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## Greenberg functors and jumps of tori

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# Preface

This thesis comprises two parts, both pertaining to arithmetic geometry. In the first one we give a formal axiomatisation of Greenberg functors; to this end, we introduce and study a notion of geometric category which abstracts the properties of the category of schemes that are relevant for Greenberg’s construction, allowing us to treat other categories, like various subcategories of that of logarithmic schemes, at the same time. In this context, we study the representability of functors “à-la Greenberg”, as well as their preservations of some properties like quasi-compactness or quasi-separatedness. We then apply this formalism both to examples already studied in the literature and to new ones: among these, we develop and study logarithmic versions of the Weil restriction and of the classical mixed-characteristic Greenberg functor.

The second part mainly deals with Edixhoven’s jumps and Chai’s conductors of tori, which are invariants related to the behaviour of their Néron models under base-change. The main results concern Edixhoven’s jumps of induced tori over an arbitrary discretely valued strictly henselian field  $K$  and, more generally, of  $K$ -tori which are direct factors of  $K$ -rational varieties.

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I enjoyed many fruitful exchanges of ideas with Art Waetershoot on the subject of Edixhoven’s jumps and logarithmic geometry; I also had useful discussions on matters related to this work with Davide Gori, Margot Bruneaux, Lars Halle and Ehud de Shalit. Portions of the section on the logarithmic Greenberg functor were developed as a part of ongoing work in collaboration with Leo Herr.

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Several of the commutative diagrams appearing in this work were drawn using the website <https://tikzcd.yichuanshen.de/>.

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## Part I

# Greenberg functors



## Introduction

A functor of categories  $F: \mathcal{C} \rightarrow \mathcal{D}$  induces by pullback a (contravariant) functor on presheaf categories, which, following [AGV71, Tome 1, exposé i, 5.0] we denote by  $F^*: \mathbf{PSh}(\mathcal{D}) \rightarrow \mathbf{PSh}(\mathcal{C})$ ,  $\mathcal{F} \mapsto \mathcal{F} \circ F$ . Many presheaves which one encounters in algebraic geometry arise as pullbacks of representable presheaves along appropriate functors of categories of schemes: explicitly, these are of the form

$$F^*X := \mathrm{Hom}_{\mathrm{Sch}/S}(F(\cdot), X),$$

where  $X \rightarrow S$  is a morphism of schemes and  $F: \mathrm{Sch}/T \rightarrow \mathrm{Sch}/S$  is a functor ( $T$  being a scheme as well); for example, if  $S \rightarrow T$  is a morphism and  $F = (\cdot) \times_T S$  is the base-change along it,  $F^*$  is the Weil restriction along  $S \rightarrow T$ .

The Greenberg functor, introduced by its eponymous in [Gre61], is another important instance occurring in arithmetic geometry: given a local Artin ring  $R$  with residue field  $k$ , assumed to be perfect if  $\mathrm{char} R \neq \mathrm{char} k$ , Greenberg constructs a natural lifting  $F: \mathrm{Sch}/k \rightarrow \mathrm{Sch}/R$  such that  $F(\mathrm{Spec} k) = \mathrm{Spec} R$  and proves that  $F^*X$  is representable for each finite type  $R$ -scheme  $X$  (this finiteness condition is actually superfluous: see [BGA18a]). Over the years, several new instances of “Greenberg functors” - functors which are similar in spirit and in their main properties to the one in [Gre61] - were introduced (among which [BS24], [Her24] and others) with various applications, and the main aim of this chapter is to develop a framework which encompasses all of them.

In order to do so, we first introduce and study a (to our knowledge) new notion of geometric category, namely that of categories with open immersions and gluings, which abstracts the properties of the category of schemes that are relevant for Greenberg’s construction, allowing us to treat other categories, like that of logarithmic schemes, at the same time.

One considers categories  $\mathcal{C}$  equipped with a functor  $|\cdot|$  from  $\mathcal{C}$  to the category  $\mathbf{Top}$  of topological spaces such that for each  $X$  in  $\mathcal{C}$  and  $U \subseteq |X|$  open,  $X$  has a pullback along  $U \hookrightarrow |X|$  (Definition 1.2.2); this determines an obvious notion of open immersion (Definition 1.2.8), and one requires a gluing condition for objects and morphisms along these open immersions (Definition 1.2.15).

Section 1.3 is devoted to a representability theorem (Theorem 1.3.7) for functors of the form  $F^*: \mathcal{D} \rightarrow \mathcal{C}$ , where  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a functor between categories with open immersions and gluings which respects open immersions: we shall see that under a technical condition one can check the representability of  $F^*X$  on an open cover of  $X$ .

A natural transformation  $\tau: G \rightarrow F$  of functors as above induces a morphism of functors  $F^* \rightarrow G^*$ ; in Section 1.4, we study these and deduce, under reasonable conditions, that  $F^*$  preserves some properties (like quasi-compactness or quasi-separatedness) of morphisms in  $\mathcal{D}$ .

The second chapter presents a non-exhaustive list of instances to which this formalism applies: we start with the classical version by Greenberg [Gre61] as described in its modern language in [BGA18a], also addressing the non-trivial issue of the well-definedness of the fpqc sheafification in its definition, and move on to the relatively perfect transform by Bertapelle and Suzuki [BS24]. The last two sections deal with new constructions, both belonging to the field of logarithmic geometry: after a recollection of the basics of this subject, in Section 2.5 we examine the logarithmic Weil restriction: for a log scheme  $S$  with underlying scheme  $\underline{S}$ , the main result of this section, Theorem 2.5.11, deals with the representability of the logarithmic Weil restriction of a log scheme  $X$  over  $S$  along the map  $S \rightarrow \underline{S}$ . Finally, we consider a logarithmic version of the mixed-characteristic Greenberg transform (which we are studying in collaboration with Leo Herr), and prove its representability under rather mild conditions (Theorem 2.6.16).

# Chapter 1

## A general formalism for Greenberg functors

### 1.1 Some recollections on Grothendieck topologies

The present section is a recollection of preliminary notions concerning Grothendieck topologies, mainly aimed at fixing notation and terminology.

*Notation 1.1.1.* All categories will be locally small.

We shall mainly adopt the same terminology as in [MM12], with some exceptions which will be pointed out.

**Definition 1.1.2.** A Grothendieck pretopology  $\text{Cov}$  on a category  $\mathcal{X}$  is an assignment, to each  $X$  in  $\mathcal{X}$ , of a collection  $\text{Cov}(X)$  of families of arrows  $\{X_i \rightarrow X\}_i$ , called a *cover* (or *covering*) of  $X$ , such that

- for any isomorphism  $X' \rightarrow X$ ,  $\{X' \rightarrow X\}$  is a covering;
- for any morphism  $f: Y \rightarrow X$  and any covering  $\{X_i \rightarrow X\}_{i \in I}$  of  $X$ , the pullback  $X_i \times_X Y$  exists for each  $i \in I$  and  $\{X_i \times_X Y \rightarrow Y\}_i$  is a covering of  $Y$ ;
- if  $\{X_i \rightarrow X\}_{i \in I}$  is a covering of  $X$  and, for each  $i \in I$ ,  $\{X_{ij} \rightarrow X_i\}_{j \in J_i}$  is a covering of  $X_i$ , then  $\{X_{ij} \rightarrow X_i \rightarrow X\}_{i \in I, j \in J_i}$  is a covering of  $X$ .

*Remarks 1.1.3.* • When comparing with the algebraic geometric literature, one should note that what, following [MM12], we call a Grothendieck pretopology is often just called a Grothendieck topology (for example in [Vis05, Definition 2.24] - see also [Vis05, Remark 2.25] - and [Ols16, Definition 2.1.2]); this is

because the notion of a Grothendieck topology, while important in several areas (one prime example being topos theory) is unnecessary for many applications in algebraic geometry - for instance, the theories of étale cohomology and of algebraic stacks as developed in [Mil80] and [Ols16], respectively - and arguably less intuitive. From our point of view, its main advantage resides in the fact that it makes the notion of sheaves on a basis much more elegant and easier to handle (see Proposition 1.1.11, and compare for example with [Autb, 009J]).

- The definition of pretopology in [AGV71, Tome 1, Exposé ii, Definition 1.3] is weaker than the one above, in that the first axiom is replaced by the condition that  $\{\text{id}_X\}$  be a covering of  $X$ : so for example if  $\mathcal{C}$  is a category with two objects  $X$  and  $Y$ , whose only automorphisms are the identities and with precisely one morphism  $X \rightarrow Y$  and one morphism  $Y \rightarrow X$  (which are then isomorphisms), the assignment  $\text{Cov}(Z) = \{\text{id}_Z\}$  for  $Z \in \{X, Y\}$  is a pretopology according to the definition in [AGV71], but not according to the one above. The one we use seems to be more standard now, witness [MM12], [Mil80], [Vis05], [Ols16]. The difference is not essential, because a pretopology in the sense of [AGV71] is contained inside a minimal pretopology in the sense above, and they both generate the same topology (the meaning of which will be explained shortly).

Recall the following

**Definition 1.1.4.** Let  $\mathcal{X}$  be a category,  $X \in \mathcal{X}$ . A sieve on  $X$  in  $\mathcal{X}$  is a collection of morphisms  $S = \{X_i \rightarrow X\}_i$  in  $\mathcal{X}$  such that for each  $X_i \rightarrow X$  in  $S$  and each  $Y \rightarrow X_i$  in  $\mathcal{X}$ , the composition  $Y \rightarrow X_i \rightarrow X$  is in  $S$  as well.

*Notation 1.1.5.* Given a family  $A$  of morphisms in  $\mathcal{X}$  with common target, we denote by  $(A)_{\mathcal{X}}$ , or simply by  $(A)$  if  $\mathcal{X}$  is clear from the context, the smallest sieve in  $\mathcal{X}$  containing  $A$ .

Let  $X \in \mathcal{X}$ ,  $S$  be a sieve on  $X$  and  $f: Y \rightarrow X$  be a morphism. We denote by  $f^*S$  the pullback sieve on  $Y$ , i.e. the sieve defined by

$$f^*S = \{g: Z \rightarrow Y \mid f \circ g \in S\}.$$

**Definition 1.1.6.** A Grothendieck topology on a category  $\mathcal{X}$  is an assignment, to each  $X \in \mathcal{X}$ , of a collection  $J(X)$  of sieves on  $X$ , called *covering sieves*, such that the following axioms are met:

- (i) for each  $X \in \mathcal{X}$ , the maximal sieve on  $X$ , i.e. the sieve  $\text{Obj}(\mathcal{X}/X)$ , is in  $J(X)$ ;
- (ii) for each  $f: Y \rightarrow X$  and each  $S \in J(X)$ ,  $f^*S \in J(Y)$ ;
- (iii) for each  $X \in \mathcal{X}$  and each sieve  $R$  on  $X$ , if  $S \in J(X)$  is such that  $f^*(R) \in J(Y)$  for all  $Y$  and all  $f: Y \rightarrow X$  in  $S$ , then  $R \in J(X)$ .

A couple  $(\mathcal{X}, J)$  as above is called a site.

A family  $A$  of arrows in a site having a common target  $X$  will be called a covering presieve for  $X$  if  $(A) \in J(X)$ .

Each Grothendieck pretopology  $\text{Cov}$  on  $\mathcal{X}$  generates a Grothendieck topology  $J$ , whose covering presieves are exactly the coverings of the pretopology (more explicitly, for each  $X$  in  $\mathcal{X}$ ,  $J(X)$  is the collection of sieves on  $X$  containing some covering in  $\text{Cov}(X)$ ). Conversely, if  $\mathcal{X}$  has fibred products, then each Grothendieck topology on  $\mathcal{X}$  is obtained in this fashion from a unique maximal pretopology; in other words, there exists a unique pretopology such that the coverings of any  $X$  in  $\mathcal{X}$  are precisely the covering presieves of  $X$  for  $J$  [MM12, Chapter III.2, p. 112], [AGV71, Tome 1, Expose ii, 1.3.1].

*Remark 1.1.7.* Because of the previous fact, what we call a covering presieve could be just called a covering (as, for example, in [Lur, Definition B.1.3]), and we sometimes do so: notice that there is no conflict between these usages (a covering presieve is indeed a presieve which is also a covering for the maximal pretopology generating it). Nevertheless, as we shall often switch between topologies and pretopologies, it is sometimes helpful to underline which point of view one is taking.

A presheaf  $P: \mathcal{X}^{\text{op}} \rightarrow \mathbf{Set}$  on a category  $\mathcal{X}$  is called a sheaf for a topology  $J$  if for each  $X \in \mathcal{X}$  and each covering sieve  $S \in J(X)$ , the canonical map

$$P(X) \rightarrow \lim_{\substack{f \in S^{\text{op}} \\ f: Y \rightarrow X}} P(Y), \quad (1.1)$$

where  $S$  is regarded as a full subcategory of the slice category  $\mathcal{X}/X$ , is a bijection [Lur, Appendix B.2]. Explicitly, an element in the limit (called a *matching family for  $S$  of elements of  $P$* ) is a collection

$$(s_f)_f \in \prod_{\substack{f \in S \\ f: Y \rightarrow X}} P(Y)$$

such that, given any commutative triangle

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow \alpha & \nearrow f' & \\ Y' & & \end{array}$$

with  $f$  and  $f'$  in  $S$ , one has  $P(\alpha)(s_{f'}) = s_f$ ; the map (1.1) being a bijection means that for each such family, there exists a unique element  $s \in P(X)$  (called an *amalgamation* for such family) such that  $P(f)(s) = s_f$  for any  $f \in S$ .

If  $(\mathcal{X}, J)$  is a site, we denote by  $\mathrm{Sh}_J(\mathcal{X})$  or, more commonly, simply by  $\mathrm{Sh}(\mathcal{X})$ , the full subcategory of  $\mathrm{PSh}(\mathcal{X})$  of sheaves of sets with respect to the  $J$ -topology. One can replace  $\mathbf{Set}$  by an arbitrary category with limits  $\mathcal{A}$  (usually a category whose objects are sets with some extra structure); we then write  $\mathrm{Sh}_J(\mathcal{X}, \mathcal{A})$  or  $\mathrm{Sh}(\mathcal{X}, \mathcal{A})$  for the resulting category of sheaves.

*Remark 1.1.8.* Notice that if  $A = \{X_i \rightarrow X\}_i$  is a covering presieve with  $(A) = S$ , then the limit in the definition of a sheaf remains the same if one replaces  $S$  with  $A$ : indeed, a map  $h$  is in  $(A)$  if and only if there exist  $f_i: X_i \rightarrow X$  in  $A$  and a factorisation

$$\begin{array}{ccc} Y & \xrightarrow{h} & X \\ \downarrow g & \nearrow f_i & \\ X_i & & \end{array} ;$$

by definition, for any matching family  $(s_f)_{f \in S}$  one then has  $s_h = P(g)(s_{f_i})$ , which shows that  $(s_f)_{f \in S}$  is determined univocally by  $(s_{f_i})_{f_i \in A}$ . In other words, the sheaf condition for  $P$  on a covering sieve can be checked on any covering presieve generating it.

In particular, if  $\mathcal{X}$  has fibred products, a presheaf  $P$  on  $\mathcal{X}$  is a sheaf if and only if for any  $X$  in  $\mathcal{X}$  and any covering  $\{X_i \rightarrow X\}_{i \in I}$  the familiar diagram

$$P(X) \longrightarrow \prod_{i \in I} P(X_i) \rightrightarrows \prod_{(i,j) \in I \times I} P(X_i \times_X X_j)$$

is an equaliser.

**Definition 1.1.9.** Let  $(\mathcal{X}, J)$  be a site. A full subcategory  $\mathcal{X}'$  of  $\mathcal{X}$  is called a *basis* for  $\mathcal{X}$  if each  $X \in \mathcal{X}$  has a covering presieve  $\{X'_i \rightarrow X\}_i$  such that  $X'_i$  is in  $\mathcal{X}'$  for each  $i$ .

*Remark 1.1.10.* At least in [MM12, Definition III.2.2], a basis for a topology is what we call a pretopology generating it; the term does not seem to appear in the other sources mentioned above. Our use of the term, borrowed from [Lur] (where a smallness condition, which we have ignored as we will not focus on set-theoretic problems, is also included) is thus probably non-standard, but we have adopted it for its convenience.

As mentioned above, the next result, proved in [Lur, Propositions B.6.3 and 4], is the reason why we use bases.

**Proposition 1.1.11.** *Let  $(\mathcal{X}, J)$  be a site and  $\mathcal{X}'$  a basis.*

- (i) *There exists a unique Grothendieck topology  $J'$  on  $\mathcal{X}'$  such that a collection  $A'$  of morphisms in  $\mathcal{X}'$  is a covering presieve for  $J'$  if and only if it is a covering presieve for  $J$ .*
- (ii) *Equipping  $\mathcal{X}'$  with the topology defined at the previous point, the restriction  $\mathrm{Sh}(\mathcal{X}) \rightarrow \mathrm{Sh}(\mathcal{X}')$  is an equivalence of categories.*

A functor between sites  $F: (\mathcal{X}, J) \rightarrow (\mathcal{Y}, K)$  is said to preserve covering presieves if for each covering presieve  $A = \{X_i \rightarrow X\}_i$  in  $\mathcal{X}$  for  $J$ ,  $F(A) := \{F(X_i \rightarrow X)\}_i$  is a covering presieve for  $K$ .

## 1.2 Categories with open immersions and gluings

In the present section we shall isolate the kind of geometric categories to which our results apply and study their main properties. After insulating categories in which a sensible (for our purposes) notion of open immersions can be defined (Definitions 1.2.2 and 1.2.8), we introduce the main objects of this section (Definition 1.2.15), that are categories in which one can glue objects and morphisms along such open immersions, and prove some preliminary results about them.

*Notation 1.2.1.* • Recall that, given a category  $\mathcal{C}$ , the arrow category of  $\mathcal{C}$ , which will be denoted by  $\mathrm{Ar}(\mathcal{C})$ , is the category whose objects are morphisms of  $\mathcal{C}$  and such that morphisms are commutative squares, i.e. for each  $f: A \rightarrow B$ ,  $g: C \rightarrow D$  in  $\mathcal{C}$ ,

$$\mathrm{Hom}_{\mathrm{Ar}(\mathcal{C})}(f, g) = \{(a, b) \in \mathrm{Hom}_{\mathcal{C}}(A, C) \times \mathrm{Hom}_{\mathcal{C}}(B, D) \mid b \circ f = g \circ a\},$$

compositions and identities being the expected ones.

We shall write  $s, t: \mathrm{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$  for the source and target functors.

- Let  $\mathrm{Top}_o^2$  be the category of pairs  $(A, B)$  where  $B$  is a topological space and  $A$  an open subset of  $B$ . We shall regard  $\mathrm{Top}_o^2$  as a subcategory of  $\mathrm{Ar}(\mathrm{Top})$ ; in particular, the target functor  $t: \mathrm{Top}_o^2 \rightarrow \mathrm{Top}$  induces the projection  $(A, B)$  to  $B$ .
- The category  $\mathrm{Top}$  will be always endowed with its natural Grothendieck pre-topology (namely, the coverings of a topological space  $A$  are its coverings by open subspaces). If  $f: A \rightarrow B$  is a continuous map of topological spaces and

$V \subseteq B$  is open, we identify the fibred product  $A \times_B V$  with the open subset  $f^{-1}(V)$  of  $A$ .

We shall use the language of fibred categories, as developed for example in [Vis05].

### Categories with open immersions

**Definition 1.2.2.** We call a pair  $(\mathcal{C}, |\cdot|)$  consisting of a locally small category  $\mathcal{C}$  and a functor  $|\cdot|: \mathcal{C} \rightarrow \mathbf{Top}$  a category with open immersions if for all  $X \in \mathcal{C}$  and all  $U \subseteq |X|$  open,  $X$  has a pullback along  $U \hookrightarrow |X|$ .

*Remark 1.2.3.* Unwinding the definition of pullbacks [Vis05, Definition 3.1], a pair  $(\mathcal{C}, |\cdot|)$  as above is a category with open immersions if and only if for all  $X \in \mathcal{C}$  and all  $U \subseteq |X|$  open,

(i) the functor

$$F_{X,U}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}, \quad Z \mapsto \{f \in \text{Hom}_{\mathcal{C}}(Z, X) \mid |f| \text{ factors through } U \hookrightarrow |X|\}$$

is representable by an object  $U_X$  in  $\mathcal{C}$  together with a universal morphism of  $\mathcal{C}$

$$\mathfrak{o}_U^X: U_X \rightarrow X;$$

(ii) the universal morphism  $\mathfrak{o}_U^X: U_X \rightarrow X$  is mapped by the functor  $|\cdot|$  to the inclusion  $U \hookrightarrow |X|$ .

Indeed, the second condition amounts precisely to  $\mathfrak{o}_U^X$  being over  $U \hookrightarrow |X|$ ; as for the first one, if *ii*) holds, morphisms in  $F_{X,U}(Z)$  are the same as diagrams

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & \searrow & \downarrow \\ |Z| & \xrightarrow{\quad} & |X| \\ & \searrow & \downarrow \\ & & U \end{array} \quad ;$$

a pair  $(U_X, \mathfrak{o}_U^X)$  represents  $F_{X,U}$  if and only if for each  $Z$ ,  $g \mapsto \mathfrak{o}_U^X \circ g$  is a bijection  $\text{Hom}_{\mathcal{C}}(Z, U_X) \rightarrow F_{X,U}(Z)$ , i.e. if and only if each such diagram has a unique filler  $Z \rightarrow U_X$ .

In particular, it follows immediately from the universal property of  $U_X$  that  $\mathfrak{o}_U^X$  is a monomorphism. Note also that  $U_X \rightarrow X$  is uniquely determined up to unique isomorphism.

*Remarks 1.2.4.* Let  $\mathcal{C}$  be a category with open immersions.

- (i) Given  $Y \in \mathcal{C}$  and open subsets  $U \subseteq V \subseteq |Y|$  and setting  $X = V_Y$ , the composition  $\mathfrak{o}_V^Y \circ \mathfrak{o}_U^X$  factors through  $\mathfrak{o}_U^Y$  and the induced map  $U_X \rightarrow U_Y$  has an inverse, as depicted in the diagram

$$\begin{array}{ccccc}
 & & & \mathfrak{o}_U^Y & \\
 & & & \curvearrowright & \\
 U_Y & & & & \\
 \uparrow \text{dashed} & & & & \\
 U_X & \xrightarrow{\mathfrak{o}_U^X} & V_Y & \xrightarrow{\mathfrak{o}_V^Y} & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 U & \hookrightarrow & V & \hookrightarrow & |Y|
 \end{array}$$

The existence of a factorisation follows since by assumption  $|\mathfrak{o}_U^X|(|U_X|) = U$ ; as  $U \subseteq V$  one can lift  $\mathfrak{o}_U^Y$  to  $X$  and subsequently to  $U_X$ ; by the universal properties of  $U_X$  and  $U_Y$ , the resulting map is the desired inverse.

- (ii) If  $\mathcal{C}$  admits an initial element  $E$ , then by the universal property of  $\mathfrak{o}_\emptyset^X$  one has  $|E| = \emptyset$  (but there can exist non-initial objects  $X$  with  $|X| = \emptyset$ : for example, consider the category with open immersions  $(\mathcal{C}, |\cdot|)$  with objects  $E$ ,  $A$  and  $B$  with  $E$  initial,  $|A| = *$ ,  $B = \emptyset_A$ , exactly one morphism from  $B$  to  $A$  and no other non-identical morphisms starting from  $A$  or  $B$ ). This already poses some restrictions on the category  $\mathcal{C}$ : for example, if  $\mathcal{C}$  has an object  $X$  which is not initial, it cannot be additive (for if it were, it would have a zero object  $E$  and a map  $X \rightarrow E$ , inducing a map  $\emptyset \neq |X| \rightarrow \emptyset$ ).

*Remark 1.2.5.* If  $(\mathcal{C}, |\cdot|)$  is a category with open immersions, one can, if one so wishes, pick a representing element  $U_X$  as above for all  $X$  in  $\mathcal{C}$  and all  $U \subseteq |X|$  open. Then taking such a pair  $(U, X) \in \mathbf{Top}_o^2 \times_{t, \mathbf{Top}, |\cdot|} \mathcal{C}$  to  $\mathfrak{o}_U^X$  defines a functor

$$\mathfrak{o}: \mathbf{Top}_o^2 \times_{t, \mathbf{Top}, |\cdot|} \mathcal{C} \rightarrow \mathbf{Ar}(\mathcal{C})$$

satisfying  $t \circ \mathfrak{o} = \text{pr}_2$ : indeed, given morphisms  $(U, X) \rightarrow (V, Y) \rightarrow (W, Z)$  in  $\mathbf{Top}_o^2 \times_{t, \mathbf{Top}, |\cdot|} \mathcal{C}$ , i.e. maps  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  in  $\mathcal{C}$  such that  $|f|(U) \subseteq V$  and  $|g|(V) \subseteq W$ , the universal property of  $W_Z$  implies that  $\mathfrak{o}(g \circ f) = \mathfrak{o}(g) \circ \mathfrak{o}(f)$ , and similarly  $\mathfrak{o}$  preserves identities.

Unless indicated otherwise, we shall always posit that  $\mathfrak{o}_{|X|}^X = \text{id}_X$ .

*Notation 1.2.6.* (i) For any  $X \in \mathcal{C}$ , we will also refer to  $|X|$  as the *underlying topological space of  $X$* .

- (ii) Assume we are in the situation of the previous remark, i.e. a choice of a functor  $\mathfrak{o}$  as above has been given. For  $X \in \mathcal{C}$  and  $U \subseteq |X|$  open, we shall refer to  $\mathfrak{o}_U^X$  (or also to  $U_X$ ) as an open subobject of  $X$ . This does not conflict too severely with the usage in category theory, where a subobject of  $X$  is an *isomorphism class* of monomorphisms to  $X$ , since each subobject (in the categorical sense)  $X' \rightarrow X$  which is an open immersion has a unique representative which is an open subobject in the above sense.

*Example 1.2.7.* Here is the prototypical example of a category with open immersions: let  $\mathcal{C} = \text{Sch}$ ,  $|\cdot|: \mathcal{C} \rightarrow \mathbf{Top}$  the functor associating to a scheme its underlying topological space; for any scheme  $X$  and any open subset  $U \subseteq |X|$ ,  $\mathfrak{o}_U^X$  is the canonical morphism  $(U, \mathcal{O}_{X|_U}) \rightarrow X$ .

**Definition 1.2.8.** Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions and fix some  $\mathfrak{o}$  as above.

- (i) We define an open immersion in  $(\mathcal{C}, |\cdot|)$  to be a morphism  $X' \rightarrow X$  such that there exists  $U \subseteq |X|$  open such that  $X' \rightarrow X$  factors through  $\mathfrak{o}_U$  and the (unique) induced morphism  $X' \rightarrow U_X$  is an isomorphism.
- (ii) For any object  $X$  of  $\mathcal{C}$ , we declare a covering of  $X$  to be a collection of open immersions  $\{X_i \rightarrow X\}_i$  such that  $|X| = \bigcup_i \text{Im}(|X_i| \rightarrow |X|)$ .

*Remark 1.2.9.* Notice that whether a map is an open immersion (and thus whether a collection of maps is a covering) is independent of the particular choice of  $\mathfrak{o}$ .

The previous definition recovers the notions of open immersion of schemes and of Zariski open covering. Just as in this case, the class of open immersions enjoys some favourable properties: most importantly, it is closed under composition and for all  $Y$  in  $\mathcal{C}$  and  $U \subseteq |Y|$  open, the base-change  $X' \rightarrow X$  of  $\mathfrak{o}_U^Y$  along any morphism  $f: X \rightarrow Y$  exists and is again an open immersion; moreover, the underlying topological space of  $X'$  can be identified with  $|f|^{-1}(U) \subseteq |X|$ . In particular, once  $\mathfrak{o}$  is fixed, one has a canonical choice for fibred products along open immersions (compare with the last point of Notation 1.2.1).

**Lemma 1.2.10.** *Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions.*

- (i) *The composition of two open immersions is again an open immersion.*
- (ii) *If a composition  $T \rightarrow X \rightarrow Y$  is an open immersion and  $X \rightarrow Y$  is an open immersion, then so is  $T \rightarrow X$ .*

(iii) If  $Y' \rightarrow Y$  is an open immersion and  $X \rightarrow Y$  is any morphism, then, setting  $U = |X| \times_{|Y|} |Y'|$ , the square

$$\begin{array}{ccc} U_X & \xrightarrow{\mathfrak{o}_U^X} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

(where the left vertical map is the unique one making the diagram commute) is cartesian.

(iv) A morphism  $f: X \rightarrow Y$  is an open immersion if and only if  $\text{Im } |f| \subseteq |Y|$  is open and any morphism  $g: T \rightarrow Y$  with  $\text{Im } |g| \subseteq \text{Im } |f|$  factors uniquely through  $f$ .

*Proof.* (i): Let  $T \rightarrow X$  and  $X \rightarrow Y$  be open immersions, so that there exist open subsets  $U \subseteq |X|$ ,  $V \subseteq |Y|$  and a commutative diagram

$$\begin{array}{ccccc} U_X & \longrightarrow & V_Y & & \\ \uparrow \wr & \searrow & \uparrow \wr & \searrow & \\ T & \longrightarrow & X & \longrightarrow & Y \end{array},$$

where the slanted arrows are  $\mathfrak{o}_U^X$  and  $\mathfrak{o}_V^Y$ ; the map  $|U_X \rightarrow V_Y|$  induces an identification of  $U$  with an open subset of  $V$ . This reduces the statement to that of Remark 1.2.4(i).

(ii): The situation is depicted in the diagram

$$\begin{array}{ccccc} & & U_Y & & \\ & \nearrow \sim & \downarrow & \searrow & \\ T & \longrightarrow & U_X & & \\ & \searrow & \downarrow & \nearrow \sim & \\ & & X & & \\ & & \nearrow & \searrow & \\ & & V_Y & \longrightarrow & Y \end{array},$$

where we have to show that  $T \rightarrow U_X$  is an isomorphism. Again, this follows from Remark 1.2.4(i), which implies that  $U_Y \rightarrow U_X$  is an isomorphism.

(iii): Explicitly,  $Y' \rightarrow Y$  is isomorphic (in  $\mathcal{C}/Y$ ) to  $\mathfrak{o}_V^Y$  for some  $V \subseteq |Y|$  open, whereby the composition  $U_X \xrightarrow{\mathfrak{o}_U^X} X \rightarrow Y$  factors through it by the definition of  $U$ , yielding the vertical left map. Without loss of generality,  $Y' = V_Y$  for some  $V \subseteq |Y|$

open. Consider a commutative diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{a} & X \\
 \downarrow b & \searrow \text{dotted} & \downarrow \\
 & U_X & \longrightarrow X \\
 & \downarrow & \downarrow \\
 & V_Y & \xrightarrow{\mathfrak{o}_V^Y} Y
 \end{array}$$

since  $U = |f|^{-1}(V)$ ,  $a$  factors uniquely through  $\mathfrak{o}_U^X$ ; as  $\mathfrak{o}_V^Y$  is a monomorphism, the left triangle commutes as well.

(iv): The forward implication follows straightly from the definition. For the converse, setting  $V := \text{Im } |f| \subseteq |Y|$ , the assumption implies that  $\mathfrak{o}_V^Y$  factors through  $f$ ; by the universal property of  $\mathfrak{o}_V^Y$ , the resulting map has an inverse. Thus  $f$  is isomorphic to  $\mathfrak{o}_V^Y$  in  $\mathcal{C}/Y$ .  $\square$

*Remark 1.2.11.* In particular, if  $X' \rightarrow X$  is an open immersion, then  $X' \times_X X'$  can be identified with  $X'$  itself, and the two maps  $X' \times_X X' \rightarrow X'$  with the identity.

*Notation 1.2.12.* • If  $f: X \rightarrow Y$  is a morphism in  $\mathcal{C}$  and  $X' \rightarrow X$  an open subobject of  $X$ , we denote by  $f|_{X'}$ , the composition  $X' \rightarrow X \xrightarrow{f} Y$ . For an open immersion  $\iota: X'' \rightarrow X$ , we shall sometimes use the notation  $f|_{\iota}$  for the composition.

- If  $\iota: X' \rightarrow X$  is an open immersion, we shall sometimes identify  $|X'|$  with an open subset of  $|X|$  via  $|\iota|$ , and so write  $|X'|$  for  $\text{Im } |\iota| \subseteq |X|$ .

**Definition/Proposition 1.2.13.** Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions and choose a functor  $\mathfrak{o}$  be as in Remark 1.2.5. Then the set of coverings defined in Definition 1.2.8(ii) is a Grothendieck pretopology on  $\mathcal{C}$ , which is independent of the particular choice of  $\mathfrak{o}$  and will be referred to as the  $|\cdot|$ -pretopology.

*Proof.* Isomorphisms are clearly open immersions and if  $X' \rightarrow X$  is an isomorphism,  $\{X' \rightarrow X\}$  is an open covering of  $X$ .

If  $\{X_i \rightarrow X\}$  is an open covering and  $Y \rightarrow X$  a morphism, Lemma 1.2.10(iii) implies that  $\{X_i \times_X Y \rightarrow Y\}_i$  is an open covering of  $Y$ .

Finally, if  $\{X_i \rightarrow X\}_i$  is a covering of  $X$  and for each  $i$   $\{X_{ij} \rightarrow X_i\}_j$  is a covering of  $X_i$ , then  $X_{ij} \rightarrow X_i \rightarrow X$  is an open immersion by Lemma 1.2.10(i) and  $\{X_{ij} \rightarrow X\}_{i,j}$  is a covering by point 2) in Remark 1.2.3.

The fact that different choices of  $\mathfrak{o}$  produce the same pretopology is immediate from Remark 1.2.9.  $\square$

Henceforth, a category with open immersions  $(\mathcal{C}, |\cdot|)$  will be endowed with the  $|\cdot|$ -pretopology unless otherwise stated.

The following notions will be useful later.

**Definition 1.2.14.** Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions. We call a full subcategory  $\mathcal{E}$  of  $\mathcal{C}$  a *subcategory with open immersions* if for any open immersion  $X \rightarrow Y$  in  $\mathcal{C}$ , if  $Y$  is in  $\mathcal{E}$ , then so is  $X$ .

In particular,  $(\mathcal{E}, |\cdot|_{|\mathcal{E}}|)$  is itself a category with open immersions.

## Gluings

In this subsection we explain what we mean for a category with open immersions to have gluings and prove some consequences of this assumption, to be used in later sections; in particular, Proposition 1.2.19, which extends Grothendieck's classical representability criterion for sheaves on scheme categories; we then define and study some properties of morphisms in categories with open immersions and gluings, like quasi-compactness.

In a category with open immersions, one can make sense of the notion of gluing data.

**Definition 1.2.15.** Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions.

(i) A gluing datum in  $\mathcal{C}$  is a triple

$$((X_i)_{i \in I}, (\{X_{ij} \hookrightarrow X_i\}_{j \in I})_{i \in I}, (\phi_{ij}: X_{ij} \rightarrow X_{ji})_{i, j \in I})$$

such that for each  $i, j, k \in I$

- $X_{ii} = X_i$ ;
- $X_{ij} \rightarrow X_i$  is an open immersion;
- $\phi_{ij}: X_{ij} \rightarrow X_{ji}$  is an isomorphism;
- $(\phi_{ij})|_{X_{ij} \times_{X_i} X_{ik}}$  factors through  $X_{jk}$  and the cocycle condition

$$\phi_{jk} \circ (\phi_{ij})|_{X_{ij} \times_{X_i} X_{ik}} = (\phi_{ik})|_{X_{ij} \times_{X_i} X_{ik}}$$

holds.<sup>1</sup>

---

<sup>1</sup>Notice that, thanks to Remark 1.2.11, we do not need to worry about the circumstance when two or more indices are the same: for example, if  $j = k$ , then the projections  $\text{pr}_1, \text{pr}_2: X_{ij} \times_{X_i} X_{ik} \rightarrow X_{ij} = X_{ik}$  can be identified (compare with [Vis05, Section 2.3.3]).

(ii) We say that a gluing datum as above is *effective* if there exist an object  $X$  of  $\mathcal{C}$  and open immersions  $\psi_i: X_i \rightarrow X$  such that

- $\{\psi_i: X_i \rightarrow X\}_i$  is an open covering;
- for all  $i, j \in I$ ,  $\psi_j \circ \phi_{ij} = (\psi_i)_{|_{U_{ij}}}$ .

(iii) We say that  $(\mathcal{C}, |\cdot|)$  has gluings if the  $|\cdot|$ -topology is subcanonical (i.e. for all  $X$  in  $\mathcal{C}$ , the presheaf  $\text{Hom}_{\mathcal{C}}(-, X)$  is a sheaf for the  $|\cdot|$ -topology) and any gluing datum in  $\mathcal{C}$  is effective.

Notice that, in general, if the  $|\cdot|$ -topology is subcanonical and a gluing datum is effective, the datum of  $(X, (\psi_i)_i)$  as above is unique up to unique isomorphism.

If the  $|\cdot|$ -topology is subcanonical, the property of being an open immersion is local on the target.

**Lemma 1.2.16.** *Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions such that the  $|\cdot|$ -topology is subcanonical. If  $f: X \rightarrow Y$  is a morphism in  $\mathcal{C}$  and there exists a covering  $\{Y_i \rightarrow Y\}_{i \in I}$  of  $Y$  such that each  $f_i: X_i := X \times_Y Y_i \rightarrow Y_i$  is an open immersion, then  $f$  is an open immersion as well.*

*Proof.* In order to verify that  $X$  satisfies the universal property of Lemma 1.2.10(iv), consider a morphism  $g: T \rightarrow Y$  such that  $|g|$  factors through  $|X|$  and for each  $i, j \in I$ , let  $U_i = |g|^{-1}(|Y_i|)$ ,  $T_i = (U_i)_T$  and  $T_{ij} = (U_i \cap U_j)_T$ . Each map  $f|_{X_i}: X_i \rightarrow Y$  is an open immersion as a composition of the open immersions  $X_i \hookrightarrow Y_i$  and  $Y_i \hookrightarrow Y$ ; consequently,  $g|_{T_i}$  factors uniquely through  $X_i$  and the induced morphisms  $T_i \rightarrow X$  agree on the  $T_{ij}$ 's, thus gluing to a unique map  $T \rightarrow X$  through which  $g$  factors.  $\square$

We now aim to prove that, as in the case of schemes, a sheaf on a category with open immersions and gluings which admits an open covering by representable sheaves is itself representable. More precisely:

**Definition 1.2.17.** Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions and  $\mathcal{C}'$  a basis. Given a presheaf  $G: (\mathcal{C}')^{\text{op}} \rightarrow \mathbf{Set}$ , we say that a subpresheaf  $H \hookrightarrow G$  is an open subpresheaf, or that  $H \hookrightarrow G$  is an open immersion of subpresheaves, if for any  $T'$  in  $\mathcal{C}'$  and any every natural transformation  $\xi: T' \rightarrow G$  of functors  $(\mathcal{C}')^{\text{op}} \rightarrow \mathbf{Set}$  the pullback  $\bar{T}$  in

$$\begin{array}{ccc} \bar{T} & \longrightarrow & T' \\ \downarrow & & \downarrow \xi \\ H & \longrightarrow & G \end{array}$$

is representable in  $\mathcal{C}$  and the morphism  $\bar{T} \rightarrow T'$  is an open immersion.

We say that a family of subpresheaves  $\{G_i \hookrightarrow G\}_{i \in I}$  is an open covering if each  $G_i \rightarrow G$  is an open immersion of presheaves and  $\{G_i \times_G T' \rightarrow T'\}_i$  is an open covering for each  $T'$  and  $\xi$ .

The case one is mostly interested in is that when  $G$  is the restriction to  $\mathcal{C}'$  of a sheaf on  $\mathcal{C}$ , because, as we shall prove in Proposition 1.2.19, the representability of a sheaf can be tested on open coverings and, as the next lemma shows, the condition that a family of subsheaves on  $\mathcal{C}$  be an open covering can in turn be checked on their restrictions to  $\mathcal{C}'$ .

For ease of notation, in the following lemma given a presheaf  $G$  on  $\mathcal{C}$  we shall denote by  $G'$  its restriction to  $\mathcal{C}'$ .

**Lemma 1.2.18.** *Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions and gluings with basis  $\mathcal{C}'$ , and let  $G: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  be a sheaf.*

- (i) *A subsheaf  $H \hookrightarrow G$  is an open subsheaf if and only if  $H' \rightarrow G'$  is an open subsheaf.*
- (ii) *A family  $\{G_i \rightarrow G\}_{i \in I}$  of subsheaves on  $\mathcal{C}$  is an open covering if and only if  $\{G'_i \rightarrow G'\}_{i \in I}$  is an open covering.*

*Proof.* One implication is clear, observing that  $T' \times_G H = T' \times_{G'} H'$  as sheaves on  $\mathcal{C}'$  for any  $T'$  in  $\mathcal{C}'$  and any natural transformation  $T' \rightarrow G$ .

For the other one, consider a subsheaf  $H \hookrightarrow G$  such that  $H' \rightarrow G'$  is an open subsheaf. We have to show that for any  $T$  in  $\mathcal{C}$  and any natural transformation  $\xi: T \rightarrow G$ , the pullback of sheaves on  $\mathcal{C}$

$$\begin{array}{ccc} \bar{T} & \hookrightarrow & T \\ \downarrow & & \downarrow \xi \\ H & \longrightarrow & G \end{array}$$

is representable and  $\bar{T} \rightarrow T$  is an open immersion.

Pick an open cover  $\{T'_m \rightarrow T\}_{m \in J}$  by elements in  $\mathcal{C}'$  and let, for  $m, n \in J$ ,  $\xi_m$  and  $\xi_{mn}$  be the “restrictions”  $\xi_m: T'_m \rightarrow T \xrightarrow{\xi} G$  and  $\xi_{mn}: T'_m \times_T T'_n \rightarrow T'_m \rightarrow T \xrightarrow{\xi} G$ ; since each  $T'_m$  is in  $\mathcal{C}'$ , for each  $m \in J$  there is an open immersion  $\bar{T}_m \hookrightarrow T'_m$  fitting in a pullback diagram of sheaves on  $\mathcal{C}'$

$$\begin{array}{ccc} \bar{T}_m & \hookrightarrow & T'_m \\ \downarrow & & \downarrow \xi \\ H & \longrightarrow & G \end{array}$$

and there are canonical isomorphisms (again of sheaves on  $\mathcal{C}'$ )

$$\bar{T}_m \times_T T'_n \xrightarrow{\sim} H \times_G \times T'_m \times_T T'_n.$$

By Proposition 1.1.11(ii), the previous diagram is cartesian in  $\mathrm{Sh}(\mathcal{C})$  and, similarly, the isomorphisms are isomorphisms of sheaves on  $\mathcal{C}$ . One therefore gets a gluing datum

$$((\bar{T}_m)_{m \in J}, (\bar{T}_m \times_T T'_n \xrightarrow{\sim} \bar{T}_n \times_T T'_m)_{m,n \in J})$$

and thus an object  $\bar{T}$  in  $\mathcal{C}$ , together with obvious natural transformations  $\bar{T} \rightarrow T$  and  $\bar{T} \rightarrow H$  making the square

$$\begin{array}{ccc} \bar{T} & \longrightarrow & T \\ \downarrow & & \downarrow \xi \\ H & \longrightarrow & G \end{array}$$

commute. Since the morphisms  $\bar{T}_m \rightarrow T'_m$  are open immersions, by Lemma 1.2.16  $\bar{T} \rightarrow T$  is an open immersion as well.

It remains to prove that the previous diagram is cartesian. Using Yoneda lemma, it is easy to see that for any  $Z$  in  $\mathcal{C}$ ,

$$(T \times_G H)(Z) = \{f: Z \rightarrow T \mid f^*(\xi) \in H(Z) \subseteq G(Z)\}$$

and the induced morphism  $\bar{T}(Z) \rightarrow (T \times_G H)(Z)$  is just post-composition with  $\bar{T} \rightarrow T$ ; equivalently, setting  $Z_m := Z \times_T T'_m$ , this can be described as the datum of morphisms  $\{Z_m \rightarrow T'_m\}_m$  such that  $\xi_{m|Z_m} \in H(Z_m)$  and such that for all  $m$  and  $n$  in  $J$  the compositions  $Z_m \rightarrow T'_m \rightarrow T$  and  $Z_n \rightarrow T'_n \rightarrow T$  agree on  $Z_m \times_Z Z_n$ .

This datum determines morphisms  $Z_m \rightarrow T'_m$  which glue together to a unique map  $Z \rightarrow \bar{T}$ , and this construction determines an inverse to  $\bar{T} \rightarrow T \times_G H$ .

If  $\{G_i \rightarrow G\}_{i \in I}$  is a family of subsheaves such that  $\{G'_i \rightarrow G'\}_{i \in I}$  is an open covering, one gets in this fashion an open covering  $\{\bar{T}^{(i)} \rightarrow T'_m\}_{i \in I}$  for all  $m \in J$ ; since  $\{T'_m \rightarrow T\}_{m \in J}$  is an open cover,  $\{\bar{T}^{(i)} \rightarrow T\}$  is an open covering as well.  $\square$

**Proposition 1.2.19.** *Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions such that the  $|\cdot|$ -topology is subcanonical and  $\mathcal{E}$  be a subcategory with open immersions of  $\mathcal{C}$  (Definition 1.2.14) which has gluings. If a sheaf  $G: \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$  for the  $|\cdot|$ -topology admits an open covering  $\{G_i \hookrightarrow G\}_{i \in I}$  such that each  $G_i$  is representable (as a sheaf on  $\mathcal{C}$ ) by an object of  $\mathcal{E}$ , then  $G$  is representable.*

*Proof.*

**Step 1.** *Construction of a would-be representing element  $X$  for  $G$ .*

For each  $i$  and  $j$  in  $I$ , let  $U_i$  represent  $G_i$  (more precisely, fix an isomorphism of sheaves  $U_i \xrightarrow{\sim} G_i$ , which will be used to identify the two in the following) and let  $U_{ij} \rightarrow U_i$  be an open immersion making the diagram

$$\begin{array}{ccc} U_{ij} & \hookrightarrow & U_j \\ \downarrow & & \downarrow \\ G_i & \longrightarrow & G \end{array}$$

cartesian. For each  $Z \in \mathcal{C}$ , there is an obvious identification  $U_{ij}(Z) \xrightarrow{\sim} U_{ji}(Z)$  induced by the given isomorphisms  $U_i \xrightarrow{\sim} G_i$ , yielding an isomorphism  $U_{ij} \rightarrow U_{ji}$ ; the isomorphisms thus constructed satisfy the cocycle condition, whereby the  $U_i$ 's glue to an object  $X$  in  $\mathcal{C}$ . Given indices  $i, j$ , write  $U_{\{i,j\}}$  for the open subobject of  $X$  corresponding to either  $U_{ij}$  or  $U_{ji}$ ; this is “the” fibre product  $U_i \times_X U_j$  (compare with Lemma 1.2.10(iii)).

**Step 2.** *Construction of an isomorphism of sheaves  $X \xrightarrow{\sim} G$ .*

Let us show that the morphisms  $\psi^{(i)}: U_i \cong G_i \rightarrow G$  assemble to a morphism of sheaves  $\psi: X \rightarrow G$ . Using Yoneda lemma to identify  $\psi^{(i)}$  with an element of  $G(U_i)$ , this follows by the sheaf condition on  $G$ : one has

$$\psi^{(i)}_{|_{U_{\{i,j\}}}} = \psi^{(j)}_{|_{U_{\{i,j\}}}}$$

because  $U_{\{i,j\}}$  is, by definition, the locus where  $\psi^{(i)}$  and  $\psi^{(j)}$  agree. Explicitly, for each  $T$  in  $\mathcal{C}$ ,  $\psi_T: X(T) \rightarrow G(T)$  is defined as follows: for each  $f \in X(T)$  and  $i \in I$ , let  $T_i := |U_i \times_X T|_T$  (compare with Lemma 1.2.10(iii)); then  $\psi_T(f)$  is the unique element in  $G(T)$  restricting to  $\psi^{(i)}_{T_i}(f|_{T_i})$  on each  $T_i$ .

Finally, let us show that  $\psi_T$  is bijective for every  $T \in \mathcal{C}$ . Let  $\xi \in G(T)$  and, for each  $i$ ,  $P_i := T \times_G G_i$ ; by definition, the structure morphism  $p_i: P_i \rightarrow U_i$  is the unique morphism of sheaves such that  $\psi^{(i)}_{P_i}(p_i) = \xi|_{p_i}$  (it is the unique such since  $\psi^{(i)}_{P_i}$  is injective). Since by assumption the  $P_i$ 's cover  $T$ , the uniqueness on double intersections guarantees that the  $p_i$ 's glue to a unique morphism  $T \rightarrow X$ , which is the desired preimage of  $\xi$ .  $\square$

*Remark 1.2.20.* We required that  $\mathcal{E}$  has gluings for ease of formulation (and because this holds in all the instances we shall deal with), but it is immediate to see that one just needs that every gluing datum in  $\mathcal{E}$  is effective in  $\mathcal{C}$ .

As a first application, for  $\mathcal{E} = \mathcal{C}$  one gets the following

**Proposition 1.2.21.** *Let  $\mathcal{C}'$  be a basis of  $(\mathcal{C}, |\cdot|)$  such that for each pair of morphisms  $X' \rightarrow Z'$  and  $Y' \rightarrow Z'$  in  $\mathcal{C}'$ , the fibre product  $X' \times_{Z'} Y'$  (in  $\mathcal{C}$ ) exists. Then  $\mathcal{C}$  has fibred products.*

*Proof.* Let  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  be morphisms in  $\mathcal{C}$ . By a slight abuse of notation, we write  $X \times_Z Y$  for the fibre product presheaf

$$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X) \times_{\mathrm{Hom}_{\mathcal{C}}(T, Z)} \mathrm{Hom}_{\mathcal{C}}(T, Y).$$

First, it is easy to see that the presheaf  $X \times_Z Y$  is actually a sheaf: indeed, let  $\{T_i \rightarrow T\}_i$  be an open covering and  $(\alpha_i, \beta_i)_{i \in I} \in \prod_{i \in I} X \times_Z Y(T_i)$  be a compatible family of sections; since  $X$  and  $Y$  are sheaves the  $\alpha_i$ 's and the  $\beta_i$ 's (separately) glue to unique  $\alpha \in \mathrm{Hom}_{\mathcal{C}}(T, X)$  and  $\beta \in \mathrm{Hom}_{\mathcal{C}}(T, Y)$ , and it is enough to show that  $f \circ \alpha = g \circ \beta$ . As these compositions agree, by assumption, on the open covering  $T_i$  of  $T$ , this follows because  $Z$  is a separated presheaf.

Next, if  $U \rightarrow X$ ,  $V \rightarrow Y$  and  $W \rightarrow Z$  are open immersions such that  $f|_U$  and  $g|_V$  factor through  $W$ , then the canonical natural transformation  $U \times_W V \rightarrow X \times_Z Y$  is an open immersion of functors. For a natural transformation  $(x, y): T \rightarrow X \times_Z Y$ , consider the pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & T \\ \downarrow & & \downarrow (x, y) \\ U \times_W V & \longrightarrow & X \times_Z Y \end{array}$$

we claim that  $P$  is represented by the open subobject  $S$  of  $T$  with underlying topological space  $|x|^{-1}(|U|) \cap |y|^{-1}(|V|)$ . For any  $A$  in  $\mathcal{C}$ , an element of  $\mathrm{Hom}_{\mathcal{C}}(A, P)$  is a triple  $(a, b, t) \in \mathrm{Hom}_{\mathcal{C}}(A, U) \times \mathrm{Hom}_{\mathcal{C}}(A, V) \times \mathrm{Hom}_{\mathcal{C}}(A, T)$  such that the squares

$$\begin{array}{ccc} A & \xrightarrow{a} & U \\ \downarrow b & & \downarrow \\ V & \longrightarrow & W \end{array}, \quad \begin{array}{ccc} A & \xrightarrow{a} & U \\ \downarrow t & & \downarrow \\ T & \xrightarrow{x} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{b} & V \\ \downarrow t & & \downarrow \\ T & \xrightarrow{y} & Y \end{array}$$

commute. A morphism  $A \rightarrow S$  yields such a triple: taking  $t$  equal to  $A \rightarrow S \rightarrow T$ , by the definition of  $S$   $x \circ t$  and  $y \circ t$  factor through the open immersions  $U \hookrightarrow X$  and  $V \hookrightarrow Y$ , respectively, and since the square

$$\begin{array}{ccc} T & \xrightarrow{x} & X \\ \downarrow y & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

commutes, the pair of maps  $(A \rightarrow U, A \rightarrow V)$  thus defined determines a morphism  $A \rightarrow U \times_W V$ . Conversely, given a triple  $(a, b, t) \in P(A)$ , the commutative squares above show that  $t$  factors (uniquely) through  $S$ .

Finally, if  $\{Z_i \rightarrow Z\}_{i \in I}$  and, for each  $i \in I$ ,  $\{X_{ij} \rightarrow X \times_Z Z_i\}_{j \in J}$  and  $\{Y_{ik} \rightarrow Y \times_Z Z_i\}_{k \in K}$  are open covers,

$$\{X_{ij} \times_{Z_i} Y_{ik} \rightarrow X \times_Z Y\}_{\substack{i \in I, \\ j \in J, k \in K}}$$

is an open cover of functors; since  $\mathcal{C}'$  is a basis, one can take all the  $Z_i$ ,  $X_{ij}$  and  $Y_{ik}$  in  $\mathcal{C}'$ , and then by assumption  $X_{ij} \times_{Z_i} Y_{ik}$  is representable for all  $i, j$  and  $k$ . Therefore, the assumptions of Proposition 1.2.19 are met, whereby  $X \times_Z Y$  is representable.  $\square$

*Remark 1.2.22.* More precisely, the proof shows that if  $\{Z_i \rightarrow Z\}_{i \in I}$  and, for each  $i \in I$ ,  $\{X_{ij} \rightarrow X \times_Z Z_i\}_{j \in J}$  and  $\{Y_{ik} \rightarrow Y \times_Z Z_i\}_{k \in K}$  are open covers,

$$\{X_{ij} \times_{Z_i} Y_{ik} \rightarrow X \times_Z Y\}_{\substack{i \in I, \\ j \in J, k \in K}}$$

is an open cover, too.

### A few properties of morphisms

Given the topological nature of the definition of categories with open immersions and gluings, it comes as no surprise that one can make sense of topological properties of morphisms as quasi-compactness and quasi-separatedness in this context, too. Assuming the existence of a good basis (Definition 1.2.29), these notions are particularly well-behaved (i.e. they are what Vakil [Vak, Definition 8.1.1] calls “reasonable” properties, see 1.2.25).

Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions and gluings.

**Definition 1.2.23.** • An object  $X$  of  $\mathcal{C}$  is called quasi-compact if  $|X|$  is quasi-compact.

- A morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  is called quasi-compact if  $|f|$  is quasi-compact.
- A morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$  is called quasi-separated if the diagonal  $X \rightarrow X \times_Y X$  is quasi-compact.

*Remark 1.2.24.* By Lemma 1.2.10(iii),  $f$  is quasi-compact if and only if for each quasi-compact open subobject  $W \rightarrow Y$  of  $Y$ ,  $W \times_Y X$  is quasi-compact.

Thanks to the results developed above, the results in [Vak, Section 11.1] concerning “reasonable” properties carry through verbatim for categories with open immersions and gluings, as soon as  $\mathcal{C}$  has fibred products. We record them here without proof.

**Definition 1.2.25** ([Vak, Definition 8.1.1]). Let  $\mathcal{C}$  have fibred products. A class  $\mathcal{P}$  of morphisms of  $\mathcal{C}$  is called “reasonable” if it contains all isomorphisms and

- (i)  $\mathcal{P}$  is closed under composition;
- (ii)  $\mathcal{P}$  is closed under fibred products, i.e. if  $f: X \rightarrow Y$  is in  $\mathcal{P}$  and  $T \rightarrow Y$  is any morphism, the structure map  $T \times_Y X \rightarrow T$  is again in  $\mathcal{P}$ .
- (iii)  $\mathcal{P}$  is local on the target, i.e. if  $f: X \rightarrow Y$  is a morphism such that there exists an open cover  $\{V_i \rightarrow Y\}_i$  with each  $V_i \times_Y X \rightarrow V_i$  in  $\mathcal{P}$ , then  $f$  is in  $\mathcal{P}$  as well.

For example, Lemmata 1.2.16, 1.2.10(i) and 1.2.10(iii) together show that the class of open immersions of  $\mathcal{C}$  is “reasonable”.

**Proposition 1.2.26** (Cancellation property, [Vak, 11.1.1]). *Let  $\mathcal{P}$  be a class of morphisms of  $\mathcal{C}$  which is closed under composition and arbitrary base-change. Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be morphisms whose composition is in  $\mathcal{P}$ ; if the diagonal map  $Y \rightarrow Y \times_Z Y$  is in  $\mathcal{P}$ , then  $f$  is in  $\mathcal{P}$ , too.*

**Proposition 1.2.27** ([Vak, Theorem 11.1.2]). *If  $\mathcal{P}$  is a “reasonable” class of morphisms of  $\mathcal{C}$ , then the class of morphisms  $X \rightarrow Y$  in  $\mathcal{C}$  such that the diagonal  $X \rightarrow X \times_Y X$  is in  $\mathcal{P}$  is again “reasonable”.*

**Proposition 1.2.28** ([Vak, 8.1.A]). *Let  $\mathcal{P}$  be a class of morphisms of  $\mathcal{C}$  which is closed under composition and arbitrary base-change. Let  $Z$  be an object of  $\mathcal{C}$ ,  $f: X \rightarrow Y$ ,  $f': X' \rightarrow Y'$  morphisms in  $\mathcal{P}$  over  $Z$ ; then the induced map  $X \times_Z X' \rightarrow Y \times_Z Y'$  is in  $\mathcal{P}$  as well.*

In order for the class of quasi-compact morphisms to be “reasonable”, we assume that  $\mathcal{C}$  has a good basis, in the following sense.

**Definition 1.2.29.** We call a basis  $\mathcal{C}'$  of  $\mathcal{C}$  good if it is closed under isomorphisms and

- (i) For all  $X \rightarrow Z$  and  $Y \rightarrow Z$  in  $\mathcal{C}'$ , the pullback  $X \times_Z Y$  exists and is again in  $\mathcal{C}'$ ;
- (ii)  $|X|$  is quasi-compact for all  $X \in \mathcal{C}'$ .

For any  $X$  in  $\mathcal{C}$ , an open cover  $\{X_i \rightarrow X\}_i$  with all the  $X_i$  belonging to  $\mathcal{C}'$  will be called good.

For the rest of this section, assume that  $\mathcal{C}$  has a good basis  $\mathcal{C}'$ , which we regard as fixed.

**Lemma 1.2.30.** *For a morphism  $f: X \rightarrow Y$ , the following are equivalent:*

- (i) *The map  $f$  is quasi-compact.*
- (ii) *There exists an open covering  $\{V_i \rightarrow Y\}_i$  with  $V_i$  in  $\mathcal{C}'$  such that each  $V_i \times_Y X$  is quasi-compact.*

*Proof.* Let  $W$  be a quasi-compact open subobject of  $Y$  and assume there exists a good open covering  $\mathcal{V} = \{V_i \rightarrow Y\}_{i \in I}$  such that each  $V_i \times_Y X$  is quasi-compact, so that for each  $i$  there is a *finite* good covering  $\{U_{ij} \rightarrow V_i \times_Y X\}_{j \in J_i}$ .

Assume first that  $|W| \subseteq |V_i|$  for some  $i$ , so that  $W$  is an open subobject of  $V_i$ . As  $\mathcal{C}'$  is a basis, one can pick a good covering  $\{W_k \rightarrow W\}_{k \in K}$ , which, by the compactness of  $W$ , we can assume to be finite; by Proposition 1.2.21 and Remark 1.2.22,  $W \times_Y X = W \times_{V_i} (V_i \times_Y X)$  admits the open covering

$$\{W_k \times_{V_i} U_{ij} \rightarrow W \times_Y X\}_{\substack{j \in J_i, \\ k \in K}},$$

which is finite. Moreover, as  $V_i$  and all the  $W_k$  and  $U_{ij}$  are in  $\mathcal{C}'$ , their fibred products  $W_k \times_{V_i} U_{ij}$  are, too, and in particular are quasi-compact. Therefore  $|W \times_Y X|$  is quasi-compact, as a finite union of quasi-compact topological spaces.

In general, again by the compactness of  $W$  and since  $\mathcal{C}'$  is a basis, one can consider a finite good covering  $\{W_l \rightarrow W\}_{l \in L}$  of  $W$  such that for each  $l \in L$ ,  $|W_l| \subseteq |V_{i(l)}|$  for some  $i(l) \in I$ . Replacing  $W$  by  $W_l$ ,  $Y$  by  $V_{i(l)}$  and  $X$  by  $V_{i(l)} \times_Y X$ , we are then reduced to the situation at the previous paragraph, which concludes the proof.  $\square$

**Lemma 1.2.31.** *The classes of quasi-compact and of quasi-separated morphisms in  $\mathcal{C}$  are “reasonable”.*

*Proof.* By Proposition 1.2.27, it suffices to deal with quasi-compactness. Stability under composition amounts to the corresponding fact for quasi-compact maps of topological spaces, which is obvious.

In order to show locality on the target, let  $X \rightarrow Y$  be a morphism and  $\{Y_i \rightarrow Y\}_i$  be an open covering such that each  $X_i := X \times_Y Y_i \rightarrow Y_i$  is quasi-compact. For each  $i$ , there exists a good open covering  $\{Y_{ij} \rightarrow Y_i\}_j$  such that  $Y_{ij} \times_{Y_i} X_i$  is quasi-compact; but then  $\{Y_{ij} \rightarrow Y_i \rightarrow Y\}_{i,j}$  is a good open covering such that  $Y_{ij} \times_Y X \cong Y_{ij} \times_{Y_i} X_i$  is quasi-compact, whereby  $X \rightarrow Y$  is quasi-compact by Lemma 1.2.30.

Finally, let  $f: X \rightarrow Y$  be a quasi-compact morphism,  $f': X' \rightarrow Y'$  its base-change along a morphism  $Y' \rightarrow Y$ . Pick a good open cover  $\{V_i \rightarrow Y\}_i$  and, for

each  $i$ , good open covers  $\{V'_{ij} \rightarrow V_i \times_Y Y'\}_{j \in J}$  and  $\{X_{ik} \rightarrow V_i \times_Y X\}_{k \in K}$ ; since  $\mathcal{C}'$  is a basis and quasi-compactness is local on the target, without loss of generality we may assume that  $Y' = V'_{ij}$  and  $X = V_i \times_Y X$  for some  $i, j$ . In particular,  $X$  is quasi-compact because  $f$  is quasi-compact, whereby one can assume  $K$  to be finite. By Proposition 1.2.21 and the remark after it,  $|X'| = \bigcup_{k \in K} |X_{ik}| \times_{|Y'|} |Y'|$ , which is quasi-compact because  $X_{ik} \times_Y Y'$  is in  $\mathcal{C}'$  by construction.  $\square$

**Lemma 1.2.32.** *Let  $f: X \rightarrow Y$  be a morphism in  $\mathcal{C}$ . If there exists a good open cover  $\{Y_i \rightarrow Y\}_i$  such that  $\{X \times_Y Y_i \rightarrow X\}_i$  is again a good open cover, then  $f$  is quasi-compact and quasi-separated.*

*Proof.* Quasi-compactness is immediate by Lemma 1.2.30. For quasi-separatedness, take a good open cover  $\{Y_i \rightarrow Y\}_i$  such that for each  $i$   $X_i := X \times_Y Y_i$  is again in  $\mathcal{C}'$ . By the definition of a good basis,  $X_i \times_{Y_i} X_i$  is again in  $\mathcal{C}'$  for all  $i$  and by Remark 1.2.22 and the definition of  $X_i$ ,  $\{X_i \times_{Y_i} X_i \rightarrow X \times_Y X\}_i$  is again a good open covering. Moreover,  $X \times_{X \times_Y X} (X_i \times_{Y_i} X_i) \cong X_i$  (as follows from the proof of Proposition 1.2.21), whence the claim, again by Lemma 1.2.30.  $\square$

### 1.3 A representability result

Let  $(\mathcal{C}, |\cdot|_{\mathcal{C}})$  and  $(\mathcal{D}, |\cdot|_{\mathcal{D}})$  be categories with open immersions and gluings and let  $\mathcal{C}'$  be a basis of  $\mathcal{C}$ . Recall that for a functor  $F: \mathcal{C}' \rightarrow \mathcal{D}$ ,  $F^*: \text{PSh}(\mathcal{D}) \rightarrow \text{PSh}(\mathcal{C}')$  denotes the pullback along  $F$ , i.e. the functor defined on objects by  $F^*(\alpha) = \alpha \circ F$  for any  $\alpha \in \text{PSh}(\mathcal{D})$  and correspondingly on morphisms.

The reader can keep in mind the following example, which is the simplest case of the mixed-characteristic Greenberg functor mentioned in the introduction: let  $k$  be a perfect field,  $n$  a positive integer,  $W_n(k)$  the ring of  $p$ -typical Witt vectors of length  $n$ ,  $\mathcal{C} = \text{Sch}/k$  and  $\mathcal{C}' = \text{Aff}/k$  its full subcategory of affine schemes,  $\mathcal{D} = \text{Sch}/W_n(k)$  and  $F(\text{Spec } A) = \text{Spec } W_n(A)$ ;  $F^*X$  is then the  $n$ -th loop functor in mixed characteristic. Several other instances will be discussed in Chapter 2.

Returning to the general case, we shall be interested in the representability of the functors  $F^*X$ , for  $X$  in  $\mathcal{D}$ . Of course, at this level of generality the question is hopeless; taking a cue from the classical version of the Greenberg functor [BGA18a, section 9], we require the existence of a natural transformation  $\gamma: |\cdot|_{\mathcal{C}'} \rightarrow |F(\cdot)|_{\mathcal{D}} :=$

$|\cdot|_{\mathcal{D}} \circ F$ , as depicted in the diagram (where  $a: Z \rightarrow T$  is any morphism in  $\mathcal{C}'$ )

$$\begin{array}{ccc} |Z| & \xrightarrow{|a|_{\mathcal{C}'}} & |T| \\ \downarrow \gamma_Z & & \downarrow \gamma_T \\ |F(Z)|_{\mathcal{D}} & \xrightarrow{|F(a)|_{\mathcal{D}}} & |F(T)|_{\mathcal{D}} \end{array}$$

For example, in the case of the  $n$ -th loop functor in mixed characteristic, for a  $k$ -algebra  $A$   $\gamma_{\mathrm{Spec} A}$  is the natural identification of topological spaces

$$|\mathrm{Spec} A| \xrightarrow{\sim} |\mathrm{Spec} W_n(A)|.$$

Assuming that  $\gamma$  is a natural isomorphism (i.e.  $\gamma_T$  is a homeomorphism for any  $T$  in  $\mathcal{C}'$ ) and  $F$  respects open immersions, for  $X$  in  $\mathcal{D}$  we shall see in Theorem 1.3.7 that the representability of  $F^*X$  is reduced to that of  $F^*X_i$  for an open cover  $\{X_i \rightarrow X\}_i$ .

In order to make the notation a bit less unpleasant, in the following we shall sometimes omit round brackets, and if  $u: X \rightarrow Y$  is an open immersion, we shall write  $|X|$  for the open subspace  $\mathrm{Im}(|u|) \subseteq |Y|$ . Moreover, for an object  $T$  in  $\mathcal{C}$  and a presheaf  $\alpha$  on  $\mathcal{C}$ , we shall tacitly identify a natural transformation  $T \rightarrow \alpha$  with the element of  $\alpha(T)$  corresponding to it via Yoneda's Lemma.

**Lemma 1.3.1.** *Let  $(\mathcal{C}, |\cdot|_{\mathcal{C}})$  and  $(\mathcal{D}, |\cdot|_{\mathcal{D}})$  be categories with open immersions,  $\mathcal{C}' \subseteq \mathcal{C}$  a basis,  $F: \mathcal{C}' \rightarrow \mathcal{D}$  a functor.*

*Assume that there exists a natural transformation*

$$\gamma: |\cdot|_{\mathcal{C}'} \rightarrow |F(\cdot)|_{\mathcal{D}}$$

*such that for any morphism  $a: Z \rightarrow T$  in  $\mathcal{C}'$  and any open subset  $V \subseteq |F(T)|$ , if  $\gamma_T \circ |a|$  factors through  $V$ , then so does  $|F(a)|$  (so that, in the situation depicted by the diagram*

$$\begin{array}{ccc} |Z| & \xrightarrow{|a|} & |T| \\ \downarrow \gamma_Z & \searrow & \downarrow \gamma_T \\ |F(Z)| & \xrightarrow{|F(a)|} & |F(T)| \\ & \nearrow & \\ & V & \hookrightarrow \end{array},$$

*if one of the dashed arrows can be filled, so can the other one).*

*Then  $F^*$  respects open immersions and sends open covers to open covers, i.e.*

- for any open immersion  $f: X \rightarrow Y$  in  $\mathcal{D}$ , the map of presheaves on  $\mathcal{C}'$   $F^*(f): F^*(X) \rightarrow F^*(Y)$  is an open immersion (Definition 1.2.17);
- for any open covering  $\{X_i \rightarrow X\}_i$  in  $\mathcal{D}$ , the family of subpresheaves on  $\mathcal{C}'$   $\{F^*X_i \rightarrow F^*X\}_i$  is an open covering.

It is easy to show (see the proof of Theorem 1.3.7) that the assumptions are met if  $\gamma$  is a natural isomorphism.

*Proof.* Notice first that  $F^*: \mathbf{PSh}(\mathcal{D}) \rightarrow \mathbf{PSh}(\mathcal{C}')$  sends subpresheaves to subpresheaves, so that the statement makes sense.

Let  $u: X \rightarrow Y$  be an open immersion in  $\mathcal{D}$ ,  $T \in \mathcal{C}'$ ,  $\xi: T \rightarrow F^*Y$  a natural transformation,  $P$  the fibred product  $P := T \times_{F^*Y} F^*X$ . Set  $W = (|\xi|^{-1}(|X|))_{F(T)}$ ,  $U = \gamma_T^{-1}(|W|) \subseteq |T|$ . We claim that  $P$  is represented by  $U_T$ .

Let  $Z$  be an object of  $\mathcal{C}'$ . Since  $u$  and  $W \rightarrow F(T)$  are open immersions, unwinding the definitions one sees that  $P(Z) = \{a \in T(Z) | F(a) \text{ factors through } W\}$  (equivalently, the morphisms  $a \in T(Z)$  such  $|F(a)|$  factors through  $|W|$ ). If  $a \in T(Z)$  factors through  $\mathfrak{o}_U^T$ , by assumption  $|F(a)|$  factors through  $|W|$ . This yields a map of presheaves  $U_T \rightarrow P$ .

Conversely, if  $F(a)$  factors through  $W$ , since

$$\begin{array}{ccc} U & \longrightarrow & |T| \\ \downarrow (\gamma_T)|_U & & \downarrow \gamma_T \\ |W| & \longrightarrow & |F(T)| \end{array}$$

is cartesian by construction, the diagram

$$\begin{array}{ccccc} & & |a| & & \\ & \nearrow & & \searrow & \\ |Z| & \cdots \longrightarrow & U & \hookrightarrow & |T| \\ \downarrow \gamma_Z & & \downarrow (\gamma_T)|_U & & \downarrow \gamma_T \\ |F(Z)| & \longrightarrow & |W| & \hookrightarrow & |F(T)| \\ & \searrow & & \nearrow & \\ & & |F(a)| & & \end{array}$$

admits a filler, whence, by the universal property of  $U_T$ ,  $a$  factors through  $\mathfrak{o}_U^T$ . Since for each  $T$  and  $\xi$  as above, the map  $F^*X \times_{F^*Y} T \rightarrow T$  is an open immersion in  $\mathcal{C}$ ,  $F^*X \rightarrow F^*Y$  is an open immersion of functors.

Let now  $\{V_i\}_i$  be an open cover of some  $X$  in  $\mathcal{D}$ . We have to prove that for every  $T \in \mathcal{C}'$  and every  $\xi: T \rightarrow F^*Y$ ,  $\{F^*V_i \times_{F^*Y} T\}_i$  is an open cover of  $T$ . We have just proved that for each  $i$ ,  $F^*V_i \times_{F^*Y} T$  is represented by  $((|\xi| \circ \gamma_T)^{-1}|V_i|)_T$ ;

since the sets  $|V_i|$  cover  $|Y|$ , their preimages under  $|\xi| \circ \gamma_T$  cover  $|T|$ , which is what had to be shown.  $\square$

*Remark 1.3.2.* One might compare and contrast this with the case of Weil restriction along an arbitrary morphism of schemes  $S' \rightarrow S$ : denoting by  $s: \text{Sch}/S \rightarrow S$  the source functor, there one has instead a natural transformation

$$\begin{array}{ccc} & s & \\ \text{Sch}/S & \begin{array}{c} \uparrow \\ \parallel \\ \uparrow \end{array} & \text{Sch} \\ & (\cdot) \times_S S' & \end{array},$$

which is instrumental in the proof that, under some conditions on  $S \rightarrow S'$ , Weil restriction respects open immersions and coverings [BLR90, Proposition 2 and Theorem 4].

**Lemma 1.3.3.** *Let  $(\mathcal{C}, |\cdot|_{\mathcal{C}})$  and  $(\mathcal{D}, |\cdot|_{\mathcal{D}})$  be categories with open immersions,  $\mathcal{C}' \subseteq \mathcal{C}$  a basis,  $F: \mathcal{C} \rightarrow \mathcal{D}$  a functor preserving open immersions and  $\gamma: |\cdot|_{\mathcal{C}'} \rightarrow |F(\cdot)|_{\mathcal{D}}$  a natural transformation.*

- (i) *If  $\gamma_T$  is surjective and open for all  $T$  in  $\mathcal{C}'$ , then  $F$  maps open coverings to open coverings.*
- (ii) *If  $\gamma_T$  is a homeomorphism for all  $T$  in  $\mathcal{C}'$ , then  $F^*: \text{PSh}(\mathcal{D}) \rightarrow \text{PSh}(\mathcal{C}')$  takes sheaves to sheaves.*

*Proof.* 1): Let  $T \in \mathcal{C}'$  and  $\mathcal{U} = \{T_i \rightarrow T\}_{i \in I}$  be an open covering in  $\mathcal{C}'$ . As  $\gamma_T$  is surjective, for any  $y \in |F(T)|$  there exists some  $i \in I$  and a point  $x \in |T|$  such that  $\gamma_T(x) = y$  and  $x \in |T_i|$ ; hence, as  $\gamma$  is a natural transformation,  $y$  is in the image of  $|F(T_i)| \rightarrow |F(T)|$ . Therefore, the family  $\{|F(T_i)| \rightarrow |F(T)|\}_i$  is jointly surjective.

2): Let  $\mathcal{U} = \{T_i \rightarrow T\}_i$  be an open covering of  $T$  in  $\mathcal{C}'$ ; a matching family on  $\mathcal{U}$  of elements of  $F^*P$  consists of elements  $s_i \in P(F(T_i))$  such that for each morphism  $\alpha: T_i \rightarrow T_j$  over  $X$ ,  $P(\alpha)(s_j) = s_i$ ; explicitly, such a family consists of elements  $s_i \in P(F(T_i))$  such that for each morphism  $\alpha: T_i \rightarrow T_j$  over  $X$ ,  $P(\alpha)(s_j) = s_i$ .

We can assume without loss of generality that  $\mathcal{U} = \{\mathfrak{o}_{\mathcal{C}, U_i}^X\}_i$  and  $F(\mathcal{U}) = \{\mathfrak{o}_{\mathcal{D}, U_i}^{FX}\}_i$  for some open subsets  $U_i$  of  $|X|$ , where we are using  $\gamma_X$  to identify each  $U_i$  with an open subset of  $|F(X)|$ . For any  $i, j$ ,  $\mathfrak{o}_{\mathcal{D}, U_i}^{FX}$  factors (necessarily in a unique way) through  $\mathfrak{o}_{\mathcal{D}, U_j}^{FX}$  if and only if  $U_i \subseteq U_j$ , which in turn is equivalent to  $\mathfrak{o}_{\mathcal{C}, U_i}^X$  factoring (again, uniquely) through  $\mathfrak{o}_{\mathcal{C}, U_j}^X$ ; this implies that  $(s_i)_i$  is a matching family on  $F(\mathcal{U})$  of elements of  $P$ , and thus, since  $F(\mathcal{U})$  is an open covering and  $P$  a sheaf, it has an amalgamation  $s$ . As any amalgamation of  $(s_i)_i$  as a matching family on  $\mathcal{U}$  of

elements of  $F^*P$  is also an amalgamation of  $(s_i)_i$  as a matching family on  $F(\mathcal{U})$  of elements of  $P$ , there is a unique such  $s$ , whereby  $F^*P$  is a sheaf.  $\square$

We record the following corollary; as it will be irrelevant for the following, we will not explain the terms appearing in its statement, just referring to [MM12] for their definitions.

**Corollary 1.3.4.** *In the notation of the previous lemma,  $F^*: \mathcal{D} \rightarrow \mathrm{Sh}(\mathcal{C}')$  is a flat [MM12, Definition VII.7.1] and continuous [MM12, pag. 393] functor and thus  $F$  induces a geometric morphism  $\mathrm{Sh}(\mathcal{C}') \rightarrow \mathrm{Sh}(\mathcal{D})$  with inverse image  $F^*(\alpha) = \alpha \circ F$  [MM12, Definition VII.1.1] for any sheaf  $\alpha$  on  $\mathcal{D}$ .*

*Proof.* This is immediate, combining the previous lemma and [MM12, Theorem VII.10.5].  $\square$

We make a small detour and observe that, under a mild additional assumption, the conclusion of Lemma 1.3.3(ii) can also be recovered from the following lemmata.

**Lemma 1.3.5.** *Let  $F: \mathcal{X} \rightarrow \mathcal{Y}$  be a functor between categories with Grothendieck pretopologies such that for each open covering  $\{X_i \rightarrow X\}$  in  $\mathcal{X}$ ,*

$$F(X_i \times_X X_j) \cong F(X_i) \times_{F(X)} F(X_j).$$

*If  $F$  maps open coverings to open coverings, then  $F^*: \mathrm{PSh}(\mathcal{Y}) \rightarrow \mathrm{PSh}(\mathcal{X})$  preserves sheaves.*

*Proof.* Let  $\mathcal{F} \in \mathrm{Sh}(\mathcal{Y})$ ,  $\{X_i \rightarrow X\}$  an open covering in  $\mathcal{X}$  and  $(s_i)$  a matching family of elements of  $F^*\mathcal{F}$ . Since  $\{F(X_i) \rightarrow F(X)\}_i$  is, by assumption, again an open covering and as  $F(X_i) \times_{F(X)} F(X_j)$  is isomorphic to  $F(X_i \times_X X_j)$ , the sheaf condition on  $\mathcal{F}$  implies that there  $(s_i)_i$  has a unique amalgamation.  $\square$

**Lemma 1.3.6.** *If there exists a natural isomorphism  $\gamma: |\cdot|_{\mathcal{C}'} \rightarrow |F(\cdot)|_{\mathcal{D}}$ , let  $Y \rightarrow X$  be a morphism in  $\mathcal{C}'$  and  $\mathfrak{o}_U^X$  an open immersion in  $\mathcal{C}'$ . If  $U_X \times_X Y$  is in  $\mathcal{C}'$ , then  $F(U_X \times_X Y) \cong F(U_X) \times_{F(X)} F(Y)$ .*

*Proof.* Since  $F$  respects open immersions,  $F(U_X) \rightarrow F(Y)$  is an open immersion, whereby by Lemma 1.2.10(iii) one has

$$F(Y) \times_{F(X)} F(U_X) \cong (|F(Y)| \times_{|F(X)|} |F(U_X)|)_{F(Y)}.$$

As  $\gamma$  is a natural isomorphism, we can identify  $|F(Y)| \times_{|F(X)|} |F(U_X)|$  with  $|Y| \times_{|X|} |U|$  via  $\gamma_Y$ . Similarly,  $|F(Y \times_X U_X)| \xrightarrow{\sim} |Y| \times_{|X|} |U|$ , whereby

$$(|F(Y)| \times_{|F(X)|} |F(U_X)|)_{F(Y)} \cong (|F(Y \times_X U_X)|)_{F(Y)} \cong F(Y \times_X U_X),$$

because  $(|F(Y \times_X U_X)|)_{F(Y)} \rightarrow F(Y)$  and  $F(Y \times_X U_X) \rightarrow F(Y)$  are open immersions into the same object and their sources have the same underlying topological space.  $\square$

If for any open covering  $\{X_i \rightarrow X\}_{i \in I}$  in  $\mathcal{C}'$  and any two indices  $i, j \in I$  the fibred product  $X_i \times_X X_j$  is again in  $\mathcal{C}'$ , then one recovers Lemma 1.3.3(ii), as anticipated.

Coming back to the main thread of the discourse, we now present the representability result we were after:

**Theorem 1.3.7.** *Let  $(\mathcal{C}, |\cdot|_{\mathcal{C}})$  and  $(\mathcal{D}, |\cdot|_{\mathcal{D}})$  be categories with open immersions whose topologies are subcanonical,  $\mathcal{C}'$  a basis of  $\mathcal{C}$  and  $\mathcal{E}$  a subcategory with open immersions of  $\mathcal{C}$  which has gluings.*

*Let  $F: \mathcal{C}' \rightarrow \mathcal{D}$  be a functor which respects open immersions and*

$$\gamma: |\cdot|_{\mathcal{C}'} \rightarrow |F(\cdot)|_{\mathcal{D}}$$

*be a natural isomorphism.*

*If  $\{X_i \rightarrow X\}_{i \in I}$  is an open cover of an object  $X \in \mathcal{D}$  and for all  $i \in I$  the presheaf  $F^*X_i$  is represented by some  $Z_i$  in  $\mathcal{E}$ , then the  $Z_i$ 's glue to a representing object for  $F^*X$ .*

*Proof.* Since  $\gamma$  is a natural isomorphism and  $F$  respects open immersions,  $F^*X$  is a sheaf on  $\mathcal{C}'$  for all  $X$  in  $\mathcal{D}$  by Lemma 1.3.3 and, consequently, it can be extended in a unique way to a sheaf on  $\mathcal{C}$  (Proposition 1.1.11). Notice that if we show that  $\{F^*X_i \rightarrow F^*X\}_i$  is an open covering, we can conclude by Proposition 1.2.19. To this end, it is enough to check that the hypothesis of Lemma 1.3.1 holds; but this is implied by the surjectivity of  $\gamma_T$ : indeed, given  $Z, T$  in  $\mathcal{C}'$ ,  $V \subseteq |F(T)|$  open and  $a: Z \rightarrow T$  such that  $|a|$  factors through  $\gamma_T^{-1}(V)$ , one has

$$V \supseteq \text{Im}(\gamma_T \circ |a|) = \text{Im}(|F(a)| \circ \gamma_Z) = \text{Im}|F(a)|,$$

which is precisely the required condition.  $\square$

It is often useful to combine this result with the following elementary observation:

*Remark 1.3.8.* The functor  $F^*: \mathcal{D} \rightarrow \text{PSh}(\mathcal{C}')$  respects limits (indeed, limits in presheaf categories are computed pointwise, whereby the statement just amounts to the fact that Hom-functors commute with limits). In particular, if  $\mathcal{A}$  is a class of objects of  $\mathcal{D}$  and  $\Phi: I \rightarrow \mathcal{D}$  a functor such that  $F^*(\Phi(i))$  is representable for all  $i \in I$ , then  $F^* \lim \Phi_i$  is representable if and only if  $\lim_i F^*(\Phi(i))$  exists in  $\mathcal{C}$ . Thus,

assuming  $F^*$  respects open immersions and coverings (and in particular, under the assumptions of the previous theorem), if every object of  $\mathcal{D}$  can be obtained by gluing objects of a certain full subcategory  $\mathcal{D}' \subseteq \mathcal{D}$  and a  $\mathcal{A}$  is a class of objects of  $\mathcal{D}'$  generating it under limits (in  $\mathcal{D}'$ ) of a certain shape, then, as soon as the relevant limits exist in  $\mathcal{C}$ , one is reduced to checking the representability of  $F^*A$  for  $A \in \mathcal{A}$ ; in many concrete cases, doing so is much easier than the general case.

*Example 1.3.9.* Let  $S = \operatorname{Spec} R$  be an affine scheme, and take  $\mathcal{D} = \operatorname{Sch}/S$  in the previous remark. Any object of  $\mathcal{D}$  can be obtained by glueing objects of the subcategory  $\mathcal{D}'$  of affine schemes over  $S$ . In turn,  $\mathcal{D}'$  is generated by the class  $\mathcal{A}$  consisting of the single element  $\mathbb{A}_S^1$  under products and equalisers: indeed, it is clear that for any (possibly infinite) index set  $\Lambda$ ,  $\mathbb{A}_S^\Lambda$  can be obtained as an infinite product of  $\mathbb{A}_S^1$ ; moreover, for any affine  $S$ -scheme  $Z = \operatorname{Spec} A$ , one can write  $A$  as a quotient  $R[x_\alpha]_{\alpha \in \Lambda} / (f_\beta)_{\beta \in \Xi}$  for some index sets  $\Lambda$  and  $\Gamma$ , whence an equaliser diagram

$$X = \operatorname{Spec} A \rightarrow \mathbb{A}_S^\Lambda \rightrightarrows \mathbb{A}_S^\Xi,$$

where the parallel arrows correspond on coordinate rings to the arrows

$$R[y_\beta]_{\beta \in \Xi} \rightarrow R[x_\alpha]_{\alpha \in \Lambda}$$

determined one by  $y_\beta \mapsto f_\beta$  and the other by  $y_\beta \mapsto 0$ . The upshot is that, in the situation of the previous remark,  $F^*X$  is representable for any  $S$ -scheme  $X$  as soon as  $F^*\mathbb{A}_S^1$  is representable.

*Remark 1.3.10.* Again (compare with Remark 1.2.20), we assumed a bit more than we could have: as a matter of fact, for the conclusion of Theorem 1.3.7 to hold it is enough to ask that any gluing datum of  $\mathcal{E}$  is effective in  $\mathcal{C}$ , or, actually, even just that the gluing datum

$$((U_i)_i, (\{U_{ij} \rightarrow U_i\}_j)_i, (U_{ij} \xrightarrow{\sim} U_{ji}))$$

of Proposition 1.2.19 (where, in the notation of the proof of the theorem,  $Z_i = U_i$ ) is effective.

*Remark 1.3.11.* If  $(\mathcal{C}, |\cdot|_{\mathcal{C}})$  and  $(\mathcal{D}, |\cdot|_{\mathcal{D}})$  are categories with open immersions and gluings and  $\mathcal{C}'$  is a basis of  $\mathcal{C}$ , any functor  $F: \mathcal{C}' \rightarrow \mathcal{D}$  that respects open immersions can be extended to a functor  $\tilde{F}: \mathcal{C} \rightarrow \mathcal{D}$ , and if  $F^*X$  is representable for any  $X$  in  $\mathcal{D}$ , then  $F^*$  is a right adjoint to  $\tilde{F}$ . In general,  $\tilde{F}$  is, however, not canonically defined, but only up to isomorphism (since the construction of  $\tilde{F}(T)$  for an arbitrary  $T$  in  $\mathcal{C}$  involves taking an open cover  $\{T_i \rightarrow T\}_i$  and gluing the various  $F(T_i)$ ).

## 1.4 Preservation of properties of morphisms and change of functors

In this brief section we revisit the contents of [BGA18a, Sections 8 and 9] from our perspective. As before, we denote by  $\mathcal{C}$  and  $\mathcal{D}$  categories with open immersions and gluings and by  $\mathcal{C}'$  a basis of  $\mathcal{C}$ .

Consider two functors  $F, G: \mathcal{C}' \rightarrow \mathcal{D}$  preserving open immersions, connected by a natural transformation  $\tau: G \rightarrow F$ ;  $\tau$  clearly induces a natural transformation of functors

$$\tau^*: F^* \rightarrow G^*$$

(keep in mind that both  $F^*$  and  $G^*$  are functors  $\mathcal{D} \rightarrow \text{PSh}(\mathcal{C}')$ ).

**Definition 1.4.1.** Let  $\mathcal{X}$  be a category and  $\Lambda$  a class of morphisms of  $\mathcal{X}$ . We call a morphism  $f: X \rightarrow Y$  in  $\mathcal{X}$   $\Lambda$ -smooth/unramified/étale if each commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow l & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

with  $A \rightarrow B$  in  $\Lambda$  has at least/at most/exactly one lift  $l$  (i.e. a morphism making the two triangles commute).<sup>2</sup>

If  $\mathcal{X}$  has a terminal object  $*$ , we call an object  $X$   $\Lambda$ -smooth/unramified/étale if the unique morphism  $X \rightarrow *$  is.

In the situation above, we also say (somehow improperly, but rather conveniently) that  $f$  is  $\tau$ -smooth/unramified/étale if it has the relevant lifting property for  $\Lambda = \{\tau_X\}_{X \in \mathcal{C}'}$ .

**Lemma 1.4.2.** (i) *If  $f: X \rightarrow Y$  is  $\tau$ -smooth/unramified/étale in  $\mathcal{D}$ , then the induced map of presheaves  $F^*X \rightarrow P$  in the pullback diagram*

$$\begin{array}{ccccc} F^*X & & & & F^*(f) \\ & \searrow & & \searrow & \\ & & P & \longrightarrow & F^*Y \\ & \searrow \tau_X & \downarrow \lrcorner & & \downarrow \tau_Y^* \\ & & G^*X & \xrightarrow{G^*(f)} & G^*Y \end{array}$$

*is an epimorphism/monomorphism/isomorphism.*

---

<sup>2</sup>The property of being  $\Lambda$ -smooth in the definition above is usually called the right lifting property with respect to  $\Lambda$ ; we shall not use this terminology because we do not know of analogous names for  $\Lambda$ -unramifiedness and  $\Lambda$ -étaleness.

- (ii) Let  $f: X \rightarrow Y$  be a morphism in  $\mathcal{D}$  and assume that  $F^*X$  and  $F^*Y$  are representable; if  $f$  is  $F(\Lambda)$ -smooth/unramified/étale for some class of morphisms  $\Lambda$  in  $\mathcal{C}'$ , then  $F^*(f)$  is  $\Lambda$ -smooth/unramified/étale.

*Proof.* 1): For any  $T$  in  $\mathcal{C}'$ ,

$$P(T) = \{(a, b) \in \text{Hom}_{\mathcal{D}}(F(T), Y) \times \text{Hom}_{\mathcal{D}}(G(T), X) \mid \begin{array}{ccc} G(T) & \xrightarrow{b} & X \\ \downarrow \tau_T & & \downarrow f \\ F(T) & \xrightarrow{a} & Y \end{array} \text{ commutes}\}$$

and the induced map  $F^*X(T) \rightarrow P(T)$  takes  $g \in \text{Hom}_{\mathcal{D}}(F(T), X)$  to the pair  $(f \circ g, g \circ \tau_T)$ . The statement then follows from the definitions.

2) is immediate, as a lifting diagram

$$\begin{array}{ccc} A & \longrightarrow & F^*X \\ \downarrow a & \nearrow & \downarrow F^*(f) \\ B & \longrightarrow & F^*Y \end{array}$$

is equivalent to a lifting diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & X \\ \downarrow F(a) & \nearrow & \downarrow f \\ F(B) & \longrightarrow & Y \end{array}$$

□

**Proposition 1.4.3.** *Assume that*

- *There exist good bases  $\mathcal{C}'$  and  $\mathcal{D}'$  of  $\mathcal{C}$  and  $\mathcal{D}$  such that  $F^*X'$  is in  $\mathcal{C}'$  for all  $X'$  in  $\mathcal{D}'$ .*
- *$F^*$  respects open immersions and open coverings.*

*If  $f: X \rightarrow Y$  is a quasi-compact/quasi-separated morphism in  $\mathcal{D}$ , then  $F^*(f)$  has the same property.*

*Proof.* Notice that the assumptions imply in particular that  $F^*X$  is representable for all  $X$  in  $\mathcal{D}$  (Remark 1.3.8).

Let  $f: X \rightarrow Y$  be quasi-compact, and let  $\{V_i \rightarrow Y\}_i$  be a good open covering. Since  $F^*$  respects open immersions and open coverings,  $\{F^*V_i \rightarrow F^*Y\}_i$  is again an open covering; moreover, for each  $i$

$$F^*V_i \times_{F^*Y} F^*X \cong F^*(V_i \times_Y X)$$

because  $F^*$  preserves limits. By assumption  $V_i \times_Y X$  is quasi-compact and thus admits a finite good covering, which is sent by  $F^*$  to a finite good covering of  $F^*(V_i \times_Y X)$ ; we conclude by Lemma 1.2.30.

If  $f$  is quasi-separated, then the diagonal  $\Delta_f: X \rightarrow X \times_Y X$  is quasi-compact; since  $F^*$  respects limits,  $\Delta_{F^*(f)} = F^*(\Delta_f)$ , which is again quasi-compact.  $\square$

*Remark 1.4.4.* As the above proof shows, it is actually sufficient that  $\mathcal{D}$  and  $\mathcal{C}$  have good bases and there exist a quasi-compact basis  $\mathcal{D}'$  of  $\mathcal{D}$  and a good basis  $\mathcal{C}'$  of  $\mathcal{C}$  such that  $F^*X'$  is in  $\mathcal{C}'$  for any  $X'$  in  $\mathcal{D}'$ .



## Chapter 2

# Concrete instances

In this chapter we present some examples of Greenberg functors to which the formalism developed so far applied. The list is not exhaustive, but can hopefully illustrate the versatility of the abstract construction; as a first instance, we start with the classical version, which we introduce in detail using the language of Greenberg algebras as developed by Lipman [Lip76], also taking care to show that their definition, which involves an fpqc sheafification, is well-posed; next, we review the relatively perfect Greenberg transform by Bertapelle and Suzuki [BS24], which is an example when  $\mathcal{C}'$  is a strict subcategory of  $\mathcal{C}$ . After this, we move to applications in logarithmic geometry, which are new: specifically, after a section on some review material, we first study a logarithmic version of Weil restriction, focussing mainly (but not exclusively) on the logarithmic Weil restriction along morphisms  $S \rightarrow \underline{S}$ , where  $S$  is a log scheme and  $\underline{S}$  its underlying scheme endowed with trivial log structure; the last section, coming in part from work in collaboration with Leo Herr, defines a (mixed-characteristic) logarithmic Greenberg transform and proves its representability under fairly general conditions. As usual, a more detailed description of the contents of each section can be found in its introduction.

### 2.1 Stability properties of categories with open immersions and gluings

Many relevant geometric categories  $\mathcal{C}$  have a forgetful functor  $|\cdot|$  such that  $(\mathcal{C}, |\cdot|)$  is a category with open immersions and gluings: this is true, for example, for the category of smooth manifolds (without requiring second-countability) and those of schemes. For another example, the category of logarithmic schemes with the

forgetful functor to  $\mathbf{Top}$  has open immersions and gluings (Lemma 2.4.25). The subcategories of  $\mathbf{Sch}$  or of the category of logarithmic schemes comprising the affine schemes form good bases.

A localisation  $\mathcal{C}/S$  of a category with open immersions (and gluings)  $\mathcal{C}$  is again a category with open immersions (and gluings), with the obvious functor to  $\mathbf{Top}$ ; the same is true for full subcategories which are closed under open immersions (and gluings).

**Lemma 2.1.1.** *Let  $(\mathcal{C}, |\cdot|)$  be a category with open immersions.*

(i) *For each  $S \in \mathcal{C}$ ,  $(\mathcal{C}/S, \mathcal{C}/S \rightarrow \mathcal{C} \xrightarrow{|\cdot|} \mathbf{Top})$  is again a category with open immersions, whose topology coincides with the localised topology of  $\mathcal{C}$ .*

- *If the  $|\cdot|$ -topology is subcanonical on  $\mathcal{C}$ , it is also on  $\mathcal{C}/S$ .*
- *If  $(\mathcal{C}, |\cdot|)$  has gluings, then so does  $(\mathcal{C}/S, \mathcal{C}/S \rightarrow \mathcal{C} \xrightarrow{|\cdot|} \mathbf{Top})$ .*
- *If in addition  $\mathcal{C}$  has a good basis  $\mathcal{C}'$ , then  $\mathcal{C}'/S$  is a good basis of  $\mathcal{C}/S$ .*

(ii) *Let  $\mathcal{E}$  be a subcategory with open immersions of  $\mathcal{C}$ .*

- *If the  $|\cdot|$ -topology is subcanonical on  $\mathcal{C}$ , it is also on  $\mathcal{E}$ .*
- *If  $(\mathcal{C}, |\cdot|)$  has gluings, then so does  $(\mathcal{E}, |\cdot|_{|\mathcal{E}})$  if and only if the property  $\mathcal{P}$  of objects of  $\mathcal{C}$*

$$\mathcal{P}(X) = "X \text{ is an object of } \mathcal{E}"$$

*is local (i.e.  $\mathcal{P}(X)$  holds if and only if there exists an open covering  $\{X_i \rightarrow X\}_i$  such that  $\mathcal{P}(X_i)$  holds for each  $i$ ).*

*Proof.* i): For any  $X \rightarrow S$  and  $Z \rightarrow S$  in  $\mathcal{C}/S$  and any  $U \subseteq |X|$  open,  $\mathrm{Hom}_{\mathcal{C}/S}(Z, U_X)$  is the set of morphisms  $f: Z \rightarrow X$  such that

$$\begin{array}{ccccc} Z & \longrightarrow & U_X & \longrightarrow & X \\ & & \searrow & & \swarrow \\ & & S & & \end{array}$$

commutes and  $\mathrm{Im} |f| \subseteq U$ ; this coincides with the set of those  $f \in \mathrm{Hom}_{\mathcal{C}/S}(Z, X)$  for which  $|f|$  factors through  $U$ . Hence  $F_{X \rightarrow S, U}$  is represented by  $U_X \rightarrow X \rightarrow S$ , and the assertion about topologies is trivial, as is the second point; since morphisms of  $\mathcal{C}$  glue, any gluing datum on the slice category is effective.

ii) is immediate. □

## 2.2 The “classical” Greenberg realisation

*Notation 2.2.1.* We fix a prime number  $p$ . As customary, we denote by  $W(-)$  the functor of  $p$ -typical Witt vectors, and for any positive integer  $n$ , by  $W_n(-)$  the functor of  $n$ -truncated  $p$ -typical Witt vectors, with the same indexing as in [Ser13, II, Section 6] (so, in particular, for any perfect field  $k$  of characteristic  $p$   $W_n(k) = W(k)/(p^n)$ ). The corresponding group schemes will be denoted by  $\mathbb{W}$  and  $\mathbb{W}_n$ .

**Introduction** Since this is the first and prototypical example, we would like to spend a few words to introduce it.

Given a complete discrete valuation ring  $\mathcal{O}$  with perfect residue field  $k$ , the starting motivation for Greenberg was the construction of a functor  $\mathrm{Gr}^{\mathcal{O}}: \mathrm{Sch}/\mathcal{O} \rightarrow \mathrm{Sch}/k$  such that for each  $\mathcal{O}$ -scheme  $X$ ,<sup>1</sup>  $(\mathrm{Gr}^{\mathcal{O}}X)(k) = X(\mathcal{O})$ ; in order to do so, he first constructs functors  $\mathrm{Gr}_n^{\mathcal{O}}: \mathrm{Sch}/\mathcal{O}_n \rightarrow \mathrm{Sch}/k$  ( $n$  a positive integer), with an analogous property for  $\mathcal{O}_n$ -schemes, where  $\mathcal{O}_n = \mathcal{O}/\mathfrak{m}^n$  is the truncation by the  $n$ -th power of the maximal ideal  $\mathfrak{m}$ , together with natural transformations  $\mathrm{Gr}_{n+1}^{\mathcal{O}} \rightarrow \mathrm{Gr}_n^{\mathcal{O}}$ , and then proves that the projective limit of the system

$$\cdots \rightarrow \mathrm{Gr}_{n+1}^{\mathcal{O}} \rightarrow \mathrm{Gr}_n^{\mathcal{O}} \rightarrow \cdots$$

defines a functor  $\mathrm{Sch}/\mathcal{O} \rightarrow \mathrm{Sch}/k$  (since, after reducing modulo  $\mathfrak{m}^n$ ,  $\mathrm{Gr}_n^{\mathcal{O}}$  can be applied to  $\mathcal{O}$ -schemes as well).

The most natural route to the definition of  $\mathrm{Gr}_n^{\mathcal{O}}$  is as a right adjoint, so the question arises of how to (naturally) lift  $k$ -algebras to  $\mathcal{O}_n$ -algebras. In the equal-characteristic case, this is straightforward: using that  $\mathcal{O}_n$  is in this case a  $k$ -algebra, one can simply associate to the  $k$ -algebra  $A$  the  $\mathcal{O}_n$ -algebra  $\mathcal{O}_n \otimes_k A$ ; the case when  $\mathcal{O}$  is a mixed-characteristic discrete valuation ring of absolute ramification 1 is similarly easy: one associates to  $A$  the  $W_n(k)$ -algebra  $W_n(A)$ . The general mixed-characteristic case is, however, more complicated: the most obvious attempt would be to take  $A$  to  $W(A) \otimes_{W(k)} \mathcal{O}_n$ , but this does not yield a representable functor [CLNS<sup>+</sup>18, Chapter IV, §2.2].

The solution involves the (now) so-called Greenberg algebras. After observing that his construction works generally for a local Artin ring  $R$  with residue field  $k$ , Greenberg places himself in this setup; the reason why this is possible is that such a ring  $R$  is canonically a  $k$ -algebra if  $\mathrm{char} R = \mathrm{char} k$  and a  $W(k)$ -algebra of finite length over  $W(k)$  otherwise.

<sup>1</sup>Greenberg actually only considered schemes of finite type, but the construction is the same.

As we have already mentioned how to deal with the former case, let us focus on the latter (so  $\text{char } k = p > 0$  and  $\text{char } R$  is a power of  $p$ ). By the structure theorem for modules over a principal ideal domain, for any  $W(k)$ -module of finite length  $M$  there exist integers  $N, n_1, \dots, n_N$  and a canonical isomorphism

$$M \xrightarrow{\sim} \prod_{i=1}^N W_{n_i}(k),$$

and if  $M$  has  $W(k)$ -algebra structure, this induces one on the right-hand side, which is usually different from the product structure. Since the multiplication map on the right-hand side is defined by polynomials with coefficients in  $W(k)$ , for any  $k$ -algebra  $A$  one obtains an algebra structure on the  $R$ -module  $\prod_{i=1}^N W_{n_i}(A)$ , and the association  $A \mapsto \prod_{i=1}^N W_{n_i}(A)$  is the desired functorial lift.

This definition is hands-on, but not very practical to work with, as any construction on the lift must then be proved to be independent of the chosen decomposition of  $R$  as a product of truncated rings of Witt vectors. A neater (but equivalent) definition was provided by Lipman [Lip76, Appendix A]: he noticed that while, as remarked above, for a  $W(k)$ -module  $M$  of finite length the set-valued functor of  $k$ -algebras  $A \mapsto W(A) \otimes_{W(k)} M$  might fail to be representable, its fpqc sheafification  $\mathcal{M}$  does not.<sup>2</sup> Moreover, as will be explained below, denoting by  $\mathbb{W}$  the group  $k$ -scheme of Witt vectors, the  $W(k)$ -module structure on  $M$  induces a  $\mathbb{W}$ -module structure on  $\mathcal{M}$ ; similarly, an algebra structure induces a  $\mathbb{W}$ -algebra structure, whereby this construction applied to  $R$  produces the desired lift  $\mathcal{R}$ .

One then proves that the functor  $\text{Aff}/k \rightarrow \text{Aff}/R$ ,  $\text{Spec } A \mapsto \text{Spec } \mathcal{R}(A)$  extends to a functor of schemes, which moreover admits a right adjoint, the Greenberg functor

$$\text{Gr}^R: \text{Sch}/R \rightarrow \text{Sch}/k.$$

A presentation of the Greenberg functor based on Lipman's idea can be found in [BGA18a], which also provides a scheme-theoretic generalisation of other constructions in [Gre61], and in [CLNS<sup>+</sup>18], which focusses on the geometry of the Greenberg realisations  $\text{Gr}_n^{\mathcal{O}} X$  for  $\mathcal{O}$  as above.

Let us conclude this brief overview of the classical Greenberg functor by pointing out that, coming back to Greenberg's original motivation and considering the ring

<sup>2</sup>The issue of whether the fpqc sheafification of this functor actually exists seems to have been ignored up to now, despite the fact that, unlike for other topologies, fpqc sheafification is not a well-defined operation on the class of functors of  $k$ -algebras [Wat75, Theorem 5.5]; we shall prove later that it does exist.

of integers  $\mathcal{O}$  of a finite extension of  $\mathbb{Q}_p$ , there is now a second, much simpler way to lift  $k$ -algebras to  $\mathcal{O}$ - or  $\mathcal{O}_n$ -algebras, which is provided by the *ramified* Witt vectors  $W_{\mathcal{O}}$ . A comparison between the two is given in [BC20, Theorem 6.2], where the authors construct a morphism  $\mathcal{R}_{\mathcal{O}} \rightarrow W_{\mathcal{O}}$  between the Greenberg algebra  $\mathcal{R}_{\mathcal{O}}$  of  $\mathcal{O}$  and the functor  $W_{\mathcal{O}}$  and prove that it induces an isomorphism on inverse perfection.

Moreover, for each positive integer  $n$  the functor  $\mathrm{Aff}/k \rightarrow \mathrm{Aff}/\mathcal{O}_n$ ,  $\mathrm{Spec} A \mapsto \mathrm{Spec} W_{\mathcal{O},n}(A)$  extends to a functor of schemes which, again, admits a right adjoint, which one might call the ramified Greenberg functor. It is then natural to ask if and how the ramified Greenberg functor is different from the original one; as shown in [BPS23, Theorem 4.9 and Example 4.10], they are not isomorphic, but they become isomorphic after taking inverse perfection, at least for those indices  $n$  which are multiples of the absolute ramification index of  $\mathcal{O}$  (notice that the indexing convention in *ibid.* is different from ours). It would be interesting to study whether using ramified Witt vectors instead of Greenberg algebras might yield simpler proofs (in the particular cases when they are applicable) of results in the literature.

**Greenberg modules and algebras** Let us now recall the construction of Greenberg modules and algebras.

Let  $k$  be a perfect field of positive characteristic  $p$ .

Since the issue of the existence of the fpqc sheafification in the definition of the Greenberg module associated to a  $W(k)$ -module of finite length has not, as far as we know, been addressed in the literature, we would like to argue why this is well-defined, despite the fact that the fpqc sheafification of an arbitrary presheaf on  $k$ -algebras might not exist [Wat75, Theorem 5.5].<sup>3</sup> The author thanks Dajano Tossici for raising the issue with him.

More generally, we shall prove the following

**Lemma 2.2.2.** *For any  $W(k)$ -module  $M$ , the presheaf on  $k$ -algebras  $A \mapsto A \otimes_{W(k)} M$  admits an fpqc sheafification.*

Notice that no finiteness condition on  $M$  is necessary.

Let us first recall some terminology from [Wat75, Section 2].

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<sup>3</sup>Lipman [Lip76] does refer to [DG70] for the existence of an fpqc sheafification; however, Demazure and Gabriel assume the existence of universes, and, beside the fact that this is arguably best avoided when possible, as remarked after [Wat75, *ibid.*] the resulting fpqc sheafification might in principle depend on the chosen universe, which is rather undesirable.

**Definition 2.2.3.** Let  $\mathbb{k}$  ( $R$  in *ibid.*) be a ring and  $\aleph$  ( $m$  in *ibid.*) be an infinite cardinal not less than the cardinality  $\#\mathbb{k}$ . Denote by  $\mathbb{k}\text{-Alg}(\aleph)$  the full subcategory of  $\mathbb{k}\text{-Alg}$  comprising the  $\mathbb{k}$ -algebras whose underlying set is contained inside some fixed set of cardinality  $\aleph$ .

A set-valued functor  $G$  on  $\mathbb{k}$ -algebras is called  $\aleph$ -based if it is the left Kan extension of its restriction to  $\mathbb{k}\text{-Alg}(\aleph)$  along the inclusion functor  $\mathbb{k}\text{-Alg}(\aleph) \hookrightarrow \mathbb{k}\text{-Alg}$ , i.e. if there is an isomorphism, natural in the  $\mathbb{k}$ -algebra  $A$ ,

$$G(A) \xrightarrow{\sim} \varinjlim_{S \rightarrow A} G(S),$$

where the limit ranges over all the morphisms (or equivalently, the injections) of  $\mathbb{k}$ -algebras  $S \rightarrow A$  with  $S$  in  $\mathbb{k}\text{-Alg}(\aleph)$ .

*Remark 2.2.4.* Waterhouse presents an apparently more general definition of  $\aleph$ -based functor than the previous one and then observes that the two are actually equivalent [Wat75, Theorem 2.1.b].

By [Wat75, Corollaries 5.2 and 5.3], any set-valued of  $\mathbb{k}$ -algebras which is  $\aleph$ -based for some cardinal  $\aleph$  as above has an fpqc sheafification, so Lemma 2.2.2 follows from the next

**Lemma 2.2.5.** *Let  $n \in \mathbb{N}_{>0} \cup \{\infty\}$  and let  $M$  be a  $k$ -module. The functor*

$$W_n(-) \otimes_{W(k)} M : k\text{-Alg} \rightarrow \mathbf{Set}$$

*is  $\aleph$ -based for any infinite cardinal  $\aleph$  greater or equal to  $\#k$ .*

*Proof.* Any  $k$ -algebra  $A$  is the direct limit of the direct system  $\{A_i\}_{i \in I}$  of its finitely generated subalgebras and any finitely generated  $k$ -algebra is a quotient of a polynomial ring in a finite number of variables over  $k$ , which has the same cardinality as  $k$ . Therefore, it is enough to show that the canonical morphism

$$\varinjlim_i \left( W_n(A_i) \otimes_{W(k)} M \right) \rightarrow W_n\left(\varinjlim_i A_i\right) \otimes_{W(k)} M$$

is bijective.

Since tensor products commute with colimits, it suffices to observe that the formation of Witt vectors over  $k$  commutes with filtered colimits: since the forgetful functor  $\mathbf{Fg} : \mathbf{Ring} \rightarrow \mathbf{Set}$  commutes with limits and filtered colimits and  $\mathbf{Fg} \circ W_n = (-)^{n+1}$ , there are canonical isomorphisms

$$\mathbf{Fg}\left(\varinjlim_i W_n(A_i)\right) \xrightarrow{\sim} \varinjlim_i \mathbf{Fg} W_n(A_i) \xrightarrow{\sim} \varinjlim_i (A_i)^{n+1} \xrightarrow{\sim} \left(\varinjlim_i A_i\right)^{n+1} \xrightarrow{\sim} \mathbf{Fg}\left(W_n\left(\varinjlim_i A_i\right)\right),$$

where the penultimate map is an isomorphism because limits commute with filtered colimits. As the forgetful functor reflects filtered colimits, the claim follows.  $\square$

*Remark 2.2.6.* Alternatively, if one were set on avoiding the set-theoretic arguments in [Wat75], one could, at least if  $M$  is of finite length over  $W(k)$ , run the same proof as in [Lip76, Proposition A.1.i] for the fppf sheafification, which presents no set-theoretic difficulties, and apply [Lip76, Corollary 0.2], which is possible since the formation of Witt vectors commutes with filtered colimits. One would then obtain that the fppf sheafification is representable, and thus an fpqc sheaf. We prefer the approach above since it yields a more general result and fits better with the presentation in [Lip76, Appendix A.1].

Notice that a finite product of  $\aleph$ -based functors is again  $\aleph$ -based, and that by [Wat75, Corollary 5.2] the fpqc sheafification of  $\aleph$ -based functors is functorial.

**Definition 2.2.7.** For a  $W(k)$ -module  $M$ , denote by  $\mathcal{M}$  the fpqc sheafification of the (set-valued) functor of  $k$ -algebras  $W(-) \otimes_{W(k)} M$ .

The  $W(k)$ -module structure on  $(M, +, \cdot)$  determines a natural  $W(-)$ -module structure on  $\mathcal{M}$ : more precisely, the morphisms  $+: M \times M \rightarrow M$  and  $(-, -): W(k) \times M \rightarrow M$  induce morphisms

$$(W(-) \otimes_{W(k)} M) \times (W(-) \otimes_{W(k)} M) \xrightarrow{\sim} W(-) \otimes_{W(k)} (M \times M) \xrightarrow{\text{id} \otimes +} W(-) \otimes_{W(k)} M$$

and

$$W(-) \times (W(-) \otimes_{W(k)} M) \xrightarrow{\sim} W(-) \otimes_{W(k)} (W(-) \times M) \xrightarrow{\text{id} \otimes W(k) \cdot} W(-) \otimes_{W(k)} M.$$

Since fpqc sheafification is functorial and respects finite products, these in turn yield morphisms  $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  and  $W(-) \times \mathcal{M} \rightarrow \mathcal{M}$  which make  $\mathcal{M}$  into a  $W(-)$ -module.

If in addition  $M$  has the structure of a  $W(k)$ -algebra, this similarly induces the structure of a  $W(-)$ -algebra on  $\mathcal{M}$ .

If  $M$  has finite length as a  $W(k)$ -module, then the  $W(-)$ -module (or algebra)  $\mathcal{M}$  is called the *Greenberg module* (or *algebra*) associated to  $M$ .

**The Greenberg functor** We finally recall the definition of the (unramified) Greenberg functor in its modern version, as expounded in [BGA18a], and show that it fits into the framework developed in the first part.

Let  $R$  be a local Artin ring  $R$  with residue field  $k$ , which is assumed to be perfect in the mixed-characteristic case. If  $\text{char } R = \text{char } k$ , then  $R$  is naturally a  $k$ -algebra; if  $\text{char } R \neq \text{char } k$ , there exist an integer  $m$ , which we take to be the minimal such for definiteness, and a canonical  $W_m(k)$ -algebra structure on  $R$  [Gre61, Section I]:

in particular, the construction of Definition 2.2.7 can be applied to  $R$ , yielding the Greenberg algebra  $\mathcal{R}$ .

Let  $F_R: \text{Aff}/k \rightarrow \text{Sch}/R$  be the functor with

$$F_R(\text{Spec } A) = \begin{cases} \text{Spec } A \otimes_k R & \text{char } k = \text{char } R \\ \text{Spec } \mathcal{R}(A) & \text{char } k \neq \text{char } R \end{cases}.$$

*Example 2.2.8.* If  $R = W_n(k)$  for some positive integer  $n$ , then  $F_R$  is canonically isomorphic to the restriction to  $\text{Aff}/k$  of the functor

$$W_n: \text{Sch}/k \rightarrow \text{Sch}/W_n(k)$$

with

$$W_n(X) = (|X|, W_n(\mathcal{O}_X)),$$

where  $W_n(\mathcal{O}_X)$  is the sheaf with  $W_n(\mathcal{O}_X)(U) = W_n(\mathcal{O}_X(U))$  for all  $U \subseteq |X|$  open: indeed, there is a natural surjection

$$W_n(k) \otimes_{W(k)} W_n(A) = W(A)/(p^n) \rightarrow W_n(A)$$

for any  $k$ -algebra  $A$ , which is even bijective if  $A = A^p$ ; by [Lip76, Corollary 0.2], this implies the claim.

We shall soon see that  $F_R^*X$  is representable for all  $W_n(k)$ -schemes  $X$ ; therefore,  $F_R^*$  is right adjoint to the functor  $W_n(-)$  of schemes (compare with Remark 1.3.11). A similar, though more elaborate, *canonical* extension of  $F$  can be performed also for an arbitrary mixed-characteristic local Artin ring  $R$  with perfect residue field [BGA18a, §6].

The construction of  $F_R$  satisfies an obvious functoriality property: more precisely, a map  $R \rightarrow R'$  of local Artin rings with residue field  $k$  induces a natural transformation

$$W_n(-) \otimes_{W(k)} R \rightarrow W_n(-) \otimes_{W(k)} R',$$

and thus a natural transformation

$$\tau = \tau_{R \rightarrow R'}: F_{R'} \rightarrow F_R.$$

As shown in Section 2.3 of [BGA18a],  $\tau_A: F_{R'}(A) \rightarrow F_R(A)$  is a nilpotent immersion for all  $k$ -algebras  $A$  if  $R \rightarrow R'$  is a nilpotent immersion.

The formalism of Greenberg functors developed in the previous chapter can be applied to recover the following known theorem ([Gre61], [BGA18a]):

**Theorem 2.2.9.** *Let  $R$  be a local Artin ring with perfect residue field  $k$ ,  $F_R$  the functor associated to  $R$  defined above.*

- (i) *The presheaf  $F_R^*X: (\text{Sch}/k)^{\text{op}} \rightarrow \text{Set}$  is representable for any  $R$ -scheme  $X$ .*
- (ii) *The functor  $F_R^*: \text{Sch}/R \rightarrow \text{Sch}/k$  respects open immersions and coverings and the properties of a morphism of  $R$ -schemes of being:*
  - (i) *formally smooth/unramified/étale;*
  - (ii) *quasi-compact;*
  - (iii) *quasi-separated;*
  - (iv) *affine.*

*Proof.* i): First, let us check the assumptions of Theorem 1.3.7. In the mixed-characteristic case, the first part of the proof of [BGA18a, Proposition 6.1] shows that  $F_R$  respects open immersions, while the corresponding statement is obvious in equal characteristic. Letting  $\mathfrak{m}$  be the maximal ideal of  $R$ , since  $R$  is Artinian there exists  $m$  such that  $\mathfrak{m}^m = 0$ ; taking  $R \rightarrow R'$  to be the natural projection to  $k$  in the previous paragraph, one thus obtains a natural nilpotent immersion

$$\tau_{\text{Spec } A}: F_k(A) = \text{Spec } A \rightarrow F_R(A)$$

for all  $k$ -algebras  $A$ , which in particular induces a natural homeomorphism of the underlying topological spaces.

Theorem 1.3.7 and Remark 1.3.8 then reduce us to proving that  $F_R^*\mathbb{A}_R^1$  is representable (see Example 1.3.9). It is easy to see that  $F_R^*\mathbb{A}_R^1 \cong \mathbb{A}_k^{\dim_k R}$  in equal characteristic, so let us just recall the proof of the mixed-characteristic case: decomposing  $M$  as a (finite) product of rings of truncated Witt vectors, without loss of generality  $R = W_n(k)$  for some  $n$ ; then by Example 2.2.8  $(F_R^*\mathbb{A}_R^1)(A) = W_n(A)$  for all  $k$ -algebras  $A$ , which easily implies that  $F_R^*\mathbb{A}_R^1$  is represented by  $\mathbb{A}_k^n$ .

ii): Preservation of open immersions and coverings was already shown as a part of the proof of Theorem 1.3.7. As for formal smoothness/unramifiedness/étaleness, taking  $\Lambda$  to be the class of nilpotent immersions of  $k$ -schemes in Lemma 1.4.2ii) it suffices to observe that  $F_R$  preserves nilpotent immersions.

Letting  $\mathcal{C}' = \text{Aff}/k$  and  $\mathcal{D}' = \text{Aff}/R$ ,  $F_R^*X'$  is in  $\mathcal{D}'$  for all  $X'$  in  $\mathcal{C}'$  by Remark 1.3.8, because  $F_R^*\mathbb{A}_R^1$  is affine. By Proposition 1.4.3,  $F_R^*$  preserves quasi-compactness and quasi-separatedness.

Since  $F_R^*X$  preserves open immersions, open coverings and fibred products, if  $f: X \rightarrow Y$  is an affine morphism of  $R$ -schemes and  $\{V_i \rightarrow Y\}_i$  an open affine

covering,  $\{F_R^*(V_i) \times_{F_R^*(Y)} F_R^*(X) \rightarrow F_R^*(X)\}_i$  is an open affine covering, whereby  $F^*(f)$  is affine.  $\square$

*Remark 2.2.10.* The functor  $F_R^*$  also preserves other properties, like quasi-projectivity, that we were not able to sense of in the abstract setup of categories with open immersions and gluing; see Section 9 in [BGA18a].

**Definition 2.2.11.** The functor  $F_R^*: \text{Sch}/R \rightarrow \text{Sch}/k$  is called the Greenberg functor associated to  $R$ . For any  $R$ -scheme  $X$ , the  $k$ -scheme  $F_R^*X$  is called the Greenberg realisation of  $X$  (associated to  $R$ ).

### 2.3 The relatively perfect Greenberg transform

The construction of the classical Greenberg functor seen above relies heavily on the good behaviour of the functor  $W_m: \text{Sch}/k \rightarrow \text{Sch}/W_m(k)$  when  $k$  is perfect. Introduced by Bertapelle and Suzuki in [BS24], the relatively perfect Greenberg transform circumvents some of the difficulties that arise when  $[k : k^p]$  is greater than 1, but still finite.

We show how the abstract formalism applies to this case as well, also yielding some permanence properties of morphisms “for free”. This is an example where  $\mathcal{C}$  is a *proper* subcategory of some  $\text{Sch}/S$ , namely, that of relatively perfect schemes over some  $\mathbb{F}_p$ -scheme  $S$ . The role of  $F$  will be played by Kato’s canonical lifting. Let us recall some definitions.

Let  $p \in \mathbb{N}$  be a fixed prime number.

**Frobenii and relatively perfect schemes** For any  $\mathbb{F}_p$ -algebra  $R$ , we denote by  $\varphi_R: R \rightarrow R$  the Frobenius morphism, i.e. the morphism of rings (but not, usually, of  $\mathbb{F}_p$ -algebras!)  $r \mapsto r^p$ .

Notice that  $\varphi$  is functorial: for any morphism of  $\mathbb{F}_p$ -algebras  $f: R \rightarrow R'$ , the square

$$\begin{array}{ccc} R & \xrightarrow{\varphi_R} & R \\ \downarrow f & & \downarrow f \\ R' & \xrightarrow{\varphi_{R'}} & R' \end{array}$$

commutes. Consequently, for any  $\mathbb{F}_p$ -scheme  $T$  there is a morphism  $\varphi_T$ , the absolute Frobenius of  $T$ , such that for any open subscheme  $\text{Spec } A \subseteq T$  one has  $\varphi_T(\text{Spec } A) = \text{Spec } A$  as topological spaces and  $\varphi_T|_{\text{Spec } A} = \text{Spec}(\varphi_A)$ . The absolute Frobenius of schemes is again functorial and is a universal homeomorphism.

Fix now an  $\mathbb{F}_p$ -scheme  $T$ . For any  $T$ -scheme  $Y$ , define the relative Frobenius  $\varphi_{Y/T}$  of  $Y$  over  $T$  by the diagram

$$\begin{array}{ccccc}
 Y & & \xrightarrow{\varphi_Y} & & Y \\
 \searrow \varphi_{Y/T} & & & & \downarrow \\
 & Y^{(p)} & \xrightarrow{\quad} & Y & \\
 & \downarrow \lrcorner & & & \downarrow \\
 & T & \xrightarrow{\varphi_T} & & T
 \end{array}$$

Unlike  $\varphi_Y$ ,  $\varphi_{Y/T}$  is a morphism over  $T$ .

It is useful to describe how  $\varphi_{Y/T}$  looks like on affine schemes: given an  $\mathbb{F}_p$ -algebra  $R$  and an  $R$ -algebra  $A$ , the relative Frobenius of  $\operatorname{Spec} A$  over  $\operatorname{Spec} R$  is the morphism corresponding on global sections to the map

$$\varphi_{R/A}: A \otimes_{R,\varphi} R \rightarrow A, \quad a \otimes r \mapsto a^p r.$$

**Definition 2.3.1.** Let  $T$  be an  $\mathbb{F}_p$ -scheme. A  $T$ -scheme  $Y$  is called relatively perfect if  $\varphi_{Y/T}$  is an isomorphism.

By the construction of the relative Frobenius, relative perfectness is a local property, and thus by Lemma 2.1.1(ii) the category of relatively perfect  $T$ -schemes is a category with open immersions and gluings when endowed with the forgetful functor to  $\mathbf{Top}$ .

**Kato's canonical lifting** Let  $A$  be a ring,  $I \trianglelefteq A$  an ideal containing  $p$  and such that  $I^n = 0$  for some  $n \in \mathbb{N}$ ,  $R = A/I$ .

*Construction 2.3.2.* Observe that for any  $N \geq n$  there is a ring homomorphism

$$W_N(R) \rightarrow A, \quad (\bar{a}_0, \dots, \bar{a}_N) \mapsto \bar{a}_0^{p^N} + p\bar{a}_1^{p^{N-1}} + \dots + p^N \bar{a}_N :$$

it a well-defined ring homomorphism because  $p \in I$  and the ghost map

$$W_N(A) \rightarrow A, \quad (a_0, \dots, a_N) \mapsto a_0^{p^N} + pa_1^{p^{N-1}} + \dots + p^N a_N$$

is a ring homomorphism. This makes  $A$  into a  $W_N(R)$ -algebra.

**Lemma 2.3.3** ([Kaz82, Lemma 1]). *For any relatively perfect  $R$ -algebra  $R'$  there exists a unique (up to unique isomorphism) formally étale  $A$ -algebra  $B$  such that  $B/IB \cong R'$  as  $R$ -algebras. It is isomorphic to  $W_N(R') \otimes_{W_N(R)} A$ , where  $N$  is any integer greater or equal to  $n$ .*

*Proof.* The uniqueness part follows from the fact that  $A \rightarrow B$  is formally étale, because  $I$  is nilpotent.

Since  $R \rightarrow R'$  is relatively perfect,  $W_N(R) \rightarrow W_N(R')$  is formally étale [Kaz82, Lemma 2] and thus so is  $A \rightarrow W_N(R') \otimes_{W_N(R)} A$ . Finally, as  $p \in I$ , the composition  $W_N(R) \rightarrow A \rightarrow R$  is simply the map truncation map  $(\bar{a}_0, \dots, \bar{a}_N) \mapsto \bar{a}_0$ , whereby

$$W_N(R') \otimes_{W_N(R)} A \otimes_A A/I \cong W_N(R') \otimes_{W_N(R)} \left( \frac{W_N(R)}{V^{N-1}(W_N(R))} \right) \cong R',$$

where the last isomorphism follows from [BS24, Proposition 2.6(1)].  $\square$

**Definition 2.3.4.** With the same notation of the previous lemma, the  $A$ -algebra  $B$  is called Kato's canonical lifting (or lift) of  $R'$  to  $A$  (along  $I$ ).

As in the case of Greenberg algebras, this can be globalised to schemes [BS24, Proposition 3.2], though this fact is not particularly relevant for what follows. Following [BS24], we could also work in a more general situation than that considered below, but we avoid doing that, so as to keep the notation as simple as possible and to highlight the parallelism with the classical Greenberg functor.

**Definition 2.3.5.** Let  $A$  be a local Artin ring with maximal ideal  $\mathfrak{m}$  and residue field  $k$  such that  $[k : k^p] < \infty$ . For any positive integer  $i$ , let  $F_{A,i} : \text{Aff}/k \rightarrow \text{Sch}/A$  be the functor associating to an affine scheme  $\text{Spec } R'$  the scheme  $\text{Spec } B$ , where  $B$  is Kato's canonical lifting of  $R'$  to  $A$  along  $\mathfrak{m}^i$ .

In Section 5 of [BS24], the authors prove (a more general version of) point  $i$ ) of the following

**Theorem 2.3.6.** *With the same notation as in the previous definition,*

- (i) *The functor  $F_{A,i}^* X$  is representable for any  $A$ -scheme  $X$ .*
- (ii) *The functor  $F_{A,i}^* : \text{Sch}/A \rightarrow \text{Sch}/k$  respects open immersions and coverings and properties of a morphism of  $A$ -schemes of being:*
  - (i) *quasi-compact;*
  - (ii) *quasi-separated;*
  - (iii) *affine.*

*Proof.*  $i)$ : As for the classical Greenberg, the projection  $A \rightarrow k$  induces, for any  $k$ -algebra  $R'$ , natural nilpotent immersions  $F_{k,i}(R') = \text{Spec } R' \rightarrow F_{A,i}(R')$ ; by Theorem 1.3.7 and Remark 1.3.8, one is then reduced to the representability of  $F_{A,i}^* \mathbb{A}_A^1$  (compare with Example 1.3.9), which is established in [BS24, Lemma 5.1].

ii): *Loc. cit.* shows that  $F_{A,i}^* \mathbb{A}_A^1$  is represented by the relative perfection of  $\mathbb{A}_k^1$ , whose definition [BS24, pag. 7] implies that it is affine. The same argument as in the proof of Theorem 2.2.9(ii) then applies.  $\square$

**Definition 2.3.7** ([BS24, Section 4]). With the same notation as in Definition 2.3.5, the functor  $F_{A,i}^*: \text{Sch}/A \rightarrow \text{Sch}/k$  is called the relatively perfect Greenberg functor with respect to  $\text{Spec } k \rightarrow \text{Spec } A$  (along the ideal  $\mathfrak{m}^i$ ).

## 2.4 Interlude: recollections on logarithmic geometry

As the last two applications will regard logarithmic geometry, some recollections on this subject are warranted.

### Monoids

All monoids (and in particular all groups) will be unital, associative and commutative; monoids will usually be written in additive notation, though we shall sometimes switch to multiplicative notation when doing so is convenient. The category of such monoids will be denoted by  $\mathbf{Mon}$ , that of (abelian) groups by  $\mathbf{Ab}$ .

A monoid  $P$  is called *finitely generated* if there exist a non-negative integer  $r$  and a surjection of monoids  $\mathbb{N}^r \rightarrow P$ ; in other words, if there exist  $p_1, \dots, p_r \in P$  such that any  $p \in P$  can be written as  $p = \sum_{i=1}^r n_i p_i$  for some non-negative integers  $n_1, \dots, n_r$ .

**Limits and colimits** The category of monoids has all limits and colimits [Ogu18, Section I.1.1]; the formation of limits and coproducts commutes with the forgetful functor to  $\mathbf{Set}$ , while that of arbitrary colimits is more subtle. Let us briefly recall some of its aspects, after *loc. cit.* Following Ogus [Ogu18], we use the symbol  $\oplus$  to denote direct sums and pushouts of monoids.

The construction of coequalisers uses the notion of congruence relation.

**Definition 2.4.1.** Let  $P$  be a monoid. A congruence relation on the monoid  $P$  is an equivalence relation  $E \subseteq P \times P$  which is a submonoid of  $P \times P$ .

Any subset  $S$  of  $P \times P$  generates a congruence relation, the smallest one containing  $S$ , and a congruence relation  $E$  is called *finitely generated* if it is generated by a finite set  $S$ ; notice that we are not requiring  $S$  to generate  $E$  as a monoid.

Congruence relations on monoids play a role analogous to that of kernels in other algebraic categories (they are “the monoids by which one can quotient”); as

a matter of fact, kernels are not very well-behaved in the category of monoids (for example, the kernel of  $\mathbb{N} \oplus \mathbb{N} \rightarrow \mathbb{N}$ ,  $(m, n) \mapsto m + n$  is zero, but the map is not injective). More precisely:

**Proposition 2.4.2** ([Ogu18, Proposition I.1.1.1]). *Let  $P$  be a monoid.*

(i) *For any surjective morphism of monoids  $P \rightarrow Q$ , the equaliser  $E$  of the maps*

$$P \times P \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} P \rightarrow Q$$

*is a congruence relation on  $P$ , and  $P \rightarrow Q$  is the coequaliser of the maps*

$$E \rightarrow P \times P \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} P.$$

(ii) *For any congruence relation  $E \subseteq P \times P$  on  $P$ , the set  $Q = P/E$  of equivalence classes can be endowed with a unique monoid structure making the natural projection  $P \rightarrow Q$  into a morphism of monoid; with this structure on  $Q$ ,  $P \rightarrow Q$  is the coequaliser of*

$$E \hookrightarrow P \times P \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} P$$

*and  $E \hookrightarrow P$  is the equaliser of*

$$P \times P \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} P \rightarrow Q.$$

If  $E$  is a congruence relation on a monoid  $P$  and  $p, p' \in P$ , the expression  $p \sim_E p'$  means that  $p$  is congruent to  $p'$  modulo  $E$ .

The coequaliser of a pair of parallel morphisms of monoids  $\alpha, \beta: P \rightrightarrows Q$  can be constructed as the quotient of  $Q$  by the congruence relation on  $Q$  generated by the set of pairs  $\{(\alpha(p), \beta(p)) \in Q \times Q | p \in P\}$ .

By a standard categorical result, it follows that the category of monoids has all colimits, since those can be assembled out of coproducts and coequalisers. Pushouts will be of particular importance, given their use in the construction of fibred products of logarithmic schemes. In general, their construction is rather unwieldy [Ogu18, Proposition I.1.1.5], but it somewhat simplifies in cases of practical interest:

**Proposition 2.4.3** ([Ogu18, Proposition I.1.1.5]). *Let  $\alpha_1: P \rightarrow Q_1$  and  $\alpha_2: P \rightarrow Q_2$  be morphisms of monoids.*

(i) *If any of  $P$ ,  $Q_1$  or  $Q_2$  is a group, the pushout  $Q_1 \oplus_P Q_2$  is the quotient of  $Q_1 \oplus Q_2$  by the congruence relation  $E$  defined by requiring that  $(q_1, q_2) \sim_E (q'_1, q'_2)$  if and only if there exist  $p, p' \in P$  with  $q_i + \alpha_i(p) = q'_i + \alpha_i(p')$ .*

(ii) If  $P$  is a group, then  $(q_1, q_2) \sim_E (q'_1, q'_2)$  if and only if there exists  $p \in P$  such that

$$q'_1 = q_1 + \alpha_1(p), \quad q'_2 = q_2 + \alpha_2(-p).$$

In particular, for  $Q_2 = 0$  one obtains a description of the cokernel of  $\alpha_1: P \rightarrow Q_1$  as the quotient of  $Q_1$  by the equivalence relation which identifies  $q$  and  $q'$  if and only if there exist  $p, p' \in P$  with  $q + \alpha_1(p') = q' + \alpha_1(p)$ . If  $\alpha_1$  is an injection, one writes  $Q_1/P$  for its cokernel.

It should be noted that  $Q_1/P$  might be the zero monoid even if  $P$  is strictly contained in  $Q$  [Ogu18, Example I.1.1.6], and in particular, an epimorphism of monoids might fail to be surjective. It is immediate to see that, however, a surjective morphism of monoids is an epimorphism.

**Some properties of monoids** For a monoid  $P$ , denote by  $P^\times$  the subgroup of units of  $P$  and by  $\bar{P}$  the quotient  $P/P^\times$ .

The inclusion of  $\mathbf{Mon}$  in  $\mathbf{Ab}$  has both a left and a right adjoint: the right adjoint is simply  $P \mapsto P^\times$ , while the left adjoint, called groupification and denoted by  $(\cdot)^{\text{gp}}$ , is constructed by letting, for any monoid  $P$ ,  $P^{\text{gp}}$  be the cokernel of the diagonal  $P \rightarrow P \times P$ . Indeed, using the explicit description of cokernels recalled above it is easy to see that it is a group, and for any group  $G$ , a morphism of groups  $\text{coker}(P \rightarrow P \times P) \rightarrow G$  is the same as a pair of maps  $f_1, f_2: P \rightarrow G$  such that  $f_1 + f_2 = 0$ ; since  $G$  is a group, either of these maps determines the other, which establishes a natural bijection  $\text{Hom}_{\mathbf{Grp}}(\text{coker}(P \rightarrow P \times P), G) \xrightarrow{\sim} \text{Hom}_{\mathbf{Mon}}(P, G)$ .

**Definition 2.4.4** ([Ogu18, Definition I.1.3.1 and ff]). A monoid  $P$  is called

- (i) sharp if  $P^\times = \{0\}$ ;
- (ii) u-integral if for any  $p \in P$  and  $u \in P^\times$ ,  $p + u = p$  if and only if  $u = 0$ ;
- (iii) integral, or cancellative, if for any  $p, q_1, q_2 \in P$ ,  $p + q_1 = p + q_2$  if and only if  $q_1 = q_2$ ;
- (iv) fine if it is finitely generated and integral;
- (v) saturated if it is integral and if for any  $p \in P^{\text{gp}}$  such that  $np \in P$  for some positive integer  $n$ , then  $p \in P$ .
- (vi) fs if it is fine and saturated.

By the description of  $P^{gp}$  as the cokernel of the diagonal,  $P$  is u-integral if and only if  $P^\times \rightarrow P^{gp}$  is injective and integral if and only if  $P \rightarrow P^{gp}$  is injective.

The full subcategory of integral (resp. saturated) monoids will be denoted by  $\mathbf{Mon}^{int}$  (resp.  $\mathbf{Mon}^{sat}$ ). The association  $M \mapsto M^{int} := \text{Im}(M \rightarrow M^{gp})$  extends to a functor  $\mathbf{Mon} \rightarrow \mathbf{Mon}^{int}$ , called *integralisation*, which is left adjoint to the inclusion  $\mathbf{Mon}^{int} \hookrightarrow \mathbf{Mon}$ : indeed, for any monoid  $M$  and any integral monoid  $N$ , one has  $\text{Hom}(M, N^{gp}) = \text{Hom}(M^{gp}, N^{gp})$ , and any map  $M^{gp} \rightarrow N^{gp}$  determines a unique map  $M^{int} \rightarrow N$  since  $N$  is integral; conversely, by construction any map  $M^{int} \rightarrow N$  extends uniquely to  $M^{gp} \rightarrow N^{gp}$ . Similarly, the inclusion  $\mathbf{Mon}^{sat} \hookrightarrow \mathbf{Mon}^{int}$  admits a left adjoint  $(\cdot)^{sat}$ , called *saturation*; for any integral monoid  $P$ , by [Ogu18, Proposition I.1.3.5.3]  $P^{sat}$  can be identified with the submonoid of  $P^{gp}$  consisting of those  $x \in P^{gp}$

$$\{x \in P^{gp} \mid \exists n \in \mathbb{Z}, n > 0 : nx \in P\}$$

The universal properties of these functors imply that  $\mathbf{Mon}^{int}$  and  $\mathbf{Mon}^{sat}$  have colimits: the colimit of a diagram in  $\mathbf{Mon}^{int}$  is obtained by applying the functor  $(\cdot)^{int}$  to the colimit constructed in  $\mathbf{Mon}$ ; similarly, a colimit of a diagram in  $\mathbf{Mon}^{sat}$  is constructed by applying the functor  $(\cdot)^{sat}$  to the colimit in  $\mathbf{Mon}^{int}$ .

**Some properties of morphisms of monoids** A technical inconvenience one often has to face in logarithmic geometry is the fact that the colimits computed in the categories  $\mathbf{Mon}$ ,  $\mathbf{Mon}^{int}$  and  $\mathbf{Mon}^{sat}$  usually differ; this leads to the following

**Definition 2.4.5.** A morphism of integral/saturated monoids  $P \rightarrow Q$  is called integral/saturated if for every morphism of integral/saturated monoids  $P \rightarrow P'$ , the pushout  $Q \oplus_{P'} Q$  is integral/saturated.

There are useful characterisations for a morphism  $P \rightarrow Q$  of integral/saturated monoids to be integral/saturated; although we will not need them, we mention them because they are the properties taken as definitions in [Ogu18] (doing so also allows one to speak saturated morphisms between integral non-saturated monoids, but this is unnecessary for our purposes).

**Proposition 2.4.6.** • [Ogu18, Definition I.4.6.2] A morphism  $\theta: P \rightarrow Q$  of integral monoids is integral if and only if for any  $p_1, p_2 \in P$  and  $q_1, q_2 \in Q$  with

$$\theta(p_1) + q_1 = \theta(p_2) + q_2,$$

there exist  $q' \in Q$  and  $p'_1, p'_2 \in P$  with  $q_i = \theta(p'_i) + q'$  and  $p_1 + p'_1 = p_2 + p'_2$ .

- [Ogu18, Corollary I.4.8.10 and Definition I.4.8.2] A morphism  $\theta: P \rightarrow Q$  of saturated monoids is saturated if and only if it is integral and for all positive integers  $n$ , the map  $h_{\theta,n}$  defined in the pushout diagram

$$\begin{array}{ccc}
 P & \xrightarrow{\cdot n} & P \\
 \downarrow & \lrcorner & \downarrow \\
 Q & \longrightarrow & Q_{\theta,n} \\
 & \searrow \cdot n & \searrow h_{\theta,n} \\
 & & Q
 \end{array}
 \quad \begin{array}{c} \theta \\ \downarrow \end{array}$$

is exact, i.e. the square

$$\begin{array}{ccc}
 Q_{\theta,n} & \xrightarrow{h_{\theta,n}} & Q \\
 \downarrow & & \downarrow \\
 Q_{\theta,n}^{gp} & \xrightarrow{h_{\theta,n}^{gp}} & Q^{gp}
 \end{array}$$

is cartesian.

*Example 2.4.7.* For any integral/saturated monoid  $P$ , the inclusion  $P^\times \hookrightarrow P$  is integral/saturated: by [Ogu18, Proposition I.4.6.3.2] and [Ogu18, Proposition I.4.8.5.4], we can assume that  $P$  is sharp, in which case the statement is trivial.

Finally, recall the following properties of morphisms of monoids:

**Definition 2.4.8** ([Ogu18, Definition I.4.1.1]). A morphism of monoids  $\alpha: P \rightarrow Q$  is called

- (i) local if  $\alpha^{-1}(Q^\times) = P^\times$ ;
- (ii) sharp if the induced morphism  $P^\times \rightarrow Q^\times$  is an isomorphism;
- (iii) logarithmic if the induced morphism  $\alpha^{-1}(Q^\times) \rightarrow Q^\times$  is an isomorphism.

Observe that a morphism  $\alpha: P \rightarrow Q$  is logarithmic if and only if it is local and sharp: the backward implication is immediate, and for the forward it is clearly enough to prove that a logarithmic morphism is local; let  $p \in P$  be such that  $\alpha(p)$  is invertible, say  $\alpha(p) + q = 0$ . As  $\alpha$  is logarithmic, there exists some  $p' \in P$  such that  $\alpha(p') = q$ , whence  $\alpha(p + p') = 0$  and, as the restriction of  $\alpha$  to  $\alpha^{-1}(Q^\times)$  is injective,  $p + p' = 0$ ; in particular,  $p \in P^\times$ .

### Sheaves of monoids

In order to define logarithmic schemes and their properties, it is necessary to globalise the previous notions and introduce sheaves of monoids on schemes. There are

two main standard choices for the Grothendieck pretopology on schemes in this context, the Zariski and étale topology. For a scheme  $T$ , we denote by  $T_{Zar}$  and  $T_{ét}$  the small Zariski and étale sites on  $T$ .

Any statement in which no specific topology is mentioned should be read as holding for either of them, with the proviso that, when working with the étale topology, a point is a geometric point and an open subset of  $X$  is an étale morphism  $U \rightarrow X$ , and so on.

**Some facts on étale sheaves** Let us first review some important facts about étale sheaves, which will be often used without mention. The corresponding statements for Zariski sheaves are either obvious or basic.

Recall first ([Mil80, Section II.2], [Autb, 03PZ]) that a morphism of schemes  $f: T \rightarrow Z$  determines an adjoint pair of functors

$$f_{pre}^{-1}: \mathbf{PSh}(Z_{ét}) \rightleftarrows \mathbf{PSh}(T_{ét}) : f_*$$

between presheaves on the small étale sites. Explicitly, for  $\mathcal{P} \in \mathbf{PSh}(T_{ét})$  and an étale morphism  $W \rightarrow Z$ ,  $(f_*\mathcal{P})(W/Z) = \mathcal{P}(W \times_Z T/T)$ , while for  $\mathcal{F} \in \mathbf{PSh}(Z_{ét})$  and an étale morphism  $U \rightarrow T$ ,

$$(f_{pre}^{-1}\mathcal{F})(U/T) = \operatorname{colim}_V \mathcal{F}(V/Z),$$

where the colimit runs over the commutative squares

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ T & \longrightarrow & Z \end{array}$$

with  $V \rightarrow Z$  étale.

Sheafifying  $f_{pre}^{-1}$ , one obtains an adjoint pair

$$f^{-1}: \mathbf{Sh}(Z_{ét}) \rightleftarrows \mathbf{Sh}(T_{ét}) : f_*$$

between categories of sheaves.

A nilpotent immersion  $\iota: T \rightarrow Z$  induces an isomorphism  $\iota_*: \mathbf{Sh}(T_{ét}) \xrightarrow{\sim} \mathbf{Sh}(Z_{ét})$  between small étale sites [Autb, 04DZ, 03SI], a fact that is referred to as the *topological invariance of the small étale site*; moreover, for any  $\mathcal{G} \in \mathbf{Sh}(T_{ét})$  and any geometric point  $\bar{z}$  of  $Z$ ,  $(\iota_*\mathcal{G})_{\bar{z}} = \mathcal{G}_{\bar{z}}$  [Autb, 04FX].

Finally, recall that the exactness of a sequence of sheaves on the small étale site of a scheme can be checked on stalks [Autb, 03PT].

Mutatis mutandis, the same results hold for (pre)sheaves of monoids.

**Properties and constructions of sheaves of monoids and of their morphisms** Let  $X$  be a scheme.

**Definition 2.4.9.** Let  $\mathcal{P}$  be one of the properties of monoids defined above, with  $\mathcal{P} \neq \text{“fine”}, \text{“fs”}$ . A sheaf of monoids  $\mathcal{M}$  on  $X$  is said to have  $\mathcal{P}$  if the monoid  $\mathcal{M}_x$  has  $\mathcal{P}$  for all point  $x$  of  $X$ . Similarly, for a property  $\mathcal{P}$  of morphisms among those defined above, a morphism  $f: \mathcal{M} \rightarrow \mathcal{N}$  is said to have  $\mathcal{P}$  if  $f_x: \mathcal{M}_x \rightarrow \mathcal{N}_x$  has  $\mathcal{P}$  for each  $x$ .

Importantly, the properties of being integral and saturated can be checked on covers:

**Proposition 2.4.10** ([Ogu18, Proposition II.1.1.3]). *A sheaf of monoids  $\mathcal{M}$  on a scheme  $X$  is integral/saturated if and only if for every open subset  $U \rightarrow X$ ,  $\mathcal{M}(U)$  is integral/saturated.*

The category of sheaves of monoids on  $X$  has limits and colimits, which are computed in the presheaf category and sheafifying if necessary (i.e. for colimits).

Any morphism of sheaves of monoids on  $X$  has a canonical factorisation through a sharp/logarithmic morphism:

**Proposition 2.4.11** ([Ogu18, Proposition II.1.1.15]). *Let  $\ell$  be the property (of morphism of sheaves of monoids) “sharp” or “logarithmic”. Any morphism  $\theta: \mathcal{P} \rightarrow \mathcal{M}$  of sheaves of monoids on  $X$  can be factored as a composition*

$$\mathcal{P} \rightarrow \mathcal{P}^\ell \xrightarrow{\theta^\ell} \mathcal{M},$$

where  $\theta^\ell$  has  $\ell$  and for any other factorisation  $\mathcal{P} \rightarrow \mathcal{P}' \xrightarrow{\theta'} \mathcal{M}$  such that  $\theta'$  has  $\ell$ , there exists a unique morphism  $\mathcal{P}^\ell \rightarrow \mathcal{P}'$  such that the left triangle in the diagram

$$\begin{array}{ccccc} & & \theta & & \\ & \nearrow & & \searrow & \\ \mathcal{P} & \xrightarrow{\quad} & \mathcal{P}^\ell & \xrightarrow{\theta^\ell} & \mathcal{M} \\ & \searrow & \vdots & \nearrow & \\ & & \mathcal{P}' & & \end{array}$$

commutes; moreover, the right triangle commutes as well.

Abbreviating “sharp” and “logarithmic” to “sh” and “log”, it is straightforward to check that  $\mathcal{P}^{sh} \rightarrow \mathcal{M}$  can be constructed as the canonical map

$$\mathcal{P}^{sh} := \mathcal{P} \oplus_{\mathcal{P}^\times} \mathcal{M}^\times \rightarrow \mathcal{P}$$

and  $\mathcal{P}^{log} \rightarrow \mathcal{M}$  as the canonical map

$$\mathcal{P}^{log} := \mathcal{P} \oplus_{\theta^{-1}(\mathcal{P}^\times)} \mathcal{M}^\times \rightarrow \mathcal{P}.$$

Since, as observed above, being logarithmic is equivalent to being local and sharp, if  $\theta$  is already local, then  $\theta^{log} = \theta^{sh}$ , a fact that is often useful to simplify computations.

*Notation 2.4.12.* For any scheme  $X$  and monoid  $P$ , denote by  $P_X$ , or simply by  $P$  if  $X$  is clear from the context, the constant sheaf on  $X$  whose stalks are equal to  $P$ .

### Prelog schemes and log schemes

**Prelog and log structures** Moving towards the notion of log scheme, we recall that of a (pre)log structure on a scheme  $X$ .

**Definition 2.4.13.** Let  $X$  be a scheme.

- A prelogarithmic structure on  $X$  is a pair  $(\mathcal{M}, \alpha)$  consisting of a sheaf of monoids  $\mathcal{M}$  and a morphism of sheaves of monoids  $\alpha: (\mathcal{M}, +) \rightarrow (\mathcal{O}_X, \cdot)$ .
- A morphism of prelogarithmic structures  $(\mathcal{M}, \alpha) \rightarrow (\mathcal{N}, \beta)$  on  $X$  is a morphism of sheaves of monoids  $\theta: \mathcal{M} \rightarrow \mathcal{N}$  such that  $\beta \circ \theta = \alpha$ .
- A logarithmic structure on  $X$  is a prelogarithmic structure  $(\mathcal{M}, \alpha)$  such that  $\alpha$  is logarithmic, i.e. the induced map  $\alpha^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{M}^\times$  is an isomorphism of sheaves of monoids. A morphism of logarithmic structures is just a morphism of prelogarithmic structures.

We denote by  $\mathbf{Log}_X$  ( $\mathbf{PLog}_X$ ) the category of (pre)logarithmic structures on  $X$ .

The words “prelogarithmic” and “logarithmic” are usually abbreviated to “prelog” and “log”. It is also customary to refer to a sheaf of monoids  $\mathcal{M}$  on  $X$  as a prelog structure on  $X$ , if the morphism  $\alpha: \mathcal{M} \rightarrow \mathcal{O}_X$  is clear from the context. All the notions defined for (morphisms of) sheaves of monoids are defined for (pre)log structures in the obvious way.

*Example 2.4.14.* For any scheme  $X$ , the inclusion  $\mathcal{O}_X^\times \rightarrow \mathcal{O}_X$  defines a log structure on  $X$ , which is the initial object in  $\mathbf{Log}_X$ ; it is called the *trivial log structure on  $X$* .

By Proposition 2.4.11, the inclusion of  $\mathbf{PLog}_X$  into  $\mathbf{Log}_X$  admits a left adjoint, called *logarithmification*; the logarithmification of a prelog structure  $\mathcal{M} \rightarrow \mathcal{O}_X$  is also called its *associated log structure*, and denoted by  $\mathcal{M}^a \rightarrow \mathcal{O}_X$ . Denoting by  $u$

the unit of this adjunction, for any morphism  $g: \mathcal{M} \rightarrow \mathcal{N}$  in  $\mathbf{PLog}/X$ , there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{N} \\ \downarrow u_{\mathcal{M}} & \nearrow & \downarrow u_{\mathcal{N}} \\ \mathcal{M}^a & \xrightarrow{f^a} & \mathcal{N}^a \end{array} \quad .$$

Notice that for any scheme  $X$ , the category  $\mathbf{PLog}_X$  is just the slice category of sheaves of monoids over  $(\mathcal{O}_X, \cdot)$ , and in particular has all colimits. It also ensues that  $\mathbf{Log}_X$  has all colimits, which are computed applying the logarithmification functor to the colimits computed in  $\mathbf{PLog}_X$ .

**Definition 2.4.15.** Let  $f: X \rightarrow Y$  be a morphism of schemes.

- Let  $\beta: \mathcal{N} \rightarrow \mathcal{O}_Y$  be a prelog structure. The inverse image of  $\beta$  along  $f$  is the prelog structure on  $X$

$$f^*(\beta): f^{-1}(\mathcal{N}) \xrightarrow{f^{-1}(\beta)} f^{-1}(\mathcal{O}_Y) \xrightarrow{f^\flat} \mathcal{O}_X.$$

- Let  $\alpha: \mathcal{M} \rightarrow \mathcal{O}_X$  be a prelog structure. The direct image of  $\alpha$  along  $f$  is the prelog structure  $f_*^p(\alpha)$  defined by the cartesian square

$$\begin{array}{ccc} f_*^p(\mathcal{M}) & \xrightarrow{f_*^p(\alpha)} & \mathcal{O}_Y \\ \downarrow & \lrcorner & \downarrow f^\sharp \\ f_*(\mathcal{M}) & \xrightarrow{f_*(\alpha)} & f_*\mathcal{O}_X \end{array} \quad .$$

As the notation suggests, one thus obtains an adjoint pair

$$f^*: \mathbf{PLog}_Y \rightleftarrows \mathbf{PLog}_X : f_*^p$$

[GR18, 12.1.2]. Composing the logarithmification functor to  $f^*: \mathbf{Log}_Y \rightarrow \mathbf{PLog}_X$ , one also obtains a functor  $f_{log}^*: \mathbf{PLog}_Y \rightarrow \mathbf{Log}_X$ , and the restrictions of  $f^*$  and  $f_*^p$  to  $\mathbf{Log}_Y$  and  $\mathbf{Log}_X$  form an adjoint pair denoted by

$$f_{log}^*: \mathbf{Log}_Y \rightleftarrows \mathbf{Log}_X : f_*^{log}.$$

**Prelog schemes and log schemes** We can finally come to the definition of (pre)log schemes:

**Definition 2.4.16.** • A prelog scheme is a pair  $(X, \alpha: \mathcal{M} \rightarrow \mathcal{O}_X)$  consisting of a scheme  $X$  and of a prelog structure  $\alpha$  on  $X$ .

- A morphism of prelog schemes  $(X, \alpha: \mathcal{M} \rightarrow \mathcal{O}_X) \rightarrow (Y, \beta: \mathcal{N} \rightarrow \mathcal{O}_Y)$  is a pair  $(f, \bar{f})$  consisting of a morphism of schemes  $f: X \rightarrow Y$  and a morphism of prelog structures  $\bar{f}: f^*\mathcal{N} \rightarrow \mathcal{M}$ .
- A log scheme is a prelog scheme whose prelog structure is logarithmic.
- A morphism  $(X, \alpha) \rightarrow (Y, \beta)$  between log schemes is just a morphism of prelog schemes.

The categories of prelog and log schemes are denoted by  $\mathbf{PLogSch}$  and  $\mathbf{LogSch}$ .

The forgetful functor from the category of (pre)-log schemes to  $\mathbf{Sch}$  will be denoted by an underline  $\_$ .

Endowing a scheme  $X$  with its trivial log structure, we can regard the category of schemes as a full subcategory of  $\mathbf{LogSch}$ : thus, unless otherwise specified, ordinary schemes will be regarded as log schemes with the trivial log structure. Notice that for any log scheme  $(X, \mathcal{M})$  there is a canonical map  $X \rightarrow \underline{X}$ , which is the identity at the level of schemes and the inclusion  $\mathcal{O}_X^\times \rightarrow \mathcal{M}$  at the level of log structures.

*Remark 2.4.17.* Since  $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$  is the initial object of  $\mathbf{Log}_X$ , the forgetful functor  $\mathbf{LogSch} \rightarrow \mathbf{Sch}$  is right adjoint to the inclusion  $\mathbf{Sch} \rightarrow \mathbf{LogSch}$ .

As the next lemma shows, the construction of the log structure associated to a prelog structure can be upgraded to a functor  $\mathbf{PLogSch} \rightarrow \mathbf{LogSch}$  taking a prelog scheme  $(X, \mathcal{M} \rightarrow \mathcal{O}_X)$  to  $(X, \mathcal{M}^a \rightarrow \mathcal{O}_X)$ ; we thank Maurizio Cailotto for explaining the next lemma to us.

**Lemma 2.4.18.** *There is a functor  $\mathbf{PLogSch} \rightarrow \mathbf{LogSch}$  taking a prelog scheme  $(X, \mathcal{M} \rightarrow \mathcal{O}_X)$  to  $(X, \mathcal{M}^a \rightarrow \mathcal{O}_X)$  and a morphism*

$$(f, \bar{f}): (X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X) \rightarrow (Y, \mathcal{N} \xrightarrow{\beta} \mathcal{O}_Y)$$

*to the morphism  $(f, \log(f))$ , where  $\log(f): f^{-1}\mathcal{N}^a \rightarrow \mathcal{M}^a$  is defined by the diagram*

$$\begin{array}{ccccc}
 f^{-1}(\beta^{-1}(\mathcal{O}_Y^\times)) & \longrightarrow & f^{-1}\mathcal{N} & & \\
 \downarrow & & \downarrow & \searrow \bar{f} & \\
 f^{-1}(\mathcal{O}_Y^\times) & \longrightarrow & f^{-1}(\mathcal{N}^a) & \longrightarrow & \alpha^{-1}(\mathcal{O}_X^\times) \longrightarrow \mathcal{M} \\
 & & \searrow & \downarrow \log(f) & \downarrow \\
 & & & \mathcal{O}_X^\times & \longrightarrow \mathcal{M}^a
 \end{array}$$

*Proof.* Since  $f^{-1}$  is a left adjoint, it preserves pushouts, whereby there exists a unique dotted arrow in the previous diagram making all rectangles commute. It is

over  $\mathcal{O}_X$  because all the arrows in the diagram are, too, and so it defines a morphism of log structures.

If  $(f, \bar{f}) = (\text{id}_X, \bar{\text{id}}_{(X, \mathcal{M})}: \text{id}_X^{-1} \mathcal{M} \rightarrow \mathcal{M})$ , then the induced map is  $\text{id}_{(X, \mathcal{M}^a)}$ : it is enough to show that  $\bar{\text{id}}_{(X, \mathcal{M}^a)}$  makes the right rectangle commute (as it clearly makes the lower one commute), i.e. that  $\bar{\text{id}}_{(X, \mathcal{M}^a)} \circ \text{id}^{-1}(u_{\mathcal{M}}) = u_{\mathcal{M}} \circ \bar{\text{id}}_{(X, \mathcal{M})}$ ; recall that  $u$  denotes the unit of the logarithmification-inclusion adjunction.<sup>4</sup> Since  $\text{id}_{X*}$  is the identity functor, the adjunction of sheaves of monoids  $\text{id}_X^{-1} \dashv \text{id}_{X*}$  yields a commutative diagram

$$\begin{array}{ccc}
 \text{Hom}(\mathcal{M}^a, \mathcal{M}^a) & \xrightarrow{\sim} & \text{Hom}(\text{id}_X^{-1} \mathcal{M}^a, \mathcal{M}^a) \\
 \downarrow (\cdot) \circ u_{\mathcal{M}} & & \downarrow (\cdot) \circ \text{id}_X^{-1}(u_{\mathcal{M}}) \\
 \text{Hom}(\mathcal{M}, \mathcal{M}^a) & \xrightarrow{\sim} & \text{Hom}(\text{id}_X^{-1} \mathcal{M}, \mathcal{M}^a) \\
 \uparrow u_{\mathcal{M}} \circ (\cdot) & & \uparrow u_{\mathcal{M}} \circ (\cdot) \\
 \text{Hom}(\mathcal{M}, \mathcal{M}) & \xrightarrow{\sim} & \text{Hom}(\text{id}_X^{-1} \mathcal{M}, \mathcal{M});
 \end{array}$$

the claim follows because  $\text{id}_{\mathcal{M}^a}$  and  $\text{id}_{\mathcal{M}}$  are both mapped to  $u_{\mathcal{M}}$  by the top (resp. bottom) left arrow.

Stacking the relevant diagrams and using the uniqueness in the universal property of pushouts, one readily sees that the mapping  $(f, \bar{f}) \mapsto (f, \log(f))$  also respects compositions.  $\square$

Many important instances of log schemes arise from the following

*Construction 2.4.19.* A morphism  $\theta: P \rightarrow A$  between a monoid  $P$  and the multiplicative monoid of a ring  $A$  (which is sometimes called a *log ring* [Ogu18, Definition III.1.2.3]) gives rise to a (pre)log scheme: the morphism of sheaves of monoids  $P_X \rightarrow \mathcal{O}_{\text{Spec } A}$  induced by  $\theta$  defines a prelog structure on  $\text{Spec } A$ ; endowing  $\text{Spec } A$  with the log structure associated to this one obtains a log scheme, denoted by

$$\text{Spec}(P \xrightarrow{\theta} A).$$

*Example 2.4.20.* (i) For any monoid  $P$  and any ring  $R$ , denote by  $R[P]$  the monoid algebra of  $P$  over  $R$ ; the canonical map  $P \rightarrow R[P]$  then defines a log ring.

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<sup>4</sup>For a morphism of schemes  $f: X \rightarrow Y$ , some authors define the pullback functor of sheaves  $f^{-1}$  as a left adjoint of  $f_*$ ; according to them,  $\text{id}_X^{-1} \mathcal{M}$  is then only isomorphic to  $\mathcal{M}$ , but not equal to it, and it would be improper to identify the two here.

- (ii) If  $R$  is a discrete valuation ring with uniformiser  $\pi$ , the morphism of monoids  $\mathbb{N} \rightarrow R, n \mapsto \pi^n$  is a log ring.
- (iii) For any integer  $r$  and any ring  $A$ , the morphism of monoids  $\mathbb{N}^r \rightarrow A[x_1, \dots, x_r], (a_1, \dots, a_r) \mapsto x_1^{a_1} \cdots x_r^{a_r}$  is a log ring.

The log structures corresponding the log rings at points *ii*) and *iii*) are called the *standard* ones (on their respective schemes).

One is especially interested in log schemes that are locally modelled on log rings; in order to make this precise, one introduces the notion of chart [Ogu18, pag. 275 and ff].

**Definition 2.4.21.** A morphism of log schemes  $f: (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$  is called strict if the induced morphism  $f_{\log}^* \mathcal{N} \rightarrow \mathcal{M}$  is an isomorphism.

**Definition 2.4.22.** • A chart for a log scheme  $(X, \mathcal{M})$  subordinated to a monoid  $P$  is strict morphism of log schemes  $X \rightarrow \text{Spec}(P \rightarrow \mathbb{Z}[P])$ .

- A chart for a morphism of log schemes  $f: (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$  is a triple

$$(X \xrightarrow{c_1} \text{Spec}(P \rightarrow \mathbb{Z}[P]), Y \xrightarrow{c_2} \text{Spec}(Q \rightarrow \mathbb{Z}[Q]), \theta: Q \rightarrow P)$$

consisting of charts  $c_1$  and  $c_2$  for  $X$  and  $Y$  and a morphism of monoids  $P \rightarrow Q$  such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{c_1} & \text{Spec}(P \rightarrow \mathbb{Z}[P]) \\ \downarrow f & & \downarrow \\ Y & \xrightarrow{c_2} & \text{Spec}(Q \rightarrow \mathbb{Z}[Q]), \end{array}$$

where the right vertical map is that induced by  $\theta$ , commutes.

Observe that the datum of a chart  $(X, \mathcal{M}) \rightarrow \text{Spec}(P \rightarrow \mathbb{Z}[P])$  is equivalent to that of a morphism of monoids  $P \rightarrow \Gamma(X, \mathcal{M})$  such that the induced morphism  $(X, \mathcal{M}) \rightarrow \text{Spec}(P \rightarrow \mathbb{Z}[P])$  is strict.

Notice also that a chart for a log scheme is *not* a local isomorphism, unlike in differential geometry.

Log schemes that locally admit (integral and) finitely generated charts are particularly important.

**Definition 2.4.23** ([Ogu18, Definition II.2.1.5]). A log scheme  $(X, \mathcal{M})$  (or its log structure) is called coherent (resp. fine/fs) if it admits an open covering  $\{U_i \rightarrow X\}_i$  such that each log scheme  $(U_i, \mathcal{M}|_{U_i})$  admits a chart subordinated to a finitely

generated (resp. finitely generated and integral/finitely generated, integral and saturated) monoid.

Being fine is not a stalk-local property, unlike properties like integrality or saturation, which are defined for (pre)log schemes and their morphisms in terms of those of their (pre)log structures.

The full subcategory of **LogSch** comprising the coherent/integral/fine/saturated/fs log schemes will be denoted by adding the superscript “coh”/“int”/“fin”/“sat”/“fs”; we shall similarly use these strings as sub- or superscripts when needing to differentiate between operations carried out in these categories.

**Proposition 2.4.24.** *The categories of arbitrary/fine/fs log schemes admit finite limits, whose construction preserves coherence.*

Given the particular importance of pullbacks, let us briefly recall their construction, referring the reader to [GR18, 12.2.5, 12.1.24] and [Ogu18, Proposition III.2.1.1] for the general statement. The pullback of two morphisms of log schemes  $(f, \bar{f}): (X, \alpha: \mathcal{M} \rightarrow \mathcal{O}_X) \rightarrow (Y, \beta: \mathcal{N} \rightarrow \mathcal{O}_Y)$  and  $(g, \bar{g}): (Y', \beta': \mathcal{N}' \rightarrow \mathcal{O}_{Y'}) \rightarrow (Y, \beta: \mathcal{N} \rightarrow \mathcal{O}_Y)$  is the log scheme  $(X', \alpha': \mathcal{M}' \rightarrow \mathcal{O}_{X'})$ , where  $X'$  is the fibred product  $X' = X \times_Y Y'$  in the category of schemes and, denoting by  $f': X' \rightarrow Y'$ ,  $g': X' \rightarrow X$  and  $h: X' \rightarrow Y$  the projections,  $\mathcal{M}' \rightarrow \mathcal{O}_{X'}$  is defined by the pushout diagram

$$\begin{array}{ccc}
 h_{\log}^*(\beta) & \longrightarrow & (g')_{\log}^*(\alpha) \\
 \downarrow & & \downarrow \\
 (f')_{\log}^*(\alpha') & \longrightarrow & \mathcal{M}' \\
 & \searrow & \downarrow \\
 & & \mathcal{O}_{X'}
 \end{array}
 .$$

**Lemma 2.4.25.** (i) *The category of log schemes (with either Zariski or étale topology) with the forgetful functor to  $|\cdot|: \mathbf{LogSch} \rightarrow \mathbf{Top}$  is a category with open immersions.*

(ii) *The  $|\cdot|$ -topology is subcanonical on the category of  $u$ -integral log schemes.*

(iii) *The subcategories of integral, integral, fine, saturated and fs log schemes which admit finitely generated charts Zariski locally have gluings.*

*Proof.* Let  $(X, \mathcal{M})$  be a logarithmic scheme (for either topology),  $U \subseteq |X|$  an open subset (regarded equivalently as an open subscheme  $j: U \rightarrow X$ ); then the canonical

morphism  $(U, j_{\log}^* \mathcal{M}) \rightarrow (X, \mathcal{M})$  has the universal property required in Definition 1.2.2, which follows immediately unwinding the definitions: more precisely, letting  $f: (Z, \mathcal{N}) \rightarrow (X, \mathcal{M})$  be a morphism of log schemes such that  $|f|$  factors through  $U$  and denoting by  $\bar{f}: Z \rightarrow U$  the unique morphism of schemes such that  $j \circ \bar{f} = f$ , it is enough to show that there exists a unique morphism of sheaves  $\bar{f}^{-1}j^{-1}\mathcal{M} \rightarrow \mathcal{N}$  making the diagram

$$\begin{array}{ccccc} f^{-1}\mathcal{M} & \longrightarrow & \bar{f}^{-1}j^{-1}\mathcal{M} & \cdots \cdots \longrightarrow & \mathcal{N} \\ \downarrow & & \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_X & \longrightarrow & \bar{f}^{-1}j^{-1}\mathcal{O}_X & \longrightarrow & \mathcal{O}_Z \end{array}$$

commute. This is trivial, because the two left horizontal arrows are isomorphisms.

The fact that the  $|\cdot|$ -topology is subcanonical on the category of u-integral log schemes is [Ogu18, Corollary III.1.4.5].

Gluing data of log schemes which admit charts Zariski locally are effective because Zariski and étale sheaves on schemes can be glued along any Zariski open cover; moreover, for such log schemes, the properties in the statement are Zariski local (even if one is working with sheaves of monoids for the étale topology) by [Ogu18, Corollary III.1.4.4] or [GR18, Corollary 12.2.22].  $\square$

*Remark 2.4.26.* The asymmetry, in the previous lemma, between Zariski and étale topology is due to the fact that we are working with log *schemes*, and not with log algebraic spaces, and schemes are glued from affine schemes along the Zariski topology; in a way, the asymmetry comes therefore intrinsically from the definition of log schemes.

The construction of limits in the categories of fine and of fs log schemes is grounded on the following result, which globalises the integralisation and saturation functors to the corresponding categories of log schemes [Ogu18, Proposition III.2.1.5]:

**Proposition 2.4.27.** *The inclusion functors  $\mathrm{LogSch}^{fin} \hookrightarrow \mathrm{LogSch}^{coh}$  and  $\mathrm{LogSch}^{fs} \hookrightarrow \mathrm{LogSch}^{fin}$  admit right adjoints, which are denoted by  $(\cdot)^{int}$  and  $(\cdot)^{sat}$  and called integralisation and saturation, respectively.*

For a log scheme of the form  $X = \mathrm{Spec}(P \rightarrow \mathbb{Z}[P])$  ( $P$  being any finitely generated monoid),

$$X^{int} = \mathrm{Spec}(P^{int} \rightarrow \mathbb{Z}[P^{int}]);$$

similarly, if  $P$  is, in addition, integral,

$$X^{sat} = \text{Spec}(P^{sat} \rightarrow \mathbb{Z}[P^{sat}]).$$

For any coherent log scheme  $X$  with a chart  $X \rightarrow Y := \text{Spec}(P \rightarrow \mathbb{Z}[P])$ ,

$$X^{int} = \underline{X} \times_Y Y^{int};$$

similarly, if  $X$  is fine, then

$$X^{sat} = \underline{X} \times_Y Y^{sat}$$

In general, one constructs  $X^{int}$  and  $X^{sat}$  by taking an affine cover  $\{U_i \rightarrow X\}_i$  of  $X$  whose members admit charts and patching the various  $U_i^{int}$  or  $U_i^{sat}$  together (invoking [Ogu18, Proposition III.2.1.4.3] when working with the étale topology).

One obtains the limit of a finite diagram of fine (resp. fs) log schemes applying the integralisation (resp. integralisation and saturation) functor(s) to the corresponding limit in the category of coherent log schemes.

While the construction of fibred products in the category **LogSch** commutes with the forgetful functor to **Sch**, this is no longer true for fibred products in the subcategories of fine or fs log schemes.

Integral and saturated morphisms of log schemes (defined, as per the convention above, stalkwise) behave as expected with respect to fibred products, at least if the log schemes involved are fine:

**Proposition 2.4.28** ([Ogu18, Proposition III.2.5.3]). *If  $f: X \rightarrow Y$  and  $Y' \rightarrow Y$  are morphisms of fine (resp. fs) log schemes and  $f$  is integral (resp. saturated), then  $X \times_Y Y'$  is integral (resp. saturated).*

*Notation 2.4.29.* As an exception to the convention above, for any log scheme  $S$  and any non-negative integer  $m$  we shall denote by  $\mathbb{A}_S^m$  the affine space  $\mathbb{A}_{\underline{S}}^m$  endowed with its standard log structure, i.e. the base change of the log scheme

$$\text{Spec}(\mathbb{N}^m \rightarrow \mathbb{Z}[x_1, \dots, x_m])$$

of Example 2.4.20 along the morphism of log schemes  $S \rightarrow (\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}}^\times)$ , even if  $S$  is a scheme with trivial log structure.

If  $S$  is a scheme, the log scheme  $(\mathbb{A}_S^m, \mathcal{O}_{\mathbb{A}_S^m}^\times)$  will instead be denoted by  $\underline{\mathbb{A}}_S^m$ ; we hope that this slight incongruence, aimed at easing notation, will not inconvenience the reader.

For a log scheme  $S$ , a morphism of log schemes  $T \rightarrow S$  will be also called an *S-log scheme*, or also a *log S-scheme* if  $S$  has trivial log structure.

The following remark will be extremely useful.

*Remark 2.4.30.* For any log scheme  $S = (S, \mathcal{M}_S)$ ,  $\mathbb{A}_S^1$  represents the functor of  $S$ -log schemes  $((T, \mathcal{M}_T) \rightarrow S) \mapsto \Gamma(T, \mathcal{M}_T)$ . Using the universal property of fibred products, one can reduce to the case when  $S = \operatorname{Spec} \mathbb{Z}$ ; a morphism of log schemes  $(T, \mathcal{M}_T \xrightarrow{\alpha} \mathcal{O}_T) \rightarrow (\mathbb{A}_{\mathbb{Z}}^1, \mathcal{M}_{\mathbb{A}_{\mathbb{Z}}^1})$  can equivalently be described as a pair  $(f, g)$  where  $f: T \rightarrow \mathbb{A}_{\mathbb{Z}}^1$  is a morphism of schemes and  $g: \mathcal{M}_{\mathbb{A}_{\mathbb{Z}}^1} \rightarrow f_* \mathcal{M}_T$  is a morphism of sheaves of monoids such that

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{A}_{\mathbb{Z}}^1} & \xrightarrow{g} & f_* \mathcal{M}_T \\ \downarrow & & \downarrow f_*(\alpha) \\ \mathcal{O}_{\mathbb{A}_{\mathbb{Z}}^1} & \xrightarrow{f^\flat} & f_* \mathcal{O}_T \end{array}$$

commutes. One readily sees that  $\mathcal{M}_{\mathbb{A}_{\mathbb{Z}}^1} = \mathbb{N} \oplus \mathcal{O}_{\mathbb{A}_{\mathbb{Z}}^1}^\times$ ; a morphism of sheaves  $\mathbb{N} \rightarrow f_* \mathcal{M}_T$  is univocally determined by the image of the map  $\mathbb{N} \rightarrow \Gamma(T, \mathcal{M}_T)$  it induces on global sections, and thus by a global section of  $\mathcal{M}_T$ . Moreover, to each  $t \in \Gamma(T, \mathcal{M}_T)$  there corresponds exactly one morphism of schemes  $T \rightarrow \mathbb{A}_{\mathbb{Z}}^1$  making the previous diagram commute, and this in turn defines a unique extension of the morphism of sheaves  $\mathbb{N} \rightarrow \mathcal{M}_T$  to a morphism  $\mathbb{N} \oplus \mathcal{O}_{\mathbb{A}_{\mathbb{Z}}^1}^\times \rightarrow \mathcal{M}_T$ .

## 2.5 Logarithmic Weil restriction

For a positive integer  $m$  and  $r \in \mathbb{N} \cup \{\infty\}$ , Herr defined in [Her24] the log jet space  $J_{X,m}^{(r)}$  of an fs log scheme  $X$  over  $k$  as the presheaf on  $\mathbf{LogSch}^{fs}$  with

$$J_{X,m}^{(r)}(\operatorname{Spec} A) = \operatorname{Hom}_k(\Delta_m^{(r)} \times_k \operatorname{Spec} A, X)$$

for any affine fs log  $k$ -scheme  $\operatorname{Spec} A$ , where  $\Delta_m^{(r)}$  is the log scheme associated to the log ring

$$\mathbb{N} \rightarrow k[x]/(x^m), \quad 1 \mapsto x^r$$

(with  $t^\infty = 0$ ).<sup>5</sup> He then proves that if  $X$  admits charts Zariski locally,  $J_{X,m}^{(r)}$  is representable by an fs log scheme.

Herr's functor is an instance of logarithmic Weil restriction: more precisely, one can write  $J_{X,m}^{(r)}$  as the Weil restriction of  $X \times_k \Delta_m^{(r)}$  along the map  $\Delta_m^{(r)} \rightarrow \operatorname{Spec} k$  (in particular, Herr does not work in the relative setup, but considers jet spaces of log schemes that are already over  $k$ ). One can ask more generally for some conditions on a morphism  $f: S' \rightarrow S$  of log schemes and on a log scheme  $X'$  over  $S'$  which

<sup>5</sup> Actually, in the definition of  $J_{X,m}^{(r)}$  we shifted the index  $m$  by 1 with respects to Herr's notation.

guarantee the representability of the logarithmic Weil restriction of  $X'$  along  $f$ ; this is the object of the main result of this section, Theorem 2.5.11, dealing with the case where  $f$  is of the form  $S \rightarrow \underline{S}$ ; this is “complemented” by Lemma 2.5.14, in which  $f$  comes from a nilpotent thickening of schemes.

As usual, one can make several choices for the setup; namely, one can work with either étale or Zariski log structures, and in the categories of arbitrary, of fine, of fs,... log schemes; we shall use sub- or superscripts to disambiguate: in the following statements,  $\ell$  will stand for “fin(e)” or  $\ell$  = “fs” or for a blank space (in which case, the statement is meant to hold for arbitrary log schemes); in particular,  $\times^{fin}$  and  $\times^{fs}$  denote the fibred product in  $\text{LogSch}^{fin}$  and  $\text{LogSch}^{fs}$ , respectively.

**Definition 2.5.1.** Let  $f: S' \rightarrow S$  be a morphism in  $\text{LogSch}^\ell$ . The logarithmic Weil restriction of an  $S'$ -log scheme  $X' \rightarrow S'$  in  $\text{LogSch}^\ell$  is the presheaf

$$f_*^\ell X': (\text{LogSch}^\ell/S)^{\text{op}} \rightarrow \text{Set}$$

with  $f_*^\ell X'(Z) = \text{Hom}_{S'}(Z \times_S^\ell S', X')$  for any  $Z \rightarrow S$  in  $\text{LogSch}^\ell/S$ .

It is a sheaf on the category of  $u$ -integral log schemes. When the map  $f$  is clear from the context, we shall also use the notation  $\text{Res}_{S'/S}^\ell X'$  for  $f_*^\ell X'$ . It is straightforward to see that if  $f: S' \rightarrow S$  is a morphism of *schemes* and  $X'$  is an  $S'$ -scheme, then the logarithmic Weil restriction  $f_* X'$  is represented by the schematic Weil restriction of  $X'$  along  $f$ , regarded as a log scheme with trivial log structure, whereby there is no conflict of notation.

We wish to apply Theorem 1.3.7; as a first step, following Remark 1.3.8 we show how to reduce the statement of the main theorem to simpler log schemes.

The next lemma generalises an argument in the proof of [Her24, Theorem 2.20] to the relative situation.

**Lemma 2.5.2.** *Let  $S$  be an affine log scheme in  $\text{LogSch}^\ell$  with a finitely generated global chart. Denote by  $\mathcal{A}$  the class of objects of  $\text{LogSch}^\ell/S$  comprising the log schemes  $\mathbb{A}_S^n$  for any non negative integer  $n$  and the log schemes of the form  $T \times_{\underline{S}} S$ , where  $T \rightarrow \underline{S}$  is a morphism of schemes.*

*Any log scheme in  $\text{LogSch}^\ell/S$  admitting a finitely generated global chart belongs to the class of  $S$ -log schemes generated by  $\mathcal{A}$  under pullbacks and equalisers in  $\text{LogSch}^\ell/S$ .*

Notice that by Example 2.4.7 the log schemes  $T \times_{\underline{S}} S$  with  $T \rightarrow \underline{S}$  a morphism of schemes are indeed in  $\text{LogSch}^\ell/S$ .

*Proof.* By the construction of limits in  $\mathbf{LogSch}^\ell$ , we can immediately reduce to the claim for  $\mathbf{LogSch}/S$ .

Let  $\mathcal{A}_1$  be the class of log schemes over  $S$  admitting a strict map to an affine space  $\mathbb{A}_S^m$ .

**Step 1.** *The class  $\mathcal{A}_1$  is generated by  $\mathcal{A}$  under pullbacks (i.e. any  $X$  in  $\mathcal{A}_1$  is a pullback of elements of  $\mathcal{A}$ ).*

Let  $f: X \rightarrow Y$  be a morphism of log schemes over  $S$ . Recalling that  $f$  is strict if and only if the square

$$\begin{array}{ccc} X & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \underline{Y} \end{array}$$

is cartesian [Ogu18, Proposition III.1.2.5], a purely categorical argument shows that  $f$  is strict if and only if the square

$$\begin{array}{ccc} X & \longrightarrow & \underline{X} \times_{\underline{S}} S \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \underline{Y} \times_{\underline{S}} S \end{array},$$

which is now a diagram in  $\mathbf{LogSch}/S$ , is cartesian in  $\mathbf{LogSch}/S$  (indeed,

$$\begin{array}{ccc} \underline{X} \times_{\underline{S}} S & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow \\ \underline{Y} \times_{\underline{S}} S & \longrightarrow & \underline{Y} \end{array}$$

is cartesian and fits into a commutative rectangle

$$\begin{array}{ccccc} X & \longrightarrow & \underline{X} \times_{\underline{S}} S & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & \underline{Y} \times_{\underline{S}} S & \longrightarrow & \underline{Y} \end{array} ).$$

Therefore, all log schemes  $X$  as in the statement that admit a strict  $S$ -map to some affine space  $Y = \mathbb{A}_S^m$  can be constructed out of elements of  $\mathcal{A}$  via pullbacks.

**Step 2.** *Any affine  $S$ -log scheme  $X$  which has a finitely generated global chart can be written as an equaliser (in the category of  $S$ -log schemes) of elements of  $\mathcal{A}_1$ .*

Let  $Q \rightarrow \Gamma(S, \mathcal{O}_S)$  be a finitely generated chart for  $S$ . By [Ogu18, Proposition II.2.4.2]<sup>6</sup> and its proof one can find a chart  $\theta: Q \rightarrow P$  for the structure morphism

<sup>6</sup>Recall [Ogu18, Definition II.2.1.5] that a (Zariski/étale) sheaf of monoids  $\mathcal{M}$  on a scheme  $X$  is called coherent if  $X$  has a covering  $\mathcal{U}$  such that  $\mathcal{M}|_{\mathcal{U}}$  has a chart subordinated to a finitely generated monoid.

$X \rightarrow S$ .

*Claim 2.5.3.* The morphism  $\theta$  fits into a commutative diagram

$$\begin{array}{ccccc} \mathbb{N}^n & \rightrightarrows & \mathbb{N}^m & \longrightarrow & P \\ \uparrow & & \uparrow & & \uparrow \theta \\ \mathbb{N}^b & \rightrightarrows & \mathbb{N}^a & \longrightarrow & Q \end{array},$$

where both rows are coequalisers.

In order to prove the claim, pick generators  $q_1, \dots, q_a$  of  $Q$  and  $p_1, \dots, p_l$  of  $P$ ; let  $m = a + l$  and define maps

$$\mathbb{N}^a \rightarrow Q, \quad e_i \mapsto q_i$$

and

$$\mathbb{N}^m \rightarrow P, \quad e_i \mapsto \begin{cases} \theta(q_i) & \text{if } 1 \leq i \leq a \\ p_{i-a} & \text{if } i > a \end{cases}.$$

Consider the commutative diagram

$$\begin{array}{ccccccc} E_m & \longrightarrow & \mathbb{N}^m \times \mathbb{N}^m & \xrightleftharpoons[\text{pr}_2]{\text{pr}_1} & \mathbb{N}^m & \longrightarrow & P \\ \vdots \uparrow & & \uparrow & & \uparrow & & \uparrow \theta \\ E_a & \longrightarrow & \mathbb{N}^a \times \mathbb{N}^a & \xrightleftharpoons[\text{pr}_2]{\text{pr}_1} & \mathbb{N}^a & \longrightarrow & Q \end{array}, \quad (2.1)$$

where the rows are equaliser diagrams, i.e. the map  $E_a \rightarrow \mathbb{N}^a \times \mathbb{N}^a$  is the equaliser of the compositions  $\mathbb{N}^a \times \mathbb{N}^a \rightrightarrows \mathbb{N}^a \rightarrow Q$ , and similarly for  $E_m$ ; by the universal property of equalisers, there exists a unique dotted arrow making the diagram commute. Since  $E_m$  is a congruence relation on  $\mathbb{N}^m$  [Ogu18, Proposition I.1.1.1] and every congruence relation on a finitely generated monoid is itself finitely generated [Ogu18, Lemma I.2.1.9], we can define, with the same argument as for  $\theta$ , surjections  $\mathbb{N}^n \twoheadrightarrow E_m$  and  $\mathbb{N}^b \twoheadrightarrow E_a$  such that the square

$$\begin{array}{ccc} \mathbb{N}^n & \longrightarrow & E_m \\ \uparrow & & \uparrow \\ \mathbb{N}^b & \longrightarrow & E_a \end{array}$$

commutes.

By [Ogu18, Proposition I.1.1.1],  $P$  is the coequaliser of the diagram

$$E_m \rightarrow \mathbb{N}^m \times \mathbb{N}^m \xrightleftharpoons[\text{pr}_2]{\text{pr}_1} \mathbb{N}^m,$$

and analogously for  $Q$ , whence the claim, observing that precomposing the rows of the diagram (2.1) by the epimorphisms  $\mathbb{N}^n \rightarrow E_m$  and  $\mathbb{N}^b \rightarrow E_a$  does not change their coequalisers.

In the situation above, for  $i = m, n$ , let  $X_i$  be the log scheme corresponding to the log ring  $\mathbb{N}^i \rightarrow P \rightarrow \Gamma(X, \mathcal{O}_X)$  (so that, in particular,  $\underline{X}_i = \underline{X}$ ), and define  $S_a, S_b$  analogously. By [GR18, Lemma 12.2.18i], for any log scheme  $(T, \mathcal{M}_T)$ , a morphism  $(T, \mathcal{M}_T) \rightarrow X_i$  is the same as the datum of a map of schemes  $T \rightarrow X_i$  and a map of monoids  $\mathbb{N}^i \rightarrow \Gamma(T, \mathcal{M}_T)$ ; therefore, the diagram in the claim yields a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_m & \rightrightarrows & X_n \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & S_a & \rightrightarrows & S_b \end{array}$$

where the rows are equalisers. This allows us to write the  $S$ -log scheme  $X$  as an equaliser, in the category of  $S$ -log schemes, of elements of  $\mathcal{A}_1$ : namely, we claim that the diagram

$$X \rightarrow S \times_{S_a} X_m \rightrightarrows S \times_{S_b} X_n$$

is an equaliser. This amounts to asking that for an arbitrary morphism of log schemes  $T \rightarrow S$ , the induced diagram

$$\mathrm{Hom}_S(T, X) \rightarrow \mathrm{Hom}_{S_a}(T, X_m) \rightrightarrows \mathrm{Hom}_{S_b}(T, X_n)$$

is an equaliser of sets, which is an easy diagram chase: a morphism  $f \in \mathrm{Hom}_{S_a}(T, X_m)$  whose images in  $\mathrm{Hom}_{S_b}(T, X_n)$  agree (i.e., such that the compositions

$$T \rightarrow X_m \rightrightarrows X_n$$

agree) factors uniquely through  $X$ , and by the commutativity of the previous rectangle and the fact that its bottom row is an equaliser diagram, the induced map  $T \rightarrow X$  factors uniquely through  $S$ .

It is now enough to show that a log scheme of the form  $S \times_{S_a} X_m$  admits a strict  $S$ -map to some affine space  $\mathbb{A}_S^N$ , and since a fibred product of strict morphisms is strict [Ogu18, Remark III.2.1.3], we can assume that  $S = S_a$  (and, consequently,  $X = X_m$ ). Let  $R = \Gamma(S, \mathcal{O}_S)$ ,  $A = \Gamma(X, \mathcal{O}_X)$  and  $N = l$ ; recalling that by construction  $m = l + a$  and the structure morphism  $X_m \rightarrow S_a$  is the one determined by the inclusion  $\mathbb{N}^a \rightarrow \mathbb{N}^a \oplus \mathbb{N}^l$ , it suffices to observe that there exists (exactly) one

ring morphism  $R[x_1, \dots, x_l] \rightarrow A$  making all shapes in the diagram

$$\begin{array}{ccccc}
 \mathbb{N}^a & \xrightarrow{\quad} & \mathbb{N}^a \oplus \mathbb{N}^l & \xlongequal{\quad} & \mathbb{N}^a \oplus \mathbb{N}^l \\
 \downarrow & & \downarrow & & \downarrow \\
 R & \xrightarrow{\quad} & R[x_1, \dots, x_l] & \dashrightarrow & A
 \end{array}$$

(There are also curved arrows from  $\mathbb{N}^a$  to  $A$  and from  $R$  to  $\mathbb{N}^a \oplus \mathbb{N}^l$ .)

commute. □

**Lemma 2.5.4.** *Let  $\ell = \text{“fine”}, \text{“fs”}$  or blank. Let  $f: S' \rightarrow S$  be a morphism of affine log schemes in  $\text{LogSch}^\ell$  such that*

- *$f$  is integral if  $\ell = \text{“fine”}$  and saturated if  $\ell = \text{“fs”}$ ;*
- *the morphism of schemes  $\underline{f}: \underline{S}' \rightarrow \underline{S}$  is a finite locally free universal homeomorphism.*

*Let  $X' \rightarrow S'$  be a morphism in  $\text{LogSch}^\ell$  and assume that  $X'$  and  $S'$  admit finitely generated global charts. If  $f_*^\ell \mathbb{A}_{S'}^1$  is representable (and coherent), then  $f_*^\ell X'$  is representable (and coherent).*

A quick remark before the proof:

*Remark 2.5.5.* For any integral morphism  $f: S' \rightarrow S$  of fine log schemes and any fine log  $S'$ -scheme  $X'$ , if  $f_* X'$  is representable, then by the universal property of the integralisation functor  $(\cdot)^{int}$  (Proposition 2.4.27) one has  $f_*^{fin} X' = (f_* X')^{int}$ , and in particular  $f_*^{fin} X'$  is representable as well: indeed, for any fine log scheme  $Z$  over  $S$  one has natural isomorphisms

$$\begin{aligned}
 \text{Hom}_S(Z, (f_* X')^{int}) &= \text{Hom}_S(Z, f_* X') = \text{Hom}_{S'}(Z \times_S S', X') = \\
 &= \text{Hom}_{S'}(Z \times_S^{fin} S', X') = (f_*^{fin} X')(Z),
 \end{aligned}$$

where we used the integrality of  $f$  and Proposition 2.4.28 to conclude that  $Z \times_S^{fin} S'$  equals  $Z \times_S S'$ . Similarly, if  $f$  is a saturated morphism between fine and saturated log schemes and  $X'$  is fine and saturated, if  $f_*^{fin} X'$  is representable, then  $f_*^{fs} X' = (f_*^{fin} X')^{sat}$  by the universal property of the saturation functor  $(\cdot)^{sat}$ .

*Proof of the lemma:* The previous remark reduces us to the case of arbitrary log schemes, also taking into account the fact that finite limits of coherent log schemes are again coherent [Ogu18, Proposition III.2.1.1].

In turn, by Remark 1.3.8 and Lemma 2.5.2 (applied taking  $S = S'$ ), the representability of  $f_*X'$  is reduced to that of the logarithmic Weil restrictions  $f_*\mathbb{A}_{S'}^n$  of the affine spaces  $\mathbb{A}_{S'}^n$ , once the representability of  $f_*(T' \times_{\underline{S}'} S')$  for all the  $\underline{S}'$ -schemes  $T'$  is established (compare with Example 1.3.9). This is easily done: for any  $S$ -log scheme  $Z$ , one has natural isomorphisms

$$\begin{aligned} f_*(T' \times_{\underline{S}'} S')(Z) &= \mathrm{Hom}_{\mathrm{LogSch}/S'}(Z \times_S S', T' \times_{\underline{S}'} S') = \mathrm{Hom}_{\mathrm{LogSch}/\underline{S}'}(Z \times_S S', T') = \\ &= \mathrm{Hom}_{\mathrm{Sch}/\underline{S}'}(Z \times_S S', T') = \mathrm{Hom}_{\mathrm{Sch}/\underline{S}'}(Z \times_{\underline{S}} \underline{S}', T') = \left( \mathrm{Res}_{\underline{S}'/\underline{S}} T' \right)(Z), \end{aligned}$$

where the first equality in the second row follows by the construction of fibred products in  $\mathrm{LogSch}$  and  $\mathrm{Res}_{\underline{S}'/\underline{S}} T'$  denotes the ordinary Weil restriction of schemes, which is representable by the assumption on  $\underline{f}$  [BGA18a, Corollary A.5]. Since

$$\mathrm{Hom}_{\mathrm{Sch}/\underline{S}}(Z, \mathrm{Res}_{\underline{S}'/\underline{S}} T') = \mathrm{Hom}_{\mathrm{LogSch}/S}(Z, (\mathrm{Res}_{\underline{S}'/\underline{S}} T') \times_{\underline{S}} S),$$

this implies that  $f_*(T' \times_{\underline{S}'} S')$  is represented by  $(\mathrm{Res}_{\underline{S}'/\underline{S}} T') \times_{\underline{S}} S$ .  $\square$

Since the categories of fine and fs log schemes with Zariski-local charts admit gluings, for these log schemes one can prove a more general representability result.

The proof of the next lemma is the same as for the analogous lemma for the Weil restriction of schemes, using Lemma 2.4.25.

**Lemma 2.5.6.** *Let  $\ell = \text{“fin”}$  or  $\text{“fs”}$ . Let  $S' \rightarrow S$ ,  $X' \rightarrow S'$  be morphisms in  $\mathrm{LogSch}^\ell$ ,  $\{S_i \rightarrow S\}_i$  a Zariski open covering; for each  $i$ , set  $X'_i = X' \times_S^\ell S_i$ , and  $S'_i = S' \times_S^\ell S_i$ , and let  $f_i: S'_i \rightarrow S_i$  be the induced map. If  $f_{i*}^\ell X'_i$  is representable and admits charts Zariski locally for each  $i$ , then so is  $f_*^\ell X'$ , and the family of induced morphisms  $\{f_{i*}^\ell X'_i \rightarrow f_*^\ell X'\}_i$  is an open covering.*

**Lemma 2.5.7.** *Let  $\ell = \text{“fin”}$  or  $\text{“fs”}$ ,  $S' \rightarrow S$ ,  $X' \rightarrow S'$  be morphisms in  $\mathrm{LogSch}^\ell$  and assume that*

- *$f$  is integral if  $\ell = \text{“fin”}$  and saturated if  $\ell = \text{“fs”}$ ;*
- *the morphism of schemes  $\underline{f}: \underline{S}' \rightarrow \underline{S}$  is a finite locally free universal homeomorphism.*

*Assume that  $X'$ ,  $S'$  and  $S$  admit finitely generated charts Zariski locally. If  $f_*^\ell \mathbb{A}_{S'}^1$  is representable and admits charts Zariski locally, then  $f_*^\ell X'$  is representable.*

Before proving this, let us make a few remarks on the assumptions:

*Remarks 2.5.8.* (i) The asymmetry between Zariski and étale topology in the previous statement is not simply an artifact of the proof, but is due to the fact that there is in general no reason to expect that gluing étale log schemes along an étale covering still results in a log scheme (see also Remark 2.4.26); therefore, it seems that the annoying assumption on the Zariski-local existence of charts cannot be dropped.

It does not seem unreasonable to expect that using Lemma 1.4.2 one could extend the above and prove that if  $X'$ ,  $S'$  and  $S$  admit finitely generated charts only étale locally, then - the other assumptions still standing -  $f_*^{fin} X'$  (resp.  $f_*^{fs} X'$ ) is representable by a log algebraic space. Due to our ignorance on the matter of log algebraic spaces, we have refrained from this line of inquiry.

- (ii) We restrict to (at least) fine log schemes because we do not know whether  $\mathrm{Hom}(-, X')$  is a sheaf on the category of log schemes: [Ogu18, Proposition III.1.4.5] only states that this holds on the category of u-integral log schemes. The category of fine log schemes is then the largest full subcategory  $\mathcal{C}$  of  $\mathrm{LogSch}$  for which we know that  $\mathcal{C}$  has finite limits and the  $|\cdot|$ -topology is subcanonical on  $\mathcal{C}$ .

*Proof.* Thanks to Lemma 2.5.6, we can reduce to the case when  $S$  and  $S'$  are affine and admit finitely generated global charts.

By Lemma 2.4.25 and Lemma 2.1.1  $\mathrm{LogSch}^\ell/S$  and of  $\mathrm{LogSch}^\ell/S'$  are categories with open immersions when endowed with the forgetful functor to  $\mathbf{Top}$ ; moreover their topology is subcanonical, and the full subcategory  $\mathcal{E}$  of  $\mathrm{LogSch}^\ell/S$  spanned by the objects which admit finitely generated charts Zariski locally has gluings. We claim that, taking  $\mathcal{D} = \mathrm{LogSch}^\ell/S'$  and  $\mathcal{C} = \mathcal{C}' = \mathrm{LogSch}^\ell/S$ , the assumptions of Theorem 1.3.7 are met: the fact that  $(\cdot) \times_S^\ell S'$  respects open immersions is clear, and since  $\underline{f}$  is a universal homeomorphism, there is an obvious natural isomorphism

$$|\cdot| \xrightarrow{\sim} |(\cdot) \times_S^\ell S'|.$$

This concludes the proof, because then one is reduced to the situation of Lemma 2.5.4.  $\square$

We now conclude our analysis of the representability of the logarithmic Weil restriction by studying, in the next lemma and in Lemma 2.5.13, the representability of  $f_* \mathbb{A}_{S'}^1$  for some (classes of) maps  $f$ ; our results then yield conditions on the representability of Weil restrictions along morphisms  $S \rightarrow \underline{S}$  (Theorem 2.5.11) and

along nilpotent thickenings of schemes (Lemma 2.5.14). As a particular case one recovers, with a different proof, results in [Her24, §2.1].

To disambiguate between the geometric sense of the word “local” and the one in Definition 2.4.8, in the following we shall call a chart  $P \rightarrow (S, \mathcal{M} \rightarrow \mathcal{O}_S)$  *monoid-local* if the induced morphism of sheaves of monoids  $P_S \rightarrow \mathcal{M}$  (and thus also  $P_S \rightarrow \mathcal{O}_S$ ) is local.

We pause for a quick remark related to an assumption in the next lemma.

*Remark 2.5.9.* The following conditions on a monoid  $P$  are equivalent:

- (i)  $P^\times$  is a direct summand of  $P$ .
- (ii)  $P = P^\times \oplus \bar{P}$ .
- (iii) The inclusion  $P^\times \hookrightarrow P$  has a retraction.

Indeed, if *i*) holds, then  $P = P^\times \oplus P'$  for some monoid  $P'$ , and by the universal property of cokernels and direct sums (or directly from the explicit description of a quotient of monoids)  $P' = (P^\times \oplus P')/P^\times = P/P^\times = \bar{P}$ .

Clearly *ii*) implies *iii*). Finally, if the inclusion  $i: P^\times \rightarrow P$  has a retraction  $r: P \rightarrow P^\times$ , then  $P = P^\times \oplus \ker r$ : to see this, let  $Q$  be a monoid and  $f_1: P^\times \rightarrow Q$ ,  $f_2: \ker r \rightarrow Q$  be morphisms of monoids; if  $f: P \rightarrow Q$  is a morphism of monoids such that  $f|_{P^\times} = f_1$  and  $f|_{\ker r} = f_2$ , then, noticing that  $(i \circ r)(x)$  is invertible for all  $x \in P$ , one can write  $f(x) = f(x - ir(x) + ir(x)) = f_2(x - ir(x)) + f_1(ir(x))$ ; it is easy to verify that this does define a morphism of monoids, thus proving the claim.

We do not know whether these conditions are equivalent to the existence of a section of the map  $P \rightarrow \bar{P}$  or to the fact that  $\bar{P}$  is a direct summand of  $P$  (the problem being that non-trivial elements of  $\bar{P}$  do not have inverses, whereby the usual argument breaks down).

**Lemma 2.5.10.** *If a log scheme  $S$  admits a monoid-local chart subordinated to a monoid  $P$  such that  $P = P^\times \oplus \bar{P}$ , then*

$$\mathrm{Res}_{S/\underline{S}} \mathbb{A}_S^1 \cong \bigsqcup_{\bar{P}} \mathbb{A}_{\underline{S}}^1,$$

*and in particular it is representable.*

*Proof.* By Remark 2.4.30,  $\mathrm{Res}_{S/\underline{S}} \mathbb{A}_S^1$  is isomorphic to the functor of  $\underline{S}$ -log schemes

$$(Z, \mathcal{M}_Z) \mapsto \Gamma(Z \times_{\underline{S}} S, \mathcal{M}_{Z \times_{\underline{S}} S}).$$

By the computation of fibred products in the category of log schemes, for any morphism of log schemes  $h: (Z, \mathcal{M}_Z) \rightarrow \underline{S}$ , the monoid  $\mathcal{M}_{Z \times_{\underline{S}} S}$  is the pushout  $\mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} h_{\log}^* \mathcal{M}_S$ ; since  $\mathcal{M}_S \rightarrow \mathcal{O}_S$  is the log structure associated to a prelog structure  $\alpha: P_S \rightarrow \mathcal{O}_S$ , where  $\alpha$  is, by assumption, local, it can be computed as the sharpification of  $\alpha$ , i.e. as the pushout  $P_S \oplus_{P_S^\times} \mathcal{O}_S^\times$ . Since  $P = P^\times \oplus \bar{P}$ , the previous pushout is isomorphic to  $\bar{P}_S \oplus \mathcal{O}_S^\times$ , whereby

$$\mathcal{M}_{Z \times_{\underline{S}} S} = \mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} h_{\log}^* (\bar{P}_S \oplus \mathcal{O}_S^\times) = \mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} (\bar{P}_Z \oplus \mathcal{O}_Z^\times) = \mathcal{M}_Z \oplus \bar{P}_Z.$$

Since the global section functor commutes with products and finite direct sums of monoids coincide with finite direct products,

$$(\text{Res}_{S/\underline{S}} \mathbb{A}_S^1)(Z) \cong \Gamma(Z, \mathcal{M}_Z \oplus \bar{P}_Z) \cong \Gamma(Z, \mathcal{M}_Z) \times \Gamma(Z, \bar{P}_Z)$$

naturally in  $Z$ . Again by Remark 2.4.30, the functor  $(Z, \mathcal{M}_Z) \mapsto \Gamma(Z, \mathcal{M}_Z)$  is represented by  $\mathbb{A}_{\underline{S}}^1$ , so it is enough to observe that the functor of  $\underline{S}$ -schemes  $Z \mapsto \Gamma(Z, \bar{P}_Z)$  is represented by  $\bigsqcup_{\bar{P}} \underline{S}$ . This is actually a well-known fact, but let us prove it for the sake of completeness. A global section of the constant sheaf  $\bar{P}_Z$  is a locally constant function  $\varrho: |Z| \rightarrow \bar{P}$ , where  $\bar{P}$  is endowed with the discrete topology; for any such  $\varrho$  and any  $p \in \bar{P}$ , let  $U_p$  be the open subscheme  $\varrho^{-1}(p)$  of  $Z$ , so that

$$Z = \bigsqcup_{\bar{P}} U_p$$

as a scheme. There exists a unique  $\underline{S}$ -morphism

$$Z \rightarrow \bigsqcup_{\bar{P}} \underline{S}$$

such that the image of its restriction to  $U_p$  is contained in the  $p$ -th copy of  $\underline{S}$  in the target. Conversely, given a morphism of  $\underline{S}$ -schemes  $\phi: Z \rightarrow \bigsqcup_{\bar{P}} \underline{S}$ , the function  $\varrho: |Z| \rightarrow \bar{P}$  mapping a point  $z$  to the connected component containing  $|\phi|(z)$  is locally constant.  $\square$

Since  $S \rightarrow \underline{S}$  is integral/saturated for all integral/saturated log schemes  $S$  (Example 2.4.7), the previous lemma and Lemma 2.5.7 immediately imply the following

**Theorem 2.5.11.** *Let  $(S, \mathcal{M}_S)$  be a log scheme.*

- (i) *If  $S$  is affine and admits a monoid-local chart  $P \rightarrow \Gamma(S, \mathcal{M}_S)$  subordinated to a monoid  $P$  such that  $P = P^\times \oplus \bar{P}$ , then  $\text{Res}_{S/\underline{S}} X$  is representable for all  $S$ -log schemes  $X$  admitting a finitely generated chart.*

- (ii) If  $S$  is fine (resp. fine and saturated) and the condition at the previous point holds Zariski locally on  $S$ , then  $\mathrm{Res}_{S/\underline{S}}^{\mathrm{fin}} X$  (resp.  $\mathrm{Res}_{S/\underline{S}}^{\mathrm{fs}} X$ ) is representable for all fine (resp. fine and saturated)  $S$ -log schemes  $X$  which admit a finitely generated chart Zariski locally.

*Example 2.5.12.* The log schemes associated to the log rings of Example 2.4.20ii) and iii) satisfy the assumptions of the previous result (notice that in these cases, one even has  $P = \bar{P}$ ).

We end this section by complementing the previous theorem by results which imply in particular the representability of  $\mathrm{Res}_{\Delta_m^{(r)}/k} X$  for any fine log scheme  $X$  over  $\Delta_m^{(r)}$  admitting finitely generated charts Zariski locally (Corollary 2.5.16); as a special case, this recovers Herr's result on the representability of  $J_{X,m}^{(r)}$ .

As in [Ogu18, Proposition IV.2.1.2], in the next statement we identify the Zariski/étale sites on  $Z'$  and on  $Z$  if  $Z \rightarrow Z'$  is a nilpotent immersion.

**Lemma 2.5.13.** *Let  $S \rightarrow S'$  be a nilpotent thickening of schemes which admits a section; for any  $S$ -scheme  $Z$ , let  $Z' = Z \times_S S'$  and  $\mathcal{I}_Z = \ker(\mathcal{O}_{Z'} \rightarrow \mathcal{O}_Z)$ . Then  $\mathrm{Res}_{S'/S} \mathbb{A}_{S'}^1$  is isomorphic, as a presheaf on  $\mathrm{LogSch}/S$ , to the functor*

$$(Z, \mathcal{M}_Z) \mapsto \Gamma(Z, \mathcal{I}_Z) \times \Gamma(Z, \mathcal{M}_Z).$$

*Proof.* Let  $(Z, \mathcal{M}_Z)$  be a log  $S$ -scheme. By [Ogu18, Proposition IV.2.1.2.3] applied to the nilpotent thickening  $Z \rightarrow Z'$ , the map of sheaves of monoids

$$(1 + \mathcal{I}_Z) \times \mathcal{M}_{Z'} \rightarrow \mathcal{M}_{Z'} \times_{\mathcal{M}_Z} \mathcal{M}_{Z'}$$

given on sections by  $(u, m) \mapsto (m, um)$  is an isomorphism. Since the composition  $S \rightarrow S' \rightarrow S$  is the identity, the map  $\mathcal{M}_{Z'} \rightarrow \mathcal{M}_Z$  has a section; base-changing the previous isomorphism along it and taking global sections, one obtains a natural (in  $Z$ ) isomorphism of sets

$$\Gamma(Z, \mathcal{I}_Z) \times \Gamma(Z, \mathcal{M}_Z) \xrightarrow{\sim} \Gamma(Z, \mathcal{M}_{Z'}).$$

The statement then follows recalling that, by Remark 2.4.30,  $\mathrm{Res}_{S'/S} \mathbb{A}_{S'}^1$  is just the functor  $(Z, \mathcal{M}_Z) \mapsto \Gamma(Z, \mathcal{M}_{Z'})$ .  $\square$

**Lemma 2.5.14.** *With the same notation as in the previous lemma, assume that the functor of  $S$ -schemes  $Z \mapsto \Gamma(Z, \mathcal{I}_Z)$  is representable and the morphism  $S' \rightarrow S$  is finite and locally free.*

- (i) If  $S'$  is affine, then  $\mathrm{Res}_{S'/S} X'$  is representable for all  $S'$ -log schemes  $X'$  admitting a finitely generated chart.
- (ii) The functor  $\mathrm{Res}_{S'/S}^{\mathrm{fin}} X'$  (resp.  $\mathrm{Res}_{S'/S}^{\mathrm{fs}} X'$ ) is representable for all fine (resp. fine and saturated)  $S'$ -log schemes  $X'$  which admit a finitely generated chart Zariski locally.

*Proof.* By [Ogu18, Proposition I.4.6.3.2 and Proposition I.4.8.5.4], any morphism between (sheaves of) abelian groups is integral and saturated, whereby  $S' \rightarrow S$  is integral and saturated. Observing in addition that  $S' \rightarrow S$  is a universal homeomorphism (being a section of a nilpotent immersion) we can thus apply Lemma 2.5.7 (resp. Lemma 2.5.4) to reduce the statement of ii) (resp. i)) to the representability of  $\mathrm{Res}_{S'/S} \mathbb{A}_{S'}^1$ ; this follows immediately from the previous lemma and 2.4.30, since by assumption the functor of  $S$ -schemes  $Z \mapsto \Gamma(Z, \mathcal{I}_Z)$  is representable.  $\square$

**Corollary 2.5.15.** *Let  $m$  be a positive integer. The restriction of  $\mathrm{Res}_{\underline{\Delta}_m^{(r)}/k} X$  to the category of fine/fs log schemes over  $\underline{\Delta}_m^{(r)}$  is representable for all fine/fs log schemes  $X$  over  $\underline{\Delta}_m^{(r)}$  that admit a finitely generated chart Zariski locally.*

*Proof.* The statement follows applying the previous lemma to the nilpotent thickening of schemes  $\mathrm{Spec} k \rightarrow \mathrm{Spec} k[x]/(x^m) = \underline{\Delta}_m^{(r)}$ , observing that in this case the functor  $Z \mapsto \Gamma(Z, \mathcal{I}_Z)$  of Lemma 2.5.13 is isomorphic to  $\mathbb{G}_a^{m-1}$  as a set-valued presheaf of schemes.  $\square$

Taking  $X = T \times_k \Delta_m^{(r)}$  in the next corollary, we recover a result by Herr (see [Her24, Theorem 2.20]).

**Corollary 2.5.16.** *The restriction of  $\mathrm{Res}_{\Delta_m^{(r)}/k} X$  to the category of fine/fs log schemes over  $\Delta_m^{(r)}$  is representable for all fine/fs log schemes  $X$  over  $\Delta_m^{(r)}$  that admit a finitely generated chart Zariski locally.*

*Proof.* Since, given composable morphisms of log schemes  $S'' \rightarrow S' \rightarrow S$  and a log scheme  $X''$  over  $S''$ , one has

$$\mathrm{Res}_{S''/S} X'' = \mathrm{Res}_{S'/S} \left( \mathrm{Res}_{S''/S'} X'' \right),$$

taking  $S'' \rightarrow S'$  to be the map  $\Delta_m^{(r)} \rightarrow \underline{\Delta}_m^{(r)}$  and  $S' \rightarrow S$  the map  $\underline{\Delta}_m^{(r)} \rightarrow \mathrm{Spec} k$ , the statement follows from Theorem 2.5.11 and the previous corollary.  $\square$

*Example 2.5.17.* Explicitly, by Lemma 2.5.10

$$X' := \mathrm{Res}_{\Delta_m^{(r)}/\underline{\Delta}_m^{(r)}} \mathbb{A}_{\Delta_m^{(r)}}^1 \cong \bigsqcup_{\mathbb{N}} \mathbb{A}_{\underline{\Delta}_m^{(r)}}^1;$$

the same proof as in [JLM<sup>+</sup>22, Proposition 2.8(i)] shows that

$$\mathrm{Res}_{\underline{\Delta}_m^{(r)}/k} X' = \bigsqcup_{\mathbb{N}} \mathrm{Res}_{\underline{\Delta}_m^{(r)}/k} \mathbb{A}_{\underline{\Delta}_m^{(r)}}^1,$$

and by Lemma 2.5.13 we obtain that

$$\mathrm{Res}_{\underline{\Delta}_m^{(r)}/k} \mathbb{A}_{\underline{\Delta}_m^{(r)}}^1 \cong \mathbb{A}_k^1 \times \underline{\mathbb{A}}_k^{m-1},$$

whence finally

$$\mathrm{Res}_{\underline{\Delta}_m^{(r)}/k} \mathbb{A}_{\underline{\Delta}_m^{(r)}}^1 = \bigsqcup_{\mathbb{N}} \mathbb{A}_k^1 \times \underline{\mathbb{A}}_k^{m-1}.$$

## 2.6 Logarithmic Greenberg functor

We present an application of our general formalism of Greenberg functors to a logarithmic version of the classical Greenberg, which is part of ongoing work with Leo Herr. Let  $k$  be a perfect field of positive characteristic  $p$ . Throughout this section, for any positive integer  $m$  we denote by  $\Delta_m$  the log scheme associated to the log ring

$$\mathbb{N} \rightarrow W_m(k), \quad n \mapsto p^n.$$

We still enforce Notation 2.4.29.

Analogously to the classical case, we define, for any positive integer  $m$ , a functor  $\mathrm{Gr}_{\log}^m$  associating in a natural way a log  $k$ -scheme to any logarithmic scheme over  $\Delta_m$ ; other choices would be possible, with minimal variations - for example,  $\mathrm{Spec} k$  and  $\mathrm{Spec} W_m(k)$  could be endowed with other log structures - but the present one seems to be the most natural.

As usual, the first step is the definition of a natural lift of  $k$ -log schemes to  $\Delta_m$ -log schemes, for a fixed positive natural number  $m$ . The construction is well-known; here we develop it in extreme detail and also prove some of its basic properties. The idea is very simple: recall first that, for a  $k$ -scheme  $X$ ,  $W_m(X)$  is the scheme  $(|X|, W_m(\mathcal{O}_X))$  (Example 2.2.8); given a prelog structure  $\mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X$  on  $X$ , one can regard  $\mathcal{M}$  and  $\mathcal{O}_X$  as sheaves on  $W_m(X)$  and define a prelog structure on  $W_m(X)$  as the composition

$$\mathcal{M} \oplus \mathbb{N} \xrightarrow{\alpha \oplus \mathrm{id}} \mathcal{O}_X \oplus \mathbb{N} \rightarrow W_m(\mathcal{O}_X),$$

where  $W_m(\mathcal{O}_X)$  is identified with the structure sheaf of  $W_m(X)$  and the map

$$\mathcal{O}_X \oplus \mathbb{N} \rightarrow W_m(\mathcal{O}_X)$$

takes a local section  $(x, n)$  to  $[x]p^n$ , where

$$[-]: \mathcal{O}_X \rightarrow W_m(\mathcal{O}_X)$$

is the Teichmüller lift. Applying the logarithmification functor of Lemma 2.4.18, one obtains a log scheme  $W_m^{\log}(X)$  over  $\Delta_m$ .

The main result of this section, Theorem 2.6.16, implies in particular that the presheaf on fine/saturated log schemes  $\mathrm{Gr}_{\log}^m X := \mathrm{Hom}_{\Delta_m}(W_m^{\log}(\cdot), X)$  is representable for any fine/saturated log  $k$ -scheme  $X$  which admits charts Zariski locally.

### A general construction

In the sketch of definition in the previous paragraph we considered a single prelog scheme  $(X, \mathcal{M})$ ; in order to rigorously prove that the assignment

$$(X, \mathcal{M}) \mapsto (W_m(X), \mathcal{M} \oplus \mathbb{N} \rightarrow W_m(\mathcal{O}_X))$$

can be upgraded to functor and to study the properties of this, it is convenient to take a step back and consider a more general situation, which also clarifies some aspects of the construction.

We first start with a “local” version working for a single scheme  $X$ , and then show, in Construction 2.6.6, how to turn this into a functor.

*Construction 2.6.1.* For any scheme  $S$ , let  $\mathrm{Sh}(S, \mathrm{Mon})$  be the category of sheaves of monoids on  $S$ . Let  $X', X$  be schemes,  $\Psi: \mathrm{Sh}(X, \mathrm{Mon}) \rightarrow \mathrm{Sh}(X', \mathrm{Mon})$  a functor,  $\tau: \Psi(\mathcal{O}_X) \rightarrow \mathcal{O}_{X'}$  a morphism of sheaves of monoids. These data determine a functor  $\Psi^{\mathrm{pre}}: \mathrm{PLog}/X \rightarrow \mathrm{PLog}/X'$ , taking a prelog structure  $\alpha: \mathcal{M} \rightarrow \mathcal{O}_X$  on  $X$  to the prelog structure

$$\tau \circ \Psi(\alpha): \Psi(\mathcal{M}) \xrightarrow{\Psi(\alpha)} \Psi(\mathcal{O}_X) \xrightarrow{\tau} \mathcal{O}_{X'};$$

composing this with the functor  $\mathrm{PLog}_{X'} \rightarrow \mathrm{Log}_{X'}$  of Lemma 2.4.18, one obtains a functor  $\Psi^a: \mathrm{PLog}/X \rightarrow \mathrm{Log}/X'$ .

*Example 2.6.2.* For any morphism of schemes  $f: X' \rightarrow X$ , taking

$$\Psi = f^{-1}: \mathrm{Sh}(X, \mathrm{Mon}) \rightarrow \mathrm{Sh}(X', \mathrm{Mon})$$

and  $\tau = f^\sharp: f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{X'}$  one has  $\Psi^{\mathrm{pre}} = f^*: \mathrm{PLog}/X \rightarrow \mathrm{PLog}/X'$  and  $\Psi^a = f_{\log}^*: \mathrm{PLog}/X \rightarrow \mathrm{Log}/X'$ .

*Example 2.6.3.* For any  $k$ -scheme  $X$  and any positive integer  $m$ , set  $X' := W_m(X)$  and denote by  $\iota^{X,m}: X \rightarrow X'$  the canonical nilpotent immersion; we can then take  $\Psi$  to be the direct image functor  $\iota_*^{X,m}: \mathrm{Sh}(X, \mathrm{Mon}) \rightarrow \mathrm{Sh}(X', \mathrm{Mon})$  and  $\tau$  to be the Teichmüller lift  $[-]: \iota_*^{X,m} \mathcal{O}_X \rightarrow W_m(\mathcal{O}_X) = \mathcal{O}_{W_m(X)}$ .

**Lemma 2.6.4.** *With the same notation as in Construction 2.6.1, assume that*

- $\Psi$  preserves pushouts;
- there exists a (necessarily unique) map  $\Psi(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_{X'}^\times$ , making the diagram

$$\begin{array}{ccc} \Psi(\mathcal{O}_X^\times) & \dashrightarrow & \mathcal{O}_{X'}^\times \\ \downarrow & & \downarrow \\ \Psi(\mathcal{O}_X) & \xrightarrow{\tau} & \mathcal{O}_{X'} \end{array},$$

where the vertical arrows are the obvious ones, commute.

Then the unit  $u: \mathrm{id} \Rightarrow (\cdot)^a$  of the logarithmification-inclusion adjunction induces, for any prelog structure  $\mathcal{M} \rightarrow \mathcal{O}_X$ , an isomorphism

$$\Psi^a(u_{\mathcal{M}}): \Psi^a(\mathcal{M}) \xrightarrow{\sim} \Psi^a(\mathcal{M}^a)$$

of log structures on  $X'$ .

*Proof.* Let  $\alpha: \mathcal{M} \rightarrow \mathcal{O}_X$  be a prelog structure.

We want to apply the universal property of logarithmification (Proposition 2.4.11). Observe first that there is a map  $\Psi(\mathcal{M}^a) \rightarrow \Psi^a(\mathcal{M})$  of sheaves of monoids over  $\mathcal{O}_{X'}$ , as depicted in the diagram

$$\begin{array}{ccccc} \Psi(\alpha^{-1}(\mathcal{O}_X^\times)) & \longrightarrow & \Psi(\mathcal{M}) & \xrightarrow{\quad} & \Psi(\mathcal{M}) \\ \downarrow \Psi(\alpha) & \searrow & \downarrow & \nearrow & \downarrow \\ \Psi(\mathcal{O}_X^\times) & \longrightarrow & \Psi(\mathcal{M}^a) & \xrightarrow{(\tau \circ \Psi(\alpha))^{-1}(\mathcal{O}_{X'}^\times)} & \Psi(\mathcal{M}) \\ & \searrow \tau & \downarrow \tau \circ \Psi(\alpha) & \nearrow & \downarrow \\ & & \mathcal{O}_{X'}^\times & \longrightarrow & \Psi^a(\mathcal{M}) \end{array}$$

indeed, as all the solid rectangles commute (using the second assumption for the commutativity of the upper one), so does the outer hexagon; since the back square is a pushout (because  $\Psi$  preserves pushouts), one obtains a map  $\Psi(\mathcal{M}^a) \rightarrow \Psi^a(\mathcal{M})$  as claimed.

By the universal property of logarithmification, there is a unique induced map  $\Psi^a(\mathcal{M}^a) \rightarrow \Psi^a(\mathcal{M})$  such that the left triangle in

$$\begin{array}{ccccc}
& & \tau \circ \Psi(\alpha^{log}) & & \\
& \nearrow & & \searrow & \\
\Psi(\mathcal{M}^a) & \longrightarrow & \Psi^a(\mathcal{M}^a) & \longrightarrow & \mathcal{O}_{X'} \\
& \searrow & \downarrow & \nearrow & \\
& & \Psi^a(\mathcal{M}) & & (\tau \circ \Psi(\alpha))^{log}
\end{array} \tag{2.2}$$

commutes, and this makes the triangle on the right commute as well.

On the other hand, the map  $\Psi^a(u_{\mathcal{M}})$  fits in a commutative diagram

$$\begin{array}{ccccc}
& & \tau \circ \Psi(\alpha) & & \\
& \nearrow & & \searrow & \\
\Psi(\mathcal{M}) & \xrightarrow{u_{\Psi(\mathcal{M})}} & \Psi^a(\mathcal{M}) & \longrightarrow & \mathcal{O}_{X'} \\
\downarrow & & \downarrow \Psi^a(u_{\mathcal{M}}) & \nearrow & \\
\Psi(\mathcal{M}^a) & \xrightarrow{u_{\Psi(\mathcal{M}^a)}} & \Psi^a(\mathcal{M}^a) & & :
\end{array}$$

the two universal properties for the logarithmifications of the maps  $\Psi(\mathcal{M}) \rightarrow \mathcal{O}_{X'}$  and  $\Psi(\mathcal{M}^a) \rightarrow \mathcal{O}_{X'}$  then imply that the vertical map  $\Psi^a(\mathcal{M}^a) \rightarrow \Psi^a(\mathcal{M})$  in diagram (2.2) is inverse to  $\Psi^a(u_{\mathcal{M}})$ .  $\square$

**Corollary 2.6.5.** *Let  $f: X' \rightarrow X$  be a morphism of  $S$ -schemes,  $\beta: \mathcal{M} \rightarrow \mathcal{O}_X$  a prelog structure on  $X$ . Then the unit  $u_{\mathcal{M}}: \mathcal{M} \rightarrow \mathcal{M}^a$  induces an isomorphism*

$$f_{log}^*(u_{\mathcal{M}}): f_{log}^*(\mathcal{M}) \xrightarrow{\sim} f_{log}^*(\mathcal{M}^a)$$

of log structures on  $X'$ .

*Proof.* It is enough to apply the previous lemma to Example 2.6.2, observing that  $f^{-1}$ , being a left adjoint, preserves pushouts, while the other condition in the statement clearly holds.  $\square$

In order to upgrade Construction 2.6.1 from a functor between (pre)log structures on fixed schemes to a functor of (pre)log schemes, one has to give a functor  $X \mapsto X'$  and to specify families  $(\Psi_X)_X$  and  $(\tau_X)_X$  subjected to some natural compatibility data. The reader who is willing to accept that the assignment  $W_m^{log}$  whose construction was sketched at the beginning of the section produces a well-defined functor  $\mathbf{PLogSch}/k \rightarrow \mathbf{LogSch}$  can mostly ignore the next construction and skip to Lemma 2.6.13. Anyway, the basic idea is very simple: if  $(X, \mathcal{M}) \mapsto (X', \Psi_X(\mathcal{M}))$  is to define a functor, then for a map of prelog  $S$ -schemes  $f: (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ , one has to produce a morphism  $(f')^{-1}\Psi_Y(\mathcal{N}) \rightarrow \Psi_X(\mathcal{M})$ ;

upon applying the functor  $\Psi_X$ , the morphism of monoids  $\bar{f}: f^{-1}\mathcal{N} \rightarrow \mathcal{M}$  yields a map  $\Psi_X(f^{-1}\mathcal{N}) \rightarrow \Psi_X(\mathcal{M})$ : it is thus natural to require the existence of a natural (in  $\mathcal{N}$ ) morphism  $(f')^{-1}\Psi_Y(\mathcal{N}) \rightarrow \Psi_X(f^{-1}\mathcal{N})$ . Demanding that the composition

$$(f')^{-1}\Psi_Y(\mathcal{N}) \rightarrow \Psi_X(f^{-1}\mathcal{N}) \rightarrow \Psi_X(\mathcal{M})$$

be a morphism of prelog structures leads immediately to condition *iii)* below, while *i)* and *ii)* are the “obvious” naturality conditions. More precisely:

*Construction 2.6.6.* Let  $S$  be a scheme. Assume given the following data:

- a functor of schemes  $(\cdot)': \text{Sch}/S \rightarrow \text{Sch}$ ;
- for each  $S$ -scheme  $X$ , a functor  $\Psi_X: \text{Sh}(X, \text{Mon}) \rightarrow \text{Sh}(X', \text{Mon})$  and a morphism of monoids  $\tau_X: \Psi(\mathcal{O}_X) \rightarrow \mathcal{O}_{X'}$ ;
- for each  $S$ -morphism  $f: X \rightarrow Y$ , a natural transformation

$$\begin{array}{ccc} & (f')^{-1} \circ \Psi_Y & \\ \text{Sh}(Y, \text{Mon}) & \begin{array}{c} \Downarrow \xi_f \\ \Downarrow \end{array} & \text{Sh}(X', \text{Mon}) \\ & \Psi_X \circ f^{-1} & \end{array}$$

such that

- (i) for any pair of composable morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z$  of  $S$ -schemes and any sheaf of monoids  $\mathcal{P}$  on  $Z$ , the square

$$\begin{array}{ccc} (f')^{-1}(g')^{-1}\Psi_Z(\mathcal{P}) & \xrightarrow{(f')^{-1}(\xi_{g,\mathcal{P}})} & (f')^{-1}\Psi_Y(g^{-1}\mathcal{P}) \\ \downarrow \wr & & \downarrow \xi_{f,g^{-1}\mathcal{P}} \\ (g' \circ f')^{-1}\Psi_Z(\mathcal{P}) & \xrightarrow{\xi_{g \circ f, \mathcal{P}}} & \Psi_X(f^{-1}g^{-1}\mathcal{P}) \end{array}$$

commutes;

- (ii) for any  $S$ -scheme  $X$ , under the identification of  $\text{id}_X^{-1}$  with the identity,  $\xi_{\text{id}_X}$  is the identical natural transformation  $\Psi_X \Rightarrow \Psi_X$ ;
- (iii) for any morphism of  $S$ -schemes  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccccc} (f')^{-1}\Psi_Y(\mathcal{O}_Y) & \xrightarrow{\xi_{f,\mathcal{O}_Y}} & \Psi_X(f^{-1}\mathcal{O}_Y) & \longrightarrow & \Psi_X(\mathcal{O}_X) \\ (f')^{-1}(\tau_Y) \downarrow & & & & \downarrow \tau_X \\ (f')^{-1}\mathcal{O}_{Y'} & \longrightarrow & & \longrightarrow & \mathcal{O}_{X'} \end{array}$$

commutes.

Define a functor of prelog schemes  $H^{pre}: \mathbf{PLogSch}/S \rightarrow \mathbf{PLogSch}$  as follows:

- for a prelog scheme  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$  over  $S$ ,  $H^{pre}(X, \alpha) = (X', \Psi_X^{pre}(\alpha))$ , i.e.

$$H^{pre}(X, \alpha) = (X', \Psi_X(\mathcal{M}) \xrightarrow{\Psi_X(\alpha)} \Psi_X(\mathcal{O}_X) \xrightarrow{\tau_X} \mathcal{O}_{X'});$$

- for a morphism of prelog schemes  $(f, \bar{f}): (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$ ,  $H^{pre}(f)$  is the morphism  $(f', \bar{f}')$ , where  $\bar{f}': (f')^{-1}\Psi_Y(\mathcal{N}) \rightarrow \Psi_X(\mathcal{M})$  is the composition

$$(f')^{-1}\Psi_Y(\mathcal{N}) \xrightarrow{\xi_{f, \mathcal{N}}} \Psi_X(f^{-1}\mathcal{N}) \xrightarrow{\Psi_X(\bar{f})} \Psi_X(\mathcal{M}).$$

This is well-defined by the next lemma.

Composing with the logarithmification functor  $\mathbf{PLogSch} \rightarrow \mathbf{LogSch}$ , one obtains a functor  $H: \mathbf{PLogSch}/S \rightarrow \mathbf{LogSch}$ .

*Remark 2.6.7.* Since for any  $S$ -scheme  $X$  and any log structure  $\alpha$  on  $X$  there is a canonical morphism  $(X, \alpha) \rightarrow S$ , the restrictions of  $H^{pre}$  and  $H$  to  $\mathbf{LogSch}/S$  factor through  $\mathbf{PLogSch}/H^{pre}(S)$  and  $\mathbf{LogSch}/H(S)$ , respectively.

**Lemma 2.6.8.** *With the same notation as in the previous construction,  $H^{pre}$  is indeed a well-defined functor  $H^{pre}: \mathbf{PLogSch}/S \rightarrow \mathbf{PLogSch}$ .*

*Proof.* First, it is straightforward to see that, with the same notation as in the previous construction,  $\bar{f}': (f')^{-1}\Psi_Y(\mathcal{N}) \rightarrow \Psi_X(\mathcal{M})$  is a morphism of prelog structures: indeed, the diagram

$$\begin{array}{ccccc} (f')^{-1}\Psi_Y(\mathcal{N}) & \xrightarrow{\xi_{f, \mathcal{N}}} & \Psi_X(f^{-1}\mathcal{N}) & \xrightarrow{\Psi_X(\bar{f})} & \Psi_X(\mathcal{M}) \\ \downarrow & & \downarrow & & \downarrow \\ (f')^{-1}\Psi_Y(\mathcal{O}_Y) & \xrightarrow{\xi_{f, \mathcal{O}_Y}} & \Psi_X(f^{-1}\mathcal{O}_Y) & \longrightarrow & \Psi_X(\mathcal{O}_X) \\ (f')^{-1}(\tau_Y) \downarrow & & & & \downarrow \tau_X \\ (f')^{-1}\mathcal{O}_{Y'} & \longrightarrow & & & \mathcal{O}_{X'} \end{array}$$

commutes: the commutativity of the upper left and right squares are the naturality of  $\xi_f$  and the functoriality of  $\Psi_X$ , respectively, and the lower rectangle commutes by the assumption *iii*) above.

The condition *ii*) implies that  $H^{pre}$  takes identities to identities.

As for compositions, observe that for any pair of composable morphisms of log  $S$ -schemes  $(X, \mathcal{M}) \xrightarrow{f} (Y, \mathcal{N}) \xrightarrow{g} (Z, \mathcal{P})$ , all the squares in

$$\begin{array}{ccccc} (f')^{-1}(g')^{-1}\Psi_Z(\mathcal{P}) & \xrightarrow{(f')^{-1}(\xi_{g, \mathcal{P}})} & (f')^{-1}\Psi_Y(g^{-1}\mathcal{P}) & \xrightarrow{((f')^{-1} \circ \Psi_Y)(\bar{g})} & (f')^{-1}\Psi_Y(\mathcal{N}) \\ \downarrow \wr & & \downarrow \xi_{f, g^{-1}\mathcal{P}} & & \downarrow \xi_{f, \mathcal{N}} \\ (g' \circ f')^{-1}\Psi_Z(\mathcal{P}) & \xrightarrow{\xi_{g \circ f, \mathcal{P}}} & \Psi_X(f^{-1}g^{-1}\mathcal{P}) & \xrightarrow{(\Psi_X \circ f^{-1})(\bar{g})} & \Psi_X(f^{-1}\mathcal{N}) \end{array}$$

commute (the left one by condition *i*) and the right one since  $\xi_f$  is a natural transformation). The condition for the “sheafy part” of the composition  $H^{pre}(g) \circ H^{pre}(f)$  to equal that of  $H^{pre}(g \circ f)$  is obtained by stacking the (tautologically) commutative square

$$\begin{array}{ccc} (f')^{-1}\Psi_Y(\mathcal{N}) & \xrightarrow{\xi_{f,\mathcal{N}}} & \Psi_X(f^{-1}\mathcal{N}) \\ \downarrow \xi_{f,\mathcal{N}} & & \downarrow \Psi_X(\bar{f}) \\ \Psi_X(f^{-1}\mathcal{N}) & \xrightarrow{\Psi_X(\bar{f})} & \Psi_X(\mathcal{M}) \end{array}$$

to the right of the previous diagram.  $\square$

### The functor $W_m^{log}$

We define at last the functor  $W_m^{log}$ . As anticipated in the opening paragraphs, this is based on the identification, for any  $k$ -scheme  $X$ , of sheaves on  $X$  with sheaves on  $W_m(X)$ . We shall use (sometimes tacitly) the results on étale sheaves recalled above under the heading “Some facts on étale sheaves” in Section 2.4.

**Definition 2.6.9.** Let  $k$  be a perfect field of positive characteristic  $p$ ,  $m$  a positive integer. For any  $k$ -scheme  $X$ , denote by  $\iota^{X,m}: X \rightarrow W_m(X)$  the natural nilpotent immersion. Let  $(\cdot)': \text{Sch}/k \rightarrow \text{Sch}/W_m$  be the functor  $W_m$ .

Recall that, for any morphism of  $k$ -schemes  $f: X \rightarrow Y$  and any sheaf of monoids  $\mathcal{N}$  on  $Y$ ,  $((f')^{-1} \circ \iota_*^{Y,m})(\mathcal{N})$  is the sheaf associated to the presheaf on the étale/Zariski site of  $W_m(X)$  defined by

$$(\bar{U} \rightarrow W_m(X)) \mapsto \text{colim } \mathcal{N}(\bar{V}_k),$$

the colimit being taken over the commutative squares

$$\begin{array}{ccc} \bar{U} & \longrightarrow & \bar{V} \\ \downarrow & & \downarrow \\ W_m(X) & \longrightarrow & W_m(Y) \end{array}$$

with  $\bar{V} \rightarrow W_m(Y)$  étale. Since an element of the colimit also defines an element of  $(\iota_*^{X,m} \circ f^{-1})(\mathcal{N})(\bar{U})$ , we obtain a natural transformation

$$\eta: (f')^{-1} \circ \iota_*^{Y,m} \rightarrow \iota_*^{X,m} \circ f^{-1}.$$

In the notation of Construction 2.6.6, let

$$W_m^{log}: \text{PLogSch}/k \rightarrow \text{LogSch}/\Delta_m$$

be the functor  $H$  associated to the following data (we do not introduce any notation for the prelogarithmic version, as we will not need to):

- $S = \operatorname{Spec} k$ ,  $(\cdot)'\colon \operatorname{Sch}/k \rightarrow \operatorname{Sch}/W_m(k)$  is the functor  $W_m(\cdot)$ ;
- for any  $k$ -scheme  $X$ ,

$$\Psi_X\colon \operatorname{Sh}(X, \operatorname{Mon}) \rightarrow \operatorname{Sh}(W_m(X), \operatorname{Mon}), \mathcal{M} \mapsto \iota_*^{X,m}(\mathcal{M}) \oplus \mathbb{N}_X,$$

while

$$\tau_X\colon \iota_*^{X,m}(\mathcal{M}) \oplus \mathbb{N}_X \rightarrow W_m(\mathcal{O}_X)$$

is the morphism defined on sections by  $(x, n) \mapsto [x]p^n$  (recall that  $[-]$  is the multiplicative Teichmüller section).

- for any morphism of  $k$ -schemes  $f\colon X \rightarrow Y$ ,  $\xi_f\colon (f')^{-1} \circ \Psi_Y \Rightarrow \Psi_X \circ f^{-1}$  is the natural transformation defined as follows: for any presheaf of monoids  $\mathcal{N}$  on  $Y$ ,  $\xi_{f,\mathcal{N}}\colon ((f')^{-1} \circ \iota_*^{Y,m}) \oplus \mathbb{N}_Y \rightarrow (\iota_*^{X,m} \circ f^{-1}) \oplus \mathbb{N}_X$  can be written on matrix form as

$$\xi_{f,\mathcal{N}} = \begin{pmatrix} \eta_{\mathcal{N}} & 0 \\ 0 & \operatorname{id}_{\mathbb{N}_X} \end{pmatrix}.$$

The commutativity of the relevant diagrams can be checked on stalks and follows straightforwardly (if tediously) from the definitions. The functor  $W_m^{\log}$  of prelog schemes produces log schemes over  $\Delta_m$  by Remark 2.6.7.

To save parentheses, we write  $W_m^{\log}(X, \alpha)$  instead of  $W_m^{\log}((X, \alpha))$ . Moreover, indulging in a common abuse of notation, if  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$  is a prelog  $k$ -scheme, we shall often denote the log scheme  $W_m^{\log}(X, \alpha)$  by  $W_m^{\log}(X, \mathcal{M})$  or even by  $W_m^{\log}(X)$ .

For a log scheme  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$  we shall also write

$$W_m^{\log}(\alpha)\colon W_m^{\log}(\mathcal{M}) \rightarrow W_m(\mathcal{O}_X)$$

for the log structure on  $W_m(X)$  associated to the prelog structure

$$\iota_*^{X,m} \mathcal{M} \xrightarrow{\iota_*^{X,m}(\alpha)} \iota_*^{X,m} \mathcal{O}_X \xrightarrow{[-]} W_m(\mathcal{O}_X).$$

*Remark 2.6.10.* It is useful to observe that, for any log  $k$ -scheme  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$ , the log structure on  $W_m^{\log}(X, \alpha)$  is equal to the direct sum (in  $\mathbf{PLog}_{W_m(X)}$ )

$$\beta\colon W_m^{\log}(\mathcal{M}) \oplus \mathbb{N} \rightarrow W_m(\mathcal{O}_X)$$

of the log structure  $W_m^{\log}(\alpha)\colon W_m^{\log}(\mathcal{M}) \rightarrow W_m(\mathcal{O}_X)$  and the prelog structure  $\mathbb{N} \rightarrow W_m(\mathcal{O}_X)$  associated to the map of presheaves  $n \mapsto p^n$ : indeed, as  $\mathbb{N} \rightarrow W_m(\mathcal{O}_X)$  is a local map and  $\mathbb{N}$  is sharp,  $\beta^{-1}(W_m(\mathcal{O}_X)^\times) = (W_m^{\log}(\alpha))^{-1}(W_m(\mathcal{O}_X)^\times)$ ; since  $W_m^{\log}(\alpha)$  is, by definition, logarithmic,  $(W_m^{\log}(\alpha))^{-1}(W_m(\mathcal{O}_X)^\times) = W_m(\mathcal{O}_X)^\times$ , and the claim easily follows from this.

*Notation 2.6.11.* In order to help the reader navigate this section, let us recap how the various functors that we denote with the symbol  $W_m$  (possibly with some decorations) are defined on objects:

- $W_m(\cdot): k\text{-Alg} \rightarrow W_m(k)\text{-Alg}$  is the usual functor of Witt vectors, which also extends to a functor between sheaves of  $k$ - and  $W_m(k)$ -algebras on any scheme (or actually any topological space);
- $W_m: \text{Sch}/k \rightarrow \text{Sch}/W_m(k)$  takes  $X$  to  $(|X|, W_m(\mathcal{O}_X))$  (Example 2.2.8);
- for any  $k$ -scheme  $X$ ,  $W_m^{\log}: \text{PLog}_X \rightarrow \text{Log}_{W_m(X)}$  takes a prelog structure  $\mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X$  on  $X$  to the logarithmic structure  $W_m^{\log}(\alpha): W_m^{\log}(\mathcal{M}) \rightarrow W_m(\mathcal{O}_X)$  associated to the prelog structure

$$\iota_*^{X,m}(\mathcal{M}) \xrightarrow{\iota_*^{X,m}(\alpha)} \iota_*^{X,m}(\mathcal{O}_X) \xrightarrow{[-]} W_m(\mathcal{O}_X) = \mathcal{O}_{W_m(\mathcal{O}_X)},$$

where  $\iota^{X,m}: X \rightarrow W_m(X)$  is the canonical nilpotent thickening;

- $W_m^{\log}: \text{PLogSch}/k \rightarrow \text{LogSch}/\Delta_m$  takes a prelog  $k$ -scheme  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$  to the log scheme  $(W_m(X), W_m^{\log}(\mathcal{M}) \oplus \mathbb{N} \rightarrow W_m(\mathcal{O}_X))$ , where  $\mathbb{N} \rightarrow W_m(\mathcal{O}_X)$  is the map of sheaves associated to the map of presheaves  $n \mapsto p^n$  (Remark 2.6.10).

The reader may frown at our using the same symbol for the functors at the last two points; let us point out, however, that this is analogous to the standard practice of denoting by  $W_m(\cdot)$  both the functor of schemes at the second point and, for any fixed scheme  $X$ , the functor  $\text{Sh}(X, k\text{-Alg}) \rightarrow \text{Sh}(X, W_m(k)\text{-Alg})$  at the first point. Similarly, we favour the notation  $[-]$  over  $[-]_X$  for the Teichmüller section  $\mathcal{O}_X \rightarrow W_m(\mathcal{O}_X)$ .

*Remarks 2.6.12.* (i) For a prelog  $k$ -scheme  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$ , we can use the fact that the Teichmüller section  $[-]: \mathcal{O}_X \rightarrow W_m(\mathcal{O}_X)$  is a local morphism to simplify the calculation of  $W_m^{\log}(\mathcal{M})$ : explicitly,  $W_m^{\log}(\mathcal{M})$  can be computed as the pushout

$$\iota_*^{X,m}(\mathcal{M}) \oplus_{[\iota_*^{X,m}(\alpha)]^{-1}(W_m(\mathcal{O}_X)^\times)} W_m(\mathcal{O}_X)^\times,$$

and the locality of  $[-]$  implies that

$$[\iota_*^{X,m}(\alpha)]^{-1}(W_m(\mathcal{O}_X)^\times) = \iota_*^{X,m}(\alpha)^{-1}(\mathcal{O}_X^\times);$$

now, since  $\iota_*^{X,m}$  induces isomorphisms on stalks and  $\alpha$  is a logarithmic map,  $\iota_*^{X,m}(\alpha)$  is again logarithmic (because being logarithmic is a stalk-local property) and so  $\iota_*^{X,m}(\alpha)^{-1}(\mathcal{O}_X^\times) = \iota_*^{X,m}(\mathcal{M})^\times$ , whereby

$$W_m^{\log}(\mathcal{M}) \cong \iota_*^{X,m}(\mathcal{M}) \oplus_{\iota_*^{X,m}(\mathcal{M})^\times} W_m(\mathcal{O}_X)^\times.$$

Since  $\iota_*^{X,m}$  induces isomorphisms on stalks, it also follows that the stalk of  $W_m^{\log}(\mathcal{M})$  at any point  $\iota^{X,m}(x)$  is isomorphic to the sharpification

$$\mathcal{M}_x \oplus_{\mathcal{M}_x^\times} W_m(\mathcal{O}_{X,x})^\times,$$

where the map  $\mathcal{M}_x^\times \rightarrow W_m(\mathcal{O}_{X,x})$  is the composition

$$\mathcal{M}_x^\times \xrightarrow{\alpha_x} \mathcal{O}_{X,x}^\times \xrightarrow{[-]} W_m(\mathcal{O}_{X,x})^\times.$$

- (ii) It will also be important to notice that for any two positive integers  $m$  and  $n$  with  $m < n$  the nilpotent immersions  $W_m(\cdot) \rightarrow W_n(\cdot)$  induced by the truncation are natural: for any morphism of  $k$ -schemes  $f: X \rightarrow Y$ , the square

$$\begin{array}{ccc} W_m(X) & \xrightarrow{W_m(f)} & W_m(Y) \\ \downarrow & & \downarrow \\ W_n(X) & \xrightarrow{W_n(f)} & W_n(Y) \end{array}$$

commutes.

**Lemma 2.6.13.** *Let  $(X, \mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X)$  be a log  $k$ -scheme.*

- (i) *If  $X$  is integral/saturated, then so is  $W_m^{\log}(X)$ .*
- (ii) *If  $X$  has a chart subordinated to a monoid  $P$ , then  $W_m^{\log}(X)$  has a chart subordinated to the monoid  $P \oplus \mathbb{N}$ .*
- (iii) *If  $A$  is a  $k$ -algebra and  $P \xrightarrow{f} A$  is a log ring, then  $W_m^{\log}(\mathrm{Spec}(P \rightarrow A))$  is the log scheme associated to the log ring*

$$P \oplus \mathbb{N} \rightarrow W_m(A), \quad (q, n) \mapsto [f(q)]p^n.$$

*Proof.* i): By Remark 2.6.10, the logarithmic structure of  $W_m^{\log}(X, \alpha)$  is equal to  $W_m^{\log}(\mathcal{M}) \oplus \mathbb{N} \rightarrow W_m(\mathcal{O}_X)$ , whereby it suffices to show that each of the stalks of  $W_m^{\log}(\mathcal{M})$  is integral/saturated if the stalks of  $\mathcal{M}$  are integral/saturated. By Remark 2.6.12(i), the stalk of  $W_m^{\log}(\mathcal{M})$  at a point  $\iota^{X,m}(x)$  is isomorphic to the sharpification  $\mathcal{M}_x \oplus_{\mathcal{M}_x^\times} W_m(\mathcal{O}_{X,x})^\times$ ; by [Ogu18, Proposition I.4.6.3.2] and [Ogu18,

Proposition I.4.8.5.4], the morphism  $\mathcal{M}_x^\times \rightarrow W_m(\mathcal{O}_{X,x})^\times$  is integral and saturated as a morphism between groups, which implies that  $W_m^{\log}(\mathcal{M})_x$  is integral/saturated if  $\mathcal{M}_x$  is.

ii): By definition, if  $(X, \mathcal{M})$  has a chart  $X \xrightarrow{c} \text{Spec}(P \rightarrow \mathbb{Z}[P])$ , the log structure on  $X$  is isomorphic to  $c_{\log}^* P$ , i.e. it is the log structure associated to the prelog structure given by the induced morphism of sheaves of monoids  $P_X \rightarrow \mathcal{O}_X$ . We now want to apply Lemma 2.6.4; in order to do so, it is enough to observe that the existence of a factorisation  $\iota_*^{X,m}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_{W_m(X)}^\times$  is immediate, and  $\iota_*^{X,m}$  preserves pushouts by [Mil80, Corollary II.3.6]. Invoking Corollary 2.6.5, one can therefore compute  $W_m^{\log}(c_{\log}^* P) \oplus \mathbb{N}$  as the logarithmic structure associated to the prelog structure  $P_{W_m(X)} \oplus \mathbb{N} \rightarrow \iota_*^{X,m}(\mathcal{O}_X) \rightarrow W_m(\mathcal{O}_X)$ ; but this precisely means that the induced map  $P \oplus \mathbb{N} \rightarrow \Gamma(W_m(X), W_m^{\log}(\mathcal{M}) \oplus \mathbb{N})$  is strict.

iii) follows straightforwardly from the proof of previous point and the definition of the functor  $W_m^{\log}$ .  $\square$

*Construction 2.6.14.* Let  $(X, \mathcal{M})$  be a log  $k$ -scheme. For any  $m \geq 1$ , the nilpotent immersion of schemes  $\iota: W_m(X) \hookrightarrow W_{m+1}(X)$  induces a strict nilpotent immersion  $W_m^{\log}(X, \mathcal{M}) \rightarrow W_{m+1}^{\log}(X, \mathcal{M})$ : more precisely, since  $\iota_* \circ \iota_*^{X,m} = \iota_*^{X,m+1}$ , one obtains by adjunction a morphism

$$\iota^{-1} W_{m+1}^{\log}(\mathcal{M}) \rightarrow W_m^{\log}(\mathcal{M}),$$

which combined with the identity on  $\mathbb{N}$  yields a morphism

$$\bar{\iota}: \iota^{-1} W_{m+1}^{\log}(\mathcal{M}) \oplus \mathbb{N} \rightarrow W_m^{\log}(\mathcal{M}) \oplus \mathbb{N};$$

it is then easy to see that

$$(\iota, \bar{\iota}): W_m^{\log}(X) \rightarrow W_{m+1}^{\log}(X)$$

is a strict nilpotent immersion.

Notice also that the construction of  $(\iota, \bar{\iota})$  is natural in  $X$ , since so are  $\iota^X$  and  $\iota^{X,m}$ .

### The logarithmic Greenberg functor: definition and representability

**Definition 2.6.15.** Let  $X$  be a log scheme over  $\Delta_m$  and  $m$  be a positive integer. The logarithmic Greenberg transform of  $X$  of level  $m$  is the presheaf  $\text{Gr}_{\log}^m X: (\text{LogSch}/k)^{\text{op}} \rightarrow \text{Set}$  such that, for log  $k$ -scheme  $Z$ ,

$$\text{Gr}_{\log}^m X(Z) = \text{Hom}_{\Delta_m}(W_m^{\log}(Z), X).$$

The main result of this section is the following theorem, which establishes the representability of  $\mathrm{Gr}_{\log}^m X$  in a wide variety of contexts.

**Theorem 2.6.16.** *Let  $X$  be a log scheme over  $\Delta_m$ .*

- (i) *If  $X$  admits a finitely generated global chart, then  $\mathrm{Gr}_{\log}^m X$  is representable in the category of  $k$ -log schemes, and if  $X$  is coherent, so is  $\mathrm{Gr}_{\log}^m X$ .*
- (ii) *If  $\ell = \text{“fin”}, \text{“fs”}$ ,  $X$  is in  $\mathrm{LogSch}^\ell / \Delta_m$  and admits charts Zariski locally, then  $\mathrm{Gr}_{\log}^m X : (\mathrm{LogSch}^\ell / k)^{\mathrm{op}} \rightarrow \mathrm{Set}$  is representable.*

*Proof.* i): As in Remark 1.3.8, it is enough to write  $X$  as a limit of log schemes over  $\Delta_m$  whose logarithmic Greenberg functor is representable (using, for the statement about coherence, that a finite limit of coherent log schemes is again coherent [GR18, Corollary 12.2.23]). By Lemma 2.5.2, it is enough to show that  $\mathrm{Gr}_{\log}^m X$  is representable when  $X = T \times_{W_m(k)} \Delta_m$  for some  $W_m(k)$ -scheme  $T$  and when  $X = \mathbb{A}_{\Delta_m}^1$ .

For  $X = \mathbb{A}_{\Delta_m}^1$ , the representability of  $\mathrm{Gr}_{\log}^m X$  will be established in Lemma 2.6.17, so let us now consider the former case, which is easier: denoting by  $\mathrm{Gr}^m T$  the classical (schematic) Greenberg transform of  $T$ , one has isomorphisms, natural in the log  $k$ -scheme  $Z$ ,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{LogSch}/k}(Z, \mathrm{Gr}_{\log}^m(T \times_{W_m(k)} \Delta_m)) &= \mathrm{Hom}_{\mathrm{LogSch}/\Delta_m}(W_m^{\log}(Z), T \times_{W_m(k)} \Delta_m) = \\ \mathrm{Hom}_{\mathrm{LogSch}/W_m(k)}(W_m^{\log}(Z), T) &= \mathrm{Hom}_{\mathrm{Sch}/W_m(k)}(\underline{W_m^{\log}(Z)}, T) = \\ \mathrm{Hom}_{\mathrm{Sch}/W_m(k)}(W_m(\underline{Z}), T) &= \mathrm{Hom}_{\mathrm{Sch}/k}(\underline{Z}, \mathrm{Gr}^m T) = \mathrm{Hom}_{\mathrm{LogSch}/k}(Z, \mathrm{Gr}^m T), \end{aligned}$$

where we used repeatedly Remark 2.4.17. Therefore,

$$\mathrm{Gr}_{\log}^m(T \times_{W_m(k)} \Delta_m) = \mathrm{Gr}^m T.$$

ii) We can reduce to the case of coherent log schemes, using Proposition 2.4.27 and Lemma 2.6.13.

As in the proof of Lemma 2.5.7, we wish to apply Theorem 1.3.7 for  $\mathcal{D} = \mathrm{LogSch}^\ell / S'$ ,  $\mathcal{C} = \mathcal{C}' = \mathrm{LogSch}^\ell / S$  and  $\mathcal{E}$  the full subcategory of  $\mathrm{LogSch}^\ell / S$  spanned by the objects which admit finitely generated charts Zariski locally. There is an obvious natural isomorphism (actually an equality)  $|\cdot| \rightarrow |W_m^{\log}(\cdot)|$ , whereby it just remains to prove that  $W_m^{\log}$  respects open immersions (as defined in Lemma 2.4.25). This is easy: one can immediately reduce to the case of a morphism  $j = (j, \mathrm{id}_{\mathcal{M}|_U}) : (U, \mathcal{M}|_U) \rightarrow (X, \mathcal{M})$  for some log scheme  $(X, \mathcal{M})$  and a Zariski open immersion  $j : U \rightarrow X$ ; but then, by definition, writing  $\mathcal{N}$  for the restriction of

$W_m^{\log}(\mathcal{M})$  to the open subscheme  $W_m(U)$  of  $W_m(X)$  one can identify  $W_m^{\log}(j)$  with the open immersion

$$(W_m(j), \text{id}_{\mathcal{N} \oplus \mathbb{N}}): W_m^{\log}(U, \mathcal{M}|_U) \rightarrow W_m^{\log}(X, \mathcal{M}). \quad \square$$

**Lemma 2.6.17.** *For any positive integer  $m$ ,  $\text{Gr}_{\log}^m \mathbb{A}_{\Delta_m}^1$  is represented by*

$$\bigsqcup_{\mathbb{N}} \mathbb{A}_k^1 \times_k \underline{\mathbb{A}}_k^{m-1}.$$

*Proof.* By Remarks 2.4.30 and 2.6.10,  $\text{Gr}_{\log}^m \mathbb{A}_{\Delta_m}^1$  is the functor of log  $k$ -schemes

$$(Z, \mathcal{M}_Z) \mapsto \Gamma(W_m(Z), W_m^{\log}(\mathcal{M}_Z) \oplus \mathbb{N});$$

for any log  $k$ -scheme  $(Z, \alpha: \mathcal{M}_Z \rightarrow \mathcal{O}_Z)$ , apply [Ogu18, Proposition IV.2.1.2.3] for  $S = W_1^{\log}(Z)$ ,  $T = W_m^{\log}(Z)$  and  $\iota: S \rightarrow T$  being the strict nilpotent immersion of Construction 2.6.14: identifying, as in *loc. cit.*, the Zariski/étale sites on  $Z$  and on  $W_m(Z)$ , one gets an isomorphism of sheaves of monoids on  $Z$

$$(1 + \mathcal{I}_Z) \times \mathcal{M}_T \xrightarrow{\sim} \mathcal{M}_T \times_{\mathcal{M}_S} \mathcal{M}_T, \quad (2.3)$$

where  $\mathcal{I}_Z = \ker(\iota^\sharp: W_m(\mathcal{O}_Z) \rightarrow \mathcal{O}_Z)$ .

We now claim that the morphism of sheaves of monoids  $W_m^{\log}(\mathcal{M}_Z) \rightarrow W_1^{\log}(\mathcal{M}_Z)$ , and thus also  $\mathcal{M}_T = W_m^{\log}(\mathcal{M}_Z) \oplus \mathbb{N} \rightarrow W_1^{\log}(\mathcal{M}_Z) \oplus \mathbb{N} = \mathcal{M}_S$ , has a section: by Remark 2.6.12(i), for any  $n$ ,  $W_n^{\log}(\mathcal{M}_Z)$  is the pushout  $\mathcal{M}_Z \oplus_{\mathcal{M}_Z^\times} W_n(\mathcal{O}_Z)^\times$ ; as  $W_1(\mathcal{O}_Z) = \mathcal{O}_Z$ , the Teichmüller section  $\mathcal{O}_Z \rightarrow W_m(\mathcal{O}_Z)$  induces the desired section  $W_1^{\log}(\mathcal{M}_Z) \rightarrow W_m^{\log}(\mathcal{M}_Z)$ .

Base-changing the isomorphism (2.3) along the section  $\mathcal{M}_S \rightarrow \mathcal{M}_T$  thus constructed and taking global sections, one obtains an isomorphism of monoids

$$\Gamma(Z, 1 + \mathcal{I}_Z) \times \Gamma(Z, \mathcal{M}_S) \xrightarrow{\sim} \Gamma(Z, \mathcal{M}_T).$$

Now,  $\Gamma(Z, \mathcal{M}_S) = (\text{Gr}_{\log}^1 \mathbb{A}_{\Delta_1}^1)(Z)$  and  $\Gamma(Z, \mathcal{M}_T) = (\text{Gr}_{\log}^m \mathbb{A}_{\Delta_m}^1)(Z)$  (functorially in  $Z$ );  $\text{Gr}_{\log}^1 \mathbb{A}_{\Delta_1}^1$  is the same as the Weil restriction

$$\text{Res}_{\Delta^1/\underline{\Delta}^1} \mathbb{A}_{\Delta^1}^1,$$

which by Lemma 2.5.10 is represented by  $\bigsqcup_{\mathbb{N}} \mathbb{A}_k^1$ . By definition,  $\iota: W_m(\mathcal{O}_Z) \rightarrow \mathcal{O}_Z$  is given on sections by  $(a_0, \dots, a_m) \mapsto a_0$ , whereby the functor  $(Z, \mathcal{M}_Z) \mapsto \Gamma(Z, 1 + \mathcal{I}_Z)$  is isomorphic, as a set-valued functor, to  $\underline{\mathbb{A}}_k^{m-1}$ , which is what remained to show.  $\square$

## Part II

# Néron models of tori and base-change



## Introduction

The behaviour of Néron models of (say) group schemes over a discrete valuation field  $K$  under base-change is known to be rather difficult to describe. As a way to glean some information about the base-change behaviour of Néron models of semi-abelian varieties, Chai [Cha00] and Edixhoven [Edi92], and subsequently Halle and Nicaise [HN11], introduced and studied some - as it turns out - closely related numerical invariants, Chai's base-change conductors and Edixhoven's jumps. Although well-understood for tamely ramified semi-abelian varieties, these are rather mysterious in the presence of wild ramification; in the second part of this thesis, we mainly focus on the computation of some of these invariants for certain classes of tori over a strictly henselian discretely valued field  $K$ .

Let us now describe the structure of this part and mention some of the main results we have obtained, referring the reader to the introduction of each section for a more comprehensive description of the results presented therein. The first chapter introduces Néron models of semi-abelian varieties and reviews Chai's and Edixhoven's constructions, as well as giving an overview of (some of) the literature on them and on the difficulties their study presents.

The second chapter opens with a brief section on the tame base-change conductor of tori, containing both well-known facts and new results, followed by a section collecting some results and examples on weakly unramified extensions of a henselian discretely valued fields, mainly dealing with their composita.

We then come to the computation of the jumps of some classes of tori: the main result of Section 4.3, Theorem 4.3.9, describes the jumps of an induced torus over an arbitrary strictly henselian discretely valued field  $K$ , thus extending [Ove19, Corollary 2.2.2]; this is the first non-trivial computation of the jumps of a semi-abelian  $K$ -variety where  $K$  has imperfect residue field.

The next section contains several results on tori that are direct factors of  $K$ -rational varieties, and in particular a proof that their jumps are rational, together with a bound on their denominator in terms of the ramification degree of their minimal splitting extension. The last section (which can perhaps be construed as an independent section) is devoted to a generalisation of de Shalit's description of the Lie algebra of a Néron model of a torus [CY01, A1] which is valid over any strictly henselian discretely valued field (Proposition 4.5.4), and might offer a different approach to the study of jumps.



## Chapter 3

# Preliminary notions

Let us start with some preliminary notation and conventions.

*Notation 3.0.1.* • In the following  $K$  denotes a discretely valued field with residue field  $k$  of characteristic exponent  $p$ .

- The ring of integers of any valued field  $F$  will be denoted by  $\mathcal{O}_F$ , its maximal ideal by  $\mathfrak{m}_F$  and its residue field by  $k_F$  (with an obvious exception for fields denoted by  $K'$ ). Reduction modulo the maximal ideal will be denoted by a bar  $\bar{\phantom{x}}$ .

All the separable fixed extensions of  $K$  will be considered inside some fixed separable closure  $K^{sep}$ , unless mentioned otherwise.

As the notion of short exact sequence of group schemes over a discrete valuation ring will be fundamental, let us recall its meaning (for group schemes over an arbitrary base): a morphism of schemes is called fppf if it is faithfully flat and locally of finite presentation [Poo08, Definition 3.4.1] and, for a base scheme  $S$ , an fppf group scheme over  $S$  is a group  $S$ -scheme whose structure morphism is fppf. A sequence  $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$  of fppf group schemes over  $S$  is called exact if it is exact as a sequence of fppf sheaves on  $S$  [Poo08, Definition 5.1.8 and Remark 6.3.28].

If  $S = \text{Spec } k$  is the spectrum of a field, fppf group  $S$ -schemes are the same as group  $k$ -schemes locally of finite type (=algebraic group  $k$ -schemes) and it follows from [BGA18b, Lemma 2.3] and [BGA18b, Corollary 2.5] that a sequence as above is exact if and only if the induced morphism  $G' \rightarrow \ker(G \rightarrow G'')$  is an isomorphism and  $G \rightarrow G''$  is faithfully flat.

Also recall the definition of weakly unramified extensions of discrete valuation rings:

**Definition 3.0.2.** An extension of discrete valuation fields will be called weakly unramified if it is algebraic and its ramification index is one; an extension of discrete valuation rings will be called weakly unramified if the induced extension of fraction fields is.

The requirement of algebraicity is perhaps non-standard (cfr. [Autb, 09E4]), but we include it because we shall only consider such extensions.

Finally, we recall the following

**Definition 3.0.3.** Let  $M/K$  be a finite separable extension of fields. The group schemes  $\mathrm{Res}_{M/K} \mathbb{G}_{m,M}$  and

$$\ker\left(N: \mathrm{Res}_{M/K} \mathbb{G}_{m,M} \rightarrow \mathbb{G}_{m,K}\right)$$

are called the *induced* and *norm torus relative to* (or *along*) *the extension*  $M/K$ .

Here, as usual,  $\mathrm{Res}_M K$  denotes the Weil restriction along  $\mathrm{Spec} M \rightarrow \mathrm{Spec} K$  and  $N$  is the norm map, so that for all  $K$ -algebras  $A$

$$\left(\mathrm{Res}_{M/K} \mathbb{G}_{m,M}\right)(A) = (A \otimes_K M)^\times$$

(seen as a multiplicative group) and  $N_A: (A \otimes_K M)^\times \rightarrow A^\times$  takes  $x \in (A \otimes_K M)^\times$  to the determinant of the  $A$ -isomorphism of  $(A \otimes_K M)^\times$  given by multiplication by  $x$ .

Notice that the induced torus  $T = \mathrm{Res}_{M/K} \mathbb{G}_{m,M}$ , and thus also the norm torus relative to  $M/K$ , splits over the Galois closure  $L/K$  of  $M/K$  (in particular, they are indeed tori). Its character module is the character  $\mathrm{Gal}(L/K)$ -module  $\chi(T)$  is  $\mathbb{Z}[\mathrm{Gal}(L/K)/\mathrm{Gal}(L/M)]$  [Bra03, Satz 0.4.3].

### 3.1 Generalities on Néron models of semi-abelian varieties

We recall some preliminary definitions and results concerning Néron models of semi-abelian varieties. We restrict to a local base, and let  $\mathcal{O}_K$  be a discrete valuation ring with fraction field  $K$ ; call  $j: \mathrm{Spec} K \rightarrow \mathrm{Spec} \mathcal{O}_K$  the inclusion of the generic fibre. The following presentation owes much to the books [BLR90] and especially [HN16].

**Néron models** We shall refer to what Bosch, Lütkebohmert and Raynaud [BLR90, Definition 10.1.1] call a Néron lft model of a smooth separated  $K$ -scheme simply as its Néron model.<sup>1</sup>

**Definition 3.1.1.** Let  $T$  be a smooth and separated  $K$ -scheme. A Néron model  $\mathcal{T}$  of  $T$  over  $\mathcal{O}_K$  is a smooth, separated  $\mathcal{O}_K$ -scheme  $\mathcal{T}$  such that for any smooth  $K$ -scheme  $\mathcal{S}$  and any  $K$ -morphism  $u_K: \mathcal{S}_K \rightarrow T$ , there exists a unique  $\mathcal{O}_K$ -morphism  $\mathcal{S} \rightarrow \mathcal{T}$  extending  $u$ .

By definition, a smooth separated  $\mathcal{O}_K$ -scheme  $\mathcal{T}$  is the Néron model of  $T$  if and only if it represents the direct image sheaf  $j_*^{sm} T$  on the small smooth site on  $\mathcal{O}_K$ , where  $T$  is regarded as a sheaf on the small smooth site on  $K$ . This is very useful, because it allows the use of sheaf-theoretic methods in the study of Néron models.

One is especially interested in Néron models of (smooth) group  $K$ -schemes; notice that, by the universal property of Néron models, if a smooth group  $K$ -scheme  $G$  admits a Néron model  $\mathcal{G}$ , then its group- $K$ -scheme structure extends uniquely to a group- $\mathcal{O}_K$ -scheme structure on  $\mathcal{G}$ .

Clearly, the formation of Néron models respects products, but, apart from this, it is not very well-behaved: for example, a short exact sequence of smooth group  $K$ -schemes does not necessarily induce a short exact sequence of Néron models, even when these exist [BLR90, Examples 7.5/7-9]. Moreover, its behaviour under ramified change of base-field is also extremely complicated, and probably impossible to describe completely. Since the invariants we shall study concern the base-change behaviour of the identity component of Néron models of group schemes, we do not wish to elaborate here (hoping that the rest of this part will convey a sense of the complexity of the issue); we instead mention that there is a very important case when the formation of Néron models is compatible with base-change, namely, that of unramified base-change:

**Theorem 3.1.2** ([BLR90, Proposition 10.1/3]<sup>2</sup>). *Let  $\mathcal{O}_K$  be a discrete valuation ring,  $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$  a (possibly infinite) unramified extension,  $G$  a smooth group  $K$ -*

<sup>1</sup>We find the terminology “Néron lft model” very unfortunate: any adjective or attribute should specify a quality of the term to which it refers, thereby identifying a subclass of objects; based on this general linguistic principle, one would expect the class of Néron lft models to be a subclass of that of Néron models, while in this case the terminology is chosen so that the opposite holds!

<sup>2</sup>To avoid confusion, let us remark that an extension of discrete valuation rings  $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$  is called “of ramification index 1” in *loc. cit.* if the ramification index  $e(K'/K)$  equals one *and* the residue field extension is separable (possibly transcendental). As Examples 4.3.11 and 4.3 will show, the conclusion of the theorem fails if the separability condition is omitted.

scheme. If  $\mathcal{G}$  is a Néron model of  $G$  over  $\mathcal{O}_K$ , then  $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'}$  is a Néron model of  $G \times_K K'$  over  $\mathcal{O}_{K'}$ .

The notion of identity component is well-known for group schemes over a field. More generally, if  $S$  is a scheme and  $G$  a group  $S$ -scheme, the identity component of  $G$  is defined in [ABD<sup>+</sup>66, Exposé VIb, Definition 3.1] as the subgroup functor  $G^0$  of  $G$  taking an  $S$ -scheme  $T$  to the group

$$G^0(T) := \{f \in G(T) \mid f_s(T_s) \subseteq G_s^0 \text{ for all points } s \text{ of } S\}.$$

If  $G$  is smooth over  $S$ , then this is representable by a subgroup  $S$ -scheme of  $G$ , which is smooth over  $S$  and open in  $G$  (*loc. cit.*, Theorem 3.10). Moreover,  $G^0$  is of finite type over  $S$ , and its fibre over any point  $s$  in  $S$  is (identified with)  $G_s^0$  (*loc. cit.*, 3.4-6).

If  $G$  is a smooth group  $K$ -scheme admitting a Néron model, the étale group scheme

$$\Phi(G) := \mathcal{G}_k / \mathcal{G}_k^0$$

is called the component group of  $G$ .

**Semi-abelian reduction** A group  $K$ -scheme  $G$  is called a semi-abelian  $K$ -variety if it is the extension of an abelian variety  $A$  by a torus  $T$ , i.e. there exists a short exact sequence  $0 \rightarrow A \rightarrow G \rightarrow T \rightarrow 0$ . By [BLR90, Theorem 10.2/2], any semi-abelian  $K$ -variety admits a Néron model.

A semi-abelian  $K$ -variety  $G$  is said to have multiplicative/semi-abelian reduction if  $\mathcal{G}_k^0$  is a torus/a semi-abelian  $k$ -variety. For a torus over a discretely valued field with separably closed residue field, both of these properties are equivalent to being split [HN16, 3.3.2.1].

**Theorem 3.1.3** (Semi-abelian reduction theorem). *Let  $K$  be a discretely valued field,  $G$  a semi-abelian  $K$ -variety.*

- (i) *There exists a unique minimal subextension  $L/K$  of  $K^s/K$  such that  $G \times_K L$  has semi-abelian reduction. The extension  $L/K$  is finite and Galois.*
- (ii) *The property of having semi-abelian reduction is stable under finite separable extensions: if  $G$  has semi-abelian reduction, then for any finite separable extension  $K'/K$ ,  $G_{K'}$  has semi-abelian reduction as well.*

For an abelian variety  $G$ , the first point is Grothendieck's semi-abelian reduction theorem [GRR72, Théorème IX.3.6] (almost: Grothendieck only states that there

is a minimal Galois extension after which  $G$  acquires semi-abelian reduction), the second one [GRR72, Corollaire IX.3.3]; the extension to semi-abelian varieties is in [HN16, Theorem 3.3.6.4].

A semi-abelian  $K$ -variety is called *tamely/wildly ramified* if the minimal extension of the previous theorem is a tame/wild.

## 3.2 Néron models and base-change

Let  $G$  be a semi-abelian variety over a discretely valued field  $K$ ,  $G'$  its base-change along an extension of discretely valued fields  $K'/K$ . By the universal property of the Néron model  $\mathcal{G}'$ , there is a unique map

$$h(G, K'): \mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{G}'$$

which restricts to  $\text{id}_{G'}$  on the generic fibre. By [GRR72, Exposé IX, Proposition 3.1e)],  $h(G, K')$  is an open immersion for every finite separable extension  $K'/K$  if and only if  $G$  has semi-abelian reduction. As mentioned above, however, the map  $h(G, K')$  is in general very hard to describe, and, if  $K'/K$  is ramified, far from being an isomorphism (even in the simple case of the multiplicative group  $\mathbb{G}_{m,K}$ : [HN16, 3.3.4.4]).

For a semi-abelian  $K$ -variety  $G$  and a finite separable extension  $K'/K$ , Chai's elementary divisors of  $G$  measure the failure of the base-change morphism  $h(G, K')$  to induce an open immersion of identities components; they were introduced by Chai [Cha00, 2.4] in the case when  $G$  acquires semi-abelian reduction over  $K'$  to quantify the failure of  $G$  to have semi-abelian reduction. Subsequently, they were studied for arbitrary extensions by Halle and Nicaise. After recalling their definition, we review the (or rather, a possible) definition of Edixhoven's jumps in terms of Chai's divisors (the original definition, using Edixhoven's filtration (3.2), will be explained later) and give a proof of their well-definedness which is entirely elementary and avoids the use of Edixhoven's non-trivial results.

For this, we first recall the notion and some properties of the elementary divisors of an injection of free modules of finite rank over a discrete valuation ring (these could actually be introduced in much greater generality, as does Kaplanski [Kap49], but this is unnecessary here).

### Elementary divisors of modules

First, recall a classical definition (see e.g. [Kap49, Section 9], [HN16, Definition 6.1.1.1]).

**Definition 3.2.1.** Let  $f: M \hookrightarrow N$  be an injection of free modules of finite rank over a discrete valuation ring  $R$  with normalised valuation  $\nu: (\text{Frac } R)^\times \rightarrow \mathbb{Z}$  and maximal ideal  $\mathfrak{m}$ .

- (i) The tuple of invariant factors of  $f$  is the tuple  $(f_1, \dots, f_r) \in R^r/R^\times$  of diagonal elements in the Smith normal form of any matrix associated to  $f$ , ordered so that  $f_i | f_{i+1}$ . Here  $R^\times$  acts on  $R^r$  diagonally, i.e.  $u \cdot (f_1, \dots, f_r) = (uf_1, \dots, uf_r)$ .
- (ii) The tuple of elementary divisors of  $f$  is the tuple of increasing non-negative integers  $(\nu(f_1), \dots, \nu(f_r))$ .

*Remarks 3.2.2.* (i) The valuation  $\nu(f_i)$  is indeed an integer, because the invariant factors of the injection  $f$  cannot vanish.

- (ii) It is immediate to see that the tuple of elementary divisors of  $f$  can be characterised as the unique increasing tuple of non-negative integers  $(c_1(f), \dots, c_r(f))$  such that

$$\text{coker}(f) \cong \bigoplus_{i=1}^r \frac{R}{\mathfrak{m}^{c_i(f)}}.$$

- (iii) One easily checks that

$$\sum_{i=1}^r c_i(f) = \text{length}_R(\text{coker } f).$$

- (iv) If  $R \hookrightarrow R'$  is an extension of discrete valuation rings of ramification index  $e$ ,  $c_i(f \otimes_R \text{id}_{R'}) = ec_i(f)$  for all  $i$ , because the invariant factors of  $f \otimes_R \text{id}_{R'}$  are  $(f_i \otimes_R 1)_i$  and the *normalised* valuation of  $R'$  takes value  $e\nu(f_i)$  on  $f_i \otimes_R 1$ .

One can compare the invariant factors of the composition of two maps with those of the two maps:

**Proposition 3.2.3.** *Let  $R$  be a discrete valuation ring,  $\alpha: A \hookrightarrow B$ ,  $\beta: B \hookrightarrow C$  inclusions of free  $R$ -modules of finite rank  $r$ ,  $\gamma = \beta \circ \alpha$ . For any  $i = 1, \dots, r$ , one has  $\alpha_i | \gamma_i$  and  $\beta_i | \gamma_i$ .*

This follows from a result by Kaplanski [Kap49, Theorem 10.3]; see also [New72, Theorem II.14].

However, it is unfortunately not true in general that for a composition  $h = g \circ f$  of injections of free  $R$ -modules of finite rank,  $c_i(h) = c_i(g) + c_i(f)$ ; as a matter of fact, this can fail even for  $g = f$ : for example, the Smith normal form of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

over the integers is

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

that of its square is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} :$$

letting  $f: \mathbb{Q}_2^2 \rightarrow \mathbb{Q}_2^2$  be the linear endomorphism defined by  $A$  in the canonical basis, one then has  $c_i(f^2) \neq 2c_i(f)$  for  $i = 1, 2$ .

On a positive note, one nevertheless has the following

**Theorem 3.2.4** ([New72, Theorem II.15]). *Let  $R$  be a discrete valuation ring,  $n$  a natural number,  $R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$  injections of free  $R$ -modules. If  $\det f$  and  $\det g$  are coprime, then  $c_i(g \circ f) = c_i(f) + c_i(g)$  for all  $i = 1, \dots, n$ .*

### Chai's divisors and Edixhoven's jumps: definitions and basic properties

We can finally turn to the definition of Chai's divisors.

As before, we let  $G$  be a semi-abelian variety over a discretely valued field  $K$ , and consider a separable extension of discrete valuation rings  $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$ . Since  $h(G, K')$  is the identity on generic fibres, the triangle

$$\begin{array}{ccc} \mathrm{Lie}(\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'}) & \xrightarrow{\mathrm{Lie} h(G, K')} & \mathrm{Lie} \mathcal{G}' \\ & \searrow & \swarrow \\ & \mathrm{Lie} G' & \end{array},$$

where the diagonal arrows are the natural inclusions, commutes, whereby  $\mathrm{Lie} h(G, K')$  is an injection of free  $\mathcal{O}_{K'}$ -modules of rank  $\dim G$  (because both  $\mathcal{G}'$  and  $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'}$  are smooth). Hence it makes sense to consider the elementary divisors of the map  $\mathrm{Lie} h(G, K')$ .

The “normalising factor”  $1/e(K'/K)$  in the following definition is chosen so that  $c_i(G)$  is independent of the choice of an extension over which  $G$  acquires semi-abelian reduction (Remark 3.2.7). We assume that  $K'/K$  is finite to be consistent with the literature, but remark that what follows just requires the finiteness of the ramification index  $e(K'/K)$ ; as a matter of fact, it might be interesting to study Chai’s divisors for infinite weakly unramified extensions.

**Definition 3.2.5** ([HN16, Definition 6.1.2.2]). Let  $G$  be a  $g$ -dimensional semi-abelian variety on a discretely valued field  $K$  and  $K'/K$  a finite separable extension. The numbers

$$\frac{c_1(\mathrm{Lie} h(G, K'))}{e(K'/K)}, \dots, \frac{c_g(\mathrm{Lie} h(G, K'))}{e(K'/K)}$$

are called Chai’s elementary  $K'$ -divisors of  $G$  and will be denoted by

$$c_1(G, K'), \dots, c_g(G, K').$$

If  $G$  acquires semi-abelian reduction over  $K'$ , one writes  $c_i(G)$  for  $c_i(G, K')$ ; the sum

$$c(G) := \sum_{i=1}^g c_i(G)$$

is called the base-change conductor of  $G$ .

As remarked in [HN16, 6.1.2.4], the  $K'$ -elementary divisors of  $G$  can taken as a measure of the difference between the identity components of  $\mathcal{G}_{\mathcal{O}_{K'}}$  and of  $\mathcal{G}'$ : more precisely, one has the following

**Proposition 3.2.6.** *For any semi-abelian  $K$ -variety  $G$  and any finite separable extension  $K'/K$ ,  $c(G, K') = 0$  if and only if  $h(G, K')$  is an open immersion.*

We add some details to the proof in [HN11, Proposition 4.16]. Any open immersion of group schemes induces an isomorphism of Lie algebras, the “if” direction is clear.

Conversely, assume  $c(G, K') = 0$ ; letting  $H = \ker h(T, K')$  one has  $\mathrm{Lie} H_k = 0$  and  $\mathrm{Lie} H_K = 0$  by [LLR04, Proposition 1.1(a)]. As a consequence, by [DG70, II.§5, Proposition 1.4] both  $H_k$  and  $H_K$  are étale; moreover, by [LLR04, Proposition 1.1(e)],  $H$  is smooth over  $\mathcal{O}_K$ , and in particular flat and locally of finite presentation. Furthermore, since  $H_k$  and  $H_K$  are étale, by [Autb, 02GL] they are a disjoint union of spectra of finite separable extensions of  $k$  and  $K$ , respectively, which by [Autb, 02GM] implies that  $H$  is étale over  $\mathcal{O}_K$ . Therefore,  $h(T, K')$  is étale as well; but an étale morphism of smooth  $\mathcal{O}_K$ -schemes which induces the identity on generic fibres is an open immersion [BLR90, Theorem 2.3/2’].

*Remark 3.2.7.* Let  $K \subseteq K_1 \subseteq K_2$  be separable extensions of discretely valued fields inside  $K^{sep}$  with finite ramification indices  $e_1$  and  $e_2$ ; let  $G$  be a semi-abelian  $K$ -variety and denote by  $G_i$  its base-change to  $K_i$ . By the universal property of Néron models the diagram

$$\begin{array}{ccc} (\mathcal{G}_{\mathcal{O}_{K_1}}) \times_{\mathcal{O}_{K_1}} \mathcal{O}_{K_2} & \xrightarrow{h(G, K_1) \times_{\mathcal{O}_{K_1}} \text{id}} & (\mathcal{G}_1)_{\mathcal{O}_{K_2}} \\ \downarrow \wr & & \downarrow h(G_1, K_2) \\ \mathcal{G}_{\mathcal{O}_{K_2}} & \xrightarrow{h(G, K_2)} & \mathcal{G}_2 \end{array} \quad (3.1)$$

commutes, because both compositions are the identity on generic fibres. If  $G$  acquires semi-abelian reduction after base-change to  $K_1$  (and thus to  $K_2$ ), then by Theorem 3.2.4 and Remark 3.2.2(iv) one has

$$c_i(\text{Lie } h(G, K_2)) = \frac{e_2}{e_1} c_i(\text{Lie } h(G, K_1)),$$

whence

$$c_i(G, K_2) = \frac{1}{e_2} c_i(\text{Lie } h(G, K_2)) = \frac{1}{e_1} c_i(\text{Lie } h(G, K_1)) = c_i(G, K_1)$$

for all  $i$ .

For a torus  $T$  over a discretely valued field with perfect residue field, a beautiful theorem by Chai and Yu [CY01] relates the base-change conductor of  $T$  to the Artin conductor of its character group, whose definition we briefly recall.

**Definition 3.2.8.** Let  $L/K$  be a finite Galois extension of discretely valued fields with separable residue extension,  $\Gamma = \text{Gal}(L/K)$  and, for any non-negative integer  $i$ , let  $\Gamma_i$  be the  $i$ -th ramification group with lower filtration [Ser13, §IV.1]. For any finite dimensional  $\mathbb{Q}$ -vector space  $V$  with  $\Gamma$ -action, the Artin conductor of  $V$  is the non-negative rational number

$$\text{Art}(V) = \sum_{i=0}^{\infty} \frac{1}{[\Gamma_0 : \Gamma_i]} \dim(V/V^{\Gamma_i});$$

the tame part of the Artin conductor of  $V$  is the integer  $\dim V - \dim V^{\Gamma_0}$ , and the wild part of the Artin conductor of  $V$ , also called its Swan conductor, is the difference between the Artin conductor and its tame part, i.e. the sum

$$\text{Sw}(V) = \sum_{i=1}^{\infty} \frac{1}{[\Gamma_0 : \Gamma_i]} \dim(V/V^{\Gamma_i}).$$

It is trivial to see that  $\text{Art}(V) = 0$  (resp.  $\text{Sw}(V) = 0$ ) if and only if  $\Gamma$  (resp.  $\Gamma_1$ ) acts trivially on  $V$ .

The (tame/wild part of) the Artin conductor of a torus  $T$  over a discretely valued field with (say) perfect residue field is defined in terms of that of its character  $\Gamma$ -module, where  $\Gamma$  is the Galois group of any of its Galois splitting extensions.

**Theorem 3.2.9** (Chai-Yu). *For any torus  $T$  over a discretely valued field with perfect residue field, one has*

$$c(G) = \frac{\text{Art}(T)}{2}.$$

A simpler proof in the case of finite residue field was independently obtained by de Shalit in the Appendix to [CY01]. Recently, Overkamp and Suzuki provided a shorter proof of the general result by means of more sophisticated techniques [OS23, Theorem 5.16], and also showed that the equality in the statement generally fails if the residue field of the ground field is imperfect [OS23, Remark 4.13].

Other interesting choices for  $K'$  are related to the behaviour of the Néron model under tame base-change of the ground field. Before discussing these, it is convenient to state the following easy lemma, which allows one to assume that  $K$  is strictly henselian:

**Lemma 3.2.10.** *Let  $K$  be a discretely valued field with strict henselization  $K^{sh}$ ,  $K'/K$  a finite separable extension. For any semi-abelian  $K$ -variety  $G$ , the diagram (with self-explanatory notation)*

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'}^{sh} & \xrightarrow{h(G, K') \times_{\mathcal{O}_{K'}} \mathcal{O}_{K'}^{sh}} & \mathcal{G}' \times_{\mathcal{O}_{K'}} \mathcal{O}_{K'}^{sh} \\ \downarrow h(G, K^{sh}) \times_{\mathcal{O}_K^{sh}} \mathcal{O}_{K'}^{sh} & & \downarrow h(G', (K')^{sh}) \\ \mathcal{G}^{sh} \times_{\mathcal{O}_K^{sh}} \mathcal{O}_{K'}^{sh} & \xrightarrow{h(G^{sh}, (K')^{sh})} & (\mathcal{G}')^{sh} \end{array}$$

*commutes and the vertical morphisms are isomorphisms. The same statement holds replacing  $\mathcal{O}_K^{sh}$  by its completion.*

Since the canonical morphism  $\mathcal{O}_K \hookrightarrow \mathcal{O}_K^{sh}$  is unramified, the first sentence follows trivially from Theorem 3.1.2, recalling that  $K' \cdot K^{sh} = (K')^{sh}$ ; similarly for the last one.

*For the rest of this section, we shall thus assume that  $K$  is strictly henselian with residue characteristic exponent  $p$ .*

Let  $\mathbb{N}_{(p)}$  be the set of positive integers coprime to  $p$ . For  $d \in \mathbb{N}_{(p)}$ , there is thus a unique degree- $d$  extension  $K(d)/K$  of  $K$  inside  $K^{sep}$ . We write  $G(d)$  for the

base-change

$$G(d) := G \times_K K(d),$$

$h(G, d)$  and  $c_i(G, d)$  for  $h(G, K(d))$  and  $c_i(G, K(d))$ .

**Lemma 3.2.11.** *Let  $G$  be a semi-abelian  $K$ -variety. For any sequence  $(d_n)_{n \in \mathbb{N}}$  of positive integers coprime to  $p$  with  $d_n | d_{n+1}$  for all  $n$  and any  $i = 1, \dots, \dim G$ , the sequence  $(c_i(G, d_n))_{n \in \mathbb{N}}$  is bounded and monotonically increasing.*

*If  $(d_n)_n$  and  $(d'_n)_n$  are two such sequences whose values are, in addition, cofinal in  $\mathbb{N}_{(p)}$  ordered by divisibility, one has*

$$\lim_{n \rightarrow \infty} c_i(G, d_n) = \lim_{n \rightarrow \infty} c_i(G, d'_n)$$

for all  $i$ .

Before proving the lemma, we conclude this round of definitions.

**Definition 3.2.12.** Let  $G$  be a  $g$ -dimensional semi-abelian  $K$ -variety.

- (i) For any  $d \in \mathbb{N}_{(p)}$ , the number  $j_i(G, d) := dc_i(G, d)$  is the  $i$ -th Edixhoven  $d$ -jump of  $G$ .
- (ii) The function

$$\text{ord}_G: \mathbb{N}_{(p)} \rightarrow \mathbb{N}, \quad d \mapsto \sum_{i=1}^g j_i(G, d)$$

is called the order function of  $G$ .

- (iii) For any sequence  $(d_n)_n$  as in the previous lemma whose values are cofinal in  $\mathbb{N}_{(p)}$  ordered by divisibility, the limit

$$j_i(G) := \lim_{n \rightarrow \infty} c_i(G, d_n) = \lim_{n \rightarrow \infty} \frac{j_i(G, d_n)}{d_n}$$

is called the  $i$ -th Edixhoven jump of  $G$ .

- (iv) The sum  $\sum_{i=1}^g j_i(G)$  is called the tame base-change conductor of  $G$  and denoted by  $c_{\text{tame}}(G)$ ; the difference  $c(G) - c_{\text{tame}}(G)$  is called the wild base-change conductor of  $G$  and denoted by  $c_{\text{wild}}(G)$ .

To avoid confusion, let us stress that the  $i$ -th Edixhoven jump is the limit of a sequence  $(c_i(G, d_n))_n$  of Chai's divisors, and not of Edixhoven's  $d$ -jumps  $(j_i(G, d_n))_n$  (this sequence will usually not even be bounded).

*Remarks 3.2.13.* (i) The original definition of Edixhoven's jumps [Edi92, Remark 5.4.5] uses Edixhoven's filtration, as explained below. The fact that the previous definition agrees with Edixhoven's is the content of the comparison theorem by Edixhoven, Halle and Nicaise (Theorem 3.2.17).

(ii) The introduction of the tame and wild base-change conductors is due to Halle and Nicaise [HN16, 6.1.3.3,12]. These can be regarded as refinements of the base-change conductor, and as the names suggest are related the tame and wild ramification of the semi-abelian variety; we shall come back to this aspect later in Section 4.1. The order function was defined differently - but, by [HN11, Proposition 7.5], equivalently - in [HN11, Definition 7.2].

We now turn to the proof of Lemma 3.2.11. The proof appearing in the literature [HN16, Corollary 6.1.3.11] appeals to the comparison theorem 3.2.17. We provide instead a direct proof, which is entirely elementary.

*Proof of Lemma 3.2.11:* Given positive integers  $d_1$  and  $d_2$  coprime to  $p$  with  $d_1|d_2$ , taking  $K_l = K(d_l)$  ( $l = 1, 2$ ) in diagram (3.1) and applying Proposition 3.2.3 and Remark 3.2.2(iv) one obtains

$$j_i(G, d_2) \geq \max\{(d_2/d_1)j_i(G, d_1), j_i(G(d_1), d_2)\},$$

whence monotonicity of any sequence  $(c_i(G, d_n))_n$  as in the statement.

For any  $d \in \mathbb{N}_{(p)}$ , taking instead  $K_1 = K(d)$  and  $K_2/K(d)$  to be a splitting extension of  $G(d)$  (so that, in particular,  $G$  acquires semi-abelian reduction over  $K_2$ ), the same argument, together with an application of Theorem 3.2.4, yields

$$e(K_2/K)c_i(G) = \frac{e(K_2/K)}{d}j_i(G, d) + e(K_2/K(d))c_i(G(d)),$$

whence

$$c_i(G) = c_i(G, d) + \frac{c_i(G(d))}{d}$$

for all  $i$ : in particular  $c_i(G, d) \leq c_i(G)$ , which shows that any sequence  $(c_i(G, d_n))_n$  as in the statement is bounded.

Finally, let  $(d_n)_n$  and  $(d'_n)_n$  two sequences as in the statement which are cofinal in  $\mathbb{N}_{(p)}$  ordered by divisibility: then for any  $m$  there exists  $N$  such that  $d_m|d'_N$  and  $d'_m|d_N$ , and the argument in the first paragraph above shows that  $c_i(G, d'_m) \leq c_i(G, d_N)$  and  $c_i(G, d_m) \leq c_i(G, d'_N)$ . It follows that the sequences  $(c_i(G, d_n))_n$  and  $(c_i(G, d'_n))_n$  have the same supremum.  $\square$

Using an alternative characterisation of the  $d$ -jumps (Theorem 3.2.17), we shall prove later than one can actually compute the jumps of a semi-abelian variety using *any* sequence in  $\mathbb{N}_{(p)}$  tending to infinity (Corollary 3.2.23).

If  $G$  is a tamely ramified semi-abelian variety, its jumps are easily computed:

**Proposition 3.2.14.** *The jumps of a  $g$ -dimensional semi-abelian  $K$ -variety  $G$  which acquires semi-abelian reduction after the tame extension  $K(d)/K$  are*

$$c_1(G, d), \dots, c_g(G, d).$$

This is [HN11, Corollary 4.18]; it is a non-trivial result if one takes Definition 3.2.16 as the starting definition of jumps, while it is straightforward with the definition above (of course, the comparison with the jumps of Edixhoven's filtration (3.2) remains the hard result): consider a sequence  $(d_n)_n$  in  $\mathbb{N}_{(p)}$  with  $d_n | d_{n+1}$  which is cofinal in  $\mathbb{N}_{(p)}$ ; by assumption, there exists  $N$  such that  $d | d_n$  if  $n \geq N$ . Since  $G$  acquires semi-abelian reduction over the extension  $K(d)/K$ , Remark 3.2.7 shows that  $c_i(G, d_n) = c_i(G, d)$  for  $n \geq N$ .

### Edixhoven's filtration

We now explain the definition of Edixhoven's jumps given in [Edi92] and then in [HN11, Definition 4.12] and [HN16, 6.1.3], together with some results on which it rests; we state the comparison theorem by Edixhoven, Halle and Nicaise (Theorem 3.2.17) which connects this definition with the one given above. We finally show that Edixhoven's jumps can be computed using any sequence in  $\mathbb{N}_{(p)}$  tending to infinity (Corollary 3.2.23), a fact that, as far as we know, had not been noticed before.

For the sake of precision, let us point out that the statements presented below are essentially due to Edixhoven [Edi92], who worked with abelian varieties, but many missing proofs were supplied by Halle and Nicaise in [HN11], who also extended the theory to semi-abelian varieties. For ease of writing, we shall sometimes attribute to Edixhoven results which he proved for abelian varieties and can be easily extended to the semi-abelian case. Also observe that the assumption that  $k = \bar{k}$  made at the start of [HN11] is not necessary for the proof of the next results (witness the fact that this restriction was lifted in [HN16]).

Since most of these results will not be needed in the rest of the thesis, the presentation will be a bit sketchier than the rest, but we feel an introduction to the subject which does not mention them would be incomplete. Many of the omitted

details and proofs can be found in [HN11, Section 4.1], whose presentation we roughly follow.

As above,  $K$  will be a strictly henselian discretely valued field,  $G$  a semi-abelian  $K$ -variety. Recall that  $p$  is the characteristic exponent of the residue field  $k$  of  $K$  and  $\mathbb{N}_{(p)}$  the set of positive integers coprime with  $p$ ; for  $d \in \mathbb{N}_{(p)}$ ,  $K(d)/K$  denotes the unique totally ramified extension of  $K$  in  $K^{sep}$  of degree  $d$ . It is Galois, and we identify its Galois group with the group  $\mu_d$  of  $d$ -roots of unity.

The starting point is a remarkable description, due to Edixhoven [Edi92, Theorem 4.2], of the Néron model of  $G$  in terms of the fixed points of the Weil restriction  $\text{Res}_{\mathcal{O}_{K(d)}/\mathcal{O}_K} \mathcal{G}(d)$  under the natural  $\mu_d$ -action.

More precisely, fix a generator  $\zeta$  of  $\mu_d$  and observe first that  $\mu_d$  acts on any  $K(d)$ -algebra  $A$  by multiplication (i.e.  $\zeta \cdot a = \zeta a$  for all  $a \in A$ ); for any affine  $K(d)$ -scheme  $Z$  one obtains a right  $\mu_d$ -action on  $Z$  over  $K(d)$  such that for any  $K(d)$ -algebra  $A$  and any morphism  $\phi \in \text{Hom}_{K(d)}(\text{Spec } A, Z)$ ,

$$\phi \cdot \zeta = \zeta_Z^{-1} \circ \phi \circ \zeta_A$$

(here and in the following, we use subscripts to distinguish the action by  $\zeta$  on different schemes or rings). In particular,  $\mu_d$  acts on  $G(d)$  over  $K(d)$ , and by the Néronian property this action extends to a  $\mu_d$ -action on  $\mathcal{G}(d)$  over  $\mathcal{O}_{K(d)}$ . One thereby obtains a right  $\mu_d$ -action on the Weil restriction

$$X_d := \text{Res}_{\mathcal{O}_{K(d)}/\mathcal{O}_K} \mathcal{G}(d),$$

defined on points by

$$f \cdot \zeta = \zeta_{\mathcal{G}(d)} \circ f \circ (\text{id}_Z \times_{\mathcal{O}_K} \zeta_{\mathcal{O}_{K(d)}}^{-1})$$

for all  $\mathcal{O}_K$ -schemes  $Z$  and all  $f \in \text{Hom}_{\mathcal{O}_K}(Z, X_d) = \text{Hom}_{\mathcal{O}_{K(d)}}(Z \times_{\mathcal{O}_K} \mathcal{O}_{K(d)}, \mathcal{G}(d))$ .

For any finite group  $\Gamma$  acting on an  $S$ -scheme  $Z$  (where  $S$  is any base-scheme), Edixhoven defines in [Edi92, Section 3] the presheaf  $Z^\Gamma$  on  $\text{Sch}/S$  of fixed points by  $T/S \mapsto Z(T)^\Gamma$ , and shows that it is representable by a subscheme of  $Z$  [Edi92, Proposition 3.1], which is smooth if the order of  $\Gamma$  is invertible on  $Z$  [Edi92, Proposition 3.4]. It is easy to see that the fixed point functor commutes with base-change [Edi92, Proposition 3.1].

He then proved the following

**Theorem 3.2.15** (Edixhoven, [HN11, Proposition 4.1]). *With the same notation as above,  $X_d^{\mu_d}$  is a Néron model for  $G$  over  $\mathcal{O}_K$ .*

For any  $i = 0, \dots, d$ , consider now the morphism of  $k$ -group schemes

$$\theta_i^d: (X_d)_k \rightarrow \operatorname{Res}_{\frac{\mathcal{O}_{K(d)}}{\mathfrak{m}_d^i}/k} \mathcal{G}(d) \times_{\mathcal{O}_{K(d)}} \frac{\mathcal{O}_{K(d)}}{\mathfrak{m}_d^i}$$

given on points by the natural morphism

$$\mathcal{G}(d) \left( \frac{A[x]}{(x^d)} \right) \rightarrow \mathcal{G}(d) \left( \frac{A[x]}{(x^i)} \right)$$

induced by the truncation  $A[x]/(x^d) \rightarrow A[x]/(x^i)$  for any  $k$ -algebra  $A$ . Noticing that the right-hand side is 0 for  $i = 0$ , one obtains a filtration

$$(X_d)_k = \ker \theta_0^d \supseteq \ker \theta_1^d \supseteq \dots \supseteq \ker \theta_d^d = 0;$$

taking the  $\mu_d$  invariants and observing that  $((X_d)_k)^{\mu_d} = ((X_d)^{\mu_d})_k = \mathcal{G}_k$  by Edixhoven's theorem, this finally yields a descending filtration by closed subgroup schemes, called *Edixhoven's filtration*,

$$\mathcal{G}_k = F_d^0 \mathcal{G}_k \supseteq F_d^1 \mathcal{G}_k \supseteq \dots \supseteq F_d^d \mathcal{G}_k = 0, \quad (3.2)$$

where

$$F_d^i \mathcal{G}_k = (\ker \theta_i^d)^{\mu_d}.$$

**Definition 3.2.16.** One says that a natural number  $j \in \{0, \dots, d-1\}$  is a *jump* of *Edixhoven's filtration with multiplicity  $m > 0$*  if the dimension of the quotient

$$\operatorname{Gr}_d^j \mathcal{G}_k := F_d^j \mathcal{G}_k / F_d^{j+1} \mathcal{G}_k$$

is  $m$ .

The following comparison result [HN11, Theorem 4.17] follows from [HN11, Theorem 4.10], which was stated without proof as [Edi92, Remark 5.4.6].

**Theorem 3.2.17** (Edixhoven, Halle-Nicaise). *Let  $G$  be a semi-abelian variety over a strictly henselian field  $K$ . For any  $d \in \mathbb{N}_{(p)}$ , the  $d$ -jumps of  $G$  are exactly the jumps of the filtration (3.2), counted with multiplicities.*

*Remark 3.2.18.* In comparing with [HN11], it should be noted that the definition of Chai's divisors there is a little different, in that the authors only consider the non-zero ones [HN11, Definition 4.15]. The equality between the multiplicity of 0 as a jump of  $G$  and that of 0 as a jump of Edixhoven's filtration follows from the equality for non-zero jumps, together with the fact that the sums of the multiplicities of all the jumps of  $G$  equals the dimension of  $G$ , and that the same holds for the sum of the multiplicities of the jumps of Edixhoven's filtration.

The graded pieces of Edixhoven's filtration also have an interpretation in terms of the Lie algebra of  $\mathcal{G}(d)_k$ , viewed as a  $k[\mu_d]$ -algebra:

**Proposition 3.2.19** ([HN11, Corollary 4.8]). *In the same notation as above, there are canonical isomorphisms*

$$\mathrm{Gr}_d^0(\mathcal{G}_k) \xrightarrow{\sim} \mathcal{G}(d)_k^{\mu_d}$$

and, for  $0 < i < d$ ,

$$\mathrm{Gr}_d^i(\mathcal{G}_k) \xrightarrow{\sim} \mathrm{Lie}(\mathcal{G}(d)_k)[i] \otimes_k \left( \frac{\mathfrak{m}_d}{\mathfrak{m}_d^2} \right)^{\otimes i},$$

where  $\mathrm{Lie}(\mathcal{G}(d)_k)[i]$  is the vector subspace of  $\mathrm{Lie}(\mathcal{G}(d)_k)$  on which  $\mu_d$  acts by  $v \cdot \zeta = \zeta^i v$ , and the right-hand side is viewed as a vector group  $k$ -scheme.

In particular, all the  $i$ -th graded pieces are smooth, and unipotent if  $i > 0$ .

The various Edixhoven filtrations on  $d$  ranging in  $\mathbb{N}_{(p)}$  can be combined into a single one, as follows: for any rational number  $\alpha = a/b$  with  $a < b$  and  $b \in \mathbb{N}_{(p)}$ , let

$$F^\alpha \mathcal{G}_k = F_b^a \mathcal{G}_k,$$

which is independent of the particular choice of  $a$  and  $b$  by [HN11, Lemma 4.11]. For any real number  $\rho \in [0, 1[$ , let  $F^{>\rho} \mathcal{G}_k = F^\alpha \mathcal{G}_k$ , where  $\alpha = a/b$  is any rational number as above such that for any other such rational number  $\beta \in [\rho, \alpha[$ ,  $F^\alpha \mathcal{G}_k = F^\beta \mathcal{G}_k$ . Notice that this is well-defined, because by the previous proposition and dimension reasons there cannot be infinitely many different pieces in the filtration  $F^\bullet \mathcal{G}_k$ . Defining similarly  $F^{<\rho} \mathcal{G}_k$  for any  $\rho \in ]0, 1]$  and letting further  $F^{<0} \mathcal{G}_k = \mathcal{G}_k$ ,  $F^{>1} \mathcal{G}_k = 0$ , one has:

**Definition 3.2.20.** A real number  $j \in [0, 1]$  is a *jump* of the filtration  $F^\bullet \mathcal{G}_k$  with *multiplicity*  $m > 0$  if  $F^{<\rho} \mathcal{G}_k / F^{>\rho} \mathcal{G}_k$  has dimension  $m$ .

We now relate the jumps in the filtration  $F^\bullet \mathcal{G}_k$  with those on the filtration

**Lemma 3.2.21.** *Let  $G$  be a semi-abelian variety over a strictly henselian field  $K$ . Let  $\{(\gamma_1; m_1), \dots, (\gamma_r; m_r)\}$  be the multiset of jumps in Edixhoven's filtration on  $\mathcal{G}_k$ , with  $\gamma_i < \gamma_{i+1}$  for all  $i$ . For  $d \in \mathbb{N}_{(p)}$ ,  $d \gg 0$ ,  $J_d(G)$  is of the form  $\{(a'_1; m_1), \dots, (a'_r; m_r)\}$  where  $a'_i = a_i$  or  $a'_i = a_i - 1$  and*

$$a_i = \max\{m \in \mathbb{N} \cup \{0\} \mid m/d \leq \gamma_i\};$$

moreover,  $a'_i < a'_{i+1}$ .

*Proof.* Observe first that, since in any case  $\gamma_i \leq 1$ ,  $a_i < d$  unless  $a_i/d = \gamma_i = 1$ , and if so, then  $F^{>\gamma_i} \mathcal{G}_k = 0 = F_d^{a_i} \mathcal{G}_k$ .

By definition,  $\gamma_i$  being a jump of multiplicity  $m_i$  entails that

$$\dim F^{<\gamma_i} \mathcal{G}_k = \dim F^{>\gamma_i} \mathcal{G}_k + m_i.$$

For each  $i$  there exists some  $\epsilon_i > 0$  such that  $\dim F^{<\gamma_i} \mathcal{G}_k = \dim F^\alpha \mathcal{G}_k$  for any  $\alpha \in [\gamma_i - \epsilon_i, \gamma_i]$ ; since we are only interested in large values of  $d$ , we shall henceforth assume that  $1/d < \epsilon_i$  for all  $i$ .

By the definition of  $a_i$ , one has  $\gamma_i - a_i/d < 1/d$ ; if  $a_i/d \neq \gamma_i$ , this implies in particular that  $\dim F^{<\gamma_i} \mathcal{G}_k = \dim F_d^{a_i} \mathcal{G}_k$  and, by the definition of  $a_i$ , that

$$\dim F_d^{a_i+1} \mathcal{G}_k \leq \dim F^{>\gamma_i} \mathcal{G}_k = \dim F^{<\gamma_i} \mathcal{G}_k - m_i = \dim F_d^{a_i} \mathcal{G}_k - m_i$$

(notice that the expression  $F_d^{a_i+1} \mathcal{G}_k$  makes sense by the observation at the beginning of the proof), which shows that  $a_i$  is a  $d$ -th jump of  $G$  with multiplicity at least  $m_i$ . If  $a_i/d = \gamma_i$ , two situations can occur: either  $\dim F^{<\gamma_i} \mathcal{G}_k = \dim F_d^{a_i} \mathcal{G}_k$ , in which case again  $\dim F_d^{a_i+1} \mathcal{G}_k \leq \dim F^{>\gamma_i} \mathcal{G}_k = \dim F^{<\gamma_i} \mathcal{G}_k - m_i$  and  $a_i$  is a  $d$ -jump of  $G$ , or  $\dim F^{>\gamma_i} \mathcal{G}_k = \dim F_d^{a_i} \mathcal{G}_k$ , in which case

$$\dim F_d^{a_i} \mathcal{G}_k \leq \dim F^{<\gamma_i} \mathcal{G}_k - m_i = \dim F_d^{a_i-1} \mathcal{G}_k - m_i,$$

where the last equality follows because we are taking  $1/d < \epsilon_i$ . In this case,  $a_i - 1$  is then a  $d$ -jump of  $G$  with multiplicity at least  $m_i$ . Notice also that in this situation one must have  $a_{i-1} \leq a_i - 1$ , because

$$\dim F_d^{a_i-1} \mathcal{G}_k \geq \dim F^{>\gamma_{i-1}} \mathcal{G}_k \geq \dim F^{<\gamma_i} \mathcal{G}_k = \dim F_d^{a_i-1} \mathcal{G}_k;$$

consequently, the map  $i \mapsto a'_i$  is increasing.

We now claim that (possibly after increasing  $d$ ) the map  $i \mapsto a'_i$  is actually *strictly* increasing (so that different Edixhoven jumps give rise to different  $d$ -jumps), whence the result will follow, because both the multiplicities of the jumps in Edixhoven's filtration and those of the  $d$ -jumps sum to  $\dim G$ . Since we already know that such mapping is increasing, we need to show that an equality  $a'_i = a'_{i+1}$  can only occur for small values of  $d$ . Assume thus that such equality holds:

- one cannot have  $a'_i = a_i - 1 = a_{i+1} = a'_{i+1}$ , because by definition  $a_i \leq a_{i+1}$  (since  $\gamma_i < \gamma_{i+1}$ );
- if  $a_i = a_{i+1}$ , then  $\gamma_{i+1} - \gamma_i < 1/d$ , which can only happen for small values of  $d$ ;

- the last remaining possibility is  $a'_i = a_i = a_{i+1} - 1 = a'_{i+1}$ ; by the discussion above, the equality  $a'_{i+1} = a_{i+1} - 1$  implies that  $\gamma_{i+1} = a_{i+1}/d$ . On the other hand, by the definition of  $a_i$  one has  $a_i/d \leq \gamma_i$ , whence, substituting,  $\gamma_{i+1} - 1/d \leq \gamma_i$ ; again, this can only occur for finitely many values of  $d$ .  $\square$

Since for any  $d \in \mathbb{N}_{(p)}$  and any index  $i$  one has  $j_i - a'_i/d \leq 2/d$ , one immediately obtains the following corollaries; the former was stated (without proof) in [HN16, 6.1.3.7].

**Corollary 3.2.22.** *Let  $G$  be a semi-abelian variety over a strictly henselian field  $K$ . The jumps of  $G$  are exactly the jumps of the filtration  $F^\bullet \mathcal{G}_k$ , counted with multiplicities.*

**Corollary 3.2.23.** *For sequence  $(d_n)_n$  in  $\mathbb{N}_{(p)}$  with  $d_n \xrightarrow{n \rightarrow \infty} \infty$  and any  $i = 1, \dots, \dim G$ ,*

$$\frac{j_i(G, d_n)}{d_n} \xrightarrow{n \rightarrow \infty} j_i(G).$$

### Computation of Chai's elementary divisors and Edixhoven's jumps: an overview

While the computation of the jumps of a tamely-ramified one is straightforward (Proposition 3.2.14), that of the jumps of a wildly ramified semi-abelian  $K$ -variety is extremely difficult, to the point that, even assuming  $k = \bar{k}$ , for most semi-abelian varieties it is not even known whether they are rational numbers. An affirmative answer to this question (raised already by Edixhoven [Edi92]) is known by work of Halle and Nicaise for Jacobians of smooth, projective, geometrically connected curves of positive genus over a strictly henselian field with algebraically closed residue field  $k$  [HN16, Theorem 6.3.1.3]; remarkably, it turns out that not only are the jumps rational, but they only depend on the combinatorial data of the special fibre of an sncd-model of the curve (and in particular, do not depend on  $\text{char } k$ ). Using methods from logarithmic geometry, the paper [EHN15] even gives explicit formulas for the jumps in terms of these data (*loc. cit.*, Theorem 5.4.1), while a different proof of the rationality of the jumps of such Jacobians was developed in [KMW25].

For non-split tori, the only computation to be found in the literature is (to our knowledge) that of the jumps of an induced torus, for  $k = \bar{k}$ , performed by Overkamp [Ove19, Corollary 2.2.2]. In what follows we shall extend his result in two directions: firstly, removing the assumption on the residue field (Proposition

4.3.8 and Theorem 4.3.9), and secondly by showing, thanks to a suggestion by Overkamp, that the jumps of a torus which is a direct factor of a  $K$ -rational variety are rational as well (more precisely, in Corollary 4.4.10 we bound their denominator in terms of the ramification degree of the minimal splitting extension).

It might be instructive to mention several difficulties that bar some naive approaches to the study of the rationality of jumps from working, while also describing some useful positive results that do hold; this will also allow us to introduce some of the ideas that will be used later. We mainly focus on the case of tori, but analogous difficulties arise for abelian varieties.

Since the jumps of semiabelian varieties naturally come with multiplicities, multisets are especially convenient in dealing with them.

*Notation 3.2.24.* Recall that a multiset is a pair  $\mathcal{M} = (|\mathcal{M}|, \mu)$  where  $|\mathcal{M}|$  is a set (called the support of  $\mathcal{M}$ ) and  $\mu: |\mathcal{M}| \rightarrow \mathbb{N}$  is a function, which is to be thought of as assigning a multiplicity to each element of  $|\mathcal{M}|$ . We shall also write  $\mathcal{M}$  with the compact notation  $\mathcal{M} = \{(a; \mu(a))\}_{a \in |\mathcal{M}|}$ .

A sub-multiset of  $(\mathcal{M}, \mu)$  is a multiset  $(\mathcal{N}, \nu)$  such that  $|\mathcal{N}| \subseteq |\mathcal{M}|$  and, for any  $x \in |\mathcal{N}|$ ,  $\nu(x) \leq \mu(x)$ . In this situation we write  $(\mathcal{N}, \nu) \subseteq (\mathcal{M}, \mu)$ , or simply  $\mathcal{N} \subseteq \mathcal{M}$ .

The sum of a finite collection  $\{\mathcal{M}_i\}_{i \in I}$  of multisets is the multiset

$$\bigsqcup_{i \in I} \mathcal{M}_i := \left( \bigcup_{i \in I} |\mathcal{M}_i|, \mu \right)$$

with, for any  $x$  in the support,  $\mu(x) = \sum_{i \in I} \mu_i(x)$ , where  $\mu_i: \bigcup_{i \in I} |\mathcal{M}_i| \rightarrow \mathbb{N}$  extends by 0 the multiplicity function of  $\mathcal{M}_i$ . Remark that the support of  $\bigsqcup_i \mathcal{M}_i$  is *not* the disjoint union of the supports  $|\mathcal{M}_i|$ : so, some element  $x$  in the sum of two multisets  $\mathcal{M}_1$  and  $\mathcal{M}_2$  might belong to both, and in this case its multiplicity in the multiset sum is the sum of the multiplicities.

For a semi-abelian  $K$ -variety  $G$  and a positive integer  $d$  coprime to  $p$ , denote by  $J(G)$  (resp.  $J_d(G)$ ) the multiset of jumps (resp.  $d$ -jumps) of  $G$ .

First, observing that every torus  $T$  can be written both as a subtorus and as a subquotient of products of induced tori, one might hope to exploit some additivity property of the multisets of jumps; in this direction, one has the following result, due to Overkamp:

**Proposition 3.2.25** (Overkamp). *Let  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  be a short exact sequence of semi-abelian  $K$ -varieties.*

(i) If the induced sequence of Néron models  $0 \rightarrow \mathcal{A}'(d) \rightarrow \mathcal{A}(d) \rightarrow \mathcal{A}''(d) \rightarrow 0$  is exact for some  $d \in \mathbb{N}_{(p)}$ , then  $J_d(A) = J_d(A') \uplus J_d(A'')$ .

(ii) If  $J_d(A) = J_d(A') \uplus J_d(A'')$  for infinitely many values of  $d$  in  $\mathbb{N}_{(p)}$ , then  $J(A) = J(A') \uplus J(A'')$ .

*Proof.* The first point is precisely the statement of [Ove19, Proposition 2.1.2]. For the second one, pick any sequence  $(d_n)_n$  in  $\mathbb{N}_{(p)}$  tending to infinity and such that  $J_{d_n}(A) = J_{d_n}(A') \uplus J_{d_n}(A'')$ . In particular, there exist jointly surjective monotonically increasing functions

$$\alpha_n: \{1, \dots, \dim A'\} \rightarrow \{1, \dots, \dim A\}$$

$$\beta_n: \{1, \dots, \dim A''\} \rightarrow \{1, \dots, \dim A\}$$

such that  $j_{d_n, i}(A') = j_{d_n, \alpha_n(i)}(A)$  for all  $i = 1, \dots, \dim A'$  and  $j_{d_n, i}(A'') = j_{d_n, \beta_n(i)}(A)$  for all  $i = 1, \dots, \dim A''$ . Since  $(d_n)_n$  takes infinitely many different values, while there are only finitely many such pairs of functions  $(\alpha, \beta)$ , up to passing to a subsequence we can assume that  $\alpha_n$  and  $\beta_n$  are actually independent of  $n$  (and we can therefore drop the index  $n$ ). By Corollary 3.2.23, one obtains that

$$j_{\alpha(i)}(A) = \lim_{n \rightarrow \infty} \frac{j_{d_n, \alpha(i)}(A)}{d_n} = \lim_{n \rightarrow \infty} \frac{j_{d_n, i}(A')}{d_n} = j_i(A')$$

for any  $i = 1, \dots, \dim A'$ , and similarly  $j_{\beta(i)}(A) = j_i(A'')$  for all  $i = 1, \dots, \dim A''$ .  $\square$

This will be one of the main ingredients in the computation of jumps of tori which are direct factors of  $K$ -rational varieties (Corollary 4.4.10), but its application to arbitrary tori is impeded by the existence of short exact sequences of tori which induce a non-exact sequence of Néron models (Example 4.4.14).

Despite this fact,  $c$  and  $c_{\text{tame}}$  are additive on short exact sequences of tori, if  $k = \bar{k}$ : the additivity of  $c_{\text{tame}}$  (for tori) follows from its identification with one half the unipotent rank (see Proposition 4.1.3 and Corollary 4.1.4); that of  $c$  is the content of the following theorem, conjectured by Chai [Cha00] and proved by Overkamp and Suzuki [OS23, Theorem 2.11], who also constructed counterexamples [OS23, Section 4] to the statement for general semi-abelian varieties, or if the residue field of  $K$  is imperfect:

**Theorem 3.2.26** ([OS23, Theorem 2.11]). *Let  $0 \rightarrow T \rightarrow B \rightarrow A \rightarrow 0$  be a short exact sequence of semi-abelian varieties over a complete discretely valued field  $K$  with algebraically closed residue field and assume that  $T$  is a torus. Then*

$$c(B) = c(A) + c(T).$$

One might therefore hope that the values of the single elementary divisors of Chai's are additive as well (if  $k = \bar{k}$ ), but this is not known.

As a different strategy to reduce to the case of induced tori, one could also try to use the fact that for any  $K$ -torus  $T$  there exist induced tori  $T_1, \dots, T_r$  and  $T'_1, \dots, T'_s$  and an isogeny

$$T \times \prod_{i=1}^r T_i \rightarrow \prod_{j=1}^s T'_j,$$

as proved in [Ono61, Theorem 1.5.1]: if Edixhoven's jumps were isogeny-invariant, then their rationality for tori would follow. While  $c_{tame}$  is isogeny-invariant if the residue field of  $K$  is algebraically closed (Corollary 4.1.4), Overkamp showed [Ove19, 2.3] that the individual jumps are not if the isogeny is wild. Let us mention for completeness that the individual jumps are invariant under *tame* isogenies:

**Lemma 3.2.27.** *Let  $K'/K$  be a finite separable extension of discretely valued field fields,  $\alpha: G_1 \rightarrow G_2$  an isogeny of semi-abelian  $K$ -varieties. If  $\deg \alpha$  is invertible in  $\mathcal{O}_{K'}$ , then  $c_i(G_1, K') = c_i(G_2, K')$  for all  $i$ .*

*Proof.* Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Lie} \mathcal{G}_1 \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} & \xrightarrow{\mathrm{Lie} h(G_1, K')} & \mathrm{Lie} \mathcal{G}'_1 & \longrightarrow & \mathrm{coker} \mathrm{Lie} h(G_1, K') \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Lie} \mathcal{G}_2 \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} & \xrightarrow{\mathrm{Lie} h(G_2, K')} & \mathrm{Lie} \mathcal{G}'_2 & \longrightarrow & \mathrm{coker} \mathrm{Lie} h(G_2, K') \longrightarrow 0 \end{array},$$

where the vertical arrows are those induced by  $\alpha$ .

Let  $\beta: G_2 \rightarrow G_1$  be an isogeny such that  $\alpha \circ \beta$  and  $\beta \circ \alpha$  are multiplication by  $\deg \alpha$  [BLR90, Lemma 7.3/2]. One can consider the same diagram as above, with  $G_1$  and  $G_2$  swapped and  $\beta$  playing the role of  $\alpha$ ; by the universal property of Néron models, multiplication by  $\deg \alpha$  on  $G_i$  or  $G'_i$  extends to multiplication by  $\deg \alpha$  on the group schemes  $\mathcal{G}_i$  or  $\mathcal{G}'_i$ . The claim then follows since multiplication by  $\deg \alpha$  in the Lie algebras is an isomorphism if  $\alpha$  is invertible in  $K$ .  $\square$



## Chapter 4

# Jumps and conductors of tori

### 4.1 Some results on the tame and the wild base-change conductor of tori

In this section we show that, for tori over a strictly henselian discretely valued field with perfect residue field,  $c_{\text{tame}}$  is additive and isogeny-invariant (Corollary 4.1.4) and  $c_{\text{wild}}(T) = 0$  if and only if  $T$  is tamely ramified (Corollary 4.1.6), confirming a conjecture by Halle and Nicaise. As far as we know, these results were not previously proved in the literature.

Recall the following well-known

**Proposition 4.1.1.** *Let  $K$  be any field,  $T$  a  $K$ -torus with splitting Galois extension  $L/K$  with Galois group  $\Gamma$ ; let  $\chi$  be the character group of  $T$  and  $\text{tr}: \chi \rightarrow \chi^\Gamma$  be the trace map, defined by  $\text{tr}(x) = \sum_{\gamma \in \Gamma} \gamma.x$ .*

*The torus  $T$  has a unique maximal split subtorus  $T_{\text{spl}}$  and a unique maximal anisotropic subtorus  $T_a$ , whose character  $\Gamma$ -modules are isomorphic to  $\chi/\ker(\text{tr})$  and  $\chi/\chi^\Gamma$ , respectively. Moreover, the natural morphism  $\chi/\ker(\text{tr}) \rightarrow \chi^\Gamma$  induces an isogeny of tori.*

*Proof.* The last assertion is the only one which is not explicitly proved in [Wat79, Theorem 7.4], though it is immediately derived from that: first, since  $\mathbb{Z}$  is a principal ideal domain, the submodule  $\chi^\Gamma$  of  $\chi$  is again free of finite rank, and thus is the character group of a  $K$ -torus; moreover, by definition, the trace map factors through an injection  $\chi/\ker(\text{tr}) \rightarrow \chi^\Gamma$ , whose cokernel  $\chi^\Gamma/\text{Im}(\text{tr})$  is annihilated by  $\#\Gamma$  (because  $\text{tr}|_{\chi^\Gamma}$  is just multiplication by  $\#\Gamma$ ).  $\square$

As a consequence, one can interpret the number

$$u(T) := \dim T - \operatorname{rk}_{\mathbb{Z}} \chi(T)^{\Gamma}$$

as the rank of the maximal anisotropic subtorus (or of the maximal anisotropic quotient) of  $T$ . By (the proof of) [NX91, Theorem 1.3] this is also the rank of the maximal unipotent quotient of  $\mathcal{T}_k^0$  (so the unipotent rank of  $\mathcal{T}_k^0$  if  $k$  is perfect). It is not hard to see that this quantity is additive and isogeny-invariant:

**Proposition 4.1.2.** *Let  $K$  be any field.*

(i) *For any any short exact sequence  $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$  of  $K$ -tori, one has  $u(T) = u(T'') + u(T')$ .*

(ii) *For any isogeny of  $K$ -tori  $T_1 \rightarrow T_2$ , one has  $u(T_1) = u(T_2)$ .*

*Proof.* i): Let  $L$  be a common Galois splitting extension of the tori in the short exact sequence,  $\Gamma = \operatorname{Gal}(L/K)$ .

Since the rank of a free abelian group is additive in short exact sequences, it is enough to show that  $\chi(\cdot)^{\Gamma}$  is an additive quantity on the class of  $K$ -tori that become split over  $L$ .

The first terms of the cohomology long exact sequence associated to the character groups read

$$0 \rightarrow \chi(T'')^{\Gamma} \rightarrow \chi(T)^{\Gamma} \rightarrow \chi(T')^{\Gamma} \rightarrow H^1(\Gamma, \chi(T'')) \rightarrow \cdots,$$

where  $H^1(\Gamma, \chi(T''))$  is torsion as a higher cohomology group of a finite group [Ser13, VIII.2, Corollary 1]. Tensoring with  $\mathbb{Q}$  over  $\mathbb{Z}$  and using that for a free abelian group  $A$ ,  $\operatorname{rk}_{\mathbb{Z}}(A) = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ , one obtains the desired result.

ii): For any  $\mathbb{Z}$ -free  $\Gamma$ -module  $A$ ,  $(A \otimes_{\mathbb{Z}} \mathbb{Q})^{\Gamma} = A^{\Gamma} \otimes_{\mathbb{Z}} \mathbb{Q}$ , where  $\Gamma$  acts trivially on  $\mathbb{Q}$ ; the statement then follows since the isogeny  $T_1 \rightarrow T_2$  induces an isomorphism  $\chi(T_2) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \chi(T_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ .  $\square$

Halle and Nicaise interpreted geometrically the base-change conductor of a torus over a strictly henselian field with algebraically closed residue field:

**Proposition 4.1.3** ([HN16, Proposition 7.1.1.3]). *Let  $K$  be a strictly henselian field with algebraically closed residue field; for any  $K$ -torus  $T$ , one has*

$$c_{\text{tame}}(T) = \frac{u(T)}{2}$$

and

$$c_{\text{wild}}(T) = \frac{\operatorname{Sw}(T)}{2}.$$

Let us briefly recall their proof: by Theorem 3.2.9,  $c(T)$  equals one half the Artin conductor of  $T$ , which can be written as a sum of its tame part and of the Swan conductor; by definition, the tame part equals  $\dim T - \mathrm{rk}_{\mathbb{Z}} \chi(T)^{\Gamma}$ , whereby the family  $c_{\mathrm{tame}}(T(d))_{d \in \mathbb{N}_{(p)}}$  is bounded by  $\dim T$ , while  $\mathrm{Sw}(T(d)) = d\mathrm{Sw}(T)$  for all  $d \in \mathbb{N}_{(p)}$  as a consequence of Herbrand's theorem (see [HN16, Proposition 3.1.2.1]): therefore, for any cofinal sequence  $(d_n)_n$  in  $\mathbb{N}_{(p)}$  ordered by divisibility

$$\frac{\mathrm{Sw}(T)}{2} = \lim_{n \rightarrow \infty} \frac{c(T(d_n))}{d_n}.$$

By Remark 3.2.7 and the additivity of length, one also has

$$c(T) = c(T, d_n) + \frac{c(T(d_n))}{d_n}$$

for all  $n$ , whereby

$$\frac{\mathrm{Sw}(T)}{2} = \lim_{n \rightarrow \infty} \frac{c(T(d_n))}{d_n} = \lim_{n \rightarrow \infty} \left( c(T) - c(T, d_n) \right) = c_{\mathrm{wild}}(T).$$

The identity for  $c_{\mathrm{tame}}$  is then recovered by subtraction.  $\square$

**Corollary 4.1.4.** *If  $K$  is a strictly henselian field with algebraically closed residue field, then  $c_{\mathrm{tame}}$  is an additive invariant of  $K$ -tori and is isogeny-invariant.*

As mentioned at the end of the introduction, this is false for the individual values of the jumps.

We shall later extend the equality  $c_{\mathrm{tame}}(T) = u(T)/2$  to totally ramified tori that are direct factors of  $K$ -rational varieties, even if  $k$  is imperfect (Corollary 4.4.17); however, such equality fails in general if  $k$  is not algebraically closed:

*Example 4.1.5.* Consider a strictly henselian field  $K$  with imperfect residue field  $k$  and let  $T$  be the induced torus relative to a finite separable extension  $M/K$  admitting a non-trivial weakly unramified subextension  $F/K$  such that  $M/F$  is totally ramified; denote by  $L/K$  its Galois closure in  $K^{\mathrm{sep}}$  and by  $f$  its ramification index. Letting  $\Delta = \mathrm{Gal}(L/M)$ ,  $\Gamma = \mathrm{Gal}(L/K)$ , the character  $\Gamma$ -module  $\chi(T)$  is  $\mathbb{Z}[\Gamma/\Delta]$  [Bra03, Satz 0.4.3]; since  $\Gamma$  acts transitively on  $\Gamma/\Delta$ , an element

$$\sum_{\sigma \in \Gamma/\Delta} n_{\sigma} e_{\sigma} \in \mathbb{Z}[\Gamma/\Delta]$$

is fixed by  $\Gamma$  if and only if  $n_{\sigma} = n_{\tau}$  for all  $\sigma, \tau \in \Gamma/\Delta$ . This shows that  $u(T) = [M : K] - \mathrm{rk}_{\mathbb{Z}} \chi(T)^{\Gamma} = [M : K] - 1$ ; on the other hand, we shall compute later in Proposition 4.3.8 that

$$c_{\mathrm{tame}}(T) = f(M/K) \frac{e(M/K) - 1}{2}.$$

As a consequence of Proposition 3.2.14,  $c_{\text{wild}}(G) = 0$  for any tamely ramified semi-abelian  $K$ -variety; Halle and Nicaise proved that for elliptic curves over a strictly henselian field with algebraically closed residue field, the converse holds as well [HN16, Proposition 7.2.2.4], and conjectured that this holds for an arbitrary semi-abelian variety [HN16, 6.1.3.12]. This is true for tori if  $k = \bar{k}$ :

**Corollary 4.1.6.** *Let  $T$  be a torus over a strictly henselian discrete valued field  $K$  with algebraically closed residue field. Then  $c_{\text{wild}}(T) = 0$  if and only if  $T$  is tamely ramified.*

*Proof.* Let  $L/K$  be a Galois splitting extension of  $T$  and  $\Gamma$  its Galois group. We can assume that the residue characteristic  $p$  is positive (if not,  $T$  is tamely ramified and the statement follows from Proposition 3.2.14). By Proposition 4.1.3,  $c_{\text{wild}}(T) = 0$  if and only if  $\text{Sw}(\chi_{\mathbb{Q}}) = 0$ , which happens if and only if the action of the wild inertia subgroup  $\Gamma_1$  on  $\chi$  is trivial.

We have already remarked that if  $T$  is tamely ramified,  $c_{\text{wild}} = 0$  (for tori, this also follows simply because  $\Gamma_1$  is trivial if  $T$  is tamely ramified). Conversely, if the  $\Gamma_1$ -action on  $\chi$  is trivial, then the  $\Gamma$ -action factors through the quotient  $\Gamma/\Gamma_1$ , which is just the Galois group of the maximal tame subextension  $K^t/K$  of  $L/K$ : therefore,  $K^t/K$  is a splitting extension for  $T$ , meaning that  $T$  is tamely ramified.  $\square$

*Remark 4.1.7.* It is instead false that if  $c_{\text{tame}}(T) = 0$ , then  $T$  is wildly ramified: assuming  $k$  to be algebraically closed, if  $c_{\text{tame}}(T) = 0$ , then  $u(T) = 0$ ; as recalled before the statement of Theorem 3.1.3, this implies that  $T$  has multiplicative reduction, and thus that it is split.

## 4.2 Weakly unramified extensions and composita

In this section we study the behaviour of composita of separable algebraic extensions of a discretely valued henselian field  $K$ , at least one of which is weakly unramified (i.e. has ramification index equal to 1); we do not require that the residue field be separably closed. Without assuming the separability of the residue field extensions, many of the pleasant properties of unramified extensions fail: for example, different (even linearly disjoint!) extensions might induce the same extension of residue fields, and a compositum of weakly unramified extensions might be totally ramified. We exhibit examples of these phenomena, isolate some conditions on two such extensions

guaranteeing that their composita are weakly unramified (Proposition 4.2.10), and collect some related results, some of which will be applied in later chapters.

Recall from [Bou90, V.12.4] that a compositum of two field extensions  $M/K$  and  $K'/K$  is the datum of two field extensions  $M \hookrightarrow M'$ ,  $K' \hookrightarrow M'$  fitting into a commutative square

$$\begin{array}{ccc} & M' & \\ M & \swarrow \searrow & K' \\ & K & \end{array}$$

and such that the union of the images of  $K'$  and  $M$  in  $M'$  generates it as a  $K$ -algebra. As customary, we shall also say that  $M'$  is a compositum of  $K'$  and  $M$  over  $K$  if the maps into it are clear from the context, and denote it by  $M \cdot K'$ .

A morphism of composita

$$\begin{array}{ccc} & M' & \\ M & \swarrow \searrow & K' \end{array} \rightarrow \begin{array}{ccc} & M'' & \\ M & \swarrow \searrow & K' \end{array}$$

is a field extension  $M' \hookrightarrow M''$  which is a morphism of both  $M$ - and  $K'$ -algebras. Given field extensions  $M/K$  and  $K'/K$ , we shall denote by  $\mathbb{C}_K(M, K')$  a non-redundant set of representatives of the isomorphism classes of composita of  $M$  and  $K'$  over  $K$ .

Let now  $K$  be a henselian discretely valued field.

The following remark will be often used without mention.

*Remark 4.2.1.* For any finite field extension  $M/K$  of a discretely valued (strictly) henselian ring, combining [Neu13, Theorem II.6.2] and [Autb, 04GH], one sees that the integral closure of  $\mathcal{O}_K$  in  $M$  is again a (strictly) henselian discrete valuation ring with fraction field  $M$ .

**Lemma 4.2.2.** *For any (strictly) henselian discretely valued field  $K$ , the colimit  $F := \varinjlim_{i \in I} F_i$  of a filtered system of weakly unramified extensions of  $K$   $\{(K \hookrightarrow F_i, \nu_{ij}: F_i \rightarrow F_j)\}_{i,j \in I}$  defines a weakly unramified extension with residue field  $k_F := \varinjlim_{i \in I} k_{F_i}$  (i.e.  $\varinjlim_{i \in I} \mathcal{O}_{F_i}$  is a (strictly) henselian discrete valuation ring with fraction field  $F$  and residue field  $k_F$ ; the valuation on  $F$  is the unique one extending that on  $K$ , it is discrete and the extension  $F/K$  is weakly unramified).*

*Proof.* For each  $i, j$  in  $I$ , since  $\nu_{ij}$  is a  $K$ -algebra morphism it defines a local injection  $\mathcal{O}_{F_i} \hookrightarrow \mathcal{O}_{F_j}$ ; applying [Autb, 04GI] to the resulting filtered system  $\{(\mathcal{O}_{F_i}, \mathcal{O}_{F_i} \hookrightarrow$

$\mathcal{O}_{F_j})\}$  one obtains that its colimit  $\mathcal{O}_F$  is again a (strictly) henselian ring with maximal ideal  $\varinjlim_{i \in I} \mathfrak{m}_{F_i}$  and residue field  $k_F$ . Since the colimit of a filtered system of domains is a domain whose fraction field is the colimit of the induced system of fraction fields, one has  $\text{Frac } \mathcal{O}_F = \varinjlim_{i \in I} \text{Frac } \mathcal{O}_{F_i} = \varinjlim_{i \in I} F_i$ .

Define a function  $\nu_F: F^\times \rightarrow \mathbb{Z}$  by  $\nu_F(x) = \nu_{F_i}(x)$ , where  $i$  is such that  $x \in F_i$  and  $\nu_{F_i}$  is the unique discrete valuation on  $F_i$  extending that of  $K$ ; as all the extensions  $\iota_{ij}$  are weakly unramified,  $\nu_F(x)$  is independent of the choice of  $i$ , and is visibly a discrete valuation; moreover,  $F/K$  is weakly unramified, since by definition  $\nu_F(\pi) = 1$  for any uniformiser  $\pi$  of  $K$ .  $\square$

**Lemma 4.2.3.** *Let  $K$  be a discretely valued henselian field and  $M/K$  be a separable algebraic extension.*

- (i) *There exists a unique maximal unramified subextension  $K_M^{ur}/K$  of  $M/K$ .*
- (ii) *Any weakly unramified subextension of  $M/K$  is contained inside a maximal weakly unramified subextension  $K_M^{wur}/K$  of  $M$ .*
- (iii) *If  $M = K^{sep}$ , then  $K_M^{wur}$  has residue field  $\bar{k}$ .*

Notice that, as Example 4.2.8 implies, the set of weakly unramified subextensions of a separable extension of  $K$  might have more than one maximal element.

*Proof.* The statement of point i) is well-known; see for example [Ser13, Theorem III.5.3] for the case when  $M/K$  is finite, whence the general case follows immediately by a limit argument.

As for points ii) and iii), decomposing the residue field extension into a separable and a purely inseparable extension [Bou90, Propositions V.7.12-13], we can easily reduce to the case  $K_M^{ur} = K$ . By Lemma 4.2.2, for any weakly unramified subextension  $E/K$  of  $M/K$ , every chain in the non-empty set of weakly unramified subextensions of  $M/E$  ordered by inclusion has an upper bound, whereby by Zorn's lemma such set has a maximal element.

Let now  $M = K^{sep}$ , with residue field  $\bar{k}$ . If  $K_M^{wur}$  is a maximal element, its residue field is equal to  $\bar{k}$ : if not, up to replacing  $K$  by  $K_M^{wur}$ , we can assume that  $K = K_M^{wur}$ ; as the extension  $\bar{k}/k$  is then purely inseparable, it admits a subextension  $k \subseteq k_F \subseteq \bar{k}$  such that  $k_F/k$  has degree  $p$ . One can write  $k_F = k(\bar{\alpha})$  for some  $\alpha \in \mathcal{O}_M$ , which can be chosen so that  $F := K(\alpha) \subseteq M$  has residue field  $k_F$ , thereby justifying the notation: indeed, by [Ser13, Proposition I.1.15 and Corollary 1] it is enough to let  $\alpha$  be a root in  $K^{sep}$  of any separable degree- $p$  lift of the minimal polynomial of  $\bar{\alpha}$ . It then follows that the extension  $F/K$  is weakly unramified, a contradiction.  $\square$

**Definition 4.2.4.** Following Overkamp<sup>1</sup> we call an extension  $M/K$  *decomposable* if it admits a maximal weakly unramified subextension  $K_M^{wur}/K$  such that  $M/K_M^{wur}$  is totally ramified.

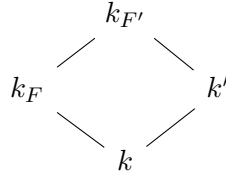
Observe that the previous result does not claim that any extension is decomposable (as a matter of fact, we suspect it is not).

*Question 4.2.5.* Assume that one maximal weakly unramified subextension of  $M/K$  is Galois. Are the others Galois, too?

It is probably not hard to extend the following results to extensions inside an algebraic closure of  $K$  (so possibly inseparable), and as a matter of fact it might be interesting to do so, but due to lack of time we only work with separable extensions.

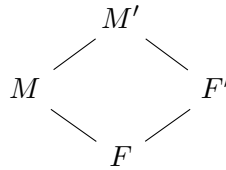
**Proposition 4.2.6.** *Let  $M/F/K$  and  $K'/K$  be separable extensions of discretely valued fields inside  $K^{sep}$  with  $F/K$  and  $K'/K$  weakly unramified and  $M/F$  totally ramified. Letting  $Z = K' \cap F (= K' \cap M)$ , assume that  $k_Z = k' \cap k_F$  and  $k'$  and  $k_F$  are linearly disjoint over  $k_Z$ . Then*

- (i)  *$F$  and  $K'$  are linearly disjoint over their intersection, and their compositum  $F'$  inside  $K^{sep}$  defines a weakly unramified extension of  $K$ . Furthermore,*



*is a compositum diagram;*

- (ii)  *$M$  and  $K'$  are linearly disjoint over their intersection, and parallel sides of the pushout square*



*have the same ramification;*

- (iii) *assume that the extension  $F/K_F^{ur}$  can be decomposed as*

$$K_F^{ur} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n,$$

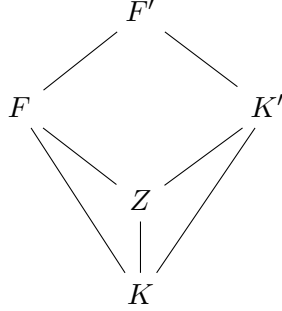
*where each  $F_{i+1}/F_i$  is simple of degree  $p$ . Then the natural morphism*

*$\mathcal{O}_M \otimes_{\mathcal{O}_Z} \mathcal{O}_{K'} \rightarrow \mathcal{O}_{M'}$  is an isomorphism of  $\mathcal{O}_{K'}$ -algebras.*

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<sup>1</sup>Personal communication.

*Proof.* *i)*: The diagram



allows us to reduce to the case when  $Z = K$ . Assume first that both extensions are finite. Since  $k_{F'}$  contains both  $k'$  and  $k_F$ , it then contains their compositum  $k' \cdot k_F$ , and as  $k_F/k$  and  $k'/k$  are  $k$ -linearly disjoint,

$$f(F'/K) \leq [F' : K] \leq [K' : K][F : K] = [k' : k][k_F : k] = [k_F \cdot k' : k] \leq [k_{F'} : k] = f(F'/K);$$

therefore, all inequalities are equalities, and in particular  $k_{F'} = k_F \cdot k'$  and

$$[K' : K][F : K] = [F' : K],$$

whereby  $K'/K$  and  $F/K$  are linearly disjoint and  $F'/K$  is weakly unramified.

For the general case, write  $K'/K$  and  $F/K$  as limits of directed systems of finite extensions  $\{K'_i/K\}_i$  and  $\{F_i/K\}_i$ , respectively; for each  $i$ ,  $k'_i$  and  $k_{F_i}$  are linearly disjoint over  $k$ , whereby each compositum  $F'_i$  is weakly unramified over  $K$ ,  $k_{F'_i} = k' \cdot k_{F_i}$  and  $F_i$  is linearly disjoint from  $K'_i$  over  $K$ . As  $F' = \varinjlim_i F'_i$  and these properties are all stable under direct limits (the former by Lemma 4.2.2), the statement follows.

*ii)*: By the previous point, we can assume that  $F = K$ . There canonical isomorphism  $M \cong K[x]/(G)$  for some Eisenstein polynomial  $G \in \mathcal{O}_K[x]$  [Ser13, Proposition I.1.6.18]; as  $K'/K$  is weakly unramified,  $G$  remains an Eisenstein, thus irreducible, polynomial over  $K'$ , whereby  $M' = K'[x]/(G) = K' \otimes_K M$  and  $M/K$  and  $K'/K$  are linearly disjoint.

*iii)*: We can as above assume that  $Z = K$ . Observe first that if  $M_1/K$  is a subextension of  $M/K$  and  $M'_1$  is its compositum with  $K'$  inside  $K^{sep}$ , then  $M/M_1$  and  $M'_1/M_1$  are again linearly disjoint [Bou90, Proposition V.5.8], and if  $\mathcal{O}_{M'} \cong \mathcal{O}_M \otimes_{\mathcal{O}_{M_1}} \mathcal{O}_{M'_1}$  and  $\mathcal{O}_{M'_1} \cong \mathcal{O}_{M_1} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$  via the canonical maps, then

$$\mathcal{O}_{M'} \cong \mathcal{O}_M \otimes_{\mathcal{O}_{M_1}} \mathcal{O}_{M'_1} \cong \mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} :$$

therefore, taking for  $M_1$  a maximal weakly unramified subextension of  $M/K$ , we can assume that  $M/K$  is either totally ramified or weakly unramified, and in the

latter case, taking for  $M_1$  the maximal unramified subextension of  $M/K$ , we can assume that  $k_F/k$  is either separable or purely inseparable. By the same reduction step and the assumption, without loss of generality  $M/K$  is simple.

Consider first the case when  $M = F$ . Let  $G \in K[x]$  be the minimal polynomial of an element  $\alpha \in F$  whose image in  $k_F$  generates it as a  $k$ -extension, so that  $F = K[x]/(G)$  and  $\mathcal{O}_F = \mathcal{O}_K[x]/(G)$  [Ser13, Proposition I.1.6.16]. Since  $k'/k$  and  $k(\bar{\alpha})/k$  are linearly disjoint,  $G \bmod \mathfrak{m}'$  remains irreducible in  $k'$ , whereby by [Ser13, Proposition I.1.6.15] the ring of integers of  $F' = K'(\alpha)$  is isomorphic to  $\mathcal{O}_{K'}[x]/(G)$  via the canonical map; the next remark yields a natural isomorphism  $\mathcal{O}_{K'}[x]/(G) \cong \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathcal{O}_F$ .

If, instead,  $F = K$ , we have already seen that  $M' = K'[x]/(G)$  for a polynomial  $G \in \mathcal{O}_K[x]$  which is Eisenstein over  $K'$ ; thus we conclude as before, applying [Ser13, Proposition I.1.6.17] instead of [Ser13, Proposition I.1.6.15].  $\square$

*Remark 4.2.7.* Let  $R$  be a ring,  $A, B$   $R$ -algebras,  $G \in A[x]$  a polynomial. Then there are canonical isomorphisms of  $A \otimes_R B$ -algebras

$$A[x]/(G) \otimes_R B \cong A[x]/(G) \otimes_A (A \otimes_R B) \cong (A \otimes_R B)[x]/(G).$$

As the next examples shows, the condition  $k_Z = k' \cap k_F$  is not vacuous, i.e. unlike for unramified extensions, there can be different weakly unramified extensions inducing the same extension of residue fields, and their compositum might fail to be weakly unramified.

*Example 4.2.8.* Let  $K$  be a mixed-characteristic complete discretely valued field with imperfect residue field  $k$ . Assume that  $K$  has  $p$ -th roots of unity and it has absolute ramification greater than one, so that  $\nu(p) > \nu(\pi)$  for any uniformiser  $\pi$ ; such  $K$  can always be obtained by taking suitable extensions of a Cohen ring. Let  $\lambda \in \mathcal{O}_K$  be an element such that  $\bar{\lambda} \in k \setminus k^p$  and, fixing a separable closure  $K^{sep}$  of  $K$ , consider  $\alpha, \beta \in K^{sep}$  such that  $\alpha^p = \lambda$ ,  $\beta^p = \pi - \lambda$  for a uniformiser  $\pi$  of  $K$ .

The two extensions  $F = K(\alpha)$  and  $K' = K(\beta)$  are Galois over  $K$ , and we claim that their compositum  $F' = K(\alpha, \beta)$  is totally ramified over  $K'$ : one has

$$(\alpha + \beta)^p \equiv \alpha^p + \beta^p = \pi \pmod{p},$$

and since  $\nu_{F'}(p) = \nu(p) > \nu(\pi) = \nu_{F'}(\pi)$ , one obtains that  $p\nu_{F'}(\alpha + \beta) = \nu(\pi)$ , whence the claim. This also implies that  $F \cap K' = K$  (or  $F/K$  and  $K'/K$  would be the same extension), and by Galoisness, that the two extensions are linearly disjoint [Coh79, Corollary 2].

We do not know whether the assumption that  $k'/k$  and  $k_F/k$  be linearly disjoint is necessary; remark that, as the next example illustrates, this is stronger than the assumption that they intersect trivially!

*Example 4.2.9* (Two finite purely inseparable extensions which intersect trivially but are not linearly disjoint). All field extensions of a field  $k$  will be considered inside a fixed algebraic closure of  $k$ .

By a classical result in field theory [Coh79, Corollary 2], given two algebraic field extensions  $k'/k$  and  $l/k$  such that at least one of them is normal and at least one of them (possibly the same) is separable,  $k'/k$  and  $l/k$  are linearly disjoint if and only if  $k' \cap l = k$ . It is elementary to see that this fails for two separable but non-normal extensions, and one can show that the separability condition cannot be removed either [Auta]: recall first that by Mac Lane's theorem [Bou90, Corollary V.4.1], an extension  $l/k$  of a field of characteristic exponent  $p$  is separable if and only if it is linearly disjoint from  $k^{p^{-\infty}}$  over  $k$ . Considering an inseparable extension  $l/k$  with  $\text{char } k > 0$  which does not have a purely inseparable element, one has that the two extensions  $l/k$  and  $k^{p^{-\infty}}/k$  intersect trivially, but by Mac Lane's theorem are not linearly disjoint. It is interesting to ask whether this can also happen for finite purely inseparable extensions; results by Sweedler [Swe68] imply that, indeed, there exist finite purely inseparable extensions of some field  $k$  which are not linearly disjoint, despite intersecting trivially (more explicit or easy counterexamples have so far eluded us).<sup>2</sup>

Sweedler calls a purely inseparable extension  $l/k$  of finite exponent *modular* if  $l^{p^n}$  and  $k$  are linearly disjoint over their intersection for all  $n$  (here  $l^{p^n} \subseteq l$  is the subfield of  $p^n$ -th powers); he then shows in *loc. cit.*, Example 1.1 that, letting  $k = \mathbb{F}_p(X^p, Y^p, Z^{p^2})$ , the extension  $l = k(Z, XZ + Y)/k$  is not modular. In particular, there exists some  $n$  such that the two finite and purely inseparable extensions  $l^{p^n}/(l^{p^n} \cap k)$  and  $k/(l^{p^n} \cap k)$  are not linearly disjoint, despite intersecting trivially by definition.

In the next result we consider all the different isomorphism classes of composita of weakly unramified extensions.

**Proposition 4.2.10.** *Let  $K'/K$  and  $F/K$  be weakly unramified extensions of a henselian field  $K$ . Consider the following condition:*

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<sup>2</sup>Let the reader beware that Section 2 in *loc. cit.* contains several mistakes: see [GW74],[Swe69] and [Che07]; the result we are interested in is contained in Section 1 and unaffected by these mistakes.

(★) for any  $\sigma \in \text{Hom}_K(F, K^{\text{sep}})$ , letting  $Z_\sigma := K' \cap \sigma(F)$ , one has  $k_{Z_\sigma} = k' \cap k_{\sigma(F)}$ .

If (★) holds and, in addition, for any  $\sigma \in \text{Hom}_K(F, K^{\text{sep}})$  the extensions  $k_{\sigma(F)}/k_{Z_\sigma}$  and  $k'/k_{Z_\sigma}$  are linearly disjoint, then any compositum  $F' \in \mathbf{C}_K(K', F)$  is weakly unramified over  $K$ .

*Proof.* Consider a compositum  $F'$  of  $F/K$  and  $K'/K$  inside some separable closure  $E$  of  $K$ , and let  $\phi: E \rightarrow K^{\text{sep}}$  be a  $K$ -isomorphism (which exists by [Bou90, Proposition V.8.4]); applying  $\phi$  to the compositum diagram of  $F'$  inside  $E$ , one obtains a diagram of field extensions inside  $K^{\text{sep}}$

$$\begin{array}{ccc} & \phi(F') & \\ \swarrow & & \searrow \\ \phi(F) & & \phi(K') \\ \searrow & & \swarrow \\ & K & \end{array} ,$$

and by Artin's extension theorem, one can find a  $K$ -automorphism  $K^{\text{sep}} \rightarrow K^{\text{sep}}$  making the diagram of  $K$ -algebras

$$\begin{array}{ccc} K^{\text{sep}} & \xrightarrow{\quad \quad \quad} & K^{\text{sep}} \\ \uparrow & \phi|_{\phi(K')}^{-1} & \uparrow \\ \phi(K') & \xrightarrow{\quad \quad \quad} & K' \end{array}$$

commute. Postcomposing this isomorphism to  $\phi$ , one obtains a compositum of  $\sigma(F)$  and  $K'$  over  $K$  in  $K^{\text{sep}}$  for some  $\sigma \in \text{Hom}_K(F, K^{\text{sep}})$ , which is weakly unramified over  $K$  if and only if  $F'/K$  is. By assumption, we can apply Proposition 4.2.6 and conclude that  $F'/K$  is weakly unramified.  $\square$

If one of the two extensions is normal, then without loss of generality (as the statement is symmetrical in  $F$  and  $K'$ )  $F/K$  is normal and it suffices to check the two conditions in the previous statement when  $\sigma$  is the given inclusion  $F \hookrightarrow K^{\text{sep}}$ . We record this as a separate corollary:

**Corollary 4.2.11.** *In the previous notation, assume that either  $K'/K$  or  $F/K$  is normal and let  $Z = F \cap K'$ . If  $k_Z = k' \cap k_F$  and  $k'$  is linearly disjoint from  $k_F$  over  $k_Z$ , then any compositum  $F' \in \mathbf{C}_K(F, K')$  is weakly unramified over  $K$ .*

As the next example shows, if neither  $F/K$  nor  $K'/K$  is normal, it might happen that the condition in (★) is verified for some  $\sigma$ , but not for others, so it is really necessary to consider all the different  $K$ -automorphisms of  $K^{\text{sep}}$  in general.

*Example 4.2.12.* Let  $k$  be an imperfect field of characteristic  $p \geq 3$  and  $K$  a mixed-characteristic complete strictly henselian field with residue field  $k$  such that  $K$  does not have elementary  $p$ -th roots of unity. For example, “the” Cohen ring with residue field  $k$  satisfies this assumption: if some  $p$ -th root of unity  $\zeta_p$  belonged to  $K$ , then  $K$  would contain the cyclotomic field  $\mathbb{Q}_p(\zeta_p)$ , which is ramified of degree  $p-1$  over  $\mathbb{Q}_p$ . Therefore, denoting by  $\nu: K^\times \rightarrow \mathbb{Z}$  the normalised valuation of  $K$ , one would have

$$\nu(\zeta_p) = \frac{\nu(p)}{p-1} = \frac{1}{p-1},$$

which is absurd.

Let  $\lambda \in \mathcal{O}_K$  be such that  $\bar{\lambda} \in k \setminus k^p$ ; the polynomial  $x^p - \lambda$  defines a weakly unramified extension of  $K$ , and fixing one of its roots  $\alpha$  inside a separable closure  $K^{sep}$  of  $K$ , there is a compositum diagram

$$\begin{array}{ccc} & K(\alpha, \zeta_p) & \\ & \swarrow \quad \searrow & \\ K(\alpha) & & K(\zeta_p \alpha) \\ & \searrow \quad \swarrow & \\ & K & \end{array} \cdot$$

The degree of the extension  $K(\alpha, \zeta_p)/K(\alpha)$  divides  $p-1$ , whereby the extension is totally ramified.

Let us also observe that the conditions in Proposition 4.2.6 are not sharp:

*Example 4.2.13.* Let  $k = \mathbb{F}_p(t_0, t_1)$ ,  $K$  be the field of Laurant series in one variable  $\pi$  with coefficients in  $k$ . Let  $l, m, n$  be non-negative integers with  $l \geq 1$  and  $m > n$  and consider the two subextensions  $F = K(\lambda)$  and  $K' = K(\mu)$  of  $K^{sep}/K$ , where  $\mu^{p^l} + \pi^m \mu - t_0 + \pi^n t_1 = 0$  and  $\lambda^{p^l} + \pi^m \lambda - t_0 = 0$ . Visibly,  $k_F = k' = k(t_0^{1/p^l})$ , and letting  $F'$  be the compositum of  $K'$  and  $F$  inside  $K^{sep}$ , the ramification index  $e(F'/K')$  depends on the exponents  $l$  and  $n$ : observe first that since  $(\lambda - \mu)^{p^l} + \pi^m(\lambda - \mu) = t_1 \pi^n$ , comparing the valuations in  $F'$  of the two sides and using that  $m > n$ , one has that

$$p^l \nu_{F'}(\lambda - \mu) = n \nu_{F'}(\pi).$$

The previous equality implies immediately that if  $p^l \nmid n$ , then  $p \mid \nu_{F'}(\pi)$  and  $e(F'/K') > 1$ . Assume instead that  $p^l \mid n$  and let  $k = n/p^l$ ,  $\beta = (\lambda - \mu)/\pi^k$ , which is in  $\mathcal{O}_{F'}^\times$  by the displayed equality.

By the definitions of  $k$  and  $\beta$ , one has

$$\beta^{p^l} = \frac{(\lambda - \mu)^{p^l}}{\pi^{kp^l}} = \frac{-\pi^m(\lambda - \mu) + t_1 \pi^n}{\pi^{kp^l}} = -\pi^{m-k(p^l-1)} \beta + t_1,$$

which implies that  $\bar{\beta}^{p^l} = t_1$  since  $m - k(p^l - 1) \geq 1$ . Consequently,  $F'/K'$  is weakly unramified of degree  $p^l$ , and in particular  $F/K$  and  $K'/K$  are linearly disjoint, despite inducing the same extension of residue fields.

However, one does have the following partial converse to Proposition 4.2.6:

**Lemma 4.2.14.** *Let  $F/K$  and  $K'/K$  be weakly unramified extensions,  $Z = F \cap K'$ . Assume that the inclusion of residue fields  $k_Z \subseteq k' \cap k_F$  is strict. If  $\mathcal{O}_{F'} = \mathcal{O}_{K'} \cdot \mathcal{O}_F$  in  $F'$ , then  $F'/K$  has non-trivial ramification.*

*Proof.* Since both  $F/K$  and  $K'/K$  are algebraic, there are finite subextensions  $K'_1/K$  and  $F_1/K$  such that the residue field of  $K'_1 \cap F_1$  does not coincide with  $k'_1 \cap k_{F_1}$ , so without loss of generality both  $F/K$  and  $K'/K$  are finite.

The ring of integral elements of  $K' \cap F$  is the intersection  $\mathcal{O}_F \cap \mathcal{O}_{K'}$ , so by hypothesis there exist  $\alpha \in \mathcal{O}_F \setminus \mathcal{O}_{K'}$  and  $\beta \in \mathcal{O}_{K'} \setminus \mathcal{O}_F$  with the same image in  $k_{F'}$ . It is enough to prove that  $K(\alpha, \beta)$  is ramified over  $K$ , whereby without loss of generality  $F = K(\alpha)$ ,  $K' = K(\beta)$  and  $k' = k_F$ ; but then  $k' \cdot k_F = k'$ , and by degree reasons and the next lemma  $F'/K$  cannot be weakly unramified.  $\square$

**Lemma 4.2.15.** *Let  $F/K$  and  $K'/K$  be weakly unramified extensions inside a separable closure  $K^{\text{sep}}$ . If their compositum  $F'$  is weakly unramified over  $F$  and  $\mathcal{O}_{F'} = \mathcal{O}_{K'} \cdot \mathcal{O}_F$ , then the diagram of residue fields*

$$\begin{array}{ccc} & k_{F'} & \\ & \swarrow \quad \searrow & \\ k_F & & k' \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

*is a compositum as well.*

*Proof.* Let  $\alpha \in \mathcal{O}_{F'}$ ; by assumption, there exist a positive integer  $r$  and  $\lambda_1, \dots, \lambda_r \in \mathcal{O}_F$ ,  $\mu_1, \dots, \mu_r \in \mathcal{O}_{K'}$  such that  $\alpha = \sum_{i=1}^r \lambda_i \mu_i$ ; as  $F'/F$  and  $F/K$  are weakly unramified, any uniformiser  $\pi$  of  $K$  is a uniformiser of  $F'$ ,  $F$  and  $K'$  as well. Therefore, reducing the previous equation modulo  $\pi$ , one can express  $\bar{\alpha}$  as a sum of products  $\bar{\lambda}_i \bar{\mu}_i$  with  $\bar{\lambda}_i \in k_F$  and  $\bar{\mu}_i \in k'$ . Consequently,  $k_{F'} \subseteq k_F \cdot k'$ , which proves the claim (as the reverse inclusion always holds).  $\square$

We conclude with the following two lemmata, also to be used below.

**Lemma 4.2.16.** *Let  $K'/K$  and  $M/K$  be algebraic extensions of a field  $K$ . If one of them is separable and  $\text{Spec } M \otimes_K K'$  is finite, then there is an isomorphism of*

$K'$ -algebras

$$M \otimes_K K' \xrightarrow{\sim} \prod_{M' \in \mathcal{C}_K(M, K')} M'$$

*Proof.* Proposition V.12.4 in [Bou90] identifies the right-hand side with

$$\prod_{\mathfrak{p} \in \text{Spec}(M \otimes_K K')} \frac{M \otimes_K K'}{\mathfrak{p}}$$

as a  $K'$ -algebra; by assumption  $\text{Spec } M \otimes_K K'$  has only a finite number of prime ideals, all of which are maximal by the main theorem in [Sha77]. In particular, different prime ideals are coprime, and applying the Chinese remainder theorem one obtains a surjection of  $K'$ -algebras from  $M \otimes_K K'$  to the product in the formula above, whose kernel is the nilradical of  $M \otimes_K K'$ . By assumption, either  $M/K$  or  $K'/K$  is separable, which by [Bou90, Example V.119.2.3] means that the tensor product  $M \otimes_K K'$  is reduced, whence the statement.  $\square$

**Lemma 4.2.17.** *Let  $M/K$  a finite separable extension of a discretely valued field  $K$ ,  $K'/K$  a weakly unramified extension. Let  $F/K$  be a maximal weakly unramified subextension of  $M/K$  and assume that  $F'/K$  is weakly unramified over  $K$  for all  $F' \in \mathcal{C}_K(F, K')$ . Then*

$$e(M'/K') = e(M/K)$$

for all  $M' \in \mathcal{C}_K(M, K')$  and

$$\sum_{M' \in \mathcal{C}_K(M, K')} f(M'/K') = f(M/K).$$

*Proof.* Any compositum of  $M/K$  and  $K'/K$  is the compositum of  $F'/F$  and  $M/F$  for some  $F' \in \mathcal{C}_K(F, K')$ , and since by assumption  $F'/K$  is weakly unramified, the first equality is a direct consequence of Proposition 4.2.6ii).

As a consequence, one also has

$$\begin{aligned} e(M/K) \sum_{M' \in \mathcal{C}_K(M, K')} f(M'/K') &= \sum_{M' \in \mathcal{C}_K(M, K')} [M' : K'] = \\ &= \dim_{K'} \left( \prod_{M' \in \mathcal{C}_K(M, K')} M' \right) = \dim_{K'} (M \otimes_K K') = [M : K] = e(M/K) f(M/K), \end{aligned}$$

whence the second equality.  $\square$

### 4.3 Jumps of induced tori and their motivic zeta function

*Notation 4.3.1.* Recall some notation from the introduction:  $\mathcal{O}_K$  denotes a strictly henselian discrete valuation ring with uniformizer  $\pi$ , residue field  $k$  of characteristic exponent  $p$ , and field of fractions  $K$ . All field extensions of  $K$  will be considered inside some fixed separable closure  $K^{sep}$  of  $K$ . The ring of integers of a valued field  $F$  will be denoted by  $\mathcal{O}_F$ , its residue field by  $k_F$  (with the usual exception of fields with a primed symbol).

If  $d$  is a positive integer coprime to  $p$ ,  $K(d)$  will be the unique degree- $d$  totally ramified extension of  $K$  inside  $K^{sep}$ , and objects referring to this will carry a subscript  $d$  (for example,  $\pi_d$  will be a uniformiser). If  $L/K$  is an extension inside  $K^{sep}$  with  $K(d) \cap L = K$ , the compositum  $K(d) \cdot L$  will be denoted by  $L(d)$ . Notice that this notation is justified, since  $L(d)/L$  is then totally ramified of degree  $d$ .

In this section we extend a computation by Overkamp [Ove19, Corollary 2.2.2] of the jumps of an induced torus, i.e. a torus of the form  $\text{Res}_{M/K} \mathbb{G}_{m,M}$  for a finite separable extension  $M/K$ , to the case of extensions of arbitrary strictly henselian discretely valued fields, by proving that they are rational numbers of the form  $a/e$ , where  $e$  is the ramification index of the Galois closure of  $M$  over  $K$  and  $a$  a non-negative integer smaller than  $e$  (Theorem 4.3.9); as far as we know, this is the first non-trivial calculation of Edixhoven's jumps of a semi-abelian variety over a valued field with non-algebraically closed residue field.

Our result follows from a reduction to the case when  $M/K$  is Galois, in which we are able to explicitly compute the jumps. Here, a new phenomenon appears, which is invisible when the residue field is perfect: interestingly enough, the inertia index of  $M/K$  does not affect the values of the jumps, but it controls instead their multiplicity.

Finally, we use Overkamp's computation of the jumps of an induced torus to compute the motivic zeta function of induced tori along purely wild extensions (Proposition 4.3.15).

Let us start by collecting some preliminary results. First, in order to reduce the rationality of the jumps of any induced torus to the case of an induced torus relative to a Galois extension, we need two results due to Lorenzini and Liu [LL01, Lemma 4.3] and Chai [Cha00, Corollary 4.7 and Remark 4.8].

**Proposition 4.3.2.** [LL01, Lemma 4.3] *Let  $S$  be a scheme,  $\phi: \mathcal{G}_2 \rightarrow \mathcal{G}_3$  a morphism smooth group  $S$ -schemes. If  $\phi$  induces an epimorphism of sheaves on the smooth*

site on  $S$ , then  $\ker \phi$  is a smooth group  $S$ -scheme.

**Corollary 4.3.3.** *Let  $j: \operatorname{Spec} K \rightarrow \operatorname{Spec} \mathcal{O}_K$  be the inclusion of the generic fibre into a spectrum of a discrete valuation ring  $\mathcal{O}_K$ . Let  $0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$  be a short exact sequence of smooth  $K$ -group schemes admitting Néron models. If  $R^1 j_*^{\text{sm}} G_1 = 0$ , then the complex of Néron models*

$$0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_3 \rightarrow 0$$

is exact as a sequence of group  $\mathcal{O}_K$ -schemes.

*Proof.* Let  $f_K: G_2 \rightarrow G_3$  the morphism in the statement and  $f: \mathcal{G}_2 \rightarrow \mathcal{G}_3$  its extension to the Néron models. By the previous proposition,  $\ker f$  is a smooth group  $\mathcal{O}_K$ -scheme, whereby it suffices to show that it is separated and satisfies the Néronian property. Separatedness is equivalent to the unit morphism  $e_{\ker f}: \operatorname{Spec} \mathcal{O}_K \rightarrow \ker f$  being a closed immersion [BLR90, Lemma 7.1/2]; by definition, the  $e_{\ker f}$  fits into a diagram

$$\begin{array}{ccccc} \operatorname{Spec} \mathcal{O}_K & & \xrightarrow{e_{\mathcal{G}_2}} & & \mathcal{G}_2 \\ & \searrow e_{\ker f} & & \searrow f & \\ & \ker f & \longrightarrow & \mathcal{G}_2 & ; \\ & \downarrow & \lrcorner & \downarrow & \\ & \operatorname{Spec} \mathcal{O}_K & \xrightarrow{e_{\mathcal{G}_3}} & \mathcal{G}_3 & \end{array}$$

the units  $e_{\mathcal{G}_2}$  and  $e_{\mathcal{G}_3}$  are closed immersions because  $\mathcal{G}_2$  and  $\mathcal{G}_3$  are separated; in particular,  $\ker f \rightarrow \mathcal{G}_2$  is separated, whereby  $e_{\ker f}$  is a closed immersion.

As for the Néronian property, by the universal property of  $\mathcal{G}_2$ , for any smooth  $\mathcal{O}_K$ -scheme  $\mathcal{H}$  and any  $K$ -morphism  $g_K: H_K \rightarrow \ker f_K$  the composition

$$\mathcal{H}_K \rightarrow \ker f_K \rightarrow G_2$$

admits a unique extension  $h: \mathcal{H} \rightarrow \mathcal{G}_2$ ; the composition  $f \circ h$  is zero, i.e. the identity map  $e_{\mathcal{G}_3}: \operatorname{Spec} \mathcal{O}_K \rightarrow \mathcal{G}_3$ , by the universal property of  $\mathcal{G}_3$ , since this is true on the generic fibre. Consequently,  $h$  factors uniquely through  $\ker f$ , and the resulting map  $g: \mathcal{H} \rightarrow \mathcal{G}_2$  coincides with  $g_K$  on the generic fibre by the universal property of  $\ker f_K$ .  $\square$

We shall also need some information about the Galois group of a finite Galois extension of strictly henselian discretely valued fields, as provided by the next result, which is well-known if the residue field  $k$  is perfect (see e.g. [Ser13, §IV.2]).

**Proposition 4.3.4.** *The Galois group of any finite Galois extension  $L/K$  of strictly henselian discretely valued fields is isomorphic to a semi-direct product  $H \rtimes \mathbb{Z}/(l)$ , where  $(l, p) = 1$  and  $H$  is a  $p$ -group.*

*Proof.* By [ZS58, §V.10, Theorem 24] (see in particular summary table after the Corollary)  $\text{Gal}(L/K)$  is equal to the inertia group;<sup>3</sup> by [ZS58, §V.10, Theorem 25], there exists a normal subgroup  $H \trianglelefteq \text{Gal}(L/K)$  such that  $\text{Gal}(L/K)/H$  is isomorphic to a subgroup of  $k^\times$  (using the same notation as in *ibid.*,  $H$  is the kernel of  $\text{Gal}(L/K) \rightarrow G_1/G'_1$ ), and therefore cyclic of order  $l$  for some integer  $l$ . Observe that  $l$  is coprime to  $p$ , because otherwise  $k^\times$  would contain an element of order  $p$ , which is impossible since the Frobenius on  $k$  is injective ( $k$  being a field).

It remains to see that  $H$  is a  $p$ -group (then one has  $\text{Gal}(L/K) = H \rtimes \mathbb{Z}/(l)$  as in the proof of [Ser13, Corollary 4 to Proposition IV.2.7]); this follows by [ZS58, §V.10, Theorem 25], which shows that, keeping the same notation as in *ibid.* to minimise confusion, the group  $G'_1 := H/G_{V_2}$  is a  $p$ -group and, moreover, that  $G_{V_2}$  is a  $p$ -group as well, as ensues from [ZS58, §V.10, Theorem 25] using descending induction on  $n$ .  $\square$

*Remark 4.3.5.* A proof that  $\text{Gal}(L/K)$  is the semi-direct product  $H \rtimes H'$  for a  $p$ -group  $H$  and a group of prime-to- $p$  order  $H'$  (a weaker statement which is actually enough for our purposes) can also be obtained as follows: by [Neu13, Corollary II.7.9]  $L/K$  admits a maximal tamely ramified subextension  $T/K$ , which, as  $K$  is strictly henselian, is Galois by [Neu13, Proposition II.7.7]; finally, by [Neu13, Proposition II.7.11], if  $e(L/K) = lp^s$  for some integer  $l$  coprime to  $p$ , then the value group of  $T$  is  $\frac{1}{l}\mathbb{Z}$ , whereby  $\text{Gal}(L/T)$  is a normal  $p$ -group; the decomposition as a semi-direct product is a consequence of [Hal59, Theorem 15.2.2].

Since, as a consequence of Cauchy's theorem for groups, a group of order  $p^n$  admits a normal subgroup of order  $p^m$  for any  $m \leq n$ , one immediately obtains:

**Corollary 4.3.6.** *With the same notation as above, there exist subextensions  $K \subseteq L_0 \subseteq L_1 \cdots \subseteq L_r = L$  such that*

- $L_i/K$  is Galois for each  $i$ ,
- $L_0/K$  is tamely ramified,
- $[L_{i+1} : L_i] = p$  for all  $i$ .

---

<sup>3</sup>The inertia group of  $\text{Gal}(L/K)$  is denoted by  $G_T$  in [ZS58]; note also that the indices of the ramification groups  $G_{V_n}$  in [ZS58] are shifted by 1 with respect to those in [Ser13].

We can now turn to our main computation. Recall that we call an extension  $M/K$  decomposable if it admits a maximal weakly unramified subextension  $F/K$  such that  $M/F$  is totally ramified (Definition 4.2.4).

**Proposition 4.3.7.** *Let  $L/K$  be a finite extension of strictly henselian discretely valued fields which is either separable and decomposable or Galois; denote by  $e$  and  $f$  be the ramification and inertia indexes of  $L/K$ , respectively, and let  $d$  be a positive integer such that  $d \equiv 1 \pmod{e}$  and  $d > e$ . If  $K(d)/K$  a totally ramified extension of  $K$  of degree  $d$ , then the cokernel of the natural map*

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} \rightarrow \mathcal{O}_{L \otimes_K K(d)}$$

is isomorphic to

$$\bigoplus_{\nu=0}^{e-1} \left( \mathcal{O}_{K(d)} / \mathfrak{m}_d^{\nu \frac{d-1}{e}} \right)^{\oplus f}$$

as an  $\mathcal{O}_{K(d)}$ -module.

The proof of the case of a Galois extension also yields the case of a decomposable extension (which is enough to obtain the rationality of the jumps of induced tori, if not a bound on their denominator; however, since the former is notationally a bit heavy, it might be useful to read this simpler case first.

*Proof ( $L/K$  decomposable):* We follow the outline of the proof of [Ove19, Lemma 2.2.1]. Let  $K \subseteq F \subseteq L$  be a subextension of  $L/K$  such that  $F/K$  is weakly unramified and  $L/F$  totally ramified; by Proposition 4.2.6, we know that  $F/K$  and  $K(d)/K$  are linearly disjoint; moreover, denoting by  $F(d)$  their compositum in  $K^{sep}$  and by  $P$  an Eisenstein polynomial in  $F[x]$  such that  $L = F[y]/(P)$  [Ser13, Proposition I.6.18], we see that  $L \otimes_K K(d) = F(d) \otimes_F L = F(d)[y]/(P)$  is a field as well, because  $F(d)$  has the same residue field as  $F$  and thus  $P$  remains irreducible over  $F(d)$ .

Recalling that  $e(F(d)/F) = d$ , since  $e < d$  and  $d \equiv 1 \pmod{e}$ , for each choice of a uniformiser  $\pi_{F(d)}$  of  $F(d)$  there exists an Eisenstein polynomial  $Q \in \mathcal{O}_{F(d)}[t]$  such that

$$P(\pi_{F(d)}^{\frac{d-1}{e}} t) = \pi_{F(d)}^{d-1} Q(t).$$

Then

$$L \otimes_K K(d) \cong F(d)[y]/(P) \xrightarrow[y \mapsto \pi_{F(d)}^{\frac{d-1}{e}} t]{\sim} F(d)[t]/\left(P(\pi_{F(d)}^{\frac{d-1}{e}} t)\right) \cong F(d)[t]/(Q(t))$$

as  $F(d)$ -algebras and by [Ser13, Corollary 1 to Proposition I.6.17]

$$\mathcal{O}_{L \otimes_K K(d)} \cong \mathcal{O}_{F(d)}[t]/(Q) \quad (4.1)$$

as  $\mathcal{O}_{F(d)}$ -algebras. On the other hand, Proposition 4.2.6 yields isomorphisms of  $\mathcal{O}_{F(d)}$ -algebras

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} \cong \left( \mathcal{O}_F \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} \right)[y]/(P) \cong \mathcal{O}_{F(d)}[y]/(P). \quad (4.2)$$

Under the isomorphisms (4.1) and (4.2) constructed above,  $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)}$  and  $\mathcal{O}_{L \otimes_K K(d)}$  have  $\mathcal{O}_{F(d)}$ -bases  $\{1, \dots, y^{e-1}\}$  and  $\{1, \dots, t^{e-1}\}$ , respectively, and the natural map  $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} \rightarrow \mathcal{O}_{L \otimes_K K(d)}$  corresponds to the morphism of  $\mathcal{O}_{F(d)}$ -algebras  $\mathcal{O}_{F(d)}[y]/(P(y)) \rightarrow \mathcal{O}_{F(d)}[t]/(Q(t))$  sending  $y$  to  $\pi_{F(d)}^{\frac{d-1}{e}} t$ , whereby its cokernel is identified with

$$\bigoplus_{\nu=0}^{e-1} \mathcal{O}_{F(d)} / \pi_{F(d)}^{\nu \frac{d-1}{e}}.$$

The statement now follows observing that one has  $\mathcal{O}_{F(d)} \cong \mathcal{O}_{K(d)}^{\oplus f}$  as  $\mathcal{O}_{K(d)}$ -modules by Proposition 4.2.6(iii) and that  $\pi_{F(d)}$  can be chosen to lie in  $\mathcal{O}_{K(d)}$  because  $e(L/K(d)) = 1$ .  $\square$

We now examine the case of a Galois extension  $L/K$ ; the proof follows the same strategy, but one encounters an additional obstacle: namely, we do not know whether  $L/K$  is decomposable. This problem can be remedied by means of Corollary 4.3.6, at the price of a more technical computation.

Since any degree- $p$  extension is either totally ramified or weakly unramified with simple residue field extension, one can write  $L/K$  as a tower of subextensions (some of which possibly trivial)

$$K = F_0 \subseteq L_0 \subseteq F_1 \subseteq L_1 \subseteq \dots \subseteq F_r \subseteq L_r = L$$

with

- $L_i/F_i$  totally ramified of degree  $e_i$ ,
- $F_{i+1}/L_i$  weakly unramified of degree  $f_i$  inducing a simple residue field extension,
- $[L_i : F_i]$  a power of  $p$  for all  $i > 0$ .

Recall that our aim is computing the cokernel of the natural map

$$\phi: \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} \rightarrow \mathcal{O}_{L(d)}.$$

**Step 1.** *Identifying  $\phi$  as a map between polynomial rings with coefficients in  $\mathcal{O}_{K(d)}$ .*

By construction, arguing as in the decomposable case, one obtains for each  $i = 0, \dots, r$  and  $j = 1, \dots, r$  polynomials  $P_i \in \mathcal{O}_{F_i}[y_i]$  and  $G_j \in \mathcal{O}_{L_{j-1}}[z_j]$  such that  $P_i$  is Eisenstein,  $\mathcal{O}_{L_i} \cong \mathcal{O}_{F_i}[y_i]/(P_i)$  and  $\mathcal{O}_{F_j} = \mathcal{O}_{L_{j-1}}[z_j]/(G_j)$ : there follows an identification of  $\mathcal{O}_K$ -algebras

$$\mathcal{O}_L = \frac{\mathcal{O}_K[y_0, \dots, y_r, z_1, \dots, z_r]}{(P_0, \dots, P_r, G_1, \dots, G_r)}. \quad (4.3)$$

Moreover, for each index  $i$  and any uniformiser  $\pi_{F_i(d)}$  of  $\mathcal{O}_{F_i(d)}$  there exists an Eisenstein polynomial  $Q_i \in \mathcal{O}_{F_i(d)}[t_i]$  such that

$$P_i(\pi_{F_i(d)}^{\frac{d-1}{e_i}} t) = \pi_{F_i(d)}^{d-1} Q_i(t_i),$$

whereby the isomorphism

$$L_i(d) = F_i(d) \otimes_{F_i} L_i = \frac{F_i(d)[y_i]}{(P_i)} \rightarrow \frac{F_i(d)[t_i]}{(Q_i)} \\ y_i \mapsto \pi_{F_i(d)}^{\frac{d-1}{e_i}} t_i$$

provides, when restricted to the ring of integers of  $L_i(d)$ , an identification

$$\mathcal{O}_{L_i(d)} \cong \frac{\mathcal{O}_{F_i(d)}[t_i]}{(Q_i)}. \quad (4.4)$$

On the other hand,  $\mathcal{O}_{F_i(d)} \cong \mathcal{O}_{L_{i-1}(d)} \otimes_{\mathcal{O}_{L_{i-1}}} \mathcal{O}_{F_i}$ , and since  $\mathcal{O}_{F_i} \cong \mathcal{O}_{L_{i-1}}[z_i]/(G_i)$ ,

$$\mathcal{O}_{F_i(d)} \cong \frac{\mathcal{O}_{L_{i-1}(d)}[z_i]}{(G_i)}. \quad (4.5)$$

Combining equations (4.5) and (4.4) together, one obtains by induction that

$$\mathcal{O}_{L(d)} \cong \frac{\mathcal{O}_{K(d)}[t_0, \dots, t_r, z_1, \dots, z_r]}{(P_0, \dots, P_r, G_1, \dots, G_r)}$$

as  $\mathcal{O}_{K(d)}$ -algebras, where we inductively identify the polynomials  $Q_i$  with polynomials in

$$\left( \frac{\mathcal{O}_{K(d)}[t_0, \dots, t_{i-1}, z_1, \dots, z_i]}{(Q_0, \dots, Q_{i-1}, G_1, \dots, G_i)} \right) [t_i],$$

and similarly for  $G_i$ .

Notice that, since  $F_i(d)/L_{i-1}(d)$  is weakly unramified, we can regard  $t_{i-1}$  as a uniformiser of  $\mathcal{O}_{F_i(d)}$  via (4.4); recalling (4.3), the map  $\phi: \mathcal{O}_{K(d)} \otimes_{\mathcal{O}_K} \mathcal{O}_L \rightarrow \mathcal{O}_{L(d)}$

can be naturally identified with the map of  $\mathcal{O}_{K(d)}$ -algebras

$$\begin{aligned} \frac{\mathcal{O}_{K(d)}[y_0, \dots, y_r, z_1, \dots, z_r]}{(P_0, \dots, P_r, G_1, \dots, G_r)} &\rightarrow \frac{\mathcal{O}_{K(d)}[t_0, \dots, t_r, z_1, \dots, z_r]}{(Q_0, \dots, Q_r, G_1, \dots, G_r)} \\ y_0 &\mapsto \pi_d^{\frac{d-1}{e_0}} t_0 \\ y_i &\mapsto t_{i-1}^{\frac{d-1}{e_i}} t_i \quad (i \geq 1) \\ z_i &\mapsto z_i \end{aligned}$$

**Step 2.** *Expressing  $\phi$  in terms of explicit  $\mathcal{O}_{K(d)}$ -bases.*

Let  $\mathcal{I}$  (resp.  $\mathcal{J}$ ) be the set of multi-indices  $I = (i_0, \dots, i_r) \in \mathbb{N}^{r+1}$  (resp.  $J = (j_1, \dots, j_r) \in \mathbb{N}^r$ ) such that  $i_l < e_l$  (resp.  $j_l < f_l$ ) for all  $l$ . Under the identifications above,  $\mathcal{O}_{K(d)} \otimes_{\mathcal{O}_K} \mathcal{O}_L$  and  $\mathcal{O}_{L(d)}$  have  $\mathcal{O}_{K(d)}$ -bases  $\{y^I z^J | I \in \mathcal{I}, J \in \mathcal{J}\}$  and  $\{t^I z^J | I \in \mathcal{I}, J \in \mathcal{J}\}$ , respectively; however, we have not described  $\phi$  as a map of  $\mathcal{O}_{K(d)}$ -modules in terms of these bases: the problem is that  $y^I z^J$  is mapped to  $(\pi_d^{\frac{d-1}{e_0}})^{i_0} t^{I'} \cdot z^J$ , with

$$I' = \left( \frac{d-1}{e_0} \cdot i_1 + i_0, \dots, \frac{d-1}{e_r} \cdot i_r + i_{r-1}, i_r \right),$$

and  $I'$  is in general no longer in  $\mathcal{I}$ . It will turn out that, in order to compute  $\text{coker } \phi$  as a  $\mathcal{O}_{K(d)}$ -module, a different choice of basis on the target is more convenient.

Let us compute  $t^{I'}$ . By our choice of  $d$ ,  $e = e_0 \cdots e_r | (d-1)$ , whereby we can write

$$t_{r-1}^{i'_{r-1}} = t_{r-1}^{\left(\frac{d-1}{e_r}\right) i_r + i_{r-1}} = t_{r-1}^{i_{r-1}} \cdot t_{r-1}^{\left(\frac{d-1}{e_r e_{r-1}}\right) i_r e_{r-1}};$$

now, since  $t_{r-1}$  is a root of the degree- $e_{r-1}$  Eisenstein polynomial  $Q_{r-1} \in \mathcal{O}_{F_{r-1}}[x]$  and, under the identification in (4.5),  $t_{r-2}$  is a uniformiser of  $\mathcal{O}_{F_{r-1}}$ , we can write  $t_{r-1}^{e_{r-1}} = u_{r-1} t_{r-2}$  for some  $u_{r-1} \in \mathcal{O}_{F_{r-1}}^\times$ . Therefore,

$$t^{I'} = u_{r-1}^{\left(\frac{d-1}{e_r e_{r-1}}\right) i_r} t^{I''},$$

with

$$I'' = \left( \frac{d-1}{e_0} \cdot i_1 + i_0, \dots, \frac{d-1}{e_r e_{r-1}} \cdot i_r + \frac{d-1}{e_{r-1}} \cdot i_{r-1} + i_{r-2}, i_{r-1}, i_r \right).$$

By induction and an easy computation, one obtains units  $\alpha_i(I) \in \mathcal{O}_{F_i}^\times$  such that

$$t^{I'} = \pi_d^{\varepsilon(I)} \alpha(I) t^I,$$

---

<sup>4</sup>Strictly speaking, we should not be using the same letter  $J$  both for multi-indices and for multi-sets of jumps, but we do not think there is any risk of confusion.

where

$$\varepsilon(I) = \frac{d-1}{e} \left( (e_1 \cdots e_r) i_0 + (e_2 \cdots e_r) i_1 + \cdots i_r \right)$$

and  $\alpha(I) = \alpha_0(I) \cdots \alpha_{r-1}(I)$ . Notice that, since  $\alpha_i(I) \in \mathcal{O}_{F_i}^\times$ , an easy argument by descending induction on  $i$  shows that the set  $\{\alpha(I)t^I z^J | I \in \mathcal{I}, J \in \mathcal{J}\}$  is still an  $\mathcal{O}_{K(d)}$ -basis of  $\mathcal{O}_{L(d)}$ . The upshot is that, in terms of the bases  $\{y^I z^J | I \in \mathcal{I}, J \in \mathcal{J}\}$  on the source and  $\{\alpha(I)t^I z^J | I \in \mathcal{I}, J \in \mathcal{J}\}$  on the target,  $\phi$  is represented by a diagonal matrix in  $\mathcal{O}_{K(d)}^{\mathcal{I} \times \mathcal{J}}$  whose  $(I, J)$ -th diagonal element is equal to  $\pi_d^{\varepsilon(I)}$ . The values assumed by the exponent as  $I$  varies in  $\mathcal{I}$  and  $J$  in  $\mathcal{J}$  are the elementary divisors of  $\phi$ ; in particular, as the exponent is actually independent of  $J$ , we can already notice that the multiplicity of each value is a multiple of the cardinality of  $\mathcal{J}$ , which, as one can easily compute, is equal to  $f$ .

It just remains to identify the possible exponents, which is now simple combinatorics:

**Step 3.** *Let  $e_0, \dots, e_r$  be positive integers,  $e = e_0 \cdots e_r$ . Then the sets  $\{0, \dots, e-1\}$  and  $\{(e_1 \cdots e_r) i_0 + (e_2 \cdots e_r) i_1 + \cdots i_r | 0 \leq i_l \leq e_l - 1 \text{ for } l = 0, \dots, r\}$  are equal.*

Induct on  $r$ . The case  $r = 0$  is tautological, so assume the statement proven for  $r-1$ ; applying the inductive assumption to the integers  $e_0, \dots, e_{r-2}, e'_{r-1} := e_{r-1}e_r$ , one obtains that

$$\{0, \dots, e_0 \cdots e_r - 1\} = \left\{ (e_1 \cdots e_r) i_0 + (e_2 \cdots e_r) i_1 + \cdots + i'_{r-1} \mid \begin{array}{l} 0 \leq i_l \leq e_l - 1 \text{ for all } l=0, \dots, r-2, \\ 0 \leq i'_{r-1} \leq e_{r-1}e_r - 1 \end{array} \right\}.$$

Comparing this equality with the one in the statement, one immediately sees that it suffices to show the claim for  $r = 1$ ; explicitly (and writing  $a, b$  for  $e_0, e_1$  so as to avoid indices), the statement amounts to the equality of sets

$$\{0, \dots, ab - 1\} = \{mb + a | 0 \leq m \leq a - 1, 0 \leq n \leq b - 1\}.$$

The inclusion “ $\supseteq$ ” is clear; for the opposite one, let  $x$  be an integer in  $[0, ab - 1]$ ; by Euclidean division, there exist integers  $m$  and  $n$ , with  $n \in [0, b - 1]$ , such that  $x = bm + n$ ; as  $x < ab$ , one has  $m \leq a - 1$ , as wished.  $\square$

**Proposition 4.3.8.** *With the same notation as in Proposition 4.3.7, the  $d$ -jumps of the induced torus  $\text{Res}_{L/K} \mathbb{G}_{m,L}$  are*

$$0, \frac{d-1}{e}, \dots, (e-1) \frac{d-1}{e},$$

*each with multiplicity  $f$ ; its jumps are  $0, \frac{1}{e}, \dots, \frac{e-1}{e}$ , each with multiplicity  $f$ .*

*Proof.* Let  $G = \text{Res}_{L/K} \mathbb{G}_{m,L}$  and let  $d$  be a positive integer greater than  $e$  and such that  $d \equiv 1 \pmod{e}$ . By [BLR90, Example 10.1/5], the identity component of Néron model of  $\mathbb{G}_{m,L}$  is just  $\mathbb{G}_{m,\mathcal{O}_L}$ , whereby by the compatibility between Weil restriction and the formation of Néron models [BLR90, Proposition 7.6/6] and the fact that the formation Lie algebras of a group scheme commutes with Weil restriction, we can identify  $\text{Lie}(\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K(d)})$  with  $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)}$  and  $\text{Lie } \mathcal{G}(d)$  with  $\mathcal{O}_{L \otimes_K K(d)}$  as  $\mathcal{O}_{K(d)}$ -modules. Under these identifications,

$$\text{Lie } h(G, d): \text{Lie}(\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K(d)}) \rightarrow \text{Lie } \mathcal{G}(d)$$

corresponds to the unique map  $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} \rightarrow \mathcal{O}_{L \otimes_K K(d)}$  making the diagram of  $\mathcal{O}_{K(d)}$ -modules

$$\begin{array}{ccc} \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K(d)} & \xrightarrow{\quad\quad\quad} & \mathcal{O}_{L \otimes_K K(d)} \\ & \searrow \quad \swarrow & \\ & L \otimes_K K(d) & \end{array},$$

where the diagonal arrows are the obvious ones, commute. Unwinding, it is straightforward to see that such map is the canonical one referred to in the statement of Proposition 4.3.7, whereby the  $d$ -jumps of  $G$  are  $0, \frac{d-1}{e}, \dots, \frac{(e-1)(d-1)}{e}$ .

By Corollary 3.2.23 we can compute the jumps of  $G$  using any sequence  $(d_n)_n$  tending to infinity and such that  $d_n \equiv 1 \pmod{e}$ ; picking any such sequence, it ensues that for  $i = 0, \dots, e-1$

$$j_i(G) = \lim_{n \rightarrow \infty} \frac{i(d_n - 1)}{d_n e} = \frac{i}{e},$$

as stated. □

**Theorem 4.3.9.** *Let  $M/K$  be a finite separable extension of strictly henselian discrete valuation rings with Galois closure  $L/K$  and let  $e = e(L/K)$ ,  $f = f(L/K)$ . Letting  $T$  be the induced torus  $T = \text{Res}_{M/K} \mathbb{G}_{m,M}$ , one has*

$$J_d(T) \subseteq \left\{ (0; f), \left( \frac{d-1}{e}; f \right), \dots, (e-1) \left( \frac{d-1}{e} \right) \right\}$$

for any  $d \equiv 1 \pmod{e}$ ,  $d > e$  and

$$J(T) \subseteq \left\{ (0; f), \left( \frac{1}{e}; f \right), \dots, \left( \frac{e-1}{e}; f \right) \right\}.$$

*In particular, the jumps of  $T$  are rational numbers.*

*Proof.* <sup>5</sup> Letting  $\Gamma = \text{Gal}(L/K)$  and  $\Delta = \text{Gal}(L/M)$ , by [Bra03, Satz 0.4.3] the canonical surjection of  $\mathbb{Z}[\Gamma]$ -modules  $\mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma/\Delta]$  induces a short exact sequence of tori

$$0 \rightarrow \text{Res}_{M/K} \mathbb{G}_{m,M} \rightarrow \text{Res}_{L/K} \mathbb{G}_{m,L} \rightarrow T' \rightarrow 0,$$

and base-changing to  $K(d)$  one obtains an exact sequence

$$0 \rightarrow \text{Res}_{M(d)/K(d)} \mathbb{G}_{m,M(d)} \rightarrow \text{Res}_{L(d)/K(d)} \mathbb{G}_{m,L(d)} \rightarrow T'(d) \rightarrow 0,$$

where, as  $K(d)/K$  and  $L/K$  are linearly disjoint,  $M(d)$  is a field and  $L(d)$  is the Galois closure of  $M(d)/K(d)$ , with ramification and inertia indices over  $K(d)$  again equal to  $e$  and  $f$ .

By [Bra03, Satz 4.2.2],<sup>6</sup> the sequence of Néron models induced by the previous sequence remains exact; therefore, we can apply Corollary 4.3.3, whereby the statement follows from Proposition 3.2.25 and the computation in Proposition 4.3.8.  $\square$

*Remark 4.3.10.* By [AS02, Corollary A.2], any finite separable extension  $M/K$  can be embedded into a finite decomposable extension  $L/K$ . One can therefore, as suggested by Overkamp, embed  $\text{Res}_{M/K} \mathbb{G}_{m,M}$  into  $\text{Res}_{L/K} \mathbb{G}_{m,L}$  for some decomposable extension  $L/K$  and deduce the rationality of the jumps of  $\text{Res}_{M/K} \mathbb{G}_{m,M}$  (but not a bound on their denominator) from the computation of the jumps of an induced torus relative to a decomposable extension, which is much simpler than the case of a Galois extension.

In view of the computation in Proposition 4.3.8, one might wonder whether the formation of identity components of Néron models commutes not only with unramified extensions, but also with weakly unramified ones, or perhaps if this is at least true at the level of identity components; if it were, by the same argument as in Lemma 3.2.10 the jumps of a semi-abelian  $K$ -variety would remain unchanged after such an extension. We now present several counterexamples to this statement.

*Example 4.3.11* (A non-split torus  $T$  with  $c_{\text{tame}} = 0$  and an extension  $K'/K$  such that  $c_{\text{tame}}(T_{K'}) \neq 0$ ). Let  $K$ ,  $K'$  and  $F$  be as in Example 4.2.8. The torus  $T = \text{Res}_{F/K} \mathbb{G}_{m,F}$  is not split, but  $c_{\text{tame}}(T) = 0$  (notice in passing that this can only happen if  $k$  is not algebraically closed, by Remark 4.1.7); on the other hand,  $T_{K'} = \text{Res}_{F'/K'} \mathbb{G}_{m,F'}$ , and since  $F'/K'$  is totally ramified of degree  $p$ , by the previous computation its jumps are  $0, \frac{1}{p}, \dots, \frac{p-1}{p}$ . This shows in particular that the

<sup>5</sup>Our original proof contained a mistake; we thank Otto Overkamp for helping us fix it.

<sup>6</sup>Notice that the author calls a field “local” if it is a complete non-archimedean discretely valued field; in particular the result also holds in the case of a separably closed imperfect residue field.

formation of identity components of Néron models of tori is not compatible with weakly unramified base-change, even along extensions which are linearly disjoint from the splitting extension.

We shall see in Proposition 4.4.15 that, however, at least for a torus  $T$  which is a direct factor of  $K$ -rational varieties, if  $K'/K$  is a weakly unramified extension such that  $k'/k$  is linearly disjoint from the residue field extension of the splitting extension of  $T$ , then the base-change morphism  $h(T, K')$  is an open immersion.

*Example 4.3.12.* This example is in a sense orthogonal to the previous one, in that it shows that the formation of identity components of Néron models of tori is not compatible with weakly unramified base-change along the splitting extension.

Let  $F/K$  be a non-trivial weakly unramified extension of (say) strictly henselian fields of finite degree equal to the residue characteristic  $p$  of  $K$ : for an example in mixed characteristic, let  $k$  be any imperfect and separably closed field,  $\mathcal{O}_K$  be a Cohen ring with residue field  $k$ ,  $\lambda \in \mathcal{O}_K$  such that  $\bar{\lambda} \in k \setminus k^p$  and  $F = K[x]/(x^p - \lambda)$ ; by [Ser13, Proposition 15]  $F/K$  is weakly unramified and

$$\mathcal{O}_F = \mathcal{O}_K[x]/(x^p - \lambda).$$

In equal characteristic, let  $\mathcal{O}_K = k[[t]]$ ,  $\lambda \in k \setminus k^p$  and  $F = K[x]/(x^p - x + \lambda)$ , which again by *loc. cit.* is weakly unramified and has ring of integers

$$\mathcal{O}_F = \mathcal{O}_K[x]/(x^p - x + \lambda)$$

(as the Artin-Schreier polynomial  $x^p - x + \bar{\lambda}$  is irreducible over  $k$ ).

Let  $G = \text{Res}_{F/K} \mathbb{G}_{m,F}$ ; as in [NX91, Proposition 3.1],  $\mathcal{G}^o = \text{Res}_{\mathcal{O}_F/\mathcal{O}_K} \mathbb{G}_{m,\mathcal{O}_F}$ , and by [ABD<sup>+</sup>66, VI.A, Proposition 3.3]

$$(\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_F)^o = \mathcal{G}^o \times_{\mathcal{O}_K} \mathcal{O}_F;$$

on the other hand,  $G \times_K F = \mathbb{G}_{m,F}^p$ , whereby the identity component of its Néron model is simply  $\mathbb{G}_{m,\mathcal{O}_F}^p$ . It is easy to see that there exists no isomorphism of group  $\mathcal{O}_F$ -schemes

$$\left( \text{Res}_{\mathcal{O}_F/\mathcal{O}_K} \mathbb{G}_{m,\mathcal{O}_F} \right)_{\mathcal{O}_F} \rightarrow \mathbb{G}_{m,\mathcal{O}_F}^n :$$

indeed, the special fibre of the term on the left is isomorphic to

$$\text{Res}_{k_F/k} \mathbb{G}_{m,k_F},$$

which is unipotent, while that of the term on the right is a (split) multiplicative group.

### The motivic zeta function of an induced torus

For the rest of this section we assume that  $k = \bar{k}$  (since as far as we know, the motivic zeta function of a semi-abelian variety has only been defined under this assumption). In [Ove19, Theorem 2.4.5], the author used the computation of the jumps of an induced torus to identify a subring of  $K_0(\text{Var}_k)[[X]]$  containing the motivic zeta function of an induced torus relative to a purely wild extension  $M/K$ , without explicitly computing it, except for the case when  $p = e(M/K) = 2$ . Here we extend his computation to arbitrary purely wild extensions. Later we shall also compute the motivic zeta function of the maximal anisotropic quotient of such an induced torus (Proposition 4.4.12).

Let us first recall some definitions.

**Definition 4.3.13.** The Grothendieck ring of varieties over  $k$ , denoted by  $K_0(\text{Var}_k)$ , is defined as follows. The underlying abelian group of  $K_0(\text{Var}_k)$  is the free abelian group on the isomorphism classes  $[X]$  of  $k$ -varieties (by which we mean reduced, separated  $k$ -schemes of finite type) quotiented by the relation  $\sim$  generated by

$$[X] \sim [X \setminus Y] + [Y]$$

for any  $k$ -variety  $X$  and any closed subvariety  $Y$  of  $X$ ; multiplication extends linearly the operation  $[X] \cdot [Y] = [X \times_k Y]$ .

As customary, we shall denote by  $\mathbb{L}$  the class of  $\mathbb{A}_k^1$  in  $K_0(\text{Var}_k)$ .

For a thorough survey on the Grothendieck ring of varieties, the reader may consult [NS11].

Notice that since, by definition, the objects of  $K_0(\text{Var}_k)$  are isomorphism classes of  $k$ -varieties, for a semi-abelian variety  $G$  it would not usually make sense to consider the equivalence class of  $\mathcal{G}_k$  inside it; this issue can be circumvented by considering instead the maximal quasi-compact open subgroup scheme  $\mathcal{G}^{qc}$  of  $\mathcal{G}$ , whose existence is established in [HN11, Section 3].

We can now present the definition of the motivic zeta function of a semi-abelian variety, as formulated in [HN16, 8.1.1].

**Definition 4.3.14.** Let  $G$  be a semi-abelian  $K$ -variety. The motivic zeta function  $Z_G(X)$  of  $G$  is the power series

$$Z_G(X) = \sum_{d: (d,p)=1} [\mathcal{G}(d)_k^{qc}] \mathbb{L}^{\text{ord}_G(d)} X^d \in K_0(\text{Var}_k)[[X]].$$

**Proposition 4.3.15.** *Let  $M/K$  be a purely wild separable extension of degree  $e$ . The motivic zeta function of the induced torus  $T = \text{Res}_{M/K} \mathbb{G}_{m,M}$  is rational and belongs to the subring*

$$\mathcal{R}_k^{\text{ctame}(T)} := K_0(\text{Var}_k) \left[ X, \frac{1}{1 - \mathbb{L}^a X^b} \right]_{\substack{(a,b) \in \mathbb{Z} \times \mathbb{N}_{>0} \\ a/b = \text{ctame}(T)}}$$

of  $K_0(\text{Var}_k)[[X]]$ ; more precisely, one can write

$$Z_T(X) = \frac{(\mathbb{L} - 1) \mathbb{L}^{e-1} X \left( 1 - \mathbb{L}^{\frac{(e-1)(p-1)}{2}} X^{p-1} \right)}{\left( 1 - \mathbb{L}^{\frac{p(e-1)}{2}} X^p \right) \left( 1 - \mathbb{L}^{\frac{e-1}{2}} X \right)} \\ \in K_0(\text{Var}_k) \left[ X, \frac{1}{\left( 1 - \mathbb{L}^{\frac{p(e-1)}{2}} X^p \right) \left( 1 - \mathbb{L}^{\frac{e-1}{2}} X \right)} \right],$$

where we stipulate that  $\frac{1 - \mathbb{L}^{\frac{e-1}{2}} X}{1 - \mathbb{L}^{\frac{e-1}{2}} X} = 1$  also for even  $e$ .

*Proof.* For each  $d$  not divisible by  $p$  one has

$$\mathcal{T}(d)^{qc} = \mathcal{T}(d)^0 = \text{Res}_{\mathcal{O}_{M(d)}/\mathcal{O}_{K(d)}} \mathbb{G}_{m, \mathcal{O}_{M(d)}},$$

whose special fibre is easily seen to be isomorphic to  $\mathbb{G}_{m,k} \times \mathbb{A}_k^{e-1}$  as a  $k$ -scheme. Therefore,

$$[\mathcal{T}(d)_k^{qc}] = \mathbb{L}^{e-1} (\mathbb{L} - 1).$$

The  $d$ -jumps of  $T$  have already been computed in [Ove19, Corollary 2.4.3] to be  $0, \left\lfloor \frac{d}{e} \right\rfloor, \dots, \left\lfloor \frac{(e-1)d}{e} \right\rfloor$ ; from this one gets

$$\text{ord}_T(d) = \sum_{i=0}^{e-1} \left\lfloor \frac{id}{e} \right\rfloor = \frac{(d-1)(e-1)}{2},$$

the last equality following from [GKP89, 9.4].

It is now enough to find a closed formula for the sum

$$\sum_{d: (d,p)=1} (\mathbb{L} - 1) \mathbb{L}^{\frac{(d+1)(e-1)}{2}} X^d.$$

It is convenient to treat the case  $p = 2$  separately. If  $p = 2$ , one has

$$Z_T(X) = (\mathbb{L} - 1) \sum_{m=0}^{\infty} \mathbb{L}^{\frac{(2m+2)(e-1)}{2}} X^{2m+1} = \\ (\mathbb{L} - 1) \mathbb{L}^{e-1} X \sum_{m=0}^{\infty} \left( \mathbb{L}^{e-1} X^2 \right)^m = \frac{(\mathbb{L} - 1) \mathbb{L}^{e-1} X}{1 - \mathbb{L}^{e-1} X^2}.$$

If  $p$  is odd, then so is  $e$ , and the number  $e^* := \frac{e-1}{2}$  is an integer; write

$$Z_T(X) = (\mathbb{L} - 1) \sum_{\alpha=1}^{p-1} Z_T^{(\alpha)}(X),$$

where

$$Z_T^{(\alpha)}(X) := \sum_{m=0}^{\infty} \mathbb{L}^{(\alpha+mp+1)e^*} X^{\alpha+mp},$$

as

$$Z_T^{(\alpha)}(X) = X^{\alpha} \mathbb{L}^{(\alpha+1)e^*} \sum_{m=0}^{\infty} (\mathbb{L}^{pe^*} X^p)^m = \frac{X^{\alpha} \mathbb{L}^{(\alpha+1)e^*}}{1 - \mathbb{L}^{pe^*} X^p},$$

one has

$$\begin{aligned} Z_T(X) &= \frac{\mathbb{L} - 1}{1 - \mathbb{L}^{pe^*} X^p} \sum_{\alpha=1}^{p-1} X^{\alpha} \mathbb{L}^{(\alpha+1)e^*} = \\ &= \frac{(\mathbb{L} - 1) \mathbb{L}^{e-1} X}{1 - \mathbb{L}^{pe^*} X^p} \sum_{\alpha=0}^{p-2} (\mathbb{L}^{e^*} X)^{\alpha} = \frac{(\mathbb{L} - 1) \mathbb{L}^{e-1} X \left(1 - \mathbb{L}^{\frac{(e-1)(p-1)}{2}} X^{p-1}\right)}{\left(1 - \mathbb{L}^{\frac{p(e-1)}{2}} X^p\right) \left(1 - \mathbb{L}^{\frac{e-1}{2}} X\right)}. \quad \square \end{aligned}$$

#### 4.4 Jumps of $K$ -tori which are direct factors of $K$ -rational varieties

This section is devoted to the study of jumps of a particular class of  $K$ -tori, namely, those which are direct factors of  $K$ -rational varieties (Definition 4.4.6): Corollary 4.4.10 describes the jumps of such a torus  $T$  in terms of a flasque resolution of  $T$  (a notion which will be recalled below) and shows in particular that they are rational, also giving an optimal bound for their denominators in terms of the ramification index of the minimal splitting extension of  $T$ . We thank Otto Overkamp for suggesting the main results Proposition 4.4.9 and Corollary 4.4.10, on which the whole section hinges, in the case of tori split by a metacyclic extension.

Combined with some results of the previous section, these have several further applications: we use the equality  $c_{tame}(T) = u(T)/2$  of Proposition 4.1.3 to tori which are direct factors of  $K$ -rational varieties and have totally ramified splitting extension, even if the residue field  $k$  is imperfect (Corollary 4.4.17) and compute the motivic zeta function of the quotient of an induced torus by its maximal split subtorus (Proposition 4.4.12); we then examine the problem of whether base-change along an extension  $K'/K$  which is linearly disjoint from the minimal splitting extension  $L/K$  of a  $K$ -torus  $T$  as above changes the jumps of  $T$ , and show that if their residue field extensions are linearly disjoint,  $h(T, K')$  is an open immersion

(Proposition 4.4.15). We also provide an example (Example 4.4.14) of a short exact sequence of tori with split rightmost term whose associated sequence of Néron models is not exact, which prevents a direct application of the arguments used in this section to more general tori (e.g. norm tori).

Let us review some definitions and results from [CTS77] and [EM75] which form the basis for this section.

Let  $G$  be a finite group (which for us will be the Galois group of a Galois splitting extension of a torus). For any  $G$ -module  $A$  and any integer  $i$ , denote by  $\hat{H}^i(G, A)$  the  $i$ -th Tate cohomology group [Ser13, §VIII.1].

**Definition 4.4.1** ([CTS77, §1]). A  $\mathbb{Z}$ -free  $G$ -module of finite rank (over  $\mathbb{Z}$ ) is called

- (i) a *permutation module* if it admits a  $\mathbb{Z}$ -basis permuted by  $G$ ;
- (ii) *invertible* if it is a direct factor of a permutation module;
- (iii) *flasque* (resp. *coflasque*) if  $\hat{H}^{-1}(H, A) = 0$  (resp.  $\hat{H}^1(H, A) = 0$ ) for all subgroups  $H$  of  $G$ .

*Remark 4.4.2.* A  $G$ -module  $A$  is a permutation module if and only if  $A \cong \prod_{i=1}^r \mathbb{Z}[G/H_i]$  for some subgroups  $H_1, \dots, H_r$  of  $G$ : clearly any  $G$ -module of this form is a permutation module; conversely, if  $\mathcal{B} = \{a_1, \dots, a_s\}$  is a  $\mathbb{Z}$ -basis of  $A$  which is permuted by  $G$ , considering the different  $G$ -orbits we can assume that  $G$  acts on  $\mathcal{B}$  transitively. Then  $A \cong \mathbb{Z}[G/G_a]$  as a  $G$ -module, where  $G_a$  is the stabiliser of some element  $a \in \mathcal{B}$ .

**Definition 4.4.3.** Let  $A$  be a  $\mathbb{Z}$ -free  $G$ -module of finite rank.

- (i) A flasque resolution of  $A$  is a short exact sequence

$$0 \rightarrow A \rightarrow P \rightarrow F \rightarrow 0$$

with  $P$  a permutation  $G$ -module and  $F$  a flasque  $G$ -module.

- (ii) A coflasque resolution of  $A$  is a short exact sequence

$$0 \rightarrow C \rightarrow R \rightarrow A \rightarrow 0$$

with  $R$  a permutation  $G$ -module and  $C$  a coflasque  $G$ -module.

Any  $\mathbb{Z}$ -free  $G$ -module of finite rank admits a (co)flasque resolution:

**Lemma 4.4.4** ([CTS77, Lemme 3,5], [EM75, Lemma 1.1]). *Any  $\mathbb{Z}$ -free  $G$ -module of finite rank  $A$  admits a flasque resolution*

$$0 \rightarrow A \rightarrow P \rightarrow F \rightarrow 0$$

*and a coflasque resolution*

$$0 \rightarrow C \rightarrow R \rightarrow A \rightarrow 0;$$

*moreover, the  $G$ -modules  $F$  and  $C$  are unique up to addition of permutation modules.*

The previous notions can be also formulated for tori, by the duality between  $K$ -tori split that split over a Galois extension  $L/K$  with Galois group  $G$  and  $\mathbb{Z}$ -free  $G$ -modules of finite rank: so for example, for any such torus  $T$  one says that  $T$  is flasque if its character  $G$ -module is flasque, a flasque resolution of  $T$  is a short exact sequence  $0 \rightarrow T_1 \rightarrow T_2 \rightarrow T \rightarrow 0$  with  $T_1$  and  $T_2$  a flasque and a permutation torus, respectively, both split over  $L$ , and so on.

*Remark 4.4.5.* By Remark 4.4.2, a torus  $T$  is a permutation torus if and only if it is isomorphic to a finite product of induced tori. In particular, both the properties of being a permutation or an invertible torus are stable under base-change along a finite separable extension of the base-field. See [CTS87, Proposition 1.4] for a much more general statement.

The reason we are interested in a flasque resolution  $0 \rightarrow F \rightarrow P \rightarrow T \rightarrow 0$  of a  $K$ -torus  $T$  is that if  $F$  is invertible, then by Corollary 4.3.3 the induced sequence of Néron models

$$0 \rightarrow \mathcal{F}(d) \rightarrow \mathcal{P}(d) \rightarrow \mathcal{T}(d) \rightarrow 0$$

is exact for any  $d$  coprime to  $p$ ; a result by Overkamp allows one to obtain the multiset of jumps of  $T$  as the difference between that of  $P$  and that of  $F$ . We recall some conditions on  $T$ , due to Colliot-Thélène and Sansuc, which are equivalent to the invertibility of the torus  $F$ ; several more are listed in [CTS87, Proposition 7.4].

**Definition 4.4.6.** A  $K$ -rational variety is a variety which is  $K$ -birational to a finite dimensional affine space over  $K$ ; a  $K$ -torus  $T$  is a direct factor of a  $K$ -rational variety if there exist a  $K$ -variety  $Y$  such that  $Y \times_K T$  is a  $K$ -rational variety.

**Proposition 4.4.7** ([CTS87, Proposition 7.4]). *Let  $0 \rightarrow F \rightarrow P \rightarrow T \rightarrow 0$  be a flasque resolution of a torus  $T$  over a field  $K$ . The following conditions are equivalent:*

- (i)  $F$  is an invertible  $K$ -torus;
- (ii)  $T$  is a direct factor of a  $K$ -rational variety;
- (iii) for any field extension  $K \hookrightarrow K'$ ,  $H^1(K', F_{K'}) = 0$ .

One has in addition the following theorem by Endo and Miyata, which implies that the conditions above are satisfied whenever  $T$  becomes split over a metacyclic extension (i.e. a Galois extension whose Galois group is metacyclic, as defined below):

**Theorem 4.4.8** ([EM75, Theorem I]). *For a finite group  $G$ , the following conditions are equivalent:*

- (i)  $G$  is metacyclic, i.e. its Sylow  $p$ -subgroup is cyclic;
- (ii) any flasque  $G$ -module is invertible.

To put this into some context, recall that if  $L/K$  is a finite Galois extension of discretely valued fields inducing a separable extension of residue fields, the inertia group  $\Gamma_0$  is cyclic if the residue characteristic  $p$  is 0 [Ser13, Corollary 1 to Proposition IV.2.7] and is a semi-direct product of a cyclic group of order coprime to  $p$  and a (normal)  $p$ -subgroup if  $p > 0$  [*ibid.*, Corollary 4]. Since for our purposes  $K$  can be assumed to be strictly henselian, one can assume that  $\Gamma_0 = \Gamma$  if  $\bar{k} = k$ ; the case of a metacyclic Galois group is then the simplest non-trivial case.

The next result will be fundamental for us: it shows that if  $T$  is a direct factor of a  $K$ -rational variety, any flasque resolution of  $T$  induces an exact sequence of Néron models.

**Proposition 4.4.9.** *Let  $T$  be a torus over a discretely valued field  $K$ . If  $T$  is a direct factor of a  $K$ -rational variety, then the sequence of Néron models*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{P} \rightarrow \mathcal{T} \rightarrow 0$$

*induced by any flasque resolution of  $T$  is exact.*

*Proof.* Let  $L/K$  be a Galois splitting extension of  $T$ . Consider the long exact sequence of sheaves on the smooth site over  $\mathcal{O}_K$

$$0 \rightarrow j_*^{sm} F \rightarrow j_*^{sm} P \rightarrow j_*^{sm} T \rightarrow R^1 j_*^{sm} F \rightarrow R^1 j_*^{sm} P \rightarrow R^1 j_*^{sm} T \rightarrow \cdots ;$$

if  $T$  is a direct factor of a  $K$ -rational variety, there exist a flasque  $K$ -torus  $F'$  and a permutation  $K$ -torus  $P'$ , both split by  $L$ , such that  $P' \cong F \times F'$  (Proposition 4.4.7).

Therefore, the identity of  $R^1 j_*^{sm} F$  factors through  $R^1 j_*^{sm} P'$ , which vanishes since any permutation torus is a finite direct product of induced tori  $T_i$  and  $R^1 j_*^{sm} T_i = 0$  for all  $i$  [Bra03, Satz 4.2.2]; Proposition 4.3.3 then implies the statement.  $\square$

The next result states in particular that the jumps of any  $K$ -torus which is a direct factor of a  $K$ -rational variety are rational.

**Corollary 4.4.10.** *(i) Let  $T$  be a  $K$ -torus which is a direct factor of a  $K$ -rational variety and  $0 \rightarrow F \rightarrow P \rightarrow T \rightarrow 0$  flasque resolution; then*

$$J_d(P) = J_d(F) \uplus J_d(T)$$

*for any  $d$  coprime to  $p$ .*

*(ii) More precisely, all the  $d$ -jumps of  $T$  are of the form  $a(d-1)/b$  with  $a, b \in \mathbb{N}$ ,  $b$  a divisor of the ramification index of a subextension of the minimal splitting extension  $L/K$  of  $T$ ,  $a < b$  and  $(a, b) = 1$ .*

*(iii) All the jumps of  $T$  are rational numbers of the form  $a/b$ , with  $a$  and  $b$  as in the previous point. In particular, each jump is equal to  $h/e(L/K)$  for some non-negative integer  $h$  smaller than  $e(L/K)$ .*

*Proof.* Since the property of being an invertible/permutation torus is stable under base-change, for any  $d$  coprime to  $p$

$$0 \rightarrow F(d) \rightarrow P(d) \rightarrow T(d) \rightarrow 0$$

is a flasque resolution of  $T(d)$  and  $F(d)$  is invertible. By the previous proposition, the sequence

$$0 \rightarrow \mathcal{F}(d) \rightarrow \mathcal{P}(d) \rightarrow \mathcal{T}(d) \rightarrow 0$$

is again exact, whereby one can apply Proposition 3.2.25(i) and obtain the first statement.

The second point follows directly from this and the computation of the jumps of an induced torus in Theorem 4.3.9; applying 3.2.25(ii) one also obtains the last statement.  $\square$

If a flasque resolution of  $T$  is known, one can hope to obtain more precise results; as an easy example, let us mention the following

**Corollary 4.4.11.** *Let  $M/K$  be a finite separable extension of ramification and inertia indices  $e, f$ , respectively. Assume that  $M/K$  is either decomposable or Galois.*

For any  $d \equiv 1 \pmod{e}$ ,  $d > e$ , the  $d$ -jumps of the quotient torus

$S = \left( \text{Res}_{M/K} \mathbb{G}_{m,M} \right) / \mathbb{G}_{m,K}$  are

$$\frac{d-1}{e}, \dots, (e-1)\frac{d-1}{e},$$

each with multiplicity  $f$ , and 0 with multiplicity  $f-1$ . Therefore, the jumps of  $S$  are  $\frac{1}{e}, \dots, \frac{e-1}{e}$ , each with multiplicity  $f$ , and 0 with multiplicity  $f-1$ .

We can use this result to compute the motivic zeta function of  $S$ , if  $M/K$  is purely wild and  $k$  is algebraically closed; recall that, with the same notation as in Proposition 4.3.15,  $\mathcal{R}_k^{c_{tame}(S)}$  is the subring

$$\mathcal{R}_k^{c_{tame}(S)} = K_0(\text{Var}_k) \left[ X, \frac{1}{1 - \mathbb{L}^a X^b} \right]_{\substack{(a,b) \in \mathbb{Z} \times \mathbb{N}_{>0} \\ a/b = c_{tame}(S)}}$$

of  $K_0(\text{Var}_k)[[X]]$  (notice that, by the previous computation,  $c_{tame}(S) = c_{tame}(T)$ ).

**Proposition 4.4.12.** *Assume that  $k$  is algebraically closed and let  $M/K$  be a purely wild separable extension of degree  $e$ . Denoting by  $T$  the induced torus  $T = \text{Res}_{M/K} \mathbb{G}_{m,M}$  and by  $S$  the quotient torus  $S = T / \mathbb{G}_{m,K}$ , the motivic zeta function of  $S$  is rational and belongs to the ring  $\mathcal{R}_k^{c_{tame}(S)}$ ; more precisely, one has*

$$Z_S(X) = \frac{e \mathbb{L}^{e-1} X \left( 1 - \mathbb{L}^{\frac{(e-1)(p-1)}{2}} X^{p-1} \right)}{\left( 1 - \mathbb{L}^{\frac{p(e-1)}{2}} X^p \right) \left( 1 - \mathbb{L}^{\frac{e-1}{2}} X \right)} \\ \in K_0(\text{Var}_k) \left[ X, \frac{1}{\left( 1 - \mathbb{L}^{\frac{p(e-1)}{2}} X^p \right) \left( 1 - \mathbb{L}^{\frac{e-1}{2}} X \right)} \right],$$

where we stipulate that  $\frac{1 - \mathbb{L}^{\frac{e-1}{2}} X}{1 - \mathbb{L}^{\frac{e-1}{2}} X} = 1$  also for even  $e$ .

*Proof.* We want to reduce to the computation performed in the proof of Proposition 4.3.15. The previous corollary implies that  $\text{ord}_S(d) = \text{ord}_T(d)$  for each  $d$  coprime to  $p$ , so it remains to compute  $[\mathcal{S}(d)_k^{qc}]$ . In [HN16, Proposition 8.1.2.2] it is proved that for any semi-abelian  $K$ -variety  $G$  one has

$$[\mathcal{G}(d)_k^{qc}] = |\Phi(G(d))_{\text{tors}}| (\mathbb{L} - 1)^{t(G(d))} \mathbb{L}^{u(G(d))} [B_G(d)],$$

where  $t$  and  $u$  denote the toric and unipotent rank, respectively, and  $B_G(d)$  is the abelian part in the Chevalley decomposition of  $\mathcal{G}(d)_k^0$ . This is easily computed for  $G = S$  thanks to [Ove19, Corollary 3.0.1]: one sees immediately that  $[B_S(d)] = 1$

and, recalling that if  $G$  is a torus,  $t(G(d))$  is also equal to the rank of the maximal split subtorus of  $G(d)$  [HN10, 3.13], one has  $t(G(d)) = 0$  by point i), whence  $u(G(d)) = e - 1$ ; finally,  $|\Phi(G(d))_{\text{tors}}| = [M : K] = e$  by point ii).

The upshot is that

$$[\mathcal{S}(d)_k^{qc}] = \frac{e}{\mathbb{L} - 1} [\mathcal{T}(d)_k^{qc}]$$

for all  $d$ , whence the result, by the definition of the motivic zeta function and Proposition 4.3.15.  $\square$

*Remark 4.4.13.* If  $L/K$  is a cyclic extension, the norm torus

$$\ker\left(N: \text{Res}_{L/K} \mathbb{G}_{m,L} \rightarrow \mathbb{G}_{m,K}\right)$$

is isomorphic to the quotient torus  $(\text{Res}_{L/K} \mathbb{G}_{m,L})/\mathbb{G}_{m,K}$  [LL01, Lemma 4.1.b] (and, at least if  $k = \bar{k}$ , the reverse implication holds as well, by [LL01, Proposition 4.17.4]). However, the jumps for an arbitrary norm torus seem to be quite more difficult to compute: as we shall show in the following example, unlike for a short exact sequence of tori whose leftmost term is split, the sequence of Néron models associated to a short exact sequence of tori whose *rightmost* term is split is not necessarily exact. This answers a question posed to the author by Nicaise. After writing it down, we learnt that this had been already shown by Brahm [Bra03, Satz 4.1.11 and following], with a different proof; nevertheless, we include ours, as the geometric description of the terms involved might be interesting.

*Example 4.4.14.* [A short exact sequence of tori with split right-most term whose associated sequence on Néron models is not exact as a sequence of group schemes] Let  $K$  be a valued field mixed characteristic  $(0, 2)$  with uniformiser  $\pi$ , consider the degree-two extension  $L = \frac{K[x]}{(x^2 - \pi)}/K$  and let  $T_2 = \text{Res}_{L/K} \mathbb{G}_{m,L}$ ,  $T_1$  the norm torus  $\ker(N: T_2 \rightarrow \mathbb{G}_{m,K})$ , and  $j: \text{Spec } K \rightarrow \text{Spec } \mathcal{O}_K$  be the inclusion of the generic fibre. Denote by  $\mathcal{G}$  and  $\mathcal{G}'$  the Néron models of  $\mathbb{G}_{m,K}$  and  $\mathbb{G}_{m,L}$ , respectively.

We shall explicitly describe  $\mathcal{T}_2$  and the induced map  $\mathcal{N}: \mathcal{T}_2 \rightarrow \mathcal{G}$ , and from this deduce that the the Néron model of  $T_1$  is not equal to  $\ker \mathcal{N}$  in the category of group schemes. Recall that  $\mathcal{G}$  is obtained by gluing copies of  $\mathbb{G}_{m,\mathcal{O}_K}$ , one for each integer: more precisely, for any two integers  $i, j$ , the  $i$ -th copy  $\mathbb{G}_{m,\mathcal{O}_K}^{(i)}$  is glued to  $\mathbb{G}_{m,\mathcal{O}_K}^{(j)}$  via the identification of their generic fibres given on points by multiplication by  $\pi^{j-i}$ .

Since Weil restriction commutes with  $j_*$ , one has in addition  $\mathcal{T}_2 = \text{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathcal{G}'$ ; we shall show that  $\mathcal{T}_2$  is obtained by gluing copies of  $\text{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m,\mathcal{O}_L}$  in a way analogous to the above. To make the previous statement more precise, observe first that the gluing datum defining  $\mathcal{G}'$  also determines a gluing datum upon applying

the Weil restriction along  $\mathrm{Spec} \mathcal{O}_L \rightarrow \mathrm{Spec} \mathcal{O}_K$ , because the latter functor preserves open immersions [BLR90, Proposition 7.6/2] as  $\mathcal{O}_L$  is free of finite rank over  $\mathcal{O}_K$ . Denoting by  $\mathcal{T}$  the corresponding  $\mathcal{O}_K$ -scheme, we claim that  $\mathcal{T}$  is isomorphic to  $\mathcal{T}_2$ .

Observe first that each copy of  $\mathbb{G}_{m, \mathcal{O}_L}$  is open in  $\mathcal{G}'$ , because it is a translate of the identity component, which is open (or simply because its complement is equal to the special fibre minus one copy of  $\mathbb{G}_{m, k}$ , which is closed since the special fibre is a disjoint union of such copies); therefore, each copy of  $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}$  is open in  $\mathcal{T}_2$ , whereby the open immersions  $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}^{(i)} \hookrightarrow \mathcal{T}_2$  glue (by the very construction of  $\mathcal{G}'$ ) to an open immersion  $\alpha: \mathcal{T} \rightarrow \mathcal{T}_2$ . We shall check fibrewise that  $\alpha$  is an isomorphism.

Denoting, as usual, by  $k$  the residue field of  $\mathcal{O}_K$  and by  $k[\epsilon]$  the ring of dual numbers over  $k$  and using the compatibility of Weil restriction with fibre products, we see that  $\alpha$  restricts to the canonical map

$$\bigsqcup_{i \in \mathbb{Z}} \mathrm{Res}_{k[\epsilon]/k} \mathbb{G}_{m, k[\epsilon]}^{(i)} \rightarrow \mathrm{Res}_{k[\epsilon]/k} \bigsqcup \mathbb{G}_{m, k[\epsilon]}^{(i)}$$

on the special fibre, which is an isomorphism either by [JLM<sup>+</sup>22, Proposition 2.8] (observe that the assumption there that the disjoint union be finite is only used to ensure that its Weil restriction is representable by a scheme, but plays no role in the proof) or by [BGA16, Proposition 2.2] (as we already know that  $\alpha_k$  is an open immersion). Similarly, the restriction of  $\alpha$  to the generic fibre is the identity of  $\mathrm{Res}_{L/K} \mathbb{G}_{m, L}$ .

We now claim that the restriction of  $\mathcal{N}: \mathcal{T}_2 \rightarrow \mathcal{G}$  to  $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}^{(n)}$  is the norm map  $N_{\mathcal{O}_L/\mathcal{O}_K}: \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}^{(n)} \rightarrow \mathbb{G}_{m, \mathcal{O}_K}^{(2n)}$ . Assuming this, it is clear that the sequence of Néron models is not exact on the right, and it is not hard to see that the sequence of Néron models cannot be exact on the left either: indeed, since topologically every point of the kernel is mapped to  $\mathcal{G}^o$  and the claim implies that  $\mathcal{N}^{-1}(\mathcal{G}^o) = \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}^{(0)}$ , one has

$$\ker \mathcal{N} = \ker(N_{\mathcal{O}_L/\mathcal{O}_K}: \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}^{(0)} \rightarrow \mathbb{G}_{m, \mathcal{O}_K});$$

as for any  $\mathcal{O}_K$ -algebra  $A$ ,  $N_{\mathcal{O}_L/\mathcal{O}_K, A}$  takes  $a_0 + a_1x$  to  $a_0^2 - a_1^2\pi$ ,  $\ker N_{\mathcal{O}_L/\mathcal{O}_K}$  is represented by the non-smooth scheme  $\mathrm{Spec} \frac{\mathcal{O}_K[x_0, x_1]}{(x_0^2 - \pi x_1^2 - 1)}$ , and cannot therefore be the Néron lft-model of  $U$ .

In order to prove the claim, observe that the only non-trivial part is showing that the norm maps  $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, \mathcal{O}_L}^{(n)} \rightarrow \mathbb{G}_{m, \mathcal{O}_K}^{(2n)}$  glue. Notice first that the restriction of  $N_{\mathcal{O}_L/\mathcal{O}_K}$  to  $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m, L}$  factors through the open subscheme  $\mathbb{G}_{m, K}$  of  $\mathbb{G}_{m, \mathcal{O}_K}$ .

Indeed, for any  $\mathcal{O}_K$ -algebra  $A$  one has

$$\mathbb{G}_{m,K}(A) = \mathrm{Hom}_{\mathcal{O}_K}(K[x^{\pm 1}], A) = \begin{cases} A^\times & \text{if } A \text{ is a } K\text{-algebra} \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathbb{G}_{m,\mathcal{O}_L}(A) = \begin{cases} (A \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times & \text{if } A \otimes_{\mathcal{O}_K} \mathcal{O}_L \text{ is an } L\text{-algebra} \\ \emptyset & \text{otherwise} \end{cases};$$

now  $A \otimes_{\mathcal{O}_K} \mathcal{O}_L$  is an  $L$ -algebra if and only if  $A$  is a  $K$ -algebra: one implication is clear; for the other one, write  $A \otimes_{\mathcal{O}_K} \mathcal{O}_L = A[y]/(y^2 - \pi)$ . This is an  $L$ -algebra precisely if  $y$  is invertible, i.e. if there exist  $f, g \in A[y]$  such that  $yf(y) = (y^2 - \pi)g(y) + 1$ , which is only possible if the degree-zero coefficient of  $g$  is an inverse of  $\pi$ .

The claim now follows from the commutativity, for any  $K$ -algebra  $A$  and for any integer  $n$ , of the diagram

$$\begin{array}{ccc} (A \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times & \xrightarrow{N} & A^\times \\ \downarrow \cdot \pi^n & & \downarrow \cdot \pi^{2n} \\ (A \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times & \xrightarrow{N} & A^\times \end{array}.$$

As another application of Proposition 4.4.9, we show that if  $T$  is a torus which is a direct factor of a  $K$ -rational variety and  $K'/K$  a weakly unramified extension such that  $k'/k$  is linearly disjoint from the residue field extension of the minimal splitting extension of  $T$ , then the base-change morphism  $h(T, K')$  is an isomorphism. This is not a trivial statement, as the examples collected in the previous section witness.

**Proposition 4.4.15.** *Let  $T$  be a  $K$ -torus which is a direct factor of a  $K$ -rational variety; let  $L/K$  be the minimal splitting extension of  $T$ . If  $K'/K$  is a weakly unramified extension such that  $l/k$  is linearly disjoint from  $k'/k$ , then the base-change morphism*

$$h(T, K'): \mathcal{T}^0 \times_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow (\mathcal{T}')^0$$

*is an open immersion of group schemes.*

*Proof.* By Proposition 3.2.6, it is enough to prove that  $\mathrm{Lie} h(T, K')$  is an isomorphism. Consider the exact (by Proposition 4.4.9) sequence of Néron models induced by a flasque resolution  $0 \rightarrow F \rightarrow P \rightarrow T \rightarrow 0$  of  $T$ ; since  $\mathcal{F}$  is smooth, by descent the map  $\mathcal{P} \rightarrow \mathcal{T}$  is smooth as well, whereby [LLR04, Proposition 1.1c),e)] implies that the induced sequence of Lie algebras

$$0 \rightarrow \mathrm{Lie} \mathcal{F} \rightarrow \mathrm{Lie} \mathcal{P} \rightarrow \mathrm{Lie} \mathcal{T} \rightarrow 0$$

is exact; by the naturality of the base-change morphism  $h$ , we can then assume that  $T$  is an induced torus  $\text{Res}_{M/K} \mathbb{G}_{m,M}$ .

As in the proof of Proposition 4.3.8,  $h(T, K')$  induces the canonical morphism

$$\mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{O}_{M \otimes_K K'}$$

between the Lie algebra of  $\mathcal{T}_{\mathcal{O}_{K'}}$  and that of  $\mathcal{T}'$ . Observe that  $K' \cap M = K$  by the assumption on residue field extensions, whereby  $K'/K$  and  $M/K$  are linearly disjoint by Proposition 4.2.6*i*): therefore,  $M \otimes_K K'$  is canonically identified with the compositum  $M'$  of  $M/K$  and  $K'/K$  inside  $K^{\text{sep}}$ . Proposition 4.2.6*iii*) then implies that the previous map is an isomorphism.  $\square$

**Corollary 4.4.16.** *In the notation of the previous proposition, one has  $J_d(T_{K'}) = J_d(T)$  for all  $d \in \mathbb{N}_{(p)}$ .*

*Proof.* The  $d$ -jumps of  $T' = T_{K'}$  are the elementary divisors of the bottom horizontal map in the diagram

$$\begin{array}{ccc} \text{Lie } \mathcal{T} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'(d)} & \xrightarrow{\text{Lie } h(T, K(d)) \otimes \mathcal{O}_{K'(d)}} & \text{Lie } \mathcal{T}(d) \otimes_{\mathcal{O}_{K(d)}} \mathcal{O}_{K'(d)} \\ \downarrow \text{Lie } h(T, K') \otimes \mathcal{O}_{K'(d)} \wr & & \downarrow \wr \text{Lie } h(T(d), K'(d)) \\ \text{Lie } \mathcal{T}' \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{K'(d)} & \xrightarrow{\text{Lie } h(T', K'(d))} & \text{Lie } \mathcal{T}'(d) \end{array} \quad ,$$

and, since the extension  $K(d) \rightarrow K'(d)$  is weakly unramified, the  $d$ -jumps of  $T$  are those of the top horizontal map. The statement then follows by the vertical isomorphisms.  $\square$

Interestingly enough, despite the cautionary Example 4.1.5, this also yields an extension the formula  $c_{\text{tame}}(T) = \frac{1}{2}u(T)$  of Proposition 4.1.3 to totally ramified  $K$ -tori (i.e. tori whose minimal splitting extension is totally ramified) which are direct factors of  $K$ -rational varieties, even if  $k$  is imperfect:

**Corollary 4.4.17.** *Let  $T$  be a  $K$ -torus which is a direct factor of a  $K$ -rational variety. If  $T$  is totally ramified, then*

$$c_{\text{tame}}(T) = \frac{1}{2}u(T).$$

*Proof.* By Lemma 4.2.3*iii*), there exists a weakly unramified extension  $K'/K$  such that  $k'$  is algebraically closed. Since the minimal splitting extension  $L/K$  of  $T$  is, by assumption, totally ramified,  $K'/K$  and  $L/K$  trivially induce linearly disjoint residue fields extensions, and base-changing to  $K'$  one reduces the statement to that

of Proposition 4.1.3: indeed, the formula  $u(T) = \dim T - \mathrm{rk}_{\mathbb{Z}} \chi(T)^{\mathrm{Gal}(L/K)}$  shows that  $u$  remains unchanged after a base-change to an extension which is linearly disjoint from  $L/K$ ; on the other hand, by the previous corollary  $c_{\mathrm{tame}}(T_{K'}) = c_{\mathrm{tame}}(T)$ .  $\square$

It seems reasonable to conjecture that the failure of the previous formula to hold for a torus  $T$  is related to the presence of some non-trivial inertia in the minimal splitting extension of  $T$ , but we do not know how to prove this for general tori. It would also be interesting to understand whether one can introduce a natural corrective term and obtain a formula for  $c_{\mathrm{tame}}$  which holds for arbitrary tori.

We conclude with some remarks on further developments which seem to be within reach using the results obtained so far.

In the proof of Corollary 4.4.16, we used the fact that  $\mathrm{Lie} h(T, K')$  is an isomorphism to conclude that base-change to  $K'$  does not alter the  $d$ -jumps of  $T$ ; on the other hand, it is not hard to find examples of  $K$ -tori  $T$  and extensions  $K'/K$  such that  $J_d(T_{K'}) = J_d(T)$ , but  $\mathrm{Lie} h(T, K')$  is not an isomorphism: for example, since by Proposition 4.3.8 the jumps of an induced torus  $\mathrm{Res}_{M/K} \mathbb{G}_{\mathrm{m}, M}$  only depend on the ramification and inertia degrees of  $M/K$  if  $M/K$  is decomposable or Galois, one immediately obtains the following fact, which we state as a lemma for reference:

**Lemma 4.4.18.** *Let  $M/K$  be a finite separable totally ramified extension of a strictly henselian field  $K$  with algebraically closed residue field,  $T = \mathrm{Res}_{M/K} \mathbb{G}_{\mathrm{m}, M}$ . If  $M/K$  is either decomposable or Galois, then for any extension  $K'/K$  which is linearly disjoint from  $M/K$  and any  $d \in \mathbb{N}_{(p)}$ , one has  $J_d(T_{K'}) = J_d(T)$ .*

Recall that this is no longer true if  $M/K$  has non-trivial inertia, as Example 4.3.11 illustrates. We do not know whether the statement remains true if  $k$  is imperfect and both  $M/K$  and  $K'/K$  are totally ramified (still assuming their linear disjointness), because we do not know whether this condition implies that their compositum is again totally ramified.

Perhaps more surprisingly, if  $k$  is not algebraically closed, then the jumps of an induced torus  $\mathrm{Res}_{M/K} \mathbb{G}_{\mathrm{m}, M}$  can remain the same even after base-change to an extension that intersects  $M$  non-trivially:

**Lemma 4.4.19.** *Let  $M/K$  be a finite separable extension,  $T = \mathrm{Res}_{M/K} \mathbb{G}_{\mathrm{m}, M}$ ,  $F/K$  a maximal weakly unramified subextension of  $M/K$ . If  $K'/K$  is a weakly unramified extension such that  $F'/K$  is weakly unramified for all  $F' \in \mathcal{C}_K(F, K')$ , then  $J_d(T_{K'}) = J_d(T)$  for all  $d \in \mathbb{N}_{(p)}$ .*

*Proof.* Let  $d \in \mathbb{N}_{(p)}$ . By Lemma 4.2.16,

$$T' = \prod_{M' \in \mathcal{C}_K(M, K')} \text{Res}_{M'/K'} \mathbb{G}_{m, M'},$$

whereby

$$J_d(T') = \bigsqcup_{M' \in \mathcal{C}_K(M, K')} J_d(\text{Res}_{M'/K'} \mathbb{G}_{m, M'});$$

by Proposition 4.3.8,

$$J_d(\text{Res}_{M'} K') = \left\{ (0; f(M'/K')), \dots, \left( \frac{(d-1)(e(M'/K') - 1)}{e(M'/K')}; f(M'/K') \right) \right\}$$

for all  $M'$  and

$$J_d(T) = \left\{ (0; f(M/K)), \dots, \left( \frac{(d-1)(e(M/K) - 1)}{e(M/K)}; f(M/K) \right) \right\}.$$

By Lemma 4.2.17,  $e(M'/K') = e(M/K)$ , implying that  $|J_d(T)| = |J_d(T')|$  for all  $M'$  (recall that  $|J_d(T)|$  denotes the support of the multiset  $J_d(T)$ ); since in addition

$$\sum_{M' \in \mathcal{C}_K(M, K')} f(M'/K') = f(M/K),$$

the multiplicities of each jump agree as well, whence the statement.  $\square$

It would be desirable to extend Lemmas 4.4.18 and 4.4.19 to tori that are direct factors of  $K$ -rational varieties; for this, it would be enough to bootstrap said results to direct factors of induced tori, but doing so is not immediate: consider an induced torus  $T$  that can be written as a product  $T = T_1 \times T_2$  and an extension  $K'/K$  as in either lemma. We just computed that the jumps of the base-change  $T'$  are the same as those of  $T$ , whereby those of  $T'_1$  are a subset of those of  $T$ , but we cannot exclude that some of the jumps of  $T_1$  get swapped with some of  $T_2$ : a natural approach to solve this problem would be to prove that  $\text{Lie } h(T(d), K'(d))$  induces an isomorphism

$$\text{coker } \text{Lie } h(T, K(d)) \otimes_{\mathcal{O}_{K(d)}} \text{id}_{\mathcal{O}_{K'(d)}} \xrightarrow{\sim} \text{coker } \text{Lie } h(T', K'(d)),$$

which would imply the desired statement by a diagram chase, but doing so seems to require the computation of the rings of integers involved.

## 4.5 The Lie algebra of the Néron model of a torus

In view of the relation (Proposition 3.2.19) between the eigenvalues of the  $\mu_d$ -action on the Lie algebra of  $\mathcal{T}(d)_k$  and the  $d$ -jumps, the following relatively concrete description of the Lie algebra of the Néron model of a torus  $T$  over a valued field

$K$ , due to de Shalit [CY01, proposition A.1.7] under the assumption that  $\mathcal{O}_K$  be compact, could be an interesting tool for computations. Thanks to a suggestion by Johannes Nicaise, the compactness condition can be relaxed to that of  $\mathcal{O}_K$  being a G-ring (Definition 4.5.3), which, since the formation of Néron models is compatible with unramified base-change and since any complete discrete valuation ring is a G-ring, is not restrictive. Informally speaking, if  $L/K$  is a Galois splitting extension of  $T$ , de Shalit shows that  $\mathrm{Lie} \mathcal{T}$  is embedded canonically into the  $\mathcal{O}_K$ -module of  $\mathrm{Gal}(L/K)$ -homomorphisms from  $\chi(T)$  to  $\mathcal{O}_L$ .

*Notation 4.5.1.* For a ring  $R$  and an  $R$ -algebra  $R'$ , we shall denote by  $R'[[t]]_1$  the subgroup  $1 + tR'[[t]]$  of  $R'[[t]]^\times$ ; if  $X$  is a group  $R$ -scheme, we set

$$X(R'[[t]])_1 = \ker(X(R'[[t]]) \rightarrow X(R')),$$

and define similarly  $R'[\epsilon]_1$ , where  $R'[\epsilon]$  is the ring of dual numbers  $R'[x]/(x^2)$ . There is a natural group morphism  $R'[[t]]_1 \rightarrow R'$  mapping a power series to the coefficient of  $t$  in it; this is the map considered in the next statement.

For an  $\mathcal{O}_K$ -algebra or module  $A$ , let us denote by  $A_K$  the  $K$ -algebra or module  $A \otimes_{\mathcal{O}_K} K$ .

**Lemma 4.5.2** ([CY01, Lemma A.1.4]). *Let  $K$  be a discrete valuation ring,  $T$  a  $K$ -torus splitting over a finite Galois extension  $L/K$  with Galois group  $\Gamma$ ,  $\mathcal{X}$  the  $\Gamma$ -module  $\chi(T)$ ,  $\tilde{T}$  the split torus  $T \times_K L$ ,  $\mathcal{Y}$  the schematic closure of  $T$  inside  $(\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T})^0$ .*

(i) *For any  $\mathcal{O}_K$ -algebra  $A$  such that  $A \rightarrow A_K$  is injective,*

$$\mathcal{Y}(A) = \mathrm{Hom}_\Gamma(\mathcal{X}, (A \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times),$$

*naturally in  $A$ .*

(ii) *The identity component of  $\mathcal{T}$  can be constructed as the identity component of the group smoothening of  $\mathcal{Y}$ .*

Since the proof in [CY01] is somewhat terse, let us provide some more details.

*Proof.* i): Let  $\mathfrak{X}$  be the functor of  $\mathcal{O}_K$ -algebras defined by

$$\mathfrak{X}(A) = \mathrm{Hom}_\Gamma(\mathcal{X}, (A \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times).$$

We have to show that for any  $\mathcal{O}_K$ -algebra  $A$  such that  $A \rightarrow A_K$  is injective,  $\mathfrak{X}(A) = \mathcal{Y}(A)$ . One has

$$\left( \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T} \right)^0 = \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T}^0$$

(with the same proof as in [NX91, Lemma 3.1]), whereby  $\left(\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T}\right)^0$  is just the Weil restriction of a split torus of the same rank as  $T$ . Let  $C$  be the  $\mathcal{O}_K$ -algebra such that

$$\left(\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T}\right)^0 = \mathrm{Spec} C;$$

the scheme  $T$ , being a subscheme of the generic fibre of  $\mathrm{Spec} C$ , will be of the form  $T = \mathrm{Spec} C_K/I_T$  for some ideal  $I_T \trianglelefteq C_K$ . In order to compute the points of  $\mathcal{Y}$ , observe first that the torus  $T$  represents the functor of  $K$ -algebras

$$B \mapsto \mathrm{Hom}_\Gamma(\mathcal{X}, (B \otimes_K L)^\times),$$

whereas  $\tilde{T}^0 = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}, (\cdot)^\times)$  (because  $\tilde{T}$  is split) and consequently

$$\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T}^0 = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{X}, (- \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times).$$

Using that localisation commutes with integral closure and with tensor products, one has an equality, for any  $K$ -algebra  $A$ ,  $A \otimes_K L = A \otimes_{\mathcal{O}_K} \mathcal{O}_L$ ; hence, under the identifications above, the ideal  $I_T$  just expresses the condition that a morphism of abelian groups  $\mathcal{X} \rightarrow (A \otimes_{\mathcal{O}_K} \mathcal{O}_L)^\times$  be  $\Gamma$ -invariant.

We claim that if  $I \trianglelefteq C$  is an ideal such that  $\mathrm{Spec} C/I$  represents the functor  $\mathfrak{X}$ , then  $I_T = I_K$ : indeed, letting  $r$  be the rank of  $T$ , for any  $\mathcal{O}_K$ -algebra  $A$  the  $A$ -valued points of  $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \tilde{T}^0$  are the morphisms of abelian groups from  $\mathcal{X}$  to  $(\mathcal{O}_L \otimes_{\mathcal{O}_K} A)^\times$ , which, after fixing an  $\mathcal{O}_K$ -basis  $\{x_1, \dots, x_m\}$  of  $\mathcal{O}_L$ , are determined precisely by  $rm$  elements  $a_{ij} \in A$  such that  $\sum_{j=1}^m a_{ij} \otimes x_j$  is invertible for each  $i$ . Therefore, one can write

$$C = \mathcal{O}_K[z_{ij}]_{\substack{i=1,\dots,r \\ j=1,\dots,m}} / J$$

for a certain ideal  $J$  expressing such invertibility condition; fixing a  $\mathbb{Z}$ -basis  $\{e_1, \dots, e_r\}$  of  $\mathcal{X}$  and writing, for every  $\gamma \in \Gamma$ ,  $\gamma.e_i = \sum \gamma_i^k e_k$  for some (uniquely determined) integers  $\gamma_i^k$ , the requirement that the morphism  $\mathcal{X} \rightarrow (\mathcal{O}_L \otimes_{\mathcal{O}_K} A)^\times$  determined by some  $a_{ij}$ 's be  $\Gamma$ -equivariant translates to the condition that  $a_{ij} = \sum_{k=1}^m \gamma_i^k a_{kj}$  for all  $i$  and  $j$ . It follows that both  $I$  and  $I_T$  are generated by  $\left\{ z_{ij} - \sum_{k=1}^m \gamma_i^k z_{kj} \right\}_{\substack{\gamma \in \Gamma \\ i,j}}$ , which proves the claim.

As

$$\mathcal{Y} = \mathrm{Spec} \frac{C}{\ker(C \rightarrow C_K/I_T)},$$

one has to prove that for any  $\mathcal{O}_K$ -algebra  $A$  as in the statement and any morphism  $\phi: C/I \rightarrow A$ , if the natural map  $C/I \rightarrow C_K/I_T$  takes some  $c \in C$  to 0, then

$\phi(c + I) = 0$ . Now the previous claim implies that the map  $C/I \rightarrow C_K/I_T$  is just the canonical map  $C/I \rightarrow (C/I)_K$ , whereby the diagram

$$\begin{array}{ccc} C/I & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ C_K/I_T & \xrightarrow{\phi_K} & A_K \end{array},$$

where the vertical maps are the obvious ones, commutes. As the right vertical map is injective, the statement follows.

ii): See [CY01, Section 3]; alternatively, combine [CY01, 3.1] with [Bra03, Theorem 3.1.3] and [Bra03, Satz 3.1.4].  $\square$

In the next statement, we shall implicitly be using the fact that for any group scheme  $H$ , the canonical map  $\mathrm{Lie} H^0 \rightarrow \mathrm{Lie} H$  is an isomorphism [LLR04, Proposition 1.1(d)].

Recall first the following

**Definition 4.5.3.** A  $G$ -ring is a noetherian ring  $R$  such that the completion map  $R_{\mathfrak{p}} \rightarrow \hat{R}_{\mathfrak{p}}$  is regular for each prime ideal  $\mathfrak{p}$  of  $R$ .

**Proposition 4.5.4.** *With the same notation as in the previous lemma, assume now that  $\mathcal{O}_K$  is a  $G$ -ring. Under the natural identification of  $\mathcal{O}_L[\epsilon]_1$  with  $\mathcal{O}_L$ , the group smoothening of  $\mathcal{Y}$  induces an injection of  $\mathcal{O}_K$ -submodules of  $\mathrm{Lie} T = \mathrm{Hom}_{\Gamma}(\mathcal{X}, L)$ , functorial in the  $K$ -torus  $T$  split by  $L$ ,*

$$\mathrm{Lie} \mathcal{T} = \mathcal{T}(\mathcal{O}_L[\epsilon])_1 \hookrightarrow \mathcal{Y}(\mathcal{O}_L[\epsilon])_1 = \mathrm{Hom}_{\Gamma}(\mathcal{X}, \mathcal{O}_L), \quad (4.6)$$

whose image consists of the morphisms  $v \in \mathrm{Hom}_{\Gamma}(\mathcal{X}, \mathcal{O}_L)$  lifting to an element of  $\mathrm{Hom}_{\Gamma}(\mathcal{X}, \mathcal{O}_L[[t]]_1)$ .

*Proof.* Let  $\mathcal{Y}' \rightarrow \mathcal{Y}$  be the group smoothening of  $\mathcal{Y}$ ; notice first that by the previous lemma the statement does make sense, i.e. the equality  $\mathcal{Y}(\mathcal{O}_L[\epsilon])_1 = \mathrm{Hom}_{\Gamma}(\mathcal{X}, \mathcal{O}_L)$  holds and  $\mathrm{Lie} \mathcal{T} = \mathrm{Lie} \mathcal{Y}'$ .

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{Y}'(\mathcal{O}_K[[t]])_1 & \longrightarrow & \mathcal{Y}(\mathcal{O}_K[[t]])_1 = \mathrm{Hom}_{\Gamma}(\mathcal{X}, \mathcal{O}_L[[t]]_1) \\ \downarrow & & \downarrow \\ \mathrm{Lie} \mathcal{Y}' = \mathcal{Y}'(\mathcal{O}_K[\epsilon])_1 & \longrightarrow & \mathcal{Y}(\mathcal{O}_K[\epsilon])_1 = \mathrm{Hom}_{\Gamma}(\mathcal{X}, \mathcal{O}_L), \end{array} \quad (4.7)$$

where the horizontal arrows are induced by the group smoothening map and the vertical ones are the obvious projections.

The bottom horizontal map is an injection of  $\mathcal{O}_K$ -submodules of

$$\mathrm{Lie} T = \mathrm{Hom}_\Gamma(\mathcal{X}, L) :$$

indeed, the functorial identification, for any group  $\mathcal{O}_K$ -scheme  $\mathcal{H}$ ,

$$\mathrm{Lie} \mathcal{H}_K = \mathrm{Lie} \mathcal{H} \otimes_{\mathcal{O}_K} K$$

implies the existence of a commutative diagram

$$\begin{array}{ccc} \mathrm{Lie} \mathcal{Y}' & \longrightarrow & \mathrm{Lie} \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Lie} \mathcal{Y}'_K & \longrightarrow & \mathrm{Lie} \mathcal{Y}_K, \end{array}$$

where the horizontal arrows are induced by the group smoothening; the assertion follows because the group smoothening reduces to the identity on the generic fibre. Since the left vertical arrow in (4.7) is surjective by formal smoothness, by a straightforward diagram chase, it is now enough to show that the top horizontal map is surjective.

We shall actually show that it is even an isomorphism. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Y}'(\mathcal{O}_K[[t]])_1 & \longrightarrow & \mathcal{Y}'(\mathcal{O}_K[[t]]) & \longrightarrow & \mathcal{Y}'(\mathcal{O}_K) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{Y}(\mathcal{O}_K[[t]])_1 & \longrightarrow & \mathcal{Y}(\mathcal{O}_K[[t]]) & \longrightarrow & \mathcal{Y}(\mathcal{O}_K) \end{array} ;$$

since by assumption the homomorphism  $\mathcal{O}_K \rightarrow \mathcal{O}_K[[t]]$  is regular, by Popescu's theorem [Autb, 07GC] one can write  $\mathcal{O}_K[[t]]$  as a direct limit of smooth  $\mathcal{O}_K$ -algebras, say  $\mathcal{O}_K[[t]] = \varinjlim R_i$ ; for each  $i$ , the map  $\mathcal{Y}'(R_i) \rightarrow \mathcal{Y}(R_i)$  induced by the group smoothening is an isomorphism by the universal property of the group smoothening, whereby, passing to the limit, also  $\mathcal{Y}'(\mathcal{O}_K[[t]]) \rightarrow \mathcal{Y}(\mathcal{O}_K[[t]])$  is an isomorphism; the statement follows, observing that the right-most vertical map is an isomorphism as well.

Functoriality is clear, because all the identifications involved are functorial.  $\square$



# Bibliography

- [ABD<sup>+</sup>66] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Alexander Grothendieck, Pierre Gabriel, Michel Raynaud, and Jean-Pierre Serre. *Schémas en groupes. Seminaire de Geometrie Algebrique du Bois Marie 1962/64 (SGA 3)*, volume 151-153 of *Lecture Notes in Mathematics*. Springer Berlin, 1963/1966.
- [AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III (SGA4)*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer, 1971.
- [AS02] Ahmed Abbes and Takeshi Saito. Ramification of local fields with imperfect residue fields. *American Journal of Mathematics*, 124(5):879–920, 2002.
- [Auta] MathStackexchange Authors. When are nonintersecting finite degree field extensions linearly disjoint? Mathematics Stack Exchange. User ll\_n (<https://math.stackexchange.com/users/617152/ll-n>) URL: <https://math.stackexchange.com/q/3009197> (version: 2024-06-09).
- [Autb] The Stacks Project Authors. Stacks Project. URL: <https://stacks.math.columbia.edu/>.
- [BC20] Alessandra Bertapelle and Maurizio Candilera. Greenberg algebras and ramified Witt vectors. *Journal de théorie des nombres de Bordeaux*, 32(3):901–921, 2020.
- [BGA16] Alessandra Bertapelle and Cristian D. González-Avilés. Galois sets of connected components and Weil restriction. *Mathematische Zeitschrift*, 285(1–2):607–612, June 2016.

- [BGA18a] Alessandra Bertapelle and Cristian D. González-Avilés. The Greenberg functor revisited. *European Journal of Mathematics*, 4(4):1340–1389, January 2018.
- [BGA18b] Alessandra Bertapelle and Cristian D. González-Avilés. On the perfection of schemes. *Expositiones Mathematicae*, 36(2):197–220, June 2018.
- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*. Springer Berlin Heidelberg, 1990.
- [Bou90] Nicolas Bourbaki. *Algebra II: Chapters 4-7 (translated from the French by P. M. Cohn and J. Howie)*. Springer Science & Business Media, 1990.
- [BPS23] Alessandra Bertapelle, Emma Previato, and Arnab Saha. Arithmetic jet spaces. *Journal of Algebra*, 623:127–153, June 2023.
- [Bra03] Bernhard Brahm. *Néron-Modelle algebraischer Tori*. PhD thesis, Westfälischen Wilhelms-Universität Münster, 2003.
- [BS24] Alessandra Bertapelle and Takashi Suzuki. The relatively perfect Greenberg transform and cycle class maps. *Manuscripta mathematica*, 175(1–2):365–407, June 2024.
- [Cha00] Ching-Li Chai. Néron models for semiabelian varieties: Congruence and change of base field. *Asian Journal of Mathematics*, 4(4):715–736, December 2000.
- [Che07] Mustapha Chellali. Erratum on Structure of Inseparable Extensions. In *International Mathematical Forum*, volume 2, pages 3269–3272. Cite-seer, 2007.
- [CLNS<sup>+</sup>18] Antoine Chambert-Loir, Johannes Nicaise, Julien Sebag, Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag. *Motivic integration*. Springer, 2018.
- [Coh79] P. M. Cohn. On the decomposition of a field as a tensor product. *Glasgow Mathematical Journal*, 20(2):141–145, 1979.
- [CTS77] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. La R-équivalence sur les tores. *Annales scientifiques de l’École Normale Supérieure*, 10(2):175–229, 1977.

- [CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. Principal homogeneous spaces under flasque tori: Applications. *Journal of Algebra*, 106(1):148–205, March 1987.
- [CY01] Ching-Li Chai and Jiu-Kang Yu. Congruences of Neron models for tori and the Artin conductor, with an appendix by E. de Shalit. *Annals of Mathematics-Second Series*, 154(2):347–382, 2001.
- [DG70] Michiel Demazure and Pierre Gabriel. *Groupes Algébriques, Tome 1. Mason&Cie*. North-Holland Publishing Company, The Netherlands, 1970.
- [Edi92] Bas Edixhoven. Néron models and tame ramification. *Compositio Mathematica*, 81(3):291–306, 1992.
- [EHN15] Dennis Eriksson, Lars Halvard Halle, and Johannes Nicaise. A logarithmic interpretation of Edixhoven’s jumps for Jacobians. *Advances in Mathematics*, 279:532–574, 2015.
- [EM75] Shizuo Endo and Takehiko Miyata. On a classification of the function fields of algebraic tori. *Nagoya Mathematical Journal*, 56:85–104, 1975. See also erratum [EM80].
- [EM80] Shizuo Endo and Takehiko Miyata. Corrigenda on a classification of the function fields of algebraic tori: (nagoya math. j. 56 (1975), 85–104). *Nagoya Mathematical Journal*, 79:187–190, 1980.
- [GKP89] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, Reading, 1989.
- [GR18] Ofer Gabber and Lorenzo Ramero. Foundations for almost ring theory – Release 7.5, 2018. URL: <https://arxiv.org/abs/math/0409584>.
- [Gre61] Marvin J Greenberg. Schemata over local rings. *Annals of Mathematics*, 73(3):624–648, 1961.
- [GRR72] Alexander Grothendieck, Michel Raynaud, and Dock Sang Rim. Groupes de monodromie en géométrie algébrique. I (SGA 7 I), 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969.

- [GW74] Robert Gilmer and Brian Wesselink. On a purely inseparable extension of a normal extension of a field. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 40, pages 94–99. Springer, 1974.
- [Hal59] Marshall Jr. Hall. *The Theory of Groups*. Macmillan, New York, 1959.
- [Her24] Leo Herr. The log tangent space of the log jet space. *Michigan Mathematical Journal*, 1(1):1–49, 2024.
- [HN10] Lars Halvard Halle and Johannes Nicaise. The Néron component series of an abelian variety. *Mathematische Annalen*, 348(3):749–778, February 2010.
- [HN11] Lars Halvard Halle and Johannes Nicaise. Motivic zeta functions of abelian varieties, and the monodromy conjecture. *Advances in Mathematics*, 227(1):610–653, 2011.
- [HN16] Lars Halvard Halle and Johannes Nicaise. *Néron Models and Base Change*. Springer International Publishing, 2016.
- [JLM<sup>+</sup>22] Lena Ji, Shizhang Li, Patrick McFaddin, Drew Moore, and Matthew Stevenson. *Weil restriction for schemes and beyond*, page 194–221. London Mathematical Society Lecture Note Series. Cambridge University Press, 2022.
- [Kap49] Irving Kaplansky. Elementary divisors and modules. *Transactions of the American Mathematical Society*, 66(2):464–491, 1949.
- [Kaz82] Kato Kazuya. A generalization of local class field theory by using K-groups III. *Journal of the Faculty of Science, the University of Tokyo: Tōkyō Daigaku Rigakubu Kiyō. Dai 1-rui, Sūgaku. Mathematics. Section 1A*, 29:31, 1982.
- [KMW25] Enis Kaya, Michaël Maex, and Art Waeterschoot. Jumps of Jacobians via orthogonal canonical forms. *Proceedings of the American Mathematical Society*, 2025.
- [Lip76] Joseph Lipman. The Picard group of a scheme over an Artin ring. *Publications Mathématiques de l’IHÉS*, 46:15–86, 1976.

- [LL01] Qing Liu and Dino Lorenzini. Special fibers of Néron models and wild ramification. *Journal für die reine und angewandte Mathematik*, 2001.
- [LLR04] Qing Liu, Dino Lorenzini, and Michel Raynaud. Néron models, Lie algebras, and reduction of curves of genus one. *Inventiones mathematicae*, 157(3), 2004.
- [Lur] Jacob Lurie. Ultracategories. URL: <https://people.math.harvard.edu/~lurie/papers/Conceptual.pdf>.
- [Mil80] James S Milne. *Etale cohomology*. Number 33. Princeton University Press, 1980.
- [MM12] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.
- [Neu13] Jürgen Neukirch. *Algebraic number theory (translated from the German original by N. Schappacher)*, volume 322. Springer Science & Business Media, 2013.
- [New72] Morris Newman. *Integral matrices*, volume 45. Academic Press, 1972.
- [NS11] Johannes Nicaise and Julien Sebag. *The Grothendieck ring of varieties*, page 145–188. London Mathematical Society Lecture Note Series. Cambridge University Press, 2011.
- [NX91] Enric Nart and Xavier Xarles. Additive reduction of algebraic tori. *Archiv der Mathematik*, 57:460–466, 1991.
- [Ogu18] Arthur Ogus. *Lectures on Logarithmic Algebraic Geometry*. Cambridge University Press, October 2018.
- [Ols16] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016.
- [Ono61] Takashi Ono. Arithmetic of algebraic tori. *Annals of Mathematics*, 74(1):101–139, 1961.
- [OS23] Otto Overkamp and Takashi Suzuki. Chai’s conjectures on base change conductors. *arXiv preprint arXiv:2310.01289*, 2023.

- [Ove19] Otto Overkamp. Jumps and motivic invariants of semiabelian Jacobians. *International Mathematics Research Notices*, 2019(20):6437–6479, 2019.
- [Poo08] Bjorn Poonen. Rational points on varieties. *Graduate Studies in Mathematics*, 186(337):5, 2008.
- [Ser13] Jean-Pierre Serre. *Local fields (translated from the French by M. J. Greenberg)*, volume 67. Springer Science & Business Media, 2013.
- [Sha77] Rodney Sharp. The dimension of the tensor product of two field extensions. *Bulletin of the London Mathematical Society*, 9(1):42–48, 1977.
- [Swe68] Moss Eisenberg Sweedler. Structure of inseparable extensions. *Annals of Mathematics*, 87(3):401–410, 1968. See also errata in [Swe69], [Che07] and [GW74].
- [Swe69] Moss Eisenberg Sweedler. Correction to “Structure of inseparable extensions”. *Annals of Mathematics*, 89(1):206–207, 1969.
- [Vak] Ravi Vakil. The rising sea: Foundations of Algebraic Geometry. URL: <https://math.stanford.edu/~vakil/216blog/FOAGjul2724public.pdf>.
- [Vis05] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.
- [Wat75] William Waterhouse. Basically bounded functors and flat sheaves. *Pacific Journal of Mathematics*, 57(2):597–610, 1975.
- [Wat79] William C Waterhouse. *Introduction to affine group schemes*, volume 66. Springer Science & Business Media, 1979.
- [ZS58] Oscar Zariski and Pierre Samuel. *Commutative Algebra, Volume I*. Springer, 1958.