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Post-flight Trajectory, Wind and Aerodynamic Angles Reconstruction for Launch Vehicles in Atmospheric Flight

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Abstract

The research deals with the topics of trajectory reconstruction (TR) and post-flight estimation of aerodynamic angles and wind components for launch vehicles (LV) during the atmospheric phase of the ascending flight. The aim is to investigate state-of-the-art methods used in TR and post-flight analysis, assessing their limitations to improve existing approaches and developing suitable and effective algorithms.

A complete nonlinear simulation model of the LV motion, featuring a realistic control system, comprehensive of all relevant aspects for the post-flight analysis is first developed to generate synthetic sensor data to be used for testing and validation of the proposed estimation methodologies. The simulation environment is completed with models of the atmosphere, wind, and sensors

Among the algorithms for TR referenced in the literature, one of the most widely used for TR is NewSTEP. Starting from the NewSTEP kinematic formulation for the process model, an Extended Kalman Filter (EKF) and an Unscented Kalman Filter (UKF) featuring a Fraser-Potter smoother are developed to achieve the best-estimated trajectory (BET). The filters make use of vehicle and ground-based measurements, where acceleration and angular rates measured by an Inertial Measurement Unit (IMU) are treated as inputs in the filter model; uncertainty parameters such as IMU biases are also included as filter states. The consistency and performance of smoothed-UKF and EKF estimates are evaluated and compared by means of Monte Carlo analyses for the case study involving the first-stage flight of the Ares-I LV, in a mission to the International Space Station (ISS) starting from the Kennedy Space Centre Complex LC-39B.

The estimation of aerodynamic angles and wind components is realized by exploiting a technique based on a Consider Extended Kalman filter (CEKF). The filter uses measurements provided by an IMU, a GNSS sensor, and an Air Data System (ADS), while first-order Gauss-Markov processes allow for the evaluation of elastic deformations at the gimbal and the wind modeling. The formulation takes into account the systematic errors of the IMU as biases and scale factors (selected as *parameters* for the CEKF), whereas the airspeed components are considered as filter states in place of the aerodynamics angles, which are computed *a-posteriori*. A multiplicative error quaternion formulation is adopted to provide an accurate yet robust reconstruction of the LV attitude.

The filter performance for the Consider Multiplicative Extended Kalman filter (CMEKF) is assessed by specific simulations and Monte Carlo campaigns. Results are presented and discussed for the aforementioned phase of the mission of the

reference LV model. The methodology appears suitable for the estimation of the angle of attack, sideslip angle, and wind speed components, in spite of high levels of uncertainty on model parameters and wind data.

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Nomenclature

Acronyms and abbreviations

AP	=	Aerodynamic reference point
AEKF	=	Additive EKF
BET	=	Best Estimated Trajectory
CEKF	=	Consider Extended Kalman Filter
CKF	=	Consider Kalman Filter
CFD	=	Computational fluid dynamics
CMEKF	=	Consider Multiplicative Extended Kalman Filter
DOF	=	Degree of freedom
EDL	=	Entry Descent and Landing
EKF	=	Extended Kalman Filter
ECEF	=	Earth-fixed Earth-centered reference frame
FADS	=	Flush Air Data System
FCS	=	Flight control system
FEM	=	Finite Element Model
GNC	=	Guidance Navigation Control
GNSS	=	Global Navigation Satellite System
GM	=	Gauss-Markov
GS	=	Ground-based radar tracking station
HIAD	=	Hypersonic Inflatable Aerodynamic Decelerator
INS	=	Inertial Navigation System
IMU	=	Inertial measurement Unit
ISS	=	International Space Station
KF	=	Kalman Filter
KSC	=	Kennedy Space Center
LOFTID	=	Low-Earth Orbit Flight Test of an Inflatable Decelerator
LV	=	Launch Vehicle
MEKF	=	Multiplicative Extended Kalman Filter
MSL	=	Mars Science Laboratory
MEADS	=	MSL Entry Atmospheric Data System
NewSTEP	=	New Statistical Estimation Programm
PD	=	Proportional Derivative
PDF	=	Probability density function
RF	=	Reference frame

RMS	=	root-mean-square
RMSE	=	root-mean-square error
RHS	=	Right-hand side
STEP	=	Statistical Estimation Programm
STS	=	Space Transportation System
TAS	=	True Airspeed
TR	=	Trajectory reconstruction
UKF	=	Unscented Kalman Filter
ZLGT	=	zero lift gravity turn maneuver

List of Symbols

$\mathbf{a}_{\text{IMU}_{[B]}}$	=	true non-gravitational acceleration
b	=	LV diameter
$b(k)$	=	Fourier coefficient
$\mathbf{B}_\omega$	=	gyro bias diagonal matrix
C_{aero}	=	lateral aerodynamic force component
C_A	=	axial force aerodynamic coefficient
$C_{l_\alpha}, C_{l_\beta}, C_{m_\alpha}, \text{ and } C_{n_\beta}$	=	aerodynamic derivatives
C_{N_α}	=	normal force aerodynamic coefficient
C_{Y_β}	=	side-force aerodynamic coefficient
d	=	Padè model state variable
$\mathbf{d}_{IMU}$	=	IMU position vector from the cg
$\mathcal{D}$	=	dissipation function
D_{aero}	=	axial aerodynamic force component
DM	=	delay margin
e_0	=	scalar part of quaternion
$\mathbf{e}$	=	vector part of quaternion
$EI(l)$	=	bending inertia function for the Euler Bernoulli beam
$\mathcal{F}_A$	=	indicates the system of coordinate denoted with the subscript A
$\mathbf{F}_{base}$	=	base force term (part of drag)
$\mathbf{f}_{ext}$	=	external force vector
f_{wext}	=	external forces for the Euler Bernoulli beam
$\mathbf{F}_A$	=	aerodynamic force vector
$\mathbf{F}_g$	=	gravity force vector
$\mathbf{F}_N$	=	TWD force vector
$G(s)$	=	system plant

g	=	earth gravity constant
$\mathbf{g}_{ext}$	=	external torque vector
$\mathbf{f}_g$	=	earth gravity acceleration vector
h	=	altitude
h_{ato}	=	target orbit apogeo height
h_{pto}	=	target orbit perigeo height
$\mathbf{h}$	=	observation function
$H_{LP}(s)$	=	low-pass filter
H_N	=	notch filter
H_p	=	part of the measurement matrix associated with the <i>parameters</i> of the CMEKF
$H_{\mathbf{x}_c}$	=	part of the measurement matrix associated with the state $\mathbf{x}_c$ of the CMEKF
I_{sp}	=	Specific impulse
i_{to}	=	target orbit inclination
J	=	integrated LV moment of inertia in $\mathcal{F}_B$
J_v	=	LV moment of inertia of the vehicle base body in $\mathcal{F}_B$
J_e	=	moment of inertia of the nozzle in $\mathcal{F}_B$
$J_{x_e}, J_{y_e}, J_{z_e}$	=	diagonal elements of the moment of inertia of the nozzle in $\mathcal{F}_B$
J_{xx}, J_{yy}, J_{zz}	=	diagonal elements of the integrated LV moment of inertia in $\mathcal{F}_B$
$\mathcal{K}$	=	Kalman gain in UKF formulation
$\mathbf{K}$	=	Kalman gain
$\widetilde{K}$	=	stiffness matrix
K_{BC}	=	controller
K_D	=	derivative gain of PID controller
K_P	=	proportional gain of PID controller
l	=	structural fixed axial coordinate along $\vec{\mathbf{x}}_S$
L	=	LV height (fully assembled)
$L(h)$	=	turbulence length scale as a function of the altitude h
$L(s)$	=	loop gain transfer function
$L_h(h)$	=	horizontal severe turbulence length scale as a function of the altitude h at KSC
$L_v(h)$	=	vertical severe turbulence length scale as a function of the altitude h at KSC
L_{UKF}	=	size of the augmented state vector in UKF

$l_{x_{Cp}}[S]$	=	center of pressure x-coordinate location in $\mathcal{F}_S$
$l_{x_{cg}}[S]$	=	x-coordinate of cg location in $\mathcal{F}_S$
l_{xe}	=	distance between cg and nozzle
$l_{x_{INS}}[S]$	=	IMU x-coordinate location in $\mathcal{F}_S$
l_{xG}	=	distance between cg and the nozzle pivot point
m	=	integrated LV mass
$\bar{m}$	=	LV Weight total excluding payload
M	=	Mach number
$\widetilde{M}$	=	Mass matrix
M_a	=	accelerometer misalignment error matrix
M_ω	=	gyro misalignment error matrix
M_A	=	Aerodynamic torque vector
M_N	=	TWD torque vector
M_T	=	Thrust torque vector
m_v	=	LV base body mass
m_e	=	nozzle mass
N_{aero}	=	normal aerodynamic force component
N_a	=	accelerometer non-orthogonality error matrix
N_ω	=	gyro non-orthogonality error matrix
$\mathcal{O}$	=	origin of the $\mathcal{F}_B$ RF
P	=	error covariance matrix
P_b	=	backward error covariance matrix in the context of smoothing
P_f	=	forward error covariance matrix in the context of smoothing
p	=	parameters vector in the context of CMEKF
p	=	x component of ω_{BI}
PM	=	phase margin
q	=	y component of ω_{BI}
$\mathbf{q}$	=	quaternion vector
q_i	=	generalized coordinate in Lagrange's equation
$\mathbf{Q}$	=	process noise covariance matrix
$\mathbf{Q}_{(.)}$	=	represents a part of the diagonal process noise covariance matrix associated with the variable defined in the subscript $(.)$
Q_{dyn}	=	dynamic pressure
Q_i	=	generalized forces in Lagrange's equation

$\mathbf{Q}_{\eta_f}$	=	generalized forces vector for the bending equation
r	=	z component of $\boldsymbol{\omega}_{BI}$
$\mathbf{R}_{AB}$	=	The generic rotation matrix from the generic frame $\mathcal{F}_B$ to the generic frame $\mathcal{F}_A$
$\mathbf{R}$	=	measurement noise covariance matrix
$\mathbf{R}_{(.)}$	=	represents a part of the diagonal measurement noise covariance matrix associated with the variable defined in the subscript $(.)$
$\mathbf{r}_I$	=	inertial position of the LV expressed in $\mathcal{F}_B$
$\mathbf{r}_e$	=	position of the nozzle cg w.r.t. $\mathcal{O}$ in $\mathcal{F}_B$
R_E	=	mean terrestrial radius
$\mathbf{r}_G$	=	position of the gimbal point w.r.t. $\mathcal{O}$ in $\mathcal{F}_B$
$\mathbf{r}_{GNSS}$	=	true position of the GNSS receiver
$\mathbf{r}_{gs}$	=	position vector of the GS in $\mathcal{F}_F$
$\mathbf{r}_v$	=	inertial position of the LV base body expressed in $\mathcal{F}_B$
s	=	Laplace variable
S	=	LV cross-section
$\mathbf{T}_{rig}$	=	thrust vector in $\mathcal{F}_B$ without the elastic component
$\mathbf{T}$	=	thrust vector in $\mathcal{F}_B$
T	=	thrust magnitude
$\mathcal{T}$	=	kinetic energy
t	=	time
t_f	=	total time in which the RMSE is evaluated
T_s	=	IMU sample time
t_{va}	=	vertical ascent duration
$\mathbf{u}$	=	input vector
$\mathcal{V}$	=	potential energy
V_a	=	airspeed modulus
$\mathbf{V}_a$	=	airspeed vector
$\mathbf{v}_I$	=	LV inertial velocity expressed in $\mathcal{F}_B$
$\mathbf{v}_{GNSS}$	=	GNSS receiver velocity vector
$\mathbf{v}_e$	=	inertial velocity of the LV mass nozzle expressed in $\mathcal{F}_B$
$\mathbf{v}_g$	=	ground speed
$\mathbf{v}_w$	=	wind velocity
v_{wp}	=	generic wind speed for the Dryden filter
$\mathbf{v}_v$	=	inertial velocity of the LV base body expressed in $\mathcal{F}_B$
W	=	Steady state wind speed vs altitude KSC profile

$W_{\{\}}$	= weight for the sigma point transformation in UKF
$w(l, t)$	= transverse displacement or deflection for the Euler Bernoulli beam
$\mathbf{w}_{(.)}$	= white random noise associated with the measured variable defined in the subscript $(.)$
$\mathbf{w}_{d,w}$	= driven white random noise associated with GNSS position output signal
$\mathbf{w}_{r,w}$	= uncorrelated white random noise associated with GNSS position output signal
$\mathbf{w}_{r,c}$	= correlated noise sequence associated with GNSS position output signal
$\mathbf{w}_{v,w}$	= uncorrelated white random noise associated with GNSS velocity output signal
$\mathbf{x}_b$	= forward state in the context of smoothing
$\mathbf{x}_c$	= estimated state in CEKF
$\mathbf{x}_f$	= backward state in the context of smoothing
$\bar{\mathbf{x}}$	= reference condition used for the Taylor series expansion of the EKF
$\underline{\mathbf{x}}$	= estimation error
$\vec{\mathbf{x}}_B, \vec{\mathbf{y}}_B, \vec{\mathbf{z}}_B$	= Body fixed axes base vectors
$\vec{\mathbf{x}}_E, \vec{\mathbf{y}}_E, \vec{\mathbf{z}}_E$	= NED fixed axes base vectors
$\vec{\mathbf{x}}_F, \vec{\mathbf{y}}_F, \vec{\mathbf{z}}_F$	= ECEF fixed axes base vectors
$\vec{\mathbf{x}}_I, \vec{\mathbf{y}}_I, \vec{\mathbf{z}}_I$	= ECI fixed axes base vectors
$\vec{\mathbf{x}}_S, \vec{\mathbf{y}}_S, \vec{\mathbf{z}}_S$	= Structural fixed axes base vectors
$\vec{\mathbf{x}}_t, \vec{\mathbf{y}}_t, \vec{\mathbf{z}}_t$	= Trajectory fixed axes base vectors
$\mathbf{y}$	= true measurement vector (without the noise)
$\mathbf{Y}$	= smoothed output vector
$\mathbf{z}$	= measurement vector

Greek Symbols

α	= angle of attack
α_{GM}	= GM model parameter
α_{UKF}	= constant parameter of the UKF
α_w	= wind induced angle of attack
β	= angle of sideslip
$\beta_{a,x}, \beta_{a,y}, \beta_{a,z}$	= accelerometer bias components
β_{UKF}	= constant parameter of the UKF

β_ω	=	accelerometer bias vector
$\beta_{\omega,x}, \beta_{\omega,y}, \beta_{\omega,z}$	=	gyro bias components
γ	=	elevation w.r.t. the radar tracking station ($\mathcal{F}_G$ RF)
δ_c	=	generic nozzle commanded rotation angle
$\delta_{\tilde{c}}$	=	generic delayed nozzle commanded rotation angle
δ_{GNSS}	=	position vector of the GNSS receiver from cg
δ_q	=	quaternion error
δ_y	=	nozzle rotation angle around y-axis of the $\mathcal{F}_N$
δ_z	=	nozzle rotation around z-axis of the $\mathcal{F}_N$
ζ	=	bending mode damping
ζ_p	=	damping parameter of the low-pass filter $H_{LP}(S)$
ζ_z	=	damping parameter of the low-pass filter $H_{LP}(S)$
ζ_N	=	damping parameter of the notch filter $H_{notch}(S)$
ζ_D	=	damping parameter of the notch filter $H_{notch}(S)$
ζ_{TVC}	=	damping ratio of the TVC actuator model
η	=	process noise vector
$\eta(\cdot)$	=	Process noise associated with the variable defined in the subscript ($\cdot$)
η_f	=	generalized modal coordinate vector
η_{f_n}	=	generalized modal coordinate vector for the nth bending mode of the LV
Θ	=	pitch angle
θ	=	longitude
θ_{gs}	=	radar Ground station longitude
θ_{LB}	=	launch base longitude
Λ	=	wind direction
λ	=	scale factor error vector
Λ_a	=	accelerometer scale factor error matrix
$\lambda_{gp}, \lambda_{gq}, \lambda_{gr}$	=	gyro scale factor
Λ_ω	=	gyro scale factor error matrix
λ_{UKF}	=	scaling parameter in UKF
μ	=	misalignment error vector
$\mu(l)$	=	linear mass density function for the Euler Bernoulli beam
μ_g	=	gravitational parameter
$\eta_{u_w}, \eta_{v_w}, \eta_{w_w}$	=	wind acceleration process noise components
ν	=	non-orthogonality error vector

ξ	=	Kalman filter measurement noise vector
$\xi_{(.)}$	=	measurement noise associated with the variable defined in the subscript $(.)$
ρ	=	atmospheric density
ρ_R	=	radar range
$\bar{\Sigma}$	=	normalized mode slopes matrix
$\bar{\sigma}$	=	normalized mode slope
$\sigma_{(.)}$	=	Gaussian noise standard deviation associated with the variable defined in the subscript $(.)$
$\sigma(h)$	=	turbulence magnitude as a function of the altitude h
$\bar{\Sigma}_{IMU}$	=	mass normalized mode slopes matrix for the first three bending modes at the IMU location
$\bar{\Sigma}_G$	=	mass normalized mode slopes matrix for the first three bending modes at the location of the mass gimbal element
$\bar{\Sigma}_v$	=	mass normalized mode slopes matrix for the first three bending modes at the location of the mass body element
$\sigma_{w(.)}$	=	noise standard deviation associated with the measurement defined in the subscript $(.)$
σ_G	=	elastic rotation vector at pivot point
$\sigma_h(h)$	=	horizontal severe turbulence magnitude as a function of the altitude h for the KSC
σ_{nINS}	=	elastic rotation associated with the nth elastic mode at IMU location
$\sigma_v(h)$	=	vertical severe turbulence magnitude as a function of the altitude h for the KSC
τ	=	GM autocorrelation time constant
τ_σ	=	GM autocorrelation time constant for the elastic angular displacements model at pivot point
τ_c	=	Gauss-Markov time constant for the GNSS correlated noise model
τ_p	=	second order Padè model delay time constant
τ_{wd}	=	GM autocorrelation time constant for the down component of the wind model
τ_{wl}	=	GM autocorrelation time constant for the north and east components of the wind model

Φ_s	=	Wiener filter
Φ	=	roll angle
ϕ	=	latitude
ϕ_{gs}	=	Radar Ground station latitude
ϕ_{LB}	=	Launch base latitude
χ	=	sigma point vector
χ	=	azimuth w.r.t. the radar tracking station ($\mathcal{F}_G$ reference frame)
χ_h	=	inverse of the observation function
$\bar{\Psi}$	=	mass normalized mode shapes matrix
$\bar{\psi}$	=	mass normalized mode shape
Ψ	=	yaw angle
$\bar{\Psi}_G$	=	mass normalized mode shapes matrix for the first three bending modes at the location of the mass gimbal element
ψ_G	=	elastic displacement vector at pivot point
ψ_{nINS}	=	elastic displacement associated with the nth elastic mode at IMU location
$\bar{\Psi}_v$	=	mass normalized mode shapes matrix for the first three bending modes at the location of the mass body element along the LV longitudinal axis
ω	=	generic angular rate vector
$\omega_{\oplus}$	=	Earth angular rate
$\underline{\omega}_{BM}$	=	lower bound of dispersed frequency of the bending mode
$\bar{\omega}_{BM}$	=	upper bound of dispersed frequency of the bending mode
ω_{BM}	=	bending mode natural frequency
ω_{cross}	=	cross-over frequency
ω_D	=	natural frequency parameter of the notch filter $H_{notch}(s)$
ω_G	=	angular rates vector of $\mathcal{F}_N$ w.r.t $\mathcal{F}_B$ in $\mathcal{F}_B$
$\dot{\omega}_G$	=	angular acceleration vector of $\mathcal{F}_N$ w.r.t $\mathcal{F}_B$ in $\mathcal{F}_B$
ω_N	=	natural frequency parameter of the notch filter $H_{notch}(s)$
ω_p	=	natural frequency parameter of the low-pass filter $H_{LP}(s)$

ω_{TVC}	=	natural frequency of the TVC actuator model
ω_z	=	natural frequency parameter of the low-pass filter $H_{LP}(s)$
Ω	=	attitude matrix related to the angular rate vector
Ω_ω	=	measured diagonal angular rate matrix

Subscripts

$(\cdot)_a$	=	designates a quantity associated with the accelerometer (except for the airspeed $\mathbf{V}_a$ and its component u_a, v_a, w_a)
$(\cdot)_i$	=	designates the iteration index
$(\cdot)_k$	=	designates the time increment
$(\cdot)_\times$	=	designates the cross product matrix
$(\cdot)_\omega$	=	designates a quantity associated with the gyro

Superscripts

$(\cdot)^a$	=	a designates the augmented state vector of the UKF
$(\cdot)^{cmd}$	=	cmd designates a commanded variable (usually the output of a control system)
$\tilde{(\cdot)}$	=	$\sim$ designates a measured variable $(\cdot)$
$\hat{(\cdot)}$	=	$\wedge$ designates the estimated variable $(\cdot)$
$(\cdot)^+$	=	$^+$ designates the <i>a posteriori</i> estimate of a state variable $(\cdot)$ within the context of Kalman filtering
$(\cdot)^-$	=	$^-$ designates the <i>a priori</i> estimate of a state variable $(\cdot)$ within the context of Kalman filtering

Chapter 1

Introduction

1.1 Research motivation

This thesis focuses on the development of trajectory reconstruction (TR) and postflight estimation techniques for launch vehicles (LVs) during the atmospheric ascending flight.

Post-flight data analysis is an essential element for the success of space missions. The extensive expenses involved in designing, building, launching, and supporting aerospace vehicles require investing significant effort into examining the gathered data in order to maximize the mission's value. During the last years, the trend in global rocket launches has shown significant growth, particularly in 2023, where 211 successful launches have been registered w.r.t. 178 in 2022 and 135 in 2021, indicating an upward trend in space activities [1].

Therefore, the development of new projects in the space sector in particular for LVs such as Vega C and Vega E brought the Italian Space Agency (ASI) to show interest in several topics including post-flight analysis for TR and data fusion as an important subject to support the development of the Italian rocket industry. This interest aligns with the strategies of the European industry to improve safety, and mission performance in terms of costs and payload capabilities, and increasing payload delivery precision. In order to achieve these goals, the industry shows interest in the application of advanced navigational systems such as hybrid Global Navigation Satellite System/Inertial Navigation System (GNSS/INS) on LV [2], with the goal of realizing a significant improvement in positioning accuracy w.r.t. the performance of inertial navigation system (INS) currently used on LV of the class of VEGA.

Other challenges concern integrating Air Data Systems (ADS) [3] or Flush Air Data Systems (FADS) [4] on LVs. Flight demonstrators such as Ares-I [5] have shown

practical problems such as the difficulty in allocating the sensor on the LV body, its calibration, and operations in adverse weather conditions, making this kind of sensor challenging to use in LV missions. There is considerable interest in improving safety during the atmospheric phase of the mission, and, in this respect, the availability of accurate estimations of dynamic pressure and/or aerodynamic angles may contribute to mitigate the aerodynamic load especially during the maximum dynamic pressure phase of flight or in non-nominal situations that can occur as a result of a failure.

In this framework, accurate modeling of LVs is not trivial as, for instance, execution of dedicated flight tests with specific maneuvers is ruled out for tight security and cost constraints. For these reasons, system identification for LVs is one of the goals of the space industry, which has further interest in assessing/improving the accuracy of vehicle aeromechanical models exploiting flight data in order to better understand theoretical predictions, validate CFD and/or wind tunnel test results, and get more accurate and reliable mathematical models for the purpose of simulation, stability analysis, and control system design and evaluation.

In this respect, two challenging areas of research are the reduction of aerodynamic model uncertainty, usually carried out through the post-processing of trajectory data, and, the development of aerodynamic load relief control schemes during the ascent phase. Results on those topics can significantly contribute to GNC algorithms improvement. Moreover, a reliable estimator of aerodynamic angles and wind velocity may improve the accuracy of the current balloon-based wind data collection method before launch. This improvement could be facilitated through data fusion techniques, integrating sensor data obtained during the mission with the pre-flight balloon-based data. Moreover, it is possible to increase the effectiveness and suitability of atmospheric state reconstruction for missions where balloon-based wind data are not available [6].

1.2 Main objectives and original contributions

Post-flight analysis in a wide sense aims at the characterization of uncertainty, the determination (or correction) of vehicle aeromechanical model parameters, and, more generally, the validation of the system model and guidance navigation and control (GNC) algorithms based on flight data. This thesis was carried out in the framework of the research agreement n. 2019-4-HH.0 CUP F86C17000080005 “Technical Assistance on Launch Vehicles and Propulsion” between the Italian Space Agency (ASI) and the Department of Mechanical and Aerospace Engineering (DIMA) at Sapienza University of Rome, with ASI aiming to enhance its expertise in the

field of TR and post-flight analysis.

The main goals of this work are

- To analyze the limitations of the classical approaches to TR
- To explore opportunities for enhancing the performance of current best-estimated trajectory (BET) algorithms providing improved performance, increased robustness, scalability, and extendability of TR codes
- To consider a novel approach for aerodynamic angles and wind components estimation, by the use of postflight data.

The initial task involves generating synthetic data for a launch mission. For this purpose an extensive simulation environment is developed to facilitate control system design through Monte Carlo analyses, allowing for the testing of various off-nominal flight conditions, including scenarios with windy conditions or deviations in model parameters. . The environment includes the mathematical model of the LV that captures the key interactions between the vehicles structure, engine, and rigid body dynamics. Furthermore, it integrates specific features that are typical of a launch mission, thereby providing a medium-fidelity simulation framework. Specifically, an atmospheric and wind model is developed, along with a reference trajectory, a flight control system for vehicle stabilization, and sensor models.

Next, an analysis of key literature works, such as those on the Ares-I LV [7], the hypersonic vehicle HXRV [8], and the Mars entry, descent, and landing probe [9], has identified the main methods for trajectory estimation, while a number of studies on aerodynamic parameters identification and wind correction for reusable re-entry vehicle [6] or real-time angle of attack estimator for commercial aircraft [10] have provided important insights into the methodologies currently adopted for aerodynamic angles and wind estimation.

The activities carried out in this work focus on Kalman filter-based methods [11] for nonlinear systems. Building on the kinematic model of NewSTEP [7], two filters are developed for TR using an EKF and UKF [12] with a smoother to refine the estimate. A different approach is taken for attitude estimation, employing a MEKF. A filter for the estimation of aerodynamic angles and wind components by using a *consider* version of the EKF is developed in order to accommodate a substantial number of additional, albeit predominantly unobservable system states or sometimes hard to estimate, due to the short flight duration of the first stage mission. Monte Carlo analyses validate filters performance, consistency, and robustness to parameter uncertainties.

As for the original contributions of the work: (i) a systematic procedure for initializing the TR filter proved to be effective in reducing the time spent on trial-and-error tuning, (ii) the use of a Multiplicative Extended Kalman Filter (MEKF) [13] has demonstrated better consistency and is promising for TR applications within a purely kinematic filter architecture for improved attitude estimation w.r.t. the additive formulation [14] and (iii) the suitability of a UKF for TR is demonstrated in cases where the uncertainty regarding initial conditions is significant, as well as where large off-nominal conditions are encountered.

The original contributions for the topic of aerodynamic angle and wind estimation concern the development of a filter that retains the model of the attitude kinematic and position from the previous BET filter and estimates the airspeed components from the lateral force equations compensating for the absence of wind accelerations and measurements. A Gauss-Markov noise model accounts for uncertainties due to systematic inertial measurement unit (IMU) error, bending dynamics, thrust, mass, atmospheric densities, and aerodynamic parameters.

1.3 Research framework

1.3.1 Trajectory Reconstruction

The TR process combines onboard, ground-based, and atmospheric measurements to produce the trajectory estimates. In short, it can be defined as the process of determining the position, velocity, and attitude of a vehicle with high accuracy, as well as evaluating the uncertainty of the estimate. Firstly applied to space missions, the TR found application also in the field of atmospheric flight.

There have been two fundamental approaches to reconstructing trajectories within Earth's atmosphere. The first method involves using radar or optical tracking data to determine the vehicle position, while the velocity is determined from Doppler measurements or by numerically differentiating the position data. This technique demands continuous radar coverage, without providing vehicle attitude information, and yielding inaccurate velocity information when numerical differentiation is used [15].

The second method relies on integrating the data from onboard accelerometers and gyroscopes. Double integration of the accelerations gives the position over time, with the integrated gyro data used to project the acceleration components in the appropriate reference system [16]. However, precise initial information about the vehicle position, velocity, and attitude is required in order to initiate the integration process. Furthermore, errors can propagate rapidly due to initial condition

inaccuracies or systematic errors in the measurements.

The use of estimation theory to determine trajectory characteristics was already used in interplanetary navigation and guidance problems [17], as well as in studies dealing with orbit determination [18, 19]. However, at that time the application of estimation theory to in-atmosphere flight was relatively new and not without many unresolved difficulties. In interplanetary space and orbital flight, the principal force acting on spacecraft is gravitational and its mathematical description is usually accurate, whereas the effects of aerodynamic force and moment acting on vehicles operating in the Earth's atmosphere, which were modeled rather imprecisely, can be relevant. Furthermore, in the atmosphere, the trajectory also depends on vehicle attitude which, in turn, is governed by complex guidance and autopilot systems through actuators. Thus, the principal difficulty in applying estimation theory to problems of vehicles in atmospheric flight lies in the development of adequately accurate mathematical models.

The TR problem was addressed for the Space Shuttle mission [20] by employing a Kalman-Schmidt filter [21] also known as *Consider Kalman filter* CKF. The approach involves numerically integrating data from onboard accelerometers and gyroscopes to predict the spacecraft state and update it with external measurements such as ground-based radio-tracking data. Errors in gyro and accelerometer measurements (such as biases and misalignments) and noise in ground-based tracking data can reduce the accuracy of TR and, in this respect, the study provides insights into the potential magnitude of trajectory state errors, and demonstrates how these errors also impact the estimation of aerodynamic coefficients.

A second work concerning the TR of the Space Transportation System (STS-1), specifically focused on the reentry trajectory, is reported in [22]. This study places significant emphasis on the pre-processing of data and the computation of the BET. The BET is then integrated with atmospheric measurements obtained in close temporal and spatial proximity to the entry phase, to determine trajectory parameters relative to the atmosphere and to compute the aerodynamic coefficients for lift L and drag D of the Shuttle vehicle.

The research work [23] on Titan IIIC was principally aimed to provide an overview of the pre-flight and post-flight trajectory analyses of the LV, which was in charge of placing payloads into geosynchronous orbits. The paper provides a basic description of the LV itself, the mission sequence, and the sensors used to measure acceleration and attitude during the flight.

One of the best-known tools for TR, extensively used in the time frame 1960-1980, is the Statistical Trajectory Estimation Program STEP [24], [16] developed by

NASA/LaRC. The STEP methodology can be described as follows: the algorithm starts with a dynamic model for the simulation of vehicle motion and, next, the equations are integrated in time to yield a predicted trajectory. The 6-DOF model of the vehicle is based on the classic set of 12 ordinary first-order nonlinear differential equations, categorized into four groups

- Three equations handle the translation of the vehicle, based on the balance of external forces, so as to determine the velocity vector
- Three equations deal with translational kinematics, for the calculation of the vehicle position
- Three equations are related to the attitude dynamics, where the external torque balance is taken into consideration, and are solved for the angular rate;
- The final set of three equations deals with the rotational kinematics, for attitude evaluation.

At this point, STEP employs the linear filter theory [11] to iteratively obtain the solution in terms of position and velocity as functions of time, that minimizes variance, commonly referred to as the least-squares solution for uncorrelated data [16]. Two main versions of the STEP algorithm were developed, the first one dubbed STEP1 [24] differs from the second (STEP2) because it requires a precise description of the aerodynamic force on the vehicle, required for the translational dynamic equations, and is restricted to non-thrusting vehicles. To this end, precise modeling of aerodynamic coefficients, atmospheric density, and vehicle mass is required, parameters that are usually uncertain. Consequently, the STEP2 model [16] avoids the need for the description of the aerodynamic actions by directly incorporating the measurements from airborne accelerometers into the equations of motion. Therefore it can be applied to any flight of vehicles where linear accelerations and angular rates components are recorded, and at least partial radar coverage is available. The systematic errors present in the accelerometer data are accommodated within the modeling process.

The STEP code has been applied in a variety of LV and re-entry vehicle programs, including those of the Viking project [25], [26], [27], where a solution to the problem of estimating an entry trajectory into an uncertain Mars atmosphere is obtained by exploiting the available measurements, that is, accelerometer and gyro rate readings along three axes, and altimeter data, and assuming that an initial condition is available. In addition to the time evolution of trajectory variables, the filter also calculates an estimate of the atmospheric density as a function of altitude.

One of the most widely used tools for TR in the last twenty years is the New Statistical Trajectory Estimation Programme known as NewSTEP [28], which allows for the solution of the BET problem. NewSTEP is a generalization of the STEP method as it borrows heavily from the STEP formulation while featuring many enhancements to either the algorithms or implementation. In particular, NewSTEP is coded entirely in Matlab® in order to make use of built-in capabilities such as numerical integration routines. The NewSTEP code has been applied in a variety of programs, including the Hyper-X Mach 7 and Mach 10 flights [8], the Mars InSight mission [29], the Orion Launch Abort flight tests (PA-1) [30], and the Ares I-X flight tests [31],[7], [32].

NewSTEP implements an Iterative Extended Kalman filter (IEKF) designed for solving TR problems in flight test experiments. The IEKF is a combination of two filtering passages, the first one (forward filter) is based on the Extended Kalman Filter (EKF) algorithm [13], while the second one (backward filter) uses the output of the first step as the initial condition of a further estimation process that propagates the state prediction through all the observed data from the final t_{k_n} through the initial time t_{k_1} . In order to obtain the combined estimate from the two filtering processes, a Fraser-Potter smoother [33] is used, which consists of a weighted average over the inverse of the covariance of the two filters.

The estimation process is realized in two distinct phases

- flight-data pre-processing
- data fusion by means of the filtering algorithm

Note that the first one relies on the analysis and processing of data collected from various sources, including onboard sensors such as IMU, GNSS, ADS, and ground-based instruments such as radar, interferometers, and optical systems, as well as atmospheric measurements from weather balloons.

The ARES I-X launcher BET programme [31], [7] is among the most interesting research work related to the TR topic using NewSTEP. The TR involves the estimation of vehicle trajectories, velocities, and other quantities such as dynamic pressure, Mach number, and angle of attack, assuming known the wind velocity during the ascent phase. The reconstruction process integrates data from onboard, ground-based, and atmospheric measurements.

The Ares I-X vehicle BET was determined using measurements from the classic suite of sensors, including IMUs and ADS, whereas data from radar and weather balloons were also employed. The data provided by the ADS sensor were excluded from the filtering process because the total air temperature measurements were

unreliable, due to the effects of rain-falls before launch and calibration issues. The paper provides details about flight-data processing and, in particular, it is relevant to observe that the LV mounted redundant IMUs, one on the first stage and the second one on the upper stage, allowing for a higher quality of angular velocity data. As a matter of fact, measurements of pitch and yaw rates were blended in order to improve the accuracy of estimates, whereas the acceleration data provided from the sensor located in the first stage were ruled out due to the solid rocket motor oscillations, and only the upper stage acceleration components were considered in the estimation process.

The aforementioned studies also addressed methods for smoothing the estimated variables and determining the accelerations and rate uncertainties due to the low-frequency structural modes of the vehicle.

One of the most complete and exhaustive applications of the NewSTEP tool is reported in [8]. The primary focus of the Hyper-X program was on developing a hypersonic airbreathing research vehicle utilizing an integrated scramjet propulsion system. Key goals included collecting data on scramjet system performance, demonstrating controlled hypersonic flight capabilities, and the identification of a set of parameters. After the engine test, the vehicle followed a descent trajectory that included various maneuvers that were optimized for the post-flight extraction of stability and control derivatives and other aerodynamic characteristics.

The TR process uses in-flight telemetry data combined with a variety of redundant measurements to generate maximum-likelihood estimates of the vehicle position, velocity, and orientation time histories, along with the estimation of their levels of uncertainty. In the estimation process, systematic error sources that affect both the IMU and radar are taken into consideration, and the filter state vector is augmented with systematic error states in order to include them in a mathematical formulation that typically includes only position, velocity, and attitude. These systematic error terms are assumed to remain constant throughout the mission. The last two works presented in this Section concern aspects about navigation for LV.

Reference [2] focuses on the practical application of Kalman filtering in the context of rocket navigation. Specifically, the integration of data from global navigation satellite systems (GNSS) and inertial sensors to improve rocket navigation accuracy was taken into consideration. A Schmidt-Kalman filter or CKF is developed to account for parameter uncertainty within the sensor models. This is essential for improving the accuracy of rocket navigation. The estimator for the rocket navigation problem was designed as follows

- The estimated filter states included the attitude increment, gyro bias, inertial

position, inertial velocity, and accelerometer bias

- The filter considered states [13] are the uncertainty of the gyro scale factor, gyro misalignment, and nonorthogonality, accelerometer scale factor, accelerometer misalignment and nonorthogonality, as well as the GNSS autocorrelated noise sequence.

It is worth noting that the latter uncertain parameters were not directly estimated but their impact on the system was explicitly considered. The paper includes also a comparison between different grades of inertial sensors to assess their performance and suitability for the fusion scheme. In the end, the work demonstrated the effectiveness and consistency of the proposed estimation technique, the validation of which was carried out through simulations and real-world flight data obtained from the fourth flight of the Vega LV.

The work of Zanetti *et al.* [34] deals with the fine-alignment phase of the Orion spacecraft navigation system. The primary goal of the work is to enhance the accuracy of the attitude estimation of the Orion spacecraft just before launch. The only available measurement data during fine alignment came from the IMU, used in conjunction with pseudo-measurements, that is, theoretical values dubbed integrated velocity, zero velocity, and pad position, which indicated that the vehicle was stationary w.r.t. the launch pad before liftoff. An EKF was employed in both fine-alignment operations and atmospheric flight phases (ascent, ascent aborts, and entry), the filter states being position, velocity, attitude, clock bias, and drift of the GPS receiver, together with IMU systematic errors modeled as first-order Gauss Markov processes. The paper also reports how measurement noise was modeled for each type of pseudo-measurement, taking into account variations caused by the motion of the launch stack. In the end, a robustness analysis was conducted through simulations to assess the filter performance under various conditions of twist and sway motion, which were treated as white noise, demonstrating the effectiveness of the estimation system.

1.3.2 Estimation of aerodynamic angles and atmospheric parameters

In the challenging phase of a LV first-stage atmospheric flight, the dynamics and the vehicle control exhibit unique characteristics. These include rapid and substantial variations in inertial and aerodynamic parameters, as well as interactions between rigid and structural modes. This class of vehicles also experiences high Mach numbers as for instance experimental vehicles such as the HXRV from the Hyper-X

program [8] or re-entry vehicles such as Mars InSight [29], where the aerodynamic load is to be accurately controlled. A classic approach for load alleviation involves the control of aerodynamic angles error, to align the LV with the airstream.

A strategy for the real-time measurement of aerodynamic angles is to use an ADS [3], which provides quantities relative to the air such as TAS, Mach number, and aerodynamic angles. This sensor has been designed in a special version to be employed on vehicles operating under high Mach number conditions (FADS). The FADS makes use of surface pressure measurements from the nose cap of the vehicle for deriving air data parameters such as angle of attack, angle of sideslip, and Mach number. Conceptually, it uses a matrix of flush surface ports to infer air data [35, 36]. FADS are mainly employed in re-entry vehicles such as the Hypersonic Inflatable Aerodynamic Decelerator (HIAD) [37] in the Low-Earth Orbit Flight Test of an Inflatable Decelerator (LOFTID) project, where flush air data sensors mounted in the vehicle nose cone provided a means to reconstruct the atmospheric-relative trajectory during entry. The reconstruction results are compared with a trajectory simulation that was based on the day of flight entry conditions.

In the work by Lugo *et al.* [38] a method to integrate FADS-derived density and wind estimates with a numerical predictor-corrector guidance algorithm is presented. It is shown that by incorporating a FADS system model into the onboard GNC platform, an enhancement in guidance performance and a decrease in trajectory uncertainties can be realized, compared to relying on corrupted accelerometers, mitigating large angle of attack oscillations and improving the overall accuracy of the landing site miss distance.

Another relevant work [35] outlines how FADS utilizes pressure measurements from a reusable re-entry vehicle nose cap to derive essential air data parameters such as angle of attack, sideslip angle, and Mach number. The tests are conducted in a wind tunnel to verify the FADS system performance, at a wind speed of 65 m/s (Mach 0.2) to assess sensor functionality in low-speed conditions. The article also covers the system-level engineering concepts for integrating FADS onboard a vehicle.

For the HXRV [4] vehicle of the Hyper-X program the problem of reducing the aerodynamic load in flight control laws has been addressed and three methods to estimate the aerodynamic angles in real-time have been proposed as an alternative to the FADS approach. The first one is the "model approach" and concerns the evaluation of the angle of attack rate from point mass kinematic equations. This approach combines the pitching motion measurements with vertical acceleration measurements to produce an estimated angle of attack. This technique may be prone to inaccuracies in the aerodynamic parameters for transforming normal acceleration

into the angle of attack. The second approach is the "IMU only", which combines the inertial velocity components as freestream velocity components ignoring the air mass velocity and the wind velocity to calculate the aerodynamic angles. The third one, a "complementary filter approach" exploits a complementary filter [39], that tracks a low-frequency input (α) and integrates the high-frequency input ($\dot{\alpha}$ rate), these two outputs are combined to estimate the α . In the end, the "IMU only" and a hybrid approach that blends FADS and IMU have been taken into account for the design of the control system.

Various methodologies are proposed in the literature to estimate aerodynamic angles and wind profiles for transatmospheric vehicles. In particular, relevant for wind estimation is the work by Vitale *et al.* [6] where a wind profile correction is achieved during the flight test of a reusable re-entry vehicle using Gauss-Markov models and an Unscented Kalman Filter. The paper also reports a methodology to estimate and correct the aerodynamic coefficients by exploiting post-flight data. Karlgaard *et al.* [40] combine flight data from weather balloons, meteorological sounding rockets, and onboard pressure sensors to reconstruct the atmospheric density and winds, together with Mach number, dynamic pressure, angle of attack, and sideslip, is reconstructed. One of the most relevant works for aerodynamic angle estimation is another work of Vitale *et al.* [10], where a virtual sensor for the angle of attack applicable to commercial aircraft is developed. The methodology involves a first phase of calibration in order to identify the lift curve of the aircraft, and a second phase where the EKF is exploited to estimate the aerodynamic angles and the wind using the lift curve and the measurements of TAS, inertial velocity components, and normal load factor given, respectively, by ADS, GPS and IMU.

Other studies solve the same estimation problem in real-time, with the aim of improving the management of structural loads for LVs in load-alleviation systems. In Ref. [41] the aerodynamic angles are determined for load-relief feedback by a Kalman filter that accounts for a simplified LV dynamical model and uses measurements of lateral accelerations provided by the inertial navigation system (INS). Within the same application area, Simplicio *et al.* [42], [43] present a disturbance observer for onboard wind anticipation that drives a load-relief compensator, while the use of wind sensors, either true or virtual, as well as a LIDAR wind sensor, is proposed by Boelitz *et al.* [44] in order to provide near real-time wind measurements. An interesting work on post-flight estimation of aerodynamic angles is [45] on the Mars Science Laboratory (MSL) free-flight aeroballistic experiment involving instrumented scale models of the MSL, which was developed at the Aberdeen Proving Ground in Maryland. The MSL was equipped with a FADS similar to the MSL Entry

Atmospheric Data System (MEADS) [46], and the primary goal of the work is to determine the values of Mach number, and aerodynamic angles that best match the observed pressures with those predicted by a computational fluid dynamics (CFD) model. The estimation process was carried out sequentially, focusing on one set of parameters at a time. The considered quantities are Mach number M , angle of attack α , and angle of sideslip β . The process can be regarded as an attempt to find a point in the M - α - β space that best aligns with the 10 pressure values measured in different locations of the vehicle. This was carried out by adjusting these parameters to minimize the differences (residuals) between the pressure readings and the predictions made by the CFD model. The FADS/model-based estimation process featured two nested loops

- The inner loop focused on estimating the local parameters (Mach number, α , and β) at each data point in time because they changed at every time step during the flight.
- Once the inner loop had processed every data point across the entire flight, the outer loop was executed to batch-estimate the global parameters (e.g., pressure transducer biases, scale factors, quadratic, and cubic terms).
- The two-loop process was repeated iteratively until convergence was achieved. Convergence indicates that the estimated parameters closely matched the observed pressures.

Unlike the local parameters, global parameters remain constant across the entire dataset.

Another relevant work based on the NewSTEP method is [29], which describes the reconstruction of the entry, descent, and landing (EDL) as-flown trajectory of the Mars InSight entry vehicle in the Mars atmosphere of the Mars InSight entry vehicle. Mars InSight uses several data sources including on-board acceleration and angular rate components from the IMU at a rate of 200 Hz and radar altimeter measurements recorded at a rate of 10 Hz after heatshield jettison from an altitude of approximately 2.4 km to the surface. The goal of the work is the TR and parameter identification in the Martian atmosphere and, in this respect, the vehicle did not have a sensor suite for direct measurements of atmospheric data during entry. The atmosphere characteristics were determined using a classical approach [47] that combines the nominal aerodynamic database [48] with measured accelerations and reconstructed wind-relative velocity (assuming mean atmospheric winds) to simultaneously estimate the freestream density, and the angles of attack and sideslip. The density estimation, along with the reconstructed altitude and gravity model,

is used to integrate the hydrostatic equation, so as to obtain an estimation of atmospheric pressure. Temperature is determined using the ideal gas law with a specific gas constant corresponding to an atmospheric composition representative of Mars [49]. Uncertainty in the reconstruction is computed through a linear covariance analysis [47].

1.4 Structure of the work

In this section, the organization of the thesis is briefly illustrated.

After recalling the main reference systems used, the non-linear six-degree-of-freedom dynamic model of the LV is described in Chap. 2, including the elastic DOFs. This model is necessary in order to generate the synthetic flight data on which the TR is applied. The model is developed taking into account all the physical aspects relevant for representing the dynamics of the LV during the atmospheric phase of flight, which goes from lift-off to first-stage separation, such as the rigid-body dynamics, aerodynamic and propulsion forces and moments, the elastic degrees of freedom, the inertial coupling terms due to the nozzle rotation and the dynamics of the Thrust Control Vector (TVC). Next, the measurement models of the vehicle sensors used in the TR process and the wind simulated model are described.

In Chapter 3 the linearized equations of motion are reported, developed as small deviations of the vehicle motion from a reference trajectory, in order to synthesize the controller for the flight control system (FCS). The controller features two proportional derivative (PD) components for attitude and translational motion control, plus notch and low-pass filters to gain and phase stabilize the bending mode. A Gain Scheduling of the control law is then developed to handle the variation of the LV parameters during the atmospheric phase of flight. Finally, the reference trajectory is discussed.

In Chapter 4 the estimation methodologies used for TR, and for aerodynamic and wind estimation are reported. In particular, the theory of the nonlinear filtering algorithms, such as the Extended Kalman filter (EKF) and the Unscented Kalman filter (UKF), is described. Moreover, stochastic models more often used to compensate for unmodeled dynamics are presented.

The filter architectures for BET and aerodynamic angles and wind components are presented in Chap. 5 and 6.

This includes an in-depth examination of both process and measurement model equations. In particular, the results of the performance analysis of two different estimation algorithms for the BET problem are also presented and discussed in Chap. 5, with the objective of investigating and comparing the effectiveness of different

filtering algorithms. Next, in Chap. 6 the performance of the filter architecture proposed for the estimation of the aerodynamic angles and wind components is discussed. The performance and suitability of the proposed approaches are evaluated through time-domain simulations and Monte Carlo campaigns for scenarios that involve the scattering of several model parameters, sensor errors, and/or wind disturbances.

Finally, Chapter 7 presents the conclusions and the major contributions of this thesis.

Chapter 2

LV Model

The chapter delves into the motion equations for rockets with a single gimbaled engine. The model formulation follows the guidelines proposed in *Barrows* [50, 51] to obtain the LV motion equations through a Lagrangian formulation.

The effects of the elastic DOFs are taken into consideration, in particular, the lateral vibration modes can impact post-flight trajectory reconstruction (because detected by the IMU). Moreover, the influence of elastic deformations on the thrust axis direction, particularly in terms of elastic rotations at the gimbal, are also dealt with.

A simplified representation of the LV elastic DOF, described in Sec. 2.2 allows for modeling the interactions between vehicle structure, engine, and rigid dynamics (Sec. 2.3), the external forces and moments being modeled according to the work in [52] for the Ares-I LV. The models of the sensor error, actuators, and atmospheric wind end the chapter.

2.1 Reference frames and notation

The reference frames used in the mathematical model of the LV, are recalled in this section. Specific symbols and notation are employed in what follows. In particular, the angular rate of frame $\mathcal{F}_A$ relative to $\mathcal{F}_C$ is represented by the vector $\boldsymbol{\omega}_{AC}$. When a vector is denoted with a subscript $[A]$, it is expressed in the frame $\mathcal{F}_A$. If not explicitly mentioned, vectors are assumed to be represented in the body frame $\mathcal{F}_B$. The notation $[\mathbf{a}_\times]$ refers to the skew-matrix form of vector $\mathbf{a}$ for cross product.

Earth-centered inertial (ECI)

The not-rotating ECI frame, $\mathcal{F}_I$, has the z -axis normal to the equatorial plane, coinciding with the Earth rotation axis and pointing to the North Pole. The x -axis

lies in the equatorial plane pointing towards the vernal equinox γ . The y -axis is in the equatorial plane and completes the right-handed Cartesian system. $\mathcal{F}_I$ is shown in Fig. 2.1 where $\hat{\mathbf{x}}_I, \hat{\mathbf{y}}_I, \hat{\mathbf{z}}_I$ represents its basis vectors.

Earth-centered Earth-fixed (ECEF)

The rotating ECEF reference frame, $\mathcal{F}_F$, has the z -axis defined as for the ECI frame. The x -axis lies in the equatorial plane pointing towards the Greenwich meridian, and the y -axis is in the equatorial plane. The frame $\mathcal{F}_F$ is illustrated in Fig. 2.1, with its basis vectors represented by $\hat{\mathbf{x}}_F, \hat{\mathbf{y}}_F, \hat{\mathbf{z}}_F$.

North-East-Down (NED)

The NED frame, $\mathcal{F}_E$, has the origin located at the vehicle center of gravity (cg), and is oriented with the x -axis pointing in the direction of North Pole, The z -axis is perpendicular to the x -axis and oriented along the local vertical, and the y -axis completes the right-hand frame. $\mathcal{F}_E$ is shown in Fig. 2.1 where $\hat{\mathbf{x}}_E, \hat{\mathbf{y}}_E, \hat{\mathbf{z}}_E$ represents its basis vectors.

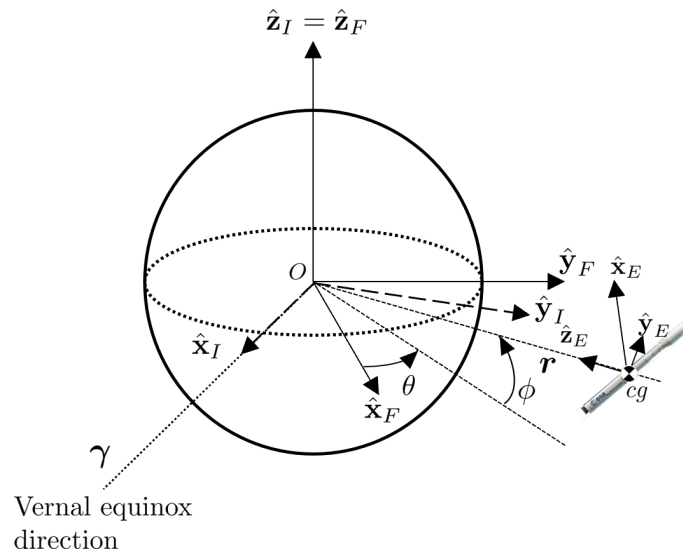


Figure 2.1. Representation of the ECI, ECEF and NED reference frames

Body-fixed (B)

The B frame, $\mathcal{F}_B$, is fixed to the principal axes of inertia of the vehicle, assumed rigid, with origin in the cg. The x -axis is along the vehicle longitudinal axis, the y -axis is arbitrarily chosen in the plane orthogonal to the x -axis, due to the axial

symmetry of the LV model, while the z -axis completes the right-handed frame. The frame $\mathcal{F}_B$ is illustrated in Fig. 2.2, with its basis vectors represented by $\vec{x}_B, \vec{y}_B, \vec{z}_B$.

Nozzle-fixed (N)

The N frame, $\mathcal{F}_N$, is fixed to the nozzle with the origin at the gimbal point. The x -axis points along the nozzle pivot point and the y -axis is perpendicular to the axis of symmetry of the nozzle and is parallel to the y -axis of the frame $\mathcal{F}_B$, pointing to the same direction, while the z -axis completes the right-handed frame.

Structural-fixed (S)

The S frame, $\mathcal{F}_S$ has its origin on the top of the nose of the LV with x -axis direct to the cg and the y -axis aligned with the y -axis in $\mathcal{F}_B$, the z -axis completes the right-handed frame. $\mathcal{F}_S$ is shown in Fig. 2.2 where $\vec{x}_S, \vec{y}_S, \vec{z}_S$ represents its basis vectors. It is worth pointing out that $\mathcal{F}_S$ is defined in the undeformed configuration as specified in [52].

Trajectory reference frame (t)

The trajectory reference frame $\mathcal{F}_T$ with basis vectors $\vec{x}_t, \vec{y}_t, \vec{z}_t$ is a right-hand non-stationary frame that translates axially with the vehicle but remains tangent to the ascent trajectory. The $\vec{x}_t, \vec{y}_t$, axes are, respectively, tangent and normal to the nominal trajectory with $\vec{x}_t$ pointing in the LV direction of motion. $\vec{z}_t$ completes the right-hand frame.

Ground Station reference frame (G)

This is a topocentric reference $\mathcal{F}_G$ centered at the Ground Station (GS), with axes aligned in the Up-East-North directions and assigned longitude (θ_{gs}) and latitude (ϕ_{gs}) coordinate for the GS.

2.2 Structural dynamics

For post-flight analysis purposes, the lateral vibration modes hold significance, as these motions are detected by the IMU. At a conceptual level, a structural dynamic model is composed of masses and springs. In the current application, a simplified approach has been taken to describe the bending modes of the Ares-I LV that involves addressing the free vibration and solving the homogeneous problem of the free-free Euler-Bernoulli beam. By assuming a linear mass density $\mu(l)$ and the

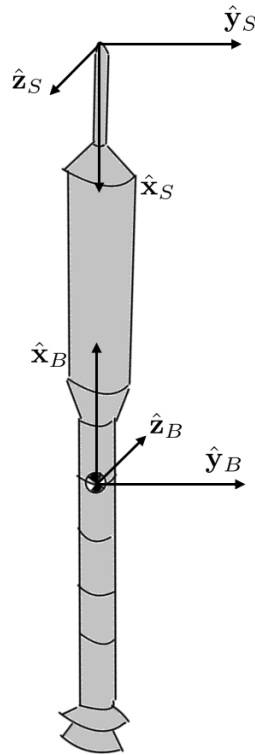


Figure 2.2. Representation of the body and structural reference frames

bending inertia $EI(l)$, the method aligns with the elastic bending mode solutions referenced in [52] for the elastic rotations and [53] for the elastic displacements. This solution is necessary for defining *mode shapes* used to project the thrust force from $\mathcal{F}_B$ to the modal base, so as to obtain the forced solution for the free-free flexible beam. Therefore, a simplified representation of the LV elastic DOFs is adopted, using the following Euler-Bernoulli beam model

$$(EI(l)w(l,t))'' + \mu(l)\ddot{w}(l,t) = f_{w_{ext}} \quad (2.1)$$

where $w(l,t)$ is the transverse displacement or deflection, l is the axial coordinate along the LV in $\mathcal{F}_S$ and t is the time. This approach neglects rotational inertia and shear distortion. A Gal rkin method [54] is considered here to deal with the case of a non-homogeneous beam with known $\mu(l)$ and $EI(l)$, and possibly concentrated mass l_j at $l = l_j$. Applying the separation principle, one expresses the displacement w as the product of space and time functions

$$w(l,t) = \sum_{n=1}^{\infty} \Phi_n(l)q_n(t) \quad (2.2)$$

Substituting Eq. (2.2) into Eq. (2.1), where the forcing term on the right-hand side (RHS) of the equation is set to zero first in order to evaluate the homogeneous solution, one has

$$\frac{d^2}{dx^2} \left(EI(l) \sum_{n=1}^{\infty} \Phi_n(l) q_n(t) \right) = \mu(l) \sum_{n=1}^{\infty} \Phi_n(l) \ddot{q}_n(t) \quad (2.3)$$

The series is truncated to N . Next, the expression on both sides is multiplied by Φ_m , and an integration over dl between 0 and L (the total height of the LV) is performed, resulting in the following integral expression

$$\int_0^L \sum_{n=1}^N q_n \frac{d^2}{dl^2} [EI(l) \Phi_n(l)] \Phi_m(l) dl = \sum_{n=1}^N \ddot{q}_n \int_0^L \mu(l) \Phi_n(l) \Phi_m(l) dl \quad \forall m \in [1, N] \quad (2.4)$$

forming a system of N equations in N unknowns.

The set of equations (2.4) is written as, in matrix form

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{0} \quad (2.5)$$

with $\mathbf{q} = [q_1 \ \dots \ q_N]^T$, $\mathbf{M}$ is a (dense) mass matrix of elements

$$M_{n,m} = \int_0^L \mu(l) \Phi_n(l) \Phi_m(l) dl \quad (2.6)$$

and $\mathbf{K}$ is a (dense) stiffness matrix of elements

$$K_{n,m} = \int_0^L EI(l) \Phi_n(l)'' \Phi_m(l)'' dl \quad (2.7)$$

This formulation allows for N_c concentrated masses m_j at the coordinates l_j . One may rewrite the linear mass density by $\tilde{\mu}(l)$ using Dirac functions

$$\tilde{\mu}(l) = \mu(l) + \sum_j m_j \delta(l - l_j) \quad (2.8)$$

where $\tilde{\mu}(l)$ is the continuous function, In that case, the elements of the mass matrix are

$$M_{n,m} = \int_0^L \mu(l) \Phi_n(l) \Phi_m(l) dl + \sum_j m_j \Phi_n(l_j) \Phi_m(l_j) \quad (2.9)$$

Figures 2.3 and 2.4 show the bending inertia and the constant piece-wise linear

mass density assumed in the present work as input data to obtain the desired solution (bending modes).

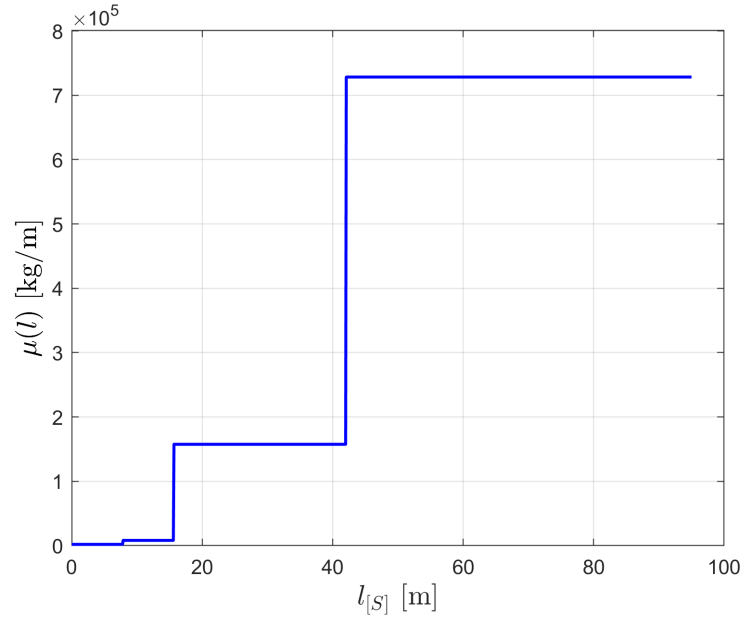


Figure 2.3. Bending inertia vs. axial coordinate

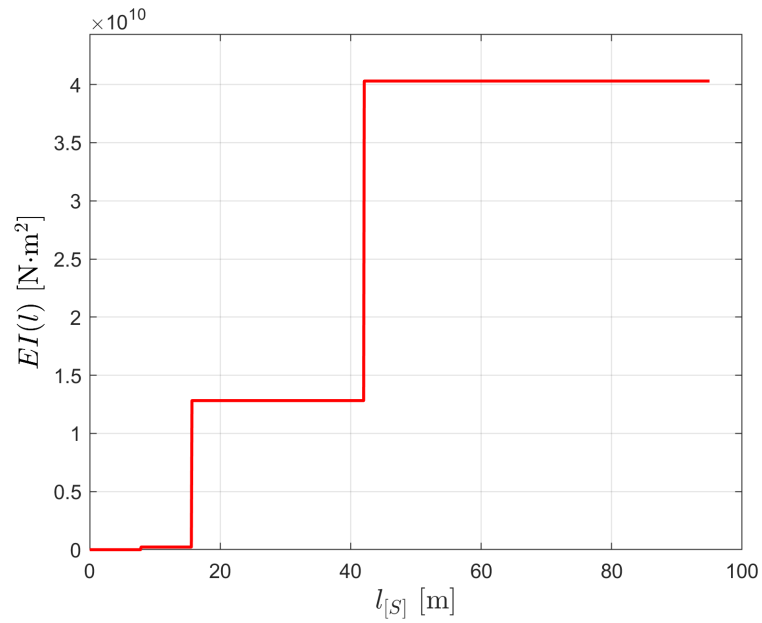


Figure 2.4. Linear mass density vs. axial coordinate

Suppose that ψ_r modes of vibration exist so that

$$w(l, t) = \sum_{r=1}^{\infty} \Psi_r(l) \eta_{f_r}(t) \quad (2.10)$$

where η_{f_r} represents the r th generalized modal coordinate, then

$$\psi_r(l) = \sum_{n=1}^{\infty} \Phi_n(l) z_n^r \quad (2.11)$$

$$w(l, t) = \sum_{r=1}^{\infty} \psi_r(l) \eta_{f_r}(t) \quad (2.12)$$

$$w(l, t) = \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \phi_r(l) z_n^r \eta_{f_r}(t) \quad (2.13)$$

$$w(l, t) = \sum_{r=1}^{\infty} \phi_n(l) z_n^r \sum_{n=1}^{\infty} \eta_{f_r}(t) \quad (2.14)$$

$$w(l, t) = \sum_{n=1}^{\infty} \phi_n(l) \sum_{r=1}^{\infty} z_n^r \eta_{f_r}(t) \quad (2.15)$$

and, by comparison with Eq. (2.2), one has

$$q_n(t) = \sum_r z_n^r \eta_{f_r}(t) \quad (2.16)$$

Now, substituting in $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$, we obtain

$$\mathbf{M}\mathbf{Z}\ddot{\boldsymbol{\eta}}_f(t) + \mathbf{K}\mathbf{Z}\boldsymbol{\eta}_f(t) = \mathbf{0} \quad (2.17)$$

This system of equations has a nontrivial solution if $Z \neq 0$. Assuming that the system is rewritten as

$$\widetilde{\mathbf{M}}\ddot{\boldsymbol{\eta}}_f + \widetilde{\mathbf{K}}\boldsymbol{\eta}_f = \mathbf{0} \quad (2.18)$$

a generalized symmetric eigenvalue problem is defined as discussed in Refs. [52] and [50] the eigenvectors solution of which represents a complete set of N normalized *mode shapes* $\bar{\boldsymbol{\Psi}} = [\bar{\boldsymbol{\psi}}_1, \dots, \bar{\boldsymbol{\psi}}_N]$, so that

$$\bar{\boldsymbol{\Psi}}^T \widetilde{\mathbf{M}} \bar{\boldsymbol{\Psi}} = \mathbf{I} \quad (2.19)$$

and

$$\bar{\boldsymbol{\Psi}}^T \widetilde{\mathbf{K}} \bar{\boldsymbol{\Psi}} = \Omega^2 \quad (2.20)$$

where Ω^2 is the eigenvalues matrix. By computing the spatial derivatives of $\bar{\Psi}$ with respect to the undeformed body axes, one obtains the *mode slopes* $\bar{\Sigma} = [\bar{\sigma}_1, \dots, \bar{\sigma}_N]$. The bar symbol ($\bar{}$) is used to indicate mass-normalized modes. This designation is derived from the selection of the eigenvector $\bar{\Psi}$, which satisfies Eq. (2.19). The solution of the eigenvalue problem provides the first three *mode shapes* and *mode slopes* represented in Figs. 2.5 and 2.6 respectively. In particular, a linear second-

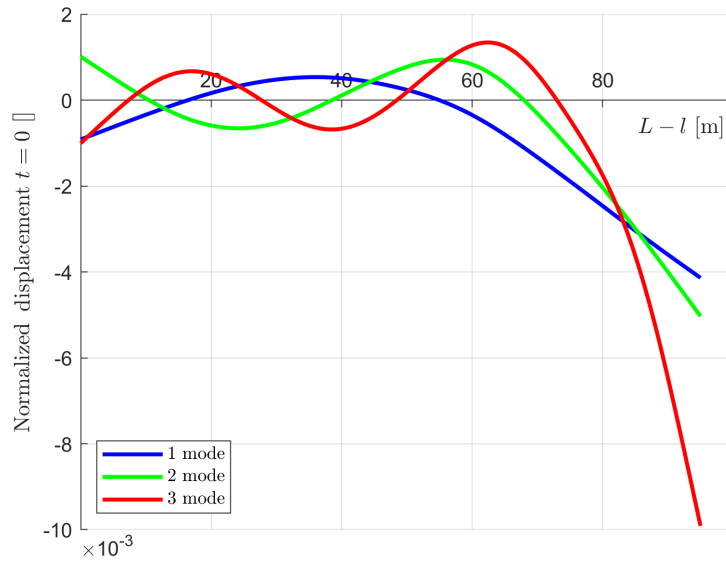


Figure 2.5. Mode shapes at flight time $t = 0$

order system with a damping ratio ζ_{BM} of 0.5% [50] is adopted, according to

$$\ddot{\eta}_f + 2\zeta_{BM}\Omega\dot{\eta}_f + \Omega^2\eta_f = \bar{\Psi}_G\mathbf{T} \quad (2.21)$$

The matrix $\bar{\Psi}_G$ represents the first three *mode shapes* at the pivot point, and $\mathbf{T}$ denotes the thrust vector. For the sake of simplicity in this study, it is assumed that the thrust is applied at the gimbal point. Consequently, the influence of torques on the structural node corresponding to the gimbal point is zero. Additionally, the aerodynamic forces, acting as external forces on the elastic dynamics, have been neglected for the sake of simplicity [52]. The first stage flight lasts for 120 seconds, for which an eigenvector solution is computed at each second in the time interval. This computation assumes that the stiffness matrix remains constant while the mass matrix undergoes changes over time. Figure 2.7 represents the variation in mass versus the bending mode frequency over time.

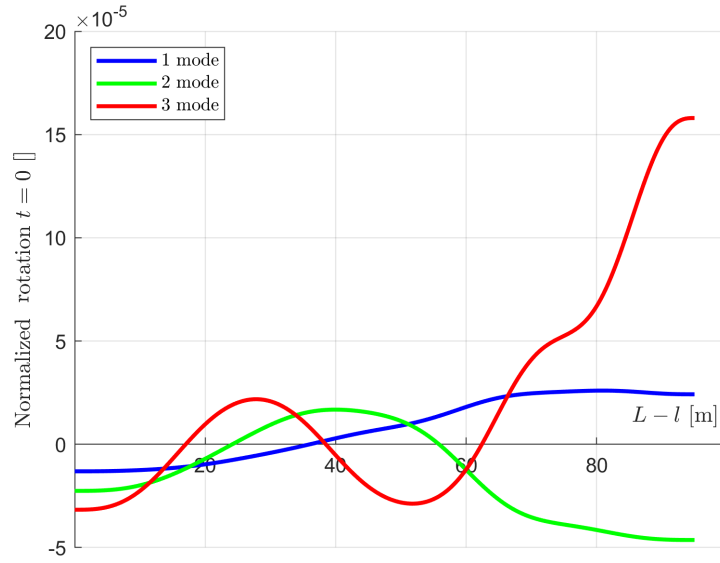


Figure 2.6. Mode slopes at flight time $t = 0$

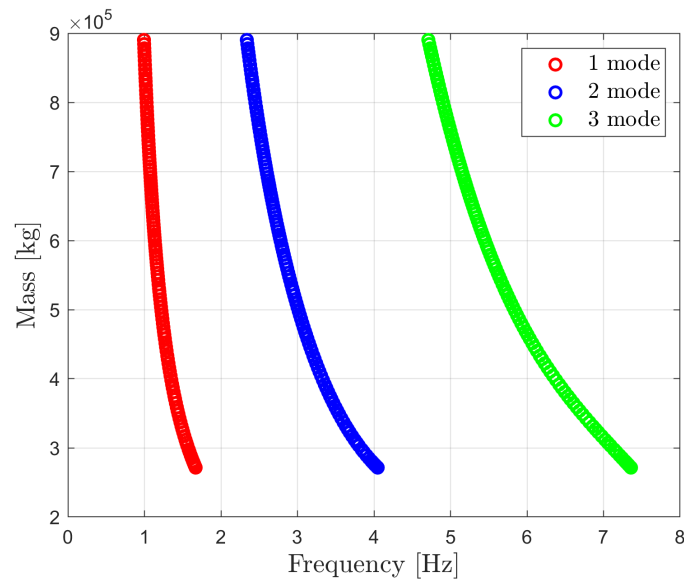


Figure 2.7. Mass vs. modes frequency

2.3 LV dynamics

2.3.1 Assumptions

All the vectors and matrices in the following formulation are expressed in $\mathcal{F}_B$. The body reference frame $\mathcal{F}_B$ is assumed with origin in $\mathcal{O}$ (see Fig. 2.8) that is the cg of the integrated vehicle (base body and nozzle as two connected rigid bodies) and is

placed on the undeformed centerline remaining stationary while the elastic vibrations take place. The motion of the rigid body vehicle is characterized by its translational position $\mathbf{r}_I$, the attitude of $\mathcal{F}_B$ w.r.t. $\mathcal{F}_I$ and the nozzle angles, which are the generalized coordinates q_i , while their derivatives, that is, translational velocity $\mathbf{v}_I$, angular rate $\boldsymbol{\omega}_{BI}$ and nozzle angular rate $\boldsymbol{\omega}_G$ are the generalized velocities $\dot{q}_i$. For the elastic DOFs, LV is represented by p masses called *nodes* and collocated along its axis of symmetry, each one characterized by 6-DOFs where $m_v = \sum_p m_p$ represent the total mass of the vehicle not including the mass of the nozzle (it is sketched in Fig. 2.8) . Each elastic mass-motion in time is described by the generalized coordinate $\boldsymbol{\eta}_f \in \mathbb{R}^{p \times 1}$ and generalized velocity $\dot{\boldsymbol{\eta}}_f \in \mathbb{R}^{p \times 1}$ along with the associated set of the first 3 *mode shapes* and 3 *mode slopes*. The first three bending modes are the most relevant for the purposes of the present work and, it is in line with literature on medium-fidelity LV models [55],[52], also, considering that their impact on IMU measurement output is higher at low frequencies, particularly when measurements are preprocessed to remove high-frequency noise.

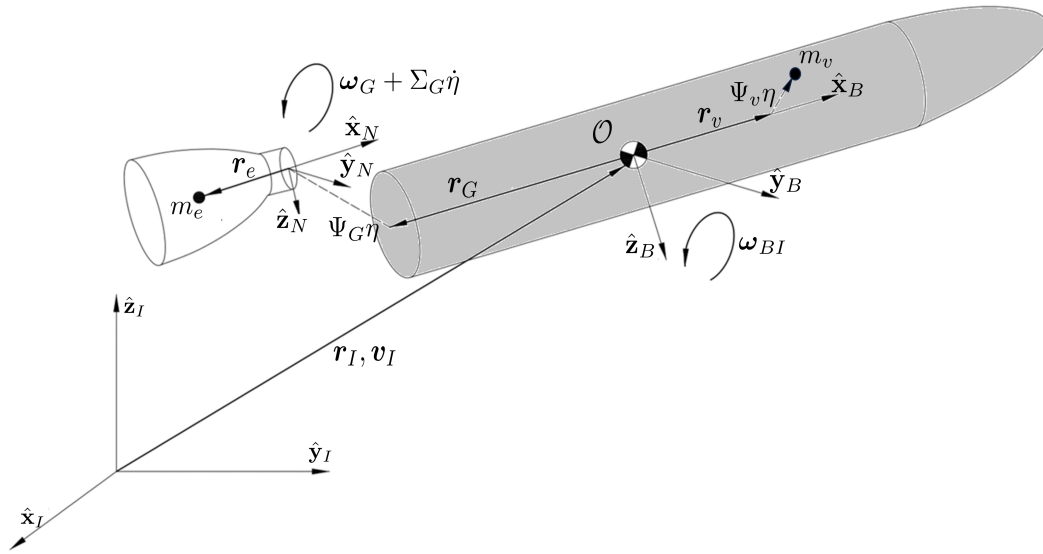


Figure 2.8. Quasi-rigid body with nozzle and elastic perturbations

The Lagrange's equation is employed to derive the equations of motion for a mechanical system undergoing vibrations, given by

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial \mathcal{D}}{\partial \dot{q}_i} = Q_i \quad (2.22)$$

where $\mathcal{T}$ is the kinetic energy, V is the potential energy, and $\mathcal{D}$ is the dissipation function, and Q_i are the generalized external forces. It is shown [50] that, when

applied to the problem under consideration, the second term in Eq. (2.22) is always zero. The third and fourth terms in (2.22) can be moved to the RHS of the equation. Thus, the Lagrange's equation is expressed as

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i} \right) = Q_i - \frac{\partial V}{\partial q_i} - \frac{\partial \mathcal{D}}{\partial \dot{q}_i} \quad (2.23)$$

Strictly speaking, Lagrange's equation is not valid in a rotating body frame. Hughes [56] acknowledges this fact and provides a set of "quasi-Lagrangian" equations, specifically applicable to a rigid body, when the second, third, and fourth terms in Eq. (2.22) are zero, given by

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \mathbf{v}_I} \right) + \boldsymbol{\omega}_{BI} \times \frac{\partial \mathcal{T}}{\partial \mathbf{v}_I} = \mathbf{f}_{ext} \quad (2.24)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \boldsymbol{\omega}_{BI}} \right) + \boldsymbol{\omega}_{BI} \times \frac{\partial \mathcal{T}}{\partial \boldsymbol{\omega}_{BI}} + \mathbf{v}_I \times \frac{\partial \mathcal{T}}{\partial \mathbf{v}_I} = \mathbf{g}_{ext} \quad (2.25)$$

In this respect, the V and $\mathcal{D}$ terms in Eq. (2.23) are used for the purpose of representing the inter-body forces, included as a part of the external forces Q_i . Here, $\mathbf{f}_{ext}$ represents the external force vector, and $\mathbf{g}_{ext}$ denotes the external torque vector about the origin $\mathcal{O}$.

2.3.2 Kinetic energy and Lagrangian accelerations

The integrated body kinetic energy is represented through translational and angular velocities in $\mathcal{F}_I$ expressed in $\mathcal{F}_B$, where there is also the nozzle energy arising from its local velocity relative to $\mathcal{O}$; therefore the kinetic energy $\mathcal{T}$ is the sum of contributions from both the base body vehicle and the nozzle, given by

$$\mathcal{T} = \mathcal{T}_v + \mathcal{T}_e \quad (2.26)$$

In $\mathcal{F}_I$, the velocity of the base body mass of the vehicle consist of the velocity of $\mathcal{F}_B$, the velocity due to elastic effects and the velocity due to rotation of $\mathcal{F}_B$. It is expressed as

$$\mathbf{v}_v = \mathbf{v}_I + \bar{\boldsymbol{\Psi}}_v \dot{\boldsymbol{\eta}}_f + \boldsymbol{\omega}_{BI}^\times \mathbf{r}_v \quad (2.27)$$

where $\bar{\boldsymbol{\Psi}}_v \in \mathbb{R}^{3 \times p}$. In the same way the velocity of the nozzle is derived as follows

$$\mathbf{v}_e = \mathbf{v}_I + \bar{\boldsymbol{\Psi}}_G \dot{\boldsymbol{\eta}}_f + \boldsymbol{\omega}_{BI}^\times (\mathbf{r}_G + \mathbf{r}_e) + \boldsymbol{\omega}_{G[B]}^\times \mathbf{r}_e + \bar{\boldsymbol{\Sigma}}_G \dot{\boldsymbol{\eta}}_f^\times \mathbf{r}_e \quad (2.28)$$

where $\mathbf{r}_e$ is the distance between the nozzle pivot point and the nozzle cg, and the angular rate $\boldsymbol{\omega}_G$ in $\mathcal{F}_B$ of $\mathcal{F}_N$ is taken into account and recalled as $\boldsymbol{\omega}_{G[B]}$ to remember that this quantity is expressed in body coordinates. The kinetic energy for the two contributions is written as

$$\mathcal{T}_v = \frac{1}{2} m_v (\mathbf{v}_I + \bar{\boldsymbol{\Psi}}_v \dot{\boldsymbol{\eta}}_f + \boldsymbol{\omega}_{BI}^\times \mathbf{r}_v)^2 \quad (2.29)$$

$$\mathcal{T}_e = \frac{1}{2} m_e (\mathbf{v}_I + \bar{\boldsymbol{\Psi}}_G \dot{\boldsymbol{\eta}}_f + \boldsymbol{\omega}_{BI}^\times (\mathbf{r}_G + \mathbf{r}_e) + \boldsymbol{\omega}_{G[B]}^\times \mathbf{r}_e + \bar{\boldsymbol{\Sigma}}_G \dot{\boldsymbol{\eta}}_f^\times \mathbf{r}_e)^2 \quad (2.30)$$

and the generalized momentum of the vehicle and nozzle is written as

$$\nabla_{\mathbf{v}_I} \mathcal{T}_v = m_v (\mathbf{v}_I + \bar{\boldsymbol{\Psi}}_v \dot{\boldsymbol{\eta}}_f + \boldsymbol{\omega}_{BI}^\times \mathbf{r}_v) \quad (2.31)$$

$$\nabla_{\mathbf{v}_I} \mathcal{T}_e = m_e (\mathbf{v}_I + \bar{\boldsymbol{\Psi}}_G \dot{\boldsymbol{\eta}}_f + \boldsymbol{\omega}_{BI}^\times (\mathbf{r}_G + \mathbf{r}_e) + \boldsymbol{\omega}_{G[B]}^\times \mathbf{r}_e + \bar{\boldsymbol{\Sigma}}_G \dot{\boldsymbol{\eta}}_f^\times \mathbf{r}_e) \quad (2.32)$$

known that $\mathcal{O}$ is the cg of the integrated vehicle, the following relation is true

$$m_e (\mathbf{r}_G + \mathbf{r}_e) + m_v \mathbf{r}_v = 0 \quad (2.33)$$

In the absence of external forces, as shown in Equation (2.34), neither linear momentum nor net angular velocity w.r.t. the cg is produced. This principle holds for any body experiencing free-free vibrations, as detailed in [50]. It follows that

$$m_v \bar{\boldsymbol{\Psi}}_v \dot{\boldsymbol{\eta}}_f + m_e \bar{\boldsymbol{\Psi}}_G \dot{\boldsymbol{\eta}}_f - m_e \mathbf{r}_e^\times \bar{\boldsymbol{\Sigma}}_G \dot{\boldsymbol{\eta}}_f = 0 \quad (2.34)$$

The linear momentum of the integrated body is thus

$$\nabla_{\mathbf{v}_I} \mathcal{T} = m \mathbf{v}_I - m_e \mathbf{r}_e^\times \boldsymbol{\omega}_{G[B]} \quad (2.35)$$

where $m = m_v + m_e$. The second term in Eq. (2.35) denotes the modification in linear momentum caused by the angular velocity of the nozzle and is associated with the lateral effect of inertial coupling known as *tail-wags-dog* (TWD). The components of the generalized momentum concerning the angular velocity $\boldsymbol{\omega}_{BI}$ are expressed as

$$\begin{aligned} \nabla_{\boldsymbol{\omega}_{BI}} \mathcal{T}_v &= m_v \mathbf{r}_v^\times \mathbf{v}_I + m_v \mathbf{r}_v^\times \bar{\boldsymbol{\Psi}}_v \dot{\boldsymbol{\eta}}_f + m_v \mathbf{r}_v^\times \boldsymbol{\omega}_{BI}^\times \mathbf{r}_v \\ \nabla_{\boldsymbol{\omega}_{BI}} \mathcal{T}_e &= m_e (\mathbf{r}_G + \mathbf{r}_e)^\times \mathbf{v}_I + m_e (\mathbf{r}_G + \mathbf{r}_e)^\times \bar{\boldsymbol{\Psi}}_G \dot{\boldsymbol{\eta}}_f \\ &\quad + m_e (\mathbf{r}_G + \mathbf{r}_e)^\times \boldsymbol{\omega}_{BI}^\times (\mathbf{r}_G + \mathbf{r}_e) \\ &\quad + m_e (\mathbf{r}_G + \mathbf{r}_e)^\times \boldsymbol{\omega}_{G[B]}^\times \mathbf{r}_e + m_e (\mathbf{r}_G + \mathbf{r}_e)^\times (\bar{\boldsymbol{\Sigma}}_G \dot{\boldsymbol{\eta}}_f)^\times \mathbf{r}_e. \end{aligned} \quad (2.36)$$

Next, we organize the terms, recalling the conservation of angular momentum during normal vibrations, also making use of the identity [57]

$$\sum_i = m_i \mathbf{r}_i^\times \boldsymbol{\omega}^\times \mathbf{r}_i = \mathbf{J} \boldsymbol{\omega} \quad (2.37)$$

so that Eq. (2.36) is simplified as follows

$$\nabla_{\boldsymbol{\omega}_{BI}} \mathcal{T}_v = \mathbf{J}_v \boldsymbol{\omega}_{BI} \quad (2.38)$$

$$\nabla_{\boldsymbol{\omega}_{BI}} \mathcal{T}_e = \mathbf{J}_e \boldsymbol{\omega}_{BI} + m_e (\mathbf{r}_G + \mathbf{r}_e)^\times \boldsymbol{\omega}_{G[B]}^\times \mathbf{r}_e \quad (2.39)$$

Equations (2.39) can be rewritten as

$$\nabla_{\boldsymbol{\omega}_{BI}} \mathcal{T}_e = \mathbf{J}_e \boldsymbol{\omega}_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \boldsymbol{\omega}_{G[B]} \quad (2.40)$$

and, assuming $\mathbf{J} = \mathbf{J}_v + \mathbf{J}_e$, the total angular momentum w.r.t. the cg is

$$\nabla_{\boldsymbol{\omega}_{BI}} \mathcal{T} = \mathbf{J} \boldsymbol{\omega}_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \boldsymbol{\omega}_{G[B]} \quad (2.41)$$

As for Eq. (2.35), the last term in Eq. (2.41) represents the angular momentum variation due to the angular velocity of the nozzle, that is, the angular effect of the TWD. The derivation of the generalized momentum for the bending dynamics follows a similar process, and a comprehensive presentation can be found in [50] and [51], yielding the following results

$$\nabla_{\dot{\boldsymbol{\eta}}_f} \mathcal{T}_v = \bar{\boldsymbol{\Psi}}_v^T m_v \bar{\boldsymbol{\Psi}}_v \dot{\boldsymbol{\eta}}_f \quad (2.42)$$

$$\nabla_{\dot{\boldsymbol{\eta}}_f} \mathcal{T}_e = (\bar{\boldsymbol{\Sigma}}_G^T \mathbf{J}_e - \bar{\boldsymbol{\Psi}}_G^T m_e \mathbf{r}_e^\times) \boldsymbol{\omega}_{G[B]} \quad (2.43)$$

The total generalized momentum for the bending dynamics is

$$\nabla_{\dot{\boldsymbol{\eta}}_f} \mathcal{T} = \dot{\boldsymbol{\eta}}_f + (\bar{\boldsymbol{\Sigma}}_G^T \mathbf{J}_e - \bar{\boldsymbol{\Psi}}_G^T m_e \mathbf{r}_e^\times) \boldsymbol{\omega}_{G[B]} \quad (2.44)$$

Due to the adoption of a *prescribed engine motion* [50], there is no need to compute the derivatives of kinetic energy w.r.t. $\boldsymbol{\omega}_{G[B]}$. This is because both this quantity and its time derivative result from the TVC dynamic, which have been modeled in this study using a second-order differential model (see Sec 2.4). Taking the time derivatives of Eqs. (2.35), (2.41), and (2.44), the Lagrangian accelerations are expressed as

$$\frac{d}{dt}(\nabla_{\mathbf{v}_I} \mathcal{T}) = m \dot{\mathbf{v}}_I - m_e \mathbf{r}_e^\times \dot{\boldsymbol{\omega}}_{G[B]} \quad (2.45)$$

$$\frac{d}{dt}(\nabla_{\omega_{BI}} \mathcal{T}) = \mathbf{J} \dot{\omega}_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \dot{\omega}_{G[B]} \quad (2.46)$$

$$\frac{d}{dt}(\nabla_{\dot{\eta}_f} \mathcal{T}) = \ddot{\eta}_f + (\bar{\Sigma}_G^T \mathbf{J}_e - \bar{\Psi}_G^T m_e \mathbf{r}_e^\times) \dot{\omega}_{G[B]} \quad (2.47)$$

Now, by substituting Eqs. (2.45) and (2.46) into Eqs. (2.24) and (2.25), gives

$$m \dot{\mathbf{v}}_I - m_e \mathbf{r}_e^\times \dot{\omega}_{G[B]} + \omega_{BI}^\times (m \mathbf{v}_I - m_e \mathbf{r}_e^\times \omega_{G[B]}) = \mathbf{f}_{ext} \quad (2.48)$$

$$\mathbf{J} \dot{\omega}_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \dot{\omega}_{G[B]} + \omega_{BI}^\times (\mathbf{J} \omega_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \omega_{G[B]}) = \mathbf{g}_{ext} \quad (2.49)$$

In order to rewrite the Eq. (2.48) in the rotating reference frame $\mathcal{F}_F$ we define $\mathbf{r}_I = \mathcal{R}_{IF} \mathbf{r}$ with its time derivatives $\mathbf{v}_I$ is $\mathbf{v} + \omega_{\oplus[B]}^\times \mathbf{r}$ and $\dot{\mathbf{v}}_I = \dot{\mathbf{v}} + \omega_{\oplus[B]}^\times \mathbf{v} + \omega_{\oplus[B]}^\times \omega_{\oplus[B]}^\times \mathbf{r}$, where the Earth angular rate is $\omega_{\oplus} = [0, 0, \omega_{\oplus}]^T$, with $\omega_{\oplus} = 7.291\,985 \times 10^{-5}$ rad/s, whereas $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity of the LV w.r.t. $\mathcal{F}_F$ expressed in $\mathcal{F}_B$. Substituting $\mathbf{v}_I$ and $\dot{\mathbf{v}}_I$ in Eq. (2.48) the momentum for the LV is obtained as follows

$$m \left(\dot{\mathbf{v}} + (\omega_{BI}^\times + \omega_{\oplus[B]}^\times)^\times \mathbf{v} + \omega_{\oplus[B]}^\times \omega_{\oplus[B]}^\times \mathbf{r} \right) - m_e \mathbf{r}_e^\times \dot{\omega}_{G[B]} - \omega_{BI}^\times m_e \mathbf{r}_e^\times \omega_{G[B]} \quad (2.50)$$

As for the elastic dynamics, using Eq. (2.21) we derive the generalized forces vector as $\mathbf{Q}_{\eta_f} = -(2\zeta \Omega \dot{\eta}_f + \Omega^2 \eta_f) + \bar{\Psi}_G^T \mathbf{T}$. It is worth noting, as mentioned earlier, that only the thrust force is considered as the excitation of the elastic DOFs, although other effects in terms of forces and torques, as for instance the aerodynamic actions may play a role [50]. Accordingly, the elastic dynamics is modeled as

$$\ddot{\eta}_f + (\bar{\Sigma}_G^T \mathbf{J}_e - \bar{\Psi}_G^T m_e \mathbf{r}_e^\times) \dot{\omega}_{G[B]} = \mathbf{Q}_{\eta_f} \quad (2.51)$$

2.3.3 Compendium of the equations of motion

In this section, the mathematical model of the LV is recalled for the sake of clarity.

Translational motion

$$m(\dot{\mathbf{v}} + (\omega_{BI}^\times + \omega_{\oplus[B]}^\times)^\times \mathbf{v} + \omega_{\oplus[B]}^\times \omega_{\oplus[B]}^\times \mathbf{r}) - m_e \mathbf{r}_e^\times \dot{\omega}_{G[B]} - \omega_{BI}^\times m_e \mathbf{r}_e^\times \omega_{G[B]} = \mathbf{f}_{ext} \quad (2.52)$$

Attitude motion

$$\mathbf{J}\dot{\boldsymbol{\omega}}_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \dot{\boldsymbol{\omega}}_{G[B]} + \boldsymbol{\omega}_{BI}^\times (\mathbf{J}\boldsymbol{\omega}_{BI} + (\mathbf{J}_e - \mathbf{r}_G^\times m_e \mathbf{r}_e^\times) \boldsymbol{\omega}_{G[B]}) = \mathbf{g}_{ext} \quad (2.53)$$

Elastic DOFs

$$\ddot{\boldsymbol{\eta}}_f + (\bar{\boldsymbol{\Sigma}}_G^T \mathbf{J}_e - \bar{\boldsymbol{\Psi}}_G^T m_e \mathbf{r}_e^\times) \dot{\boldsymbol{\omega}}_{G[B]} = \mathbf{Q}_{\eta_f} \quad (2.54)$$

The term $m\mathbf{v}$ can be interpreted as an additional force on the rocket due to the gas expulsion [58], and it is included in $\mathbf{f}_{ext}$, while $\mathbf{J}\boldsymbol{\omega}_{BI}$ is considered negligible w.r.t. the other contribution. Further details on the expressions of $\mathbf{f}_{ext}$ and $\mathbf{g}_{ext}$ that, as said, represent aerodynamic and thrust external forces and moments, will be provided in the following section, the TWD terms will be explicitly defined in scalar form.

The model describes an aerospace vehicle that is *quasi-rigid* and features a nozzle the mass of which is non-negligible. Although the bending and rigid body DOFs are decoupled through the orthogonal bending dynamics definition, the nozzle dynamics introduce coupling among all DOFs.

2.3.4 External and TWD forces and moments

The external forces for the LV are expressed in $\mathcal{F}_B$ as

$$\mathbf{f}_{ext} = \mathbf{F}_g + \mathbf{T} + \mathbf{F}_A \quad (2.55)$$

$$\mathbf{g}_{ext} = \mathbf{M}_A + \mathbf{M}_T \quad (2.56)$$

While $\mathbf{F}_N$ represents the TWD force term and $\mathbf{M}_N$ represents the TWD torque term, respectively.

Weight force

Assuming a simple inverse square model for the gravity field, the weight force is given by

$$\mathbf{F}_g = \mathcal{R}_{BE} \begin{bmatrix} 0 \\ 0 \\ g(\frac{R_E}{r})^2 \end{bmatrix} \quad (2.57)$$

R_E is the Earth radius and $\mathbf{r}$ the LV position in $\mathcal{F}_F$. The rotation matrix $\mathcal{R}_{BE}$ is expressed as

$$\mathcal{R}_{BE} = \mathbf{I} - 2e_0[\mathbf{e}_\times] + 2[\mathbf{e}_\times]^2 \quad (2.58)$$

being e_0 and $\mathbf{e}$, respectively, the scalar and vector part of the quaternion $\mathbf{q} = [e_0, \mathbf{e}]$, and $[\mathbf{e}_\times]$ the skew-symmetric matrix formed from the vector part of quaternion $\mathbf{e}$.

Thrust force and torque

The TVC is used to orient the thrust direction for control and stabilization purposes. The thrust force $\mathbf{T}$ is applied at the nozzle pivot point and directed along the x -axis of a frame $\mathcal{F}_N$ fixed to the nozzle. The nozzle orientation with respect to the LV airframe $\mathcal{F}_B$ is given by the matrix $\mathbf{R}_{NB} = \mathbf{R}_2(\delta_z)\mathbf{R}_3(\delta_y)$, that reads

$$\mathbf{R}_{NB} = \begin{bmatrix} \cos(\delta_z) & 0 & \sin(\delta_z) \\ 0 & 1 & 0 \\ -\sin(\delta_z) & 0 & \cos(\delta_z) \end{bmatrix} \begin{bmatrix} \cos(\delta_y) & -\sin(\delta_y) & 0 \\ \sin(\delta_y) & \cos(\delta_y) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.59)$$

being δ_y and δ_z the nozzle rotation angles (also referred for the sake of simplicity as TVC angles).

As a result of the nozzle rotation commanded by the TVC, the components of the thrust vector in the body frame are expressed as

$$\mathbf{T} = \mathbf{T}_{rig} + \begin{bmatrix} 0 \\ T_{rig_y}\sigma_{Gy} \\ T_{rig_z}\sigma_{Gz} \end{bmatrix} \quad (2.60)$$

where T is the thrust magnitude, $\mathbf{T}_{rig}$ is the ideal thrust not considering the elastic deformation of the vehicle, given by

$$\mathbf{T}_{rig} = \begin{bmatrix} T_{rig_x} \\ T_{rig_y} \\ T_{rig_z} \end{bmatrix} = \mathbf{R}_{NB}^T \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix} \quad (2.61)$$

The angles σ_{Gy} and σ_{Gz} denote the (additional) gimbal rotations with respect to the undeformed axis due to the flexible DOFs. They represent the pitch and yaw components of the elastic rotation at the nozzle pivot point $\boldsymbol{\sigma}_G(t) = \bar{\boldsymbol{\Sigma}}_G \boldsymbol{\eta}(t)$, while $\boldsymbol{\psi}_G(t) = \bar{\boldsymbol{\Psi}}_G \boldsymbol{\eta}(t)$ is the displacement at the nozzle pivot point.

Note that σ_{Gx} and ψ_{Gx} are zero as the elastic deformation in the axial direction is neglected. The moment due to the thrust is expressed as

$$\mathbf{M}_T = (\mathbf{r}_G + \boldsymbol{\psi}_G) \times \mathbf{T} \quad (2.62)$$

where $\mathbf{r}_G$ is the position vector of the nozzle pivot point.

Aerodynamic force and torque

The aerodynamic force and moment read

$$\mathbf{F}_A = \begin{bmatrix} -D_{aero} \\ C_{aero} \\ -N_{aero} \end{bmatrix}, \quad \mathbf{M}_A = \mathbf{r}_{ap} \times \begin{bmatrix} -D_{aero} \\ C_{aero} \\ -N_{aero} \end{bmatrix} + \begin{bmatrix} Q_{dyn} S b [C_{l_\alpha}(M)\alpha + C_{l_\beta}(M)\beta] \\ Q_{dyn} S b C_{m_\alpha}(M)\alpha \\ Q_{dyn} S b C_{n_\beta}(M)\beta \end{bmatrix} \quad (2.63)$$

where $\mathbf{r}_{ap}$ is the position vector of the aerodynamic reference point (AP), and D_{aero} , C_{aero} , and N_{aero} are the axial (drag), lateral, and normal force components, respectively, given by

$$\begin{aligned} D_{aero} &= Q_{dyn} S C_A(M) - F_{base} \\ C_{aero} &= Q_{dyn} S C_{Y\beta}(M)\beta \\ N_{aero} &= Q_{dyn} S C_{N\alpha}(M)\alpha \end{aligned} \quad (2.64)$$

where $C_A(M)$ is the axial-force coefficient, and $C_{Y\beta}(M)$ and $C_{N\alpha}(M)$ are the slopes of the side-force and normal force coefficients with respect to β and α , respectively, expressed in tabular form as functions of the Mach number (M). The aerodynamic derivatives $C_{l_\alpha}(M)$, $C_{l_\beta}(M)$, $C_{m_\alpha}(M)$, and $C_{n_\beta}(M)$ also depend on M (see Appendix A for further details). In Eqs. (2.64), $Q_{dyn} = \frac{1}{2}\rho V_a^2$ is the dynamic pressure, ρ the atmospheric density, and V_a the airspeed modulus, b and S are the LV diameter and cross-section, respectively, and F_{base} is the base force term resulting from the aerodynamic interaction between the LV base and the surrounding fluid and/or exhaust flow, which depends on altitude. The aerodynamic effects associated with the elastic deformation are not considered in this model.

The aerodynamic angles are

$$\alpha = \arctan\left(\frac{w_a}{u_a}\right) \quad (2.65)$$

$$\beta = \arctan\left(\frac{v_a}{\sqrt{u_a^2 + w_a^2}}\right) \quad (2.66)$$

and the airspeed $\mathbf{V}_a = \mathbf{v}_g - \mathbf{v}_w$, where $\mathbf{v}_g$ and $\mathbf{v}_w$ are the ground and wind speeds, respectively, is written as

$$\mathbf{V}_a = \begin{bmatrix} u_a \\ v_a \\ w_a \end{bmatrix} = V_a \begin{bmatrix} \cos \alpha \cos \beta \\ \sin \beta \\ \sin \alpha \cos \beta \end{bmatrix} \quad (2.67)$$

TWD force and torque

The inertial effects of nozzle rotation in terms of force and moment in $\mathcal{F}_b$

$$\mathbf{F}_N = m_e \mathbf{r}_e \times \dot{\boldsymbol{\omega}}_G + \boldsymbol{\omega}_{BI} \times (m_e \mathbf{r}_e \times \boldsymbol{\omega}_G) \quad (2.68)$$

$$\mathbf{M}_N = (\mathbf{J}_e - \mathbf{r}_G \times m_e \mathbf{r}_e \times) \dot{\boldsymbol{\omega}}_G + \boldsymbol{\omega}_{BI} \times (\mathbf{J}_e - \mathbf{r}_G \times m_e \mathbf{r}_e \times) \boldsymbol{\omega}_G \quad (2.69)$$

Given the nozzle mass, m_e , and the position vectors $\mathbf{r}_G$ and $\mathbf{r}_e$ in $\mathcal{F}_b$, it is

$$\mathbf{r}_G = \begin{bmatrix} -l_{xG} \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{r}_e = \begin{bmatrix} -l_{xe} \cos(\delta_z) \cos(\delta_y) \\ -l_{xe} \sin(\delta_z) \\ l_{xe} \sin(\delta_y) \cos(\delta_z) \end{bmatrix} \quad (2.70)$$

the nozzle angular rate $\boldsymbol{\omega}_G$ and its derivative $\dot{\boldsymbol{\omega}}_G$ are, respectively

$$\boldsymbol{\omega}_G = \begin{bmatrix} \sin(\delta_y) \dot{\delta}_z \\ \dot{\delta}_y \\ \cos(\delta_y) \dot{\delta}_z \end{bmatrix} \quad (2.71)$$

$$\dot{\boldsymbol{\omega}}_G = \begin{bmatrix} \sin(\delta_y) \ddot{\delta}_z + \cos(\delta_y) \dot{\delta}_z \dot{\delta}_y \\ \ddot{\delta}_y \\ -\sin(\delta_y) \dot{\delta}_z \dot{\delta}_y + \cos(\delta_y) \ddot{\delta}_z \end{bmatrix} \quad (2.72)$$

Assuming a diagonal inertia matrix of the nozzle $\mathbf{J}_e = \mathcal{R}_{BN} \mathbf{J}_{e[N]} \mathcal{R}_{NB} \approx \mathbf{J}_{e[N]}$, valid for small nozzle rotations between the frames $\mathcal{F}_b$ and $\mathcal{F}_N$, that is

$$\mathbf{J}_e = \begin{bmatrix} J_{xe} & 0 & 0 \\ 0 & J_{ye} & 0 \\ 0 & 0 & J_{ze} \end{bmatrix} \quad (2.73)$$

the components of $\mathbf{F}_N$ and $\mathbf{M}_N$ are expressed as [59]

$$F_{N_x} = m_e l_{xe} \left[\left(\sin(\delta_y) \dot{\delta}_z \dot{\delta}_y - \cos(\delta_y) \ddot{\delta}_z \right) \sin(\delta_z) \right. \\ \left. + q \sin(\delta_z) \sin(\delta_y) \dot{\delta}_z - q \cos(\delta_z) \cos(\delta_y) \dot{\delta}_y - r \cos(\delta_z) \dot{\delta}_z \right. \\ \left. - \sin(\delta_y) \cos(\delta_z) \ddot{\delta}_y \right], \quad (2.74a)$$

$$F_{N_y} = m_e l_{xe} \left[- \left(\sin(\delta_z) \sin(\delta_y) \dot{\delta}_z - \cos(\delta_z) \cos(\delta_y) \dot{\delta}_y \right) p \right. \\ \left. - \left(\sin(\delta_z) \cos(\delta_y) \dot{\delta}_z + \sin(\delta_y) \cos(\delta_z) \dot{\delta}_y \right) r + \cos(\delta_z) \ddot{\delta}_z \right], \quad (2.74b)$$

$$F_{N_z} = m_e l_{xe} \left[\left(\sin(\delta_y) \ddot{\delta}_z + \cos(\delta_y) \dot{\delta}_z \dot{\delta}_y \right) \sin(\delta_z) + p \cos(\delta_z) \dot{\delta}_z \right. \\ \left. + q \sin(\delta_z) \cos(\delta_y) \dot{\delta}_z + q \sin(\delta_y) \cos(\delta_z) \dot{\delta}_y - \cos(\delta_z) \cos(\delta_y) \ddot{\delta}_y \right] \quad (2.74c)$$

$$M_{N_x} = -J_{xe} \left(\sin(\delta_y) \ddot{\delta}_z + \cos(\delta_y) \dot{\delta}_z \dot{\delta}_y \right) \\ + \left[J_{ye} \dot{\delta}_y - m_e l_{xe} l_{xG} \left(\sin(\delta_z) \sin(\delta_y) \dot{\delta}_z - \cos(\delta_z) \cos(\delta_y) \dot{\delta}_y \right) \right] r \\ - \left(J_{ze} \cos(\delta_y) + m_e l_{xe} l_{xG} \cos(\delta_z) \right) q \dot{\delta}_z, \quad (2.74d)$$

$$M_{N_y} = -J_{xe} r \sin(\delta_y) \dot{\delta}_z - J_{ye} \ddot{\delta}_y + \\ + m_e l_{xe} l_{xG} \left[\left(\sin(\delta_y) \ddot{\delta}_z + \cos(\delta_y) \dot{\delta}_z \dot{\delta}_y \right) \sin(\delta_z) + \right. \\ \left. - \cos(\delta_z) \cos(\delta_y) \ddot{\delta}_y \right] + \left(J_{ze} \cos(\delta_y) + m_e l_{xe} l_{xG} \cos(\delta_z) \right) p \dot{\delta}_z, \quad (2.74e)$$

$$M_{N_z} = J_{xe} q \sin(\delta_y) \dot{\delta}_z + J_{ze} \left(\sin(\delta_y) \dot{\delta}_z \dot{\delta}_y - \cos(\delta_y) \ddot{\delta}_z \right) + \\ - m_e l_{xe} l_{xG} \cos(\delta_z) \ddot{\delta}_z + \\ - \left[J_{ye} \dot{\delta}_y - m_e l_{xe} l_{xG} \left(\sin(\delta_z) \sin(\delta_y) \dot{\delta}_z - \cos(\delta_z) \cos(\delta_y) \dot{\delta}_y \right) \right] p \quad (2.74f)$$

where p , q , and r are the components of $\boldsymbol{\omega}_{BI}$.

2.4 TVC actuator dynamics

The actuator dynamics is described by the following second-order system

$$\ddot{\delta} + 2\zeta_{TVC}\omega_{TVC}\dot{\delta} = \omega_{TVC}^2(\delta_{\tilde{c}} - \delta) \quad (2.75)$$

where $\delta_{\tilde{c}} = \dot{d} + \delta_c$, and δ_c is the TVC command, whereas $\delta_{\tilde{c}}$ is the delayed version of the same signal. ζ_{TVC} and ω_{TVC} represent the damping ratio and natural frequency, respectively. The hardware processing time is described using a pure delay modeled

by a second-order Padé approximation, according to

$$\begin{bmatrix} \dot{d} \\ \ddot{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{12}{\tau_p^2} & -\frac{6}{\tau_p} \end{bmatrix} \begin{bmatrix} d \\ \dot{d} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{12}{\tau_p^2} \end{bmatrix} \delta_c \quad (2.76)$$

$$\delta_{\bar{c}} = \dot{d} + \delta_c \quad (2.77)$$

The two cascaded systems (Eqs. (2.75) and (2.76)) generate control inputs for the LV system model, specifically δ and its time derivatives $\dot{\delta}$ and $\ddot{\delta}$. Equation (2.75) is applied to both the pitch channel, giving δ_y , $\dot{\delta}_y$, and $\ddot{\delta}_y$, and the yaw channel, yielding δ_z , $\dot{\delta}_z$, and $\ddot{\delta}_z$. Consequently, the engine angular rate and angular acceleration are $\omega_{G[N]} = [0, \dot{\delta}_y, \dot{\delta}_z]^T$ and $\dot{\omega}_{G[N]} = [0, \ddot{\delta}_y, \ddot{\delta}_z]^T$ respectively.

2.5 Wind model

A classic model for the wind speed profile as a function of the altitude has been adopted [60]. The model uses data derived from statistical analysis of wind speed measurements at the Kennedy Space Center (KSC) site, that accounts for different conditions and/or ranges of wind speed. The wind profile is represented in Fig. 2.9 as a *steady-state wind profile* and its values are reported in Table 2.1.

Table 2.1. Scale wind speed (W) as a function of altitude h (km)

Altitude [km]	Wind speed [m/s]
0	0
1	8
6	23
11	43
12	45
13	43
20	7
23	7
40	43
50	75

The vertical wind speed is negligible except for that associated with gusts or turbulence. Given the steady-state wind profile, a white noise process is passed through a Dryden filter to model the turbulence [60], according to the transfer function

$$G_w(s, h) = \frac{v_w(s, h)}{n_w(s)} = \frac{\sqrt{\frac{2}{\pi} \frac{V_a(h) - v_w(h)}{L(h)} \sigma^2(h)}}{s + \frac{V_a(h) - v_w(h)}{L(h)}} \quad (2.78)$$

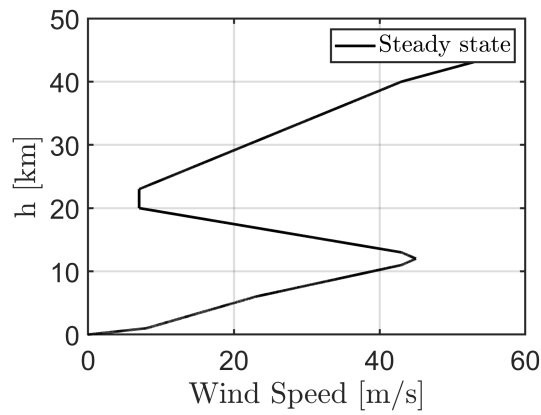


Figure 2.9. Steady-state wind speed vs. altitude

where $v_w(s, h)$ is the stochastic wind component, $n_w(s)$ is the white noise, $V_a(h)$ is the true airspeed, $v_w(h)$ the wind speed and $L(h)$ and $\sigma(h)$ are, respectively the turbulence length scale and the turbulence magnitude versus altitude h the values of which are given in Table 2.2. The wind direction is assigned and the steady state

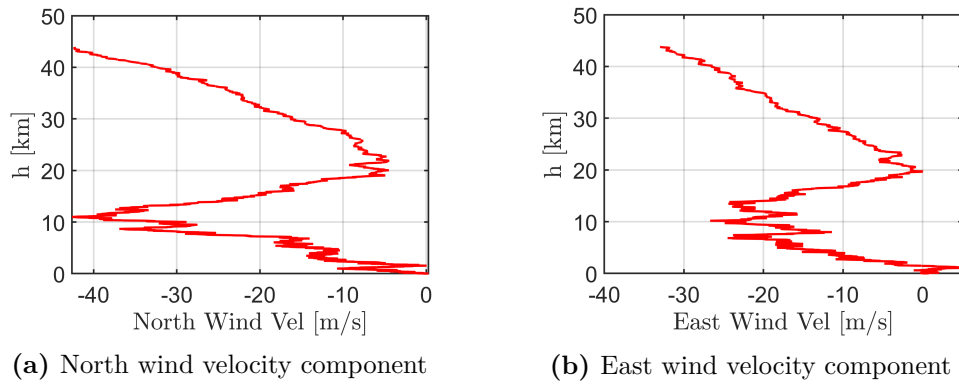


Figure 2.10. Horizontal wind velocity components vs. altitude

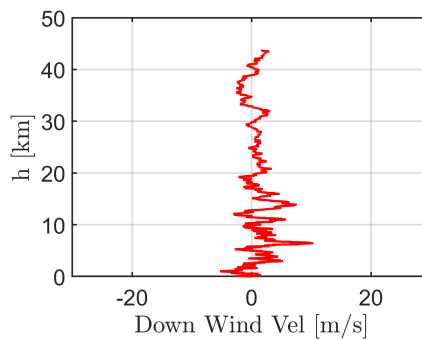


Figure 2.11. Vertical wind velocity component vs. altitude

wind profile is projected on the North and East directions as shown in Figs. 2.10a and

Table 2.2. Horizontal and vertical severe turbulence magnitudes $\sigma_h(h)$ and $\sigma_v(h)$, and turbulence length scales $L_h(h)$ and $L_v(h)$

Altitude [km]	$\sigma_h(h)$ [m/s]	$\sigma_v(h)$ [m/s]	$L_h(h)$ [km]	$L_v(h)$ [km]
1	5.70	4.67	0.832	0.624
2	5.80	4.75	0.902	0.831
4	6.24	5.13	1.040	0.972
6	7.16	5.69	1.040	1.010
8	7.59	5.98	1.040	0.980
10	7.72	6.00	1.230	1.100
12	7.89	5.71	1.800	1.540
14	6.93	5.05	2.820	2.120
18	5.00	4.31	3.400	2.600
20	4.07	3.81	5.000	3.340
25	3.85	3.38	8.640	4.410
30	4.34	3.34	12.000	6.560
35	5.60	3.59	28.600	8.880
40	6.89	3.87	35.400	8.330
45	7.89	4.30	42.600	6.200
50	9.33	4.88	50.100	5.200

2.10b. The output of Eq. (2.78) is added independently for each wind component, particularly for the down component (Fig. 2.11) the wind speed is only governed by the turbulence noise because the wind speed for the steady state wind profile is assumed zero.

A simplification of the steady state wind speed for altitude from 0 to 1 km has been adopted. Indeed, from an altitude of 1 km, the wind speed is extrapolated linearly to zero speed at ground level, whereas a specific wind shear envelope that takes into account the ground interaction should be adopted in this flight region following the guidelines reported in [60].

2.6 Sensor models

Measurements are provided by vehicle sensors, that is an IMU, a GNSS receiver, an ADS, and a ground-based radar tracking station (GS).

The IMU gives inertial non-gravitational acceleration $\mathbf{a}_{\text{IMU}}$ and body-to-inertial angular rate $\boldsymbol{\omega}_{BI}$ at sensor location in the body frame $\mathcal{F}_B$, with $\bar{\boldsymbol{\Sigma}}_{IMU}$ that is the modal slope matrix at IMU location. The measurements are affected by systematic errors and noise. Following Ref. [2], modeled systematic errors are bias, scale factor, misalignment, and non-orthogonality. Accelerometer and gyro data are thus

expressed as

$$\tilde{\mathbf{a}}_{\text{IMU}} = (\mathbf{I} + \mathbf{\Lambda}_a + \mathbf{M}_a + \mathbf{N}_a)\mathbf{a}_{\text{IMU}} + \boldsymbol{\beta}_a + \mathbf{w}_a \quad (2.79)$$

$$\dot{\boldsymbol{\beta}}_a = \mathbf{w}_{b_a} \quad (2.80)$$

$$\tilde{\boldsymbol{\omega}}_{BI} = (\mathbf{I} + \mathbf{\Lambda}_\omega + \mathbf{M}_\omega + \mathbf{N}_\omega)(\boldsymbol{\omega}_{BI} + \bar{\boldsymbol{\Sigma}}_{\text{IMU}}\dot{\boldsymbol{\eta}}_f) + \boldsymbol{\beta}_\omega + \mathbf{w}_\omega \quad (2.81)$$

$$\dot{\boldsymbol{\beta}}_\omega = \mathbf{w}_{b_\omega} \quad (2.82)$$

where $\mathbf{a}_{\text{IMU}[B]}$ is the true non gravitational acceleration at IMU location in $\mathcal{F}_B$, while the superscript $\sim$ indicates the measured quantities, $\mathbf{w}$ is a zero mean white Gaussian noise, and the subscripts a and ω refer to accelerometer and gyro, respectively. The matrices $\mathbf{\Lambda}_i = \text{diag}(\boldsymbol{\lambda}_i)$, $\mathbf{M}_i = [\boldsymbol{\mu}_i \times]$, and $\mathbf{N}_i = \mathbf{N}(\boldsymbol{\nu}_i)$, represent scale factor, misalignment, and non-orthogonality, respectively. $\mathbf{N}$ is given by

$$\mathbf{N}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & \nu_z & \nu_y \\ \nu_z & 0 & \nu_x \\ \nu_y & \nu_x & 0 \end{bmatrix} \quad (2.83)$$

Finally, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, are scale factor, misalignment, and non-orthogonality errors, respectively, all modeled as constant parameters, whereas the bias errors $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}_\omega$ are modeled as a zero-order Gauss Markov process.

A simplified model of the GNSS receiver is considered [2], where the position error is expressed as the sum of an uncorrelated (white) random noise $\mathbf{w}_{r,w}$ and a correlated noise sequence $\mathbf{w}_{r,c}$, given by a first-order Markov process with time constant τ_c , driven by the white noise $\mathbf{w}_{d,w}$ with standard deviation $\sigma_{w_{d,c}}$. Therefore, the GNSS output signal is written as

$$\tilde{\mathbf{r}}_{\text{GNSS}[F]} = \mathbf{r}_{\text{GNSS}[F]} + \mathbf{w}_{r,c} + \mathbf{w}_{r,w} \quad (2.84)$$

$$\dot{\mathbf{w}}_{r,c} = -\frac{1}{\tau_c}\mathbf{w}_{r,c} + \mathbf{w}_{d,w} \quad (2.85)$$

where $\mathbf{r}_{\text{GNSS}[F]}$ and $\tilde{\mathbf{r}}_{\text{GNSS}[F]}$ are, respectively, the true and measured positions of the receiver in $\mathcal{F}_F$. An additive random noise $\mathbf{w}_{v,w}$ is considered for the velocity measurement, so that

$$\tilde{\mathbf{v}}_{\text{GNSS}[F]} = \mathbf{v}_{\text{GNSS}[F]} + \mathbf{w}_{v,w} \quad (2.86)$$

where $\mathbf{v}_{\text{GNSS}[F]}$ and $\tilde{\mathbf{v}}_{\text{GNSS}[F]}$ represent the true and measured planet-relative velocity vectors, respectively, with components in the $\mathcal{F}_F$ frame.

Radar tracking station returns measurements of vehicle range ρ_R , azimuth χ ,

and elevation γ with respect to the topocentric reference frame $\mathcal{F}_G$, given by [8]

$$\tilde{\rho}_R = \rho_R + w_{\rho_R} \quad \tilde{\gamma} = \gamma + w_\gamma \quad \tilde{\chi} = \chi + w_\chi \quad (2.87)$$

where the superscript $\sim$ has the known meaning, and w_{ρ_R} , w_k , and w_χ represent, respectively, Gaussian noises with zero mean and standard deviations $\sigma_{w_{\rho_R}}$, σ_{w_γ} , and σ_{w_χ} .

Table 2.3. Sensor location

Sensor	Location	Reference frame	Unit
IMU	[45 -0.14 1.52]	$\mathcal{F}_S$	[m]
GNSS	[62.91 1.05 -0.92]	$\mathcal{F}_S$	[m]
ADS	/	/	
Radar	[28.72857681 -80.67478864 0]	$\mathcal{F}_F$ as $[\phi_{gs}, \theta_{gs}, h_{gs}]$	[deg, deg, m]

Table 2.4. Parameters of sensor error models.

Sensor	Parameter	Name	Value	Unit
IMU	σ_{w_a}	rate random walk	[100 100 100]	$\mu\text{g}/\sqrt{\text{Hz}}$
	σ_{w_ω}	angular random walk	[0.005 0.005 0.005]	$\text{deg}/\sqrt{\text{h}}$
	$\sigma_{w_{b_a}}$	bias stability	[1.38 1.38 1.38] · 10 ⁻⁵	$\mu\text{g}/\sqrt{\text{h}}$
	$\beta_a _{t=0}$	bias constant, accelerometer	[100 100 100]	μg
	$\sigma_{w_{b_\omega}}$	bias rate, random walk	[6.94 6.94 6.94] · 10 ⁻⁹	$\text{deg}/\text{h}^{3/2}$
	$\beta_\omega _{t=0}$	bias constant, gyro	[0.02 0.02 0.02]	deg/h
	λ_a	scale factor, accelerometer	[50 50 50]	ppm
	λ_ω	scale factor, gyro	[30 30 30]	ppm
	μ_a	misalignment, accelerometer	[60 60 60]	μrad
	μ_ω	misalignment, gyro	[60 60 60]	μrad
	ν_a	non-orthogonality, accelerometer	[30 30 30]	μrad
	ν_ω	non-orthogonality, gyro	[30 30 30]	μrad
GNSS	τ_c	autocorrelation time	1800	s
	$\sigma_{w_{d,w}}$	driving noise on position	1	m/s
	$\sigma_{w_{r,w}}$	noise on position	6	m
	σ_{w_v}	noise on velocity	1	m/s
ADS	σ_{w_T}	noise on TAS	5	m/s
Radar	$\sigma_{w_{\rho_R}}$	noise range	10	m
	σ_{w_γ}	noise elevation	0.01	deg
	σ_{w_χ}	noise azimuth	0.01	deg

The last sensor is an ideal ADS, where uncertainties on Mach number and total air temperature, are ignored for the sake of simplicity [61]. Therefore, it is assumed that the TAS measurement is

$$\tilde{V}_T = \sqrt{u_a^2 + v_a^2 + w_a^2} + w_{V_T} \quad (2.88)$$

subjected to random noise w_{V_T} white and Gaussian in nature. Correction factors used to compensate for effects related to the location of the sensor [31] are also neglected.

Table 2.3 shows the locations of the IMU and GNSS in the $\mathcal{F}_s$ frame, and the latitude, longitude, and altitude of the ground station. The numerical values of parameters adopted for the error models are reported in Table 2.4.

Chapter 3

LV Control and reference mission

In this chapter, a detailed description of the flight control system (FCS) design and the reference mission chosen as the main to simulate the first stage flight of the Ares-I LV is presented. The purpose of the simulated first-stage flight presented in this Section is to generate a synthetic data-set for post-flight analysis that is object of this work. First the chapter presents the linearized (around the reference trajectory) model for the LV, which incorporates the effects of the elastic dynamics on the commanded control input (because the thrust direction is deviated due to elastic rotation at pivot point) and on the controlled states, indeed rotations and displacements at IMU location are detected and fed back from the Inertial Navigation System (INS) to the controller.

Following, the control design process for a reference mission is detailed, outlining the approach for each phase.

3.1 Control problem statement

The development of the control system concern two primary objectives: The first is related to the stabilization of short-period motion (due to the aerodynamic instability of the launcher and its elastic dynamics), and the second concern to the guidance problem, namely, ensuring the launcher follows a reference trajectory. As a remark, the trajectory is flown in open-loop guidance and the control sistem manage only perturbation minimizing trajectory deviations in terms of attitude error and lateral drift. Achieving both requirements necessitates simultaneous minimization of aerodynamic loading (especially during the maximum dynamic pressure trajectory phase) and model robustness to parameter variations and disturbances (such as wind

gusts) [59]. After stabilizing the short-period dynamics and adjusting the deviation from the nominal trajectory using PD controllers, a low-pass filter is required to stabilize the phase of the bending dynamics associated with the first elastic mode, while a series of notch filters are used to attenuate the gain of the first three elastic modes. In the end of the chapter a summary of the reference mission simulated is provided, while the development of the control system fully adheres to the guidelines outlined by [59].

3.2 State-space model

A linear LV model, tailored for the objectives of this study, is developed by considering the rotational dynamics around the pitch axis and transverse drift dynamics relative to the vehicle reference trajectory. Typically, when designing attitude control systems, the interaction between pitch and yaw dynamics can be decoupled. This is often due to the low roll rate of the Launch Vehicle (LV) and its symmetrical configuration about the roll axis. As a result, a simplified single-axis, 3-degree of freedom (3-DoF) linear model can be used to characterize the LV's motion for the purposes outlined in this thesis. The model features contributions from the first three bending modes (described by Eq. (2.21)), while neglecting the effects of aerodynamic damping and tail-wags-dog (TWD) [55]. For the sake of simplicity, we denote z and $\dot{z}$ as the drift and drift rate of the center of mass along the normal axis of the trajectory frame, respectively. The perturbed pitch angle is represented by Θ , and η denotes the generalized coordinate of the first three elastic modes. δ stands for the nozzle angle. Next, ω_{BM} , ζ_{BM} , and ψ_{TVC} are, respectively, to the bending modes natural frequency, damping ratio, and displacement component over the generalized mass at the TVC location. Finally, $l_{x_{CP}[S]}$, $l_{x_{cg}[S]}$, $l_{x_{INS}[S]}$ are the x-coordinate in $\mathcal{F}_S$ of the center of pressure, cg and INS location, respectively. According to the typical notation for LV dynamic models [62], the following parameters are defined

$$A_6 = \frac{Q_{\text{dyn}} S C_{N_\alpha} (l_{x_{CP}[S]} - l_{x_{cg}[S]})}{J_{yy}} \quad K_1 = -\frac{T(l_{x_G[S]} - l_{x_{cg}[S]})}{J_{yy}} \quad (3.1)$$

$$a_1 = -\frac{Q_{\text{dyn}} S C_{N_\alpha}}{m V_a} \quad a_4 = \frac{T - D}{m} \quad L_{cg-INS} = l_{x_{cg}[S]} - l_{x_{INS}[S]} \quad (3.2)$$

where, A_6 and K_1 are dubbed as the aerodynamic and control moment coefficients, respectively, while $\alpha_w = \frac{v_w}{V_a}$ represents the wind-induced angle of attack, acting as a disturbance on the system.

The linear formulation of LV model dynamics in state-space form is given [59],[62]

$$\begin{aligned}
 \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \dot{\Theta} \\ \ddot{\Theta} \\ \ddot{\delta}_y \\ \dot{\delta}_y \\ \dot{\eta}_{f1} \\ \ddot{\eta}_{f1} \\ \dot{\eta}_{f2} \\ \ddot{\eta}_{f2} \\ \dot{\eta}_{f3} \\ \ddot{\eta}_{f3} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_1 V_a - a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & A_6/V_a & A_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_{TVC}^2 & -2\zeta_{TVC}\omega_{TVC} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_{BM1}^2 & -2\zeta_{BM}\omega_{BM1} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_{BM2}^2 & -2\zeta_{BM}\omega_{BM2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_{BM3}^2 & -2\zeta_{BM}\omega_{BM3} \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \\ \Theta \\ \dot{\Theta} \\ \ddot{\delta}_y \\ \dot{\delta}_y \\ \eta_{f1} \\ \dot{\eta}_{f1} \\ \eta_{f2} \\ \dot{\eta}_{f2} \\ \eta_{f3} \\ \dot{\eta}_{f3} \end{bmatrix} + \\
 &+ \begin{bmatrix} 0 \\ -\frac{T}{m} \\ 0 \\ K_1 \\ 0 \\ \omega_{TVC}^2 \\ 0 \\ \psi_{1GT} \\ 0 \\ \psi_{2GT} \\ 0 \\ \psi_{3GT} \end{bmatrix} \delta_{\tilde{c}} + \begin{bmatrix} 0 \\ -a_1 V_a \\ 0 \\ -A_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \alpha_w
 \end{aligned} \tag{3.3}$$

The elements of the output vector $\mathbf{y} = [\Theta_{INS} \ \dot{\Theta}_{INS} \ z_{INS} \ \dot{z}_{INS}]^T$ are the INS measurements, include the effect of flexible motion and is expressed as

$$\mathbf{y} = \begin{bmatrix} \Theta_{INS} \\ \dot{\Theta}_{INS} \\ z_{INS} \\ \dot{z}_{INS} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & \sigma_{1INS} & 0 & \sigma_{2INS} & 0 & \sigma_{3INS} & 0 \\ 0 & 0 & 0 & 1 & 0 & \sigma_{1INS} & 0 & \sigma_{2INS} & 0 & \sigma_{3INS} \\ 1 & 0 & 0 & 0 & \psi_{1INS} & 0 & \psi_{2INS} & 0 & \psi_{3INS} & 0 \\ 0 & 1 & 0 & L_{cg-INS} & 0 & \psi_{1INS} & 0 & \psi_{2INS} & 0 & \psi_{3INS} \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \\ \Theta \\ \dot{\Theta} \\ \eta_{f1} \\ \dot{\eta}_{f1} \\ \eta_{f2} \\ \dot{\eta}_{f2} \\ \eta_{f3} \\ \dot{\eta}_{f3} \end{bmatrix}$$

(3.4)

where, ψ_{INS} and σ_{INS} are the displacement and rotation components, respectively, of the bending modes at the IMU location.

3.3 FCS design

The first step in designing a LV FCS involves determining the mission profile, reference trajectory, and overall vehicle configuration. This task is carried out by the guidance function, which specifies the reference trajectory in terms of the vehicle position, velocity, and thrust vector orientation. Typically, guidance commands are translated into pitch and yaw (or pitch and yaw rates) commands, aiming to orient the vehicle in a predefined direction. Consequently, the LV guidance system provides pitch and yaw commands as the reference input for the FCS. Unlike traditional approaches to FCS design, where the study of short-period dynamics is often conducted independently of the guidance problem, the two aspects are inseparable when considering attitude control in a LV [63]. Attitude control plays a crucial role in minimizing trajectory deviations in terms of attitude error and lateral

drift, and, is necessary for reducing the angle of attack to ensure the structural integrity of the vehicle. As a result, FCS design involves multiple objectives, the foremost being the stabilization of short-period motion, considering the aerodynamic instability and high flexibility of the LV.

LVs are usually aerodynamically unstable because the location of the aerodynamic center of pressure is forward of the vehicle cg position. Hence, in order to provide adequate short-period stability, traditional FCS architectures that control attitude angle using proportional and derivative (PD) actions, as reported in Refs. [55],[64] and [62]. Once the short-period dynamics of the vehicle have been stabilized, the performance of the FCS on aerodynamic loads and trajectory tracking is to be assessed. To this end, the variations of the angle of attack due to wind gusts must be limited particularly in the flight phase of high dynamic pressure, to prevent loss of control. The angle of attack is also to be managed in order to mitigate aerodynamic moments, and in this respect, the control system incorporates a load relief feature, designed to orient the vehicle into the wind direction and reduce the angle of attack. However, it's important to observe that achieving load relief goal may cause errors on attitude and position w.r.t to the reference trajectory, so that address the conflicting requirements of maintaining the prescribed trajectory while reducing structural load is required in order to design an appropriate control architecture. further detail about load relief strategy for the adopted FCS design can be found in [62]. The FCS relies on PD controllers for attitude and lateral drift dynamics, plus filters to phase-stabilize and attenuate the bending modes as detailed in [62]. In particualre it has

- a PD on pitch Θ with gains K_{P_Θ} and $K_{D_{\dot{\Theta}}}$
- a PD on lateral drift z (with gains K_{P_z}) and drift velocity $\dot{z}$ (with gain $K_{D_{\dot{z}}}$)
- a second-order low-pass filters $H_{LP}(s)$ to phase-stabilize the bending modes and notch filters $H_N(s)$ to attenuate the bending modes magnitude.

Figure 3.1 shows the control architecture, described in the frequency domain as

$$K_{FCS}(s) = \begin{bmatrix} K_{P_\Theta} & K_{D_{\dot{\Theta}}} & K_{P_z} & K_{D_{\dot{z}}} \end{bmatrix} H_{LP}(s) H_N(s) \quad (3.5)$$

Table 3.1 shows a classical set of requirements for robust stability of LVs [65], in terms of stability margins, where the requirement in terms of delay margin (DM) can be converted in phase margin (PM) as $DM = PM / \omega_{cross}$ with ω_{cross} the cross-over frequency of the system. They are also shown in the Nichols chart of Fig. 3.2 that shows the loop gain transfer function (TF) of the system, calculated as an

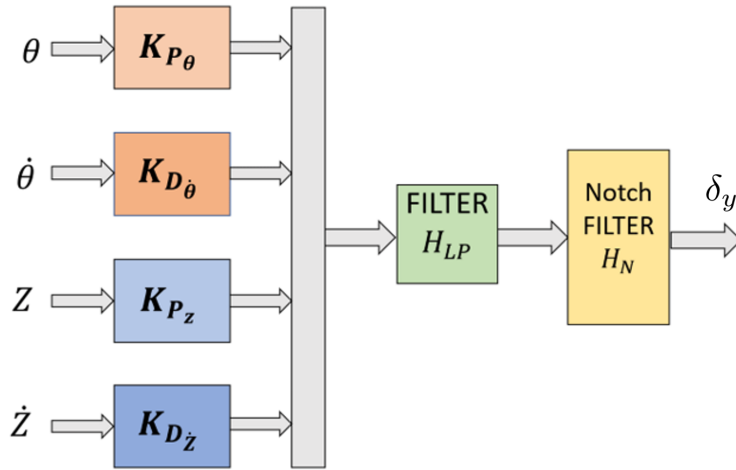


Figure 3.1. TVC schematic control architecture

input-output TF obtained by breaking the loop at the control generation point (plant input) and injecting a signal, then the return signal before the breaking point is $-K_{FCS}(s)G(s)$, so $L(s) = K_{FCS}(s)G(s)$ is the loop gain transfer function where $G(s)$ represent the system plant.

Table 3.1. FCS stability requirements.

Rigid-Body margins			Flexible-body margins	
Aero GM	Rigid DM	Rigid GM	Flex GM	Flex DM
≥ 6 dB	≥ 100 ms	≤ -6 dB	≤ -3 dB	≥ 50 ms

Table 3.2. FCS performance requirements.

Requirements	Metrics	Bounds
Aerodynamic load	$Q\alpha$	$< Q\alpha$ safety envelope
Lateral drift	z_{max}	< 500 m
Lateral rate drift	$\dot{z}_{max}$	< 15 m/s
TVC angle deflection	δ_{max}	< 6 deg

To address variations of LV parameters throughout the duration of the flight, the process of FCS synthesis needs to be reiterated at specific design points along the reference trajectory. Consequently, the scheduling of controller gains and filter parameters relies on a chosen variable, such as time or non-gravitational velocity. It is essential for the scheduling time grid to be sufficiently dense to ensure that the FCS is effective in addressing the pertinent variations in LV dynamical characteristics throughout the entirety of the flight.

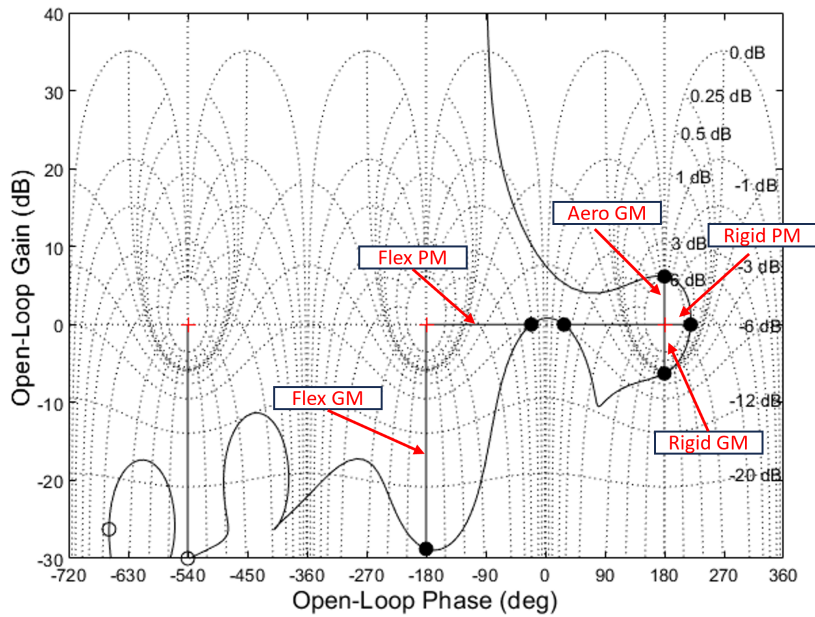


Figure 3.2. Nichols plot of the open-loop response

3.4 PD gain syntesis and bending modes stabilization

As, said, the first requirement of FCS on LV stabilization is satisfied by means a PD controller on the attitude feedback channel, the design of which is conducted by considering the rotational dynamics $\ddot{\Theta} = A_6\Theta + K_1\delta_c$ having neglected lateral, TVC and bending mode dynamics. By imposing stability requirements of 6 dB for the GM and 30 deg for the PM the following expression are obtained for the proportional and derivative gain respectively [59]

$$K_{P_\Theta} = \frac{2A_6}{K_1} \quad (3.6)$$

$$K_{D_\Theta} = \frac{\sqrt{A_6}}{K_1} \quad (3.7)$$

Afterward, the TVC actuator dynamics must be taken into account, also considering the additional phase shift due to the computational delay time. The result is a slight loss of PM. Finally, addressing the stabilization of bending modes presents a complex challenge in the realm of LV attitude control design due to the uncertain knowledge and time-varying nature of bending mode properties. In particular, achieving a satisfactory response in the rigid-body behavior while concurrently stabilizing elastic dynamics becomes progressively difficult when there is insufficient

frequency separation between bending and rigid-body modes.

The INS provides both the attitude angles and angular rates. However these output are affected by the elastic dynamics detected from the IMU at its location, consequently their effect is directly fed back in the attitude control loop by means of the controlled attitude angles and angular rates. This typically results in a deterioration of control system stability, and in certain circumstances, it may drive the system towards instability.

Given the possible instability arising from the bending modes, in particular the first one, a form of compensation is essential to stabilize the system. Consequently, the conventional approach involves implementing second-order structural filters to achieve appropriate gain-phase stabilization of LV flexible modes. Examples of such filters include low-pass and notch filters. In control system, the design of low-pass and notch filters follows the guidelines provided by Wie [66]. Phase stabilization guarantees the desired gain and phase characteristics at a designated frequency. A bending mode is considered phase-stabilized when it possesses a finite phase margin, and , to this end, a non-minimum phase second-order, low-pass filter is employed

$$H_{LP}(s) = \frac{s^2 - 2\zeta_z\omega_z s + \omega_z^2}{s^2 + 2\zeta_p\omega_p s + \omega_p^2} \quad (3.8)$$

Pole and zero locations are specified so that they share the same natural frequency ($\omega_z = \omega_p = 1.3\omega_{BM1}$) with $\zeta_p > \zeta_z$. For the considered LV model, where the phase of the first bending mode is close to 0° (typically within the range of -180° to 180°), achieving phase stabilization involves shifting the phase of the open-loop response curve relative to the flexible dynamics to the left of the critical point [0dB, 180 deg]. Figure. 3.3 shows the Nichols of the unstable system, while applying a low-pass filter (Eq.(3.8)) to the open-loop response, the system become stable (the critical point is encircled clockwise), and it is shown in Fig. 3.4. Moreover, Figs. 3.3 and 3.4 reveal that the magnitude of the bending modes is excessively high, necessitating attenuation.

A notch filter is thus adopted in order to stabilize the gain of the bending mode. Notch filtering prevent the destabilization of a specific mode through feedback control facilitating gain stabilization by reducing the control loop gain at the desired frequency, ensuring stability in the face of phase uncertainties. As reported in [66], the notch filter is represented by

$$H_{notch}(s) = \frac{s^2 + 2\zeta_N\omega_N s + \omega_N^2}{s^2 + 2\zeta_D\omega_D s + \omega_D^2} \quad (3.9)$$

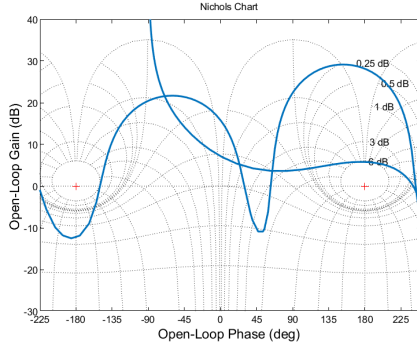


Figure 3.3. Unstable system

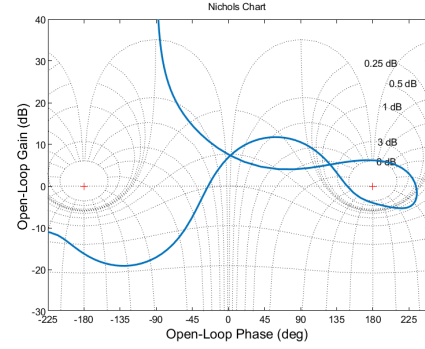


Figure 3.4. Phase stabilized system

Figure 3.5. Phase stabilization of the LV model

with $\omega_N = \omega_D = \omega_{BM}$. The notch filter is designed to position its zeros in proximity to the undesirable, lightly damped poles of the system for approximate cancellation, while the poles are located to enhance system damping. Consequently, the parameters ζ_N and ζ_D , where $\zeta_N < \zeta_D$, define the bandwidth and peak level of filter attenuation. As reported in [67], in order to deal with the uncertainty of the bending mode frequency, the notch filter is realized by three cascaded second-order filters centered, respectively, at the nominal bending mode frequency ω_{BM} , and at the minimum and maximum expected dispersed frequencies of the bending mode due to modeling uncertainties ($\underline{\omega}_{BM}$ and $\bar{\omega}_{BM}$). The overall transfer function is

$$H_N(s) = \frac{s^2 + 2\zeta_N \underline{\omega}_{BM} s + \underline{\omega}_{BM}^2}{s^2 + 2\zeta_D \underline{\omega}_{BM} s + \underline{\omega}_{BM}^2} \cdot \frac{s^2 + 2\zeta_N \omega_{BM} s + \omega_{BM}^2}{s^2 + 2\zeta_D \omega_{BM} s + \omega_{BM}^2} \cdot \frac{s^2 + 2\zeta_N \bar{\omega}_{BM} s + \bar{\omega}_{BM}^2}{s^2 + 2\zeta_D \bar{\omega}_{BM} s + \bar{\omega}_{BM}^2} \quad (3.10)$$

where ω_{BM} is the (nominal) bending mode frequency, and considering an uncertainty of $\omega_{BM} \pm 10\%$, it is $\underline{\omega}_{BM} = 0.9\omega_{BM}$ and $\bar{\omega}_{BM} = 1.1\omega_{BM}$. Moreover, it also provides additional phase stabilization, that increases the Flex-PM (see Fig. 3.2) relative to the bending mode. In the end, to address the tasks of constraining the aerodynamic load and lateral drift concerning the reference trajectory, additional feedback on drift z and drift rate $\dot{z}$ is introduced, obtaining the following control law

$$\mathbf{K}_{FCS}(s) = [K_{P_\Theta} \quad K_{D_{\dot{\Theta}}} \quad K_{P_z} \quad K_{D_{\dot{z}}}] H_{LP}(s) H_{N_1}(s) H_{N_2}(s) H_{N_3}(s)$$

where a combination of three notch filters ($H_N(s) = H_{N_1}(s), H_{N_2}(s), H_{N_3}(s)$) with the form of Eq.(3.10), are used so as to notch the first three bending mode of the system. Here, as mentioned earlier, K_{P_z} and $K_{D_{\dot{z}}}$ represent the control gains for

the feedback on z and $\dot{z}$, respectively.

It is important to note that, although the proportional control on Θ affects the angle of attack, selecting an appropriate K_{P_Θ} value to meet stability requirements while limiting the variation of the angle of attack and, consequently, the aerodynamic load is challenging. Usually, once Eqs. (3.6) and (3.7) are used, only minor adjustments of K_{P_Θ} and K_{D_Θ} are required for achieving robustness with respect to lateral dynamics, whereas of K_{P_z} and K_{D_z} provide an effective means of reducing lateral drift and aerodynamic loads without compromising attitude tracking performance. It is noteworthy that utilizing large values of K_{P_z} and K_{D_z} can lead to system instability.

3.5 Gain scheduling

Many of the parameters of the LV are time-dependent, experiencing rapid variations throughout the flight. These include mass and inertial properties, aerodynamic coefficients, bending mode frequency, and thrust. To effectively handle these parameter changes and ensure the desired stability margin and performance during the atmospheric phase, a suitable strategy is employed known as Gain Scheduling of the control law [62].

The gain scheduling approach involves breaking down the LV control problem into several linear designs conducted at different operational points along the flight envelope. This results in obtaining a globally scheduled controller through interpolation of these design points. In the scope of this study, PD control gains are synthesized with a time step of 5 seconds along the trajectory. They are then linearly interpolated as functions of flight time, as illustrated in Fig. 3.6

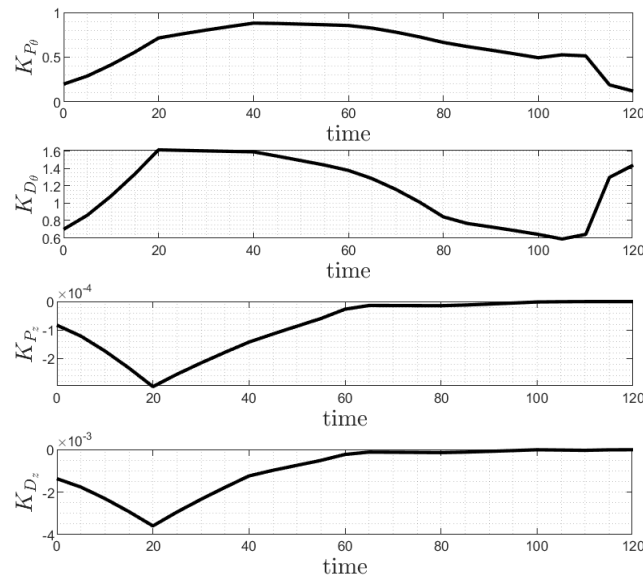


Figure 3.6. PD controller gains versus time.

Under nominal conditions, achieving system stability requires placing all closed-loop poles on the left side of the complex plane. This criterion can be readily assessed in the frequency domain through Nichols plots. In this context, the Nichols plots of open-loop responses, obtained at intervals of 10 s from $t = 0$ to 120 s, as shown in Fig. 3.7, indicate that the LV equipped with the FCS developed in this chapter remains stable throughout its flight.

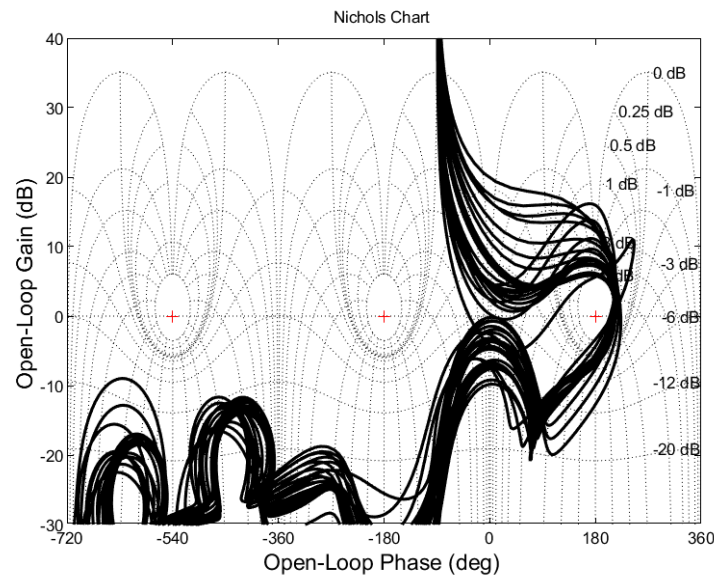


Figure 3.7. Nichols plots at design points.

3.6 Reference Mission

The purpose of the Ares-I flight simulation is to generate a syntetic data-set such as sensors output and true output (because it is a simulated flight) in order to develop and validate in simulation TR and/or estimation algorithm to employ in post flight analysis. In this respect the aim of this section is to describe the reference mission.

The first-stage flight of a servicing mission to the ISS is simulated (total flight time around 120 s). The main parameters of the considered LV (taken from the literature [52]), are reported in Table 3.3.

Table 3.3. Principal parameters of the ARES I model

Quantity	Value	Unit
Height (full assembled)	95	m
Reference length	3.71	m
Aerodynamic reference area	10.8	m ²
Weight at liftoff (including payload)	896.24	ton
Position of c.g. at lift-off [†]	67.15	m
Position of aerodynamic reference point at lift-off [†]	83.82	m
Position of gimbal attachment point at lift-off [†]	90.22	m

[†] Axial positions are referred to vehicle nose.

The launch starts from the Kennedy Space Centre Complex LC-39B. The reference trajectory from lift-off to target orbit has been determined by means of an optimisation tool based on the method presented in Refs. [68] and [69]. At the basis of the optimisation tool there is the EOS (Evolutionary Optimisation at Sapienza) optimisation code [68], used for nonlinear constrained problems (i.e. LV trajectory optimisation), that requires data on

- mission (launch base location, target orbit parameters)
- stages (specific impulse, propellant mass)
- the whole LV (C_A , thrust and weights)
- atmospheric parameters.

Table 3.4 shows the target orbit and mission data. The equations of motion (see Sec. 2.3) are integrated using a fourth-order Runge-Kutta algorithm with a constant time step of 0.005 s, the model being implemented in Simulink®. The time histories of attitude angles in Fig. 3.8 and 3.9 in particular the pitch angle vs time, show the first phases of the mission

Table 3.4. Target orbit data

Code	Value	Definition
θ_{LB}	-80.604333 [deg]	Launch Base Longitude
ϕ_{LB}	28.608389 [deg]	Launch Base Latitude
h_{pto}	417 [km]	Target Orbit Perigee Height
h_{ato}	417 [km]	Target Orbit Apogee Height
i_{to}	51 [deg]	Target Orbit Inclination
t_{va}	5 [s]	Vertical Ascent Duration

- phase 1: vertical liftoff ($0 - 5$ s)
- phase 2: pitch-over ($5 - 10$ s)
- phase 3: zero-lift gravity-turn maneuver (ZLGT) ($10 - 120$ s)

After a vertical liftoff (phase 1), the LV performs a programmed rotation (pitch-over, phase 2) that allows the vehicle to begin a ZLGT maneuver (phase 3). The ZLGT exploits gravity to further steer the rocket velocity and, at the same time, minimize the transverse aerodynamic load by keeping the angle of attack, as low as possible, that is, aligning the vehicle to the relative velocity vector. It is apparent in Fig. 3.14 that the LV reaches a speed of about Mach 3.7 at the end of the first stage burn, at an altitude of 44.000 m. Dynamic pressure is maximum between 50 s and 60 s, while the maximum relative velocity is around 1,250 m/s.

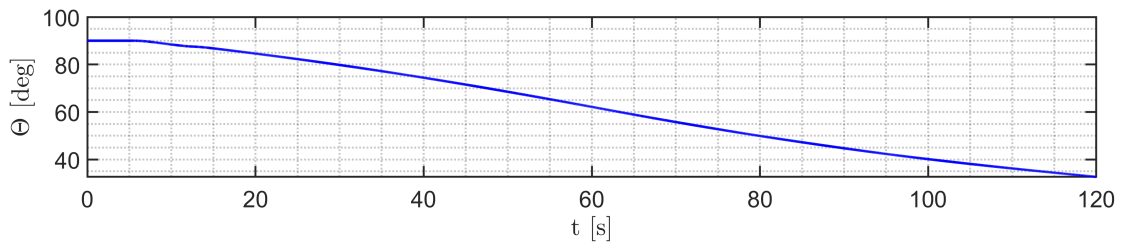


Figure 3.8. Pitch angle

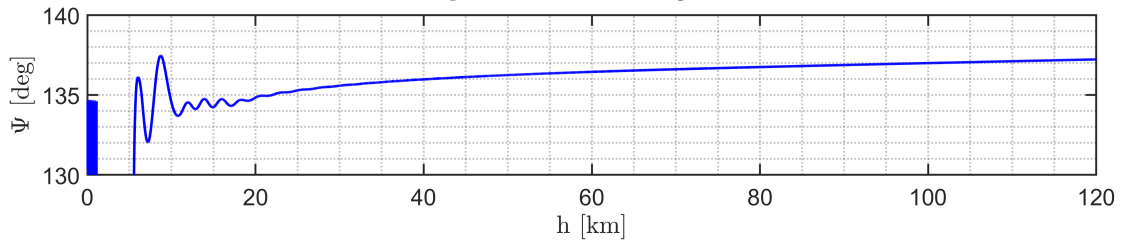


Figure 3.9. Yaw angle

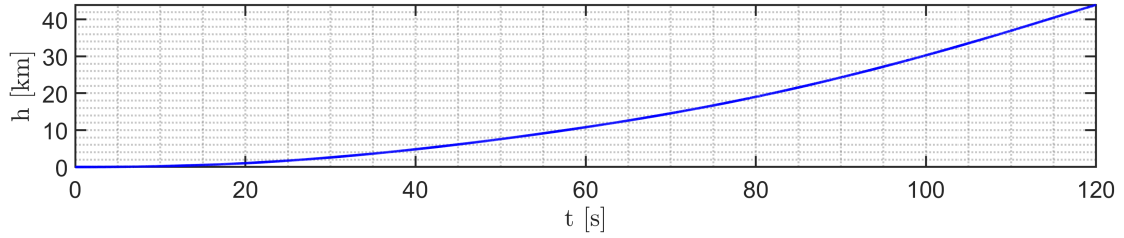


Figure 3.10. Altitude

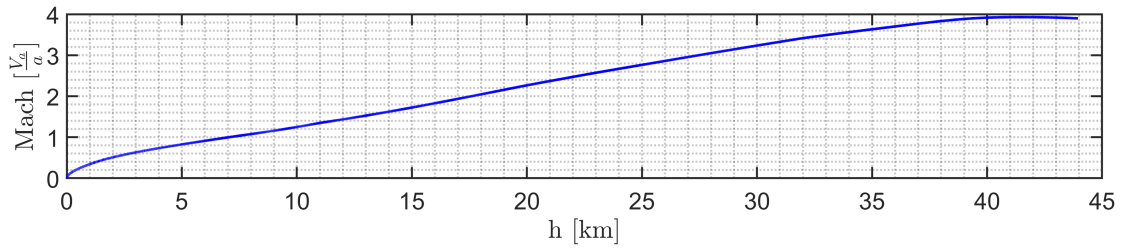


Figure 3.11. Mach

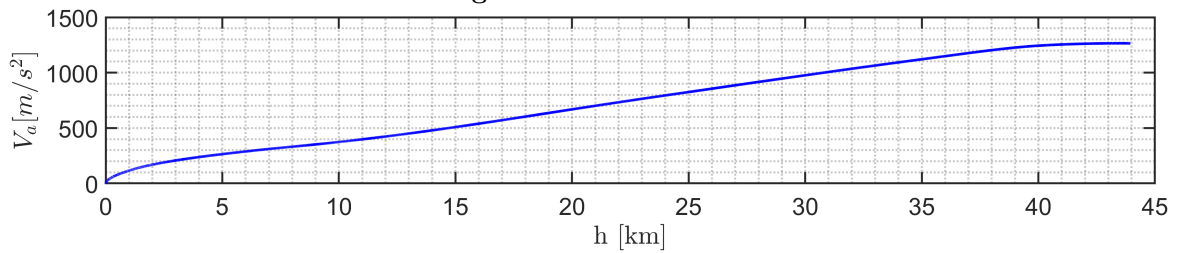


Figure 3.12. Velocity

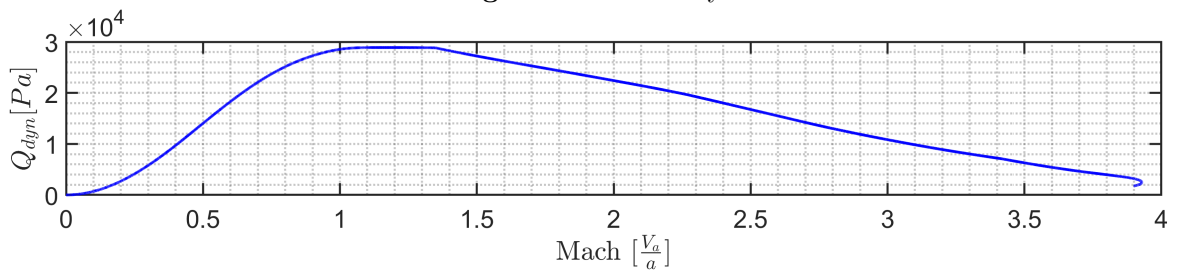


Figure 3.13. Dynamic pressure

Figure 3.14. Reference trajectory: pitch (a), yaw (b), altitude (c), Mach (d), velocity (e), dynamic pressure (f)

Chapter 4

Estimation Methodologies

In this chapter, we introduce the main nonlinear estimation methodologies for TR and the estimation of aerodynamic angles and wind components. As shown in the current literature in the field of TR for flight vehicles [28], LVs [31], re-entry vehicles [70] and in applications concerning the estimation of aerodynamic parameters [6], the main estimation methodologies are based on the Kalman filter (KF) [11].

KF achieves optimality by minimizing the mean square error and maximizing likelihood estimation, given specific conditions. These conditions are met when both noise sources and the initial estimation error conform to Gaussian distributions, and the system dynamics and measurement processes are linear. Even if these conditions are not fully met, the KF continues to be regarded as the best option among linear unbiased estimators for achieving minimum variance [71]. The Extended Kalman Filter (EKF) is the most used variant in the aerospace industry for state estimation. EKF is the nonlinear estimation problem variant of the traditional KF, designed to handle small estimation errors by employing a first-order approximation to compute their covariance matrix. The Chapter starts with an introduction about the adopted notation, then the continuous and the continuous-discrete formulation of the EKF are presented, however, to address the limitations imposed by the first-order approximation assumption for the EKF and better handle nonlinearities, alternative algorithms have been introduced because often used in aerospace applications and in this work such as the Iterated Extended Kalman Filter (IEKF) [13] (NewSTEP core algorithm [31]) or the Unscented Kalman Filter (UKF) [12]. To address cases concerning state estimation with a high number of auxiliary states a Consider EKF (CEKF) formulation is reported, then the task of determining attitude is approached through a Multiplicative Extended Kalman Filter (MEKF) formulation [13]. At the end of the Chapter, a batch method also known as "smoother" [13] useful in all the cases that do not require a real-time estimation is presented. Sections concerning the

stochastic models and signal processing are included to provide a description of noise models often used in the filters to compensate for unmodelled dynamics and the possible application of these signal processing techniques in the data pre-processing step of a post-flight analysis. About signal pre-processing, an example application is shown, where first angular acceleration and linear acceleration are filtered by means of a Wiener filter in order to eliminate the high-frequency elastic contribution detected from the IMU, then angular acceleration is obtained by means of a smoothed differentiation of the angular rate.

4.1 Notation

At the beginning of this chapter, it is worth adding some details on notation. Specifically, $(\mathbf{a}(k))$ denotes a generic vector $(\mathbf{a})$ at the discrete time instant $(t = t_k)$. Time dependence is occasionally expressed using a subscript, such that $(\mathbf{a}(k))$ is equivalently represented as $(\mathbf{a}_k)$. About the filtering algorithm, $\hat{\mathbf{x}}_k^-$ represents the a priori estimate of the state $\mathbf{x}$ at the k -th time, while $\hat{\mathbf{x}}_k^+$ is the a posteriori estimate. Similarly, $\mathbf{P}_k^-$ represents the a priori estimate of the error covariance $\mathbf{P}$ at the k -th time interval, while $\mathbf{P}_k^+$ is the a posteriori estimate. In the end, the symbol $\hat{}$ refers to the estimated quantity.

4.2 Extended Kalman Filter

The Extended Kalman Filter (EKF) is widely employed in the aerospace industry for state estimation, serving as the foundation for the navigation and control systems of aerospace vehicles. Its adaptability in dealing with nonlinear dynamics and measurement uncertainties makes it indispensable for missions that demand high accuracy and reliability, from satellite orbit determination [71] to autonomous aerial vehicle guidance [72]. The EKF is a natural extension of the classic KF that can handle nonlinear systems [13]. While a standard KF is designed to operate on linear systems with Gaussian noise, real-world systems are often nonlinear, making the straightforward application of a KF inappropriate. Consider a stochastic nonlinear dynamic system with non-additive Gaussian noise and a nonlinear measurement equation also affected by non-additive Gaussian noise, such as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\eta}(t), t) \quad (4.1)$$

$$\mathbf{z}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\xi}(t), t) \quad (4.2)$$

with

- $\mathbf{x}(t) \in \mathbb{R}^{n_x}$: state vector
- $\mathbf{u}(t) \in \mathbb{R}^{n_u}$: input vector
- $\mathbf{z}(t) \in \mathbb{R}^{n_z}$: output vector
- $\mathbf{f} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$: state evolution function vector
- $\mathbf{h} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$: observation vector
- $\boldsymbol{\eta}(t) \in \mathbb{R}^{n_x}$: process noise vector
- $\boldsymbol{\xi}(t) \in \mathbb{R}^{n_z}$: measurement noise vector

The process and measurement noises are characterized by a Gaussian probability density function (PDF) with zero mean and known covariance matrix $\boldsymbol{\sigma}$

- $\boldsymbol{\eta}(t) \sim \mathcal{N}\left(\mathbf{0}_{n_x \times 1}, \boldsymbol{\sigma}_{\boldsymbol{\eta}}(t)\right)$
- $\boldsymbol{\xi}(t) \sim \mathcal{N}\left(\mathbf{0}_{n_z \times 1}, \boldsymbol{\sigma}_{\boldsymbol{\xi}}(t)\right)$

where $\mathbf{0}_{a \times b}$ represents a zero matrix of dimensions $a \times b$. The PDF related to the initial state is known and Gaussian expressed as

- $\mathbf{x}_0 \sim \mathcal{N}\left(\hat{\mathbf{x}}_0, \mathbf{P}_0\right)$

with $\mathbf{P}_0$ representing the covariance matrix of the error at the initial time. The equations at the basis of the EKF are obtained assuming that the true state is sufficiently close to the current estimate and the function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ and $\mathbf{h}(\mathbf{x}(t), t)$ are continuously differentiable. Consider a first-order Taylor series expansion of Eq.s (4.1) and (4.2) around a reference condition $\bar{\mathbf{x}}(t)$. It is written

$$\begin{aligned} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\eta}(t), t) \approx & \mathbf{f}(\bar{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\eta}(t), t) \Big|_{\boldsymbol{\eta}(t)=0} + \\ & + \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\eta}(t)=0} [\mathbf{x}(t) - \bar{\mathbf{x}}(t)] + \frac{\partial \mathbf{f}}{\partial \boldsymbol{\eta}} \Big|_{\bar{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\eta}(t)=0} \boldsymbol{\eta}(t) \end{aligned} \quad (4.3)$$

$$\begin{aligned} \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\xi}(t), t) \approx & \mathbf{h}(\bar{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\xi}(t), t) \Big|_{\boldsymbol{\xi}(t)=0} + \\ & + \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\xi}(t)=0} [\mathbf{x}(t) - \bar{\mathbf{x}}(t)] + \frac{\partial \mathbf{h}}{\partial \boldsymbol{\xi}} \Big|_{\bar{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\xi}(t)=0} \boldsymbol{\xi}(t) \end{aligned} \quad (4.4)$$

In the EKF, it is assumed that the reference condition $\bar{\mathbf{x}}(t)$ is the current estimate $\hat{\mathbf{x}}(t)$; therefore, applying the operation of expected value to both sides of Eq. (4.3)–(4.4) yields

$$E\{\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\eta}(t), t)\} = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) \quad (4.5)$$

$$E\{\mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\xi}(t), t)\} = \mathbf{h}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) \quad (4.6)$$

Following a similar process to the derivation of the KF equations [13], the formulation for the EKF yields the following results:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) + \mathbf{K}(t)[\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t)] \quad (4.7)$$

$$\hat{\mathbf{z}}(t) = \mathbf{h}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) \quad (4.8)$$

Replacing Eq. (4.3) and (4.4), with $\hat{\mathbf{x}}(t) = \bar{\mathbf{x}}(t)$, in Eq. (4.7) and using Eq. (4.1) gives

$$\dot{\underline{\mathbf{x}}}(t) = [\mathbf{F}(t) - \mathbf{K}(t)\mathbf{H}(t)]\underline{\mathbf{x}}(t) - \mathbf{G}(t)\boldsymbol{\eta}(t) + \mathbf{K}(t)\mathbf{D}(t)\boldsymbol{\xi}(t) \quad (4.9)$$

where $\underline{\mathbf{x}}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$ defines the error, while the matrices $\mathbf{F}(t)$, $\mathbf{H}(t)$, $\mathbf{G}(t)$ and $\mathbf{D}(t)$ are defined as

$$\mathbf{F}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t)}, \quad \mathbf{H}(t) \equiv \left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t)}, \quad (4.10)$$

$$\mathbf{G}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \boldsymbol{\eta}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\eta}(t)=0}, \quad \mathbf{D}(t) \equiv \left. \frac{\partial \mathbf{h}}{\partial \boldsymbol{\xi}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\xi}(t)=0} \quad (4.11)$$

assuming uncorrelated process noise and measurement noise

$$E\{\boldsymbol{\eta}(t)\boldsymbol{\xi}^T(\tau)\} = 0 \quad (4.12)$$

and defining the covariance matrices of the process and measurement noise as

$$\begin{aligned} E\{\boldsymbol{\eta}(t)\boldsymbol{\eta}^T(\tau)\} &= \mathbf{Q}(t)\delta(t - \tau) \\ E\{\boldsymbol{\xi}(t)\boldsymbol{\xi}^T(\tau)\} &= \mathbf{R}(t)\delta(t - \tau) \end{aligned} \quad (4.13)$$

The steps leading to the expression of the error covariance variation $\mathbf{P}(t) \equiv E\{\underline{\mathbf{x}}(t)\underline{\mathbf{x}}^T(t)\}$ are analogous to those performed for the KF [13]. The EKF filter equations in the continuous-time version are

Estimation equations

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) + \mathbf{K}(t)[\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t)] \quad (4.14)$$

$$\hat{\mathbf{z}}(t) = \mathbf{h}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) \quad (4.15)$$

Covariance

$$\begin{aligned} \dot{\mathbf{P}}(t) = & \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) - \mathbf{P}(t)\mathbf{H}^T(t)[\mathbf{D}(t)\mathbf{R}(t)\mathbf{D}^T(t)]^{-1}\mathbf{H}(t)\mathbf{P}(t) + \\ & + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \end{aligned} \quad (4.16)$$

$$\mathbf{F}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t)}, \quad \mathbf{H}(t) \equiv \left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(t)} \quad (4.17)$$

Gain

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T(t)[\mathbf{D}(t)\mathbf{R}(t)]^{-1} \quad (4.18)$$

where the gain matrix $\mathbf{K}(t)$, as for the KF, is defined to minimize the trace of $\dot{\mathbf{P}}(t)$ [13].

In all cases of practical interest, the measurements are available only at discrete times. This leads to the need for a continuous-discrete EKF, in which the prediction of the system state occurs continuously (i.e., with a high frequency), while the correction (or update) due to the presence of new measurements occurs at discrete times. The derivation of the equations for the continuous-discrete EKF (reported below) follows the same procedure shown for the continuous EKF and is omitted here for the sake of brevity. In what follows the *Joseph's form* for the covariance update is used (see Eq. (4.28)), which is particularly useful to ensure numerical stability and maintaining symmetry and positive-definiteness of the covariance matrix [72].

Model

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\eta}(t), t) \quad (4.19)$$

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\xi}_k) \quad (4.20)$$

Initial condition

$$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 \quad \mathbf{P}_0 = E \left\{ \underline{\mathbf{x}}(t_0) \underline{\mathbf{x}}^T(t_0) \right\} \quad (4.21)$$

Estimation and error propagation

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t) \quad (4.22)$$

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \quad (4.23)$$

$$\mathbf{F}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t)}, \quad \mathbf{G}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \boldsymbol{\eta}} \right|_{\hat{\mathbf{x}}(t), \mathbf{u}(t), \boldsymbol{\eta}(t)=0} \quad (4.24)$$

Gain

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T (\hat{\mathbf{x}}_k^-) \left[\underbrace{\mathbf{H}_k (\hat{\mathbf{x}}_k^-) \mathbf{P}_k^- \mathbf{H}_k^T (\hat{\mathbf{x}}_k^-) + \mathbf{D}_k \mathbf{R}_k \mathbf{D}_k^T}_{\mathbf{S}_k} \right]^{-1} \quad (4.25)$$

$$\mathbf{H}(t) \equiv \left. \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_k^-, \mathbf{u}_k}, \quad \mathbf{D}(t) \equiv \left. \frac{\partial \mathbf{h}}{\partial \boldsymbol{\xi}} \right|_{\hat{\mathbf{x}}_k^-, \mathbf{u}_k, \boldsymbol{\xi}_k=0} \quad (4.26)$$

Update

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k \left[\underbrace{\mathbf{z}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-, \mathbf{u}_k)}_{\Delta_{y_k}} \right] \quad (4.27)$$

$$\mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{D}_k \mathbf{R}_k \mathbf{D}_k^T \mathbf{K}_k^T \quad (4.28)$$

4.2.1 Iterated EKF

In this work, the IEKF [13] is used to enhance the state estimation accuracy in systems exhibiting strongly nonlinear dynamics. The IEKF iteratively refines the estimates by recalculating $\hat{\mathbf{x}}_k^+$, $\mathbf{P}_k^+$, and $\mathbf{K}_k$ at each step, with each iteration linearizing about the most recent estimate. Iterations proceed according to the sequence defined by Eq.s (4.29)-(4.31), starting with the initial prediction $\hat{\mathbf{x}}_{k_0}^+ = \hat{\mathbf{x}}_{k_0}^-$. These iterations are continued until the estimates converge to a steady state, so when the error between $\hat{\mathbf{x}}_{k_i}^- - \hat{\mathbf{x}}_{k_i}^+$ drops below a certain threshold, indicating optimal improvement.

$$\hat{\mathbf{x}}_{k_{i+1}}^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_{k_i} \left[\mathbf{z}_k - \mathbf{h}(\hat{\mathbf{x}}_k^+) - \mathbf{H}_k(\hat{\mathbf{x}}_{k_i}^+)(\hat{\mathbf{x}}_{k_i}^- - \hat{\mathbf{x}}_{k_i}^+) \right] \quad (4.29)$$

$$\mathbf{K}_{k_i} = \mathbf{P}_k^- \mathbf{H}_k^\top(\hat{\mathbf{x}}_{k_i}^+) \left[\mathbf{H}_{k_i}(\hat{\mathbf{x}}_{k_i}^+) \mathbf{P}_k^- \mathbf{H}_k^\top(\hat{\mathbf{x}}_k^+) + \mathbf{R}_k \right]^{-1} \quad (4.30)$$

$$\mathbf{P}_{k_{i+1}}^+ = \left[\mathbf{I} - \mathbf{K}_{k_i} \mathbf{H}_k(\hat{\mathbf{x}}_{k_i}^+) \right] \mathbf{P}_k^- \quad (4.31)$$

4.2.2 Consider EKF

In the preceding section (4.2), the EKF was detailed. The *Consider* EKF (CEKF) formulation extends this framework to the case of systems characterized by a significant number of auxiliary states, more often badly observable and hard to estimate. These states, often introduced to account for systematic sensor errors [2], are termed as *parameters*. While not directly estimated in the CEKF, these entities play a crucial role by influencing the accuracy of the state estimates through their uncertainties. This approach enables a more robust handling of the dynamic model's uncertainties [2].

Consider to partition the states in "estimated" states $\mathbf{x}_c$ and *parameters* $\mathbf{p}$ as follow

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_c \\ \mathbf{p} \end{bmatrix}^T \quad (4.32)$$

the covariance matrix $\mathbf{P}$ can be partitioned as

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{x_c x_c} & \mathbf{P}_{x_c p} \\ \mathbf{P}_{p x_c} & \mathbf{P}_{pp} \end{bmatrix} \quad (4.33)$$

The state estimation covariance is

$$\mathbf{P}_{x_c x_c} = E \left[(\mathbf{x}_c - \hat{\mathbf{x}}_c)(\mathbf{x}_c - \hat{\mathbf{x}}_c)^T \right] \quad (4.34)$$

and the parameter uncertainty is $(\mathbf{p} - \hat{\mathbf{p}})$, with covariance

$$\mathbf{P}_{pp} = E \left[(\mathbf{p} - \hat{\mathbf{p}})(\mathbf{p} - \hat{\mathbf{p}})^T \right] \quad (4.35)$$

and cross-covariance

$$\mathbf{P}_{x_c p} = E \left[(\mathbf{x}_c - \hat{\mathbf{x}}_c)(\mathbf{p} - \hat{\mathbf{p}})^T \right] \quad (4.36)$$

where by definition, $\mathbf{P}_{p x_c} = \mathbf{P}_{x_c p}^T$. Also partitioning the measurement matrix as

$H = [H_{x_c} \ H_p]$ is possible to define a partitioned Kalman gain

$$K = \begin{bmatrix} K_{x_c} \\ K_p \end{bmatrix} = \begin{bmatrix} P_{x_c x_c} H_{x_c}^T + P_{x_c p} H_p^T \\ P_{p x_c} H_{x_c}^T + P_{pp} H_p^T \end{bmatrix} S^{-1} \quad (4.37)$$

where as defined in Eq.(4.25)

$$S = H P^- H^T + D R D^T; \quad (4.38)$$

$$S = H_{x_c} P_{x_c x_c}^- H_{x_c}^T + H_{x_c} P_{x_c p}^- H_p^T + H_p P_{p x_c}^- H_{x_c}^T + H_p P_{pp}^- H_p^T + D R D^T \quad (4.39)$$

In the work by Zanetti et al. [73], it is shown that when calculating P^+ from Eq. (4.28) using Eq. (4.37), the cross-covariance between x_c and p does not include K_p . Therefore, the cross-covariance is independent of the choice of K_p , allowing for the simplification $K_p = 0$. Consequently, one can apply the simplification $K_p = 0$ within the EKF framework obtaining the *Consider* filter variant. As a result, the a posteriori state update is expressed as

$$\begin{bmatrix} x_c^+ \\ p^+ \end{bmatrix} = \begin{bmatrix} x_c^- \\ p^- \end{bmatrix} + \begin{bmatrix} K_c \\ 0 \end{bmatrix} (z - h(\hat{x}^-, u)). \quad (4.40)$$

In the CEKF formulation, the update of the *parameters* $p^+ = p^-$ and the posterior *parameters* covariance $P_{pp}^+ = P_{pp}^-$ thus, both the estimated value and covariance of the parameters remains constant along the whole estimation process.

4.2.3 Multiplicative Extended Kalman Filter (MEKF)

Attitude estimation for aerospace applications implies determining the orientation of a vehicle, such as a satellite or an aircraft, with respect to an inertial frame or a relative frame, and, in the context of this current study, the objective is to estimate the vehicle attitude w.r.t the NED frame $\mathcal{F}_E$. Consequently, after a general development of the kinematic quaternion error formulation, the specific attitude determination problem is addressed in Sec. 4.2.4.

The primary goal of attitude estimation is to accurately infer the orientation of a vehicle, and a robust method to achieve this is by leveraging the quaternion representation. Quaternions are composed of one real part and three imaginary parts, bound by a unit norm condition, confining their effective representation to $\mathbb{R}^3$ despite existing in $\mathbb{R}^4$.

When using quaternion to represent the attitude of a vehicle, it is mandatory

to explicitly state what convention is used. Indeed, within the realm of quaternion conventions, the Hamilton [74] convention and the JPL [75] ones are commonly adopted, and the former is considered in this work. The quaternion is arranged such that the scalar component e_0 precedes the vector component $\mathbf{e} = [e_1, e_2, e_3]^T$, that is,

$$\mathbf{q} = [e_0, e_1, e_2, e_3]^T \quad (4.41)$$

The operator $\otimes$ stands for the Hamilton quaternion product. So that given two generic quaternions $\mathbf{p}$ and $\mathbf{q}$ results

$$\mathbf{p} \otimes \mathbf{q} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2 \\ p_0 q_2 - p_1 q_3 + p_2 q_0 + p_3 q_1 \\ p_0 q_3 + p_1 q_2 - p_2 q_1 + p_3 q_0 \end{bmatrix} \quad (4.42)$$

The Additive Extended Kalman Filter (AEKF) [14] for attitude determination, requires the normalization of the quaternion following each update. However, the additive operations between quaternions may compromise the quaternions' normalization. Moreover, this technique generally suffers from a lack of physical interpretability since the addition of quaternions does not equate to a valid rotational operation [14]. To address this issue, the MEKF was developed [13]. Instead of estimating the quaternion at the current time, the MEKF estimates a small correction relative to the current attitude estimate. This quaternion correction, denoted as $\delta \mathbf{q}$, encapsulates the rotational discrepancy between the true $\mathbf{q}$ and estimated $\hat{\mathbf{q}}$ attitudes.

$$\delta \mathbf{q} = \hat{\mathbf{q}}^{-1} \otimes \mathbf{q} \quad (4.43)$$

Under the assumption of small errors, this error quaternion can be represented using Euler angles, $\delta \boldsymbol{\alpha} \approx 2[e_1, e_2, e_3]^T$, or any tri parametric representation.

$$\delta \mathbf{q} \approx \begin{bmatrix} 1 \\ \frac{\delta \boldsymbol{\alpha}}{2} \end{bmatrix} \quad (4.44)$$

The MEKF approach ensures that the state estimate, which represents the error rotation $\delta \boldsymbol{\alpha}$, is three-dimensional and unconstrained. Starting from the definition of the error in Hamilton's convention we start from Eq.(4.43). Taking the time derivative of $\delta \mathbf{q}$

$$\delta \dot{\mathbf{q}} = \dot{\hat{\mathbf{q}}}^{-1} \otimes \mathbf{q} + \hat{\mathbf{q}}^{-1} \otimes \dot{\mathbf{q}} \quad (4.45)$$

given that

$$\begin{bmatrix} \dot{\hat{e}}_0 \\ \dot{\hat{e}} \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} 0 & -\hat{\omega}^T \\ \hat{\omega} & -[\hat{\omega} \times] \end{bmatrix}}_{\Omega(\hat{\omega})} \begin{bmatrix} \hat{e}_0 \\ \hat{e} \end{bmatrix} \quad (4.46)$$

and, in compact form

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2} \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} \otimes \hat{\mathbf{q}} \quad (4.47)$$

substituting Eq. (4.47) in $\dot{\hat{\mathbf{q}}}^{-1} \otimes \hat{\mathbf{q}} + \hat{\mathbf{q}}^{-1} \otimes \dot{\hat{\mathbf{q}}}$ we obtain

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2} \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} \otimes \hat{\mathbf{q}}^{-1} \quad (4.48)$$

so, using Eq. (4.48) in Eq.(4.43), we have

$$\delta \dot{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} \hat{\mathbf{q}}^{-1} \otimes \mathbf{q} + \hat{\mathbf{q}}^{-1} \otimes \frac{1}{2} \begin{bmatrix} 0 \\ \omega \end{bmatrix} \mathbf{q} \quad (4.49)$$

By expressing Eq. (4.49) in term of quaternion errors

$$\delta \dot{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} \otimes \delta \mathbf{q} - \delta \mathbf{q} \otimes \frac{1}{2} \begin{bmatrix} 0 \\ \omega \end{bmatrix} \quad (4.50)$$

and defining the error between the estimated and true angular rate as $\delta \omega = \omega - \hat{\omega}$, Eq. (4.50) becomes

$$\delta \dot{\mathbf{q}} = \left[-\delta \mathbf{q} \otimes \frac{1}{2} \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} \otimes \delta \mathbf{q} \right] + \delta \mathbf{q} \otimes \frac{1}{2} \begin{bmatrix} 0 \\ \delta \omega \end{bmatrix} \quad (4.51)$$

Given the following equalities

$$\delta \mathbf{q} \otimes \begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\hat{\omega}^T \\ \hat{\omega} & [\hat{\omega} \times] \end{bmatrix}}_{\Gamma(\hat{\omega})} \delta \mathbf{q} \quad (4.52)$$

$$\begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix} \otimes \delta \mathbf{q} = \underbrace{\begin{bmatrix} 0 & -\hat{\omega}^T \\ \hat{\omega} & -[\hat{\omega} \times] \end{bmatrix}}_{\Omega(\hat{\omega})} \delta \mathbf{q} \quad (4.53)$$

Eq. (4.51) is written as

$$\delta\dot{\mathbf{q}} = \frac{1}{2} \left[\boldsymbol{\Omega}(\hat{\boldsymbol{\omega}}) + \boldsymbol{\Gamma}(\hat{\boldsymbol{\omega}}) \right] \delta\mathbf{q} + \frac{1}{2} \delta\mathbf{q} \otimes \begin{bmatrix} 0 \\ \delta\boldsymbol{\omega} \end{bmatrix} \quad (4.54)$$

$$\delta\dot{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & -2[\hat{\boldsymbol{\omega}} \times] \end{bmatrix} \delta\mathbf{q} + \frac{1}{2} \delta\mathbf{q} \otimes \begin{bmatrix} 0 \\ \delta\boldsymbol{\omega} \end{bmatrix} \quad (4.55)$$

where, exploiting Eq. (4.44) and being $\delta\mathbf{e} = \frac{\delta\boldsymbol{\alpha}}{2}$, the kinematic relationship follow

$$\delta\dot{\mathbf{q}} = - \begin{bmatrix} 0 \\ [\hat{\boldsymbol{\omega}} \times] \delta\mathbf{e} \end{bmatrix} + \frac{1}{2} \delta\mathbf{q} \otimes \begin{bmatrix} 0 \\ \delta\boldsymbol{\omega} \end{bmatrix} \quad (4.56)$$

The first order approximation of Eq. (4.56) is

$$\delta\dot{e}_0 = 0 \quad (4.57)$$

$$\delta\dot{\mathbf{e}} = -[\hat{\boldsymbol{\omega}} \times] \delta\mathbf{e} + \frac{1}{2} \delta\boldsymbol{\omega} \quad (4.58)$$

where it is assumed that the estimated quaternion is close to the true one, and, therefore $e_0 = 1$. The error dynamics just derived is a general result that can be rewritten as

$$\delta\dot{\boldsymbol{\alpha}} = [-\hat{\boldsymbol{\omega}} \times] \delta\boldsymbol{\alpha} + \delta\boldsymbol{\omega} \quad (4.59)$$

that shows that the error evolves in response to the estimated angular rate and from the error between the estimated and the true angular rate $\delta\boldsymbol{\omega}$. Implementing the MEKF satisfies the geometrical constraints of quaternion algebra, inherently preserving the unit norm.

4.2.4 LV attitude estimation algorithm

As shown in *Crassidis et.al* [13], the blending of position and IMU data in a EKF enables the estimation of vehicle attitude without the need for direct measurement. In the following development, Eq.(4.59) is made explicit assuming $\boldsymbol{\omega} = \boldsymbol{\omega}_{BE}$ and expressing it as a function of the rate gyro measurement in order to propagate the error dynamics from the angular rate measurement input. The EKF equations for attitude estimation are reported with some details for the sake of clarity. Consider

the gyro and accelerometer measurements modeled as

$$\tilde{\mathbf{a}}_{\text{IMU}} = (\mathbf{I} + \mathbf{\Lambda}_a)\mathbf{a}_{\text{IMU}} + \boldsymbol{\beta}_a + \boldsymbol{\eta}_a \quad (4.60)$$

$$\dot{\boldsymbol{\beta}}_a = \boldsymbol{\eta}_{b_a} \quad (4.61)$$

$$\tilde{\boldsymbol{\omega}}_{BI} = (\mathbf{I} + \mathbf{\Lambda}_\omega)\boldsymbol{\omega}_{BI} + \boldsymbol{\beta}_\omega + \boldsymbol{\eta}_\omega \quad (4.62)$$

$$\dot{\boldsymbol{\beta}}_\omega = \boldsymbol{\eta}_{b_\omega} \quad (4.63)$$

with $\boldsymbol{\eta}_a, \boldsymbol{\eta}_\omega, \boldsymbol{\eta}_{b_a}$ and $\boldsymbol{\eta}_{b_\omega}$ assumed zero-mean Gaussian white-noise processes. Now, to calculate $\delta\boldsymbol{\omega} = \boldsymbol{\omega}_{BE} - \hat{\boldsymbol{\omega}}_{BE}$, we express $\boldsymbol{\omega}_{BE}$ and $\hat{\boldsymbol{\omega}}_{BE}$ as

$$\boldsymbol{\omega}_{BE} = \boldsymbol{\omega}_{BI} - \boldsymbol{\omega}_{EI} \quad (4.64)$$

$$\hat{\boldsymbol{\omega}}_{BE} = \hat{\boldsymbol{\omega}}_{BI} - \hat{\boldsymbol{\omega}}_{EI} \quad (4.65)$$

By inverting Eq. (4.63) to obtain $\boldsymbol{\omega}_{BI}$, and assuming that $(\mathbf{I}_{3 \times 3} + \mathbf{\Lambda}_\omega)^{-1} \approx (\mathbf{I}_{3 \times 3} - \mathbf{\Lambda}_\omega)$, Eqs. (4.64)–(4.65) become, respectively

$$\boldsymbol{\omega}_{BE} = [(\mathbf{I}_{3 \times 3} - \mathbf{\Lambda}_\omega)(\tilde{\boldsymbol{\omega}}_{BI} - \boldsymbol{\beta}_\omega - \boldsymbol{\eta}_\omega)] - (\mathbf{I}_{3 \times 3} - [\delta\boldsymbol{\alpha} \times])\hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}}^-)\boldsymbol{\omega}_{EI[E]} \quad (4.66)$$

$$\hat{\boldsymbol{\omega}}_{BE} = [(\mathbf{I}_{3 \times 3} - \hat{\mathbf{\Lambda}}_\omega)(\tilde{\boldsymbol{\omega}}_{BI} - \hat{\boldsymbol{\beta}}_\omega)] - \hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}})\hat{\boldsymbol{\omega}}_{EI[E]} \quad (4.67)$$

with

$$\mathcal{R}_{BE}(\mathbf{q}) = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_0e_1 + e_2e_3) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \quad (4.68)$$

Since $\mathcal{R}_{BE}(\mathbf{q}) = \mathcal{R}_{BE}(\delta\mathbf{q})\hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}}^-)$, with $\mathcal{R}_{BE}(\delta\mathbf{q}) \approx \mathbf{I}_{3 \times 3} - [\delta\boldsymbol{\alpha} \times]$ [13]. Using Eqs. (4.66)–(4.67) and defining $\Delta_{\mathbf{\Lambda}_\omega} = \mathbf{\Lambda}_\omega - \hat{\mathbf{\Lambda}}_\omega$ and $\Delta_{\boldsymbol{\beta}_\omega} = \boldsymbol{\beta}_\omega - \hat{\boldsymbol{\beta}}_\omega$ we obtain

$$\begin{aligned} \delta\boldsymbol{\omega} = & -\Delta_{\mathbf{\Lambda}_\omega}(\tilde{\boldsymbol{\omega}}_{BI} - \hat{\boldsymbol{\beta}}_\omega) - (\mathbf{I}_{3 \times 3} - \hat{\mathbf{\Lambda}}_\omega)\Delta_{\boldsymbol{\beta}_\omega} - (\mathbf{I}_{3 \times 3} - \hat{\mathbf{\Lambda}}_\omega)\boldsymbol{\eta}_\omega \\ & - \hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}})(\boldsymbol{\omega}_{EI[E]} - \hat{\boldsymbol{\omega}}_{EI[E]}) + [\delta\boldsymbol{\alpha} \times]\hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}}^{-1})\boldsymbol{\omega}_{EI[E]} \end{aligned} \quad (4.69)$$

$$\begin{aligned}
\delta\omega = & - \left[\underbrace{\begin{bmatrix} \tilde{\omega}_{BI,x} & 0 & 0 \\ 0 & \tilde{\omega}_{BI,y} & 0 \\ 0 & 0 & \tilde{\omega}_{BI,z} \end{bmatrix}}_{\tilde{\Omega}_\omega} - \underbrace{\begin{bmatrix} \hat{\beta}_{\omega,x} & 0 & 0 \\ 0 & \hat{\beta}_{\omega,y} & 0 \\ 0 & 0 & \hat{\beta}_{\omega,z} \end{bmatrix}}_{\hat{B}_\omega} \right] \Delta_{\Lambda_\omega} + \\
& + (\mathbf{I}_{3 \times 3} - \hat{\Lambda}_\omega) \Delta_{\beta_\omega} + (\mathbf{I}_{3 \times 3} - \hat{\Lambda}_\omega) \eta_\omega \Big] - \hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}})(\omega_{EI[E]} - \hat{\omega}_{EI[E]}) + \\
& + [\delta\alpha \times] \hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}}^-) \omega_{EI[E]}
\end{aligned} \tag{4.70}$$

Considering Eq. (4.59) and substituting the estimated angular rate $\hat{\omega}_{BE}$ for this specific problem, follow

$$\delta\dot{\alpha} = -[\hat{\omega}_{BI} \times] \delta\alpha + \hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}}^-) [\hat{\omega}_{EI} \times] \delta\alpha + \delta\omega \tag{4.71}$$

and substituting $\delta\omega$ in Eq. (4.59) we obtain the final explicit form of the attitude error dynamic as follow

$$\begin{aligned}
\delta\dot{\alpha} = & - [\hat{\omega}_{BI} \times] \delta\alpha - \left[(\tilde{\Omega}_\omega - \hat{B}_\omega) \Delta_{\Lambda_\omega} + (\mathbf{I}_{3 \times 3} - \hat{\Lambda}_\omega) \Delta_{\beta_\omega} + (\mathbf{I}_{3 \times 3} - \hat{\Lambda}_\omega) \eta_\omega \right] + \\
& - \hat{\mathcal{R}}_{BE}(\hat{\mathbf{q}})(\omega_{EI[E]} - \hat{\omega}_{EI[E]})
\end{aligned} \tag{4.72}$$

In the EKF formulation, during the propagation steps the kinematic of quaternion [76] is propagated, so the dimension of the system state remains the same, whereas the EKF covariance is reduced by one state because the quaternion error dynamic $\delta\dot{\alpha}$ is propagated instead of $\dot{\mathbf{q}}$. The partial derivatives of the process and measurement functions ($\mathbf{F}$ and $\mathbf{H}$ matrices) are computed with respect to $\delta\alpha$ and evaluated at the optimal estimate, specifically when $\delta\alpha = 0$, therefore all trasformation matrices $\mathcal{R}$ that implies rotations in $\mathcal{F}_B$ are expressed as functions of $\delta\alpha$ as $\mathcal{R}(\mathbf{q}) = (\mathbf{I}_{3 \times 3} - [\delta\alpha \times]) \hat{\mathcal{R}}(\hat{\mathbf{q}}^-)$. As for the update step, the quaternion states are updated differently, as follows

$$\hat{\mathbf{q}}^+ = \hat{\mathbf{q}}^- \otimes \begin{bmatrix} 1 \\ \frac{\delta\hat{\alpha}^+}{2} \end{bmatrix} \tag{4.73}$$

with $\delta\hat{\alpha}^+ = \mathbf{K}_\alpha(\mathbf{z} - \mathbf{h}(\hat{\mathbf{x}}^-))$, where $\mathbf{K}_\alpha$ is the Kalman gain corresponding to the rows of $\delta\alpha$.

4.3 UKF

The Unscented Kalman Filter (UKF) [12] emerges as a robust alternative to the EKF for state estimation in systems characterized by significant nonlinearities. The EKF, despite its effectiveness in linear and nonlinear contexts, encounters issues with significant nonlinear dynamics. Its reliance on linearization through Jacobian matrices can introduce errors that jeopardize the accuracy and stability of the estimation process. To address these issues, the UKF offers a method that sidesteps the need for linearization and, by extension, the explicit computation of derivatives. This is achieved while maintaining performance levels comparable to those of a second-order EKF [77], albeit with a generally higher computational load.

Central to the UKF methodology is the Unscented Transformation (UT), a technique designed to capture the statistical properties of a random variable undergoing a nonlinear transformation. Unlike traditional linearization methods, the UT employs a set of deterministically chosen sample points, known as sigma points, to approximate the mean and covariance of the state distribution. These points are selected and weighted in such a manner that they provide a representation of the original distribution moments. When propagated through the nonlinear system dynamics, they allow for the evaluation of new state mean and covariance without resorting to derivative calculations. Figure 4.1 illustrates the concept of the Unscented Transformation, showing how a discrete set of points can effectively

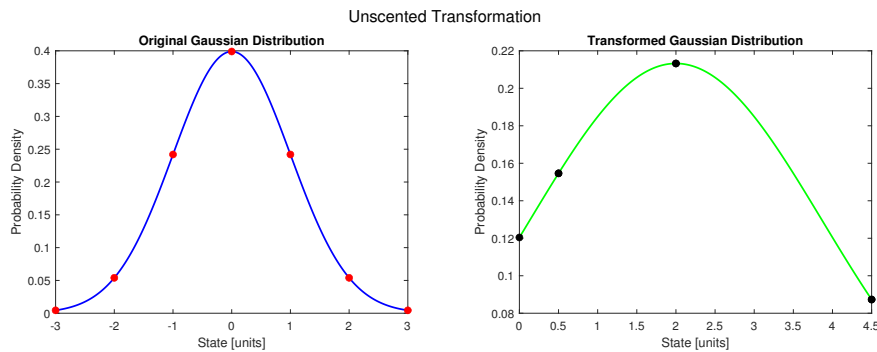


Figure 4.1. Principle of the Unscented transformation Gaussian through the function $\frac{x^2}{2}$

capture and propagate the uncertainty through nonlinear mappings (quadratic in the example).

Consider a discrete-time system with non-additive process noise, and additive

measurement noise in the form

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \boldsymbol{\eta}_{k-1}, \mathbf{u}_{k-1}, t_{k-1}) \quad (4.74)$$

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, t_k) + \boldsymbol{\xi}_k \quad (4.75)$$

where t_k is the discrete time step, $\mathbf{x}_k \in \mathbb{R}^{n_x}$ is the internal state vector, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ and $\mathbf{y}_k \in \mathbb{R}^{n_y}$ are, respectively, the vectors of measured input and output. Also, $\boldsymbol{\eta}_k \in \mathbb{R}^{n_q}$ and $\boldsymbol{\xi}_k \in \mathbb{R}^{n_\xi}$ are process and measurement noises, respectively, assumed uncorrelated, with zero means and covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$. Let $\hat{\mathbf{x}}_k^+$ and $\mathbf{P}_{x_k x_k}^+$ be the estimated mean and covariance of the state at time t_k , given the measurements up to and including time t_k , while $\hat{\mathbf{x}}_k^-$, $\mathbf{P}_{x_k x_k}^-$ are the same quantities, given the measurements up to t_{k-1} .

An augmented state vector $\mathbf{x}^a = [\mathbf{x}, \boldsymbol{\eta}]$, is introduced to deal with non-additive process noise in Eq. (4.74) [12]. The estimated mean and covariance of the augmented state at time t_k are given by

$$\hat{\mathbf{x}}_k^{a\pm} = \begin{bmatrix} \hat{\mathbf{x}}_k^\pm & \mathbf{0} \end{bmatrix}^T \quad (4.76)$$

$$\mathbf{P}_{x_k x_k}^{a\pm} = \begin{bmatrix} \mathbf{P}_{x_k x_k}^\pm & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_k \end{bmatrix} \quad (4.77)$$

The main features of the UKF algorithm are recalled as follows. The sigma points $\boldsymbol{\chi}_k^{a+}$ corresponding to the augmented state probability distribution of mean $\hat{\mathbf{x}}_k^{a+}$ and covariance $\mathbf{P}_k^{a+}$ are first calculated, as

$$\boldsymbol{\chi}_{0,k}^{a+} = \hat{\mathbf{x}}_k^{a+} \quad (4.78)$$

$$\boldsymbol{\chi}_{i,k}^{a+} = \hat{\mathbf{x}}_k^{a+} + \left(\sqrt{(L_{UKF} + \lambda_{UKF}) \mathbf{P}_{x_k x_k}^{a+}} \right)_i \quad (4.79)$$

$$\boldsymbol{\chi}_{i+L_{UKF},k}^{a+} = \hat{\mathbf{x}}_k^{a+} - \left(\sqrt{(L_{UKF} + \lambda_{UKF}) \mathbf{P}_{x_k x_k}^{a+}} \right)_i \quad (4.80)$$

where the notation $(\mathbf{A})_i$ is used to identify the i -th column of matrix $\mathbf{A}$, L_{UKF} is the size of the augmented state vector, and $\lambda_{UKF} = \alpha_{UKF}^2 (L_{UKF} + c) - L_{UKF}$ is a scaling parameter. The spread of sigma points around the mean value is specified by the constant parameter α_{UKF} , which is typically set to a small positive number (e.g. $10^{-4} \leq \alpha_{UKF} \leq 1$), while c is used to incorporate higher-order statistic moments of the probability distribution [12]. In the current application, the values $\alpha_{UKF} = 10^{-2}$ and $c = 3 - L_{UKF}$ are used. Propagation of sigma points through the system

equations gives

$$\chi_{i,k}^- = f(\chi_{i,k-1}^{a+}, \mathbf{u}_{k-1}, k) \quad (4.81)$$

and, for the corresponding measurements

$$\mathbf{y}_{i,k}^- = h(\chi_{i,k}^-, \mathbf{u}_k, k) \quad (4.82)$$

The statistics (i.e., the mean $\hat{\mathbf{x}}_k^-$ and covariance $P_{x_k x_k}^-$) of the propagated states are given as linear combinations of the transformed sigma points, as

$$\hat{\mathbf{x}}_k^- = \sum_{i=0}^{2L_{UKF}} W_i^{(m)} \chi_{i,k}^- \quad (4.83)$$

$$P_{x_k x_k}^- = \sum_{i=0}^{2L_{UKF}} W_i^{(c)} \{\chi_{i,k}^- - \hat{\mathbf{x}}_k^-\} \{\chi_{i,k}^- - \hat{\mathbf{x}}_k^-\}^T \quad (4.84)$$

where the weights are expressed as

$$W_0^{(m)} = \lambda_{UKF} / (L_{UKF} + \lambda_{UKF}) \quad (4.85)$$

$$W_0^{(c)} = \lambda_{UKF} / (L_{UKF} + \lambda_{UKF}) + (1 - \alpha_{UKF}^2 + \beta_{UKF}) \quad (4.86)$$

$$W_i^{(m)} = W_i^c = (L_{UKF} + \lambda_{UKF}) / 2 \quad \forall i = 1, \dots, 2L_{UKF} \quad (4.87)$$

and the value of the constant parameter β_{UKF} is set to 2 for a Gaussian distribution of the true states [12].

The corrected estimates are written as

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathcal{K}(\mathbf{z}_k - \hat{\mathbf{z}}_k^-) \quad (4.88)$$

$$P_{x_k x_k}^+ = P_{x_k x_k}^- - \mathcal{K} P_{z_k z_k} \mathcal{K}^T \quad (4.89)$$

where $\hat{\mathbf{z}}_k^-$ is the mean measurement at time t_k , evaluated as

$$\hat{\mathbf{z}}_k^- = \sum_{i=0}^{2L_{UKF}} W_i^{(m)} \mathbf{y}_{i,k}^- \quad (4.90)$$

and $\mathcal{K}$ is the Kalman gain

$$\mathcal{K} = P_{x_k z_k} P_{z_k z_k}^{-1} \quad (4.91)$$

that depends on the cross-correlation matrix $P_{x_k z_k}$ and the innovation covariance

matrix $\mathbf{P}_{z_k z_k}$, computed as

$$\mathbf{P}_{z_k z_k} = \sum_{i=0}^{2L_{UKF}} W_i^{(c)} \{\mathbf{y}_{i,k}^- - \hat{\mathbf{z}}_k^-\} \{\mathbf{y}_{i,k}^- - \hat{\mathbf{z}}_k^-\}^T + \mathbf{R}_k \quad (4.92)$$

$$\mathbf{P}_{x_k z_k} = \sum_{i=0}^{2L_{UKF}} W_i^{(c)} \{\mathbf{x}_{i,k}^- - \hat{\mathbf{x}}_k^-\} \{\mathbf{y}_{i,k}^- - \hat{\mathbf{z}}_k^-\}^T \quad (4.93)$$

4.4 Fraser-Potter smoother

The Fraser-Potter Smoother, also known as the Fixed-Interval Smoother [33], is an extension of the KF for smoothing data over a fixed interval. It is designed to provide optimal estimates of the state vector of a dynamic system over a past time interval, given observations up to the present time. The formulation of the Fraser-Potter Smoother involves both forward and backward differential equations to propagate state estimates and covariances. The general idea is to first run a forward pass using the KF to obtain estimates up to the current time. Then, a backward pass is conducted to refine these estimates based on future observations within the interval.

Model

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\eta}(t), t) \quad (4.94)$$

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}, \mathbf{u}_k, \boldsymbol{\xi}_k) \quad (4.95)$$

Initial condition (forward)

$$\hat{\mathbf{x}}_{f_k}(t_0) = \hat{\mathbf{x}}_0 \quad \mathbf{P}_0 = E \left\{ \underline{\mathbf{x}}(t_0) \underline{\mathbf{x}}^T(t_0) \right\} \quad (4.96)$$

Forward filter

$$\dot{\hat{\mathbf{x}}}_f(t) = \mathbf{f}(\hat{\mathbf{x}}_f(t), \mathbf{u}(t), t) \quad (4.97)$$

$$\dot{\mathbf{P}}_f(t) = \mathbf{F}(t) \mathbf{P}_f(t) + \mathbf{P}_f(t) \mathbf{F}^T(t) + \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \quad (4.98)$$

$$\mathbf{F}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}_{f_k}} \right|_{\hat{\mathbf{x}}_{f_k}(t), \mathbf{u}(t)}, \quad \mathbf{G}(t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \boldsymbol{\eta}} \right|_{\hat{\mathbf{x}}_{f_k}(t), \mathbf{u}(t), \boldsymbol{\eta}(t)=0} \quad (4.99)$$

Initial condition (backward)

$$\hat{\mathbf{x}}_b(t_0) = \hat{\mathbf{x}}_f(t_f) \quad \mathbf{P}_0 = E \left\{ \underline{\mathbf{x}}_f(t_f) \underline{\mathbf{x}}_f^T(t_f) \right\} \quad (4.100)$$

Backward filter

$$\dot{\hat{\mathbf{x}}}_b(t) = \mathbf{f}(\hat{\mathbf{x}}_b(t), \mathbf{u}(t), t) \quad (4.101)$$

$$\dot{\mathbf{P}}_b(t) = \mathbf{F}(t) \mathbf{P}_b(t) + \mathbf{P}_b(t) \mathbf{F}^T(t) - \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \quad (4.102)$$

The equations reported above are for the forward and backward passes, whereas for the update, the standard EKF, Eq.s ((4.25)–(4.28)) can be used back in time with a backward pass. This involves reversing the time direction for the error propagation and using the final estimates from the forward pass as the initial conditions for the backward pass. In the end, to obtain a better combined estimate from the two estimates, the Fraser-Potter smoother is used, which consists of a weighted average over the inverse of the covariance

$$\mathbf{P}_k = \left[(\mathbf{P}_{f_k}^+)^{-1} + (\mathbf{P}_{b_k}^-)^{-1} \right]^{-1} \quad (4.103)$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{P}}_k \left[(\mathbf{P}_{f_k}^+)^{-1} \hat{\mathbf{x}}_{f_k}^+ + (\mathbf{P}_{b_k}^-)^{-1} \hat{\mathbf{x}}_{b_k}^- \right] \quad (4.104)$$

where $\hat{\mathbf{x}}_{f_k}^+$, $(\mathbf{P}_{f_k}^+)$ and $\hat{\mathbf{x}}_{b_k}^-$, $(\mathbf{P}_{b_k}^-)$ are, respectively, the state and error covariance estimates of the forward and the backward filter, while $\hat{\mathbf{x}}_k$ and $\mathbf{P}_k$ are the smoothed state estimate and associated error covariance. This procedure allows to significantly improve the accuracy of results due to, in fact, the higher number of data available to refine the quality of the estimation [33].

4.5 Stochastic models

4.5.1 White and Gaussian noise

White Gaussian noise is a fundamental concept in signal processing and control systems. It is characterized by a constant PSD across frequencies, and its amplitude follows a Gaussian distribution at any given time, i.e., $(p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}})$, where (μ) is the mean, and (σ^2) is the variance. This implies that all frequencies are equally represented, making the process uncorrelated in time. The "Gaussian" aspect refers to the probability distribution of the noise amplitude at any given time, which follows a normal distribution with zero mean and a specified variance. This distribution is symmetrical, indicating that positive and negative deviations from the mean are equally likely, and it is fully characterized by its mean and variance.

White Gaussian noise is often used as an idealized model to represent random fluctuations observed in various physical systems, providing a basis for analyzing system responses and performance in the presence of uncertainty. Its simplicity and well-understood properties make it a cornerstone in the study of stochastic processes and filtering techniques, where it is often used to model measurement and/or process noise and other sources of randomness due to the Gaussian noise KF hypothesis [71]. Figure 4.2 reports a sample path of White Gaussian noise, showing its random fluctuations. To further understand its properties, Fig. 4.3 shows the normalized

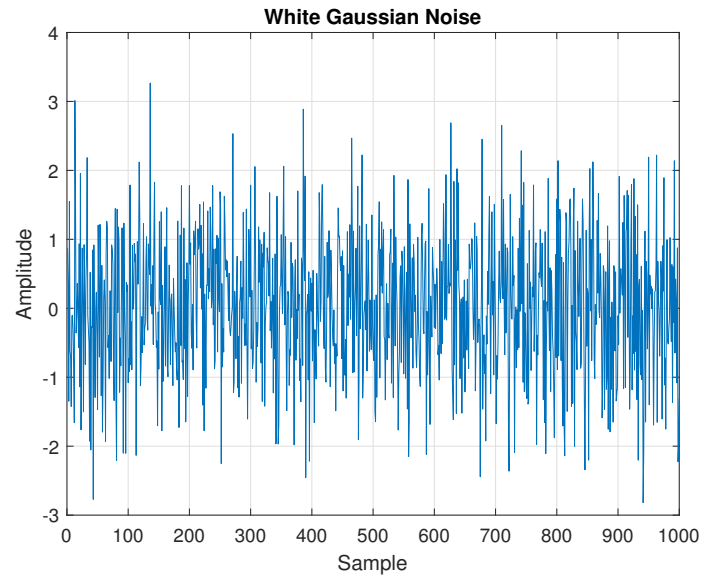


Figure 4.2. White Gaussian noise sample path

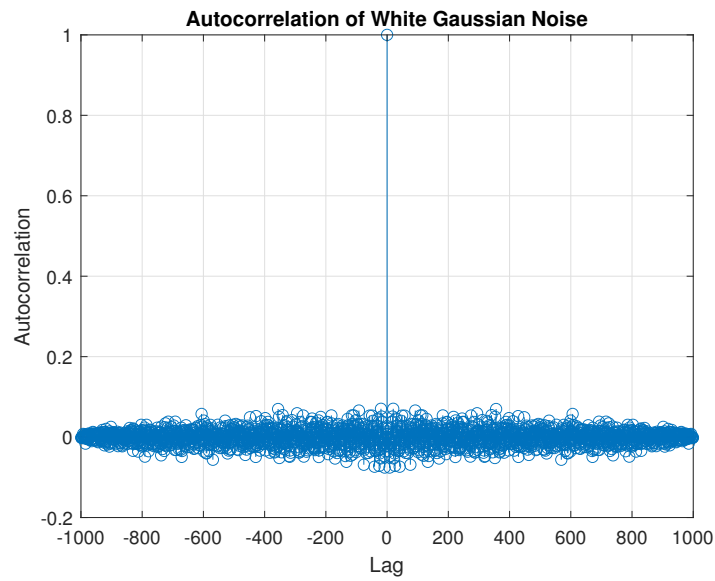


Figure 4.3. White Gaussian normalized autocorrelation function

autocorrelation function, $\rho_{aut}(\tau) = \frac{E[(x(t)-\mu)(x(t+\tau)-\mu)]}{\sigma^2} = \delta(\tau)$, with τ the time lag. This expression of the autocorrelation time indicates that white noise is uncorrelated at different times.

4.5.2 Random walk

The random walk process, fundamental for modeling cumulative randomness, is described by the equation $x_{k+1} = x_k + w_k$, where w_k is a random step taken at each instance, illustrating how the position changes over discrete time steps. Figure 4.4 reports a random walk simulation, visually capturing the essence of cumulative, independent random steps. The autocorrelation function $R(\tau) = \tau\sigma^2$ for $\tau \geq 0$ of a random walk increases linearly with the time lag τ as shown in Fig. 4.5 in its normalized form.

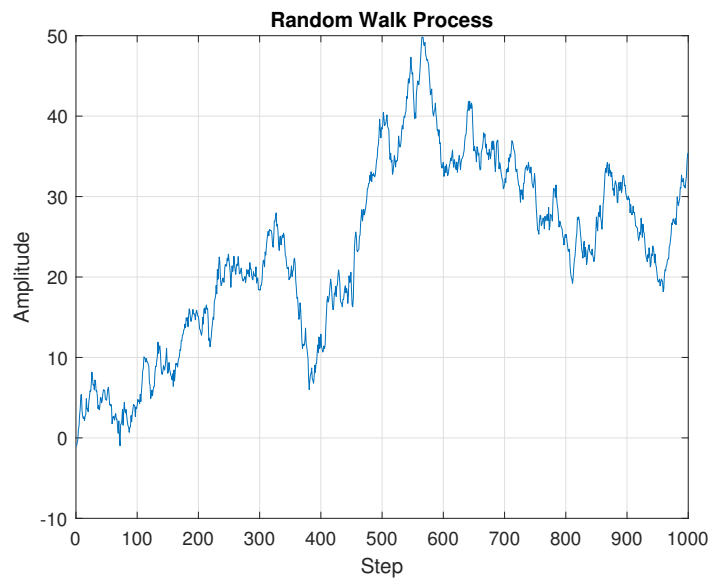


Figure 4.4. Random walk process sample path

4.5.3 Gauss-Markov process

The Gauss-Markov process models temporal correlations within stochastic processes, introducing a memory effect where the future state depends on the current one, governed by $x(t+1) = -\alpha_{GM}x(t) + w(t)$, where $0 < \alpha_{GM} < 1$ and $w(t)$ represents the process noise. Figure 4.6 depicts a Gauss-Markov process, while the autocorrelation function $R(\tau) = \sigma^2 e^{-\alpha_{GM}|\tau|}$ for a Gauss-Markov process is reported in Fig. 4.7 in its normalized form, showing how the correlation between states decays over time.

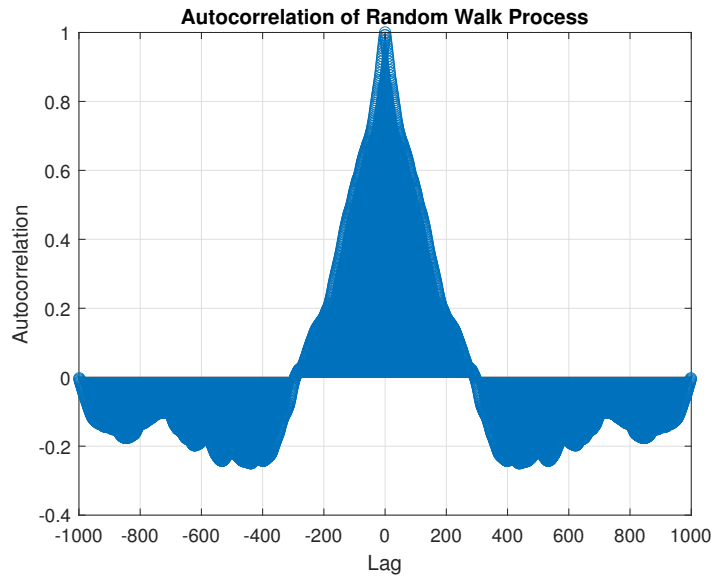


Figure 4.5. Random walk process normalized autocorrelation function

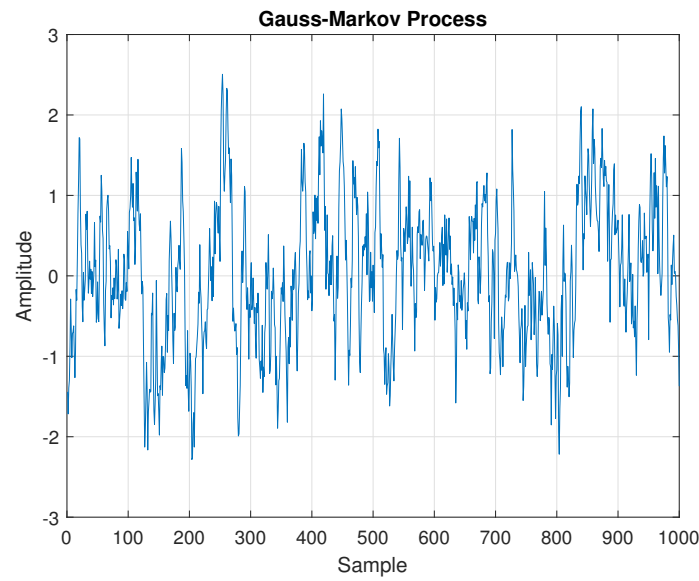


Figure 4.6. Gauss-Markov process sample path

4.6 Signal processing

Preprocessing of measurements is usually carried out in order to improve data quality by reducing noise, eliminating systematic errors, editing bad data, adjusting for latencies, and transforming the data into a format suitable for filter algorithms. In fact, data preprocessing mainly consists in [8]

1. removing duplicate points in the data series that result from differences between

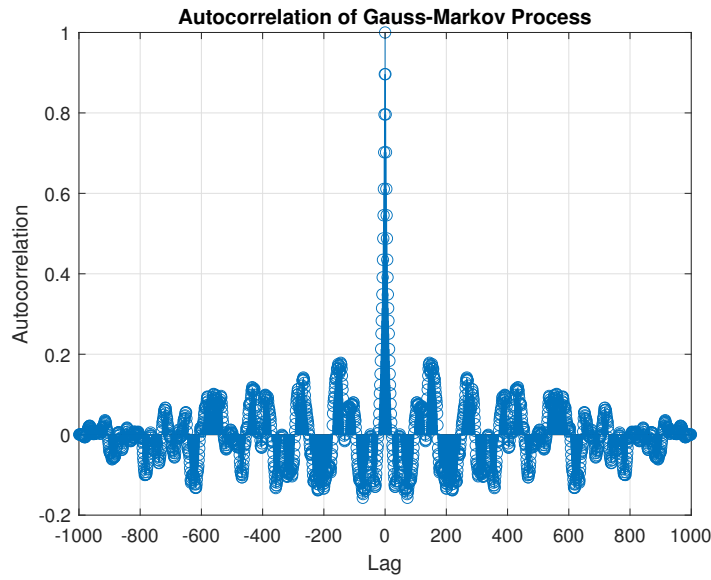


Figure 4.7. Gauss-Markov process normalized autocorrelation function

the sensor data refresh rate and the telemetry sampling rate

2. removing rogue points from the data series
3. sampling the data series at a uniform time step
4. filtering the data series

This section deals with the description of some relevant processes for filtering input data in the area of trajectory reconstruction.

4.6.1 IMU data

Measurements of acceleration components and angular rates are filtered in order to remove the effects of high-frequency structural noise and measurement noise. To this end, a Fourier series expansion and an optimal Wiener filtering method are taken into consideration [78], to separate signal y and noise ν . Given the following expression for a measurement z , where the subscript indicates the i -th time step

$$z(i) = y(i) + \nu(i) \quad (4.105)$$

the strategy for determining the noise sequence $\nu(i)$ is to find a description for the signal in the frequency domain, invert the Fourier transform to get $y(i)$, and find

$\nu(i)$ from Eq. (4.105) in the form

$$\nu(i) = z(i) - y(i) \quad (4.106)$$

The Fourier sine series expansion contains all spectral components that can be computed from the given finite time series, including both signal and noise. It is worth recalling that the Fourier expansion for a coherent signal is fundamentally different from the Fourier series for noise. A coherent signal has Fourier series coefficient amplitudes that rapidly decrease asymptotically to zero as the discrete frequency index k increases. On the other hand, noise is incoherent and in theory has constant power over the frequency range up to the Nyquist frequency, so that the Fourier coefficients have a relatively small and constant magnitude for all frequencies. Lanczos [78] reports that a coherent signal with discontinuities only in the second-order time derivative or higher at the endpoints has Fourier series coefficient amplitudes proportional to k^{-3} , whereas the same amplitudes are proportional to k^{-1} when the discontinuities are in signal amplitude and first-order derivative. Since Fourier series expansion implicitly assumes that the time series under consideration is periodic, it turns out that the majority of time series measured in flight tests present discontinuities in the amplitude and first time derivative at the endpoints.

In order to achieve the Fourier coefficient amplitudes of the coherent part of the signal proportional to k^{-3} , and so an easier visualization of the noisy part, discontinuities at the extremes of the time series can be removed. To this end, a linear trend from the measured time history is to be subtracted, taking the amplitudes at the endpoints to zero, and then the result is mirrored about the origin, allowing the slope discontinuities at the endpoints to be eliminated [78]. Accordingly, the signal $g(i)$, with $g(-N+1) = g(0) = g(N-1) = 0$, is computed as

$$g(i) = z(i) - z(0) - i \left[\frac{z(N-1) - z(0)}{N-1} \right] \quad (4.107)$$

$$g(-i) = -g(i) \quad i = 0, 1, 2, \dots, N-1 \quad (4.108)$$

Since g is an odd function of time it is expanded using a Fourier sine series

$$\hat{g}(i) = \sum_{k=1}^{N-1} b(k) \sin \left[k\pi \left(\frac{i}{N-1} \right) \right] \quad i = 0, 1, 2, \dots, N-1 \quad (4.109)$$

where $\hat{g}(i)$ indicates the approximation to $g(i)$, and $b(k)$ is the Fourier coefficient of

order k , computed as

$$b(k) = \frac{2}{N-1} \sum_{i=1}^{N-2} g(i) \sin \left[k\pi \left(\frac{i}{N-1} \right) \right] \quad (4.110)$$

where the index i runs from 1 to $|N-2|$, because $g(0)$ and $g(N-1)$ are zero. The upper limit for k is $(N-1)$, which corresponds to the Nyquist frequency. The magnitude of the coefficients of the Fourier sine series for the components of angular rate measurements (have a simulation of LV reference mission) with endpoint discontinuities removed is shown in Fig. 4.8.

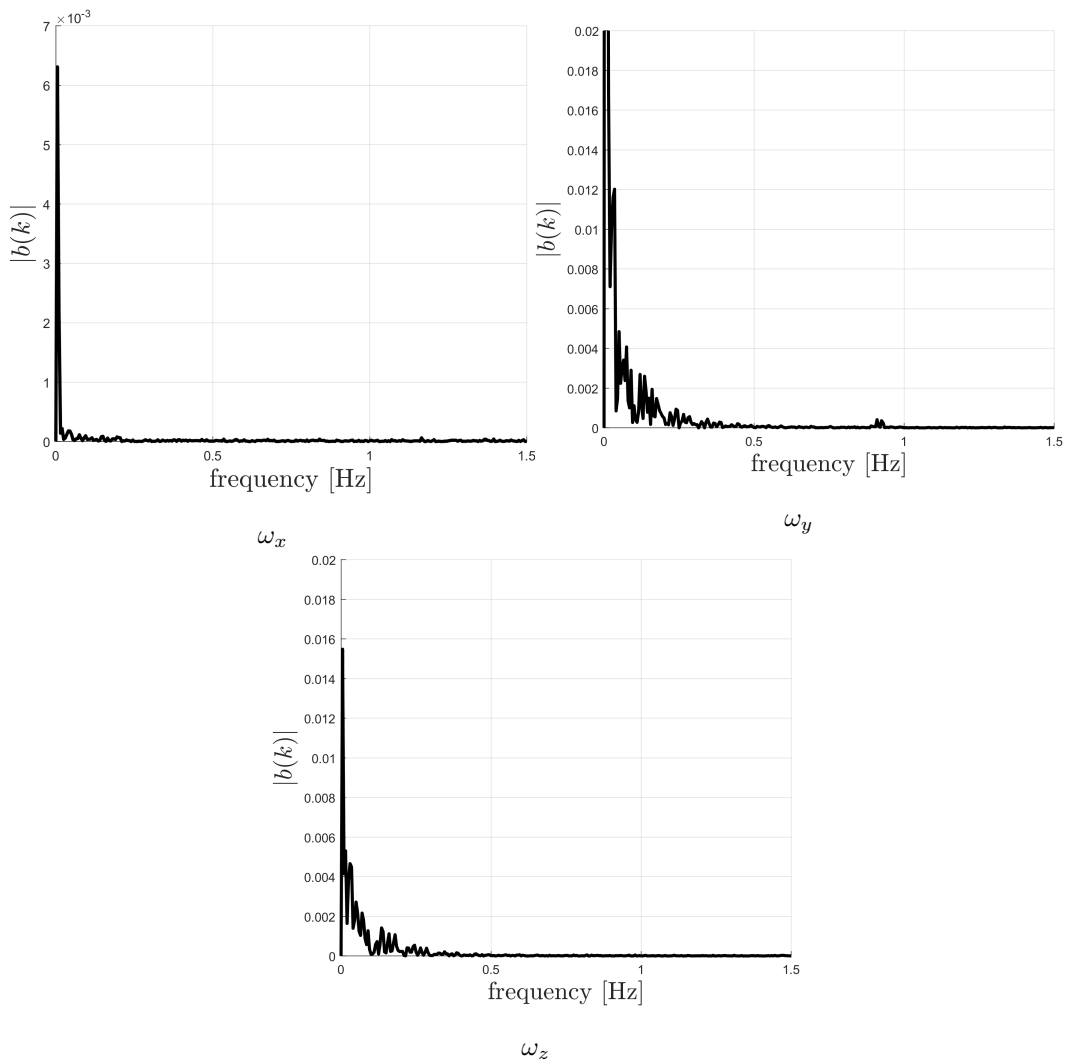


Figure 4.8. Coefficients of the Fourier sine series for the measured components of angular rate with endpoint discontinuities removed.

As far as low-pass smoothing is concerned, a simple method for implementing

the algorithm would be to visually determine the highest frequency in the cubic roll-off of the Fourier coefficients $b(k)$, defined as the frequency where the Fourier coefficient magnitudes reach the noise floor, characterized by the relatively constant magnitude of the Fourier coefficients, set all the coefficients at higher frequency equal to zero, and then compute the smoothed time history using the inverse Fourier sine transform. Figure 4.8 shows the coefficients of the Fourier sine series, from which we visually identify cutoff frequencies approximately at 0.3 Hz, 1 Hz, and 0.5 Hz for the ω_x , ω_y , and ω_z components of the angular rate, respectively.

Consider the Fourier transform of the measured time series, which is composed of signal plus noise,

$$\tilde{z}(f) = \tilde{y}(f) + \tilde{\nu}(f) \quad (4.111)$$

where f is the frequency. The goal is to design a frequency domain filter that is optimal in the sense of minimizing the squared difference between true $\tilde{y}(f)$ and estimated $\hat{\tilde{y}}(f)$ signals over the entire frequency range up to the Nyquist frequency. This is realized by minimizing the function

$$\int_0^{f_N} [\tilde{y}(f) - \hat{\tilde{y}}(f)]^2 df \quad (4.112)$$

The estimated signal in the frequency domain is obtained by multiplying a filter $\Phi_s(f)$ with $\tilde{z}(f)$,

$$\hat{\tilde{y}}(f) = \Phi_s(f)\tilde{z}(f) = \Phi_s(f)[\tilde{y}(f) + \tilde{\nu}(f)] \quad (4.113)$$

Substituting for $\hat{\tilde{y}}(f)$ from Eq. (4.113) and recognizing that $\tilde{y}(f)\tilde{\nu}(f)$ integrated over all frequencies will be approximately zero due to the incoherence of noise, Eq. (4.112) becomes

$$\int_0^{f_N} \{\tilde{y}^2(f) - 2\Phi_s(f)\tilde{y}^2(f) + \Phi_s^2(f)[\tilde{y}^2(f) + \tilde{\nu}^2(f)]\} df \quad (4.114)$$

Taking the derivative of the integrand in Eq. (4.114) with respect to $\Phi_s(f)$, setting the result equal to zero, and solving for $\Phi_s(f)$ gives, in terms of the discrete frequency index k ,

$$\Phi_s(k) = \frac{\tilde{y}^2(k)}{\tilde{y}^2(k) + \tilde{\nu}^2(k)} \quad 0 \leq k \leq N-1 \quad (4.115)$$

Equation (4.115) provides the final form of the Wiener filter, devised by assuming that the Fourier coefficients for the signal are those below the frequency cutoffs determined visually from Fig. 4.8. Figure 4.9 shows the gain plots of the filter frequency responses with the angular rate components as inputs.

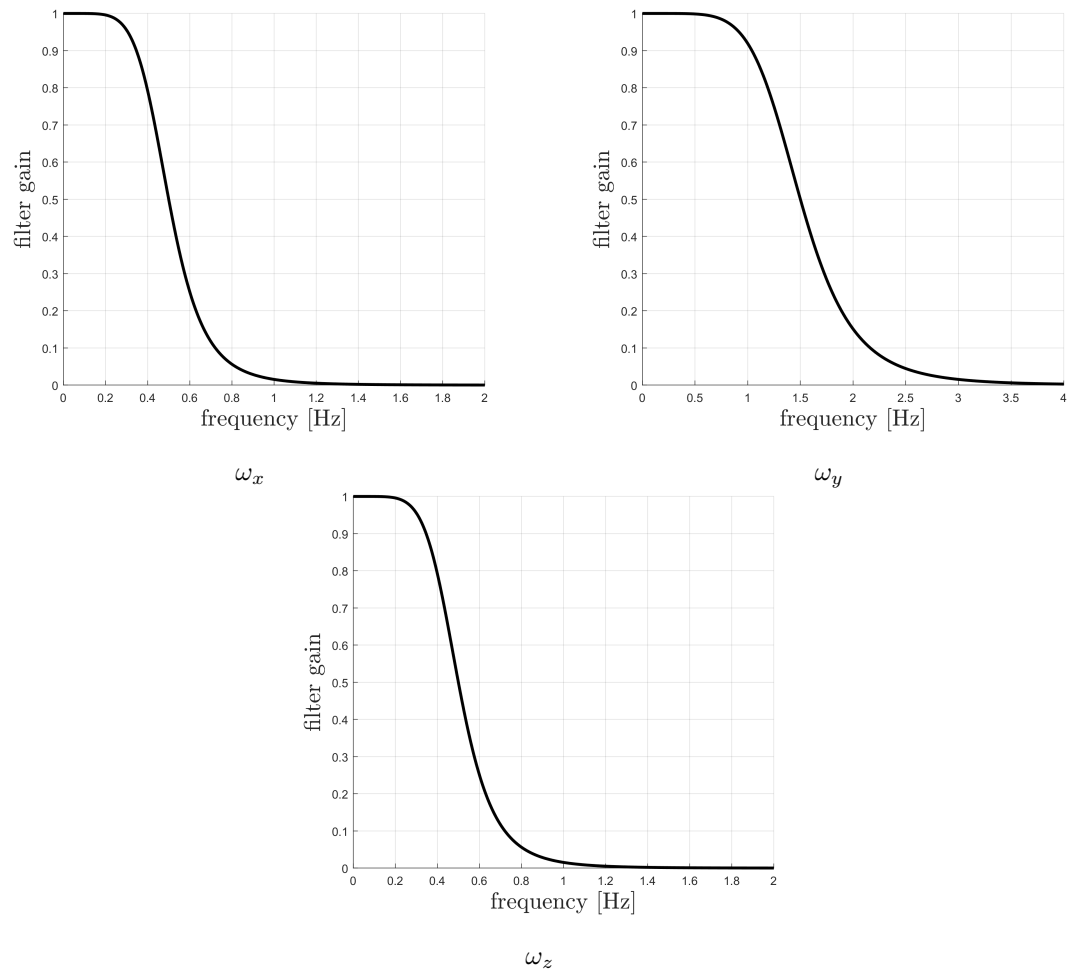


Figure 4.9. Gains of Wiener frequency-domain smoothing filter for $|b(k)|$.

The optimal filter (Eq. (4.115)) then multiplies $b(k)$ in the inverse Fourier transform, so that the smoothed signal (with endpoint discontinuities still removed) is expressed as

$$\hat{g}_s(i) = \sum_{k=1}^{N-1} \Phi_s(k)b(k) \sin \left[k\pi \left(\frac{i}{N-1} \right) \right] \quad i = 0, 1, 2, \dots, N-1 \quad (4.116)$$

The summation over the frequency index k is truncated when the values of $\Phi_s(k)b(k)$ approaches zero. As a final step, the linear trend, removed from the original time history using Eq. (4.108), is restored to the smoothed values $\hat{g}_s(i)$ from Eq. (4.116). The final smoothed time evolution of the signal is given by

$$\hat{y}(i) = z_s(i) = \hat{g}_s(i) + z(0) + i \left[\frac{z(N-1) - z(0)}{N-1} \right] \quad i = 0, 1, 2, \dots, N-1 \quad (4.117)$$

Figure 4.10 reports an example of the smoothed coherent signal computed by applying the Weiner filter to the measured angular rate.

After smoothing the angular rates, the acceleration at vehicle cg is evaluated and smoothed in order, again, to eliminate structural and measurement noises. To this end, the derivatives of the angular rate components are calculated by differentiating the sine series and the linear trend before transforming back to the time domain. The expression for the smoothed derivative is

$$\frac{d\hat{y}(i)}{dt} = \left[\frac{z(N-1) - z(0)}{N-1} \right] + \sum_{k=1}^{N-1} \Phi_s(k)b(k) \left(\frac{k\pi}{N-1} \right) \cos \left[k\pi \left(\frac{i}{N-1} \right) \right] \quad (4.118)$$

$i = 0, 1, \dots, N-1$

The smoothed components of the angular acceleration are reported in Fig. 4.11.

The linear acceleration components are then smoothed through the Weiner filter, and the result is shown in Fig. 4.12.

The noise sequence $\nu(i)$ is obtained from Eq. (4.106) using $\hat{y}(i)$ computed from Eq. (4.117). Note that it only presents wideband random components, that is, the noise is not colored by deterministic modeling error. In fact, this noise is assumed Gaussian and stationary, and the computed estimate for the mean value is close to zero. Finally, the measurement noise covariance matrix $\mathbf{R}$ is estimated as

$$\hat{\mathbf{R}} = \frac{(\mathbf{z} - \hat{\mathbf{Y}})^T (\mathbf{z} - \hat{\mathbf{Y}})}{N-1} \quad (4.119)$$

where $\mathbf{z}$ and $\hat{\mathbf{Y}}$ are, respectively, the vectors of measured and smoothed outputs.

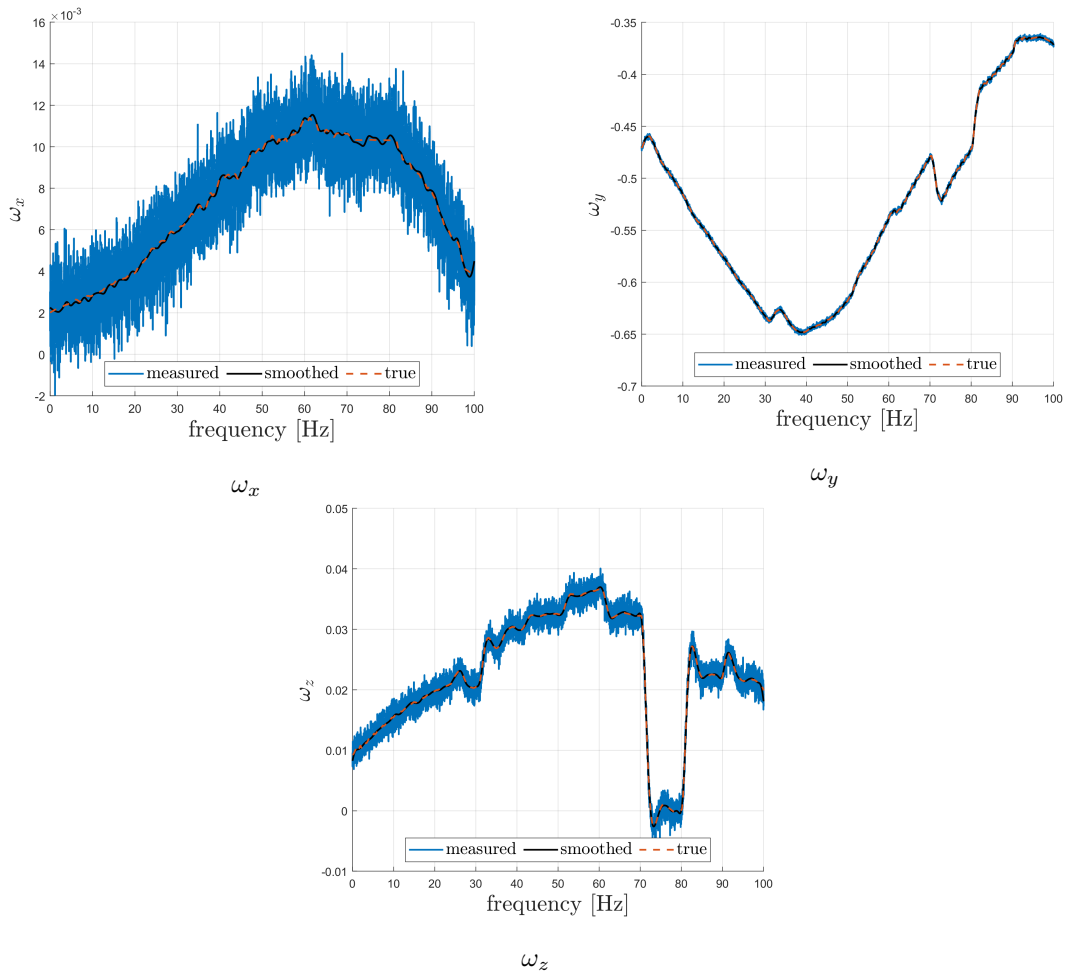


Figure 4.10. Measured, smoothed, and estimated angular rate components.

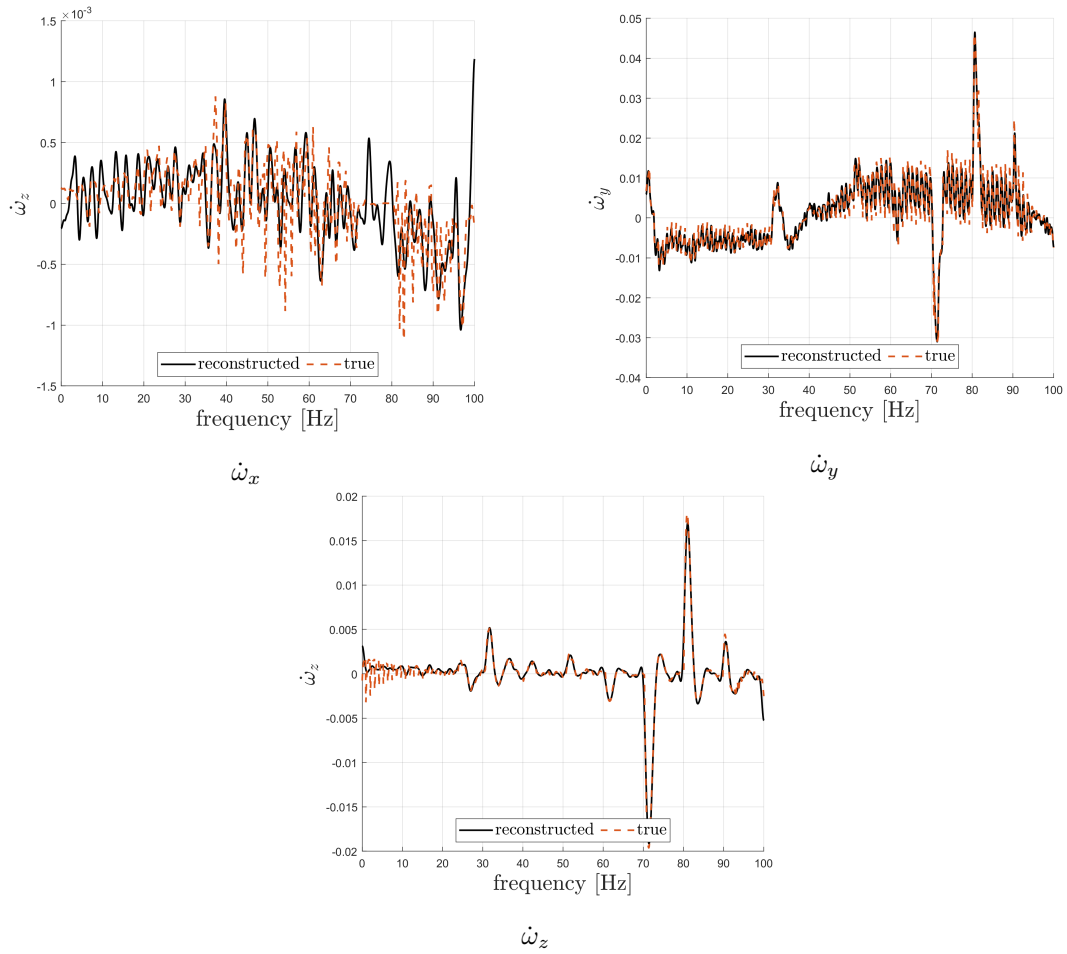


Figure 4.11. True and smoothed components of angular acceleration components.

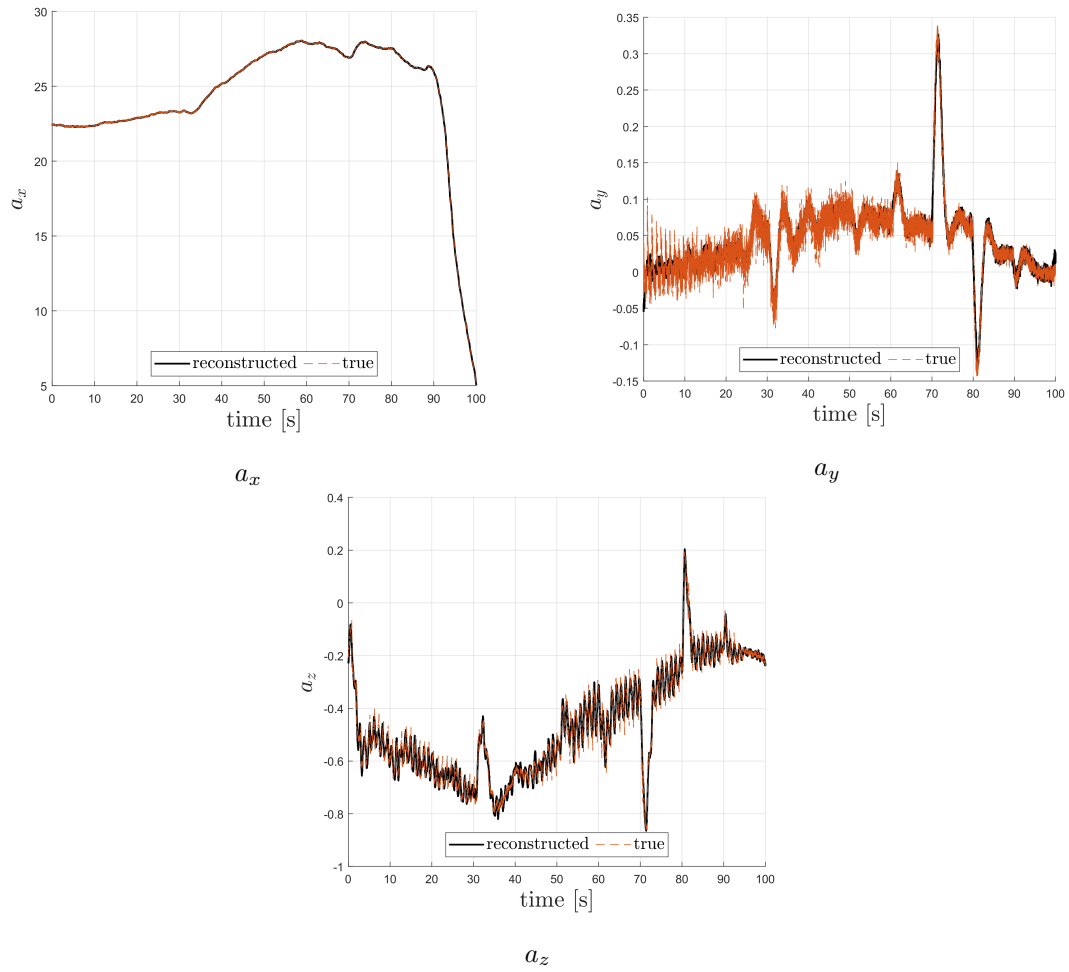


Figure 4.12. True and smoothed components of linear acceleration components.

Chapter 5

BET filter

The development of a BET filter for LV TR is discussed in this chapter. The filter formulation uses IMU sensor outputs as inputs and GNSS and radar sensor outputs as measurements, while a kinematic filter process model is adopted for the LV translational and rotational motion. The filter is developed in non-dimensional form and the attitude is represented by quaternion rotations. Performances, consistency analysis, and computational efficiency of EKF and UKF algorithm featuring a Fraser-Potter smoother [33] to enhance estimation quality are evaluated and compared.

5.1 Problem statement

Starting from the TR state-of-the-art, the New Statistical Trajectory Estimation Program (NewSTEP) was downloaded through the NASA Technology Transfer Program [79]. NewSTEP is a Matlab-based iterative EKF designed for solving problems in flight test experiments (see Chap. 1 for further information about TR solutions that use NewSTEP). The codes recalled in this section use the same dynamical model of NewSTEP and feature a Fraser-Potter smoother [33].

The formulation exploits the outputs from the IMU (accelerations and angular rates) considering them as inputs for the trajectory model, while the outputs from the remaining measurement instruments (GNSS and radar) (described in Sec. 2.6) constitute the filter measurement vector $\mathbf{z}$ (see Fig. 5.1 for the filter scheme) where for both measurement sources, a sampling frequency of 10 Hz is assumed. GNSS measurement frequency is generally lower than 10 Hz, but for the sake of simplicity, all measurement rates are chosen to be the same. It is observed that this choice doesn't affect the contribution of the work. Furthermore, is worth to remark that incorporating biases into radar and GNSS measurements and colored noise in GNSS velocity and radar position is essential for evaluating a more realistic scenario. In

the end, since the IMU is typically affected by systematic calibration errors, in cases where these are not negligible, the filter state is augmented to estimate the parameters of the error models for the input measurements (acceleration and angular rate) to the filter model equations, assigning a Random Walk model to the unknown parameters. The filter does not require the formulation of dynamic equations related to the DOFs of translation and rotation and is consequently also referred to as a *kinematic* filter.

The filter development in non-dimensional form is particularly advantageous as it circumvents issues related to the ill-conditioned inverse of the error covariance matrices, ensuring a more robust and reliable estimation process [80]. The approach of using a kinematic filter formulation in TR for probes, LVs, and flight missions in the framework of post-flight analyses is particularly valuable for achieving optimal results when limited information on the vehicle forces and moments characterization is available.

Building on the methodologies presented in References [31] and [8], the architectures for an EKF and a UKF are developed. As discussed in [9], where for a re-entry vehicle the UKF shows better results for some state variables estimation w.r.t. the EKF, in this work the same comparison aims to determine whether the UKF can provide specific advantages for BET applications in LV missions. Enhancements in BET especially in attitude estimation within this filter architecture can be achieved through the implementation of either a MEKF (see Chap.6) or an Unscented Quaternion Estimator (USQUE) [81].

The filter architecture is composed of vehicle position, expressed as geocentric radius r of cg, longitude θ , and latitude ϕ , inertial velocity components in $\mathcal{F}_E$, $\mathbf{v}_{[E]} = [u, v, w]$, and quaternion parameters $\mathbf{q} = [e_0, \mathbf{e}]$ that describe the orientation of $\mathcal{F}_b$ with respect to the $\mathcal{F}_E$ frame. Biases on IMU linear accelerations β_a and angular rates, β_ω are also included in the state vector for estimation purposes, so that it results

$$\mathbf{x} = [r, \phi, \theta, u, v, w, e_0, e_1, e_2, e_3, \beta_{a,x}, \beta_{a,y}, \beta_{a,z}, \beta_{\omega,x}, \beta_{\omega,y}, \beta_{\omega,z}]^T \quad (5.1)$$

Linear acceleration $\mathbf{a} = [a_x, a_y, a_z]$ at cg location and angular rate $\boldsymbol{\omega}_{BI} = [p, q, r]$ are inputs for the equations of motion, that is

$$\mathbf{u}(t) = [\tilde{\mathbf{a}}_{\text{IMU}}, \tilde{\boldsymbol{\omega}}_{BI}]^T$$

As already mentioned, the mathematical model for the filter (shown in Sec. 5.2 and Sec. 5.3) is formulated in non-dimensional form. To this end, the following reference

quantities for acceleration (a^*), velocity (v^*), length (s^*), angular rate (ω^*), are introduced

$$a^* = \frac{\mu_g}{R_E^2}; \quad v^* = \frac{R_E}{\sqrt{\frac{R_E^3}{\mu_g}}}; \quad s^* = R_E; \quad t^* = \sqrt{\frac{R_E^3}{\mu_g}}; \quad \omega^* = \frac{1}{\sqrt{\frac{R_E^3}{\mu_g}}}; \quad (5.2)$$

where R_E is the Earth radius.

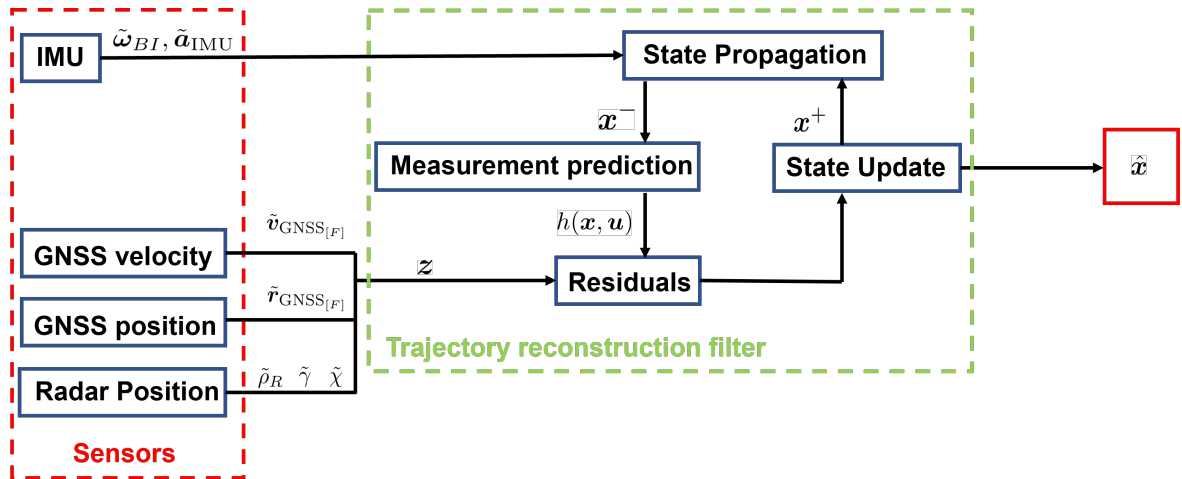


Figure 5.1. Sketch of the BET filter

5.2 Process equations

IMU systematic errors

The acceleration in body coordinates is calculated as

$$\mathbf{a}_{\text{IMU}} = \tilde{\mathbf{a}}_{\text{IMU}} - \boldsymbol{\beta}_a \quad (5.3)$$

where $\mathbf{a}$ is the acceleration transformed to the vehicle cg according to

$$\mathbf{a} = \mathbf{a}_{\text{IMU}} - \left(\boldsymbol{\omega}_{BI} \times (\boldsymbol{\omega}_{BI} \times \mathbf{d}_{\text{IMU}}) + \dot{\boldsymbol{\omega}}_{BI} \times \mathbf{d}_{\text{IMU}} + 2\boldsymbol{\omega}_{BI} \times \dot{\mathbf{d}}_{\text{IMU}} + \ddot{\mathbf{d}}_{\text{IMU}} \right) \quad (5.4)$$

being $\mathbf{d}_{\text{IMU}}$ the position vector of IMU sensor in $\mathcal{F}_B$, and $\dot{\boldsymbol{\omega}}_{BI}$ a smoothed finite-difference approximation of the true angular acceleration, expressed as

$$\dot{\boldsymbol{\omega}}_{BI} \approx \dot{\tilde{\boldsymbol{\omega}}}_{BI} \quad (5.5)$$

where $\dot{\tilde{\omega}}_{BI}$ is derived from the measured angular velocity $\tilde{\omega}_{BI}$ following the approach described in Sec. 4.6. The measured angular rates $\tilde{\omega}_{BI}$ are also corrected from the systematic errors (biases) which read as

$$\omega_{BI} = \tilde{\omega}_{BI} - \beta_{\omega} \quad (5.6)$$

The transformation matrix $\mathcal{R}_{BE}$ from frame $\mathcal{F}_E$ to the body frame $\mathcal{F}_B$ is expressed as a function of the quaternions (see Eq. 4.68). Measured acceleration and angular rate have been characterized by a Gaussian and white process noise η_{ω} and η_a with zero mean and spectral densities $\sigma_{w_{\omega}}^2 \mathbf{I}_{3 \times 3}$ and $\sigma_{w_a}^2 \mathbf{I}_{3 \times 3}$ respectively, while the accelerometer and gyro bias, β_a and β_{ω} , respectively, are modeled as a zero-order Markov process

$$\dot{\beta}_a = \eta_{\beta_a} \quad (5.7)$$

$$\dot{\beta}_{\omega} = \eta_{\beta_{\omega}} \quad (5.8)$$

where η_{β_a} and $\eta_{\beta_{\omega}}$ are zero mean Gaussian-white noise with spectral densities $\sigma_{w_{\beta_{\omega}}}^2 \mathbf{I}_{3 \times 3}$ and $\sigma_{w_{\beta_a}}^2 \mathbf{I}_{3 \times 3}$ respectively.

Inertial position and velocity

The inertial position in terms of range latitude and longitude, and velocity in $\mathcal{F}_E$ are expressed by the use of a kinematic formulation as follows

$$\dot{r} = -w \quad (5.9)$$

$$\dot{\theta} = \frac{v}{r \cos \phi} - \omega_{\oplus} \quad (5.10)$$

$$\dot{\phi} = \frac{u}{r} \quad (5.11)$$

$$\dot{\mathbf{v}}_{[E]} = -\omega_{EI} \times \mathbf{v}_{[E]} + \mathbf{f}_{g[E]} + \mathcal{R}_{BE}^T \mathbf{a} \quad (5.12)$$

where the gravity acceleration in $\mathcal{F}_E$ is

$$\mathbf{f}_{g[E]} = \frac{\mu_g}{r^2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (5.13)$$

and

$$\omega_{EI} \times \mathbf{v}_{[E]} = -\frac{1}{r} \begin{bmatrix} uw - v^2 \tan \phi \\ uv \tan \phi + vw \\ -(u^2 + v^2) \end{bmatrix} \quad (5.14)$$

Attitude

The attitude kinematics is described by

$$\begin{bmatrix} \dot{e}_0 \\ \dot{\mathbf{e}} \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} -\mathbf{e}^T \\ e_0 \mathbf{I} + [\mathbf{e} \times] \end{bmatrix} \right) \boldsymbol{\omega}_{BE} \quad (5.15)$$

where

$$\boldsymbol{\omega}_{BE} = \boldsymbol{\omega}_{BI} - \mathcal{R}_{BE} \boldsymbol{\omega}_{EI} \quad (5.16)$$

being $\boldsymbol{\omega}_{EI}$ the NED-to-ECI angular velocity in $\mathcal{F}_E$

$$\boldsymbol{\omega}_{EI} = [v/r, -u/r, -v \tan \phi/r]^T \quad (5.17)$$

Summing up, the process noise vector $\boldsymbol{\eta} \in \mathcal{R}^{12}$ is

$$\boldsymbol{\eta} = [\boldsymbol{\eta}_a, \boldsymbol{\eta}_\omega, \boldsymbol{\eta}_{\beta_a}, \boldsymbol{\eta}_{\beta_\omega}] \quad (5.18)$$

5.3 Measurement equations

The measurement equations of the GNSS sensor are written as follows

$$\mathbf{r}_{\text{GNSS}[F]} = \mathbf{r}_{[F]} + \mathcal{R}_{FB} \boldsymbol{\delta}_{\text{GNSS}[B]} \quad (5.19)$$

$$\begin{aligned} \mathbf{v}_{\text{GNSS}[F]} &= \mathcal{R}_{FE} \left(\mathbf{v}_{[E]} + [\boldsymbol{\omega}_{\oplus[E]} \times] \mathcal{R}_{EF} \mathbf{r}_{[F]} \right) + \\ &\quad + \mathcal{R}_{FB} [\boldsymbol{\omega}_{BE} \times] \boldsymbol{\delta}_{\text{GNSS}[B]} \end{aligned} \quad (5.20)$$

where

$$\mathbf{r}_{[F]} = r [\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi]^T \quad (5.21)$$

and $\boldsymbol{\omega}_{\oplus[E]} = \omega_{\oplus} [-\cos \theta \cos \phi, -\sin \theta, -\cos \phi \sin \phi]^T$, being $\omega_{\oplus} = 7.291\,985 \times 10^{-5}$ rad/s the Earth angular rate in $\mathcal{F}_E$, and $\boldsymbol{\delta}_{\text{GNSS}}$ is the receiver position vector from the LV cg. Note that $\mathcal{R}_{FB} = (\mathcal{R}_{BE} \mathcal{R}_{EF})^T$ is readily available as $\mathcal{R}_{BE}$ is given by Eq. (2.58), and $\mathcal{R}_{EF}$ is a function of LV latitude and longitude only, that is

$$\mathcal{R}_{EF} = \begin{bmatrix} -\cos \theta \sin \phi & -\sin \theta \sin \phi & \cos \phi \\ -\sin \theta & \cos \theta & 0 \\ -\cos \theta \cos \phi & -\cos \phi \sin \theta & -\sin \phi \end{bmatrix} \quad (5.22)$$

Turning to radar data, the relative position of LV with respect to the GS in $\mathcal{F}_G$,

$\boldsymbol{\rho}_{R[G]} = [\rho_x, \rho_y, \rho_z]^T$ is expressed as

$$\boldsymbol{\rho}_{R[G]} = \mathcal{R}_{GF}(\mathbf{r}_{[F]} - \mathbf{r}_{\text{gs}[F]}) \quad (5.23)$$

where the transformation matrix $\mathcal{R}_{GF}$ is given by

$$\mathcal{R}_{GF} = \begin{bmatrix} -\cos \theta_{gs} \sin \phi_{gs} & -\sin \theta_{gs} \sin \phi_{gs} & \cos \phi_{gs} \\ -\sin \theta_{gs} & \cos \theta_{gs} & 0 \\ -\cos \theta_{gs} \cos \phi_{gs} & -\cos \phi_{gs} \sin \theta_{gs} & -\sin \phi_{gs} \end{bmatrix} \quad (5.24)$$

and

$$\mathbf{r}_{\text{gs}[F]} = R_E[\cos \theta_{\text{gs}} \cos \phi_{\text{gs}}, \sin \theta_{\text{gs}} \cos \phi_{\text{gs}}, \sin \psi_{\text{gs}}]^T \quad (5.25)$$

is the GS location in the $\mathcal{F}_F$, being θ_{gs} and ϕ_{gs} its geographical coordinates. Accordingly, the measurement equations are

$$\rho_{R[G]} = \sqrt{\rho_x^2 + \rho_y^2 + \rho_z^2} \quad (5.26)$$

$$\tan(\gamma/2) = \rho_y / \left(\sqrt{\rho_x^2 + \rho_y^2} + \rho_x \right) \quad (5.27)$$

$$\sin \chi = -\rho_z / \rho_{R[G]} \quad (5.28)$$

5.4 Filter initialization and tuning

The initial conditions on the filter variables r , ϕ , and θ are assigned from the GNSS position measurement. LV cg position is written in the $\mathcal{F}_F$ according to Eq. 5.19 where the relative position of the receiver with respect to the vehicle cg in $\mathcal{F}_B$, $\boldsymbol{\delta}_{GNSS[B]}$, is known. Since the transformation matrix $\mathcal{R}_{FB}$ requires the knowledge of initial cg position, which is unknown, an uncertainty term, $\tilde{\boldsymbol{\delta}}_{\text{GNSS}[F]}$, is considered, so that

$$\mathbf{r}_{[F]} = \mathbf{r}_{\text{GNSS}[F]} - \tilde{\boldsymbol{\delta}}_{\text{GNSS}[F]} \quad (5.29)$$

The cg position, $\mathbf{r}_{[F]} = [x_{cg}, y_{cg}, z_{cg}]^T$ is given, in polar coordinates as

$$r = \sqrt{x_{cg}^2 + y_{cg}^2 + z_{cg}^2} \quad (5.30)$$

$$\sin \phi = \frac{z_{cg}}{r} \quad (5.31)$$

$$\cos \theta = \frac{x_{cg}}{r \cos \phi} \quad (5.32)$$

$$\sin \theta = \frac{y_{cg}}{r \cos \phi} \quad (5.33)$$

The initial condition on inertial velocity components in the NED frame is obtained from the GNSS velocity measurements by inverting Eq. 5.20 as

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathcal{R}_{FE}^T \left(\mathbf{v}_{\text{GNSS}_{[F]}} - \mathcal{R}_{FB}[\boldsymbol{\omega}_{BE} \times] \boldsymbol{\delta}_{\text{GNSS}_{[B]}} \right) + \begin{bmatrix} 0 \\ r\omega_{\oplus} \cos \phi \\ 0 \end{bmatrix} \quad (5.34)$$

The attitude parameters in terms of Euler angles are initialized randomly by scattering about the true values. The quaternions are thus expressed as

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos \frac{\Psi}{2} \cos \frac{\Theta}{2} \cos \frac{\Phi}{2} + \sin \frac{\Psi}{2} \sin \frac{\Theta}{2} \sin \frac{\Phi}{2} \\ \cos \frac{\Psi}{2} \cos \frac{\Theta}{2} \cos \frac{\Phi}{2} - \sin \frac{\Psi}{2} \sin \frac{\Theta}{2} \sin \frac{\Phi}{2} \\ \cos \frac{\Psi}{2} \sin \frac{\Theta}{2} \cos \frac{\Phi}{2} + \sin \frac{\Psi}{2} \cos \frac{\Theta}{2} \sin \frac{\Phi}{2} \\ \sin \frac{\Psi}{2} \cos \frac{\Theta}{2} \cos \frac{\Phi}{2} - \cos \frac{\Psi}{2} \sin \frac{\Theta}{2} \sin \frac{\Phi}{2} \end{bmatrix} \quad (5.35)$$

Accelerometer and gyro biases are initialized to zero.

Let the filter states at the initial time be expressed as $\hat{\mathbf{x}}_0 = h^{-1}(\mathbf{z}_0) = \chi_h(\mathbf{z}_0)$ where $\mathbf{x}_0$ is the initial value of the filter state and $\mathbf{z}_0$ are the observable outputs. The initial value of covariance $\mathbf{P}_0$ is evaluated in terms of measurement covariance by approximating the function that relates the filter state to the measured variables by a second-order polynomial approximation and then taking the expectation [82], as

$$\mathbf{P}_0 = E\{(\chi_h(\mathbf{z}_0) - \mathbf{x}_0)(\chi_h(\mathbf{z}_0) - \mathbf{x}_0)^T\} \quad (5.36)$$

Using a second-order polynomial approximation of the function $\chi_h(\mathbf{y})$, and taking the expectation yields

$$\begin{aligned} \mathbf{P}_0 = & \frac{1}{4\Delta^2} \sum_{p=1}^n [\chi_h(\mathbf{z}_0 + \Delta \mathbf{s}_{y,p}) - \chi_h(\mathbf{z}_0 - \Delta \mathbf{s}_{y,p})][\chi_h(\mathbf{z}_0 + \Delta \mathbf{s}_{y,p}) + \\ & - \chi_h(\mathbf{z}_0 - \Delta \mathbf{s}_{y,p})]^T + \frac{\Delta^2 - 1}{4\Delta^4} \sum_{p=1}^n [\chi_h(\mathbf{z}_0 + \Delta \mathbf{s}_{y,p}) + \chi_h(\mathbf{z}_0 - \Delta \mathbf{s}_{y,p}) + \\ & - 2\chi_h(\mathbf{z}_0)][\chi_h(\mathbf{x}_0 + \Delta \mathbf{s}_{y,p}) + \chi_h(\mathbf{z}_0 - \Delta \mathbf{s}_{y,p}) - 2\chi_h(\mathbf{z}_0)]^T \end{aligned} \quad (5.37)$$

where $\Delta = \sqrt{3}$, and $\mathbf{S}_y = [\mathbf{s}_{y,1}, \mathbf{s}_{y,2}, \dots, \mathbf{s}_{y,n}] = \mathbf{R}$ is the covariance matrix of the measurement $\mathbf{z}_0$. The covariance of the initial bias estimate is determined by considering the sensor specifications outlined in Table 2.4.

Filter tuning is performed by selecting the value of the process and measurement noise covariance matrices, namely, $\mathbf{Q}$ and $\mathbf{R}$, respectively. Matrix $\mathbf{R}$ is assumed diagonal, and its elements are the error covariance of the sensors reported in Table 2.4,

that is

$$\mathbf{R} = \text{diag} \left(\sigma_{w_v, w}^2, \sigma_{w_\rho}^2, \sigma_{w_\gamma}^2, \sigma_{w_\chi}^2, \sigma_{w_{r, w}}^2 \right) \quad (5.38)$$

The elements of $\mathbf{Q}$, also assumed diagonal, are associated with the PSD of the input signals for the EKF, as

$$\mathbf{Q}_{\text{EKF}} = \text{diag} \left(\sigma_{w_a}^2, \sigma_{w_\omega}^2, \sigma_{w_{b_a}}^2, \sigma_{w_{b_\omega}}^2 \right) \quad (5.39)$$

where the non-zero elements of Eq. (5.39) are respectively the noise velocity random walk PSD, the noise angle random walk PSD, and the bias rate random walk reported in Table 2.4. On the other hand, the UKF uses the noise covariance of the same signals [13], so that

$$\mathbf{Q}_{\text{UKF}} = \text{diag} \left(\frac{\sigma_{w_a}^2}{T_s}, \frac{\sigma_{w_\omega}^2}{T_s}, \frac{\sigma_{w_{b_a}}^2}{T_s}, \frac{\sigma_{w_{b_\omega}}^2}{T_s} \right) \quad (5.40)$$

being $T_s = 0.005$ s the IMU sample time. This is related to the different formulations of the two filters, that is continuous-discrete form for the EKF, and time-discrete formulation for the UKF. As an interesting remark, terms of the process noise covariance matrix associated with position and velocity should be considered [83]. In the present work they may be neglected due to the small integration step adopted for the *a priori* estimate.

5.5 BET filter analysis and results

As said, the filter architecture proposed in Chap. 4 applied to the simulated first-stage phase of flight of the Ares-I LV model [52], the main parameters of which are reported in Table 3.3. The portion of trajectory considered for reconstruction is the zero-lift gravity-turn (ZLGT) phase of atmospheric flight, required to steer the LV velocity vector while keeping the transverse aerodynamic load sufficiently low. It starts at the end of the pitch-over maneuver and lasts about 105 s. During the ZLGT, the LV reaches the speed of Mach 1 at $t = 48$ s while, at the stage burnout, the vehicle has an altitude of roughly 44 km and a velocity of about 1.2 km/s.

A synthetic wind generated by a Dryden model representative of the typical wind in the Atlantic region is considered as shown in Sec. 2.5. Preprocessing of sensor data, particularly for IMU, is usually carried out in TR [84] in order to remove high-frequency sensor noise and structural vibration (see Sec. 4.6). In this application, the angular rate oscillates at a relatively low frequency (below 1 Hz) as the forcing action of the transversal thrust components generated by TVC is projected onto the

dynamics of the first elastic mode (1 Hz natural frequency). Consequently, the use of an optimal Fourier filter with a cut-off frequency of about 1 Hz to smooth the filter input signals turns out to be ineffective and is ruled out.

5.5.1 Performance analysis

Performances of the four filters, that is, EKF, UKF, and their improved, smoothed counterparts, dubbed S-EKF and S-UKF, obtained by combining a forward and a backward pass via Fraser-Potter smoothing, are analyzed and compared. The S-EKF corresponds to the NewSTEP [84] formulation that, as said, is a reference methodology in BET evaluation for LVs. The process model of the filter is integrated using a fourth-order Runge-Kutta algorithm with a constant time step of 0.005 s, the same (200 Hz) frequency as the IMU sensor, while GNSS and radar measurements have a 10 Hz sample rate.

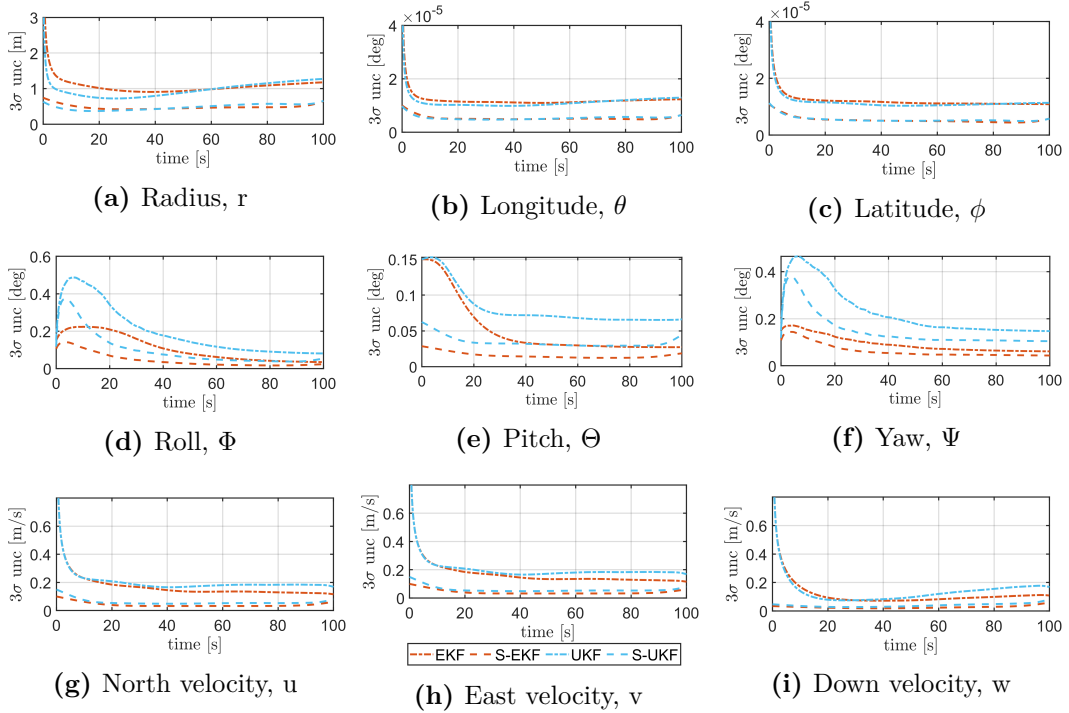


Figure 5.2. Accuracy of estimates for the different filter algorithms.

Figure 5.2 shows the 3σ uncertainties on the state estimation error obtained by the filters. For the sake of clarity, attitude errors are expressed using Euler angles, namely, roll Φ , pitch Θ , and yaw Ψ angles in place of the quaternions ($\mathbf{q}$). As expected, the two-pass smoothing improves filter performances for both S-EKF and S-UKF algorithms in terms of accuracy and speed of convergence, as the whole information contained in the measurements is exploited to improve the BET

solution. In more detail, the backward pass improves the accuracy of the estimate at the initial time points, because the results of the forward pass are affected by the uncertainty of initial conditions. It is apparent that S-EKF and S-UKF have similar performances in position and velocity estimation, while S-EKF provides lower estimation uncertainties than S-UKF for the attitude states, where the filters cannot exploit direct measurements. Filter characteristics are also analyzed in terms of averaged accuracy levels measured as the mean value of 1σ bounds over the duration of the estimation process, reported in Table 5.1. UKF-based filters present a slightly

Table 5.1. Estimated levels of accuracy.

Quantity	Averaged 1σ bound			
	EKF	UKF	S-EKF	S-UKF
u , mm/s	57.4	68.7	14	20.2
v , mm/s	53	64.1	13.1	18.9
w , mm/s	43.8	51.6	9.4	14
r , cm	36.3	36.3	16.4	16.5
ϕ , μ deg	4.8	4.6	1.8	1.9
θ , μ deg	6	5.7	1.8	1.8
Φ , mdeg	30.5	74.9	21	48.9
Θ , mdeg	16.5	26.9	5.3	11.6
Ψ , mdeg	34.9	64.5	14.5	32.6
β_a , mm/s ²	1.53	1.55	0.97	1.01
β_ω , μ rad/s	0.17	0.17	0.12	0.12

higher uncertainty on the estimation of attitude variables that are projected to other states such as u and w velocity components following vector transformation to the $\mathcal{F}_E$ frame. This behavior of the filters will be further investigated through the Monte Carlo analysis discussed in the next subsection. The modulus of residuals, that is, the differences between BET and true values for position and velocity components in the $\mathcal{F}_F$ frame, plotted in semi-log scale in Fig. 5.3 together with that of the noisy GNSS signals, shows the improved accuracy obtained by both filters. In particular, the RMS of position residuals is reduced by a factor of 28 and 27 w.r.t. the GNSS signal using, respectively, S-EKF and S-UKF, while the effect of filtering on velocity components decreases the residuals (RMS) by a factor as high as 59 and 51, respectively. Figure 5.4 shows the accelerometer and gyro error states covariance for the biases. Beyond the accelerometer bias along the x-axis, the others remain unobservable. However, they are retained within the filter due to their impact on the covariance of other states, therefore their uncertainty is taken into account in the overall estimate.

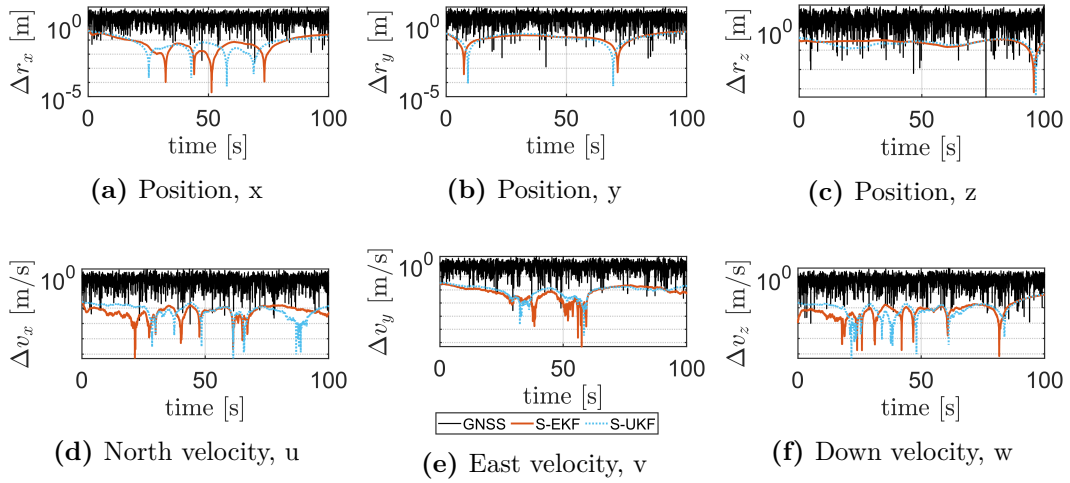


Figure 5.3. Residuals on GNSS position and velocity measurements in $\mathcal{F}_F$ (black) vs estimated values using S-EKF (red) and S-UKF (blue).

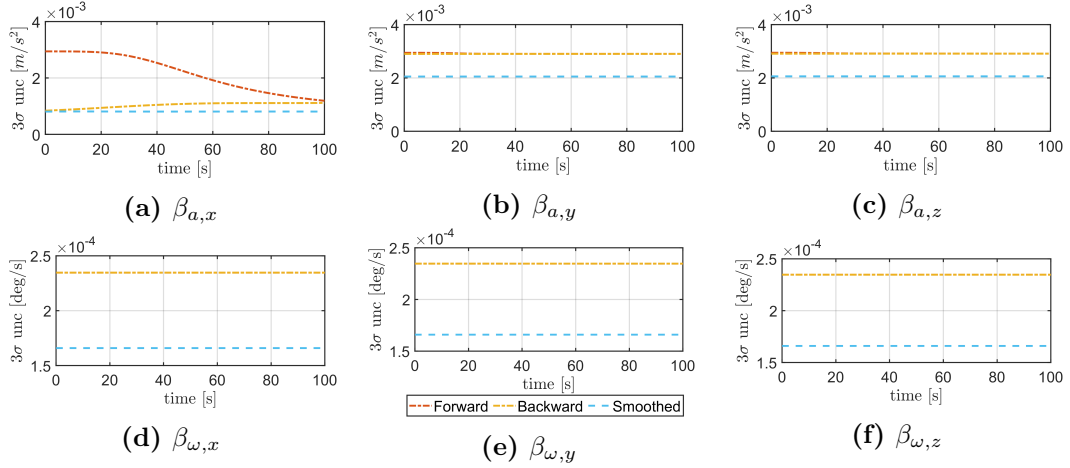


Figure 5.4. Estimated accuracy of the biases for the filtering phases of UKF.

In the end, the effect of a poor selection of initial state $\mathbf{x}_0$ is analyzed by initializing the filter with an error of 1 deg on attitude variables while keeping the initial covariance low. Figures 5.5 and 5.6 show the time evolution of the estimation error

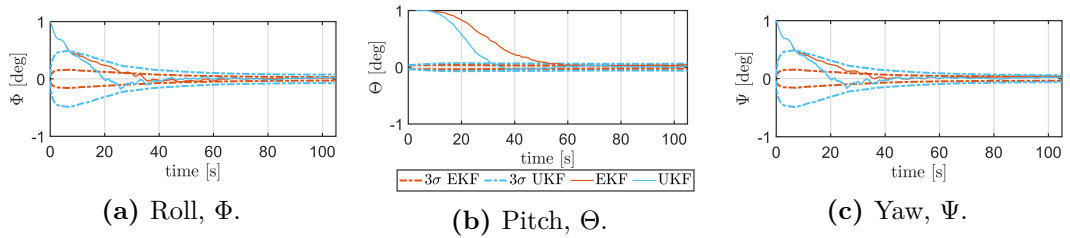


Figure 5.5. Effects of initial state uncertainty on EKF and UKF filter performances.

on the Euler angles (continuous lines) and related estimated covariance (dashed lines)

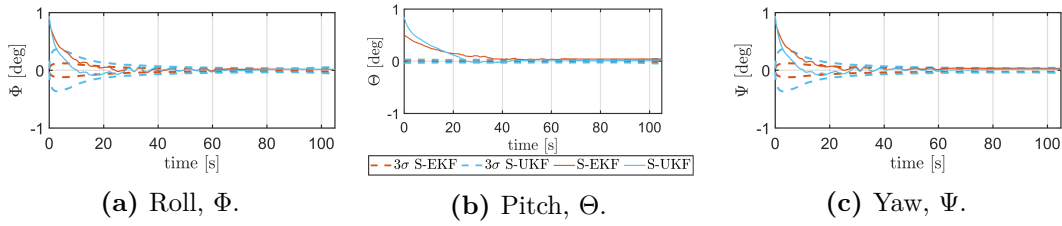


Figure 5.6. Effects of initial state uncertainty on S-EKF and S-UKF filter performances.

for forward-pass (EKF and UKF) and smoothed (S-EKF and S-UKF) algorithms, respectively. The UKF converges much more rapidly than the EKF to the true solution, and the average RMS error is significantly smaller (0.28 deg vs. 0.32 deg). The same remark holds when considering the smoothed solutions, that is S-UKF vs. S-EKF. In fact, the UKF takes advantage of the improved propagation of the mean and covariance matrix when the dispersion of the sigma points is sufficiently large, as in the present case, over the locally linear propagation of the same quantities for the EKF, the accuracy of which is somewhat limited for large initial errors. The effect of large attitude errors in the initial condition also affects the estimation of the other states, the initial errors of which increase in spite of the high frequency of IMU measurements and the correction provided by GNSS data. Filter performances on position and velocity states, not reported in figures for the sake of conciseness, are similar to that described for the attitude, that is, faster convergence of UKF with respect to EKF.

Finally, the issue of computational cost is taken into consideration. The BET is determined by algorithms implemented in Matlab®, without any specific code optimization, and requires roughly 2 and 17 s for the EKF and UKF filters, respectively, on a computer with Intel® Core™ i7-7700HQ CPU @ 2.80 GHz. Computational times double when considering the smoothed version of the filters. The higher computational burden of UKF (roughly 10 times that of EKF) mostly depends on the augmented state necessary to account for the presence of non-additive process noise, which is not considered in the EKF filters. Nevertheless, in post-flight BET applications, the computational cost is not of primary concern, as the reconstruction process is not subjected to tight constraints on the execution time.

5.5.2 Consistency analysis

The consistency of the four filters is investigated by a campaign of 200 runs, where IMU biases, and wind direction and intensity are randomly sampled in the ranges reported in Table 5.2. The initial conditions are different for each simulation due to the noise in the GNSS signals.

Table 5.2. Scattering ranges for the Monte Carlo campaign.

Scattering level			
$\beta_a _{t=0}$	$\beta_g _{t=0}$	$ v_w $	v_w direction
$\pm 100 \mu g$	$\pm 0.02 \text{ deg/h}$	$\pm 10 \text{ m/s}$	$\pm 180 \text{ deg}$

Filter consistency may be assessed by verifying that each error signal stays within the 3σ (state error covariance) bounds for more than 99.7% of time. Figures 5.7 and 5.8, where the estimation errors on the filter states for all runs (solid grey lines) vs. the 3σ covariance bounds (black dashed lines) are depicted as computed using the S-EKF and S-UKF filter, respectively, show that performance of both filters is satisfactory but the consistency criterion is not always satisfied. In Table 5.3 the percentage of the total time (over all simulations) in which the error signal stays within the 3σ bounds is reported in order to have a consistency index for both EKF and UKF; Bold values signify filter states that meet the criterion. It is apparent that the error covariance of the filter based on EKF might be underestimated, particularly for the attitude states, which would explain the differences on the 3σ bounds among the two filters observed in Fig. 5.2 for the Euler angles.

Table 5.3. Consistency results for the S-EKF and S-UKF algorithms.

State	Consistency check, %	
	S-EKF	S-UKF
East velocity, v	99.49	99.85
North velocity, u	99.07	99.84
Down velocity, w	99.07	99.72
Radius, r	99.53	99.63
Latitude, ϕ	99.33	99.44
Longitude, θ	99.47	99.39
Roll, Φ	99.51	99.99
Pitch, Θ	90.28	98.99
Yaw, Ψ	96.65	99.94

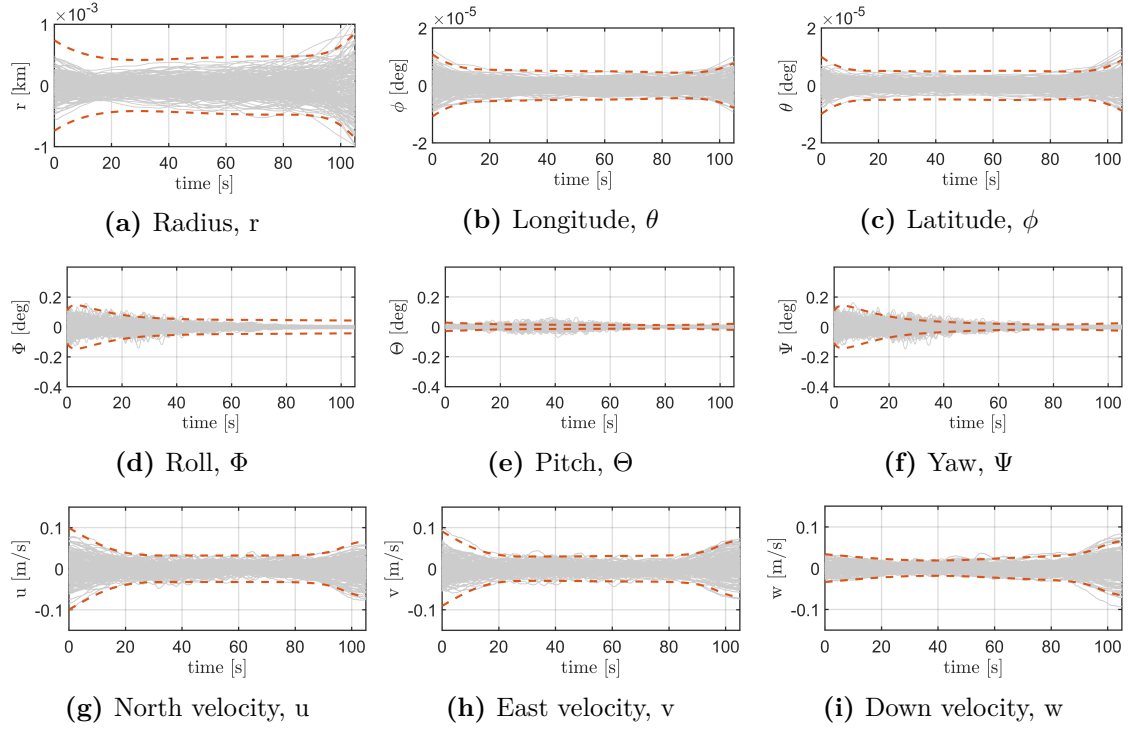


Figure 5.7. Monte Carlo campaign: S-EKF accuracy (orange lines) vs. estimation errors (grey lines).

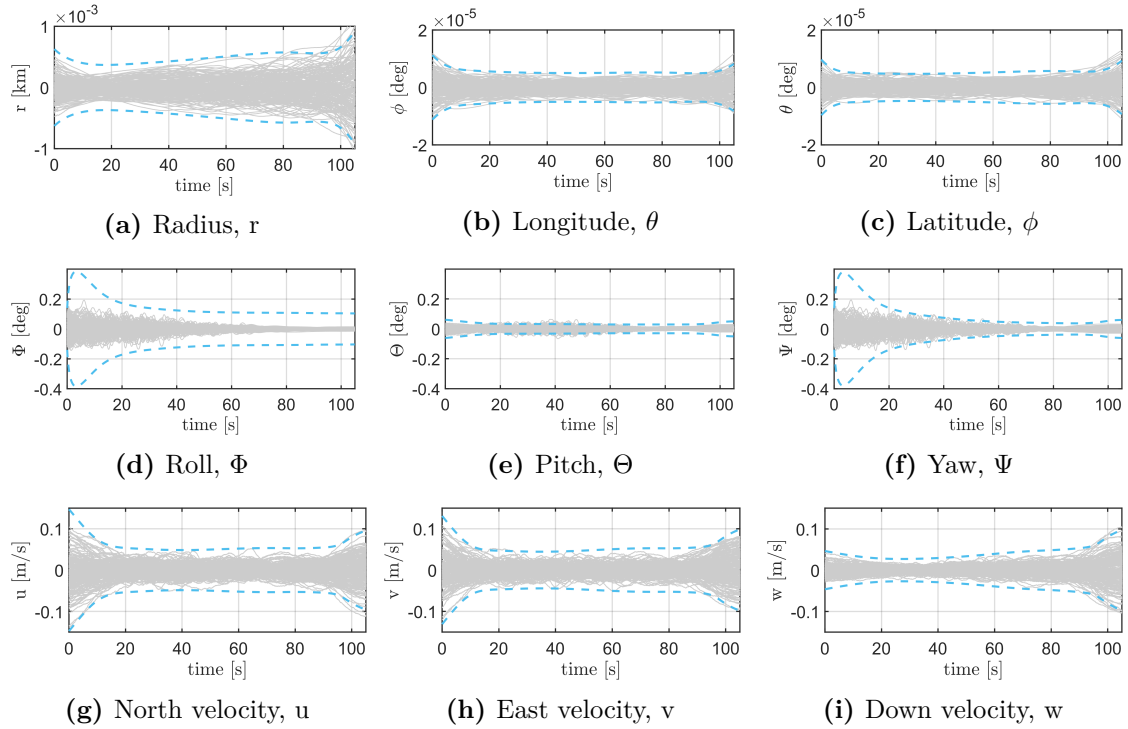


Figure 5.8. Monte Carlo campaign: S-UKF accuracy (blue lines) vs. estimation errors (grey lines).

Table 5.4. RMS of the residuals of the MC campaign.

Quantity	Time-average RMSE			
	EKF	UKF	S-EKF	S-UKF
u, mm/s	71.9	75.7	16.2	19.5
v, mm/s	56.3	61.0	15.1	18.3
w, mm/s	68.4	71.7	12.3	14.3
r, cm	34.5	35.6	17.4	17.9
ϕ , μ deg	48.5	49.8	2.12	2.08
θ , μ deg	81.1	83.3	1.88	1.96
Φ , mdeg	23.9	29.9	21.33	24.3
Θ , mdeg	26.2	27.4	9.19	10.7
Ψ , mdeg	28.1	36.0	21.2	24.9

Table 5.4 presents the RMS of the residuals for S-EKF and S-UKF, time-averaged over the whole set of simulations. Both S-EKF and S-UKF give solutions of high and comparable quality, as the values of RMSE in the two cases are similar, with a slight advantage of EKF. This result might also be related to the high accuracy of the IMU measurements, which, in turn, provide high accuracy in the *a priori* state estimate $\hat{\mathbf{x}}_k^-$ as well as in the filter initial conditions. Besides that, it is worth observing that the kinematic formulation of the filter is also designed to limit the nonlinearities in the process equations, particularly when compared to filters based on the dynamical model of the vehicle, where for instance the variation of aerodynamic force and moment is highly nonlinear. Also, the behavior of filter state variables in simulation is very smooth due to the action of the flight control system, which is highly effective in limiting the perturbations of attitude angles and lateral velocity components.

Chapter 6

Estimation of aerodynamic angles and wind components

This chapter delves into the estimation of aerodynamic angles and wind components for LVs during post-flight analysis using a Consider Multiplicative Extended Kalman Filter (CMEKF). The filter architecture models systematic IMU errors using random walk processes, while it compensates for wind velocity components and elastic rotations at the pivot point through a first-order Markov process model. Through Monte Carlo simulations performed in Sec. 6.5.1 the filter consistency, robustness, and accuracy are assessed, and the results are discussed in order to assess the filter performance.

6.1 Problem statement

The estimation of the freestream states and more specifically aerodynamic angles and wind components in post-flight analysis and/or real-time is a complex task for aerospace vehicles such as experimental flight vehicles [28], LVs [31] and re-entry vehicles [85]. Several methods to accomplish this task have been proposed, no one without drawback, (see Chap. 1 for further details) for the estimation of the freestream states in a scenario where the vehicles experience high Mach numbers in a large flight envelope and present high nonlinearity associated with the aerodynamic forces and moments.

The methodology employed in this study involves blending information such as the ADS measurement and a model of forces and moments of the LV. This strategy appears to be particularly effective for mitigating the impact of uncertainty in the aerodynamic model (modeled in the process of the filter and for the acceleration measurement model) through the utilization of air-relative sensor measurement. In

this work, the TAS has been adopted as an air-relative measurement for the filter.

The filter treats the components of airspeed as state variables, rather than directly estimating the aerodynamic angles. These angles are then determined from the estimated airspeed states. Specifically, a MEKF is employed for the attitude estimation, which, by neglecting higher-order terms, substitutes the four-component quaternion with a more manageable three-component error vector ideal for scenarios characterized by small attitude variations in time. The filter architecture incorporates first-order Gauss-Markov processes to model the wind velocity, capturing the wind speed profile variation in time. Furthermore, elastic rotations $\sigma_G(t)$ in the pitch and yaw planes, particularly at the nozzle gimbal location, are similarly characterized using these processes.

The LV angular rate components, influenced by the transversal thrust components generated by the TVC, oscillate at low frequencies (below 1 Hz), which in turn modulates the first elastic mode dynamics with a natural frequency of 1 Hz, as in Eq. (2.21). Estimation and subsequent correction of these oscillations are crucial for the accurate evaluation of the direction of the actual thrust axis.

Finally, random walk processes are featured to account for biases and scale factors in accelerometers and gyros, while the effects of misalignment and non-orthogonality errors are not accounted for in the filter architecture.

The formulation exploits the $\tilde{a}_{IMU_x}$ acceleration, the angular rate $\tilde{\omega}_{BI}$ from the IMU and the TVC commanded angles $\tilde{\delta}_y^{cmd}, \tilde{\delta}_z^{cmd}$ to consider them as inputs for the filter process model, while the remaining $\tilde{a}_{IMU_y}$ and $\tilde{a}_{IMU_z}$ components of the acceleration with the TAS, $\tilde{V}_T$ and GNSS position $\tilde{r}_{GNSS}$ and velocity $\tilde{v}_{GNSS}$ constitute the filter measurement vector $\mathbf{z}$ (see Fig. 6.1 where the logic scheme of the filter is shown). More information about the sensor error models is reported in Sec. 2.6.

The state vector $\mathbf{x}$ of the filter is

$$\mathbf{x}(t) = \left[u_a, v_a, w_a, u_{w[E]}, v_{w[E]}, w_{w[E]}, \sigma_{Gy}, \sigma_{Gz}, e_0, e_1, e_2, e_3, r, \phi, \theta, \beta_a, \beta_\omega, \lambda_a, \lambda_\omega \right]^T \quad (6.1)$$

where $[u_a, v_a, w_a]$ are the airspeed components in $\mathcal{F}_B$, $[u_w, v_w, w_w]_{[E]}$ are the wind components in $\mathcal{F}_E$ and σ_{Gy}, σ_{Gz} are the elastic pitch and yaw angular displacements at the gimbal location (see Chap. 2), respectively. The position vector of the LV cg is expressed in terms of geocentric radius r , latitude ϕ , and longitude θ . Next, the vectors $\beta_a = [\beta_{a,x}, \beta_{a,y}, \beta_{a,z}]^T$, $\lambda_a = [\lambda_{a,x}, \lambda_{a,y}, \lambda_{a,z}]^T$, $\beta_\omega = [\beta_{\omega,x}, \beta_{\omega,y}, \beta_{\omega,z}]^T$, and $\lambda_\omega = [\lambda_{\omega,x}, \lambda_{\omega,y}, \lambda_{\omega,z}]^T$ are, respectively, accelerometer biases and scale factors, and gyro biases and scale factors.

The input vector $\mathbf{u}$ is

$$\mathbf{u}(t) = [\tilde{\delta}_y^{cmd}, \tilde{\delta}_z^{cmd}, \tilde{\omega}_{BI}, \tilde{a}_{IMU_x}]^T$$

where a_x is the longitudinal component of the acceleration in $\mathcal{F}_B$ at cg location, and $\tilde{\delta}_y^{cmd} = \delta_y^{cmd} + w_\delta$ and $\tilde{\delta}_z^{cmd} = \delta_z^{cmd} + w_\delta$ are the TVC command angles in the pitch and yaw planes, respectively, affected by the noise $w_\delta \sim \mathcal{N}(0, \sigma_{w_\delta}^2)$, with $\sigma_{w_\delta} = \pm 0.1 \text{ deg}$. As usual, the notation $\mathcal{N}(0, \mathbf{Q})$ indicates a zero-mean white Gaussian noise characterized by the variance matrix $\mathbf{Q}$.

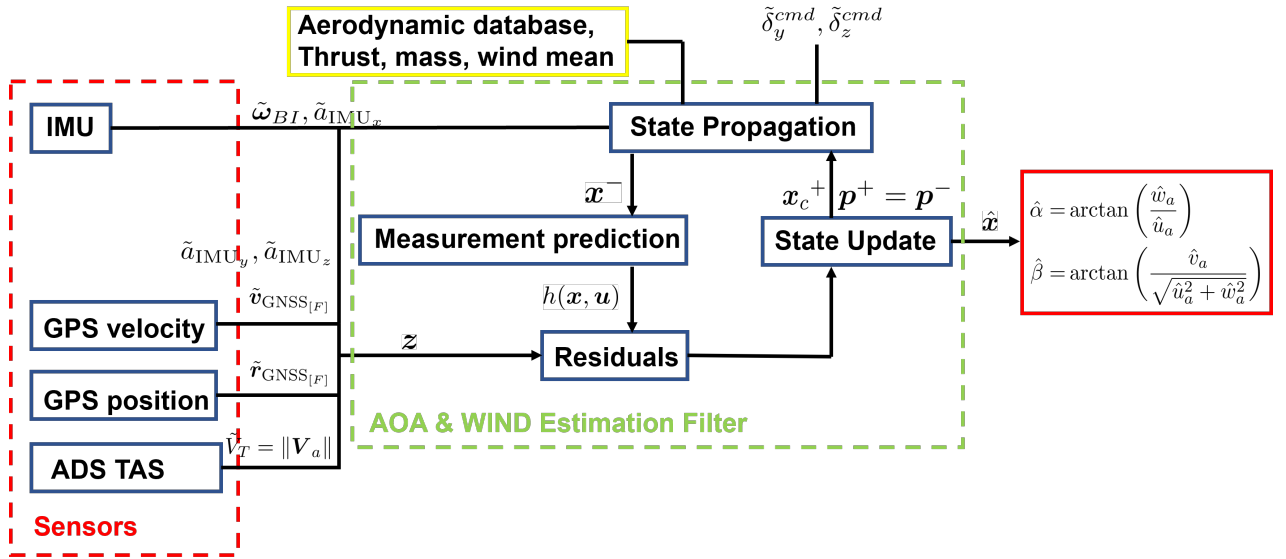


Figure 6.1. Sketch of the filter for the estimation of aerodynamic angles and wind components

6.2 Process equations

IMU systematic errors

The true angular rate ω_{BI} is expressed in terms of the measured one $\tilde{\omega}_{BI}$ provided by the rate gyros of the IMU, and reads as follows

$$\omega_{BI} = (\mathbf{I} + \mathbf{\Lambda}_\omega)^{-1}(\tilde{\omega}_{BI} - \beta_\omega) \quad (6.2)$$

while a_x is modeled as follow

$$\mathbf{a} = (\mathbf{I} + \mathbf{\Lambda}_a)^{-1}(\tilde{\mathbf{a}}_{IMU} - \beta_a) - (\omega_{BI} \times (\omega_{BI} \times \mathbf{d}_{IMU}) - \dot{\omega}_{BI} \times \mathbf{d}_{IMU}) \quad (6.3)$$

where measured acceleration and angular rate are expressed as $\tilde{\mathbf{a}}_{\text{IMU}} = \mathbf{a}_{\text{IMU}} + \boldsymbol{\eta}_a$ and $\tilde{\boldsymbol{\omega}}_{BI} = \boldsymbol{\omega}_{BI} + \boldsymbol{\eta}_\omega$ with $\boldsymbol{\eta}_\omega$ and $\boldsymbol{\eta}_a$ are zero mean Gaussian-white noise with spectral densities $\sigma_{w_\omega}^2 \mathbf{I}_{3 \times 3}$ and $\sigma_{w_a}^2 \mathbf{I}_{3 \times 3}$ respectively. $\dot{\boldsymbol{\omega}}_{BI}$ represents a smoothed finite-difference estimate of the actual angular acceleration, expressed as

$$\dot{\boldsymbol{\omega}}_{BI} = (\mathbf{I} + \boldsymbol{\Lambda}_\omega)^{-1} \dot{\tilde{\boldsymbol{\omega}}}_{BI} \quad (6.4)$$

which is derived from the measured angular velocity $\tilde{\boldsymbol{\omega}}_{BI}$ following the approach described in sec. 4.6. Accelerometer and gyro bias and scale factors $\beta_a, \beta_\omega, \lambda_a, \lambda_\omega$, respectively, are modeled as a zero-order Markov process

$$\dot{\beta}_a = \boldsymbol{\eta}_{\beta_a} \quad (6.5)$$

$$\dot{\beta}_\omega = \boldsymbol{\eta}_{\beta_\omega} \quad (6.6)$$

$$\dot{\lambda}_a = \boldsymbol{\eta}_{\lambda_a} \quad (6.7)$$

$$\dot{\lambda}_\omega = \boldsymbol{\eta}_{\lambda_\omega} \quad (6.8)$$

where $\boldsymbol{\eta}_{\beta_a} \sim \mathcal{N}(0, \mathbf{Q}_{\beta_a})$, $\boldsymbol{\eta}_{\beta_\omega} \sim \mathcal{N}(0, \mathbf{Q}_{\beta_\omega})$, $\boldsymbol{\eta}_{\lambda_a} \sim \mathcal{N}(0, \mathbf{Q}_{\lambda_a})$, $\boldsymbol{\eta}_{\lambda_\omega} \sim \mathcal{N}(0, \mathbf{Q}_{\lambda_\omega})$, and the noise covariance matrices are specified as $\mathbf{Q}_{\beta_a} = \text{diag}([\sigma_{\beta_{a,x}}^2, \sigma_{\beta_{a,y}}^2, \sigma_{\beta_{a,z}}^2])$, $\mathbf{Q}_{\beta_\omega} = \text{diag}([\sigma_{\beta_{\omega,x}}^2, \sigma_{\beta_{\omega,y}}^2, \sigma_{\beta_{\omega,z}}^2])$, $\mathbf{Q}_{\lambda_a} = \text{diag}([\sigma_{\lambda_{a,x}}^2, \sigma_{\lambda_{a,y}}^2, \sigma_{\lambda_{a,z}}^2])$, and $\mathbf{Q}_{\lambda_\omega} = \text{diag}([\sigma_{\lambda_{\omega,x}}^2, \sigma_{\lambda_{\omega,y}}^2, \sigma_{\lambda_{\omega,z}}^2])$.

Airspeed

In the filter process equations, the variations of the airspeed components are expressed as

$$\dot{u}_a = a_x - (w_a + w_w)(q + q_\oplus) + (v_a + v_w)(r + r_\oplus) - \dot{u}_w + \varpi_x + f_{g_x} \quad (6.9)$$

$$\dot{v}_a = a_y + (w_a + w_w)(p + p_\oplus) - (u_a + u_w)(r + r_\oplus) - \dot{v}_w + \varpi_y + f_{g_y} \quad (6.10)$$

$$\dot{w}_a = a_z - (v_a + v_w)(p + p_\oplus) + (u_a + u_w)(q + q_\oplus) - \dot{w}_w + \varpi_z + f_{g_z} \quad (6.11)$$

where

$$a_y = \frac{1}{m} \left(\frac{1}{2} \rho V_T^2 S C_{Y_\beta} \arctan \left(\frac{v_a}{\sqrt{u_a^2 + w_a^2}} \right) + \right. \quad (6.12)$$

$$\left. -T \sin(\tilde{\delta}_z^{cmd} + \sigma_{G_z} + \eta_{\delta_z}) \cos(\tilde{\delta}_y^{cmd} + \sigma_{G_y} + \eta_{\delta_y}) \right)$$

$$a_z = \frac{1}{m} \left(-\frac{1}{2} \rho V_T^2 S C_{N_\alpha} \arctan \left(\frac{w_a}{u_a} \right) + T \sin(\tilde{\delta}_y^{cmd} + \sigma_{G_y} + \eta_{\delta_y}) \right) \quad (6.13)$$

and $f_{g_x}, f_{g_y}, f_{g_z}$ are the components of the gravity acceleration in $\mathcal{F}_B$. Further details on the derivation of Eqs. (6.9)–(6.11) can be found in Appendix A.2.

LV position

The kinematic equations for the LV coordinates in $\mathcal{F}_E$ are written as follows

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{r \cos \phi} & 0 \\ \frac{1}{r} & 0 & 0 \end{bmatrix} \left(\mathcal{R}_{EB} \begin{bmatrix} u_a \\ v_a \\ w_a \end{bmatrix} + \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix}_{[E]} \right) \quad (6.14)$$

Attitude

The kinematics of quaternions in $\mathcal{F}_B$ is represented as follows

$$\begin{bmatrix} \dot{e}_0 \\ \dot{\mathbf{e}} \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} -\mathbf{e}^T \\ e_0 \mathbf{I} + [\mathbf{e} \times] \end{bmatrix} \right) \boldsymbol{\omega}_{BE} \quad (6.15)$$

$$\boldsymbol{\omega}_{BE} = \boldsymbol{\omega}_{BI} - \mathcal{R}_{BE} \boldsymbol{\omega}_{EI} \quad (6.16)$$

being $\boldsymbol{\omega}_{EI}$ the NED-to-ECI angular velocity in $\mathcal{F}_E$

$$\boldsymbol{\omega}_{EI} = [(\dot{\theta} + \omega_\oplus) \cos \phi, -\dot{\phi}, -(\dot{\theta} + \omega_\oplus) \sin \phi]^T \quad (6.17)$$

As shown in Sec. 4.2.3, to obtain the process equation for the attitude error, the error dynamics is rewritten in terms of $\delta \boldsymbol{\alpha} = 2\delta \mathbf{e}$, as

$$\delta \dot{\mathbf{e}}_0 = 0 \quad (6.18)$$

$$\delta \dot{\boldsymbol{\alpha}} = -[\hat{\boldsymbol{\omega}}_{BE} \times] \delta \boldsymbol{\alpha} + \delta \boldsymbol{\omega}_{BE} \quad (6.19)$$

At this point, Eq. (6.16) is considered for the expression of the error on angular rate $\delta\boldsymbol{\omega}_{BE} = \boldsymbol{\omega}_{BE} - \hat{\boldsymbol{\omega}}_{BE}$, where the matrix $\mathcal{R}_{BE}$ is associated with the one estimated through the attitude error, in the form

$$\mathcal{R}_{BE} = \mathcal{R}_{BE}(\delta\mathbf{q})\hat{\mathcal{R}}_{BE}(\mathbf{q}) \quad (6.20)$$

When the Euler's angles are sufficiently small the first-order approximation of the rotation matrix can be used, that is, $\mathcal{R}_{BE}(\delta\mathbf{q}) \approx \mathbf{I}_{3 \times 3} - [\delta\boldsymbol{\alpha} \times]$ [13]. By substituting the approximation $(\mathbf{I}_{3 \times 3} + \boldsymbol{\Lambda}_\omega)^{-1} \approx (\mathbf{I}_{3 \times 3} - \boldsymbol{\Lambda}_\omega)$ into Eq. (6.2) and assuming that scale factors are small enough, Eq. (6.19) becomes

$$\begin{aligned} \delta\dot{\boldsymbol{\alpha}} = & [(\mathbf{I}_{3 \times 3} - \hat{\boldsymbol{\Lambda}}_\omega)(\tilde{\boldsymbol{\omega}}_{BI} - \hat{\boldsymbol{\beta}}_\omega) \times] \delta\boldsymbol{\alpha} + \\ & - \left[\left(\underbrace{\begin{bmatrix} \tilde{\omega}_{BI,x} & 0 & 0 \\ 0 & \tilde{\omega}_{BI,y} & 0 \\ 0 & 0 & \tilde{\omega}_{BI,z} \end{bmatrix}}_{\hat{\boldsymbol{\Omega}}_\omega} - \underbrace{\begin{bmatrix} \hat{\beta}_{\omega,x} & 0 & 0 \\ 0 & \hat{\beta}_{\omega,y} & 0 \\ 0 & 0 & \hat{\beta}_{\omega,z} \end{bmatrix}}_{\hat{\mathbf{B}}_\omega} \right) (\boldsymbol{\lambda}_\omega - \hat{\boldsymbol{\lambda}}_\omega) + \\ & + (\mathbf{I}_{3 \times 3} - \hat{\boldsymbol{\Lambda}}_\omega)(\boldsymbol{\beta}_\omega - \hat{\boldsymbol{\beta}}_\omega) + (\mathbf{I}_{3 \times 3} - \hat{\boldsymbol{\Lambda}}_\omega)\boldsymbol{\eta}_\omega \right] - \hat{\mathcal{R}}_{BE}(\mathbf{q})(\boldsymbol{\omega}_{EI} - \hat{\boldsymbol{\omega}}_{EI}) \end{aligned} \quad (6.21)$$

obtaining the quaternion error dynamic (see Sec. 4.2.3 for more details).

Wind components

The time derivatives of the wind velocity components are represented as first-order Markov processes, where the steady-state wind profile is assumed known with W the mean and λ the wind direction. For the Down velocity component of the wind, the mean is considered equal to zero. Accordingly,

$$\begin{bmatrix} \dot{u}_w \\ \dot{v}_w \\ \dot{w}_w \end{bmatrix}_{[E]} = \begin{bmatrix} -\frac{1}{\tau_{wl}}(u_{w[E]} + W \sin(\pi/2 + \lambda)\eta_W) + \eta_{u_w} \\ -\frac{1}{\tau_{wl}}(v_{w[E]} + W \cos(\pi/2 + \lambda)\eta_W) + \eta_{v_w} \\ -\frac{1}{\tau_{wd}}(w_{w[E]}) + \eta_{w_w} \end{bmatrix} \quad (6.22)$$

with process noise that takes into account the uncertainty on the mean value of wind speed in the form $\eta_W \sim \mathcal{N}(0, Q_W)$, and the turbulence noise, namely, $\eta_{u_w} \sim \mathcal{N}(0, Q_{u_w})$, $\eta_{v_w} \sim \mathcal{N}(0, Q_{v_w})$ and $\eta_{w_w} \sim \mathcal{N}(0, Q_{w_w})$. In order to use the wind

acceleration in Eqs. (6.9)-(6.11), Eq.(6.22) is written in the $\mathcal{F}_B$ frame

$$\begin{bmatrix} \dot{u}_w \\ \dot{v}_w \\ \dot{w}_w \end{bmatrix} = \mathcal{R}_{BE} \begin{bmatrix} \dot{u}_w \\ \dot{v}_w \\ \dot{w}_w \end{bmatrix}_{[E]} + \boldsymbol{\omega}_{BE \times} \mathcal{R}_{BE} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix}_{[E]} \quad (6.23)$$

The constant values of correlation times τ_{wl} and τ_{wd} are obtained using the autocorrelation of the lateral (North and East) and Down components of the wind velocity, respectively. Figure 6.2 shows an example of the normalized autocorrelation function

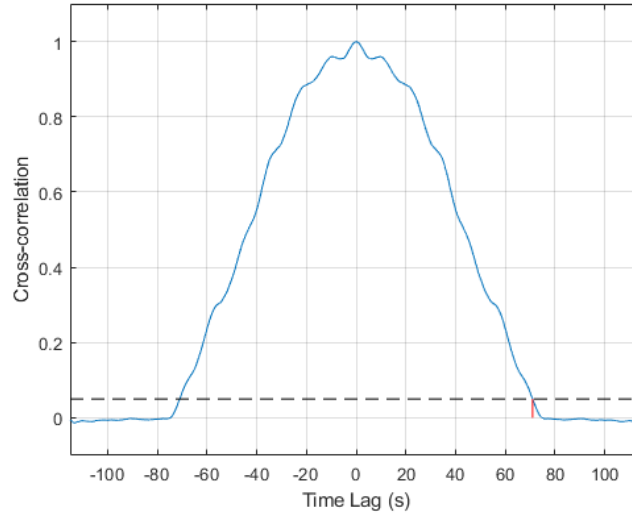


Figure 6.2. Normalized autocorrelation function of the North component of the wind velocity $u_{w[E]}$.

of the North component of wind speed. The correlation time is obtained as one-third of the time shift (marked in the figure by a dashed red line) where the function value is 0.05 [6]. This gives the same value, $\tau_{wl} = 71/3$ s, for the lateral components, and $\tau_{wd} = 28/3$ s for the downward component.

Elastic displacements at the gimbal

The last process equations deal with the angular displacements, due to the elastic DOFs at the gimbal point, which are modeled as first-order Markov processes, that is

$$\dot{\sigma}_{G_y} = -\frac{1}{\tau_{\sigma_y}} \sigma_{G_y} + \eta_{\sigma_{G_y}} \quad (6.24)$$

$$\dot{\sigma}_{G_z} = -\frac{1}{\tau_{\sigma_z}} \sigma_{G_z} + \eta_{\sigma_{G_z}} \quad (6.25)$$

The correlation times $\tau_{\sigma_y} = \tau_{\sigma_z} = 5/3$ s have been specified by a trial and error procedure, while $\eta_{\sigma_{G_y}} \sim \mathcal{N}(0, Q_{\sigma_{G_y}})$ $\eta_{\sigma_{G_z}} \sim \mathcal{N}(0, Q_{\sigma_{G_z}})$.

Summing up, the process noise vector $\boldsymbol{\eta} \in \mathcal{R}^{23}$ is

$$\boldsymbol{\eta} = \left[\eta_{a,x}, \boldsymbol{\eta}_\omega, \eta_\delta, \eta_W, \eta_{u_w}, \eta_{v_w}, \eta_{w_w}, \eta_{\sigma_{G_y}}, \eta_{\sigma_{G_z}}, \boldsymbol{\eta}_{\beta_a}, \boldsymbol{\eta}_{\beta_\omega}, \boldsymbol{\eta}_{\lambda_a}, \boldsymbol{\eta}_{\lambda_\omega} \right] \quad (6.26)$$

6.3 Measurement equations

Turning to the measurement model of the filter, the acceleration $\tilde{\mathbf{a}}_{\text{IMU}}$ provided by IMU is written as

$$\mathbf{a}_{\text{IMU}} = \mathbf{a} + \left(\boldsymbol{\omega}_{BI} \times (\boldsymbol{\omega}_{BI} \times \mathbf{d}_{\text{IMU}}) + \dot{\boldsymbol{\omega}}_{BI} \times \mathbf{d}_{\text{IMU}} + 2\boldsymbol{\omega}_{BI} \times \dot{\mathbf{d}}_{\text{IMU}} + \ddot{\mathbf{d}}_{\text{IMU}} \right) + \boldsymbol{\xi}_a \quad (6.27)$$

where only the y and z components are employed as measurements, and $\boldsymbol{\xi}_a \sim \mathcal{N}(0, \boldsymbol{\sigma}_{w_a}^2)$. Next, the terms $\dot{\mathbf{d}}_{\text{IMU}}$ and $\ddot{\mathbf{d}}_{\text{IMU}}$ are related to the sensor-shift in position due to the time variation of the cg location and the elastic DOFs.

$$a_y = \frac{1}{m} \left(\frac{1}{2} (P_1 + \xi_{P_1}) V_T^2 \arctan \left(\frac{v_a}{\sqrt{u_a^2 + w_a^2}} \right) + \right. \quad (6.28)$$

$$\left. -T \sin(\tilde{\delta}_z^{cmd} + \sigma_{G_z} + \xi_\delta) \cos(\tilde{\delta}_y^{cmd} + \sigma_{G_y} + \xi_\delta) \right) \quad (6.29)$$

$$a_z = \frac{1}{m} \left(-\frac{1}{2} (P_2 + \xi_{P_2}) V_T^2 \arctan \left(\frac{w_a}{u_a} \right) + T \sin(\tilde{\delta}_y^{cmd} + \sigma_{G_y} + \xi_\delta) \right) \quad (6.30)$$

The same model given by Eqs. (6.12)-(6.13) has been adopted for the a_y and a_z components of the true acceleration $\mathbf{a}$ at cg location, where the measurement noise has been added as shown in Eqs. (6.29) and (6.30) where, for the process equations the measurement noise on the nozzle commands ($\tilde{\delta}_y^{cmd}$ and $\tilde{\delta}_z^{cmd}$) is considered as $\xi_\delta \sim \mathcal{N}(0, \sigma_{w_\delta}^2)$. Also, a noise associated with the uncertain parameters

$$P_1 = \rho S C_{Y_\beta} \quad P_2 = \rho S C_{N_\alpha} \quad (6.31)$$

is considered, being $\xi_{P_1} \sim \mathcal{N}(0, R_{P_1})$, $\xi_{P_2} \sim \mathcal{N}(0, R_{P_2})$.

Note that in the present measurement model the terms $\dot{\mathbf{d}}_{\text{IMU}}$ and $\ddot{\mathbf{d}}_{\text{IMU}}$ are neglected, and the additive noises $\xi_{a,y}$ and $\xi_{a,z}$ are zero mean Gaussian-white noises with spectral density $\sigma_{w_a}^2$. They are used to take into consideration the effects of the (otherwise unmodeled) elastic dynamics.

The measurement equations of the GNSS sensor are written as follows

$$\begin{aligned} \mathbf{r}_{\text{GNSS}[F]} &= \mathbf{r}_{[F]} + \mathcal{R}_{FB} \boldsymbol{\delta}_{\text{GNSS}[B]} + \boldsymbol{\xi}_r \\ \mathbf{v}_{\text{GNSS}[F]} &= \mathcal{R}_{FE} \left(\mathcal{R}_{EB} \mathbf{v}_a + \mathbf{v}_w + [\boldsymbol{\omega}_{\oplus[E]} \times] \mathcal{R}_{EF} \mathbf{r}_{[F]} \right) + \mathcal{R}_{FB} [\boldsymbol{\omega}_{BE} \times] \boldsymbol{\delta}_{\text{GNSS}[B]} + \boldsymbol{\xi}_v \end{aligned} \quad (6.32)$$

where $\boldsymbol{\xi}_r \sim \mathcal{N}(0, \mathbf{R}_r)$ and $\boldsymbol{\xi}_v \sim \mathcal{N}(0, \mathbf{R}_v)$ have the known meaning.

The position vector of the LV cg in the frame $\mathcal{F}_F$ is

$$\mathbf{r}_{[F]} = r [\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi]^T \quad (6.33)$$

whereas $\boldsymbol{\omega}_{BE}$ is the vehicle angular rate with respect to the NED reference frame, with components in $\mathcal{F}_B$.

Note that $\mathcal{R}_{FB}$ is obtained as $(\mathcal{R}_{BE} \mathcal{R}_{EF})^T$, where $\mathcal{R}_{BE}$ is given by Eq. (2.58), and $\mathcal{R}_{EF}$ is a transformation matrix depending only on the LV latitude and longitude (see Eq. (5.22))

The true airspeed measurement equation is

$$V_T = \sqrt{u_a^2 + v_a^2 + w_a^2} + \xi_{V_T} \quad (6.34)$$

where $\xi_{V_T} \sim \mathcal{N}(0, \sigma_{w_T}^2)$. In conclusion, the measurement vector $\mathbf{z} \in \mathcal{R}^9$ and the measurement noise vector $\boldsymbol{\xi} \in \mathcal{R}^{12}$ are as follows

$$\mathbf{z} = [a_{\text{IMU},y}, a_{\text{IMU},z}, \mathbf{r}_{\text{GNSS}}, \mathbf{v}_{\text{GNSS}}, V_T] \quad (6.35)$$

$$\boldsymbol{\xi} = [\xi_{a,y}, \xi_{a,z}, \boldsymbol{\xi}_v, \xi_{V_T}, \xi_\delta, \xi_r, \xi_{P_1}, \xi_{P_2}] \quad (6.36)$$

6.4 Filter algorithm, initialization and tuning

The challenge in the estimation problem at hand, for the nonlinear dynamical system described in 6.2, is represented by the high number of additional states needed for dealing with the systematic errors of the IMU, i.e. β_a , β_ω , λ_a , and λ_ω . According to the Consider EKF (CEKF) formulation, these variables, heretofore recalled as parameters, are set to their nominal values and not updated, while only taken into account in the evaluation of estimation covariance (see Chap. 4), this choice is not purely due to computational burden, but allows the filter to handle situations where the covariance matrix becomes ill-conditioned or singular due to poor observability the parameters as in this case of study. In order to enhance accuracy, uncertain parameters in the model may be disregarded if their impact on

the state propagation is minimal. Alternatively, for improved precision, they can be treated as unknown values and estimated alongside other states. However, the latter approach increases the state dimension and consequently the computational burden. Since a MEKF is employed for the attitude estimation, the resulting filter algorithm is a Consider Multiplicative EKF (CMEKF) with the state vector partitioned as

$$\mathbf{x} = [\mathbf{x}_c, \mathbf{p}]^T \quad (6.37)$$

where

$$\mathbf{x}_c = [u_a, v_a, w_a, u_w, v_w, w_w, \delta\sigma_y, \delta\sigma_z, e_0, e_1, e_2, e_3, r, \phi, \theta, \lambda_{a,x}] \quad (6.38)$$

and

$$\mathbf{p} = [\beta_{a,x}, \beta_{a,y}, \beta_{a,z}, \beta_{\omega,x}, \beta_{\omega,y}, \beta_{\omega,z}, \lambda_{a,y}, \lambda_{a,z}, \lambda_{\omega,x}, \lambda_{\omega,y}, \lambda_{\omega,z}] \quad (6.39)$$

In the formulation of the process noise matrix reported in Tab. 6.1, it is worth

Table 6.1. Process noise covariance matrix $\mathbf{Q}$

diagonal index	symbol	Value	Units
1	/	$(9.81 \times 10^{-6})^2 \cdot \sigma_{a,x}^2$	$(\text{m/s})^2$
2 to 4	/	$\left(\frac{\pi}{180}\right)^2 \frac{1}{60} [\sigma_{w_\omega}^2, \sigma_{w_\omega}^2, \sigma_{w_\omega}^2]$	$(\text{rad}/\sqrt{\text{s}})^2$
5	/	$\left(\frac{\pi}{180}\right)^2 \cdot \sigma_{w_\delta}^2$	rad^2
6	Q_W	7^2	$(\text{m/s})^2$
7	Q_{u_w}	1^2	$(\text{m/s})^2$
8	Q_{v_w}	1^2	$(\text{m/s})^2$
9	Q_{w_w}	2.5^2	$(\text{m/s})^2$
10	$Q_{\delta\sigma_{Gy}}$	$(0.6 \times 10^{-3})^2$	rad^2
11	$Q_{\delta\sigma_{Gz}}$	$(1.8 \times 10^{-3})^2$	rad^2
12 to 14	Q_{β_a}	$(9.81 \times 10^{-6})^2 \frac{1}{60} [\sigma_{w_{b_a}}^2, \sigma_{w_{b_a}}^2, \sigma_{w_{b_a}}^2]$	$(\text{m/s})^2$
15 to 17	Q_{β_ω}	$\left(\frac{\pi}{180}\right)^2 \frac{1}{60^3} [\sigma_{w_{b_\omega}}^2, \sigma_{w_{b_\omega}}^2, \sigma_{w_{b_\omega}}^2]$	$(\text{rad/s}^{3/2})^2$
18 to 20	Q_{λ_a}	$[(0.1 \times 10^{-6})^2, (0.1 \times 10^{-10})^2, (0.1 \times 10^{-10})^2]$	$\square$
21 to 23	Q_{λ_ω}	$[(0.1 \times 10^{-10})^2, (0.1 \times 10^{-10})^2, (0.1 \times 10^{-10})^2]$	$\square$

recognizing the tuned components of the ones against the true ones. The process noise values are assigned knowing the PSDs of the IMU noise and the uncertainties inherent to the instruments. The performance characteristics of the IMU are typically evaluated through ground-based tests when it is mounted on the vehicle. This procedure leads to a rigorous and methodical adjustment of the filter parameters, which is based on a comprehensive understanding of existing disturbance processes. Such a systematic approach to tuning ensures that the filter performance is closely

aligned with the actual dynamical behavior of the system under observation.

The other process noise parameters are Q_W, Q_{u_w}, Q_{v_w} and Q_{w_w} for the wind, and are assigned taking into account the uncertainty of wind forecasts, while $Q_{\sigma_{G_y}}$ and $Q_{\sigma_{G_z}}$, along with the process noise corresponding to the nozzle angles δ_y and δ_z , is determined by considering the measurement uncertainty in the actuator outputs w.r.t. the commanded values. Additionally, a pre-flight Monte Carlo campaign can be carried out to establish bounds on the elastic rotation that may occur during flight, which may help in quantifying the expected variations and refining the process noise estimates to better reflect the actual flight conditions.

The covariance matrix of measurement noise $\mathbf{R}$, assumed diagonal with dimension coherent with Eq. (6.36), reads

$$\mathbf{R} = \text{diag}\left((9.81 \times 10^{-6})^2 \cdot \frac{\sigma_{a,y}^2}{T_s}, (9.81 \times 10^{-6})^2 \cdot \frac{\sigma_{a,z}^2}{T_s}, \sigma_{w_{v,x}}^2, \sigma_{w_{v,y}}^2, \sigma_{w_{v,z}}^2, \sigma_{w_T}^2, \left(\frac{\pi}{180}\right)^2 \cdot \sigma_{\sigma_\delta}^2, \sigma_{w_{r,w,x}}^2, \sigma_{w_{r,w,y}}^2, \sigma_{w_{r,w,z}}^2, R_{P_1}, R_{P_2}\right) \quad (6.40)$$

with $T_s = 0.005$, that is, the IMU sample time, and $R_{P_1} = R_{P_2} = 1.4^2$. Most of the nonzero elements of $\mathbf{R}$ correspond to the squares of the standard deviations of the noise in the measurement equations, with the σ values reported in Tab. 2.4. The initial error covariance is shown in Table 6.2. For biases and scale factors, the expected values are assumed to follow a uniform distribution, whereas the initial conditions are set to zero.

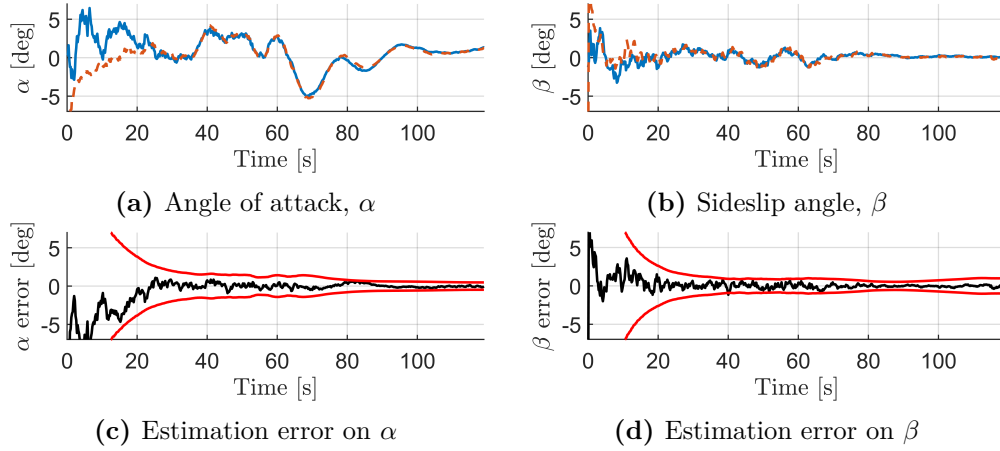
6.5 Results and discussion

In this section, the results obtained by considering the same mission already discussed in Chap. 5, are analyzed in order to assess the filter performance. Following the initial upward within the first 5 seconds of liftoff, there is a predetermined pitch-over rotation occurring between 5 and 10 seconds. This rotation initiates a ZLGT maneuver lasting from 10 to 120 seconds. During this phase, the rocket utilizes gravity to adjust its velocity trajectory while simultaneously minimizing the lateral aerodynamic force. In the simulation, the primary maneuver of the LV is executed in the pitch plane, and the direction of the wind is 44 degrees from the North. Figure 6.3 shows the results of the estimation process and, in particular, the true and estimated values of the aerodynamic angles (light-blue and orange lines, respectively) are shown in Figs. 6.3a and 6.3b. Figures 6.3c and 6.3d complement

Table 6.2. Initial error covariance matrix P_0

Diagonal Index	Value	Units
1 to 3	$[2, 2, 2]^2$	$(\text{m/s})^2$
4 to 6	$[2, 2, 2]^2$	$(\text{m/s})^2$
7 to 8	$[1 \times 10^{-4}, 1 \times 10^{-4}]^2$	$(\text{rad})^2$
9 to 11	$[1 \times 10^{-2}, 1 \times 10^{-2}, 1 \times 10^{-2}]^2$	$(\text{rad})^2$
12	6^2	m^2
13 to 14	$\frac{1}{4}[1 \times 10^{-5}, 1 \times 10^{-5}]^2$	$(\text{rad})^2$
15 to 17	$\left(\frac{2}{12}\right)^2 (1 \times 10^{-6})^2 [\beta_a _{t=0}^2, \beta_a _{t=0}^2, \beta_a _{t=0}^2]$	$(\text{m/s})^2$
18 to 20	$\left(\frac{2}{12} \frac{\pi}{180}\right)^2 \frac{1}{60^4} [\beta_\omega _{t=0}^2, \beta_\omega _{t=0}^2, \beta_\omega _{t=0}^2]$	$(\text{rad/s})^2$
21 to 23	$\left(\frac{2}{12}\right)^2 (1 \times 10^{-6})^2 [\lambda_{a,x}, \lambda_{a,y}, \lambda_{a,z}]$	$\square$
24 to 26	$\left(\frac{2}{12}\right)^2 (1 \times 10^{-6})^2 [\lambda_{\omega,x}, \lambda_{\omega,y}, \lambda_{\omega,z}]$	$\square$

this analysis by showing the estimation errors (black lines) and the 3- σ bounds (red lines), which are computed by taking the square root of the diagonal elements of the estimated covariance matrix.

**Figure 6.3.** Results of the CMEKF filter for the aerodynamic angles: true (light blue), estimated (orange), 3- σ bounds (red), and estimation error (black).

True and estimated variables, as well as estimation error and bounds, are shown in Fig. 6.4, for the North, East, and Down wind components. It is apparent that the filter is effective and provides adequate accuracy, most notably in the central portion of the flight. The accuracy of the aerodynamic angles estimation (Figs. 6.3a and 6.3b) is rather low in the first part of the simulation (about 12 s) when the LV velocity is low and significant variations of α and β are determined by relatively limited variations of wind speed. The estimation error also increases in the final

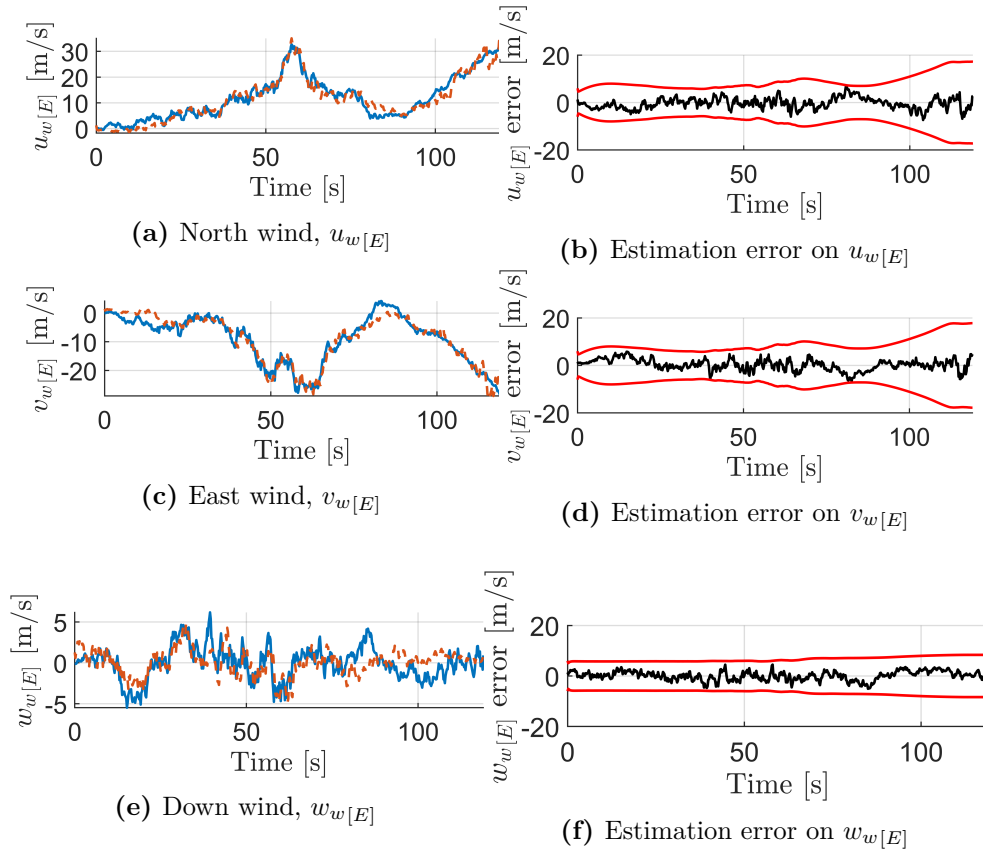


Figure 6.4. Results of the CMEKF filter for the wind components: true value (light blue), estimated (orange), 3- σ bounds (red), and estimation error (black)

part of the flight, particularly on the wind components (Figs. 6.4a and 6.4c) due, in this case, to the high speed of the vehicle (in comparison with wind speed) and the decreasing amplitude of aerodynamic actions being the air density very low at high altitude.

6.5.1 Consistency analysis

The assessment of filter consistency and performance is conducted through Monte Carlo simulations, where a scattering uniformly distributed within specified ranges is introduced across model parameters, IMU systematic error sources, and wind and atmospheric characteristics. As for the scattering of model parameters, specific attention is given to aerodynamic factors such as C_A , C_{N_α} and C_{Y_β} , that undergo variations of $\pm 10\%$ of their nominal values. The thrust (T) is scattered through a range of $\pm 2\%$, and, consequently, the LV mass is also scattered according to

$$\dot{m} = -\frac{T}{gI_{sp}} \quad (6.41)$$

where I_{sp} is the specific impulse. The time variations of thrust and mass are reported in Fig. 6.5b and Fig. 6.5c, where their nominal values (the data available to the filter) are depicted in black, and their scattered values are shown in gray. The scattering ranges for the model parameters are reported in Tab. 6.3 Furthermore, IMU biases and scale factors, atmospheric density (ρ), wind direction (λ), and mean wind speed (W), the latter shown in Fig. 6.5a (black line for the nominal value of mean wind and blue lines for its scattered values), are randomly specified within the ranges shown in Tab. 6.4.

Table 6.3. Scattering ranges for the Monte Carlo campaign.

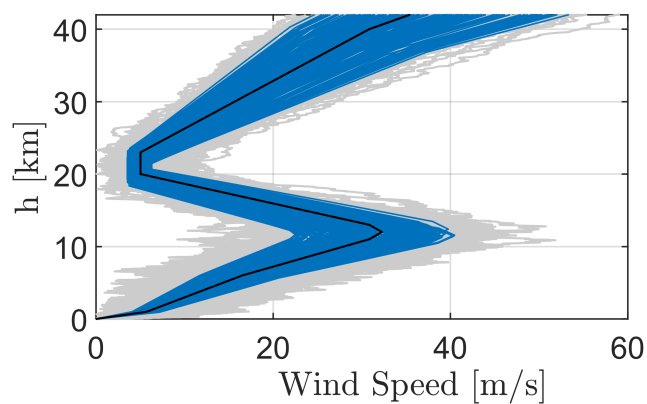
LV model parameters						
δ_y	δ_z	C_a	C_{N_α}	C_{Y_β}	T	m
± 0.1 deg	± 0.1 deg	$\pm 10\%$	$\pm 10\%$	$\pm 10\%$	$\pm 2\%$	follows the mass-flow rate Eq.(6.41)

Table 6.4. Scattering ranges for the Monte Carlo campaign.

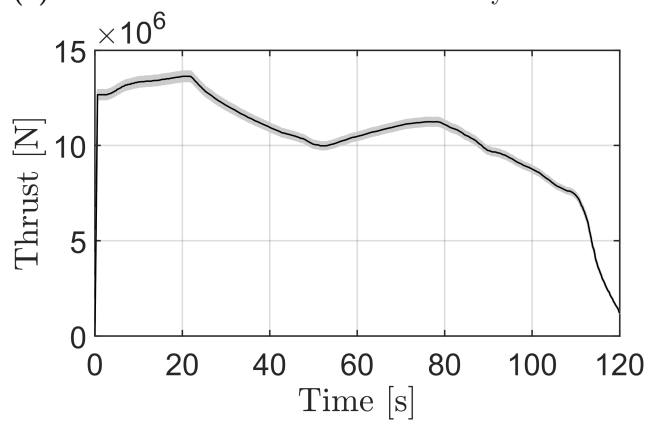
IMU and atmospheric parameters						
$\lambda_a _{t=0}$	$\lambda_\omega _{t=0}$	$\beta_a _{t=0}$	$\beta_\omega _{t=0}$	W	λ	ρ
± 50 ppm	± 30 ppm	$\pm 100 \mu g$	± 0.02 deg/h	$\pm 20\%$	± 180 deg	$\pm 5\%$

For the analysis, 200 simulation runs are conducted, with the same sampling frequency of 10 Hz for all sensors, in order to evaluate the consistency, robustness, and accuracy of the filter. Figure 6.6 shows the consistency of the airspeed velocity and wind components, whereas Fig. 6.7 shows the consistency of the attitude and position states. Figure 6.8 reports the consistency of the estimation of elastic rotations at the gimbal and scale factor on a_x . The continuous black lines represent the standard deviation of the estimation error as predicted by the filter, expressed as $3\text{-}\sigma$ values, while the dashed blue lines give the standard deviations of the errors calculated directly from the Monte Carlo samples, computed again as $3\text{-}\sigma$, the consistency of the filter being provided by the overlapping of these two lines. Finally, the continuous blue line corresponds to the mean of the samples.

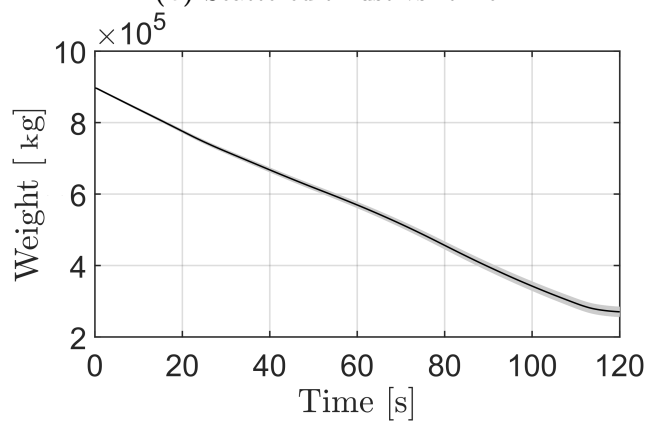
The expected value of the error is very close to zero, which means that the filter works as an unbiased filter, in line with the theoretical expectations for minimum mean square error (MMSE) estimators. It is noteworthy in Fig. 6.6 that the filter slightly underestimates the error covariance in the time range between 40 s and 60 s, while it overestimates the error covariance in the latter part of the flight (after 80 s). The former discrepancy may be caused by the dynamic pressure reaching its peak, causing uncertain aerodynamic terms to exert more influence on the force



(a) Scattered values of mean wind intensity vs. altitude



(b) Scattered thrust vs. time



(c) Scattered mass vs. time

Figure 6.5. Monte Carlo campaign: nominal values (black lines), scattered values (grey lines), scattered mean wind (blue lines).

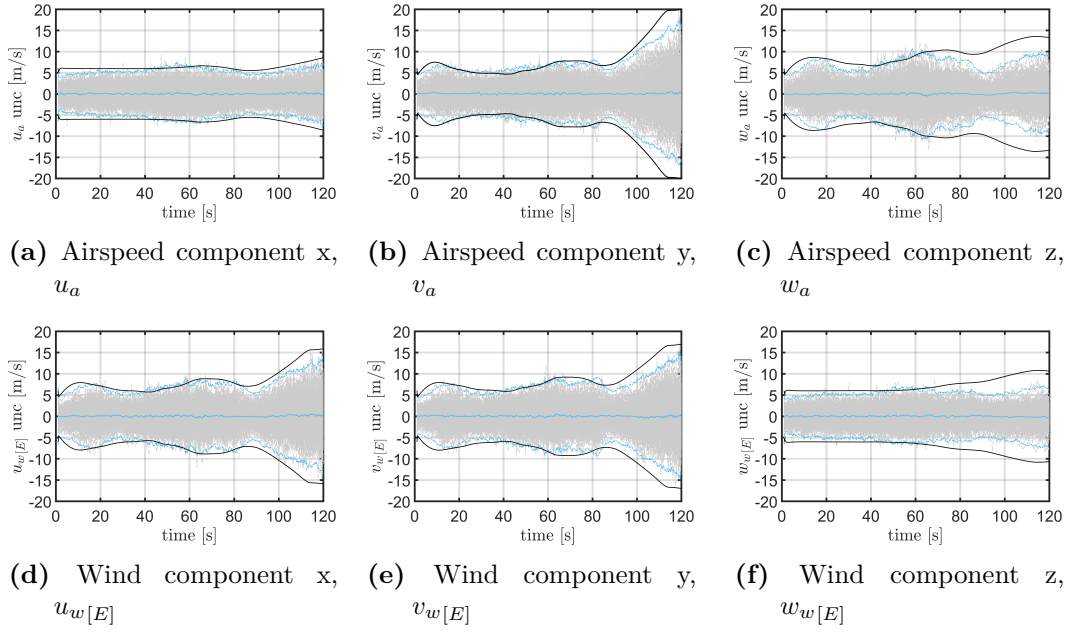


Figure 6.6. Monte Carlo campaign: 3σ covariance (black lines), estimation errors (grey lines), mean of MC samples (continuous blue lines), covariance of MC samples (dotted blue lines).

balance than other factors. On the other hand, in the last part of the flight, air turbulence is significantly reduced in comparison to the level in the troposphere and, consequently, the constant value of process noise adopted in this study proves to be inadequate to capture this effect. After approximately 80 s of flight, the low value of density makes the wind poorly observable, so that the covariance of the filter starts to increase. It is also apparent in Fig. 6.4e that the downward component of the wind is poorly estimated in the last part of the flight.

The results on the evaluation of attitude and position, reported in Fig. 6.7, show that the filter exhibits a very good consistency and convergence of the estimates. In particular, Fig. 6.7a shows that the bound of the samples is slightly underestimated between 80 s and 100 s, due to an inaccurate estimation of the bound of the scale factor $\lambda_{a,x}$ (reported in Fig. 6.8c). The errors in the states associated with elastic displacements, as illustrated in Fig. 6.8, indicate an overestimation of the filter covariance compared to the covariance of the samples. Nevertheless, these states allow the filter to deal with the related uncertainty in the estimation process. In particular, the approach to maintain them as states in the formulation of the CMEKF is seen to provide better results in comparison with treating them as parameters in the filter.

Finally, Fig. 6.9 shows the uncertainty in the estimation of aerodynamic angles, obtained *a posteriori*. It is apparent that the accuracy of the estimation increases

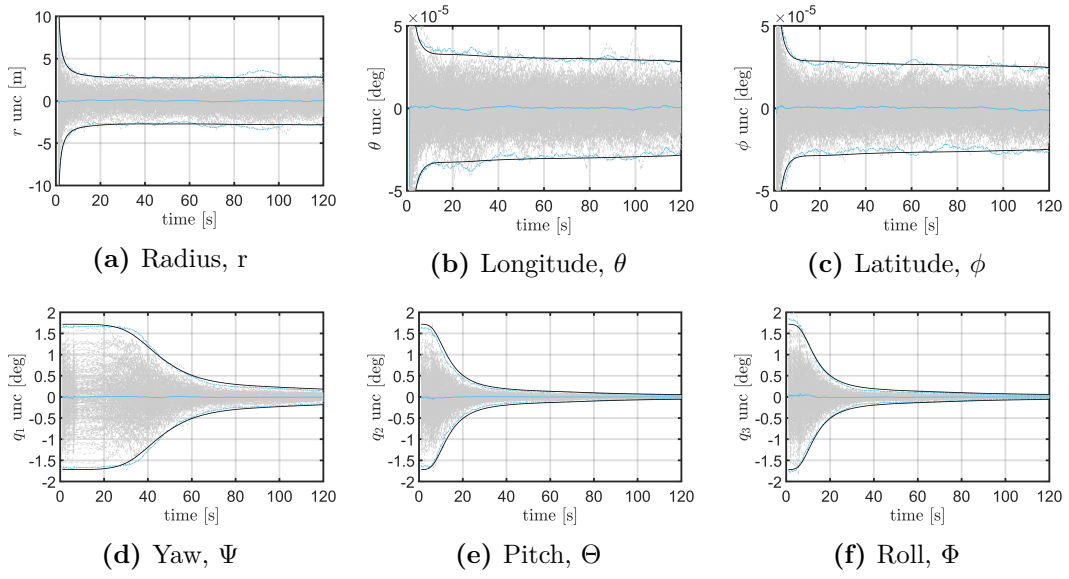


Figure 6.7. Monte Carlo campaign: 3σ covariance (black lines), estimation errors (grey lines), mean of MC samples (continuous blue lines), covariance of MC samples (dotted blue lines).

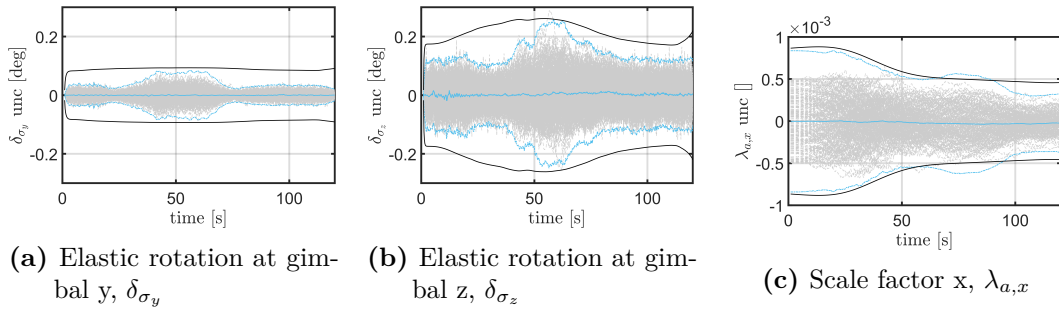


Figure 6.8. Monte Carlo campaign: 3σ covariance (black lines), estimation errors (grey lines), mean of MC samples (continuous blue lines), covariance of MC samples (dotted blue lines).

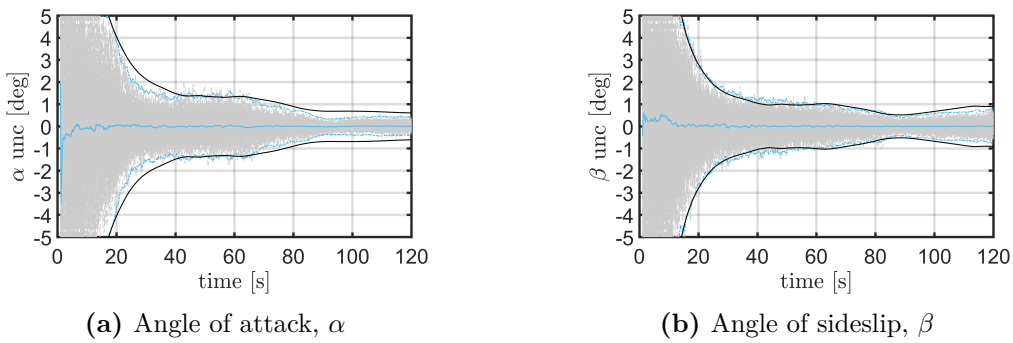


Figure 6.9. Monte Carlo campaign: 3σ covariance (black lines), estimation errors (grey lines), mean of MC samples (continuous blue lines), covariance of MC samples (dotted blue lines).

after about 20 s, while there are large errors in the initial phase of the flight because, when the longitudinal velocity is low, a discrepancy in the evaluation of the airspeed component induces large variations in the resulting values of α and β . It is worth observing that the uncertainty on the estimation of α decreases for the whole duration of the considered phase of flight, while the plot of the uncertainty on β shows that the estimation is less accurate from a flight time of about 90 s due to growing errors on the estimation of the lateral airspeed (v_a).

Table 6.5. RMSE on estimated variables for $t > 10$ s (gravity turn phase).

Aerodynamic angles, airspeed and wind velocity components								
α [deg]	β [deg]	α_{tot} [deg]	u_a [m/s]	v_a [m/s]	w_a [m/s]	$u_{w[E]}$ [m/s]	$v_{w[E]}$ [m/s]	$w_{w[E]}$ [m/s]
0.51	0.82	0.96	2.09	2.27	2.68	2.90	2.14	2.10

A better understanding of these trends can be obtained by evaluating the filter performance in terms of root-mean-square error (RMSE). In particular, Tab. 6.5 shows the RMSE values for α , β , $\alpha_{\text{tot}} = \sqrt{\alpha^2 + \beta^2}$, airspeed components u_a, v_a, w_a and wind velocity components $u_{w[E]}, v_{w[E]}, w_{w[E]}$, computed as

$$x_{\text{RMSE}} = \sqrt{\frac{1}{t_f} \int_0^{t_f} x^2(t) dt} \quad (6.42)$$

where $x(t)$ stays for the variable of interest.

In terms of the estimation of aerodynamic angles (α_{tot}), the results indicate adequate accuracy, with a maximum RMSE of approximately 0.96 degrees. It is worth observing that the estimation of the vertical wind component demonstrates a slightly higher accuracy compared to v_{w_N} and v_{w_E} because it consistently exhibits a lower RMSE this is probably due to the limited steady-state variation of the vertical wind component with altitude, in contrast to the other two components that show altitude-dependent variations. Note that the evaluation of the RMSE does not include the initial 10 seconds of simulation (from lift-off to the end of pitch-over) because of the uneven and large variations of the aerodynamic angles associated with the low speed of the vehicle. Table 6.6 reports the RMSE values of both position

Table 6.6. RMSE on estimated variables for $t > 10$ s (gravity turn phase) .

Position and attitude					
Ψ [deg]	Θ [deg]	Φ [deg]	r [m]	θ [deg]	ϕ [deg]
0.228	0.176	0.124	0.770	1.331×10^{-5}	7.755×10^{-6}

and attitude errors. Compared to the initial attitude uncertainty, with a $1\text{-}\sigma$ value

of 0.57 deg, a significant reduction in attitude error, measured in terms of RMSE, is obtained by the filtering process. In particular, during the gravity turn phase, there is a reduction of 60%, 70%, and 80% for yaw, pitch, and roll, respectively. Also, the range error is maintained below 1 m as a result of the filtering process. Considering that the GNSS measurement has a precision of approximately 6 m for each component, and a tenfold improvement in position accuracy is apparent.

Finally, measurement residuals are shown in Fig. 6.10. Grey lines represent the residual errors in measurement estimations $\Delta \mathbf{y}_k$ (see Eq.(4.27)), while the red curves are the RMS values of $\pm 3\sqrt{\text{diag}(\mathbf{S})}$ (see Eq.(4.25)), calculated at any time step over the 200 Monte Carlo simulations indicates the diagonal elements of the measurement residual covariance matrix $\mathbf{S}$. It is apparent that each vector of residuals is contained within the bounds imposed by the measurement residual covariance matrix.

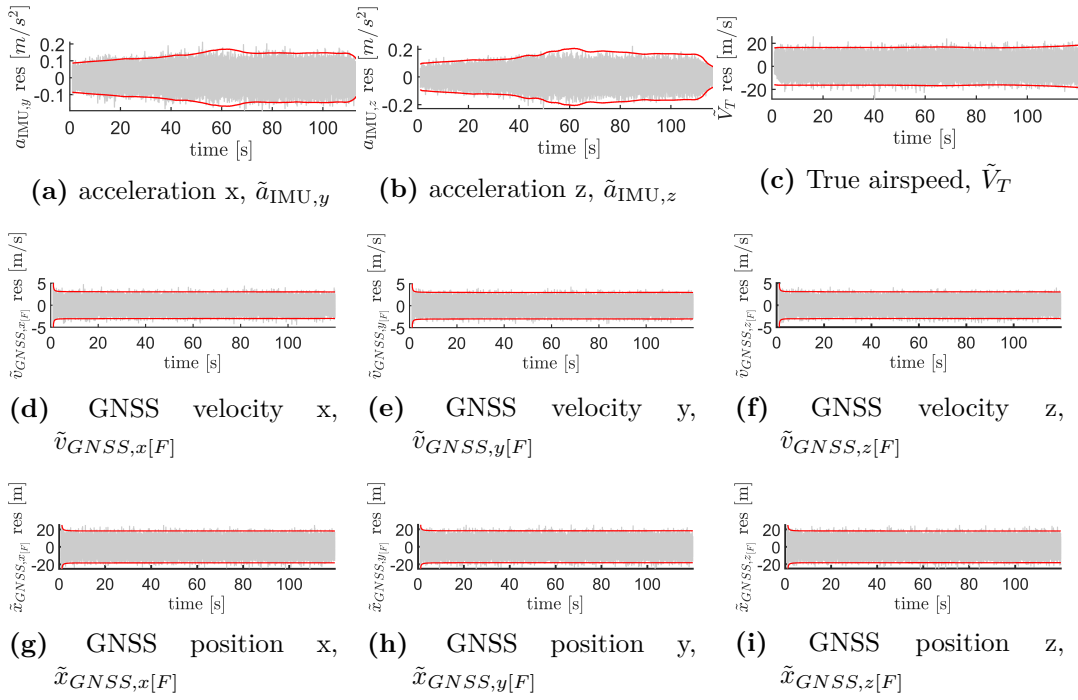


Figure 6.10. Monte Carlo campaign: 3σ measurement residual covariance (red lines), measurement residuals (grey lines).

Chapter 7

Conclusions

7.1 Main achievements

The thesis is focused on the analysis of methodologies to be applied in postflight analysis for the first stage flight of a Launch Vehicle (LV) with the following major objectives

- To analyze the limitations, and explore opportunities for enhancing the performance of the classical methods for the best-estimated trajectory (BET), providing improved suitability, robustness, scalability, and extendability of TR codes
- To explore a novel approach for aerodynamic angles and wind components estimation, by the use of post-flight data.

7.1.1 LV modeling

The initial phase of the work involved LV modeling to accurately represent its dynamic behavior from lift-off to the separation of the first stage. The development of the simulation environment for the Ares-I LV provided the framework for this part of the study because simulating the atmospheric phase of flight generates synthetic data essential for validating the post-flight estimation algorithms developed throughout this thesis.

The Ares-I model is characterized by six rigid degrees of freedom (DOFs), the elastic DOFs, and the effect of the inertial coupling between the base vehicle (vehicle without the engine) and the engine dynamic. The elastic DOFs have been represented by the first three bending modes that are solutions of a free vibration of Euler-Bernoulli beam representing the LV with non-uniform distributions of mass and inertia along the LV axis. The effects of sloshing dynamics have been neglected,

as the first stage of the considered LV is a solid rocket motor. The LV model was developed using a Lagrangian formulation, which is often used to derive the equations of motion of mechanical systems. A "quasi-Lagrangian" formulation is adopted which allows write the governing equations in body axes. The sensor measurement models for the Inertial Measurement Unit (IMU), Global Navigation Satellite System (GNSS), Radar for positioning, and Air Data System (ADS) are included in the model. A realistic model of wind from the launch pad to the end of the flight is also considered in the simulation homework.

7.1.2 Best estimated Trajectory

An approach based on a forward and backward estimation merged by the use of a Fraser-Potter smoother has been taken into consideration in this work for the BET. In particular, an Extended Kalman filter (EKF) and an Unscented Kalman Filter (UKF) have been investigated for this kind of application. The smoothed UKF (S-UKF), shows performance comparable to those obtained with other state-of-the-art methods described in the literature, such as the NASA code NewSTEP, which relies on a smoothed EKF (S-EKF). In particular, when the estimate of the initial state is very accurate, that is, the initial covariance $\mathbf{P}_0$ is small and the initial error on the state estimate is also minimal, S-EKF and S-UKF return very close solutions with a slight advantage of EKF in terms of both improved estimation accuracy and reduced uncertainty. In fact, the UKF relies on sampling a limited number of points (sigma points) to represent the state distribution. When the state covariance is small, the points are close to each other and the a-priori estimate does not show any significant difference from that obtained using the Riccati equation in the EKF for the propagation of the covariance.

Consistency analysis of the S-UKF filter appears satisfactory, whereas using the actual tuning of the EKF-based algorithm might underestimate error bounds, particularly on the attitude angles. As for the effects of initial conditions, the UKF appears more effective in terms of faster convergence when dealing with large initial estimation errors, while the two algorithms give almost identical performance when the covariance of initial estimates is erroneously determined.

In the current implementation, the UKF is significantly more computationally expensive than the EKF, as the former requires a computational time of about one order of magnitude larger than the latter to process the same raw measurements. This is mostly due to the augmented formulation here employed, and improvements are expected with a more efficient implementation of the algorithm. Eventually, the UKF demonstrates performance equal or superior to existing methods, such

as those employed by NASA’s NewSTEP code. Notably, the application of UKF may be promising when dealing with trajectory reconstruction where large off-nominal conditions are encountered. In these situations, the higher-order propagation provided by UKF through the sigma points should prove effective, offering faster convergence and robustness to initial condition inaccuracies, in spite of its higher computational load compared to the EKF.

7.1.3 Estimation of aerodynamic angles and wind components

A filtering technique based on a Consider Multiplicative Extended Kalman filter (CMEKF) has been presented with the aim of estimating the aerodynamic angles and the atmospheric wind components during the LV flight. This formulation takes into account systematic errors of the IMU (selected as *parameters* for the CMEKF), considering the airspeed components in $\mathcal{F}_B$ as state variables instead of the aerodynamic angles and employing a multiplicative error quaternion for attitude estimation to improve the accuracy over the additive formulation.

Four significant sources of uncertainty, have been taken into consideration, that is, i) the deviation of thrust direction due to elastic effects at the gimbal location, ii) the influence of non-collocated IMU and GNSS sensors, iii) the presence of IMU systematic errors, such as bias and scale factors, and iv) large discrepancies between pre-flight forecast and in-flight values of the magnitude and direction of the wind.

The results show the suitability of the filter in a simulation scenario with high-turbulence intensity and scattered (yet close-to-nominal) flight conditions. A Monte Carlo analysis has been carried out, scattering a considerable amount of structural, aero-propulsive, and flight parameters, to comprehensively assess their impact on the filter performance. In this respect, the results are encouraging as the proposed filter formulation provides an effective tool to estimate LV aerodynamic angles and wind components by using a small subset of commonly used pre-flight data together with the in-flight measurements. In particular, the wind intensity and direction are estimated with satisfying accuracy, in spite of the large variations of the specified nominal profile.

As for the aerodynamic angles, the results show high uncertainty in the initial phase of flight, when the errors in the airspeed estimation are large in comparison with the LV velocity. At the following times, the estimation converges at least up to altitudes where the effect of wind is negligible in comparison with the LV flight speed.

7.2 Future development

Future developments on the work of this dissertation will focus on enhancing the BET, with a special emphasis on improving attitude estimation through the implementation of filtering techniques such as the Multiplicative Extended Kalman Filter and the Unscented Quaternion Estimator as key tools to better account for systematic error sources of the gyro and in all cases where attitude measurements are not available. Also, adopting the UKF for estimating aerodynamic angles and wind components can more effectively address the nonlinearities associated with external aerodynamic forces and thrust moments, compared to its application for the BET, where nonlinearities in the kinematic equations are less relevant than in the modeling of forces and moments.

In the realm of system identification, in spite of the use of aerodynamic forces and moments in tabular form as a function of Mach as actually used for modeling LV, an analytical modeling approach would allow for the development of filters that not only accommodate but also correct or estimate these forces, for example including aerodynamic model parameters as states of the system. This can improve the current strategy of treating model parameters solely as uncertainties for the estimation of aerodynamic angles and wind components.

In the end, an improvement in estimation techniques to enhance both accuracy and efficiency is to make all the equations for the aerodynamic angles and wind filter non-dimensional, as for the BET filter. This approach should simplify the application of smoothing techniques, where, due to the different nature of the states currently considered in the filter, and the different magnitude of estimation errors, problems in inverting the error covariance matrix may arise.

Appendix A

Appendix

A.1 Model Data

The locations of the AP and c_p are related by Eq. A.1:

$$l_{x_{cp}} = l_{x_{AP}} - \frac{(C_{m_{\alpha_0}} + C_{m_{\alpha}}\alpha)b}{C_{N_0} + C_{N_{\alpha}}\alpha} \quad (\text{A.1})$$

where, due to symmetry,

$$\begin{aligned} C_{m_{\alpha_0}} &= 0 \\ C_{N_0} &= 0 \end{aligned}$$

AP location is not fixed but close to the considered constant value (≈ 83 m in $\mathcal{F}_S$) by Ref. [52]. Simulations have been performed considering both AP fixed and not, and results do not show significant differences. About LV mass variation, it is modeled according to

$$\dot{m} = -\frac{T}{gI_{sp}} \quad (\text{A.2})$$

where I_{sp} is the specific impulse. The time variations of thrust and I_{sp} are reported in Fig. A.13 and Fig. A.14.

A.2 Force equation

The momentum equation for the LV is written as

$$m([\dot{\mathbf{v}} + (\boldsymbol{\omega}_{BI} + \mathcal{R}_{BF}\boldsymbol{\omega}_{\oplus}) \times \mathbf{v}] + \mathcal{R}_{BF}[\boldsymbol{\omega}_{\oplus} \times (\boldsymbol{\omega}_{\oplus} \times \mathbf{p}_{[F]})]) = \mathbf{f}_{ext} \quad (\text{A.3})$$

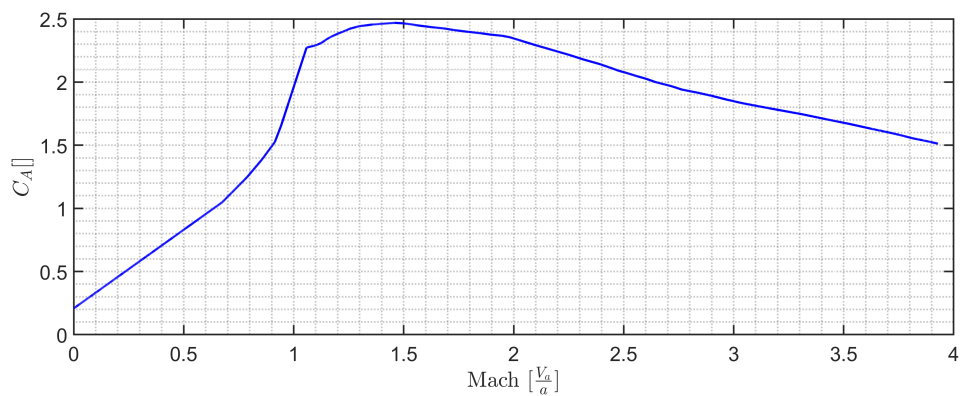


Figure A.1. Axial force coefficient vs Mach

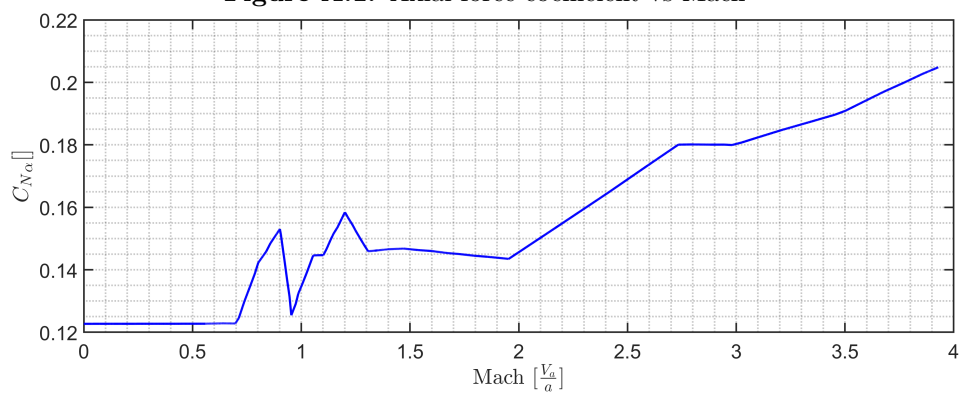


Figure A.2. slope of normal force coefficient vs Mach

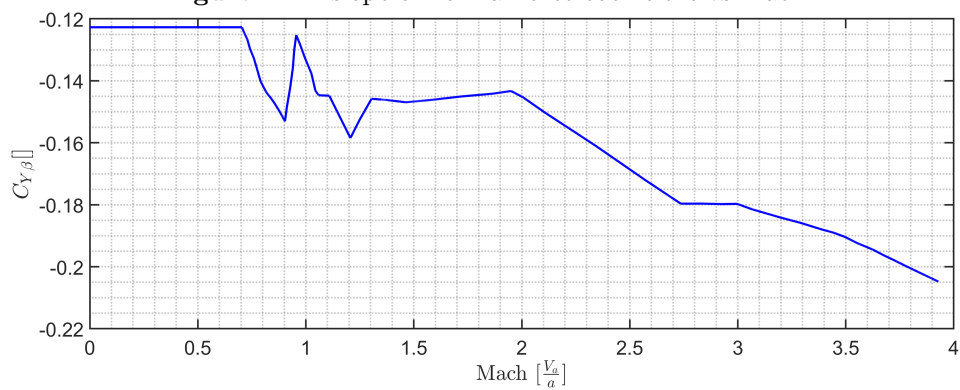


Figure A.3. slope of side force coefficient vs Mach

Figure A.4. Aerodynamic model data

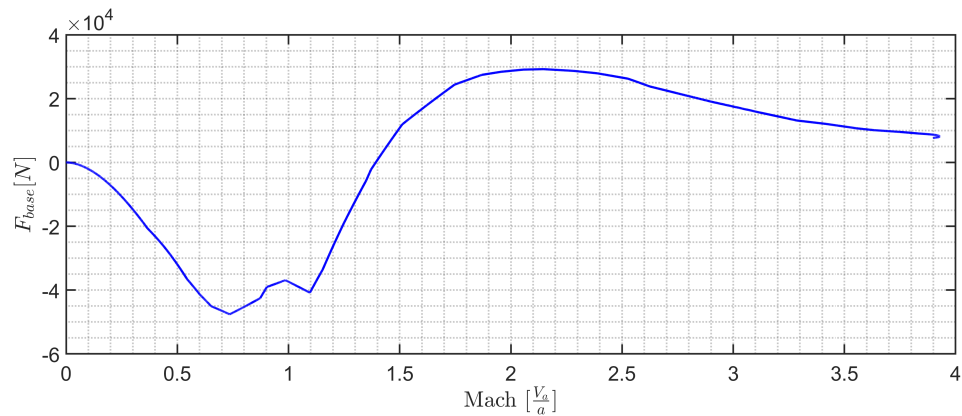


Figure A.5. Base force vs Mach

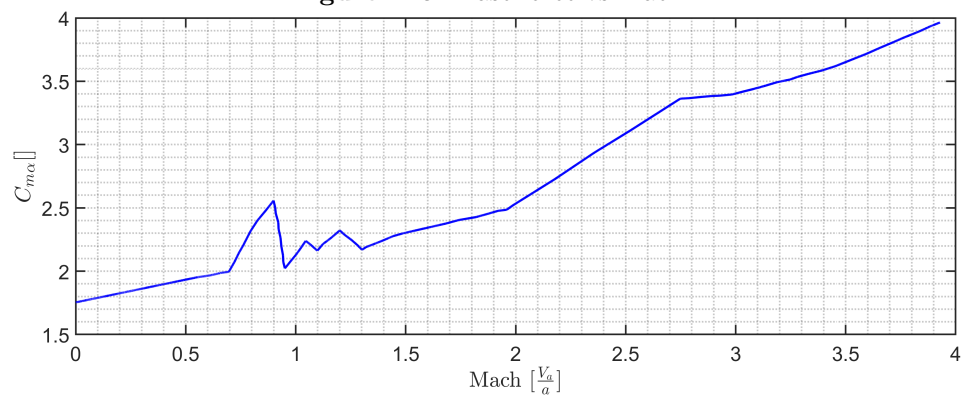


Figure A.6. Pitch moment coefficient vs Mach

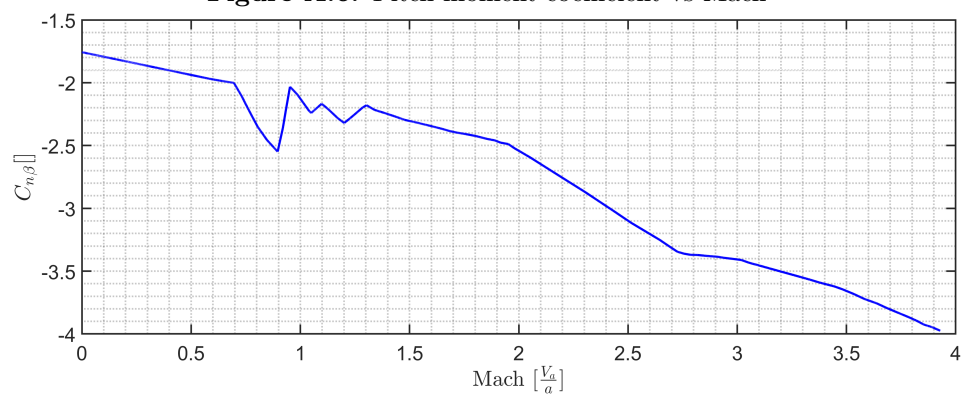


Figure A.7. Yaw moment coefficient vs Mach

Figure A.8. Aerodynamic model data

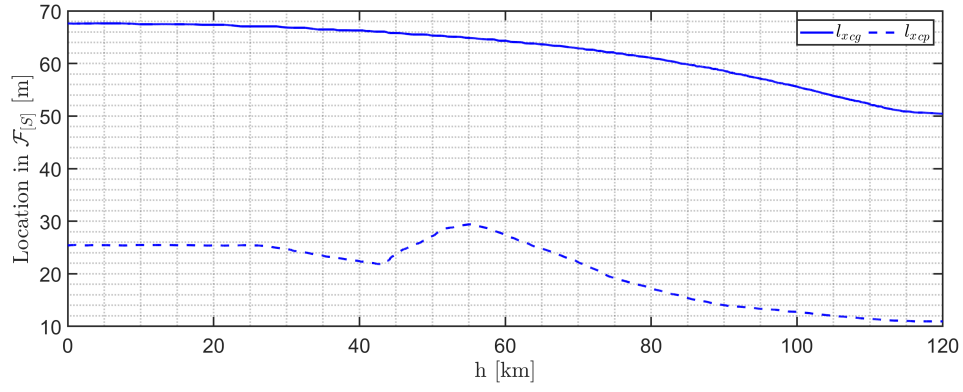


Figure A.9. cg and center of pressure vs time

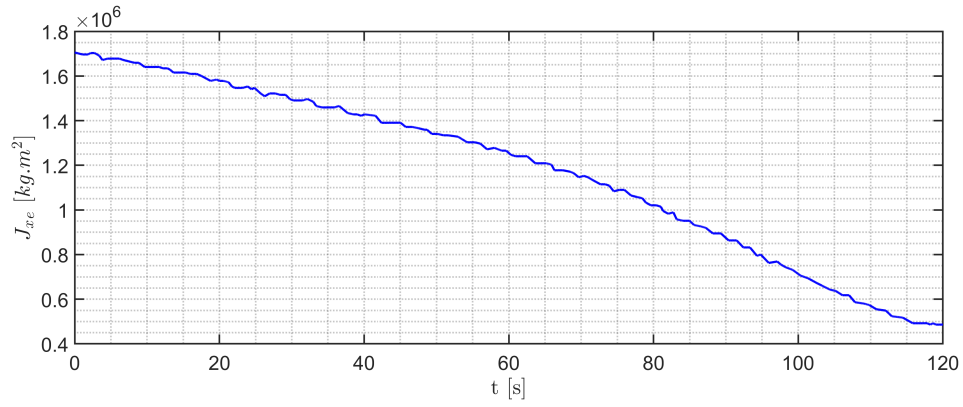


Figure A.10. Moment of inertia w.r.t. x-axis

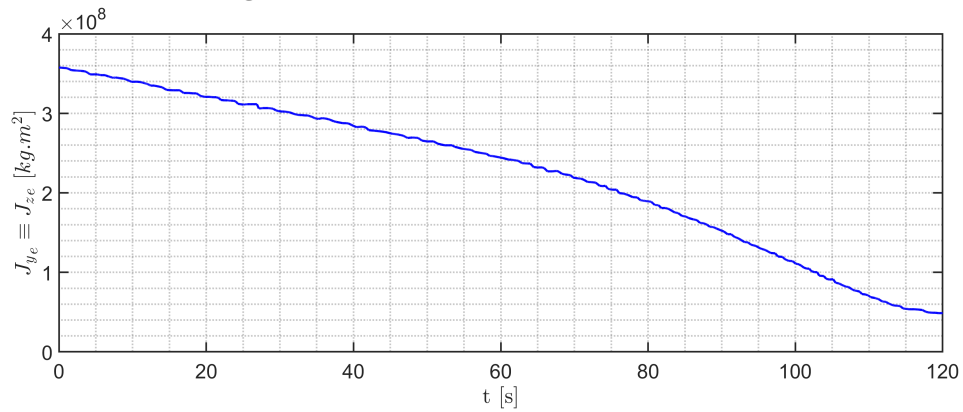


Figure A.11. Moment of inertia w.r.t. y and z axes

Figure A.12. Moments of Inertia, cg and c_p time varying location

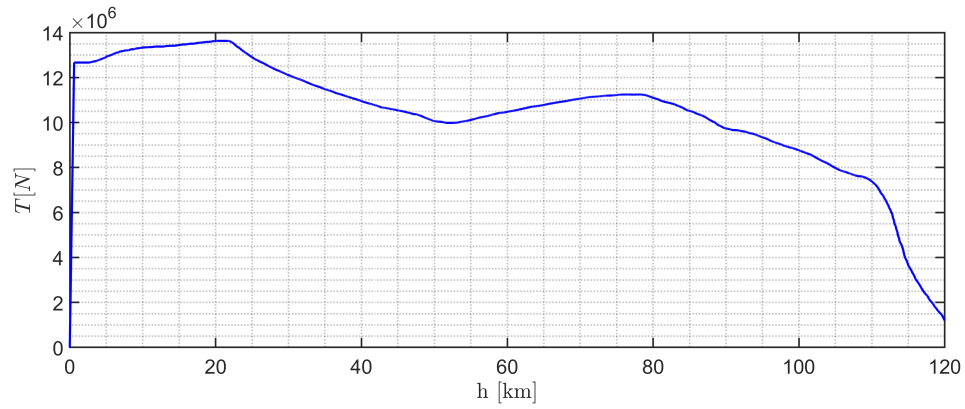


Figure A.13. Thrust force vs time

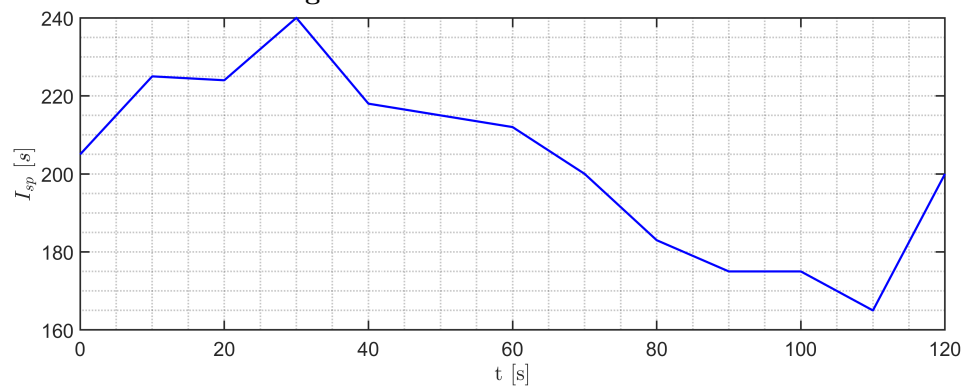


Figure A.14. Specific impulse vs time

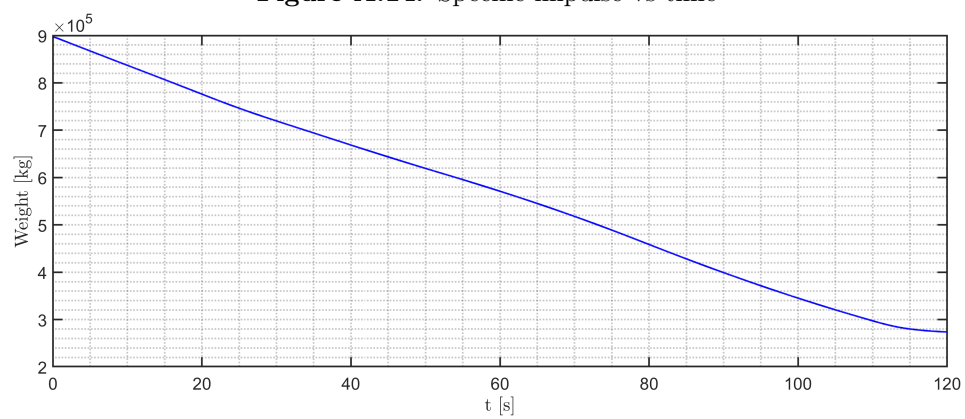


Figure A.15. Weight vs time

Figure A.16. Thrust and specific Impulse

where $\mathbf{f}_{ext} = \mathbf{F}_g + \mathbf{T} + \mathbf{F}_A$, $\mathbf{v} = \mathbf{V}_a + \mathbf{v}_w$, and the wind velocity vector in $\mathcal{F}_B$ is expressed as

$$\mathbf{v}_w = \mathcal{R}_{BE} \begin{bmatrix} u_w \\ v_w \\ w_w \end{bmatrix}_{[E]} \quad (\text{A.4})$$

Equation (A.3) is rewritten in terms of nongravitational acceleration at cg location ($\mathbf{a}$) as

$$\dot{\mathbf{v}} = \mathbf{a} - (\boldsymbol{\omega}_{BI} + \mathcal{R}_{BF}\boldsymbol{\omega}_{\oplus}) \times \mathbf{v} - \mathcal{R}_{BF}[\boldsymbol{\omega}_{\oplus} \times (\boldsymbol{\omega}_{\oplus} \times \mathbf{r}_{[F]})] + \mathbf{f}_g \quad (\text{A.5})$$

By expressing $\dot{\mathbf{v}}$ in terms of wind velocity, Eq. (A.5) reads

$$\dot{\mathbf{V}}_a = -\dot{\mathbf{v}}_w + \mathbf{a} - (\boldsymbol{\omega}_{BI} + \mathcal{R}_{BF}\boldsymbol{\omega}_{\oplus}) \times \mathbf{V}_a - (\boldsymbol{\omega}_{BI} + \mathcal{R}_{BF}\boldsymbol{\omega}_{\oplus}) \times \mathbf{v}_w + \boldsymbol{\varpi} + \mathbf{f}_g \quad (\text{A.6})$$

where $\boldsymbol{\varpi} = -\mathcal{R}_{BF}[\boldsymbol{\omega}_{\oplus} \times (\boldsymbol{\omega}_{\oplus} \times \mathbf{r}_{[F]})]$ and $\mathbf{f}_g$ the gravity acceleration.

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