algebra is the matrix $\xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the corresponding vector field is

$$\varrho(\xi) = q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2} + p_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial p_2}. \tag{9.5.5}$$

From a simple computation in coordinates, we can check that $\varrho(\xi) \lrcorner\, \omega$ is equal to

$$dJ = \frac{1}{f}\left(p_2dp_1 - p_1dp_2 - (q_2 + \frac{a}{f^2}p_1J)dp_1 + (q_1 + \frac{a}{f^2}p_2J)dp_2\right). \tag{9.5.6}$$

We now study the orbit space defined by the action of $SO(2)$ on the hypersurface $J = 0$. Since $SO(2)$ is compact, its action on $\mathcal{P}$ is proper. Moreover, the action is free because the origin $0 \in \mathbb{R}^4$ is excluded from the phase space. The map $J : \mathcal{P} \to \mathbb{R}$ has constant rank equal to 1, and thus it is a smooth submersion. Consequently, the level set $J = 0$ is a properly embedded submanifold of codimension 1 in $\mathcal{P}$.

Furthermore, the condition $J = 0$ is equivalent to $q \times p = 0$, which forces the vectors q and p to be colinear. Therefore, the constraint surface

$$S = \{(q, p) \in \mathcal{P} \mid J(q, p) = 0\}$$

is a vector bundle over the base manifold $\mathbb{R}^2 \setminus \{0\}$ with fibre $\mathbb{R}$. In fact, it can be equivalently written as

$$S = \{(q, p) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \mid p \in Span(q) \subset \mathbb{R}^2\}, \tag{9.5.7}$$

which clearly defines a submanifold of dimension 3. This vector bundle is trivial, since for any fixed $\alpha \in \mathbb{R} \setminus \{0\}$, the map $q \mapsto (q, \alpha q)$ defines a global nowhere-vanishing section.

To better understand the global structure, let us focus on the vector bundle description. The constraint surface S inherits the $SO(2)$ action from the ambient phase space. This action maps fibres to fibres, making S a $SO(2)$-invariant vector bundle. Consider the trivialization of S induced by the section

$$\begin{array}{rccc} \sigma: & \mathbb{R}^2 \setminus \{0\} & \to & S; \\ & q & \mapsto & (q, q), \end{array} \tag{9.5.8}$$

which is clearly $SO(2)$-equivariant. In this trivialization, S is identified with the product bundle $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}$, with the group action given by

$$(q, \alpha) \mapsto (R(\theta) q, \alpha),$$

where $R(\theta)$ is the standard rotation matrix in the defining representation of $SO(2)$. Hence, the quotient space is given by

$$\Big((\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R} \Big) / SO(2) \cong \mathbb{R}_+ \times \mathbb{R},$$

so the topology of the symplectic reduction is

$$\mathcal{P} /\!\!/ SO(2) \cong \mathbb{R}_+ \times \mathbb{R}. \tag{9.5.9}$$

This suggests that the reduced phase space can be covered by a global coordinate chart.

We parametrize the constraint surface by introducing coordinates $r > 0$, $\rho \in \mathbb{R}$, and $\theta \in [0, 2\pi)$ as

$$q_1 = r\cos\theta, \quad q_2 = r\sin\theta, \quad p_1 = \rho\cos\theta, \quad p_2 = \rho\sin\theta.$$

The projection $\pi : J^{-1}(0) \to \mathcal{P} /\!\!/ SO(2)$ is then simply given by $\pi(r, \rho, \theta) = (r, \rho)$, so that (r, ρ) define a global coordinate system on $\mathcal{P} /\!\!/ SO(2)$. The symplectic form on the quotient is of the form $\tilde{\omega} = w(r, \rho)\, dr \wedge d\rho$, and its pullback by π is

$$\pi^* \tilde{\omega} = w(r, \rho)\, dr \wedge d\rho.$$

To determine the function $w(r, \rho)$, we impose the condition $\omega|_{J^{-1}(0)} = \pi^* \tilde{\omega}$. Using the parametrisation of the constraint surface, we compute:

$$\begin{aligned} \omega|_{J^{-1}(0)} &= \left(-\frac{1}{f} \delta_{ij}\, dq^i \wedge dp^j + \frac{a}{f} J\, dp^1 \wedge dp^2 \right) \Big|_{J^{-1}(0)} \\ &= -\frac{1}{f(\rho)} \left(d(r\cos\theta) \wedge d(\rho\cos\theta) + d(r\sin\theta) \wedge d(\rho\sin\theta) \right) \\ &= -\frac{1}{f(\rho)}\, dr \wedge d\rho. \end{aligned} \tag{9.5.10}$$

Therefore, the symplectic form on $\mathcal{P} /\!\!/ SO(2)$ is

$$\tilde{\omega} = -\frac{1}{f(\rho)}\, dr \wedge d\rho. \tag{9.5.11}$$

It is important to note that one might be tempted to define $r = \sqrt{q_1^2 + q_2^2}$ and $\rho = \sqrt{p_1^2 + p_2^2}$. However, this is not correct: in that case, (r, ρ) would define only local coordinates on $\mathcal{P} /\!\!/ SO(2)$. In fact, they cover only the open subset of the symplectic reduction corresponding to points in S of the form $p = \alpha q$ with $\alpha > 0$ (or, alternatively, $\alpha < 0$). Not only does ρ fail to be smooth at $\alpha = 0$, but these coordinates cannot distinguish the orbit of the point $(q, \alpha q)$ from that of $(q, -\alpha q)$, despite being constant along the orbits.

A possible solution to this issue is to define coordinates on S using the following charts. On the open set where $q_1 \neq 0$, define:

$$r = \sqrt{q_1^2 + q_2^2}, \quad \rho = \operatorname{sgn}\left(\frac{p_1}{q_1}\right) \sqrt{p_1^2 + p_2^2}.$$

Similarly, on the open set where $q_2 \neq 0$, define:

$$r = \sqrt{q_1^2 + q_2^2}, \quad \rho = \operatorname{sgn}\left(\frac{p_2}{q_2}\right) \sqrt{p_1^2 + p_2^2}.$$

These two charts together cover the entire symplectic quotient. On their overlap (i.e., when both $q_1 \neq 0$ and $q_2 \neq 0$), they agree, and the expression for the symplectic form remains unchanged.

It is not immediately obvious that these maps are smooth, especially at points where $p_1 = 0$ or $p_2 = 0$. However, since p and q are colinear on S, we can write $p_1 = \alpha q_1$ and $p_2 = \alpha q_2$, for some $\alpha \in \mathbb{R}$. Then:

$$\rho = \operatorname{sgn}\left(\frac{p_1}{q_1}\right) \sqrt{(\alpha q_1)^2 + (\alpha q_2)^2} = \operatorname{sgn}\left(\frac{p_1}{q_1}\right) |\alpha| \sqrt{q_1^2 + q_2^2} = \frac{p_1}{q_1} r,$$

which shows that the map is indeed smooth. An analogous computation holds for the second chart using $\alpha = \frac{p_2}{q_2}$. On their common domain, the change of coordinates is smooth, as it is simply the identity map.

We thus see how the deformation of the algebra is inherited by the symplectic quotient when choosing a coordinate system that is naturally induced by the original one.

9.5.2 Rotational-invariant deformed algebra in three dimensions

We now provide a similar analysis for the three-dimensional case, in which the phase space is $\mathcal{P} = \mathrm{T}^*(\mathbb{R}^3 \setminus \{0\}) \cong (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$, endowed with the Poisson structure

$$\begin{aligned} &\{p_i, p_j\} = 0, \\ &\{q_i, q_j\} = a(p)\epsilon_{ijk} J_k, \\ &\{q_i, p_j\} = f(p)\delta_{ij}, \\ &\text{where } J_i = \frac{1}{f}\epsilon_{ijk} q_j p_k. \end{aligned} \tag{9.5.12}$$

Which, from a previous analysis, is the most generic form for a rotational-invariant GUP algebra. An element $g \in SO(3)$ acts via the rotation matrix $R(g)$ in the defining representation as $(q, p) \mapsto (R(g)q, R(g)p)$. The Lie algebra $\mathfrak{so}(3)$ is spanned by the matrices T_i, with dual basis $\mathfrak{t}^i$. The momentum map for this action is given by

$$\begin{array}{rccc} \mu: & \mathcal{P} & \to & \mathfrak{so}(3)^*; \\ & (q,p) & \mapsto & J_i \mathfrak{t}^i. \end{array} \tag{9.5.13}$$

Again, the functions $a(p)$ and $f(p)$ must depend only on the modulus of p. Thus, the $SO(3)$-equivariance of the momentum map yields

$$\begin{aligned} J_i(R(g)q, R(g)p)\mathfrak{t}^i &= \frac{1}{f(Rp)} \epsilon_{ijk} R_{jl} R_{km} q_l p_m \mathfrak{t}^i \\ &= \frac{1}{f(p)} R_{il} \epsilon_{ljk} q_j p_k \mathfrak{t}^i = J_i(q,p) \operatorname{Ad}_g^* \mathfrak{t}^i. \end{aligned}$$

Considering an element $\xi = \xi^i T_i$ in the Lie algebra $\mathfrak{so}(3)$, the corresponding vector field on $\mathcal{P}$ is

$$\varrho(\xi) = \epsilon_{ijk} \xi^i q_k \frac{\partial}{\partial q_j} + \epsilon_{ijk} \xi^i p_k \frac{\partial}{\partial p_j}. \tag{9.5.14}$$

Note that ξ^i are constants. A straightforward computation in linear algebra shows that $\varrho(\xi) \lrcorner\, \omega$ is equal to

$$\xi^i dJ_i = \frac{1}{f} \left(\epsilon_{ijk} \xi^i q_j dp_k - \epsilon_{ijk} \xi^i p_k dq_j + \frac{a}{f} \xi^i J_i p_j dp_j \right). \tag{9.5.15}$$

The constraint surface $\mu^{-1}(0)$ is defined by the three equations $J_i = 0$. The action is proper since $SO(3)$ is compact, but it is not free because whenever q and p are collinear, the stabiliser is isomorphic to $SO(2)$. Moreover, 0 is not a regular value for this map. Indeed, in coordinates, the tangent map $d_p\mu : \mathrm{T}_p\mathcal{P} \to \mathrm{T}_0\mathfrak{so}(3)^* \cong \mathbb{R}^3$, for $p \in \mu^{-1}(0)$, is given by

$$\begin{pmatrix} 0 & -p_3 & p_2 & 0 & -q_3 & q_2 \\ p_3 & 0 & -p_1 & q_3 & 0 & -q_1 \\ -p_2 & p_1 & 0 & -q_2 & q_1 & 0 \end{pmatrix}. \tag{9.5.16}$$

By the *bordered minors theorem*, it follows that $\operatorname{rk}(d\mu) = 2$ on $\mu^{-1}(0)$. However, the rank is constant on $\mu^{-1}(0)$, as is the dimension of the stabiliser group at each point. The constraint surface is thus a smooth 4-dimensional manifold, which is a vector bundle over $\mathbb{R}^3 \setminus \{0\}$ with fibre $\mathbb{R}$. As in the 2-dimensional case, the condition $\mu(q, p) = 0$ is equivalent to $q \times p = 0$, which forces q and p to be colinear. Hence,

$$S = \{(q,p) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3 \mid p \in \operatorname{Span}(q)\} \cong \mu^{-1}(0) \tag{9.5.17}$$

is a trivial vector bundle: $S \cong (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$. Using similar arguments as in the 2-dimensional case, we deduce the topology of the symplectic quotient $\mathcal{P} /\!\!/ SO(3)$. The constraint surface is an $SO(3)$-invariant vector bundle. The section

$$\begin{array}{rccc} \sigma: & \mathbb{R}^3 \setminus \{0\} & \to & S; \\ & q & \mapsto & (q,q), \end{array} \tag{9.5.18}$$

is an equivariant section, since the stabiliser of q coincides with that of $\sigma(q)$ as subgroups of $SO(3)$. Thus, the trivialisation induced by σ is an $SO(3)$-invariant vector bundle $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ with $SO(3)$ acting by $R(g) \times \mathrm{id}_{\mathbb{R}}$. The quotient of $\mathbb{R}^3$ by the defining representation of $SO(3)$ is well known, and we obtain

$$\mathcal{P} /\!\!/ SO(3) \cong ((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R})/SO(3) \cong \mathbb{R}_+ \times \mathbb{R}. \tag{9.5.19}$$

Moreover, every point in $\mu^{-1}(0)$ has a stabiliser conjugate to $SO(2)$, which is sufficient to ensure that $\mu^{-1}(0)$ is a manifold and that the quotient $\mu^{-1}(0)/SO(3)$ inherits a natural symplectic structure. By Theorem 2.1 in [158], the reduced space $\mathcal{P} /\!\!/ SO(3)$ consists of a unique stratum, which is naturally a symplectic manifold.

A possible parametrisation of the constraint surface is given by

$$\begin{aligned} q_1 &= r \sin\theta \cos\phi, & p_1 &= \rho \sin\theta \cos\phi, \\ q_2 &= r \sin\theta \sin\phi, & p_2 &= \rho \sin\theta \sin\phi, \\ q_3 &= r \cos\theta, & p_3 &= \rho \cos\theta, \end{aligned}$$

where $r > 0, \rho \in \mathbb{R}$, and $\theta \in [0, \pi], \phi \in [0, 2\pi)$. The induced form is $\omega|_{\mu^{-1}(0)} = \frac{1}{f(\rho)} d\rho \wedge dr$. Since the projection is $\pi(r, \rho, \theta, \phi) = (r, \rho)$, the symplectic structure on $\mathcal{P} /\!\!/ SO(3)$ is

$$\tilde{\omega} = \frac{1}{f(\rho)} d\rho \wedge dr. \tag{9.5.20}$$

Even in this physically relevant case, we can show that the deformation of the symplectic form is naturally preserved under symplectic reduction, thereby ensuring a well-posed application of the GUP framework to constrained Hamiltonian systems.

9.5.3 Hamiltonian Constraint in GUP

The case in which the theory possesses a single Hamiltonian constraint $H = 0$ must be treated differently. In this setting, we do not have the action of a Lie group. Although it is intuitive to interpret the dynamical evolution induced by the Hamiltonian vector field X_H as an $\mathbb{R}$-action, this is not strictly correct, since the vector field is not generally complete, and the associated flow cannot be defined for all times at all points. Consequently, the standard technique of symplectic reduction cannot be applied in this scenario. To address this problem, we propose an approach more closely related to gauge fixing and relational observables.

Let us consider a constrained Hamiltonian system $(\mathcal{P}, \omega, H)$, where $\mathcal{P}$ is a $(2n+2)$-dimensional symplectic manifold, ω is a symplectic form, which, in our preferred coordinate system $(q_1, \ldots, q_{n+1}, p_1, \ldots, p_{n+1})$, takes the specific form as in Def. 9.1.1, and H is the Hamiltonian function, which also defines the constraint $H = 0$. On the level set $S = \{x \in \mathcal{P} \mid H(x) = 0\}$, the Hamiltonian vector field is tangent, i.e. $X_H \in \mathfrak{X}(S)$. We aim to define our reduced symplectic manifold as a $2n$-dimensional symplectic submanifold $(N, \omega|_N)$ of S, such that X_H is nowhere tangent to N.

To induce a dynamics on N, a natural idea would be to consider the local foliation generated by evolving N under the flow of X_H. However, this does not yield a genuine dynamics on N, since the motion is identified with its initial data, i.e. the integral curve with its intersection point on N. Instead, we introduce a new vector field $T \in \mathfrak{X}(S)$, nowhere tangent to N, which will serve as an external time. The vector field T generates an integrable rank-1 distribution on $\mathrm{T}S$. By a corollary of the Frobenius theorem [124], for each $x \in N$ there exists an open neighbourhood U of x in S and coordinates $(x_0, x_1, \ldots, x_{2n})$, with $x_i \in (-\varepsilon, \varepsilon)$, in which $N \cap U$ is described by $x_0 = 0$ and $T = \frac{\partial}{\partial x_0}$. This gives rise to a natural local diffeomorphism for each fixed $x_0 = t$, namely:

$$F_t : U_{x_0=t} \to N, \quad F_t(x_0 = t, x_1, \ldots, x_{2n}) = (0, x_1, \ldots, x_{2n}).$$

Hence, we can project the Hamiltonian vector field X_H for each fixed time t, obtaining a family of time-dependent vector fields $X_t = dF_t(X_H)$ on N.

One can show that, under general regularity conditions for the Hamiltonian H and a specific assumption on the symplectic form,

$$\mathcal{L}_{X_t}\omega|_N = 0. \tag{9.5.21}$$

Thus, if N is simply connected, there exists a family of time-dependent Hamiltonians H_t on N such that $X_{H_t} = X_t$.

Proof. Suppose $H : \mathcal{P} \to \mathbb{R}$ is a constant-rank map with $\mathrm{rk}(H) = 1$. By the inverse function theorem, there exists a local coordinate chart $(y_0, y_1, \ldots, y_{2n}, x_{2n+1})$ on $\mathcal{M}$ such that the level set $S = H^{-1}(0)$ is locally described by $x_{2n+1} = f(y_0, \ldots, y_{2n})$. As discussed, we may change coordinates on S such that N is described by $x_0 = 0$. Composing these coordinate transformations yields a new system $(x_0, x_1, \ldots, x_{2n}, x_{2n+1})$ with the desired properties.

In this coordinate system, the symplectic form on $\mathcal{P}$ reads:

$$\omega = \omega_a\, dx^0 \wedge dx^a + \omega_{ab}\, dx^a \wedge dx^b + \mathring{\omega}_a\, dx^a \wedge dx^{2n+1} + \mathring{\omega}\, dx^0 \wedge dx^{2n+1},$$

where indices a, b range from 1 to $2n$.

Since X_H is tangent to S, we have $X_H|_S \lrcorner\, \omega|_S = (X_H \lrcorner\, \omega)|_S = (dH)|_S = 0$. Moreover:

$$\omega|_S = \left(\omega_a - \mathring{\omega}_a \frac{\partial f}{\partial x^0} + \mathring{\omega} \frac{\partial f}{\partial x^a}\right)\bigg|_S dx^0 \wedge dx^a + \left(\omega_{ab} + \mathring{\omega}_a \frac{\partial f}{\partial x^b}\right)\bigg|_S dx^a \wedge dx^b,$$
$$X_H|_S = X^a \frac{\partial}{\partial x^a} + X^0 \frac{\partial}{\partial x^0},$$

where X^0 and X^a depend only on $(x_0, \ldots, x_{2n})$.

The equation $X_H \lrcorner\, \omega = 0$ yields $2n + 2$ equations on S:

$$\begin{aligned} \frac{\partial H}{\partial x^0} &= -\tfrac{1}{2}\omega_a X^a, \\ \frac{\partial H}{\partial x^{2n+1}} &= \tfrac{1}{2}(\mathring{\omega} X^0 + \mathring{\omega}_a X^a), \\ \frac{\partial H}{\partial x^a} &= \tfrac{1}{2} X^0 \omega_a + \omega_{ba} X^b. \end{aligned}$$

The map $F_t : (x_0 = t, x_1, \ldots, x_{2n}) \mapsto (0, x_1, \ldots, x_{2n})$ provides the projected vector fields:

$$X_t = X^a|_{x_0=t} \frac{\partial}{\partial x^a}.$$

The induced symplectic form on N is simply the second term in $\omega|_S$, evaluated at $x_0 = 0$:

$$\omega|_N = \left(\omega_{ab} + \mathring{\omega}_a \frac{\partial f}{\partial x^b}\right)\Big|_{x_0=0} dx^a \wedge dx^b.$$

Therefore:

$$X_t \lrcorner\, \omega|_N = X^a \omega_{ab}\big|_{x_0=0} dx^b + \tfrac{1}{2}\left(\mathring{\omega}_a X^b \frac{\partial f}{\partial x^b} - \mathring{\omega}_b X^b \frac{\partial f}{\partial x^a}\right)\Big|_{x_0=0} dx^a.$$

Whenever $\omega_a = 0 = \mathring{\omega}_a$, we obtain:

$$X_t \lrcorner\, \omega|_N = \frac{\partial H}{\partial x^a}\Big|_{x_0=t} dx^a,$$

which is a closed form on N. By Cartan's formula, it follows that $\mathcal{L}_{X_t}\omega|_N = 0$, so X_t is symplectic. If N is simply connected, i.e. $H^1_{\mathrm{dR}}(N) = 0$, every closed 1-form is exact, and every symplectic vector field is Hamiltonian. □

From the proof, it is clear that the specific assumption on the symplectic form is that the flat coordinates satisfy $\{x^0, x^a\} = 0 = \{x^a, x^{2n+1}\}$.

Cosmology as a Constrained GUP Theory

An example of this procedure is provided by cosmology. We consider Bianchi models in Misner variables within a GUP framework. The phase space is $\mathcal{P} = \mathrm{T}^*\mathbb{R}^3$, with coordinates $(q_0, q_1, q_2, p_0, p_1, p_2)$, satisfying the following non-vanishing Poisson brackets:

$$\begin{aligned} \{q_i, p_j\} &= f(p)\delta_{ij}, \\ \{q_1, q_2\} &= l(p), \end{aligned} \tag{9.5.22}$$

for some suitable functions f and l, and with Hamiltonian

$$H = -p_0^2 + p_1^2 + p_2^2 + e^{4q_0} V(q_1, q_2), \tag{9.5.23}$$

where V is a positive smooth function characterising the model. The corresponding symplectic form in this coordinate system reads:

$$\omega_{ab} = \frac{1}{f}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{l}{f} \\ 0 & 0 & 1 & 0 & -\frac{l}{f} & 0 \end{pmatrix}. \tag{9.5.24}$$

By the closure condition, the function $l(p)$ must satisfy:

$$l(p) = \frac{\partial f}{\partial p_1} q_2 - \frac{\partial f}{\partial p_2} q_1.$$

We now analyse the dynamics. The Hamiltonian vector field is computed to be:

$$\begin{aligned} X_H = 2p_0 f\frac{\partial}{\partial q_0} - \left(2p_1 f + e^{4q_0} l\frac{\partial V}{\partial q_2}\right)\frac{\partial}{\partial q_1} - \left(2p_2 f - e^{4q_0} l\frac{\partial V}{\partial q_1}\right)\frac{\partial}{\partial q_2} \\ + 4e^{4q_0} fV\frac{\partial}{\partial p_0} + e^{4q_0} f\frac{\partial V}{\partial q_1}\frac{\partial}{\partial p_1} + e^{4q_0} f\frac{\partial V}{\partial q_2}\frac{\partial}{\partial p_2}. \end{aligned} \tag{9.5.25}$$

The constraint surface is a fibre bundle over the configuration space, with fibres given by two-sheeted hyperboloids. For each fixed q, the equation $p_0^2 = p_1^2 + p_2^2 + e^{4q_0} V(q_1, q_2)$ describes a hyperboloid; the positivity of V guarantees ellipticity. Since the base manifold is contractible, the bundle is trivial. Thus, we can adopt $(q_0, q_1, q_2, p_1, p_2)$ as a global coordinate system on the positive-energy branch:

$$S = \mathrm{Graph}\left(\sqrt{p_1^2 + p_2^2 + e^{4q_0} V(q_1, q_2)}\right).$$

On this surface, the Hamiltonian vector field becomes:

$$\begin{aligned} X_H|_S = 2\sqrt{p_1^2 + p_2^2 + e^{4q_0} V(q_1, q_2)} f\frac{\partial}{\partial q_0} + e^{4q_0} f\frac{\partial V}{\partial q_1}\frac{\partial}{\partial p_1} + e^{4q_0} f\frac{\partial V}{\partial q_2}\frac{\partial}{\partial p_2} \\ - \left(2p_1 f + e^{4q_0} l\frac{\partial V}{\partial q_2}\right)\frac{\partial}{\partial q_1} - \left(2p_2 f - e^{4q_0} l\frac{\partial V}{\partial q_1}\right)\frac{\partial}{\partial q_2}. \end{aligned} \tag{9.5.26}$$

We define the symplectic manifold N as the submanifold of S given by $q_0 = 0$. On N, the induced symplectic form is:

$$\omega|_N = \frac{1}{f(p)}\left(dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \frac{l(p)}{f(p)} dp_1 \wedge dp_2\right). \tag{9.5.27}$$

We choose the time vector field to be $T = \frac{\partial}{\partial q_0}$, so that our coordinate system defines a global flat chart on a tubular neighbourhood of N. The family of maps

$$F_t(q_0 = t, q_1, q_2, p_1, p_2) = (0, q_1, q_2, p_1, p_2)$$

then defines a family $\{X_t\}$ of vector fields on N:

$$\begin{aligned} X_t = dF_t(X_H|_S) = e^{4t} f\frac{\partial V}{\partial q_1}\frac{\partial}{\partial p_1} + e^{4t} f\frac{\partial V}{\partial q_2}\frac{\partial}{\partial p_2} \\ - \left(2p_1 f + e^{4t} l\frac{\partial V}{\partial q_2}\right)\frac{\partial}{\partial q_1} - \left(2p_2 f - e^{4t} l\frac{\partial V}{\partial q_1}\right)\frac{\partial}{\partial q_2}. \end{aligned} \tag{9.5.28}$$

To compute $\mathcal{L}_{X_t}\omega|_N$, it is convenient to use Cartan's magic formula:

$$\mathcal{L}_{X_t}\omega|_N = d(X_t \lrcorner\, \omega|_N),$$

where d is the exterior derivative on N. Let us first compute:

$$X_t \lrcorner\, \omega|_N = e^{4t}\frac{\partial V}{\partial q_1} dq_1 + e^{4t}\frac{\partial V}{\partial q_2} dq_2 + 2p_1 dp_1 + 2p_2 dp_2.$$

This is clearly a closed 1-form. Since $N \cong \mathbb{R}^4$ is simply connected, any closed 1-form is also exact. It is straightforward to find a family of functions $\{H_t\}$ such that $X_t = dH_t$; explicitly:

$$H_t = p_1^2 + p_2^2 + e^{4t} V(q_1, q_2). \tag{9.5.29}$$

The geometric framework developed here provides a rigorous foundation for the study of GUP-deformed constrained Hamiltonian systems. This overcomes an ambiguity in the GUP literature about cosmology, where deformations are often imposed directly on the reduced phase space without reference to the parent geometry [35, 157]. While such approaches can be justified within an effective framework, where only the physical degrees of freedom are deformed, they bypass the relation between the reduced and full symplectic structures, potentially undermining consistency. In contrast, our geometric approach clarifies this relation. When the original symplectic form deforms all degrees of freedom appropriately, its restriction to N naturally yields the effective symplectic structure used in cosmological models. This offers a solid justification for its direct use and grounds the effective method in a well-defined Hamiltonian formalism.

Importantly, the analysis highlights a structural condition on the deformed symplectic form: the deformation must preserve commutativity with the chosen "time" variable. If this condition is violated, the Lie derivative of the induced symplectic form along the Hamiltonian flow becomes non-zero, signalling a breakdown of Hamiltonian dynamics. This leads to a loss of phase space volume conservation and, by extension, time-reversal symmetry. Such a scenario seems to mirror known issues in non-commutative quantum theories where time is deformed, often resulting in non-unitarity and causality violations. In this sense, the requirement for time-space commutativity in the classical GUP framework appears as a classical analogue of the condition for unitarity in quantum theory [100, 155].

While further generalisations may exist, the current construction establishes clear geometric and physical constraints on admissible GUP deformations, offering a consistent basis for future investigations of constrained systems in deformed Hamiltonian mechanics.

Summary and Outlook

In this thesis, we addressed the long-standing issue of the relation between Loop Quantum Cosmology and Loop Quantum Gravity within the canonical framework. Our analysis focused both on the generality of the standard LQC approach and on the problem of the apparent loss of the internal $SU(2)$ gauge symmetry in the cosmological setting.

We began by extending the LQC framework to encompass more general cosmological models, namely the non-diagonal Bianchi models. We demonstrated that the main ideas of LQC can indeed be implemented in this setting by quantising three "diagonal fluxes", which admit a clear interpretation as the areas of the faces of a fiducial cell, together with three angular variables encoding the orientation of the Universe's expansion axes. In this context, we also examined alternative quantisation proposals, showing that, although they may appear to simplify the formalism at first, they encounter difficulties either in the definition of the quantum Hamiltonian operator or, more problematically, in their physical interpretation. Within the proposed quantisation scheme, we analysed the status of the $SU(2)$ gauge symmetry in these general cosmological models. We found that, although a global $SU(2)$ internal symmetry can be introduced, the imposition of homogeneity in the minisuperspace quantisation framework inevitably yields an "Abelianisation" of the theory. This is reflected in the fact that the Gauß constraint reduces to three Abelian constraints, whose implementation at the quantum level is equivalent. Consequently, the imposition of the Gauß constraint projects to states that are independent of the gauge variables, effectively recovering the Hilbert space of the non-diagonal models but without any residual gauge freedom.

This observation makes it clear that recovering the full $SU(2)$ gauge symmetry requires going beyond the minisuperspace approach. To achieve this, we introduced in Chapter 6 the appropriate geometric framework for the Ashtekar-Barbero-Immirzi formulation of General Relativity. Beyond the standard principal bundle language familiar in gauge theory, we emphasised the necessity of incorporating spin structures, which are crucial to bridge the $SU(2)$ gauge formulation with the Riemannian setting of General Relativity. In this framework, not only the phase space variables, i.e. the Ashtekar-Sen connection and the densitised triad, but also the constraints themselves can be described as geometric objects, thereby establishing a direct connection with the mathematical structures of Yang-Mills theory.

A detailed geometric analysis, employing tools from symplectic geometry and fibre bundle theory, was then carried out both for the full phase space of the theory

and its symmetry reductions. Applying these techniques to the cosmological setting, we demanded a notion of homogeneity for the Ashtekar connection consistent with Wang's theorem. This allowed us to construct a classical cosmological sector of the Ashtekar-Barbero-Immirzi formulation that restores the local $SU(2)$ gauge symmetry. The physical viability of this symmetry reduction procedure is guaranteed by the fact that the reconstructed ADM variables, namely the spatial metric and the extrinsic curvature, are homogeneous in the conventional sense, while the diffeomorphism symmetry is reduced in a controlled manner.
Moreover, we computed the classical moduli space of homogeneous Ashtekar connections modulo gauge transformations for all three-dimensional homogeneous cosmological models. We showed that these moduli spaces possess remarkable properties, overcoming the usual difficulties encountered in gauge field theories: they are finite-dimensional topological manifolds, possibly with boundary, whose structure depends only on the isotropy group of the underlying homogeneous space.

We then proceeded to quantise the reduced theory using the techniques of Loop Quantum Gravity. In this setting, spin-network states naturally emerge, but now adapted to special symmetric graphs that retain the full combinatorial richness of LQG while encoding the symmetries of cosmology. These states therefore provide a natural bridge between the kinematics of LQG and the quantum states of LQC. This construction lays a solid foundation for the implementation of quantum dynamics: while preserving the structural complexity of the full theory, the high degree of symmetry renders the problem more tractable. We expect that, by employing semiclassical states, one should be able to recover the effective LQC dynamics as an appropriate limit. On the other hand, the analysis of exact quantum solutions of the Hamiltonian constraint will likely need to be performed on a case-by-case basis, but may in turn provide valuable insights for the full theory.

The strength of the approach developed in this thesis lies in its firm geometric underpinnings, which not only illuminate the interplay between LQC and LQG but also offer a versatile framework readily generalisable to other physical settings. Potential future applications include black hole models, where symmetry reductions play a similarly fundamental role, and covariant formulations of LQG, where connections to spin foam models may be more rigorously established.

Appendix A

Geometric and spectral aspects of Lie groups and Lie algebras

In this appendix, we present a brief review of Lie groups and Lie algebras from a more geometrical perspective. Rather than offering a comprehensive survey of all relevant topics, we focus on introducing selected notions that are frequently cited in the main text, presented within the interpretative framework used in this thesis. This appendix is primarily based on references [124, 105, 106].

A *Lie group* G is a smooth manifold with a group structure such that the multiplication and inversion operations are both smooth maps. On a Lie group G, every element $g \in G$ defines maps L_g, $R_g : G \to G$ called *left translation* and *right translation*, respectively, by

$$L_g(h) = gh, \; R_g(h) = hg, \quad \text{for all } h \in G.$$

Both L_g and R_g are diffeomorphisms, with inverses given by $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$, respectively.
If G and H are Lie groups, a *Lie group homomorphism* from G to H is a smooth map $F : G \to H$ that is also a group homomorphism. It is called a Lie group isomorphism if, in addition, F is a diffeomorphism. Such maps satisfy

$$F \circ L_g = L_{F(g)} \circ F, \quad F \circ R_g = L_{R(g)} \circ F,$$

from which it follows that every Lie group homomorphism has constant rank.
A *Lie subgroup* of G is a subgroup endowed with a topology and smooth structure such that it becomes both a Lie group and an immersed submanifold of G. There is a strong correspondence between Lie subgroups and Lie group homomorphisms:

Proposition A.0.1. *Let $F : G \to H$ be a Lie group homomorphism. Then*

- *The kernel of F is a properly embedded Lie subgroup of G, whose codimension is equal to the rank of F.*
- *If $F : G \to H$ is an injective Lie group homomorphism, the image $F(G)$ inherits a unique smooth manifold structure such that it becomes a Lie subgroup of H and $F : G \to F(G)$ is a Lie group isomorphism.*

Let S be a subset of G, the *subgroup generated by S* is the smallest subgroup containing S

Proposition A.0.2. *Let $U \subset G$ be a connected neighbourhood of the identity, then it generates a connected open subgroup of G. If, in addition, G is connected, then U generates G.*

In particular, U generates the identity component G_0 of G, which is the only connected open subgroup of G. When a subgroup is generated by a subset S, it means that every element of the subgroup can be written as a finite product of elements from S and their inverses.

A vector field X on G is said to be *left invariant* if it is L_g-related to itself for every $g \in G$, namely,

$$d_{g'}L_g(X_{g'}) = X_{gg'}, \text{ for all } g, g' \in G. \tag{A.0.1}$$

We recall that $d_{g'}L_g$ is the differential of the map L_g in g', that is a linear map on tangent spaces $d_{g'}L_g : \mathrm{T}_{g'}G \to \mathrm{T}_{gg'}G$. This equation may be expressed more concisely as $(L_g)_*X = X$ for all $g \in G$, where $(L_g)_*$ denotes the pushforward by L_g. Similarly, a vector field Y is said to be *right invariant* if it satisfies $(R_g)_*Y = Y$ for all $g \in G$. Given two left-invariant vector fields X and Y, the Lie bracket $[X, Y]$ is still a left-invariant vector field. We recall that a *Lie algebra* $\mathfrak{A}$ is a vector space equipped with an alternating bilinear map $[\cdot,\cdot] : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ that satisfies the Jacobi identity. Thus, the set of all left-invariant vector fields forms a Lie subalgebra of the space of the Lie algebra of smooth vector fields on G equipped with the usual Lie brackets. We denote this subalgebra by $\mathfrak{g}$, and refer to it as the Lie algebra of G. The Lie algebra $\mathfrak{g}$ is finite-dimensional. In fact, the following fundamental result holds:

Theorem A.0.3. *The Lie algebra $\mathfrak{g}$ of G is isomorphic, as vector space, to $\mathrm{T}_{\mathbb{1}}G$.*

Proof. By construction, $\mathfrak{g}$ is a real vector space. The isomorphism is given by the evaluation map $\mathrm{ev} : X \mapsto X_e$. The map is injective, indeed, if $X_{\mathbb{1}} = 0$ then $X_g = d_{\mathbb{1}}L_g(X_{\mathbb{1}}) = 0$ for all $g \in G$, hence $X \equiv 0$. The map is also surjective, for each $v \in \mathrm{T}_{\mathbb{1}}G$, we can define a unique vector field $X^v_g = d_{\mathbb{1}}L_g(v)$, which is clearly left-invariant and smooth. □

The space of right-invariant vector fields is a Lie subalgebra of $\mathfrak{X}(G)$, and it is anti-isomorphic to $\mathfrak{g}$. It is well known that every Lie group admits a global left-invariant frame; hence, every Lie group is parallelizable. Furthermore, every Lie subgroup defines a Lie subalgebra, and also the converse is true:

Theorem A.0.4. *Suppose $H \subset G$ is a Lie subgroup, and $\iota : H \hookrightarrow G$ is the inclusion map. Then, there exists a Lie subalgebra in $\mathfrak{g}$ that is canonically isomorphic to $\mathfrak{h}$, characterized by*

$$\{X \in \mathfrak{g} \mid X_{\mathbb{1}} \in \mathrm{T}_{\mathbb{1}}H\}.$$

Conversely, if G is a Lie group and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.

An important property of a left-invariant vector field is that every left-invariant vector field is complete, meaning that its maximal integral curves are defined on the entire real line. This property allows us to establish a strong link between a Lie group and its Lie algebra.

Definition A.0.1. The exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = c(1)$, where $c : \mathbb{R} \to G$ is the maximal integral curve of X such that $c(0) = \mathbb{1}$.

The exponential map is a local diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ to a neighbourhood of the identity $\mathbb{1}$ in G. Furthermore, by Prop.A.0.2, if G is connected, then every element of G can be written as a finite product $g = \exp(X_1)\exp(X_2)\ldots\exp(X_n)$, for some finite collection $X_1, X_2, \ldots, X_n \in \mathfrak{g}$. Moreover, in terms of the exponential map, the integral curves of left-invariant vector fields admit a particularly simple form. Let X be a left-invariant vector field, the integral curve of X passing through g for $t = 0$ is given by

$$c(t) = g \exp\left(t X_{\mathbb{1}}\right).$$

A Lie group homomorphism $F : G \to H$ induces a Lie algebra homomorphism via pushforward $F_* : \mathfrak{g} \to \mathfrak{h}$. Conversely, any Lie algebra homomorphism arises from a unique Lie group homomorphism $F : G \to H$ provided that G is simply connected. Moreover, right multiplication defines an automorphism of $\mathfrak{g}$. Specifically, if X is a left-invariant vector field, then $(R_g)_* X$ is also left-invariant for any $g \in G$.

Definition A.0.2. The adjoint representation of G on $\mathfrak{g}$ is given by

$$\begin{array}{rccc} \mathrm{Ad}: & G & \to & \mathrm{Aut}(\mathfrak{g}); \\ & g & \mapsto & \mathrm{Ad}_g \doteq (R_g)_*. \end{array} \tag{A.0.2}$$

The differential of the adjoint representation at the identity, together with the identification $\mathrm{T}_{\mathrm{id}_\mathfrak{g}}\mathrm{Aut}(\mathfrak{g}) \cong \mathfrak{gl}(\mathfrak{g})$, yields a representation of the Lie algebra $\mathfrak{g}$ on itself.

Definition A.0.3. The adjoint representation of a Lie algebra $\mathfrak{g}$ on itself is given by

$$\mathrm{ad} \doteq d_{\mathbb{1}}\,\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}); \tag{A.0.3}$$

It is well known that for all X, $Y \in \mathfrak{g}$, the adjoint action satisfies $\mathrm{ad}_X Y = [X, Y]$. Considering the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, i.e. the space of linear functionals on $\mathfrak{g}$, one may also define the *coadjoint representations*:

Definition A.0.4. The coadjoint representation $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ of G is defined by

$$(\mathrm{Ad}^*_g \lambda)(X) \doteq \lambda(\mathrm{Ad}^{-1}_g X) \text{ for all } g \in G,\ X \in \mathfrak{g},\ \lambda \in \mathfrak{g}^*. \tag{A.0.4}$$

Analogously, the coadjoint representation $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ of $\mathfrak{g}$ is defined by

$$(\mathrm{ad}^*_X \lambda)(Y) \doteq -\lambda(\mathrm{ad}_X Y) \text{ for all } X,\ Y \in \mathfrak{g},\ \lambda \in \mathfrak{g}^*. \tag{A.0.5}$$

A left-invariant metric q on a Lie group G is a Riemannian metric satisfying $L_g^* q = q$ for all $g \in G$, with L_g^* denoting the pullback by L_g.. Such metrics are in one-to-one correspondence with inner products on the Lie algebra $\mathfrak{g}$. Note, however, that left-invariant vector fields are generally not Killing vector fields for q; rather, it is the right-invariant vector fields that generate isometries of left-invariant metrics. For semisimple Lie groups, there exists a canonical choice for inner product on $\mathfrak{g}$, known as the *Killing form*. The Killing form is a symmetric, bilinear, and non-degenerate form on a semisimple Lie algebra defined by

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y). \tag{A.0.6}$$

Up to a constant factor, it defines a scalar product. Moreover, it is invariant under the adjoint action: $B(\mathrm{ad}_Z X, \mathrm{ad}_Z Y) = B(X, Y)$.

On a locally compact Hausdorff group G, there always exists a *Haar measure* μ_H, which is a countably-additive Radon measure satisfying left-invariance: for every Borel subset S of G, we have $\mu_H(gS) = \mu_H(S)$. The Haar measure is unique up to a positive multiplicative constant, which, in the case of compact groups, can be fixed by requiring $\mu_H(G) = 1$. On a Lie group G, the Haar measure can be constructed as the volume form associated with any left-invariant metric; this yields a left-invariant top-form, naturally defining a measure on G. Similarly, one can construct a right-invariant measure ν on G. The left translation of a right-invariant measure is also a right-invariant measure. Since right-invariant measures are also unique up to a constant scaling factor, there exists a function $\Delta : G \to \mathbb{R}^+$ such that $\nu \circ L_{g^{-1}} = \Delta(g)\nu$. This function is called the *modular function*, and one can verify that it is a continuous homomorphism from G to the group of positive real numbers. If $\Delta(g) \equiv 1$, the group is called *unimodular*, in which case the Haar measure is both left- and right-invariant. Most of the Lie groups of interest in physics are unimodular; for instance, the Poincaré group in Special Relativity and $SU(N)$ in particle physics.

With a Haar measure, one can develop harmonic analysis on Lie groups. We begin with the following theorem regarding integrable functions on G:

Theorem A.0.5. *The space $L^1(G)$ is a Banach *-algebra, arising as the completion of the space $C_c(G)$ of continuous complex functions on G with compact support with respect to the norm*

$$||f||_{L^1} \doteq \int_G |f(g)| d\mu_H(g),$$

and with the convolution product and involution given by

$$\begin{aligned}
[f_1 \bullet f_2](g) &= \int_G f_1(s) f_2(s^{-1}g) d\mu_H(s), \\
f^*(g) &= \overline{f(g^{-1})} \Delta(g^{-1}).
\end{aligned}$$

Furthermore, the space $C_c(G)$ has a deep connection with the unitary representations of G:

Theorem A.0.6. *Let U be a strongly continuous unitary representation of G on some Hilbert space $\mathcal{H}$, then $\pi_U : C_c(G) \to \mathcal{B}(\mathcal{H})$ given by*

$$\pi_U(f) \doteq \int_G f(g) U(g) d\mu_H(g),$$

*is a non-degenerate *-representation of the algebra $C_c(G)$. Furthermore, the map $U \mapsto \pi_U$ is a bijection between the set of strongly continuous unitary representations of G and non-degenerate bounded *-representations of $C_c(G)$.*

A particular class of complex functions on G is given by the so-called *matrix coefficients*. A matrix coefficient is a function $f : G \to \mathbb{C}$ that can be written as $f = L \circ \pi$, where $\pi : G \to GL(V)$ is a finite-dimensional continuous representation of G on a vector space V, and L is a linear functional on $\mathrm{End}(V)$. The following is the foundational result for harmonic analysis on compact Lie groups:

Theorem A.0.7 (Peter-Weyl theorem)**.** *Let G be a compact Lie group.*

(i) The set of matrix coefficients of G is dense in the space of continuous complex functions $C(G)$ on G, equipped with the uniform norm.

(ii) Let π be a unitary representation of G on a complex Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G.

(iii) The Hilbert space of square-integrable functions on G decomposes as

$$L^2(G) \cong \bigoplus_{\pi \in \Pi} V_\pi \otimes V_\pi^*,$$

where Π is the set of equivalence classes of irreducible unitary representations of G, and V_π is a fixed carrier space of a representative in the class π. An orthonormal basis of $L^2(G)$ is given by the set

$$\left\{ \sqrt{\dim(V_\pi)} D^{(\pi)}_{mn} \mid \pi \in \Pi,\, 1 \le m, n \le \dim(V_\pi) \right\},$$

where, the functions $D^{(\pi)}_{mn}$ are matrix coefficients defined by

$$D^{(\pi)}_{mn}(g) = \langle e_m, \pi(g) e_n \rangle, \quad \textit{for an orthonormal basis } \{e_n\} \textit{ of } V_\pi.$$

Part (iii) follows directly from (i) and (ii). The space $L^2(G)$ can be understood as a representation of $G \times G$. In this setting, the space of matrix coefficients for a representation V_π can be identified with $\mathrm{End}(V_\pi)$, equipped with a unitary action of $G \times G$ defined by $(g,h) \cdot A = \pi(g) A \pi(h^{-1})$, identifying then $\mathrm{End}(V_\pi) \cong V_\pi \otimes V_\pi^*$, from part (ii) follows the decomposition of $L^2(G)$, while from (i), one can deduce the form of the orthonormal basis. In the special case where $G = SU(2)$, which is the case of interest for Loop Quantum Gravity, the functions $D^{(\pi)}_{mn}(g)$ are the well-known Wigner D-matrices. The last concept relevant to harmonic analysis on Lie groups is that of the *Pontryagin dual.*

Definition A.0.5. Let G be a locally compact Abelian group; its Pontryagin dual $\hat{G}$ is the collection of continuous group homomorphisms from G to the circle group $\mathbb{T}$. That is, $\hat{G} \doteq \mathrm{Hom}(G, U(1))$ equipped with the pointwise product law.

The Pontryagin dual is usually considered equipped with the natural topology induced by the compact-open topology. An important result follows for the relation between the topology of a group and its dual.

Theorem A.0.8. *There is a canonical isomorphism $G \cong \hat{\hat{G}}$ between any locally compact Abelian group and its double dual.*
Moreover, G is compact if and only if the dual group $\hat{G}$ is discrete. Conversely, G is discrete if and only if $\hat{G}$ is compact.

Notable examples include: $\hat{\mathbb{R}} = \mathbb{R}$, $\hat{\mathbb{T}} = \mathbb{Z}$, and $\hat{\mathbb{Z}}_n = \mathbb{Z}_n$. Given a Haar measure on G, one can define an abstract version of the Fourier transform. For $f \in L^1(G)$, the Fourier transform is defined as the function $\widehat{f} : \widehat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \int_G f(g)\overline{\chi(g)} d\mu_H(g). \tag{A.0.7}$$

Moreover, the inverse of this transformation always exists

Proposition A.0.9. *For every Haar measure μ_H on G, there exists a Haar measure ν_H on $\hat{G}$ such that whenever $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$, we have*

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g) d\nu_H(\chi) \qquad \mu_H\text{-almost everywhere.}$$

In particular, the Fourier transform sends L^1 functions to continuous functions that vanish at infinity, providing a homomorphism of Abelian *-algebras $L^1(G) \to C_0(\hat{G})$. This transform is well defined on the subspace $L^1(G) \cap L^2(G)$, particularly on compactly supported functions. The Plancherel theorem ensures that the Fourier transform of an L^2 function is an L^2 function.

Theorem A.0.10 (Plancherel)**.** *Suppose $f \in C_c(G)$, and let μ_H, ν_H be Haar measures on G and $\widehat{G}$ chosen in duality. Then, the Fourier transform satisfies*

$$\int_G |f(g)|^2 \, d\mu_H(g) = \int_{\widehat{G}} |\widehat{f}(\chi)|^2 \, d\nu_H(\chi).$$

That is, the Fourier transform is a norm-preserving bounded operator on a dense subset of $L^2(G)$, and therefore extends uniquely to a unitary operator:

$$\mathcal{F} : L^2(G, d\mu_H) \to L^2(\widehat{G}, d\nu_H).$$

Appendix B

Geometry of Yang-Mills fields

The historical development of principal bundle theory and gauge theory is deeply intertwined. Both have their origins in the late 19^{th} century, beginning with the discovery of Maxwell's equations and, later, with the seminal work of H. Weyl, who introduced the concept of *gauge invariance* and proposed it as a fundamental principle of physics. However, the mathematical and physical developments evolved largely independently until the 1980s.

In the early 1930s, P. Dirac and H. Hopf independently introduced the concept of principal $U(1)$-bundles: Dirac, somewhat implicitly, in his study of magnetic monopoles and the electromagnetic field; Hopf, more explicitly, through the construction of what is now known as the Hopf fibration.

During the 1950s, S. Chern and N. Steenrod formalised the theory of fibre bundles in algebraic terms, driven by growing interest in algebraic topology. This work was supported by earlier contributions from E. Cartan and C. Ehresmann, and by further developments by Chern himself, particularly in the context of characteristic classes.

Around the same period, C. N. Yang and R. L. Mills formulated what is now known as Yang-Mills theory, a non-Abelian generalisation of Maxwell's theory. These developments coincided with significant advances in quantum field theory, particularly quantum electrodynamics, which provided the first experimentally verified predictions of gauge theory.

In 1975, Yang and Chern discovered that their research in non-Abelian gauge theory and fibre bundles was describing the same theoretical structure. This revelation highlighted a deep and unexpected connection between theoretical physics and differential geometry.

It was not until the 1980s, particularly through the work of M. F. Atiyah and E. Witten, that the relationship between mathematics and physics became firmly established and mutually beneficial. One of the most notable outcomes of this collaboration was the development of Donaldson's theory, which revolutionised the study of four-manifolds using techniques inspired by gauge theory.

This appendix aims to introduce the main concepts of fibre bundle theory in view of their physical applications. The exposition is primarily based on the foundational texts [160, 29, 140, 118, 119, 73]. We begin by presenting the mathematical definitions and central ideas of principal bundle theory. Subsequently, through the introduction of the Yang-Mills action, we discuss their physical interpretation and relevance.

Finally, we turn to vector bundles, providing a parallel formulation that proves useful in physics for the description of matter fields and for understanding the covariant derivative.

Definition B.0.1. Let M be a smooth manifold and G a Lie group. A *principal G-bundle* over M is a smooth manifold P such that:

- G acts freely on P via a smooth right action $(p, g) \in P \times G \mapsto pg = R_g(p) \in P$,
- M is the quotient space $M = P/G$, and the canonical projection $\pi : P \to M$ is smooth,
- P is locally trivial: for each $x \in M$, there exists a neighbourhood U and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ such that $\mathrm{pr}_U \circ \varphi = \pi$.

We call M the *base manifold*, and G the *structure group* or *gauge group*. The preimage of a point $x \in M$ is called the *fibre* over x. Notice that G acts freely and transitively on each fibre, so the fibres are all diffeomorphic to G. A principal G-bundle is often represented by the diagram:

$$\begin{tikzcd} G \arrow[r] & P \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

A key concept in the study of principal bundles is the *local section*. A local section is a map s from an open subset $U \subset M$ into its preimage $\pi^{-1}(U)$ such that $\pi \circ s = \mathrm{id}_U$. Local sections are associated with local trivializations of the principal bundle: indeed, given a local trivialization $\varphi : \pi^{-1}(U) \to U \times G$, it is always possible to define a section $s : U \to P$ by

$$s(x) \doteq \varphi^{-1}(x, \mathbb{1}),$$

where $\mathbb{1}$ is the identity element in G.

The converse also holds: a local section $s : U \to P$ induces a local trivialization defined by

$$\varphi(p) = (\pi(p), g_{p,s}),$$

where $g_{p,s} \in G$ is the unique element such that $p = s(x)g_{p,s}$ for each $p \in \pi^{-1}(U)$.

While local sections always exist, the existence of a global section $s : M \to P$ depends on the topological structure of the bundle.

Proposition B.0.1. *A principal G-bundle over M is trivial, i.e., $P \cong M \times G$, if and only if it admits a global section.*

Without delving into details, principal G-bundles over a manifold M can be classified by studying the topological properties of the base manifold and the *classifying space BG*. In particular, isomorphism classes of principal G-bundles over M correspond bijectively to the set $[M, BG]$ of homotopy classes of continuous maps from M to BG. It follows immediately that if either M or BG is contractible, then the only principal G-bundle over M is the trivial one.

The geometric problem of describing a *horizontal space* complementary to the fibres leads to the introduction of a *connection.* While it is straightforward to define the *vertical space* at a point $p \in P$ as

$$V_p = \{v \in \mathrm{T}_p P \mid d_p\pi(v) = 0\},$$

the choice of a complementary subspace H_p such that $\mathrm{T}_p P = V_p \oplus H_p$ and $H_{pg} = (R_g)_* H_p$ is non-unique. This choice corresponds to specifying a special Lie algebra-valued 1-form ω on P, called a *connection form.*

Definition B.0.2. A connection form (or simply a connection) on a principal G-bundle P is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that:

- $\omega(X_v) = v$ for each $v \in \mathfrak{g}$, where X_v is the fundamental vector field on P induced by v:
$$X_v(p) = \left.\frac{d}{dt}\right|_{t=0} p \exp(tv),$$
- $R_g^* \omega = \mathrm{Ad}_{g^{-1}}\, \omega$ for all $g \in G$.

At each point $p \in P$, the connection form $\omega_p : \mathrm{T}_p P \to \mathfrak{g}$ defines the horizontal space by
$$H_p = \ker(\omega_p).$$
The introduction of a connection is crucial to define differentiation on principal bundles.

Definition B.0.3. Let V be a vector space carrying a representation ρ of G. A V-valued k-form α on a principal G-bundle P is *horizontal* if:

- $R_g^* \alpha = \rho(g^{-1})\alpha$ for all $g \in G$,
- $\alpha_p(X_1, \ldots, X_k) = 0$ whenever at least one vector X_i is vertical.

The *exterior covariant derivative* of a horizontal V-valued k-form α with respect to a connection ω is the horizontal V-valued $(k+1)$-form $d_\omega \alpha$ defined by

$$d_\omega \alpha(X_1, \ldots, X_{k+1}) = d\alpha(hX_1, \ldots, hX_{k+1}),$$

where h is the projector onto the horizontal subspace, given pointwise by $h_p = \mathrm{pr}_{H_p} : \mathrm{T}_p P \to H_p$.

Theorem B.0.2. *The curvature form Θ of a connection ω is defined as its exterior covariant derivative: $\Theta = d_\omega \omega$. It can be expressed as*

$$\Theta(X, Y) = d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)], \quad \textit{for any } X, Y \in \mathrm{T}_p P \textit{ and } p \in P.$$

The mathematical formulation of Yang-Mills theory, and more generally of gauge field theory, is deeply rooted in the framework of principal bundles. A Yang-Mills theory with internal symmetry group G is a local theory of the connection on a principal G-bundle over a Riemannian or Lorentzian manifold M.

The *gauge field* or *gauge potential* is closely related to the connection. Interpreting the gauge field A as a $\mathfrak{g}$-valued 1-form on an open subset of M, it arises as the local field of a given connection ω, namely as its pullback via a local section: $A = s^*\omega$. The same relation holds for the *field strength* and the curvature: the field strength F of a gauge potential A is the pullback, via the same section, of the curvature form of the associated connection: $F = s^*\Theta$.

Moreover, the usual gauge transformation law for the field is recovered naturally. A change of gauge corresponds to choosing a different local section that trivialises the principal bundle. Given two local sections s, s' defined over the same open set $U \subset M$, there exists a smooth map $g : U \to G$ such that

$$s'(x) = s(x)g(x),$$

from which it follows that, when G is a matrix Lie group,

$$A'(x) = s'^*\omega_x = (R_g s)^*\omega = \mathrm{Ad}_{g(x)^{-1}} A(x) + g(x)^{-1} d_x g(x). \tag{B.0.1}$$

The equations of motion for Yang-Mills theory on a closed four-dimensional Lorentzian manifold M can be derived via a variational principle from the action functional

$$S_{\mathrm{YM}} = -\frac{1}{2}\int_M \mathrm{Tr}(F \wedge \star F), \tag{B.0.2}$$

where $\star$ denotes the Hodge star operator.

This action is defined on the space $\mathcal{A}$ of connections on a principal G-bundle P. The space $\mathcal{A}$ is an affine space whose underlying vector space is the space of $\mathfrak{g}$-valued horizontal 1-forms on P; hence, $\mathcal{A}$ is a Fréchet manifold.

The symmetry group of the theory (i.e. the group of transformations that leave the Lagrangian, and hence the equations of motion, invariant) is the group of vertical automorphisms, which forms a Fréchet Lie group:

$$\mathcal{G} = \mathrm{Gau}(P) \doteq \{f \in \mathrm{Aut}(P) \mid \pi \circ f = \pi\}. \tag{B.0.3}$$

The action of $\mathcal{G}$ on $\mathcal{A}$ is given by pullback, $(f, \omega) \mapsto f^*\omega$. Locally, by choosing a trivialisation, the group $\mathcal{G}$ is isomorphic to the group of smooth maps $C^\infty(U, G)$. The transformation is called vertical because it moves points of the principal bundle P along the fibres while leaving the corresponding base point fixed. This formalises the notion of internal transformations: the fibre over a point represents the internal degrees of freedom, whereas the base manifold M corresponds to the usual spacetime coordinates.

The space of classically physically distinct states is the moduli space

$$\mathcal{M} = \mathcal{A}/\mathcal{G}, \tag{B.0.4}$$

namely the space of gauge orbits. Since the action of $\mathcal{G}$ is typically not free, the moduli space $\mathcal{M}$ generally exhibits singularities. Nevertheless, physical observables are expected to be functionals on $\mathcal{M}$ (although in quantum theory this need not always hold, due to the possible presence of anomalies). This is because gauge transformations encode redundancies and unphysical degrees of freedom: the true physical content is associated with the gauge orbit rather than with an individual gauge field, and hence observables should depend only on the orbit.

For this reason, it is of interest to study the principal $\mathcal{G}$-bundle over $\mathcal{M}$:

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{A} \\ & & \big\downarrow \\ & & \mathcal{A}/\mathcal{G} \end{array} \tag{B.0.5}$$

While $\mathcal{A}$ is contractible, the principal bundle is not trivial. The non-existence of a global section from $\mathcal{M}$ to $\mathcal{A}$ is known as the *Gribov problem.*

The matter content of the theory can be introduced via the notion of a vector bundle.

Definition B.0.4. Let M be a smooth manifold. A (real) *vector bundle* of rank r over M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions:

- For each $x \in M$, the fibre $E_x \doteq \pi^{-1}(x)$ is endowed with the structure of an r-dimensional real vector space,
- E is locally trivial: for each $x \in M$, there exists a neighbourhood U of x in M and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $\mathrm{pr}_U \circ \varphi = \pi$, and the restriction $\varphi|_{E_x}$ is a vector space isomorphism from E_x to $\mathbb{R}^r$.

Such a vector bundle is also denoted by $E \to M$.

A local *vector field* is a smooth local section $\sigma : U \subset M \to E$ of the vector bundle. A (global) vector field is a global smooth section $\sigma : M \to E$. The set $C^\infty(M, E)$ of vector fields, i.e., the global sections of E, forms a vector space and admits a natural system of seminorms that make it into a Fréchet space. It is also a $C^\infty(M)$-module.
Two simple examples of vector bundles are the trivial line bundle $M \times \mathbb{R} \to M$, whose vector fields are real-valued functions on M and which in physics is used to describe a chargeless scalar field, and the tangent bundle $\mathrm{T}M \to M$, whose sections are tangent vector fields which may describe either the velocity of a particle or the velocity field of a fluid.

A local *frame* for E over an open set $U \subset M$ is an ordered collection of r local sections of E, $(\sigma_1, \ldots, \sigma_r)$, such that for each $x \in U$, the tuple $(\sigma_1(x), \ldots, \sigma_r(x))$ forms a basis of the fibre E_x. If $U = M$, it is called a *global frame* or a *frame of vector fields.*

Every local frame for a vector bundle is associated with a local trivialisation. Let $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ be a local trivialisation; then one can define a local frame by

$$\sigma_i(x) \doteq \varphi^{-1}(x, \mathfrak{e}_i), \quad i = 1, \ldots, r,$$

where $\mathfrak{e}_i$ is the i-th canonical basis vector of $\mathbb{R}^r$. Conversely, given a local frame $(\sigma_1, \ldots, \sigma_r)$, every $v \in E_x$ can be uniquely written as a linear combination

$$v = v^1 \sigma_1(x) + \cdots + v^r \sigma_r(x),$$

for some $(v^1, \ldots, v^r) \in \mathbb{R}^r$. Then the corresponding local trivialisation is given by

$$\varphi(v) = (x, (v^1, \ldots, v^r)).$$

It follows immediately:

Proposition B.0.3. *A vector bundle is trivial if and only if it admits a global frame.*

Let E and F be vector bundles of ranks r_1 and r_2 respectively over a manifold M. Given any functorial operation on vector spaces, a corresponding operation can be defined on vector bundles by applying the operation fibrewise. For example, the dual bundle E^*, the direct sum bundle $E \oplus F$, and the homomorphism bundle $\mathrm{Hom}(E, F)$ are defined as:

$$(E^*)_x = (E_x)^*, \qquad (E \oplus F)_x = E_x \oplus F_x, \qquad \mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x).$$

A particularly important construction is the tensor bundle $\Lambda^k \mathrm{T}^* M \otimes E$. The global sections of this bundle are called *E-valued differential k-forms*, and their space is denoted by $\Omega^k(M, E)$.

On vector bundles, one can define the notion of *parallel transport* via the introduction of a covariant derivative.

Definition B.0.5. A *covariant derivative* D on the bundle E is a linear differential operator satisfying the following properties:

- $D : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$,
- $D(\nu \wedge \sigma) = d\nu \wedge \sigma + (-1)^k \nu \wedge D\sigma$,

for any $\nu \in \Omega^k(M)$ and $\sigma \in \Omega^q(M, E)$.

Given a section $\sigma : M \to E$, it is said to be *parallel* with respect to a vector field $X \in \mathfrak{X}(M)$ if

$$D_X \sigma = 0.$$

Principal bundles and vector bundles are two sides of the same coin. Let E be a vector bundle of rank r over a manifold M. Its *frame bundle* $L(E)$, i.e. the collection of all ordered bases for each fibre, is a principal $GL(r, \mathbb{R})$-bundle over M.

Conversely, given a principal G-bundle P and a representation ρ of G on a vector space V, one can construct an associated vector bundle defined by

$$E = P \times_\rho V = (P \times V)/\sim,$$

where the equivalence relation is given by

$$(p, v) \sim (pg, \rho(g^{-1})v), \quad \text{for } g \in G.$$

The projection map is given by $\pi_\rho([p, v]) = \pi(p)$.

A particularly important associated bundle is the *adjoint bundle*, defined via the adjoint representation of G on its Lie algebra $\mathfrak{g}$:

$$\mathrm{ad}\, P = P \times_{\mathrm{Ad}} \mathfrak{g}.$$

Sections of the associated vector bundle E correspond bijectively to G-equivariant smooth maps:

$$\left\{\Psi : P \to V \,\middle|\, \Psi(pg) = \rho(g^{-1})\Psi(p) \text{ for all } g \in G\right\}. \tag{B.0.6}$$

Given a local (or global) section s of the principal bundle, the section of E induced by such a Ψ is defined as

$$x \mapsto [s(x), \Psi(s(x))].$$

Consequently, any section of E is described locally by a V-valued function on an open set $U \subset M$. In terms of a local trivialisation, this section can be written as $\sigma|_U = [s, \psi]$, where $\psi = \Psi \circ s : U \to V$. Upon changing the trivialising section according to $s(x) \mapsto s(x)g(x)$, the function transforms as

$$\psi(x) \mapsto \rho(g(x)^{-1})\psi(x).$$

In physical terms, the matter field ψ is *gauge-covariant*, as it transforms under a gauge transformation via the appropriate representation.

This formalism includes not only matter fields, but also other geometrical objects: there exists a one-to-one correspondence between V-valued horizontal k-forms on P and elements of $\Omega^k(M, E)$. The relation is realised via pullback by a local section and, conversely, via the bundle projection.

As a direct consequence: the space of connections is an affine space modelled on $\Omega^1(M, \operatorname{ad} P)$. The curvature form Θ of a given connection ω may be viewed as an element of $\Omega^2(M, \operatorname{ad} P)$, the associated local $\mathfrak{g}$-valued 2-form is the physical field strength F.

Even the connection and covariant derivative are closely linked. Given a connection ω on the principal G-bundle P, one can induce a covariant derivative on the associated vector bundle E via the local formula:

$$D\sigma = [s,\, d\psi + \rho_*(s^*\omega) \wedge \psi], \tag{B.0.7}$$

where $\sigma \in \Omega^k(M, E)$ is a section and ψ is the corresponding local V-valued k-form. This expression clearly highlights the role of the connection, through the gauge field $A = s^*\omega$, in defining the covariant derivative, thereby recovering the standard formulation familiar from physics.

Bibliography

[1] Abbati, M. C., Cirelli, R., Mania', A., and Michor, P. The Lie group of automorphisms of a principle bundle. *Journal of Geometry and Physics*, **6** (1989), 215–235. doi:10.1016/0393-0440(89)90015-6.

[2] Agricola, I., Ferreira, A. C., and Friedrich, T. The classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$. *Differential Geometry and its Applications*, **39** (2015), 59–92. doi:10.1016/j.difgeo.2014.11.005.

[3] Agricola, I., Hofmann, J., and Lawn, M.-A. Invariant spinors on homogeneous spheres. *Differential Geometry and its Applications*, **89** (2023), 102014. doi:10.1016/j.difgeo.2023.102014.

[4] Agullo, I., Ashtekar, A., and Nelson, W. The pre-inflationary dynamics of loop quantum cosmology: confronting quantum gravity with observations. *Classical and Quantum Gravity*, **30** (2013), 085014. doi:10.1088/0264-9381/30/8/085014.

[5] Agullo, I., Wang, A., and Wilson-Ewing, E. Loop quantum cosmology: Relation between theory and observations. In *Handbook of Quantum Gravity* (edited by C. Bambi, L. Modesto, and I. Shapiro), p. 1–46. Springer Nature, Singapore (2023). doi:10.1007/978-981-19-3079-9_103-1.

[6] Alekseevsky, D., Kriegl, A., Michor, P. W., and Losik, M. Choosing roots of polynomials smoothly. *Israel Journal of Mathematics*, **105** (1998), 203–233. doi:10.1007/BF02780330.

[7] Alexandrov, S. and Livine, E. R. SU(2) loop quantum gravity seen from covariant theory. *Physical Review D*, **67** (2003), 044009. doi:10.1103/PhysRevD.67.044009.

[8] Ali, A. F., Das, S., and Vagenas, E. C. Discreteness of Space from the Generalized Uncertainty Principle. *Physics Letters B*, **678** (2009), 497–499. doi:10.1016/j.physletb.2009.06.061.

[9] Alonso-Serrano, A., Brizuela, D., and Uria, S. F. Quantum Kasner transition in a locally rotationally symmetric Bianchi II universe. *Physical Review D*, **104** (2021), 024006. doi:10.1103/PhysRevD.104.024006.

[10] Arnowitt, R., Deser, S., and Misner, C. W. Canonical Variables for General Relativity. *Physical Review*, **117** (1960), 1595–1602. doi:10.1103/PhysRev.117.1595.

[11] Arnowitt, R., Deser, S., and Misner, C. W. Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, **40** (2008), 1997–2027. doi:10.1007/s10714-008-0661-1.

[12] Ashtekar, A. New Variables for Classical and Quantum Gravity. *Physical Review Letters*, **57** (1986), 2244–2247. doi:10.1103/PhysRevLett.57.2244.

[13] Ashtekar, A. New Hamiltonian formulation of general relativity. *Physical Review D*, **36** (1987), 1587–1602. doi:10.1103/PhysRevD.36.1587.

[14] Ashtekar, A. and Bianchi, E. A short review of loop quantum gravity. *Reports on Progress in Physics*, **84** (2021), 042001. doi:10.1088/1361-6633/abed91.

[15] Ashtekar, A., Bojowald, M., and Lewandowski, J. Mathematical structure of loop quantum cosmology. *Advances in Theoretical and Mathematical Physics*, **7** (2003), 233–268.

[16] Ashtekar, A. and Campiglia, M. On the uniqueness of kinematics of loop quantum cosmology. *Classical and Quantum Gravity*, **29** (2012), 242001. doi:10.1088/0264-9381/29/24/242001.

[17] Ashtekar, A., Fairhurst, S., and Willis, J. L. Quantum gravity, shadow states, and quantum mechanics. *Classical and Quantum Gravity*, **20** (2003), 1031–1061. doi:10.1088/0264-9381/20/6/302.

[18] Ashtekar, A., Henderson, A., and Sloan, D. Hamiltonian formulation of the Belinskii-Khalatnikov-Lifshitz conjecture. *Physical Review D*, **83** (2011), 084024. doi:10.1103/PhysRevD.83.084024.

[19] Ashtekar, A. and Isham, C. J. Representations of the holonomy algebras of gravity and nonAbelian gauge theories. *Classical and Quantum Gravity*, **9** (1992), 1433. doi:10.1088/0264-9381/9/6/004.

[20] Ashtekar, A. and Lewandowski, J. Projective techniques and functional integration for gauge theories. *Journal of Mathematical Physics*, **36** (1995), 2170–2191. doi:10.1063/1.531037.

[21] Ashtekar, A. and Lewandowski, J. Quantum theory of geometry: I. Area operators. *Classical and Quantum Gravity*, **14** (1997), A55. doi:10.1088/0264-9381/14/1A/006.

[22] Ashtekar, A. and Lewandowski, J. Quantum theory of geometry II: Volume operators. *Advances in Theoretical and Mathematical Physics*, **1** (1997), 388–429. doi:10.4310/ATMP.1997.v1.n2.a8.

[23] Ashtekar, A., Lewandowski, J., Marolf, D., Mourão, J., and Thiemann, T. Quantization of diffeomorphism invariant theories of connections with local degrees of freedom. *Journal of Mathematical Physics*, **36** (1995), 6456–6493. doi:10.1063/1.531252.

[24] Ashtekar, A., Pawlowski, T., and Singh, P. Quantum Nature of the Big Bang. *Physical Review Letters*, **96** (2006), 141301. doi:10.1103/PhysRevLett.96.141301.

[25] Ashtekar, A., Pawlowski, T., and Singh, P. Quantum nature of the big bang: An analytical and numerical investigation. *Physical Review D*, **73** (2006), 124038. doi:10.1103/PhysRevD.73.124038.

[26] Ashtekar, A., Pawlowski, T., and Singh, P. Quantum nature of the big bang: Improved dynamics. *Physical Review D*, **74** (2006), 084003. doi:10.1103/PhysRevD.74.084003.

[27] Ashtekar, A. and Singh, P. Loop quantum cosmology: a status report. *Classical and Quantum Gravity*, **28** (2011), 213001. doi:10.1088/0264-9381/28/21/213001.

[28] Ashtekar, A. and Wilson-Ewing, E. Loop quantum cosmology of Bianchi type I models. *Physical Review D*, **79** (2009), 083535. doi:10.1103/PhysRevD.79.083535.

[29] Atiyah, M. F. *Geometry of Yang-Mills Fields*. Scuola Normale Superiore, Pisa (1979). ISBN 978-88-7642-303-1.

[30] Baez, J. C. Generalized measures in gauge theory. *Letters in Mathematical Physics*, **31** (1994), 213–223. doi:10.1007/BF00761713.

[31] Baez, J. C. Spin Foam Models. *Classical and Quantum Gravity*, **15** (1998), 1827–1858. doi:10.1088/0264-9381/15/7/004.

[32] Bär, C. Spin geometry. Lecture Notes, Universität Potsdam (2011). Online; accessed 25 August 2025. Available from: https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Veranstaltungen/SS11/spingeo.pdf.

[33] Bär, C. Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes. *Communications in Mathematical Physics*, **333** (2015), 1585–1615. doi:10.1007/s00220-014-2097-7.

[34] Barbero G., J. F. Real Ashtekar variables for Lorentzian signature spacetimes. *Physical Review D*, **51** (1995), 5507–5510. doi:10.1103/PhysRevD.51.5507.

[35] Battisti, M. V. and Montani, G. The Mixmaster Universe in a generalized uncertainty principle framework. *Physics Letters B*, **681** (2009), 179–184. doi:10.1016/j.physletb.2009.10.003.

[36] Baytaş, B. and Yokomizo, N. Cosmological states in loop quantum gravity on homogeneous graphs. *Physical Review D*, **107** (2023), 066009. doi:10.1103/PhysRevD.107.066009.

[37] Beetle, C., Engle, J. S., Hogan, M. E., and Mendonça, P. Diffeomorphism invariant cosmological sector in loop quantum gravity. *Classical and Quantum Gravity*, **34** (2017), 225009. doi:10.1088/1361-6382/aa89c6.

[38] Belinski, V. A. On the cosmological singularity. *International Journal of Modern Physics D*, **23** (2014), 1430016. doi:10.1142/S021827181430016X.

[39] Belinskii, V. A., Khalatnikov, I. M., and Lifshitz, E. M. Oscillatory approach to a singular point in the relativistic cosmology. *Advances in Physics*, **19** (1970), 525–573. doi:10.1080/00018737000101171.

[40] Belinskii, V. A., Lifshitz, E. M., Khalatnikov, I. M., and Agyei, A. K. The oscillatory mode of approach to a singularity in homogeneous cosmological models with rotating axes. In *Perspectives in Theoretical Physics* (edited by L. P. Pitaevski), p. 677–689. Pergamon, Amsterdam (1992). doi:10.1016/B978-0-08-036364-6.50048-X.

[41] Bianchi, E., Rovelli, C., and Vidotto, F. Towards spinfoam cosmology. *Physical Review D*, **82** (2010), 084035. doi:10.1103/PhysRevD.82.084035.

[42] Bianchi, L. *Sugli spazí a tre dimensioni che ammettono un gruppo continuo di movimenti*. Accademia dei Lincei, Roma (1897). Available from: https://media.accademiaxl.it/memorie/S3-VXI-1898/Bianchi267-352.pdf.

[43] Biswas, I. and Teleman, A. Invariant connections and invariant holomorphic bundles on homogeneous manifolds. *Central European Journal of Mathematics*, **12** (2014), 1–13. doi:10.2478/s11533-013-0330-9.

[44] Bojowald, M. Loop quantum cosmology: I. Kinematics. *Classical and Quantum Gravity*, **17** (2000), 1489. doi:10.1088/0264-9381/17/6/312.

[45] Bojowald, M. Loop quantum cosmology: II. Volume operators. *Classical and Quantum Gravity*, **17** (2000), 1509. doi:10.1088/0264-9381/17/6/313.

[46] Bojowald, M. Absence of a Singularity in Loop Quantum Cosmology. *Physical Review Letters*, **86** (2001), 5227–5230. doi:10.1103/PhysRevLett.86.5227.

[47] Bojowald, M. Loop quantum cosmology: III. Wheeler-DeWitt operators. *Classical and Quantum Gravity*, **18** (2001), 1055. doi:10.1088/0264-9381/18/6/307.

[48] Bojowald, M. Isotropic loop quantum cosmology. *Classical and Quantum Gravity*, **19** (2002), 2717. doi:10.1088/0264-9381/19/10/313.

[49] Bojowald, M. Homogeneous loop quantum cosmology. *Classical and Quantum Gravity*, **20** (2003), 2595. doi:10.1088/0264-9381/20/13/310.

[50] Bojowald, M. Spherically symmetric quantum geometry: states and basic operators. *Classical and Quantum Gravity*, **21** (2004), 3733. doi:10.1088/0264-9381/21/15/008.

[51] Bojowald, M. Mathematical Structure of Loop Quantum Cosmology: Homogeneous Models. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, **9** (2013), 082. doi:10.3842/SIGMA.2013.082.

[52] Bojowald, M. Effective Field Theory of Loop Quantum Cosmology. *Universe*, **5** (2019), 44. doi:10.3390/universe5020044.

[53] Bojowald, M. The BKL scenario, infrared renormalization, and quantum cosmology. *Journal of Cosmology and Astroparticle Physics*, **2019** (2019), 026. doi:10.1088/1475-7516/2019/01/026.

[54] Bojowald, M. Critical Evaluation of Common Claims in Loop Quantum Cosmology. *Universe*, **6** (2020), 36. doi:10.3390/universe6030036.

[55] Bojowald, M., Calcagni, G., and Tsujikawa, S. Observational Constraints on Loop Quantum Cosmology. *Physical Review Letters*, **107** (2011), 211302. doi:10.1103/PhysRevLett.107.211302.

[56] Bojowald, M., Harada, T., and Tibrewala, R. Lemaitre-Tolman-Bondi collapse from the perspective of loop quantum gravity. *Physical Review D*, **78** (2008), 064057. doi:10.1103/PhysRevD.78.064057.

[57] Bojowald, M. and Kastrup, H. A. Symmetry reduction for quantized diffeomorphism-invariant theories of connections. *Classical and Quantum Gravity*, **17** (2000), 3009. doi:10.1088/0264-9381/17/15/311.

[58] Borissov, R., Pietri, R. D., and Rovelli, C. Matrix elements of Thiemann's Hamiltonian constraint in loop quantum gravity. *Classical and Quantum Gravity*, **14** (1997), 2793. doi:10.1088/0264-9381/14/10/008.

[59] Brizuela, D., Mena Marugan, G. A., and Pawlowski, T. Big Bounce and inhomogeneities. *Classical and Quantum Gravity*, **27** (2010), 052001. doi:10.1088/0264-9381/27/5/052001.

[60] Brizuela, D. and Uria, S. F. Semiclassical study of the mixmaster model: The quantum Kasner map. *Physical Review D*, **106** (2022), 064051. doi:10.1103/PhysRevD.106.064051.

[61] Brodbeck, O. On symmetric gauge fields for arbitrary gauge and symmetry groups. *Helvetica Physica Acta*, **69** (1996), 321–324.

[62] Brügmann, B., Gambini, R., and Pullin, J. Knot invariants as nondegenerate quantum geometries. *Physical Review Letters*, **68** (1992), 431–434. doi:10.1103/PhysRevLett.68.431.

[63] Brunnemann, J. and Fleischhack, C. Non-Almost Periodicity of Parallel Transports for Homogeneous Connections. *Mathematical Physics, Analysis and Geometry*, **15** (2012), 299–315. doi:10.1007/s11040-012-9109-7.

[64] Cailleteau, T., Barrau, A., Vidotto, F., and Grain, J. Consistency of holonomy-corrected scalar, vector, and tensor perturbations in loop quantum cosmology. *Physical Review D*, **86** (2012), 087301. doi:10.1103/PhysRevD.86.087301.

[65] Choquet-Bruhat, Y. and Geroch, R. Global aspects of the Cauchy problem in general relativity. *Communications in Mathematical Physics*, **14** (1969), 329–335. doi:10.1007/BF01645389.

[66] Cianfrani, F., Marchini, A., and Montani, G. The picture of the Bianchi I model via gauge fixing in Loop Quantum Gravity. *Europhysics Letters*, **99** (2012), 10003. doi:10.1209/0295-5075/99/10003.

[67] Cianfrani, F. and Montani, G. Boost invariance of the gravitational field dynamics: quantization without time gauge. *Classical and Quantum Gravity*, **24** (2007), 4161. doi:10.1088/0264-9381/24/16/011.

[68] Cianfrani, F. and Montani, G. The role of the time gauge in the second-order formalism. *International Journal of Modern Physics A*, **23** (2008), 1214–1217. doi:10.1142/S0217751X08040081.

[69] Cianfrani, F. and Montani, G. A critical analysis of the cosmological implementation of loop quantum gravity. *Modern Physics Letters A*, **27** (2012), 1250032. doi:10.1142/S0217732312500320.

[70] Cianfrani, F. and Montani, G. Implications of the gauge-fixing loop quantum cosmology. *Physical Review D*, **85** (2012), 024027. doi:10.1103/PhysRevD.85.024027.

[71] Corichi, A., Vukašinac, T., and Zapata, J. A. Polymer quantum mechanics and its continuum limit. *Physical Review D*, **76** (2007), 044016. doi:10.1103/PhysRevD.76.044016.

[72] Daura Serrano, J., Kohn, M., and Lawn, M.-A. G-invariant spin structures on spheres. *Annals of Global Analysis and Geometry*, **62** (2022), 437–455. doi:10.1007/s10455-022-09855-z.

[73] Demailly, J.-P. *Complex analytic and differential geometry.* Grenoble: Université de Grenoble I (1997). Available from: https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[74] DeWitt, B. S. Quantum Theory of Gravity. I. The Canonical Theory. *Physical Review*, **160** (1967), 1113–1148. doi:10.1103/PhysRev.160.1113.

[75] Dirac, P. A. M. *Lectures on Quantum Mechanics.* Belfer Graduate School of Science, Yeshiva University, New York (1964).

[76] Dirac, P. A. M. Generalized Hamiltonian dynamics. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **246** (1997), 326–332. doi:10.1098/rspa.1958.0141.

[77] Dirac, P. A. M. The theory of gravitation in hamiltonian form. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **246** (1997), 333–343. `doi:10.1098/rspa.1958.0142`.

[78] Einstein, A. Die Grundlage der allgemeinen Relativitätstheorie. *Annalen der Physik*, **354** (1916), 769–822. `doi:10.1002/andp.19163540702`.

[79] Einstein, A. *The meaning of relativity*. Princeton University Press, Princeton (1923). ISBN 978-0-6911-6408-3.

[80] Engle, J., Hanusch, M., and Thiemann, T. Uniqueness of the Representation in Homogeneous Isotropic LQC. *Communications in Mathematical Physics*, **354** (2017), 231–246. `doi:10.1007/s00220-017-2881-2`.

[81] Fadel, M. and Maggiore, M. Revisiting the algebraic structure of the generalized uncertainty principle. *Physical Review D*, **105** (2022), 106017. `doi:10.1103/PhysRevD.105.106017`.

[82] Fatibene, L. and Francaviglia, M. Spin structures on manifolds and Ashtekar variables. *International Journal of Geometric Methods in Modern Physics*, **02** (2005), 147–157. `doi:10.1142/S0219887805000533`.

[83] Fatibene, L., Orizzonte, A., Albano, A., Coriasco, S., Ferraris, M., Garruto, S., and Morandi, N. Introduction to loop quantum gravity. The Holst's action and the covariant formalism. *International Journal of Geometric Methods in Modern Physics*, **21** (2024), 2440016. `doi:10.1142/S0219887824400164`.

[84] Fazzini, F., Husain, V., and Wilson-Ewing, E. Shell-crossings and shock formation during gravitational collapse in effective loop quantum gravity. *Physical Review D*, **109** (2024), 084052. `doi:10.1103/PhysRevD.109.084052`.

[85] Fleischhack, C. Representations of the Weyl Algebra in Quantum Geometry. *Communications in Mathematical Physics*, **285** (2009), 67–140. `doi:10.1007/s00220-008-0593-3`.

[86] Fleischhack, C. and Levermann, P. Ashtekar Variables: Structures in Bundles. *Vietnam Journal of Mathematics*, **44** (2016), 203–214. `doi:10.1007/s10013-016-0184-0`.

[87] Fredenhagen, K. and Reszewski, F. Polymer state approximation of Schrödinger wavefunctions. *Classical and Quantum Gravity*, **23** (2006), 6577. `doi:10.1088/0264-9381/23/22/028`.

[88] Friedman, A. Über die Krümmung des Raumes. *Zeitschrift für Physik*, **10** (1922), 377–386. `doi:10.1007/BF01332580`.

[89] Friedmann, A. Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes. *Zeitschrift für Physik*, **21** (1924), 326–332. `doi:10.1007/BF01328280`.

[90] Gamboa, J., Loewe, M., and Rojas, J. C. Noncommutative quantum mechanics. *Physical Review D*, **64** (2001), 067901. doi:10.1103/PhysRevD.64.067901.

[91] Gaul, M. and Rovelli, C. A generalized Hamiltonian constraint operator in loop quantum gravity and its simplest Euclidean matrix elements. *Classical and Quantum Gravity*, **18** (2001), 1593. doi:10.1088/0264-9381/18/9/301.

[92] Geroch, R. Domain of Dependence. *Journal of Mathematical Physics*, **11** (1970), 437–449. doi:10.1063/1.1665157.

[93] Giesel, K., Han, M., Li, B.-F., Liu, H., and Singh, P. Spherical symmetric gravitational collapse of a dust cloud: Polymerized dynamics in reduced phase space. *Physical Review D*, **107** (2023), 044047. doi:10.1103/PhysRevD.107.044047.

[94] Giesel, K. and Thiemann, T. Algebraic quantum gravity (AQG) I. Conceptual Setup. *Classical and Quantum Gravity*, **24** (2007), 2465–2497. doi:10.1088/0264-9381/24/10/003.

[95] Giesel, K. and Thiemann, T. Algebraic quantum gravity (AQG). II. Semiclassical Analysis. *Classical and Quantum Gravity*, **24** (2007), 2499–2564. doi:10.1088/0264-9381/24/10/004.

[96] Giesel, K. and Thiemann, T. Algebraic quantum gravity (AQG). III. Semiclassical perturbation theory. *Classical and Quantum Gravity*, **24** (2007), 2565–2588. doi:10.1088/0264-9381/24/10/005.

[97] Giulini, D. Group averaging and refined algebraic quantization. *Nuclear Physics B - Proceedings Supplements*, **88** (2000), 385–388. doi:10.1016/S0920-5632(00)00808-2.

[98] Giulini, D. and Marolf, D. A Uniqueness Theorem for Constraint Quantization. *Classical and Quantum Gravity*, **16** (1999), 2489–2505. doi:10.1088/0264-9381/16/7/322.

[99] Giulini, D. and Marolf, D. On the generality of refined algebraic quantization. *Classical and Quantum Gravity*, **16** (1999), 2479. doi:10.1088/0264-9381/16/7/321.

[100] Gomis, J. and Mehen, T. Space–time noncommutative field theories and unitarity. *Nuclear Physics B*, **591** (2000), 265–276. doi:10.1016/S0550-3213(00)00525-3.

[101] Guedes, T. L. M., Mena Marugán, G. A., Müller, M., and Vidotto, F. Taming Thiemann's Hamiltonian constraint in canonical loop quantum gravity: Reversibility, eigenstates, and graph-change analysis. *Physical Review D*, **112** (2025), 026024. doi:10.1103/hjdk-kdhk.

[102] Ha, K. Y. and Lee, J. B. Left invariant metrics and curvatures on simply connected three-dimensional Lie groups. *Mathematische Nachrichten*, **282** (2009), 868–898. doi:10.1002/mana.200610777.

[103] Ha, K. Y. and Lee, J. B. The isometry groups of simply connected 3-dimensional unimodular Lie groups. *Journal of Geometry and Physics*, **62** (2012), 189–203. `doi:10.1016/j.geomphys.2011.10.011`.

[104] Haag, R. *Local Quantum Physics*. Theoretical and Mathematical Physics. Springer Berlin, Heidelberg, 2 edn. (1996).

[105] Hall, B. C. *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics. Springer New York, NY, 1 edn. (2013). ISBN 978-1-4614-7115-8.

[106] Hall, B. C. *Lie Groups, Lie Algebras, and Representations*. Graduate Texts in Mathematics. Springer Cham, 2 edn. (2015). ISBN 978-3-319-13466-6.

[107] Han, M. and Liu, H. Covariant $\overline{\mu}$-scheme effective dynamics, mimetic gravity, and nonsingular black holes: Applications to spherically symmetric quantum gravity. *Physical Review D*, **109** (2024), 084033. `doi:10.1103/PhysRevD.109.084033`.

[108] Harnad, J., Shnider, S., and Vinet, L. Group actions on principal bundles and invariance conditions for gauge fields. *Journal of Mathematical Physics*, **21** (1980), 2719–2724. `doi:10.1063/1.524389`.

[109] Hawking, S. W. The occurrence of singularities in cosmology. III. Causality and singularities. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **300** (1967), 187–201. `doi:10.1098/rspa.1967.0164`.

[110] Hawking, S. W. and Ellis, G. F. R. *The Large Scale Structure of Space-Time*. Cambridge University Press, Cambridge (1973). ISBN 978-0-521-09906-6. Available from: `https://www.cambridge.org/core/books/large-scale-structure-of-spacetime/1E6B961EC9878EDDBBD6AC0AF031CC93`.

[111] Henderson, A., Rovelli, C., Vidotto, F., and Wilson-Ewing, E. Local spinfoam expansion in loop quantum cosmology. *Classical and Quantum Gravity*, **28** (2010), 025003. `doi:10.1088/0264-9381/28/2/025003`.

[112] Henneaux, M. and Teitelboim, C. *Quantization of Gauge Systems*. Princeton University Press, Princeton (1992). ISBN 978-0-691-03769-1.

[113] Hilbert, D. Die Grundlagen der Physik . (Erste Mitteilung.). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, **1915** (1915), 395–408.

[114] Holst, S. Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action. *Physical Review D*, **53** (1996), 5966–5969. `doi:10.1103/PhysRevD.53.5966`.

[115] Immirzi, G. Real and complex connections for canonical gravity. *Classical and Quantum Gravity*, **14** (1997), L177. `doi:10.1088/0264-9381/14/10/002`.

[116] Kempf, A. Uncertainty Relation in Quantum Mechanics with Quantum Group Symmetry. *Journal of Mathematical Physics*, **35** (1994), 4483–4496. doi:10.1063/1.530798.

[117] Kempf, A., Mangano, G., and Mann, R. B. Hilbert space representation of the minimal length uncertainty relation. *Physical Review D*, **52** (1995), 1108–1118. doi:10.1103/PhysRevD.52.1108.

[118] Kobayashi, S. and Nomizu, K. *Foundations of Differential Geometry, Volume 1*. A Wiley Publication in Applied Statistics. Wiley (1996). ISBN 978-0-471-15733-5.

[119] Kobayashi, S. and Nomizu, K. *Foundations of Differential Geometry, Volume 2*. Wiley Classics Library. Wiley (1996). ISBN 978-0-471-15732-8.

[120] Kontsevich, M. Deformation quantization of poisson manifolds. *Letters in Mathematical Physics*, **66** (2003), 157–216. doi:10.1023/B:MATH.0000027508.00421.bf.

[121] Kori, T. Pre-symplectic structures on the space of connections. *Differential Geometry and its Applications*, **67** (2019), 101559. doi:10.1016/j.difgeo.2019.101559.

[122] Landau, L. D. and Lifshitz, E. M. *The Classical Theory of Fields: Volume 2*. Butterworth-Heinemann (1975). ISBN 978-0-7506-2768-9.

[123] Lawson, H. B. and Michelsohn, M.-L. *Spin Geometry*. Princeton University Press, Princeton (2016). ISBN 978-1-4008-8391-2.

[124] Lee, J. M. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, New York, NY (2014). ISBN 978-1-4419-9981-8.

[125] Lemaître, A. G. A Homogeneous Universe of Constant Mass and Increasing Radius accounting for the Radial Velocity of Extra-galactic Nebulæ. *Monthly Notices of the Royal Astronomical Society*, **91** (1931), 483–490. doi:10.1093/mnras/91.5.483.

[126] Lewandowski, J., Okołów, A., Sahlmann, H., and Thiemann, T. Uniqueness of Diffeomorphism Invariant States on Holonomy–Flux Algebras. *Communications in Mathematical Physics*, **267** (2006), 703–733. doi:10.1007/s00220-006-0100-7.

[127] Libermann, P. and Marle, C.-M. *Symplectic Geometry and Analytical Mechanics*, vol. 35 of *Mathematics and Its Applications*. Springer Dordrecht, 1 edn. (1987). ISBN 978-90-277-2438-0.

[128] LIGO Scientific Collaboration and Virgo Collaboration, et al. Observation of gravitational waves from a binary black hole merger. *Physical Reviw Letters*, **116** (2016), 061102. doi:10.1103/PhysRevLett.116.061102.

[129] Maggiore, M. The algebraic structure of the generalized uncertainty principle. *Physics Letters B*, **319** (1993), 83–86. doi:10.1016/0370-2693(93)90785-G.

[130] Maggiore, M. Quantum groups, gravity, and the generalized uncertainty principle. *Physical Review D*, **49** (1994), 5182–5187. doi:10.1103/PhysRevD.49.5182.

[131] Marsden, J. E. and Ratiu, T. S. Hamiltonian systems on linear symplectic spaces. In *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems* (edited by J. E. Marsden and T. S. Ratiu), p. 61–103. Springer, New York, NY (1999). doi:10.1007/978-0-387-21792-5_2.

[132] Martín-Benito, M., Mena Marugán, G. A., and Pawlowski, T. Loop quantization of vacuum Bianchi I cosmology. *Physical Review D*, **78** (2008), 064008. doi:10.1103/PhysRevD.78.064008.

[133] Mele, F. M. and Münch, J. Physical relevance of the fiducial cell in loop quantum cosmology. *Physical Review D*, **108** (2023), 106004. doi:10.1103/PhysRevD.108.106004.

[134] Mele, F. M. and Münch, J. On the role of fiducial structures in minisuperspace reduction and quantum fluctuations in LQC. *Classical and Quantum Gravity*, **41** (2024), 245003. doi:10.1088/1361-6382/ad8c1e.

[135] Milnor, J. Curvatures of left invariant metrics on lie groups. *Advances in Mathematics*, **21** (1976), 293–329. doi:10.1016/S0001-8708(76)80002-3.

[136] Montani, G., Battisti, M. V., Benini, R., and Imponente, G. *Primordial Cosmology*. WORLD SCIENTIFIC, Singapore (2011).

[137] Mukhanov, V. F. and Chibisov, G. V. Quantum Fluctuations and a Nonsingular Universe. *JETP Letters*, **33** (1981), 532–535.

[138] Mukhanov, V. F., Feldman, H. A., and Brandenberger, R. H. Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions. *Physics Reports*, **215** (1992), 203–333. doi:10.1016/0370-1573(92)90044-Z.

[139] Müller, O. Measures in the geometric quantization of field theories. *Journal of Geometry and Physics*, **56** (2006), 1029–1041. doi:10.1016/j.geomphys.2005.06.003.

[140] Nash, C. *Differential Topology and Quantum Field Theory*. Elsevier Science (1991). ISBN 978-0-12-514076-8.

[141] Penrose, R. Gravitational Collapse and Space-Time Singularities. *Physical Review Letters*, **14** (1965), 57–59. doi:10.1103/PhysRevLett.14.57.

[142] Perez, A. The Spin-Foam Approach to Quantum Gravity. *Living Reviews in Relativity*, **16** (2013), 3. doi:10.12942/lrr-2013-3.

[143] Reisenberger, M. P. and Rovelli, C. “Sum over surfaces” form of loop quantum gravity. *Physical Review D*, **56** (1997), 3490–3508. doi:10.1103/PhysRevD.56.3490.

[144] Rellich, F. and Berkowitz, J. *Perturbation Theory of Eigenvalue Problems*. New York University, Institute of Mathematical Sciences, New York (1953). ISBN 9780677006802.

[145] Rovelli, C. and Smolin, L. Knot Theory and Quantum Gravity. *Physical Review Letters*, **61** (1988), 1155–1158. doi:10.1103/PhysRevLett.61.1155.

[146] Rovelli, C. and Smolin, L. The physical Hamiltonian in nonperturbative quantum gravity. *Physical Review Letters*, **72** (1994), 446–449. doi:10.1103/PhysRevLett.72.446.

[147] Rovelli, C. and Smolin, L. Discreteness of area and volume in quantum gravity. *Nuclear Physics B*, **442** (1995), 593–619. doi:10.1016/0550-3213(95)00150-Q.

[148] Rovelli, C. and Smolin, L. Spin networks and quantum gravity. *Physical Review D*, **52** (1995), 5743–5759. doi:10.1103/PhysRevD.52.5743.

[149] Rovelli, C. and Vidotto, F. Stepping out of homogeneity in loop quantum cosmology. *Classical and Quantum Gravity*, **25** (2008), 225024. doi:10.1088/0264-9381/25/22/225024.

[150] Rovelli, C. and Vidotto, F. *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*. Cambridge University Press, Cambridge (2014). ISBN 978-1-107-06962-6.

[151] Rovelli, C. and Vidotto, F. Planck stars. *International Journal of Modern Physics D*, **23** (2014), 1442026. doi:10.1142/S0218271814420267.

[152] Ryan, M. P. J. The oscillatory regime near the singularity in Bianchi-type IX universes. *Annals of Physics*, **70** (1972), 301–322. doi:10.1016/0003-4916(72)90269-2.

[153] Sahlmann, H. and Sherif, W. Towards quantum gravity with neural networks: solving quantum Hamilton constraints of 3d Euclidean gravity in the weak coupling limit. *Classical and Quantum Gravity*, **41** (2024), 215006. doi:10.1088/1361-6382/ad7c14.

[154] Sahlmann, H. and Thiemann, T. Towards the QFT on Curved Spacetime Limit of QGR. I: A General Scheme. *Classical and Quantum Gravity*, **23** (2006), 867–908. doi:10.1088/0264-9381/23/3/019.

[155] Salminen, T. and Tureanu, A. Noncommutative time in quantum field theory. *Physical Review D*, **84** (2011), 025009. doi:10.1103/PhysRevD.84.025009.

[156] Segal, I. E. The phase space for the yang-mills equations. In *Differential Geometric Methods in Mathematical Physics* (edited by H.-D. Doebner), p. 101–109. Springer, Berlin, Heidelberg (1981). ISBN 978-3-540-38573-8. doi:10.1007/3-540-10578-6_25.

[157] Segreto, S. and Montani, G. Dynamics of the Mixmaster universe in a non-commutative generalized uncertainty principle framework. *Journal of Cosmology and Astroparticle Physics*, **2025** (2025), 061. `doi:10.1088/1475-7516/2025/03/061`.

[158] Sjamaar, R. and Lerman, E. Stratified Symplectic Spaces and Reduction. *Annals of Mathematics*, **134** (1991), 375–422. `doi:10.2307/2944350`.

[159] Śniatycki, J. A hamiltonian analysis of Yang-Mills equations. *Reports on Mathematical Physics*, **44** (1999), 205–214. `doi:10.1016/S0034-4877(99)80162-1`.

[160] Steenrod, N. *The Topology of Fibre Bundles*. Princeton University Press, Princeton (1999). ISBN 978-0-691-00548-5.

[161] Šubin, M. A. Differential and pseudodifferential operators in spaces of almost periodic functions. *Mathematics of the USSR-Sbornik*, **24** (1974), 547. `doi:10.1070/SM1974v024n04ABEH001923`.

[162] The Event Horizon Telescope Collaboration, et al. First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole. *The Astrophysical Journal Letters*, **875** (2019), L4. `doi:10.3847/2041-8213/ab0e85`.

[163] Thiemann, T. Quantum spin dynamics (QSD). *Classical and Quantum Gravity*, **15** (1998), 839. `doi:10.1088/0264-9381/15/4/011`.

[164] Thiemann, T. Quantum spin dynamics (QSD): II. The kernel of the Wheeler - DeWitt constraint operator. *Classical and Quantum Gravity*, **15** (1998), 875. `doi:10.1088/0264-9381/15/4/012`.

[165] Thiemann, T. Quantum spin dynamics (QSD): III. Quantum constraint algebra and physical scalar product in quantum general relativity. *Classical and Quantum Gravity*, **15** (1998), 1207. `doi:10.1088/0264-9381/15/5/010`.

[166] Thiemann, T. *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2007). ISBN 978-0-521-74187-3.

[167] Thiemann, T. Symmetry reduction, gauge reduction, backreaction, and consistent higher order perturbation theory. *Physical Review D*, **110** (2024), 104025. `doi:10.1103/PhysRevD.110.104025`.

[168] Velhinho, J. M. A groupoid approach to spaces of generalized connections. *Journal of Geometry and Physics*, **41** (2002), 166–180. `doi:10.1016/S0393-0440(01)00051-1`.

[169] Wang, H.-C. On Invariant Connections over a Principal Fibre Bundle. *Nagoya Mathematical Journal*, **13** (1958), 1–19. `doi:10.1017/S0027763000023461`.

[170] Weinstein, A. Poisson geometry. *Differential Geometry and its Applications*, **9** (1998), 213–238. doi:https://doi.org/10.1016/S0926-2245(98)00022-9.