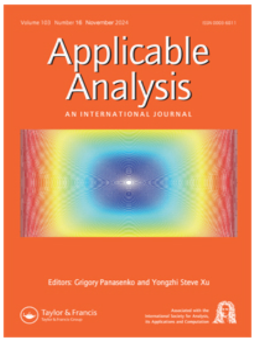

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A nonlinear elliptic system with a transport term and singular data

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ABSTRACT
We consider the nonlinear elliptic system

$$\begin{cases} u \in W_0^{\frac{N}{N-1}}(\Omega): & -\mathrm{div}(M(x)\nabla u) + u = -\mathrm{div}(u\,M(x)\,\nabla\psi) + f(x), \\ \psi \in W_0^{1,2}(\Omega): & -\mathrm{div}(M(x)\nabla\psi) + \psi = R(u) + E(x)\,\nabla\psi, \end{cases}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, $N \geq 3$; $M(x)$ is a coercive, symmetric matrix with $L^\infty(\Omega)$ coefficients; $f(x)$ and $E(x)$ belong to some Lebesgue space, and $R(s)$ is a continuous function such that

$$0 \leq R(s) \leq |s|^\theta, \quad \text{for } \theta < \frac{2}{N}.$$

Using a duality technique, we prove existence of at least a weak solution (u,ψ). Moreover, if $N=3$ or $N=4$, we prove under stronger assumptions on $f(x)$ and $E(x)$ that the solution u belongs to $W_0^{1,2}(\Omega)$.

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1. Introduction

In this paper we study the nonlinear elliptic system

$$\begin{cases} 0 \leq u \in W_0^{\frac{N}{N-1}}(\Omega): & -\mathrm{div}(M(x)\nabla u) + u = -\mathrm{div}(uM(x)\,\nabla\psi) + f(x), \\ 0 \leq \psi \in W_0^{1,2}(\Omega): & -\mathrm{div}(M(x)\nabla\psi) + \psi = R(u) + E(x)\,\nabla\psi, \end{cases} \tag{1}$$

under the following assumptions.

- Ω is a bounded, open subset of $\mathbb{R}^N$, $N \geq 3$;
- $M : \Omega \to \mathbb{R}^{N^2}$ is a symmetric matrix such that

$$M(x)\,\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta, \text{ for almost every } x \in \Omega, \tag{2}$$

  for some $0<\alpha<\beta<\infty$ and for every ξ in $\mathbb{R}^N$;
- $f(x) \geq 0$ is a function such that

$$f(x)\,\log(1+f(x)) \in L^1(\Omega); \tag{3}$$

- $E(x)$ is a vector field such that

$$E(x) \in [L^N(\Omega)]^N; \tag{4}$$

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- $R:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$0\le R(s)\le |s|^{\theta},\quad \text{for some } 0<\theta<\frac{2}{N}. \tag{5}$$

The study of the nonlinear elliptic system (1) is the natural continuation of the works [1,2]. In [1], the authors consider the problem

$$\begin{cases} u\in W_0^{\frac{N}{N-1}}(\Omega): & -\mathrm{div}(M(x)\nabla u)+u=-\mathrm{div}(u\,M(x)\,\nabla\psi)+f(x),\\ \psi\in W_0^{1,2}(\Omega): & -\mathrm{div}(M(x)\nabla\psi)+\psi=u^{\theta}, \end{cases} \tag{6}$$

and prove the existence of positive solutions u and ψ in $W_0^{1,2}(\Omega)\cap L^{\infty}(\Omega)$, with $f(x)$ in $L^m(\Omega)$, $m>\frac{N}{2}$. One of the main tools of the proof is the use of nonlinear test functions $g(u)$ (in the first equation) and $h(u)$ (in the second equation), with $h'(s)=s\,g'(s)$. This choice produces a cancellation of the terms coming from $-\mathrm{div}(u\,M(x)\,\nabla\psi)$ and $-\mathrm{div}(M(x)\,\nabla\psi)$ (see also Section 4).

Conversely, in this paper we use a linear duality approach, so that we cancel the contribution of the principal parts; even if our problem is nonlinear, here we use a duality method as in [2].

With respect to (6), system (1) presents the extra term $E(x)\,\nabla\psi$, where $E(x)$ belongs to $[L^N(\Omega)]^N$. A system like (1) has been studied in [2], where the second equation is

$$\psi\in W_0^{1,2}(\Omega):\ -\mathrm{div}(M(x)\nabla\psi)+\psi=B(u)+E(x)\,\nabla\psi,$$

with $B(s)$ is a continuous, bounded and positive function. The authors obtain the existence of at least a weak solution (u,ψ) in $W_0^{\frac{N}{N-1}}(\Omega)\times W_0^{1,2}(\Omega)$.

System (1) is related to the stationary solutions of biological models of one or several species whose movement is driven by chemical stimuli. Such systems were introduced in the literature in the 70′s after the pioneering works of Keller and Segel [3,4] and it is known as chemotaxis system or Keller-Segel model: see for instance the surveys Horstmann [5,6]; Hillen and Painter [7]; Bellomo et al. [8] and references therein for more details concerning previous results of such systems.

The Keller-Seller model describes the behaviour of the biological species 'u' in terms of a parabolic equation (see for instance [9–11] among others). System (1) represents the stationary case of such models where $f(x)$ and $E(x)$ are external data. Chemotaxis systems with external data have been already studied by different authors, see for instance Tello and Winkler [12], Black [13,14] among others.

The main result of this paper is the existence of solutions of system (1) under assumptions (2)–(5).

Due to the assumptions on the data $f(x)$ and $E(x)$, we are going to prove the existence of weak solutions, whose definition is the following.

Definition 1.1: Let Ω be as above, and let $M(x), f(x), E(x)$ and $R(s)$ be such that assumptions (2)–(5) hold. We say that a pair of functions (u,ψ) is a weak solution to system (1) if $u\,|\nabla\psi|^2$ belongs to $L^1(\Omega)$ and

$$0\le u\in W_0^{1,\frac{N}{N-1}}(\Omega):\ \int_\Omega M(x)\nabla u\nabla w+\int_\Omega u\,w=\int_\Omega uM(x)\nabla\psi\,\nabla w+\int_\Omega f(x)\,w,$$

$$0\le\psi\in W_0^{1,2}(\Omega):\ \int_\Omega M(x)\nabla\psi\nabla\eta+\int_\Omega\psi\,\eta=\int_\Omega R(u)\,\eta+\int_\Omega E(x)\,\nabla\psi\,\eta,$$

for any $w\in W_0^{1,N}(\Omega)$, $\eta\in W_0^{1,2}(\Omega)$.

Remark 1.2: In view of assumption (3), (16) and Moser-Trudinger inequality, the term

$$\int_\Omega f(x)\,w$$

is bounded.

The article is organised as follows. In Section 2, we approximate the system by modifying the unbounded terms $f(x)$, $E(x)$, $R(s)$, and the nonlinear term

$$-\mathrm{div}(uM(x)\,\nabla\psi)$$

to obtain some a priori bounds of the approximating solutions. Thanks to such a priori bounds, we obtain a sequence $\{(u_n, \psi_n)\}$ which converges to a weak solution of (1). The details of this convergence are presented in Section 3. Finally, in Section 4 we prove that if $N=3$ or $N=4$, if $f(x)$ belongs to $L^{\frac{2N}{N+2}}(\Omega)$, and if $E(x)$ belongs to $[L^\infty(\Omega)]^N$, then the solution u of the first equation of system (1) belongs to $W_0^{1,2}(\Omega)$.

Throughout the article we will use the following notation:

$$2^* = \frac{2N}{N-2}, \qquad 2_* = (2^*)' = \frac{2N}{N+2}.$$

2. Approximate solutions

Following [2,15–17], we approximate problem (1), and then obtain some a priori estimates on the solutions; these solutions will then converge to a solution of (1). Given n in $\mathbb{N}$, we first define the sequence $\{f_n\}$ as

$$f_n(x) = \frac{f(x)}{1+\frac{1}{n}f(x)}.$$

Notice that, in view of (3),

$$0 \le f_n(x) \le f(x), \quad f_n \in L^\infty(\Omega),\ \forall\, n\in\mathbb{N},$$

and

$$f_n \to f, \text{ in } L^1(\Omega).$$

Analogously, we replace the terms

$$\mathrm{div}(uM(x)\nabla\psi), \quad E(x)\nabla\psi, \quad R(u),$$

by

$$\mathrm{div}\left(\frac{u_n}{1+\frac{1}{n}\,|u_n|}M(x)\nabla\psi\right), \quad \frac{E(x)}{1+\frac{1}{n}|u_n|}\nabla\psi, \quad \frac{R(u_n)}{1+\frac{1}{n}R(u_n)}$$

to obtain the following approximating system

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla u_n) + u_n = -\mathrm{div}\left(\dfrac{u_n}{1+\frac{1}{n}\,|u_n|}M(x)\nabla\psi_n\right) + f_n(x), \\ \psi_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla\psi_n) + \psi_n = \dfrac{R(u_n)}{1+\frac{1}{n}\,R(u_n)} + \dfrac{E(x)}{1+\frac{1}{n}\,|u_n|}\,\nabla\psi_n. \end{cases} \tag{7}$$

In order to prove a priori estimates on the sequences $\{u_n\}$ and $\{\psi_n\}$ we recall the definition of weak solution for system (7).

Definition 2.1: A weak solution to (7) is a pair of functions (u_n, ψ_n) in $[W_0^{1,2}(\Omega)]^2$ such that

$$\int_\Omega M(x)\nabla u_n \nabla w + \int_\Omega u_n\, w = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} M(x)\nabla\psi_n\, \nabla w + \int_\Omega f_n(x)\, w,$$

and

$$\int_\Omega M(x)\nabla\psi_n \nabla\eta + \int_\Omega \psi_n\, \eta = \int_\Omega \frac{R(u_n)}{1+\frac{1}{n}R(u_n)}\,\eta + \int_\Omega \frac{E(x)}{1+\frac{1}{n}|u_n|}\,\nabla\psi_n\, \eta,$$

for every $w \in W_0^{1,2}(\Omega)$ and $\eta \in W_0^{1,2}(\Omega)$.

Since the function $\frac{R(s)}{1+\frac{1}{n}R(s)}$ is continuous and bounded, system (7) has at least a weak solution in the sense of Definition 2.1; such result is obtained (see [2] for the details) applying Schauder fixed point theorem to the function $S: W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ defined by

$$S(w) = z,$$

with z is the weak solution to the equation

$$z \in W_0^{1,2}(\Omega) : -\mathrm{div}(M(x)\nabla z) + z = -\mathrm{div}\left(\frac{w}{1+\frac{1}{n}|w|} M(x)\nabla\zeta\right) + f_n(x), \tag{8}$$

where ζ is the weak solution of

$$\zeta \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : -\mathrm{div}(M(x)\nabla\zeta) + \zeta = \frac{R(w)}{1+\frac{1}{n}R(w)} + \frac{E(x)}{1+\frac{1}{n}|w|}\,\nabla\zeta. \tag{9}$$

Note that existence and uniqueness of z follows from Lax-Milgram theorem, while existence and uniqueness of ζ follows from the results of [16].

We devote the following lemmas to obtain some a priori estimates on the solutions of the approximated problem. In the proofs, C, $C(f)$ and $C(f, E)$ denote positive quantities depending on the data of the problem; of course these quantities do not depend on the approximating parameter n and they may change over the curse of the proofs.

We first recall some known results.

Lemma 2.2: *Under assumptions* (2)–(5), *the solution* ψ_n *to the second equation of* (7) *satisfies*

$$\psi_n \geq 0.$$

Proof: The proof is similar to the proof of Lemma 2.3 in [2], therefore we omit the details (see also [15,18]). ■

In the next lemma, we present some estimates on the solutions u_n (which also hold under the weaker assumption that $E(x)$ belongs to $[L^2(\Omega)]^N$).

Lemma 2.3: *Under the assumptions* (2)–(5) *the following inequalities hold*

$$u_n \geq 0, \tag{10}$$

and

$$0 \leq \int_\Omega u_n \leq \int_\Omega f(x). \tag{11}$$

Proof: The proof of (10) and (11) is obtained in [15]. See also [1,2,17]. ■

As a consequence of Lemma 2.3, from now on we can write u_n instead of $|u_n|$ in every equation and formula.

Lemma 2.4: *Under assumptions* (2)–(5), *the sequence* $\{\psi_n\}$ *is such that*

$$\int_\Omega |\nabla \psi_n|^2 + \int_\Omega e^{\psi_n} \le C(f,E).$$

Thus, the sequence $\{\psi_n\}$ *is bounded in* $L^p(\Omega)$ *for any* $1 \le p < +\infty$.

Proof: Take $\eta = \mathrm{e}^{2\psi_n} - 1$ as test function in the weak formulation of the equation solved by ψ_n (see Definition 2.1) to have

$$\begin{aligned} 2\int_\Omega M(x)\nabla\psi_n\nabla\psi_n\,\mathrm{e}^{2\psi_n} + \int_\Omega \psi_n(\mathrm{e}^{2\psi_n}-1) \\ = \int_\Omega \frac{R(u_n)}{1+\frac{1}{n}R(u_n)}\,(\mathrm{e}^{2\psi_n}-1) + \int_\Omega \frac{E(x)}{1+\frac{1}{n}\,u_n}\,\nabla\psi_n\,(\mathrm{e}^{2\psi_n}-1). \end{aligned}$$

We apply Young inequality to the term

$$\int_\Omega \frac{E(x)}{1+\frac{1}{n}\,u_n}\,\nabla\psi_n\,(\mathrm{e}^{2\psi_n}-1)$$

and (2) to obtain

$$\begin{aligned} &2\,\alpha\int_\Omega |\nabla\psi_n|^2\,\mathrm{e}^{2\psi_n} + \int_\Omega \psi_n(\mathrm{e}^{2\psi_n}-1) \\ &\le \int_\Omega R(u_n)\,(\mathrm{e}^{2\psi_n}-1) + \alpha\int_\Omega |\nabla\psi_n|^2\mathrm{e}^{2\psi_n} - \alpha\int_\Omega |\nabla\psi_n|^2 + \frac{1}{4\alpha}\int_\Omega |E(x)|^2(\mathrm{e}^{2\psi_n}-1), \end{aligned}$$

which implies, for $k>0$ and $\rho>0$,

$$\begin{aligned} &\alpha\int_\Omega |\nabla\psi_n|^2\,\mathrm{e}^{2\psi_n} + \alpha\int_\Omega |\nabla\psi_n|^2 + \int_{\{0\le\psi_n\le k\}} \psi_n(\mathrm{e}^{2\psi_n}-1) + \int_{\{k<\psi_n\}} \psi_n(\mathrm{e}^{2\psi_n}-1) \\ &\le \int_{\{0\le u_n\le k\}} R(u_n)\,(\mathrm{e}^{2\psi_n}-1) + \int_{\{k<u_n\}} R(u_n)\,(\mathrm{e}^{2\psi_n}-1) \\ &\quad + \frac{1}{4\alpha}\int_{\{|E(x)|\le\rho\}} |E(x)|^2(\mathrm{e}^{2\psi_n}-1) + \frac{1}{4\alpha}\int_{\{\rho<|E(x)|\}} |E(x)|^2(\mathrm{e}^{2\psi_n}-1). \end{aligned}$$

Since $0 \le R(s) \le s^\theta$, and $\theta < \frac{2}{N} < 1$, we have

$$\begin{aligned} &\alpha\int_\Omega |\nabla\psi_n|^2\,\mathrm{e}^{2\psi_n} + \alpha\int_\Omega |\nabla\psi_n|^2 + \int_{\{0\le\psi_n\le k\}} \psi_n(\mathrm{e}^{2\psi_n}-1) \\ &\quad + k\int_\Omega (\mathrm{e}^{2\psi_n}-1) - k\int_{\{0\le\psi_n\le k\}} (\mathrm{e}^{2\psi_n}-1) \\ &\le k^\theta\int_\Omega (\mathrm{e}^{2\psi_n}-1) + \int_{\{k<u_n\}} u_n^\theta\,(\mathrm{e}^{2\psi_n}-1) \\ &\quad + \frac{\rho^2}{4\alpha}\int_\Omega (\mathrm{e}^{2\psi_n}-1) + \frac{1}{4\alpha}\int_{\{\rho<|E(x)|\}} |E(x)|^2(\mathrm{e}^{2\psi_n}-1). \end{aligned}$$

We now suppose that k is such that

$$k>k^{\theta}+\frac{\rho^2}{4\alpha}, \tag{12}$$

and use the inequality

$$t^2-1\leq 2(t-1)^2+1, \quad \text{for } t\geq 1,$$

to obtain, in view of the fact that

$$\int_{\{0\leq\psi_n\leq k\}}\psi_n(\mathrm{e}^{2\psi_n}-1)\geq 0,$$

that

$$\begin{aligned}\alpha\int_\Omega|\nabla\psi_n|^2\,\mathrm{e}^{2\psi_n}&+\alpha\int_\Omega|\nabla\psi_n|^2-C(k)\\ &\leq 2\int_{\{k<u_n\}}u_n^{\theta}\,(\mathrm{e}^{\psi_n}-1)^2+\int_\Omega u_n^{\theta}\\ &\quad+\frac{1}{2\,\alpha}\int_{\{\rho<|E(x)|\}}|E(x)|^2(\mathrm{e}^{\psi_n}-1)^2+\int_\Omega|E(x)|^2,\end{aligned} \tag{13}$$

for some positive constant $C(k)$. Since $\mathrm{e}^{\psi_n}\nabla\psi_n=\nabla(\mathrm{e}^{\psi_n}-1)$, the Sobolev embedding of $W_0^{1,2}(\Omega)$ into $L^{2^*}(\Omega)$ implies that

$$\mathcal{S}\left[\int_\Omega|\mathrm{e}^{\psi_n}-1|^{2^*}\right]^{\frac{2}{2^*}}\leq\int_\Omega|\nabla\psi_n|^2\,\mathrm{e}^{2\psi_n}.$$

Using this inequality in (13), we obtain

$$\begin{aligned}\mathcal{S}\alpha\left[\int_\Omega(\mathrm{e}^{\psi_n}-1)^{2^*}\right]^{\frac{2}{2^*}}&+\alpha\int_\Omega|\nabla\psi_n|^2\\ &\leq C(k)+2\int_{\{k<u_n\}}u_n^{\theta}\,(\mathrm{e}^{\psi_n}-1)^2+\int_\Omega u_n^{\theta}\\ &\quad+\frac{1}{2\,\alpha}\int_{\{\rho<|E(x)|\}}|E(x)|^2(\mathrm{e}^{\psi_n}-1)^2+\int_\Omega|E(x)|^2.\end{aligned} \tag{14}$$

We apply Hölder inequality with exponents $\frac{2^*}{2}$ and $\frac{N}{2}$ to the right hand side terms of (14) to have that

$$\begin{aligned}&\mathcal{S}\alpha\left[\int_\Omega(\mathrm{e}^{\psi_n}-1)^{2^*}\right]^{\frac{2}{2^*}}+\alpha\int_\Omega|\nabla\psi_n|^2\\ &\leq C(k)+\left[2\left[\int_{\{k<u_n\}}u_n^{\theta\frac{N}{2}}\right]^{\frac{2}{N}}+\frac{1}{2\,\alpha}\left[\int_{\{\rho<|E(x)|\}}|E(x)|^N\right]^{\frac{2}{N}}\right]\left[\int_\Omega(\mathrm{e}^{\psi_n}-1)^{2^*}\right]^{\frac{2}{2^*}}\\ &\quad+C(f)+\|E\|^2_{L^2(\Omega)}.\end{aligned}$$

Since the sequence $\{u_n\}$ is bounded in $L^1(\Omega)$ by Lemma 2.3, since $\theta<2/N$, and since $E(x)$ belongs to $[L^N(\Omega)]^N$, we can choose ρ large enough, and then k large enough so that (12) holds, in order to have that

$$2\left[\int_{\{k<u_n\}} u_n^{\theta\frac{N}{2}}\right]^{\frac{2}{N}} + \frac{1}{2\alpha}\left[\int_{\{\rho<|E(x)|\}} |E(x)|^N\right]^{\frac{2}{N}} \leq \mathcal{S}\frac{\alpha}{2}.$$

Therefore

$$\frac{\mathcal{S}\,\alpha}{2}\left[\int_\Omega (\mathrm{e}^{\psi_n}-1)^{2^*}\right]^{\frac{2}{2^*}} + \alpha\int_\Omega |\nabla\psi_n|^2 \leq C(k) + C(f) + \|E\|^2_{L^2(\Omega)} = C(f,E),$$

which implies the result. ■

Remark 2.5: Let $f(x) \geq 0$ and $g(x) \geq 0$ be functions in $L^\infty(\Omega)$. Then, as a consequence of the inequality

$$s\,t \leq s\log(1+s) + \mathrm{e}^t, \quad \text{for } s,\, t \in \mathbb{R}_+, \tag{15}$$

it follows that

$$\int_\Omega f(x)\log(1+g(x)) \leq \int_\Omega f(x)\log(1+f(x)) + \int_\Omega (1+g(x)). \tag{16}$$

Our next result proves the main estimates needed in order to pass to the limit in the equations satisfied by u_n and ψ_n.

Lemma 2.6: *Under assumptions* (2)–(5), *the solution* (u_n, ψ_n) *of* (7) *satisfies*

$$\int_\Omega \frac{|\nabla u_n|^2}{1+u_n} + \|u_n\|_{L^{\frac{2^*}{2}}(\Omega)} \leq C(f,E), \tag{17}$$

and

$$\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2 \leq C(f,E). \tag{18}$$

Proof: We split the proof in several steps.

Step 1 – As in [19], we start with a duality approach: we choose ψ_n as test function in the first equation in (7) and u_n in the second one, to obtain

$$\begin{cases} \displaystyle\int_\Omega M(x)\nabla u_n\nabla\psi_n + \int_\Omega u_n\psi_n = \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}M(x)\nabla\psi_n\nabla\psi_n + \int_\Omega f_n(x)\psi_n \\ \displaystyle\int_\Omega M(x)\nabla\psi_n\nabla u_n + \int_\Omega u_n\psi_n = \int_\Omega \frac{R(u_n)}{1+\frac{1}{n}R(u_n)}u_n + \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}E(x)\nabla\psi_n. \end{cases}$$

Combining both expressions we arrive to

$$\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}M(x)\nabla\psi_n\nabla\psi_n + \int_\Omega f_n(x)\psi_n = \int_\Omega \frac{u_n\,R(u_n)}{1+\frac{1}{n}R(u_n)} + \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}E(x)\nabla\psi_n.$$

Since

$$\alpha\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2 \leq \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}M(x)\nabla\psi_n\nabla\psi_n,$$

and

$$0 \le \frac{u_n R(u_n)}{1 + \frac{1}{n} R(u_n)} \le u_n^{\theta+1},$$

and since $f_n(x)\,\psi_n \ge 0$, we thus have, thanks to (2)

$$\alpha \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |\nabla \psi_n|^2 \le \int_\Omega u_n^{1+\theta} + \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} E(x) \nabla \psi_n. \tag{19}$$

Using Young inequality in the last term, we have

$$\int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} E(x) \nabla \psi_n \le \frac{\alpha}{2} \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |\nabla \psi_n|^2 + \frac{1}{2\alpha} \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |E(x)|^2,$$

so that from (19) it follows that

$$\int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |\nabla \psi_n|^2 \le C \int_\Omega u_n^{1+\theta} + C \int_\Omega u_n\, |E(x)|^2. \tag{20}$$

Step 2 – We now take $\log(1+u_n)$ as test function in the first equation of (7) to obtain, after some computations, and using (2),

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{1+u_n} + \int_\Omega u_n \log(1+u_n) \le \beta \int_\Omega \frac{u_n |\nabla \psi_n| |\nabla u_n|}{(1 + \frac{1}{n} u_n)(1+u_n)} + \int_\Omega f_n(x) \log(1+u_n). \tag{21}$$

Thanks to (16) and to Lemma 2.3, we have that

$$\int_\Omega f_n(x) \log(1+u_n) \le \int_\Omega f(x)\ \log(1 + f(x)) + \int_\Omega (1+u_n) \le C(f),$$

so that (21) becomes, dropping a positive term,

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \le \beta \int_\Omega \frac{u_n |\nabla \psi_n| |\nabla u_n|}{(1 + \frac{1}{n} u_n)(1+u_n)} + C(f). \tag{22}$$

Thanks to Young inequality, we have

$$\begin{aligned}
\beta \int_\Omega \frac{u_n |\nabla \psi_n| |\nabla u_n|}{(1 + \frac{1}{n} u_n)(1+u_n)} &= \beta \int_\Omega \frac{u_n |\nabla \psi_n|}{(1 + \frac{1}{n} u_n)\sqrt{1+u_n}} \frac{|\nabla u_n|}{\sqrt{1+u_n}} \\
&\le \frac{\beta^2}{2\alpha} \int_\Omega \frac{u_n^2}{(1 + \frac{1}{n}\, u_n)^2 (1+u_n)} |\nabla \psi_n|^2 + \frac{\alpha}{2} \int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \\
&\le \frac{\beta^2}{2\alpha} \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |\nabla \psi_n|^2 + \frac{\alpha}{2} \int_\Omega \frac{|\nabla u_n|^2}{1+u_n}.
\end{aligned}$$

Using this inequality in (22), we have

$$\frac{\alpha}{2} \int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \le \frac{\beta^2}{2\alpha} \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |\nabla \psi_n|^2 + C(f),$$

i.e. recalling (20),

$$\int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \le C \int_\Omega \frac{u_n}{1 + \frac{1}{n}\, u_n} |\nabla \psi_n|^2 + C(f) \le C \int_\Omega u_n^{1+\theta} + C \int_\Omega u_n\, |E(x)|^2 + C(f). \tag{23}$$

Since

$$\frac{|\nabla u_n|^2}{1+u_n} = 4|\nabla[(1+u_n)^{\frac{1}{2}} - 1]|^2,$$

by Sobolev embedding it follows from (23) that

$$\left[\int_\Omega [(1+u_n)^{\frac{1}{2}} - 1]^{2^*}\right]^{\frac{2}{2^*}} \le C\int_\Omega u_n^{1+\theta} + C\int_\Omega u_n\,|E(x)|^2 + C(f).$$

Now, since

$$\|a\|_{L^{2^*}(\Omega)} - \|b\|_{L^{2^*}(\Omega)} \le \|a-b\|_{L^{2^*}(\Omega)},$$

from the above inequality we deduce, after some computations, that

$$\|u_n\|_{L^{\frac{2^*}{2}}(\Omega)} \le C\int_\Omega u_n^{1+\theta} + C\int_\Omega u_n\,|E(x)|^2 + C(f). \tag{24}$$

Now, for every $\rho>0$ we have, thanks to Hölder inequality,

$$\begin{aligned}\int_\Omega u_n\,|E(x)|^2 &= \int_{\{|E(x)|\le\rho\}} u_n\,|E(x)|^2 + \int_{\{\rho<|E(x)|\}} u_n\,|E(x)|^2\\ &\le \rho^2\|f\|_{L^1(\Omega)} + \|u_n\|_{L^{\frac{2^*}{2}}(\Omega)}\left[\int_{\{\rho<|E(x)|\}} |E(x)|^N\right]^{\frac{2}{N}},\end{aligned}$$

where in the last passage we have used Lemma 2.3. On the other hand, for every $k>0$ we have (again by Hölder inequality)

$$\begin{aligned}\int_\Omega u_n^{1+\theta} = \int_\Omega u_n\,u_n^\theta &= \int_{\{0\le u_n\le k\}} u_n\,u_n^\theta + \int_{\{k<u_n\}} u_n\,u_n^\theta\\ &\le k^{1+\theta}|\Omega| + \|u_n\|_{L^{\frac{2^*}{2}}(\Omega)}\left[\int_{\{k<u_n\}} u_n^{\theta\frac{N}{2}}\right]^{\frac{2}{N}}.\end{aligned}$$

Then

$$\begin{aligned}\int_\Omega u_n^{1+\theta} + \int_\Omega u_n\,|E(x)|^2 &\le \rho^2\|f\|_{L^1(\Omega)} + k^{1+\theta}|\Omega|\\ &+ \|u_n\|_{L^{\frac{2^*}{2}}(\Omega)}\left(\left[\int_{\{\rho<|E(x)|\}} |E(x)|^N\right]^{\frac{2}{N}} + \left[\int_{\{k<u_n\}} u_n^{\theta\frac{N}{2}}\right]^{\frac{2}{N}}\right).\end{aligned}$$

We now recall that $|E(x)|^N$ belongs to $L^1(\Omega)$, that $\{u_n\}$ is a bounded sequence in $L^1(\Omega)$ and that $\theta\,\frac{N}{2}<1$. Thus, for any $\varepsilon>0$, there exist $k>0$ and $\rho>0$ large enough, such that

$$\left[\int_{\{\rho<|E(x)|\}} |E(x)|^N\right]^{\frac{2}{N}} + \left[\int_{\{k<u_n\}} u_n^{\theta\frac{N}{2}}\right]^{\frac{2}{N}} \le \varepsilon,$$

which implies that

$$\int_\Omega u_n^{1+\theta} + \int_\Omega u_n\,|E(x)|^2 \le \rho^2\|f\|_{L^1(\Omega)} + k^{1+\theta}|\Omega| + \varepsilon\,\|u_n\|_{L^{\frac{2^*}{2}}(\Omega)}. \tag{25}$$

Using this inequality in (24), and choosing ε small enough, yields

$$\|u_n\|_{L^{\frac{2^*}{2}}(\Omega)} \le C(f,E), \tag{26}$$

which is the second part of (17). From (25) it then follows that

$$\int_\Omega u_n^{1+\theta} + \int_\Omega u_n\,|E(x)|^2 \le \rho^2\|f\|_{L^1(\Omega)} + k^{1+\theta}|\Omega| + C(f,E).$$

This inequality, used in (20) and (23), yields that

$$\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}\,|\nabla\psi_n|^2 \le C(f,E), \qquad \int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \le C(f,E),$$

as desired. ■

As a consequence of the previous lemma we have the following estimate.

Corollary 2.7: *Under the assumptions of Lemma* 2.6 *we have*

$$\int_\Omega |\nabla u_n|^{\frac{N}{N-1}} \le C(f,E). \tag{27}$$

Proof: We proceed as in [20] and apply Hölder inequality with exponents $\frac{2(N-1)}{N}$ and $\frac{2(N-1)}{N-2}$ to obtain

$$\begin{aligned}
\int_\Omega |\nabla u_n|^{\frac{N}{N-1}} &= \int_\Omega \frac{|\nabla u_n|^{\frac{N}{N-1}}}{(1+u_n)^{\frac{N}{2(N-1)}}}\,(1+u_n)^{\frac{N}{2(N-1)}} \\
&\le \left[\int_\Omega \frac{|\nabla u_n|^2}{1+u_n}\right]^{\frac{N}{2(N-1)}} \left[\int_\Omega (1+u_n)^{\frac{2^*}{2}}\right]^{\frac{N-2}{2(N-1)}} \\
&\le C(f,E),
\end{aligned}$$

where in the last passage we have used (17). ■

3. Existence of solutions

In this section we prove the existence of weak solutions to (1). The proof is based on the existence of a subsequence (u_n,ψ_n) which converges to the solution. The result is presented in the following theorem.

Theorem 3.1: *Under assumptions* (2)–(5), *there exists a weak solution* (u,ψ) *to* (1) *in the sense of Definition* 1.1.

Proof: We consider the sequence (u_n,ψ_n) of weak solutions to (7). Since the sequence $\{u_n\}$ is bounded in $W_0^{\frac{N}{N-1}}(\Omega)$, there exists a subsequence (still denoted by $\{u_n\}$) and u in $W_0^{\frac{N}{N-1}}(\Omega)$ such that

$$\begin{aligned} u_n &\longrightarrow u, \quad \text{strongly in } L^s(\Omega), \text{ for any } s < \tfrac{2^*}{2} \text{ and almost everywhere,} \\ u_n &\rightharpoonup u, \quad \text{weakly in } W_0^{\frac{N}{N-1}}(\Omega). \end{aligned} \tag{28}$$

Moreover, the sequence $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega) \cap L^p(\Omega)$ (for any $1 \le p < +\infty$) and therefore there exists ψ $W_0^{1,2}(\Omega)$ and a subsequence (still denoted by $\{\psi_n\}$) such that

$$\begin{aligned} \psi_n &\rightharpoonup \psi, \quad \text{weakly in } W_0^{1,2}(\Omega), \\ \psi_n &\to \psi, \quad \text{strongly in } L^p(\Omega), \text{ for every } p < +\infty \text{ and almost everywhere.} \end{aligned} \tag{29}$$

The strong convergence of the sequence $\{u_n\}$ to u in $L^s(\Omega)$, for $s < \frac{2^*}{2}$, implies that the sequence $\{R(u_n)\}$ strongly converges to $R(u)$ in $L^q(\Omega)$, for every $q < \frac{2^*}{2\theta}$. Since $\theta < \frac{2}{N}$, one has that

$$\frac{2^*}{2\theta} > \frac{2N}{N+2},$$

so that the sequence $\{R(u_n)\}$ converges to $R(u)$ in $L^{2_*}(\Omega)$. Furthermore, by the first equation of (29), by the fact that $E(x)$ belongs to $[L^N(\Omega)]^N$, and since u_n converges to u almost everywhere, we have that

$$\frac{E(x)}{1 + \frac{1}{n} u_n} \nabla\psi_n \rightharpoonup E(x)\, \nabla\psi \quad \text{in } L^{2_*}(\Omega).$$

Thus, we can pass to the limit in the weak formulation of the second equation of system (7) to have that

$$\int_\Omega M(x)\nabla\psi\nabla w + \int_\Omega \psi\, w = \int_\Omega R(u)\, w + \int_\Omega E(x)\, \nabla\psi\, w, \quad \forall\, w \in W_0^{1,2}(\Omega),$$

so that ψ is a solution of the second equation of system (1) in the sense of Definition 1.1.

In order to prove that u is a solution of the first equation, we begin by proving that the convergence of the sequence $\{\psi_n\}$ in $W_0^{1,2}(\Omega)$ is actually strong. In order to do that, we take $\psi_n - \psi$ as test function in the weak formulation of the second equation of (7) to have

$$\begin{aligned} \int_\Omega M(x)\nabla[\psi_n - \psi]\nabla[\psi_n - \psi] &= -\int_\Omega M(x)\nabla\psi\nabla[\psi_n - \psi] \\ -\int_\Omega \psi_n[\psi_n - \psi] &+ \int_\Omega \frac{R(u_n)}{1 + \frac{1}{n}R(u_n)}[\psi_n - \psi] + \int_\Omega \frac{E(x)}{1 + \frac{1}{n} u_n} \nabla\psi_n\,[\psi_n - \psi]. \end{aligned} \tag{30}$$

Since

$$\left| \int_\Omega \frac{E(x)}{1 + \frac{1}{n} u_n} \nabla\psi_n\,[\psi_n - \psi] \right| \le \|E(x)\|_{L^N(\Omega)}\, \|\psi_n\|_{W_0^{1,2}(\Omega)}\, \|\psi_n - \psi\|_{L^{2^*}(\Omega)},$$

from the second convergence of (29) with $p = 2^*$ we have that

$$\lim_{n\to+\infty} \int_\Omega \frac{E(x)}{1 + \frac{1}{n} u_n} \nabla\psi_n\,[\psi_n - \psi] = 0.$$

Furthermore, since the sequence $\{R(u_n)\}$ is bounded in $L^{\frac{1}{\theta}}(\Omega)$ by Lemma 2.3, again from the second convergence of (29) with $p = (\frac{1}{\theta})' = \frac{1}{1-\theta}$, we have

$$\lim_{n\to+\infty} \int_\Omega \frac{R(u_n)}{1 + \frac{1}{n}R(u_n)}[\psi_n - \psi] = 0,$$

while the fact that

$$\lim_{n\to+\infty}\int_\Omega \psi_n\,[\psi_n-\psi]=0,$$

follows from the strong convergence of ψ_n to ψ in $L^2(\Omega)$. Since from the first convergence of (29):

$$\lim_{n\to+\infty}\int_\Omega M(x)\nabla\psi\nabla[\psi_n-\psi]=0,$$

from (2) and thanks to (30) it follows that

$$\lim_{n\to+\infty}\alpha\int_\Omega |\nabla[\psi_n-\psi]|^2=0,$$

so that the sequence $\{\psi_n\}$ strongly converges to ψ in $W_0^{1,2}(\Omega)$. Therefore, up to subsequences, we have that

$$\text{the sequence } \{\nabla\psi_n\} \text{ almost everywhere converges to } \nabla\psi \text{ in } \Omega. \tag{31}$$

We now follow [2]. Let $\rho>0$, and let Σ be a measurable subset of Ω. Then, recalling (18),

$$\begin{aligned}\int_\Sigma \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n| &= \rho\int_{\Sigma\cap\{|\nabla\psi_n|\le\rho\}} u_n+\frac{1}{\rho}\int_{\Sigma\cap\{\rho<|\nabla\psi_n|\}}\frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2\\ &\le \rho\int_\Sigma u_n+\frac{C(f,E)}{\rho}.\end{aligned}$$

If $\varepsilon>0$, thanks to the strong convergence of the sequence $\{u_n\}$ to u in $L^1(\Omega)$, one can choose ρ large enough, and then $|\Sigma|$ small enough, to have that

$$\sup_{n\in\mathbb{N}}\int_\Sigma \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|\le\epsilon,$$

so that

$$\text{the sequence } \left\{\frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|\right\} \text{ is equiintegrable.}$$

Therefore, from Vitali theorem it follows that

$$\text{the sequence } \left\{\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right\} \text{ strongly converges to } u\,\nabla\psi \text{ in } [L^1(\Omega)]^N.$$

Using this result, as well as the strong convergences of u_n to u, and of $f_n(x)$ to $f(x)$ in $L^1(\Omega)$, we can pass to the limit in the equation solved by u_n to have that

$$\int_\Omega M(x)\nabla u\nabla\varphi+\int_\Omega u\,\varphi=\int_\Omega u\,\nabla\psi\nabla\varphi+\int_\Omega f(x)\,\varphi,\quad \forall\,\varphi\in C_0^\infty(\Omega).$$

In order to obtain test functions in $W_0^{1,N}(\Omega)$, we need a further step. Recalling again (18), and using the almost everywhere convergence of u_n to u and of $\nabla\psi_n$ to $\nabla\psi$, as well as Fatou lemma, we obtain that

$$0 \le \int_\Omega u\,|\nabla\psi|^2 \le \liminf_{n\to+\infty} \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2 \le C(f,E),$$

so that the function $u\,|\nabla\psi|^2$ belongs to $L^1(\Omega)$. Thus, writing

$$u\,\nabla\psi = \sqrt{u}\,\sqrt{u}\,\nabla\psi,$$

we have, recalling that $\sqrt{u}$ is in $L^{2^*}(\Omega)$, and that $\sqrt{u}\,\nabla\psi$ is in $[L^2(\Omega)]^N$, that

$$u\,\nabla\psi \in [L^{\frac{N}{N-1}}(\Omega)]^N,$$

so that one has

$$u\,\nabla\psi\nabla\eta \in L^1(\Omega), \quad \forall\,\eta \in W_0^{1,N}(\Omega).$$

Since the same property is true for $M(x)\nabla u\nabla\eta$, for $u\,\eta$, and for $f(x)\,\eta$ (recall that η, being in $W_0^{1,N}(\Omega)$, is exponentially summable, and use 15), one can use the density of $C_0^\infty(\Omega)$ in $W_0^{1,N}(\Omega)$ to have that

$$\int_\Omega M(x)\nabla u\nabla\eta + \int_\Omega u\,\eta = \int_\Omega u\,\nabla\psi\nabla\eta + \int_\Omega f(x)\,\eta, \quad \forall\,\eta \in W_0^{1,N}(\Omega),$$

as desired. ∎

4. The case $N \le 4$

In this section we are going to prove that if the space dimension N is either 3 or 4, under a natural assumption on $f(x)$ and a boundedness assumption on $E(x)$, the solution u of the first equation of system (1) belongs to $W_0^{1,2}(\Omega)$.

Theorem 4.1: *Let $N=3$ or $N=4$. Let $f(x)$ be a function such that*

$$0 \le f(x) \in L^{2_*}(\Omega), \tag{32}$$

and let $E(x)$ be a vector field such that

$$E \in [L^\infty(\Omega)]^N. \tag{33}$$

The solution u of the first equation of system (1) belongs to $W_0^{1,2}(\Omega)$.

Proof: Since the solution u of the first equation of system (1) is the limit of the sequence $\{u_n\}$ of solutions of the first equation of system (7), it will be enough to prove that such sequence is bounded in $W_0^{1,2}(\Omega)$ in order to prove our result. We recall that, as a consequence of (17) and (18), and of the fact that the assumptions on $f(x)$ and $E(x)$ are now stronger, one has that

$$\int_\Omega u_n^{\frac{2^*}{2}} + \int_\Omega \frac{u_n}{1+\frac{1}{n}\,u_n}\,|\nabla\psi_n|^2 \le C(f,E). \tag{34}$$

We follow the approach of [1], choosing test functions in the equations of system (7) which will allow us to 'cancel out' two terms. More precisely, we choose u_n and

$$H_n(u_n) = \int_0^{u_n} \frac{s}{1+\frac{1}{n}s},$$

as test functions in the first and second equation of (7) respectively. We obtain

$$\int_\Omega M(x)\nabla u_n\nabla u_n + \int_\Omega u_n^2 = \int_\Omega \frac{u_n}{1+\frac{1}{n}\,u_n} M(x)\nabla\psi_n\nabla u_n + \int_\Omega f_n(x)\,u_n,$$

and

$$\int_\Omega \frac{u_n}{1+\frac{1}{n}\,u_n} M(x)\nabla\psi_n\nabla u_n + \int_\Omega \psi_n H_n(u_n)$$
$$= \int_\Omega \frac{R(u_n)}{1+\frac{1}{n}R(u_n)} H_n(u_n) + \int_\Omega \frac{H_n(u_n)}{1+\frac{1}{n}u_n} E(x)\nabla\psi_n.$$

Since the third term of the first expression is equal to the first one of the second, we have

$$\int_\Omega M(x)\nabla u_n\nabla u_n + \int_\Omega u_n^2 + \int_\Omega \psi_n\,H_n(u_n)$$
$$= \int_\Omega f_n(x)u_n + \int_\Omega \frac{R(u_n)}{1+\frac{1}{n}R(u_n)}\,H_n(u_n) + \int_\Omega \frac{H_n(u_n)}{1+\frac{1}{n}\,u_n} E(x)\nabla\psi_n.$$

Since

$$0 \le H_n(u_n) = \int_0^{u_n} \frac{s}{1+\frac{1}{n}\,s}\,\mathrm{d}s \le \int_0^{u_n} s\,\mathrm{d}s = \frac{u_n^2}{2},$$

we thus have, using (2), and dropping two positive terms in the left hand side,

$$\alpha\int_\Omega |\nabla u_n|^2 \le \int_\Omega f_n(x)u_n + \frac{1}{2}\int_\Omega u_n^{2+\theta} + \frac{1}{2}\int_\Omega \frac{u_n^2}{1+\frac{1}{n}\,u_n}\,|E(x)|\,|\nabla\psi_n|. \tag{35}$$

Let $\varepsilon>0$. We have, thanks to Young inequality, and since $2+\theta<2+\frac{2}{N}<3$,

$$\int_\Omega u_n^{2+\theta} \le C(\varepsilon) + \varepsilon\int_\Omega u_n^3. \tag{36}$$

Furthermore, we have, thanks to Young inequality and (34),

$$\begin{aligned}\int_\Omega \frac{u_n^2}{1+\frac{1}{n}\,u_n}\,|E(x)|\,|\nabla\psi_n| &= \int_\Omega u_n^{\frac{3}{2}}\,|E(x)|\,\frac{u_n^{\frac{1}{2}}\,|\nabla\psi_n|}{1+\frac{1}{n}\,u_n}\\ &\le \varepsilon\int_\Omega u_n^3\,|E(x)|^2 + C(\varepsilon)\int_\Omega \frac{u_n}{[1+\frac{1}{n}\,u_n]^2}\,|\nabla\psi_n|^2\\ &\le \varepsilon\,\|E\|^2_{L^\infty(\Omega)}\int_\Omega u_n^3 + C(\varepsilon)\int_\Omega \frac{u_n}{1+\frac{1}{n}\,u_n}\,|\nabla\psi_n|^2\\ &= \varepsilon\,\|E\|^2_{L^\infty(\Omega)}\int_\Omega u_n^3 + C(f,E)\,.\end{aligned} \tag{37}$$

Using (36) and (37) in (35), we arrive at

$$\alpha \int_\Omega |\nabla u_n|^2 \le \int_\Omega f_n(x) u_n + C(f,E)\,\varepsilon \int_\Omega u_n^3 + C(f,E). \tag{38}$$

We now have, by Hölder inequality,

$$\int_\Omega u_n^3 = \int_\Omega u_n\, u_n^2 \le \left[\int_\Omega u_n^{\frac{N}{2}}\right]^{\frac{2}{N}} \left[\int_\Omega u_n^{2^*}\right]^{\frac{2}{2^*}}. \tag{39}$$

Observing that $\frac{N}{2} \le \frac{2^*}{2}$ if $N \le 4$, thanks to (34), we have, by Hölder inequality,

$$\int_\Omega u_n^{\frac{N}{2}} \le C \left[\int_\Omega u_n^{\frac{2^*}{2}}\right]^{\frac{N-2}{2}} \le C(f,E),$$

so that from (39) follows that

$$\int_\Omega u_n^3 \le C(f,E) \left[\int_\Omega u_n^{2^*}\right]^{\frac{2}{2^*}}.$$

Using this inequality in (38), and recalling Sobolev embedding, we have that

$$\mathcal{S}\,\alpha \left[\int_\Omega u_n^{2^*}\right]^{\frac{2}{2^*}} \le \alpha \int_\Omega |\nabla u_n|^2 \le \int_\Omega f_n(x)\, u_n + C(f,E)\,\varepsilon \left[\int_\Omega u_n^{2^*}\right]^{\frac{2}{2^*}} + C(f,E). \tag{40}$$

Choosing ε small enough, we thus have that, recalling the assumptions on $f(x)$, and again by Hölder inequality,

$$\frac{\mathcal{S}\,\alpha}{2} \left[\int_\Omega u_n^{2^*}\right]^{\frac{2}{2^*}} \le \int_\Omega f_n(x)\, u_n + C(f,E) \le \|f\|_{L^{2*}(\Omega)} \left[\int_\Omega u_n^{2^*}\right]^{\frac{1}{2^*}} + C(f,E).$$

This inequality implies that

$$\|u_n\|_{L^{2^*}(\Omega)} \le C(f,E)\, \|f\|_{L^{2*}(\Omega)},$$

and from this estimate, and (40), it follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, as desired. ■

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