Sapienza Università di Roma

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A categorification of cluster algebras of type B and C through symmetric quivers

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Ai miei nonni

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* * *

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Sommario

In questa tesi presentiamo una categorificazione delle algebre cluster di tipo B e C mediante una classe specifica di quiver simmetrici associati a triangolazioni di poligoni regolari.

Sia $\mathbf{P}_{2n+2}$ il poligono regolare con $2n+2$ vertici. Sia θ la rotazione di 180°. È stato dimostrato da Fomin e Zelevinsky che le triangolazioni θ-invarianti di $\mathbf{P}_{2n+2}$ sono in biezione con i cluster delle algebre cluster di tipo B_n e C_n. Inoltre, le variabili cluster corrispondono alle orbite dell'azione di θ sulle diagonali del poligono.

Data una triangolazione θ-invariante T di $\mathbf{P}_{2n+2}$, nella trattazione introduciamo le definizioni di algebre cluster di tipo B_n e C_n con coefficienti principali in T, e dimostriamo una formula di espansione per la variabile cluster x_{ab} corrispondente alla θ-orbita $[a, b]$ della diagonale che collega i vertici a e b. La formula che presentiamo è data in modo combinatorio. Da un lato essa esprime ogni variabile cluster di tipo B_n e C_n in termini di variabili cluster di tipo A_n, dall'altro ci permette di ottenere la sua espansione in termini delle variabili cluster del seme iniziale.

Inoltre, associamo a ogni θ-orbita $[a, b]$ di $\mathbf{P}_{2n+2}$ uno "snake graph" modificato $\mathcal{G}_{ab}$, costruito incollando tra loro gli snake graph corrispondenti a particolari diagonali, ottenute da quelle di $[a, b]$ indentificando alcuni vertici del poligono. Otteniamo così l'espansione cluster di x_{ab} in termini di "perfect matching" di $\mathcal{G}_{ab}$. Questo estende il lavoro di Musiker per le algebre cluster di tipo B e C a ogni seme.

D'altra parte, la teoria delle rappresentazioni dei quiver simmetrici è stata sviluppata da Derksen e Weyman, e Boos e Cerulli Irelli. Un *quiver simmetrico* è un quiver Q con una involuzione dei vertici e delle frecce che inverte l'orientazione delle frecce. Una *rappresentazione simmetrica* è una rappresentazione ordinaria di Q, con il dato aggiuntivo di un prodotto scalare, e la richiesta che ogni coppia duale di frecce agisca in modo antiaggiunto rispetto ad esso. Le rappresentazioni simmetriche sono di due tipi: *ortogonali* e *simplettiche*. Esse formano una categoria additiva non abeliana.

Nel lavoro di tesi associamo un quiver simmetrico di tipo A_{2n-1} a ogni seme di un'algebra cluster di tipo B_n e C_n. In questo modo, le variabili cluster di tipo B_n (rispettivamente, C_n) sono in biezione con le rappresentazioni indecomponibili ortogonali (rispettivamente, simplettiche) di Q. Forniamo, inoltre, una mappa di Caldero-Chapoton in questo contesto. Diamo, infine, un'interpretazione categorica della formula di espansione cluster nel caso di quiver aciclici, e presentiamo una congettura per il caso ciclico.

Parole chiave: algebre cluster, quiver simmetrici, rappresentazioni simmetriche, poligoni triangolati, snake graph, coefficienti principali.

Abstract

In this thesis we present a categorification of cluster algebras of type B and C through a specific class of symmetric quivers arising from triangulations of polygons.

Let $\mathbf{P}_{2n+2}$ be the regular polygon with $2n+2$ vertices. Let θ be the rotation of 180°. Fomin and Zelevinsky showed that θ-invariant triangulations of $\mathbf{P}_{2n+2}$ are in bijection with the clusters of cluster algebras of type B_n and C_n. Furthermore, cluster variables correspond to the orbits of the action of θ on the diagonals of the polygon.

Given a θ-invariant triangulation T of $\mathbf{P}_{2n+2}$, we define cluster algebras of type B_n and C_n with principal coefficients in T, and we prove an expansion formula for the cluster variable x_{ab} corresponding to the θ-orbit $[a,b]$ of the diagonal which connects the vertices a and b. The formula we present is given in a combinatorial way. On the one hand, it expresses each cluster variable of type B_n and C_n in terms of cluster variables of type A_n, on the other hand, it allows us to get its expansion in terms of the cluster variables of the initial seed.

Moreover, we associate to each θ-orbit $[a,b]$ of $\mathbf{P}_{2n+2}$ a modified snake graph $\mathcal{G}_{ab}$, constructed by gluing together the snake graphs corresponding to particular diagonals, obtained from those of $[a,b]$ by identifying some vertices of the polygon. Then we get the cluster expansion of x_{ab} in terms of perfect matchings of $\mathcal{G}_{ab}$. This extends the work of Musiker for cluster algebras of type B and C to every seed.

On the other hand, the representation theory of symmetric quivers was developed by Derksen and Weyman, as well as Boos and Cerulli Irelli. A *symmetric quiver* is a quiver Q with an involution of vertices and arrows which reverses the orientation of arrows. A *symmetric representation* is an ordinary representation of Q equipped with some extra data that forces each dual pair of arrows to act anti-adjointly. Symmetric representations are of two types: *orthogonal* and *symplectic*. They form an additive category which is not abelian.

We associate a cluster tilted bound symmetric quiver Q of type A_{2n-1} to any seed of a cluster algebra of type B_n and C_n. Under this correspondence, cluster variables of type B_n (resp. C_n) are in bijection with orthogonal (resp. symplectic) indecomposable representations of Q. We find a Caldero-Chapoton map in this setting. We also give a categorical interpretation of the cluster expansion formula in the case of acyclic quivers, and we present a conjecture for the cyclic case.

Keywords: cluster algebras, symmetric quivers, symmetric representations, triangulated polygons, snake graphs, principal coefficients.

Contents

Introduction

Cluster algebras, discovered by Fomin and Zelevinsky and introduced in their seminal paper [30], are commutative algebras with a special combinatorial structure. Their first aim was to find an algebraic and combinatorial framework for understanding, on the one hand, canonical bases in certain quantum groups, on the other hand, total positivity in certain varieties. Among these algebras one finds coordinate rings of many algebraic varieties: for instance, homogeneous coordinate rings of Grassmannians, Schubert varieties, and other related varieties carry a cluster algebra structure. Since its inception, the theory of cluster algebras was extremely successful, and many exciting connections and applications to a number of different research areas have been found over the years, including representation theory of finite dimensional algebras and Lie algebras, combinatorics, preprojective algebras, Calabi-Yau algebras and categories, algebraic and hyperbolic geometry, dynamical systems, and knot theory.

A *cluster algebra* is a subalgebra of a field of rational functions in n variables $x_1, \dots, x_n$ that is generated by the so called *cluster variables*. Cluster variables are constructed recursively from an initial seed by a process called *mutation*, and they are grouped into overlapping sets of constant cardinality n, the *clusters*. A cluster algebra is said to be of *finite type* if it has a finite number of cluster variables. Cluster algebras of finite type are classified by Dynkin diagrams, in the same way as semisimple Lie algebras and finite root systems [31].

A remarkable result in the theory states that every cluster variable u is a Laurent polynomial in the cluster variables $x_1, \dots, x_n$ of the initial cluster, i.e.,

$$u = \frac{f(x_1, \dots, x_n)}{\prod_{i=1}^{n} x_i^{d_i}}, \tag{0.0.1}$$

where f is a polynomial, and $d_1, \dots, d_n$ are non-negative integers. This is usually referred to as the *Laurent phenomenon* [30], and the right hand side of 0.0.1 is the *cluster expansion* of u in the initial cluster variables.

Fomin, Shapiro and Thurston in [26, 27], and Labardini-Fragoso in [38], initiated the study of cluster algebras, and quivers with potential, arising from triangulations of surfaces with boundary and marked points. In their approach, cluster variables correspond to arcs in the surface, and clusters correspond to triangulations. Musiker and Schiffler in [40], and Musiker, Schiffler and Williams in [41], gave an expansion formula for the cluster variables in terms of perfect matchings of some labeled graphs, called *snake graphs*, that are recursively constructed from the surface by gluing together elementary pieces called tiles.

Let $\mathbf{P}_{2n+2}$ be the regular polygon with $2n+2$ vertices. Let θ be the rotation of 180°. Fomin and Zelevinsky showed in [31] that θ-invariant triangulations of $\mathbf{P}_{2n+2}$ are in bijection with the clusters of cluster algebras of type B_n and C_n. Furthermore, cluster variables correspond to the orbits of the action of θ on the diagonals of $\mathbf{P}_{2n+2}$, which can be either diameters or pairs of centrally symmetric non-diameter diagonals. In this thesis, given a θ-invariant triangulation T, we define cluster algebras of type B_n and C_n with principal coefficients in T, and we find an expansion formula for the cluster variable x_{ab} corresponding to the θ-orbit $[a, b]$ of the diagonal (a, b) which connects the vertices a and b (cf. Theorem 2.2.2, Theorem 2.3.7). The formula we present is given in a combinatorial way. On the one hand, it expresses each cluster variable of type B_n and C_n in terms of cluster variables of type A_n, on the other hand, it allows one to get its expansion in terms of the cluster variables of the initial seed.

Moreover, we associate to each θ-orbit $[a, b]$ of $\mathbf{P}_{2n+2}$ a modified snake graph $\mathcal{G}_{ab}$ constructed by gluing together the snake graphs corresponding to particular diagonals obtained from those of $[a, b]$ by identifying some vertices of the polygon. Then we get the cluster expansion of x_{ab} in terms of perfect matchings of $\mathcal{G}_{ab}$ (cf. Theorem 3.2.14, Theorem 3.2.27). This extends the work of Musiker [39] for cluster algebras of type B and C to every seed.

Several other works on cluster expansion formulas for cluster algebras of type B and C exist in the literature. In particular, Nakanishi and Stella provide in [42] a diagrammatic description of the $\mathbf{g}$-vectors of cluster algebras of type B and C, while Reading studies them in [45] using ring homomorphisms between cluster algebras of type B and C, and cluster algebras of type A, induced by the fact that exchange matrices of type B_n and C_n "dominate" exchange matrices of type A_n. Moreover, a cluster algebra of type B_n (resp. C_n) can be realized as a disk with one orbifold point of weight 2 (resp. $\frac{1}{2}$), and $n+1$ boundary marked points [23]. In [25], Felikson and Tumarkin compute $\mathbf{g}$-vectors for cluster algebras from orbifolds, including type B and C, in terms of laminations on the orbifolds. In [12], Canakci and Tumarkin introduce snake and band graphs associated to curves in a triangulated orbifold with orbifold points of weight $\frac{1}{2}$, including type C. Furthermore, a relation between skew-symmetric and skew-symmetrizable cluster algebras has been investigated in [22, 24] via folding. Finally, in [2], Banaian and Kelley extend the snake graph construction of Musiker, Schiffler and Williams [41] to generalized cluster algebras from unpunctured orbifolds, including generalized cluster algebras of type B and C.

The link between cluster algebras and representation theory gave rise to the so-called *additive categorification* of cluster algebras [6]. Additive categorification has been widely prolific, from providing the first proof of one of the foundational conjectures of the theory (the linear independence of cluster monomials [15]) for skew-symmetric cases, the description of generic bases by Geiss, Leclerc and Schröer [33] and Keller's proof of the periodicity conjecture of Zamolodchikov [37]. Furthermore, the insights from cluster theory have led to important developments in representation theory, the most important of which being tau-tilting theory, a very active field of study.

At the moment, additive categorification is an incomplete theory: it mostly exists for skew-symmetric cluster algebras. The most ambitious objective in the theory is to find an additive categorification for all skew-symmetrizable cluster algebras.

In this thesis we present an additive categorification of cluster algebras of type B and C through symmetric representations of a specific class of symmetric quivers, arising from triangulations of polygons invariant under the reflection along a fixed diameter.

The representation theory of symmetric quivers was developed by Derksen and Weyman in [19], as well as Boos and Cerulli Irelli in [5]. A *symmetric quiver* is a quiver Q with an involution σ of vertices and arrows which reverses the orientation of arrows. A *symmetric representation* is an ordinary representation equipped with some extra data that forces each dual pair $(\alpha, \sigma(\alpha))$ of arrows of Q to act anti-adjointly. Symmetric representations are of two types: *orthogonal* and *symplectic*. They form two additive categories which are not abelian.

Derksen and Weyman in [19] stated the correspondence between positive roots of a root system of type B_n (resp. C_n) and orthogonal (resp. symplectic) indecomposable representations of symmetric quivers of type A_{2n-1}. On the other hand, from the classification of finite type cluster algebras, we know that positive roots of type B_n and C_n correspond to non-initial cluster variables of type B_n and C_n. Therefore, there is a one-to-one correspondence between non-initial cluster variables of type B_n (resp. C_n) and orthogonal (resp. symplectic) indecomposable representations of symmetric quivers of type A_{2n-1}. One of the aims of this thesis is to find explicitly this bijection. In the process of doing this, we extend it to any symmetric quiver in the mutation class of a symmetric quiver of type A_{2n-1}.

Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$. The quiver naturally associated to it (cf. Definition 1.1.10) is not symmetric. In order to get a symmetric quiver, we apply to the polygon an involution which depends on the orientation of the unique diameter d of T; we call it F_d. It consists of cutting $\mathbf{P}_{2n+2}$ along d, then reflecting the right part with respect to the axis of symmetry of d, and finally gluing it again along d. F_d induces an action on isotopy classes of diagonals of the polygon. Let ρ denote the reflection of the polygon along d. Under the bijection F_d, θ-orbits correspond to ρ-orbits. In particular, diameters correspond to ρ-invariant diagonals, while pairs of centrally symmetric diagonals correspond to ρ-invariant pairs of diagonals which are not orthogonal to d. Let T' be the element in the isotopy class of $F_d(T)$ which is also a triangulation. Then T' is a ρ-invariant triangulation of $\mathbf{P}_{2n+2}$ which contains d. Therefore, the quiver $Q(T')$ associated to T' is a cluster-tilted bound symmetric quiver of type A_{2n-1} with no fixed arrows. Furthermore, indecomposable representations L_γ of $Q(T')$ correspond to diagonals γ of $\mathbf{P}_{2n+2}$ which are not in T', and indecomposable symmetric representations correspond to their ρ-orbits. It follows that, in particular, non-initial cluster variables of type B_n (resp. C_n) correspond to orthogonal (resp. symplectic) indecomposable representations of $Q(T')$.

Furthermore, formulas of Theorem 3.2.14 (type B) and Theorem 3.2.27 (type C) give the cluster expansion of each cluster variable associated to a θ-orbit in terms of the cluster variables of the initial seed. It follows from the above correspondence that, given a cluster-tilted bound symmetric quiver Q of type A_{2n-1} with no fixed arrows, they allow us to express the type B_n (resp. C_n) cluster variable that corresponds to an orthogonal (resp. symplectic) indecomposable representation of Q in terms of the initial cluster variables. In other words, we get a Caldero-Chapoton like map [7] from the categories of orthogonal and symplectic representations of cluster tilted bound symmetric quivers of type A_{2n-1} (with no fixed arrows) to cluster algebras of type B_n and C_n.

To conclude, we give a categorical interpretation of Theorem 2.2.2 in the case where $Q(T')$ has no oriented cycles. We also present a conjecture in the cyclic case.

This approach could be used to produce a categorification of other classes of non-skew-symmetric cluster algebras through the representation theory of symmetric quivers. For example, they could provide an alternative categorification of non-skew-symmetric cluster algebras associated by Felikson, Shapiro and Tumarkin [23] to surfaces with marked points and order-2 orbifold points. These algebras have been categorified in the work of Geuenich and Labardini-Fragoso [35, 36] by species with potential.

In literature there are other different categorifications of cluster algebras of type B and C. In [34] Geiss, Leclerc and Schröer use categories of locally free modules over certain Iwanaga-Gorenstein algebras; in [18] Demonet uses exact stably 2-Calabi-Yau categories endowed with the action of a finite group; in [35, 36] Geuenich and Labardini-Fragoso use species with potential. On the other hand, in [3] Bazier-Matte, Chan and Wright use symmetric representations to give a categorification of quasi-cluster algebras from non-orientable surfaces.

The dissertation is organized as follows.

- In Chapter 1 we give an overview of cluster algebras of geometric type. We recall the definition and the main results of the theory, including the Laurent phenomenon, the separation of addition, and the finite type classification. In Section 1.2.1 we report some classical examples of cluster algebras of small rank. In Section 1.3.1 we focus on cluster algebras with principal coefficients, that are the setting in which we will work in the following chapters. We give the definition of F-polynomial and $\mathbf{g}$-vector of a cluster variable, and explain their key role in its expansion in the initial cluster variables in a cluster algebra with arbitrary choice of coefficients. In Sections 1.4.1 and 1.4.2 we recall the combinatorial description of the seed patterns of type A_n, B_n and C_n in terms of diagonals of a regular polygon, and orbits of the action of θ on the diagonals, respectively. These models will be used in Chapter 2 to find the cluster expansion formulas for cluster algebras of type B_n and C_n.
- In Chapter 2, mostly based on [17], we introduce cluster algebras of type B_n and C_n with principal coefficients in a given θ-invariant triangulation of $\mathbf{P}_{2n+2}$. Moreover, we define a simple operation on the diagonals of the polygon, the *restriction* (cf. Definition 2.1.1), which allow us to relate cluster variables of type B_n and C_n to cluster variables of type A_n. The main results of the chapter are the cluster expansion formulas for type B (cf. Theorem 2.2.2) and C (cf. Theorem 2.3.7). They are given in a combinatorial way, and the proofs rely on the geometric model for cluster algebras of type A, B and C discussed in Chapter 1.
- In Chapter 3, mostly based on [16], we associate a modified snake graph $\mathcal{G}_{ab}$ to each θ-orbit

$[a,b]$, and prove that the perfect matching Laurent polynomial of $\mathcal{G}_{ab}$ is precisely the cluster variable which corresponds to $[a,b]$. The shape of $\mathcal{G}_{ab}$ has been inspired by the work of Musiker for cluster algebras of type B and C [39], which this chapter extends to every seed.

- Chapter 4 is devoted to a recollection on quiver representations, and it gives the background necessary for the last chapter. We review the notions of projective and injective modules (Section 4.1.1), quivers and path algebras (Section 4.2), representations of bound quivers (Section 4.3.1), representations of type A quivers given by diagonals of a polygon (Section 4.3.2), quiver Grassmannians (Section 4.3.3), F-polynomials and $\mathbf{g}$-vectors of quiver representations (Section 4.3.4), Auslander-Reiten translation and Auslander-Reiten formulas (Section 4.4).
- Finally, in Chapter 5, mostly based on [17], we present the categorification of cluster algebras of type B and C mentioned above. First, in Section 5.1, we give an introduction to symmetric representation theory. Then, in Section 5.4, we explain how each cluster variable of type B_n and C_n corresponds to a symmetric indecomposable representation of a cluster tilted bound symmetric quiver of type A_{2n-1}. Finally, in Section 5.5, we show a categorical interpretation of the formula relating cluster variables of type B to cluster variables of type A (cf. Theorem 2.2.2 of Chapter 2), in the case of quivers without oriented cycles. This result (cf. Theorem 5.5.15) relies on the cluster multiplication formula of Cerulli Irelli, Esposito, Franzen, Reineke [14] for acyclic quivers. We also present a conjectural extension of this formula to the case of bound quivers, and provide several examples supporting our conjecture.

Chapter 1

Cluster algebras

This chapter is devoted to give an exhaustive overview of cluster algebras of *geometric type*. We recall the definition and the main results of the theory, including the Laurent phenomenon, the separation of addition, and the finite type classification. In Section 1.2.1 we report some classical examples of cluster algebras of small rank. In Section 1.3.1 we focus on cluster algebras with principal coefficients that are the setting in which we will work in the following chapters. We give the definition of F-polynomial and $\mathbf{g}$-vector of a cluster variable, and explain their key role in its expansion in the initial cluster variables in a cluster algebra with arbitrary choice of coefficients. In Sections 1.4.1 and 1.4.2 we recall the combinatorial description of the seed patterns of type A_n, B_n and C_n in terms of diagonals of a regular polygon, and orbits of the action of θ on the diagonals, respectively. These models will be used in Chapter 2 to find the cluster expansion formulas for cluster algebras of type B_n and C_n. For the exposition we follow mainly [28–32].

1.1 Mutations of quivers and matrices

In this section we introduce the concepts of mutations of quivers and skew-symmetrizable matrices, which lie at the heart of the combinatorial framework for the general theory of cluster algebras.

Definition 1.1.1. A (finite) *quiver* $Q = (Q_0, Q_1, s, t)$ is an oriented graph, consisting of vertices and directed edges called *arrows*. Q_0 and Q_1 denote the finite set of vertices and arrows, respectively, while $s, t : Q_0 \to Q_1$ are two functions that provide the orientation $\alpha : s(\alpha) \to t(\alpha)$ of arrows. Multiple edges are allowed, while loops (i.e., arrows connecting a vertex to itself) and oriented 2-cycles (i.e. pairs of arrows with opposite orientation which connect the same pair of vertices) are not allowed. A quiver does not have to be connected. Some vertices of the quiver are designated as *frozen*. They are usually represented by square boxes in the quiver. The remaining vertices are called *mutable*. We assume that there are no arrows between pairs of frozen vertices.

The terminology in Definition 1.1.1 anticipates the role that quiver mutations play in the forthcoming definition of a cluster algebra; where the vertices of a quiver are labeled by the elements of an extended cluster, so that the frozen vertices correspond to frozen variables, and the mutable vertices to the cluster variables.

Definition 1.1.2. Let Q be a quiver. Let k be a mutable vertex of Q. The *quiver mutation* μ_k transforms Q into a new quiver $Q' = \mu_k(Q)$ via a sequence of three steps:

1. for each oriented two-arrow path $i \to k \to j$, if at least one between i and j is mutable, add a new arrow $i \to j$, otherwise do nothing;
2. reverse the orientation of all arrows incident to k;
3. remove all oriented 2-cycles.

Example 1.1.3. Let Q be the quiver on the left of Figure 1.1, where vertices 1,2,3 are mutable, while vertices 4,5,6 are frozen. We compute $\mu_2(Q)$.

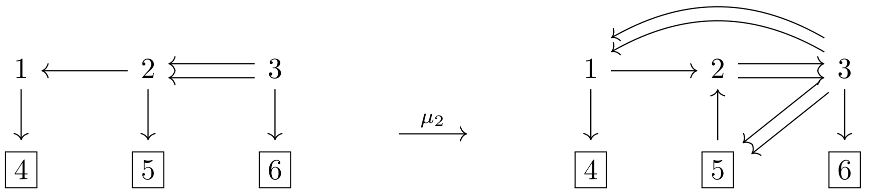


Figure 1.1: The quiver mutation μ_2 of the quiver on the left gives the quiver on the right.

Remark 1.1.4. If a vertex k of a quiver is a sink, i.e. it has no outgoing arrows, or a source, i.e. it has no incoming arrows, then the mutation at k reverses the orientations of all arrows incident to k, and does nothing else. This operation was first considered in the context of quiver representation theory (the reflection functors of Bernstein, Gelfand, Ponomarev [4]).

We now list some simple but important properties of quiver mutation.

Lemma 1.1.5.

1. *Mutation is an involution:* $\mu_k(\mu_k(Q)) = Q$.
2. *Mutation commutes with the simultaneous reversal of orientations of all arrows of a quiver.*
3. *Let k and ℓ be two mutable vertices which have no arrows between them. Then mutations at k and ℓ commute with each other:* $\mu_\ell(\mu_k(Q)) = \mu_k(\mu_\ell(Q))$. *In particular, mutations in different connected components of a quiver do not interact with each other.*

Definition 1.1.6. Two quivers Q and Q' are called *mutation equivalent* if Q can be transformed into Q' by a sequence of mutations. The *mutation equivalence class* of a quiver Q is the set of all quivers which are mutation equivalent to Q.

Example 1.1.7. Consider the quiver Q at the top left of Figure 1.2. All three vertices of Q are mutable. Figure 1.2 shows the mutation equivalence class of Q, up to labeling of vertices.

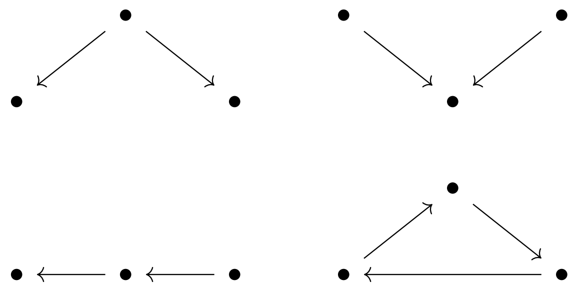


Figure 1.2: The mutation equivalence class of Q.

Example 1.1.8. The quiver Q in Figure 1.3 is known as the *Markov quiver.* Mutating Q at any of its 3 vertices produces a quiver which is equal to Q, so the mutation equivalence class of Q consists of a single element.

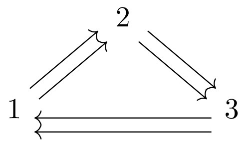


Figure 1.3: The Markov quiver.

Definition 1.1.9. A quiver Q is said to have *finite mutation type* if the mutation equivalence class of Q is finite.

1.1.1 Triangulations of polygons

Let $\mathbf{P}_m$ be a polygon with m vertices. Two diagonals of $\mathbf{P}_m$ are called *crossing* if they are distinct and have a common interior point. A triangulation of $\mathbf{P}_m$ is a maximal collection of pairwise non-crossing diagonals. In this section we associate a quiver to each triangulation of $\mathbf{P}_m$, and explain how flips of diagonals of such triangulations correspond to mutations of the associated quivers.

Definition 1.1.10. Let T be a triangulation of $\mathbf{P}_m$. The quiver $Q(T)$ associated to T is defined as follows. The vertices of $Q(T)$ are labeled by the diagonals of T. If two diagonals belong to the same triangle, we connect the corresponding vertices in $Q(T)$ by an arrow whose orientation is determined by the counterclockwise orientation of the boundary of the triangle. See Figure 1.4. $Q(T)$ has no fronzen vertices.

The following lemma explains how flipping a diagonal γ in a triangulation T, i.e., replacing γ in the quadrilateral formed by the two triangles of T which contain γ, by the other diagonal γ' of the same quadrilateral, as in Figure 1.5, corresponds to a mutation of the associated quiver:

Lemma 1.1.11. *Let T be a triangulation of $\mathbf{P}_m$ as above. Let T' be the triangulation obtained from T by flipping a diagonal γ. Then the quiver $Q(T')$ is obtained from $Q(T)$ by mutating at the vertex labeled by γ.*

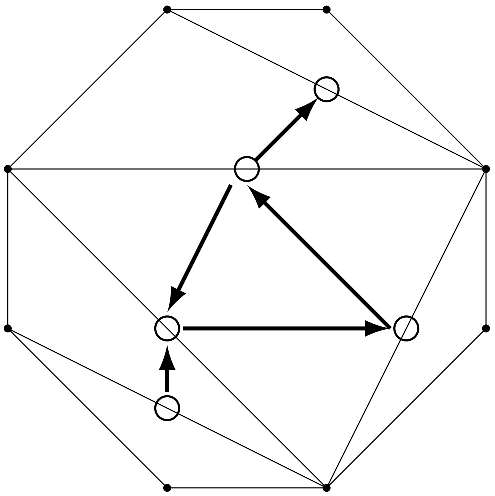


Figure 1.4: The quiver $Q(T)$ associated to a triangulation T of an octagon.

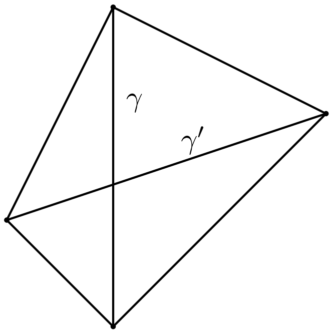


Figure 1.5: Flip of the diagonal γ.

The construction of Definition 1.1.10 can be generalized to triangulations of more general oriented surfaces with boundary and punctures. See [26] for details.

1.1.2 Matrix mutation

In this section, we extend the notion of mutation from quivers to a specific class of matrices. We begin by explaining how matrices can be viewed as generalizations of quivers, and by rephrasing quiver mutations in terms of operations on matrices.

Definition 1.1.12. Let Q be a quiver (cf. Definition 1.1.1) with m vertices, n of them mutable. We label the vertices of Q by the indices $1, \ldots, m$ so that the mutable vertices are labeled $1, \ldots, n$. The *extended exchange matrix* of Q is the $m \times n$ matrix $\tilde{B}(Q) = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} -\ell & \text{if there are } \ell \text{ arrows from vertex } i \text{ to vertex } j \text{ in } Q; \\ \ell & \text{if there are } \ell \text{ arrows from vertex } j \text{ to vertex } i \text{ in } Q; \\ 0 & \text{otherwise.} \end{cases}$$

The *exchange matrix* $B(Q)$ is the $n \times n$ skew-symmetric submatrix of $\tilde{B}(Q)$ occupying the first n rows:

$$B(Q) = (b_{ij})_{i,j=1}^{n} \, .$$

Example 1.1.13. Consider the Markov quiver Q shown in Figure 1.3. Then

$$\tilde{B}(Q) = B(Q) = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix} .$$

Remark 1.1.14. While the definition of $\tilde{B}(Q)$ depends on the choice of labeling of the vertices of Q by the integers $1, \ldots, m$, we often consider extended exchange matrices up to a simultaneous relabeling of rows and columns $1, 2, \ldots, n$, and a relabeling of the rows $n+1, n+2, \ldots, m$.

It can be easily seen that:

Lemma 1.1.15. *Let k be a mutable vertex of a quiver Q. The extended exchange matrix $\tilde{B}(\mu_k(Q)) = (b'_{ij})$ of the mutated quiver $\mu_k(Q)$ is given by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0; \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0; \\ b_{ij} & \text{otherwise.} \end{cases} \tag{1.1.1}$$

Using 1.1.1 we can extend the definition of mutation from skew-symmetric matrices, i.e., mutable parts of quivers, to a more general class of matrices:

Definition 1.1.16. An $n \times n$ matrix $B = (b_{ij})$ with integer entries is called *skew-symmetrizable* if $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$. In other words, a matrix is skew-symmetrizable if it differs from a skew-symmetric matrix by a rescaling of its rows by positive integer scalars. The diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ with diagonal entries $d_1, \ldots, d_n$ is called the *symmetrizer* of B.

An $m \times n$ integer matrix whose top $n \times n$ submatrix is skew-symmetrizable is called an *extended skew-symmetrizable* matrix.

Definition 1.1.17. Let $\tilde{B} = (b_{ij})$ be an $m \times n$ extended skew-symmetrizable integer matrix. For $k \in \{1, \ldots, n\}$, the *matrix mutation μ_k in direction k* transforms $\tilde{B}$ into the $m \times n$ matrix $\mu_k(\tilde{B}) = (b'_{ij})$ whose entries are given by 1.1.1.

By Lemma 1.1.15, matrix mutation generalizes quiver mutation.

Definition 1.1.18. Two matrices $\tilde{B}$ and $\tilde{B}'$ are called *mutation equivalent* if $\tilde{B}$ can be transformed into $\tilde{B}'$ by a sequence of matrix mutations. The *mutation equivalence class* of a matrix $\tilde{B}$ is the set of all matrices which are mutation equivalent to $\tilde{B}$.

Lemma 1.1.19. *Under the conventions of Definitions 1.1.16 and 1.1.17, we have that*

1. *the mutated matrix* $\mu_k(\tilde{B})$ *is again extended skew-symmetrizable, with the same choice of* $d_1, \dots, d_n$*;*
2. $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$*;*
3. $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$*;*
4. $\mu_k(B^T) = (\mu_k(B))^T$*, where* B^T *denotes the transpose of* B*;*
5. *if* $b_{ij} = b_{ji} = 0$*, then* $\mu_i(\mu_j(\tilde{B})) = \mu_j(\mu_i(\tilde{B}))$*.*

Remark 1.1.20. It follows from Lemma 1.1.19 that the symmetrizer is constant in the mutation class of $\tilde{B}$.

1.2 Cluster algebras of geometric type

Let m and n be two positive integers such that $m \geq n$. Let $\mathcal{F}$ be the field of rational functions over $\mathbb{C}$ in m independent variables.

Definition 1.2.1. A *labeled seed* of geometric type in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}},\tilde{B})$ where

- $\tilde{\mathbf{x}} = (x_1, \dots, x_m)$ is an m-tuple of elements of $\mathcal{F}$ forming a *free generating set*, that is, $x_1, \dots, x_m$ are algebraically independent, and $\mathcal{F} = \mathbb{C}(x_1, \dots, x_m)$;
- $\tilde{B} = (b_{ij})$ is an $m \times n$ extended skew-symmetrizable integer matrix.

One uses the following terminology:

- $\tilde{\mathbf{x}}$ is the (labeled) *extended cluster* of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$;
- the n-tuple $\mathbf{x} = (x_1, \dots, x_n)$ is the (labeled) *cluster* of this seed;
- the elements $x_1, \dots, x_n$ are its *cluster variables*;
- the remaining elements $x_{n+1}, \dots, x_m$ of $\tilde{\mathbf{x}}$ are the *frozen variables* (or *coefficient variables*);
- the matrix $\tilde{B}$ is the *extended exchange matrix* of the seed;
- its top $n \times n$ submatrix B is the *exchange matrix.*

Two labeled seeds Σ and Σ' define the same *seed* if Σ' is obtained from Σ by simultaneous relabeling of the elements of $\tilde{\mathbf{x}}$, and the corresponding relabeling of the rows and columns of B.

Definition 1.2.2. Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a labeled seed. Let $k \in \{1, \dots, n\}$. The *seed mutation* μ_k in direction k transforms $(\tilde{\mathbf{x}}, \tilde{B})$ into the new labeled seed $\mu_k(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}', \tilde{B}')$ defined as follows:

- $\tilde{B}' = \mu_k(\tilde{B})$;
- the extended cluster $\tilde{\mathbf{x}}' = \mu_k(\tilde{\mathbf{x}}) = (x'_1, \dots, x'_m)$ is given by $x'_j = x_j$ for $j \neq k$, where $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x_k\, x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}. \tag{1.2.1}$$

If the indexing set for one of the two monomials above is the empty set, then by convention the corresponding product is set equal to 1.

To keep track of the various labeled seeds one can obtain by mutation from a given one, one introduces the following combinatorial setup:

Definition 1.2.3. Let $\mathbb{T}_n$ denote the *n-regular tree* whose edges are labeled by the numbers $1, \dots, n$, so that the n edges incident to each vertex receive different labels. See Figure 1.6.

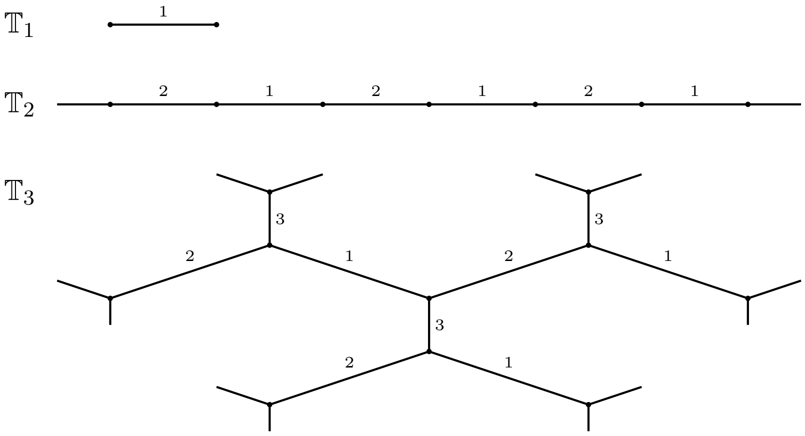


Figure 1.6: The n-regular trees $\mathbb{T}_n$ for $n = 1, 2, 3$.

We write $t \overset{k}{\text{——}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k.

Definition 1.2.4. A *seed pattern* is defined by assigning a labeled seed $(\tilde{\mathbf{x}}(t), \tilde{B}(t))$ to every vertex $t \in \mathbb{T}_n$, so that the seeds assigned to the endpoints of any edge $t \overset{k}{\text{——}} t'$ are obtained from each other by the seed mutation in direction k. A seed pattern is uniquely determined by any one of its seeds. See Figure 1.7.

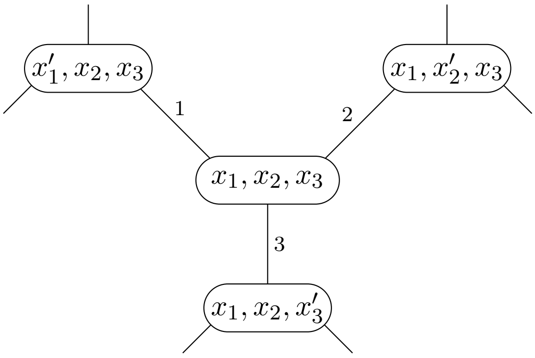


Figure 1.7: Clusters in a seed pattern.

Now we are ready to see the definition of cluster algebra.

Definition 1.2.5. Let $(\tilde{\mathbf{x}}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$ be a seed pattern as above, and let

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}(t)$$

be the set of all cluster variables appearing in its seeds. Let $R = \mathbb{C}[x_{n+1}, \dots, x_m]$ be the polynomial ring generated by the frozen variables.

The *cluster algebra* $\mathcal{A}$ (of geometric type, over R) associated with the given seed pattern is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables, i.e., $\mathcal{A} = R[\mathcal{X}]$. The *rank* of a cluster algebra is the cardinality of each of its clusters (denoted above by n). Cluster algebras without frozen variables are generally said to have *trivial coefficients.*

A common way to describe a cluster algebra is to choose an *initial seed* $(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ in $\mathcal{F}$ and build a seed pattern mutating from it. The corresponding cluster algebra, denoted $\mathcal{A}(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$, is generated over the ground ring R by all cluster variables appearing in the seeds obtained from $(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ by repeated mutations. We note that $\mathcal{A}(\tilde{\mathbf{x}}_\circ, \tilde{B}_\circ)$ does not depend on the choice of the initial seed.

The combinatorics of seed mutations is captured by the exchange graph of a cluster algebra.

Definition 1.2.6 (*Exchange graphs*)**.** The *exchange graph* of a cluster algebra $\mathcal{A}$ is the n-regular (finite or infinite) connected graph whose vertices are the seeds of the seed pattern corresponding to $\mathcal{A}$, and whose edges connect the seeds related by a single mutation.

The exchange graph can be obtained as a quotient of the tree $\mathbb{T}_n$ modulo the equivalence relation on vertices defined by setting $t \sim t'$ whenever Σ_t and $\Sigma_{t'}$ define the same seed.

Definition 1.2.7 (*Finite type*)**.** A cluster algebra $\mathcal{A}$ is of *finite type* if its exchange graph is finite, that is, $\mathcal{A}$ has finitely many distinct seeds.

We will see in Section 1.4 that cluster algebras of finite type correspond to finite root systems, and that the property of a seed to define a cluster algebra of finite type depends only on the exchange matrix B.

1.2.1 Examples of rank 1 and 2

In this section, we present some classical examples of cluster algebras of small rank.

Rank 1

This case is very simple. The tree $\mathbb{T}_1$ has two vertices, so we only have two seeds, and two clusters (x_1) and (x_1'). The extended exchange matrix $\tilde{B}_\circ$ can be any $m \times 1$ matrix whose top entry is 0. The single exchange relation has the form $x_1\, x_1' = M_1 + M_2$ where M_1 and M_2 are monomials in the frozen variables $x_2, \dots, x_m$ which do not share a common factor x_i. The cluster algebra is generated by $x_1, x_1', x_2, \dots, x_m$, subject to this relation, and lies inside the ambient field $\mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_m)$.

Example 1.2.8. The coordinate ring of the subgroup U^+ of unipotent upper-triangular 3×3 matrices

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SL}_3(\mathbb{C})$$

is $\mathbb{C}[a, b, c]$. This ring has the structure of a cluster algebra of rank 1 in which

- the ambient field is $\mathcal{F} = \mathbb{C}(a, b, c)$;

- the frozen variables are b and $P = ac - b$;
- the cluster variables are a and c;
- the single exchange relation is $ac = P + b$.

Rank 2

Any 2×2 skew-symmetrizable matrix looks like this:

$$\pm \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}, \tag{1.2.2}$$

for some integers b and c which are either both positive, or both equal to 0. Applying a mutation μ_1 or μ_2 to a matrix of the form (1.2.2) simply changes its sign.

Example 1.2.9. In the case $b = c = 0$, the two mutations commute, because each μ_k changes the sign of the entries in column k of the extended exchange matrix, and leaves the other column untouched; as the two matrix columns do not affect each other, this case reduces to the rank 1 case. We get four cluster variables x_1, x_2, x_1', x_2', four clusters (x_1, x_2), (x_1', x_2), (x_1, x_2'), and (x_1', x_2'), and two exchange relations of the form $x_1 x_1' = M_1 + M_2$ and $x_2 x_2' = M_3 + M_4$, where M_1, M_2, M_3, M_4 are monomials in the frozen variables.

For the rest of this section, we assume that $b > 0$ and $c > 0$. We denote the cluster variables in our cluster algebra $\mathcal{A}$ of rank 2 by

$$\ldots, z_{-2}, z_{-1}, z_0, z_1, z_2, \ldots,$$

so that the seed pattern looks like this:

$$\cdots \overset{1}{\text{———}} \begin{matrix} (z_1 \quad z_0) \\ \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} \end{matrix} \overset{2}{\text{———}} \begin{matrix} (z_1 \quad z_2) \\ \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} \end{matrix} \overset{1}{\text{———}} \begin{matrix} (z_3 \quad z_2) \\ \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} \end{matrix} \overset{2}{\text{———}} \begin{matrix} (z_3 \quad z_4) \\ \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} \end{matrix} \overset{1}{\text{———}} \cdots$$

where we placed each cluster on top of the corresponding exchange matrix. The extended exchange matrix may have additional rows.

We denote by $\mathcal{A} = \mathcal{A}(b, c)$ a cluster algebra of rank 2 which has exchange matrices $\pm \left[\begin{smallmatrix} 0 & b \\ -c & 0 \end{smallmatrix}\right]$ and trivial coefficients. The exchange relations in $\mathcal{A}(b, c)$ are, in the notation introduced above:

$$z_{k-1}\, z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ is even;} \\ z_k^b + 1 & \text{if } k \text{ is odd.} \end{cases} \tag{1.2.3}$$

Example 1.2.10. The cluster variables in the cluster algebra $\mathcal{A}(1, 1)$ satisfy the recurrence

$$z_{k-1}\, z_{k+1} = z_k + 1. \tag{1.2.4}$$

Expressing everything in terms of the initial cluster (z_1, z_2), we get:

$$z_3 = \frac{z_2+1}{z_1}, \quad z_4 = \frac{z_1+z_2+1}{z_1z_2}, \quad z_5 = \frac{z_1+1}{z_2}, \quad z_6 = z_1, \quad z_7 = z_2, \ \ldots, \tag{1.2.5}$$

so the sequence is periodic of period 5. Thus in this case, we have only 5 distinct cluster variables. In the seed pattern, we will have:

$$\cdots \overset{2}{\text{———}} \begin{array}{c} \begin{pmatrix} z_1 & z_2 \end{pmatrix} \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array} \overset{1}{\text{———}} \begin{array}{c} \begin{pmatrix} z_3 & z_2 \end{pmatrix} \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array} \overset{2}{\text{———}} \cdots \overset{1}{\text{———}} \begin{array}{c} \begin{pmatrix} z_7 & z_6 \end{pmatrix} \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array} \overset{2}{\text{———}} \cdots$$

Note that even though the labeled seeds containing the clusters (z_1, z_2) and (z_7, z_6) are different, the corresponding seeds coincide: just switch z_6 and z_7, and interchange the rows and the columns in the associated exchange matrix. Thus, this exchange pattern has 5 distinct seeds, and the exchange graph of $\mathcal{A}(1,1)$ is a 5-cycle.

Example 1.2.11. Consider the cluster algebra $\mathcal{A}(1,4)$. Setting $z_1 = z_2 = 1$ and applying the recurrence (1.2.3), we see that the cluster variables $z_3, z_4, \ldots$ specialize to the following values:

$$2, 3, 41, 14, 937, 67, 21506, 321, 493697, 1538, 11333521, 7369, 260177282, \ldots$$

This is not a periodic sequence, so the (unspecialized) sequence of cluster variables is also not periodic. The interesting and surprising fact is that all these numbers are integers. If we recursively compute the cluster variables $z_3, z_4, \ldots$ in terms of z_1 and z_2, we see that all of them are Laurent polynomials in z_1 and z_2, i.e., their denominators are monomials:

$$\begin{aligned}
z_3 &= \frac{z_2^4+1}{z_1}, \\
z_4 &= \frac{z_3+1}{z_2} = \frac{z_2^4+z_1+1}{z_1z_2}, \\
z_5 &= \frac{z_4^4+1}{z_3} \\
&= \frac{z_2^{12}+4z_1z_2^8+3z_2^8+6z_1^2z_2^4+8z_1z_2^4+z_1^4+3z_2^4+4z_1^3+6z_1^2+4z_1+1}{z_1^3z_2^4}, \\
z_6 &= \frac{z_5+1}{z_4} = \frac{z_2^8+3z_1z_2^4+2z_2^4+z_1^3+3z_1^2+3z_1+1}{z_1^2z_2^3}, \ \text{etc.}
\end{aligned}$$

It follows that the evaluations of these expressions at $z_1 = z_2 = 1$ are integers. This is totally unexpected: for example, the computation of z_6 involves dividing by $z_4 = \frac{z_2^4+z_1+1}{z_1z_2}$. We will see in the next section (cf. Theorem 1.3.7) that this phenomenon, known as *Laurent phenomenon*, occurs in general.

1.3 Tropical semifields

In this section, we introduce the notion of semifield, and in particular of tropical semifield, which will give us an important alternative way to encode the bottom part of an extended exchange matrix $\tilde{B}$.

Definition 1.3.1. A *semifield* is an abelian group P, written multiplicatively, endowed with an operation of *auxiliary addition* $\oplus$ which is required to be commutative and associative, and satisfy the distributive law with respect to the multiplication in P.

We emphasize that $(P, \oplus)$ does not have to be a group, just a semigroup, so it does not contain an additive identity (or "zero") element (unless P is trivial).

Definition 1.3.2. Let $\mathrm{Trop}(q_1, \dots, q_\ell)$ denote the multiplicative group of Laurent monomials in the variables $q_1, \dots, q_\ell$. We equip $\mathrm{Trop}(q_1, \dots, q_\ell)$ with the binary operation of *tropical addition* $\oplus$ defined by

$$\prod_{i=1}^{\ell} q_i^{a_i} \oplus \prod_{i=1}^{\ell} q_i^{b_i} = \prod_{i=1}^{\ell} q_i^{\min(a_i, b_i)} . \tag{1.3.1}$$

Lemma 1.3.3. *Tropical addition is commutative and associative, and it satisfies the distributive law with respect to the ordinary multiplication:*

$$(p \oplus q)r = pr \oplus qr.$$

Thus $\mathrm{Trop}(q_1, \dots, q_\ell)$ is a semifield, which we call the *tropical semifield* generated by $q_1, \dots, q_\ell$. The formalism of the tropical semifield, and its auxiliary addition, allows us to reformulate the operation of matrix mutation in the following way. Let $\tilde{B}$ be an $m \times n$ extended exchange matrix. As before, $x_{n+1}, \dots, x_m$ are the frozen variables. We encode the bottom $(m-n) \times n$ submatrix of $\tilde{B}$ by the *coefficient tuple* $\mathbf{y} = (y_1, \dots, y_n) \in \mathrm{Trop}(x_{n+1}, \dots, x_m)^n$ defined by

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}}, \quad j \in \{1, \dots, n\} . \tag{1.3.2}$$

Thus the matrix $\tilde{B}$ contains the same information as its top $n \times n$ submatrix B together with the coefficient tuple $\mathbf{y}$.

Proposition 1.3.4. *Let $\tilde{B} = (b_{ij})$ and $\tilde{B}'$ be two extended skew-symmetrizable integer matrices related by a mutation μ_k, and let $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{y}' = (y'_1, \dots, y'_n)$ be the corresponding coefficient tuples (cf.* (1.3.2)*). Then*

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leq 0; \\ y_j (y_k^{-1} \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geq 0. \end{cases} \tag{1.3.3}$$

Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a labeled seed as before. Since the extended exchange matrix $\tilde{B}$ contains the same information as the exchange matrix B together with the coefficient tuple $\mathbf{y}$ defined by (1.3.2), we can identify the seed $(\tilde{\mathbf{x}}, \tilde{B})$ with the triple $(\mathbf{x}, \mathbf{y}, B)$. Abusing notation, we will also refer to such triples as (labeled) seeds:

Definition 1.3.5. Let $\mathcal{F}$ be a field of rational functions over $\mathbb{C}$ in some m variables which include the *frozen variables* $x_{n+1}, \ldots, x_m$. A labeled *seed* (of geometric type) of rank n is a triple $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ consisting of

- a *cluster* $\mathbf{x}$, i.e. an n-tuple of elements of $\mathcal{F}$ such that the *extended cluster* $\mathbf{x} \cup \{x_{n+1}, \ldots, x_m\}$ freely generates $\mathcal{F}$;
- an *exchange matrix* B, i.e. a skew-symmetrizable integer matrix;
- a *coefficient tuple* $\mathbf{y}$, i.e. an n-tuple of Laurent monomials in the tropical semifield $\mathrm{Trop}(x_{n+1}, \ldots, x_m)$.

We can now restate the rules of seed mutation in this language.

Proposition 1.3.6. *Let* $(\mathbf{x}, \mathbf{y}, B)$ *and* $(\mathbf{x}', \mathbf{y}', B')$ *be two labeled seeds related by a mutation* μ_k. *Then* $(\mathbf{x}', \mathbf{y}', B')$ *is obtained from* $(\mathbf{x}, \mathbf{y}, B)$ *as follows:*

- $B' = \mu_k(B)$;
- $\mathbf{y}'$ *is given by* (1.3.3);
- $\mathbf{x}' = (\mathbf{x} - \{x_k\}) \cup \{x'_k\}$, *where* x'_k *is defined by the exchange relation*

$$x_k\, x'_k = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik}>0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik}<0} x_i^{-b_{ik}}\,; \tag{1.3.4}$$

Equation (1.3.4) is used in [32] to define cluster algebras over an arbitrary semifield $\mathbb{P}$. In the definition of labeled seed, the coefficient tuple $\mathbf{y}$ is allowed to be an n-tuple of elements of $\mathbb{P}$, and the mutation rule is given by Proposition 1.3.6. The cluster algebra $\mathcal{A}$ associated with a given seed pattern is defined as the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by all cluster variables, and it is denoted as $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$, where $(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ is any labeled seed in the underlying seed pattern. When $\mathbb{P} = \mathrm{Trop}(x_{n+1}, \ldots, x_m)$, we recover cluster algebras of geometric type.

The examples of Laurentness that we have seen in Section 1.2.1 are actually special cases of the following general phenomenon.

Theorem 1.3.7. *Let* $\mathcal{A}$ *be a cluster algebra over an arbitrary semifield* $\mathbb{P}$. *Then each cluster variable* x_t *can be expressed as a Laurent polynomial with coefficients in* $\mathbb{Z}\mathbb{P}$ *in the elements of any cluster* $\mathbf{x} = (x_1, \ldots, x_n)$, *i.e.* x_t *can be written as a reduced fraction*

$$x_t = \frac{f(x_1, \ldots, x_n)}{\prod_{i=1}^{n} x_i^{d_i}}, \tag{1.3.5}$$

where $f \in \mathbb{Z}\mathbb{P}[x_1, \ldots x_n]$ *and* $d_i \in \mathbb{Z}_{\geq 0}$.

This is known as the *Laurent phenomenon.* The right hand side of equation 1.3.5 is called the *cluster expansion* of x_t in $\mathbf{x}$.

1.3.1 Cluster algebras with principal coefficients

Definition 1.3.8 (*Principal coefficients*)**.** We say that a cluster algebra $\mathcal{A}$ has *principal coefficients in the initial seed* $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ if $\mathbb{P} = \mathrm{Trop}(y_1, \dots, y_n)$ and $\mathbf{y}_{t_0} = (y_1, \dots, y_n)$. In this case, we denote $\mathcal{A} = \mathcal{A}_\bullet(B_{t_0})$.

Remark 1.3.9. Definition 1.3.8 can be rephrased as follows: a cluster algebra $\mathcal{A}$ has principal coefficients in the initial seed $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ if $\mathcal{A}$ is of geometric type, and the extended exchange matrix $\tilde{B}_{t_0}$ is of size $2n \times n$, and such that the top $n \times n$ part is B_{t_0}, while the bottom part is the $n \times n$ identity matrix.

We will see a formula that expresses a cluster variable, in an arbitrary cluster algebra with a given initial exchange matrix B_{t_0}, from the cluster variable in the corresponding cluster algebra with principal coefficients. In order to write this formula, we need to introduce some notation.

Definition 1.3.10 (*The functions $X_{\ell,t}$ and $F_{\ell,t}$*)**.** Let $\mathcal{A} = \mathcal{A}_\bullet(B_{t_0})$ be the cluster algebra with principal coefficients in the initial seed $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ with

$$\mathbf{x}_{t_0} = (x_1, \dots, x_n), \quad \mathbf{y}_{t_0} = (y_1, \dots, y_n), \quad B_{t_0} = (b_{ij})\,. \tag{1.3.6}$$

Thus $\mathbb{P} = \mathrm{Trop}(y_1, \dots, y_n)$, and all coefficients in all exchange relations (1.3.4) are monomials in $y_1, \dots, y_n$. By iterating these exchange relations, we can express every cluster variable $X_{\ell;t}$ as a unique Laurent polynomial in $x_1, \dots, x_n$ whose coefficients are integer polynomial in $y_1, \dots, y_n$ (cf. Theorem 1.3.7).

Let $F_{\ell;t} = \mathbb{Z}[y_1, \dots, y_n]$ denote the polynomial with integer coefficients obtained from $X_{\ell;t}$ by specializing all the x_j to 1:

$$F_{\ell;t}(y_1, \dots, y_n) = X_{\ell;t}(1, \dots, 1; y_1, \dots, y_n). \tag{1.3.7}$$

$F_{\ell;t}$ is called the *F-polynomial* of the cluster variable $X_{\ell;t}$.

For instance, we have $X_{\ell;t_0} = x_\ell$ and $F_{\ell;t_0} = 1$ for all ℓ; and if $t_1 \overset{k}{\text{——}} t_0$, then

$$X_{k;t_1} = \frac{y_k \prod\limits_{b_{ik}>0} x_i^{b_{ik}} + \prod\limits_{b_{ik}<0} x_i^{-b_{ik}}}{x_k}, \quad F_{k;t_1} = y_k + 1\,.$$

Remark 1.3.11. Observe that $F_{\ell;t}$ is also a subtraction-free rational expression over $\mathbb{Q}$ in $y_1, \dots, y_n$, since mutations do not involve subtraction.

If F is a subtraction-free rational expression over $\mathbb{Q}$ in several variables, $\mathbb{P}$ a semifield, and $u_1, \dots, u_\ell$ some elements of $\mathbb{P}$, then we denote by $F|_{\mathbb{P}}(u_1, \dots, u_\ell)$ the evaluation of F at $u_1, \dots, u_\ell$. For example, if $F(u_1, u_2) = u_1^2 - u_1u_2 + u_2^2 = \frac{u_1^3 + u_2^3}{u_1 + u_2}$, and $\mathbb{P} = \mathrm{Trop}(y_1, y_2)$, then $F|_{\mathbb{P}}(y_1, y_2) = \frac{y_1^3 \oplus y_2^3}{y_1 \oplus y_2} = 1$.

Theorem 1.3.12. *Let $\mathcal{A}$ be a cluster algebra over an arbitrary semifield $\mathbb{P}$, with initial seed given by* (1.3.6)*. Then the cluster variables in $\mathcal{A}$ can be expressed as follows:*

$$x_{\ell;t} = \frac{X_{\ell;t}|_{\mathcal{F}}(x_1, \dots, x_n; y_1, \dots, y_n)}{F_{\ell;t}|_{\mathbb{P}}(y_1, \dots, y_n)} . \tag{1.3.8}$$

Proof. See [32], Theorem 3.7. ☐

Formula (1.3.8) exhibits the "separation of additions" phenomenon: the numerator in the right-hand side is totally independent of the auxiliary addition $\oplus$, while the denominator does not involve the ordinary addition in $\mathcal{F}$.

Remark 1.3.13. It follows from Theorem 1.3.12 that knowing the expansion formula of a cluster variable in the case where the cluster algebra has principal coefficients in the initial seed allows one to compute the expansion formula for arbitrary coefficients.

Example 1.3.14. Let $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, and let $\tilde{B} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

Let $X = \dfrac{x_1y_1y_2 + x_3y_1 + x_2}{x_1x_2} \in \mathcal{A}_\bullet(B)$, where $y_1 = \dfrac{x_4x_6}{x_5}$, $y_2 = \dfrac{x_4x_7}{x_6}$, according to (1.3.2). Then the F-polynomial of X is $F = y_1y_2 + y_1 + 1$. Theorem 1.3.12 allows us to compute the corresponding cluster variable $x \in \mathcal{A}((x_1, \dots, x_7), \tilde{B})$:

$$x = \frac{\dfrac{x_1\dfrac{x_4x_6}{x_5}\dfrac{x_4x_7}{x_6} + x_3\dfrac{x_4x_6}{x_5} + x_2}{x_1x_2}}{\dfrac{x_4x_6}{x_5}\dfrac{x_4x_7}{x_6} \oplus \dfrac{x_4x_6}{x_5} \oplus 1} = \frac{x_1\dfrac{x_4x_6}{x_5}\dfrac{x_4x_7}{x_6} + x_3\dfrac{x_4x_6}{x_5} + x_2}{x_1x_2} x_5 = \frac{x_1x_4^2x_7 + x_3x_4x_6 + x_2x_5}{x_1x_2}.$$

The next Proposition and Corollary state the existence of a natural $\mathbb{Z}^n$-grading in a cluster algebra with principal coefficients in the initial seed, with respect to which cluster variables are homogeneous elements.

Proposition 1.3.15. *Every Laurent polynomial $X_{\ell;t}$ is homogeneous with respect to the $\mathbb{Z}^n$-grading in $\mathbb{Z}[x_1^{\pm1}, \dots, x_n^{\pm1}; y_1, \dots, y_n]$ given by*

$$\deg(x_i) = \mathbf{e}_i, \quad \deg(y_j) = -\mathbf{b}_j\ , \tag{1.3.9}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis of $\mathbb{Z}^n$, and $\mathbf{b}_j = \sum_i b_{ij}\, \mathbf{e}_i$ is the j-th column of B_{t_0}.

Proof. See [32], Proposition 6.1. ☐

Proposition 1.3.15 can be restated as follows:

Corollary 1.3.16. *Under the $\mathbb{Z}^n$-grading given by* (1.3.9)*, the cluster algebra $\mathcal{A}_\bullet(B_{t_0})$ is a $\mathbb{Z}^n$-graded subalgebra of $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}; y_1, \dots, y_n]$. All cluster variables in $\mathcal{A}_\bullet(B_{t_0})$ are homogeneous elements.*

Definition 1.3.17. The vectors

$$\mathbf{g}_{\ell;t} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \deg(X_{\ell;t}) \in \mathbb{Z}^n, \tag{1.3.10}$$

given by the multi-degrees of the cluster variables in $\mathcal{A}_\bullet(B_{t_0})$ with respect to the grading defined in 1.3.9, are called **g**-*vectors*, and $\mathbf{g}_{\ell;t}$ is called the **g**-vector of the cluster variable $X_{\ell;t}$.

Example 1.3.18. The **g**-vector of the cluster variable $X = \dfrac{x_1y_1y_2 + x_3y_1 + x_2}{x_1x_2}$ of Example 1.3.14 is $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

Now the "separation of additions" result in Theorem 1.3.12 can be refined as follows.

Let $\hat{y}_1, \dots, \hat{y}_n$ be given by

$$\hat{y}_j = y_j \prod_i x_i^{b_{ij}}. \tag{1.3.11}$$

Corollary 1.3.19. *Cluster variables in an arbitrary cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ can be expressed in terms of the cluster variables of the initial seed* (1.3.6) *by the formula*

$$x_{\ell;t} = \frac{F_{\ell;t}|_{\mathcal{F}}(\hat{y}_1, \dots, \hat{y}_n)}{F_{\ell;t}|_{\mathbb{P}}(y_1, \dots, y_n)} x_1^{g_1} \cdots x_n^{g_n}. \tag{1.3.12}$$

Proof. See [32], Corollary 6.3. ☐

Remark 1.3.20. Knowing the cluster expansion of $x_{\ell;t}$ in the cluster variables of $\mathbf{x}_{t_0}$ is equivalent to knowing the F-polynomial $F_{\ell;t}$ and the **g**-vector $\mathbf{g}_{\ell;t}$ of $X_{\ell;t}$.

Example 1.3.21. The cluster expansion of the cluster variable X and x of Example 1.3.14 can be computed using Corollary 1.3.19 as follows:

$$\begin{aligned} X &= (\hat{y}_1\hat{y}_2 + \hat{y}_1 + 1)x_1^{-1} = (y_1\frac{x_3}{x_2}y_2\frac{x_1}{x_3} + y_1\frac{x_3}{x_2} + 1)x_1^{-1} = \frac{x_1y_1y_2 + x_3y_1 + x_2}{x_1x_2}, \\ x &= \frac{\hat{y}_1\hat{y}_2 + \hat{y}_1 + 1}{\frac{x_4x_6}{x_5}\frac{x_4x_7}{x_6} \oplus \frac{x_4x_6}{x_5} \oplus 1} x_1^{-1} = \big(\frac{x_4x_6}{x_5}\frac{x_3}{x_2}\frac{x_4x_7}{x_6}\frac{x_1}{x_3} + \frac{x_4x_6}{x_5}\frac{x_3}{x_2} + 1\big)x_1^{-1}x_5 \\ &= \frac{x_1x_4^2x_7 + x_3x_4x_6 + x_2x_5}{x_1x_2}. \end{aligned}$$

Let $\mathcal{A} = \mathcal{A}_\bullet(B_{t_0})$ be the cluster algebra with principal coefficients in the initial seed $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$. Let $\tilde{B}^t = (b_{ij}^t)$ be the $2n \times n$ extended exchange matrix at vertex t of the seed pattern corresponding to $\mathcal{A}$. Another family of integer vectors is defined:

Definition 1.3.22. The vectors

$$\mathbf{c}_{\ell;t} = (b^t_{n+1,\ell}, \dots, b^t_{2n,\ell}) \in \mathbb{Z}^n \tag{1.3.13}$$

are called **c**-vectors, and $\mathbf{c}_{\ell;t}$ is called the **c**-vector of the cluster variable $X_{\ell;t}$.

We denote by $C_t^{B_{t_0}}$ (resp. $G_t^{B_{t_0}}$) the integer matrix with columns $\mathbf{c}_{1;t}, \dots, \mathbf{c}_{n;t}$ (resp. with columns $\mathbf{g}_{1;t}, \dots, \mathbf{g}_{n;t}$). These two matrices depend on the exchange matrix B_{t_0} of the initial seed Σ_{t_0}, such that the cluster algebra $\mathcal{A}$ has principal coefficients in Σ_{t_0}. In particular, we have

$$C_{t_0}^{B_{t_0}} = G_{t_0}^{B_{t_0}} = I \quad \text{(the identity matrix)}. \tag{1.3.14}$$

Families of **c**-vectors and **g**-vectors are related to each other by the following result:

Theorem 1.3.23. *For any skew-symmetrizable exchange matrix B_{t_0}, and any $t \in \mathbb{T}_n$, we have*

$$(G_t^{B_{t_0}})^T = (C_t^{-B_{t_0}^T})^{-1}, \tag{1.3.15}$$

where B^T stands for the transpose matrix of B.

Proof. See [43], Theorem 1.2. ☐

Remark 1.3.24. Note that in the case when B_{t_0} is skew-symmetric, (1.3.15) takes the form $(G_t^{B_{t_0}})^T = (C_t^{B_{t_0}})^{-1}$.

1.4 Finite type classification

The subject of this section is the classification, obtained in [31], of cluster algebras of finite type, which turns out to be completely parallel to the famous Cartan-Killing classification of semisimple Lie algebras. We will see that the property of a cluster algebra with an initial seed $(\mathbf{x}, \mathbf{y}, B)$ to be of finite type depends only on the mutation class of the exchange matrix B, and not on the choice of a coefficient tuple $\mathbf{y}$. Such mutation classes are in one-to-one correspondence with Cartan matrices of finite type, and so with finite root systems.

Definition 1.4.1. A square integer matrix $A = (a_{ij})$ is called a *symmetrizable generalized Cartan matrix* if it satisfies the following conditions:

- all diagonal entries of A are equal to 2;
- all entries out of the diagonal of A are non-positive;
- there exists a diagonal matrix D with positive diagonal entries such that the matrix DA is symmetric.

We call A *positive* if DA is positive definite; this is equivalent to the positivity of all principal minors. In particular, any such matrix satisfies

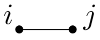
$$\det \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix} = 4 - a_{ij}a_{ji} > 0,$$

or equivalently

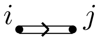
$$a_{ij}a_{ji} \leq 3 \text{ for } i \neq j. \tag{1.4.1}$$

Positive symmetrizable generalized Cartan matrices are also called *Cartan matrices of finite type.*

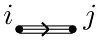
Definition 1.4.2. The *Dynkin diagram* of an $n \times n$ Cartan matrix A is a graph with vertices $1, \ldots, n$ in which vertices i and $j \neq i$ are joined by one or more edges whenever $a_{ij} \neq 0$, in the following way:



if $a_{ij} = -1$ and $a_{ji} = -1$;

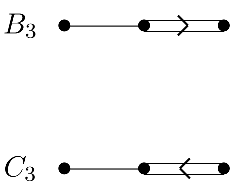


if $a_{ij} = -1$ and $a_{ji} = -2$;



if $a_{ij} = -1$ and $a_{ji} = -3$;

Example 1.4.3. Two examples of Dynkin diagrams are shown in Figure 1.8. The notation B_3 and C_3 is explained in Figure 1.9.

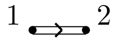


$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Figure 1.8: Dynkin diagrams and Cartan matrices of types B_3 and C_3.

Remark 1.4.4. It is important to stress that the meaning of double and triple arrows in a Dynkin diagram is very different from the meaning of multiple arrows in a quiver. A double arrow



in a Dynkin diagram corresponds to the submatrix

$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

of the associated Cartan matrix. Meanwhile, a double arrow

$$1 \rightrightarrows 2$$

in a quiver corresponds to the submatrix

$$\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

of the associated exchange matrix (cf. Definition 1.1.12).

A Cartan matrix is called *indecomposable* if its Dynkin diagram is connected. We recall the following classical result on the classification of indecomposable Cartan matrices of finite type:

Theorem 1.4.5. *Dynkin diagrams in Figure 1.9 form a complete list of connected Dynkin diagrams corresponding to indecomposable Cartan matrices of finite type.*

Proof. See for example [13], Section 6.4. ☐

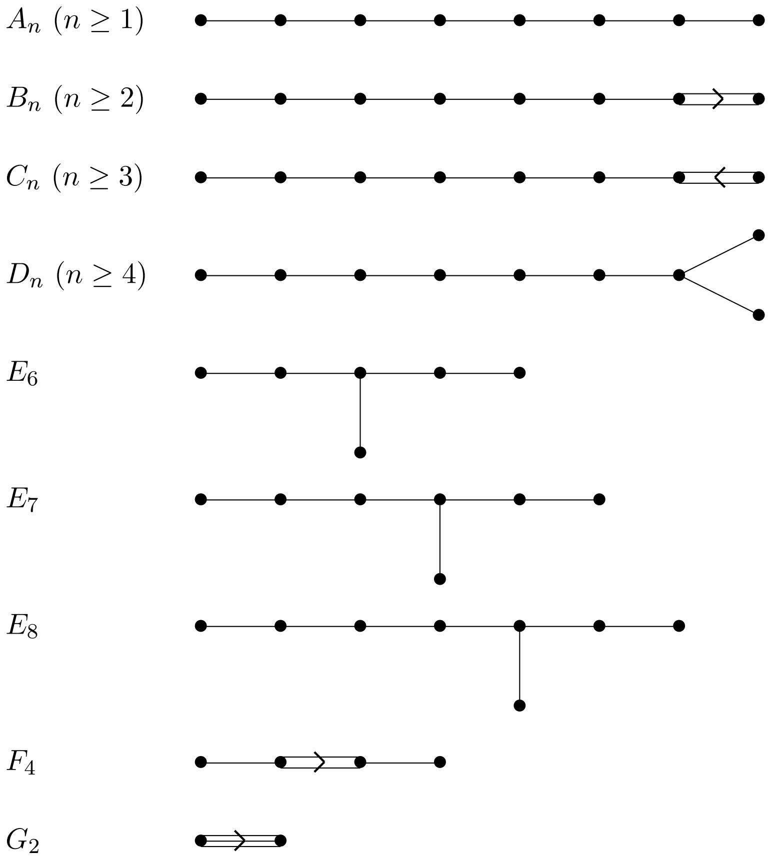


Figure 1.9: Dynkin diagrams of indecomposable Cartan matrices. The subscript n indicates the number of nodes in the diagram.

The relationship between Cartan matrices and skew-symmetrizable matrices, i.e., exchange matrices of cluster algebras, is based on the following definition:

Definition 1.4.6. Let $B = (b_{ij})$ be a skew-symmetrizable integer matrix. Its *Cartan counterpart* is the symmetrizable generalized Cartan matrix

$$A = A(B) = (a_{ij}) \tag{1.4.2}$$

of the same size, defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases} \tag{1.4.3}$$

The main result of this section is the following classification of cluster algebras of finite type:

Theorem 1.4.7. *A cluster algebra is of finite type if and only if one of the seeds of the corresponding seed pattern contains an exchange matrix B whose Cartan counterpart $A(B)$ is a Cartan matrix of finite type.*

Proof. See [31], Theorem 1.4. ☐

Remark 1.4.8. It follows from Theorem 1.4.7 that the property of a cluster algebra to be of finite type depends only on the exchange matrix B of one of its seeds, and not on the coefficient tuple $\mathbf{y}$. In particular, the top $n \times n$ submatrix B of an extended exchange matrix $\tilde{B}$ determines whether a seed pattern of geometric type has finitely many seeds or not. The bottom part of $\tilde{B}$ has no effect on this property.

Definition 1.4.9. Let X_n be a Dynkin diagram on n vertices. A cluster algebra of rank n is said to be *of type* X_n if one of the exchange matrices of its seeds has Cartan counterpart of type X_n. Abusing notation, we will say that a skew-symmetrizable matrix B is *of type* X_n if it defines a cluster algebra of type X_n, i.e., if it is mutation equivalent to a skew-symmetrizable matrix B' such that its Cartan counterpart $A(B')$ is a Cartan matrix of type X_n.

Example 1.4.10. The matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$$

are of types $A_1 \sqcup A_1$, A_2, B_2, and G_2, respectively.

Remark 1.4.11. It follows from Theorem 1.4.7 that each cluster algebra of finite type has a well-defined Cartan-Killing type (e.g., A_n, B_n, etc.).

Let $A = (a_{ij})_{i,j \in I}$ be a Cartan matrix of finite type, and $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ a cluster algebra of finite type related to A as in Theorem 1.4.7. Let Φ be the root system associated with A, with the set of simple roots $\Pi = \{\alpha_i : i \in I\}$. Simple reflections s_i act on simple roots by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. Let $\mathbf{x}_{t_0} = \{x_i : i \in I\}$ be the cluster for the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$. We will use the shorthand $x^\alpha = \prod_{i \in I} x_i^{a_i}$ for any vector $\alpha = \sum_{i \in I} a_i \alpha_i$ in the root lattice.

The following result shows that the cluster variables of $\mathcal{A}$ are naturally parameterized by the set $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi)$ of *almost positive roots*:

Theorem 1.4.12. *There is a unique bijection $\alpha \mapsto x[\alpha]$ between the almost positive roots in Φ and the cluster variables in $\mathcal{A}$ such that, for any $\alpha \in \Phi_{\geq -1}$, the cluster variable $x[\alpha]$ is expressed in terms of the initial cluster $\mathbf{x}_{t_0} = \{x_i : i \in I\}$ as*

$$x[\alpha] = \frac{P_\alpha(\mathbf{x}_{t_0})}{x^\alpha}, \tag{1.4.4}$$

where P_α is a polynomial over $\mathbb{ZP}$ with non-zero constant term. Under this bijection, $x[-\alpha_i] = x_i$.

Proof. See [31], Theorem 1.9. ☐

Remark 1.4.13. Formula (1.4.4) is an example of the Laurent phenomenon established in Theorem 1.3.7 for arbitrary cluster algebras.

1.4.1 Seed patterns of type A_n

Let Φ be a root system of type A_n. The set of almost positive roots $\Phi_{\geq -1}$ is identified with the set of all diagonals of a regular $(n+3)$-gon $\mathbf{P}_{n+3}$ in the following way: the roots in $-\Pi$ correspond to the diagonals on the "zig zag" shown in Figure 1.10, while each positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ corresponds to the unique diagonal that crosses precisely the diagonals corresponding to $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ (see Figure 1.11).

Clusters are in bijection with triangulations of $\mathbf{P}_{n+3}$, i.e., maximal collections of non-crossing diagonals. Moreover, mutations correspond to flips, so two triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other by replacing a diagonal in a quadrilateral formed by two triangles of the triangulation by the other diagonal of the same quadrilateral, as in Figure 1.12. Furthermore, the exchange matrix of the seed whose cluster corresponds to a triangulation $\bar{T} = \{\tau_1, \ldots, \tau_n\}$ of $\mathbf{P}_{n+3}$ is given by the $n \times n$ matrix $B(\bar{T}) = (b_{ij}(\bar{T}))$ defined by:

$$b_{ij}(\bar{T}) = \begin{cases} 1 & \text{if } \tau_i \text{ and } \tau_j \text{ are two sides of a triangle in } \bar{T}, \\ & \text{with } \tau_i \text{ following } \tau_j \text{ in counterclockwise order;} \\ -1 & \text{if } \tau_i \text{ and } \tau_j \text{ label two sides of a triangle in } \bar{T}, \\ & \text{with } \tau_j \text{ following } \tau_i \text{ in counterclockwise order;} \\ 0 & \text{if } \tau_i \text{ and } \tau_j \text{ do not belong to the same triangle in } \bar{T}. \end{cases} \tag{1.4.5}$$

Remark 1.4.14. The skew-symmetric matrix $B(\bar{T})$ is the exchange matrix of the quiver $Q(\bar{T})$ described in Definition 1.1.10 and Figure 1.4.

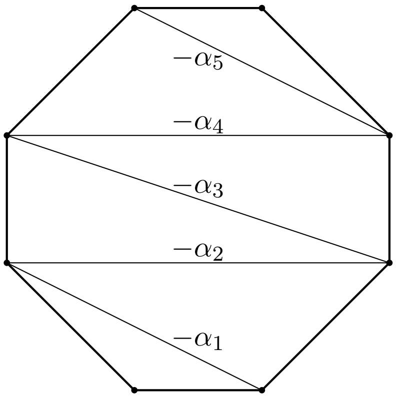


Figure 1.10: The "zig zag" in type A_5

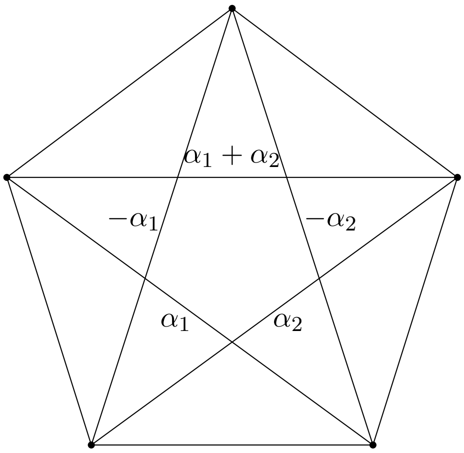


Figure 1.11: Labeling of the diagonals in type A_2

Therefore, we can describe the cluster variables and the exchange relations of cluster algebras of type A_n in concrete combinatorial terms. For a diagonal (a, b) which connects vertices a and b of $\mathbf{P}_{n+3}$, we denote by $x_{(a,b)}$ the cluster variable $x[\alpha]$ associated to the root α corresponding to (a, b), with the convention that $x_{(a,b)} = 1$ if a and b are two consecutive vertices of $\mathbf{P}_{n+3}$.

It follows from the above discussion that exchange relations in a cluster algebra of type A_n have the form

$$x_{(a,b)}x_{(c,d)} = p^+_{ab,cd}\, x_{(a,d)}\, x_{(b,c)} + p^-_{ab,cd}\, x_{(a,c)}\, x_{(b,d)} \ , \tag{1.4.6}$$

where a, d, b, c are any four vertices of $\mathbf{P}_{n+3}$ taken in counter-clockwise order, and $p^\pm_{ab,cd}$ are elements of the coefficient semifield $\mathbb{P}$. See Figure 1.12.

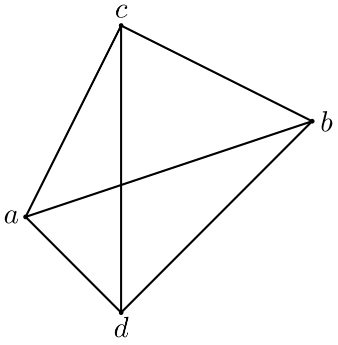


Figure 1.12: Exchanges in type A_n.

Definition 1.4.15. If $\mathcal{A} = \mathcal{A}_\bullet(B(\bar{T}))$ is a cluster algebra of type A_n with principal coefficients in the initial seed whose cluster corresponds to the triangulation $\bar{T}$ of $\mathbf{P}_{n+3}$, we say that $\mathcal{A}$ *has principal coefficients in* $\bar{T}$, and we denote it by $\mathcal{A}^A_\bullet(\bar{T})$.

In this case the coefficients $p^\pm_{ab,cd}$ can be explicitly determined from $\bar{T}$. The following definition and proposition are just a restatement of Definition 17.2 and Proposition 17.3 of [27] in the case of diagonals of a polygon.

Definition 1.4.16. Let $\gamma = (a,b)$ be a diagonal of $\mathbf{P}_{n+3}$. The *elementary lamination* associated to γ is the segment L_γ which begins at a point $a' \in \mathbf{P}$ located near a in the clockwise direction, and ends at a point $b' \in \mathbf{P}$ near b in the clockwise direction. If $\bar{T} = \{\tau_1, \dots, \tau_n\}$ is a triangulation of $\mathbf{P}_{n+3}$, then we let L_i denote L_{τ_i}.

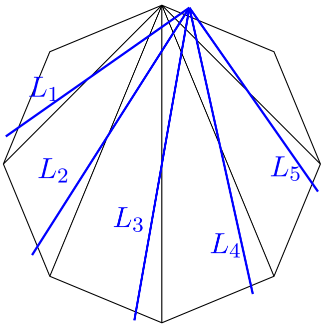


Figure 1.13: A triangulated octagon with the elementary lamination associated to each diagonal of the triangulation (in blue).

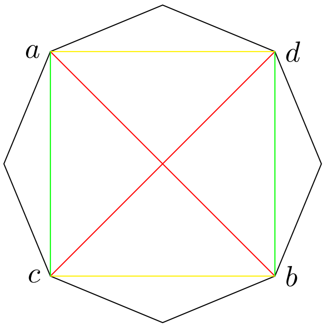
Notation 1.4.17. Let $\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{Z}^n_{\geq 0}$. We denote by $\mathbf{y}^{\mathbf{d}}$ the monomial $y_1^{d_1} \cdots y_n^{d_n}$.

Proposition 1.4.18. *Let $\mathcal{A}^A_\bullet(\bar{T})$ be a cluster algebra of type A_n with principal coefficients in a triangulation $\bar{T} = \{\tau_1, \dots, \tau_n\}$ of $\mathbf{P}_{n+3}$. Let (a,b) and (c,d) be two diagonals which intersect each other. Then*

$$x_{(a,b)} x_{(c,d)} = \mathbf{y}^{\mathbf{d}_{ac,bd}} x_{(a,d)}\ x_{(b,c)} + \mathbf{y}^{\mathbf{d}_{ad,bc}} x_{(a,c)}\ x_{(b,d)}, \tag{1.4.7}$$

where $\mathbf{d}_{ac,bd}$ (resp., $\mathbf{d}_{ad,bc}$) is the vector whose i-th coordinate is 1 if L_i crosses both (a,c) and (b,d) (resp., (a,d) and (b,c)); 0 otherwise.

Proof. See [27], Proposition 17.3. ☐



Example 1.4.19. Let $\mathcal{A}^A_\bullet(\bar{T})$ be the cluster algebra of type A_3 with principal coefficients in the triangulation of the hexagon in Figure 1.14. Using Proposition 1.4.18, we compute the cluster variables $x_{\gamma_1}, x_{\gamma_2} \in \mathcal{A}^A_\bullet(\bar{T})$, where γ_1 and γ_2 are the two diagonals depicted in blue in the picture.

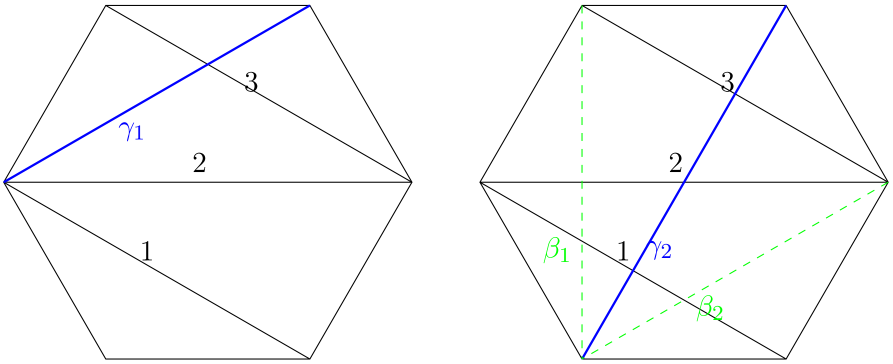


Figure 1.14: Two exchange relations in a triangulated hexagon.

We have

$$x_{\gamma_1}x_3 = y_3 + x_2,$$

and so

$$x_{\gamma_1} = \frac{y_3 + x_2}{x_3}.$$

Therefore its F-polynomial and $\mathbf{g}$-vector are

$$F_{\gamma_1} = y_3 + 1, \qquad \text{and} \qquad \mathbf{g}_{\gamma_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

On the other hand,

$$x_{\gamma_2}x_3 = y_3x_{\beta_1} + x_{\beta_2},$$

where

$$x_{\beta_1}x_2 = y_1y_2x_3 + x_{\beta_2}, \qquad \text{and} \qquad x_{\beta_2}x_1 = y_1 + x_2.$$

Therefore

$$x_{\gamma_2} = \frac{y_1y_2y_3x_1x_3 + y_1y_3 + y_1x_2 + y_3x_2 + x_2^2}{x_1x_2x_3}.$$

Moreover,

$$F_{\gamma_2} = y_1y_2y_3 + y_1y_3 + y_1 + y_3 + 1, \qquad \text{and} \qquad \mathbf{g}_{\gamma_2} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

1.4.2 Seed patterns of types B_n and C_n

Let Φ be a root system of type B_n or C_n. Let $\mathbf{P}_{2n+2}$ be the regular polygon with $2n+2$ vertices. Let θ denote the 180°-rotation of $\mathbf{P}_{2n+2}$. There is a natural action of θ on the diagonals of $\mathbf{P}_{2n+2}$. Each orbit of this action is either a diameter (i.e., a diagonal connecting antipodal vertices) or an unordered pair of centrally symmetric non-diameter diagonals of $\mathbf{P}_{2n+2}$. Almost positive roots in Φ are identified with these orbits. Under this identification, each of the roots $-\alpha_i$ for $i = 1, \dots, n-1$ is represented by a pair of diagonals on the "zig zag" shown in Figure 1.15, while $-\alpha_n$ is identified with the only diameter on the zig zag. On the other hand, each positive root $\beta = \sum_i b_i \alpha_i$ is represented by the unique θ-orbit such that each of its diagonals crosses each of the diagonals corresponding to $-\alpha_i$ at b_i points.

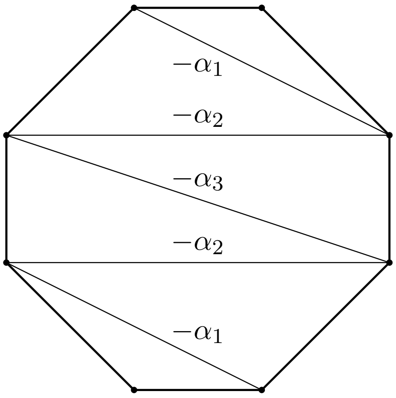


Figure 1.15: The "zig zag" for the types B_3 and C_3

Clusters are in bijection with θ-invariant triangulations of $\mathbf{P}_{2n+2}$, and θ-invariant triangulations are joined by an edge in the exchange graph if and only if they are obtained from each other either by a flip involving two diameters, or by a pair of centrally symmetric flips.

Furthermore, cluster variables correspond to θ-orbits. For a vertex a of $\mathbf{P}_{2n+2}$, let $\bar{a}$ denote the antipodal vertex $\theta(a)$. We denote by $[a, b]$ the θ-orbit of the diagonal (a, b), and by x_{ab} the cluster variable $x[\alpha]$ associated to the root α corresponding to $[a, b]$. Thus, we have $x_{ab} = x_{ba} = x_{\bar{a}\bar{b}}$. Similarly to the type A_n, we adopt the convention that $x_{ab} = 1$ if a and b are consecutive vertices of $\mathbf{P}_{2n+2}$.

It follows from the above discussion that we have a geometric description of the exchange relations also in types B_n and C_n :

Proposition 1.4.20. *The exchange relations in a cluster algebra of type B_n or C_n have the following form, where $r = 1$ for B_n and $r = 2$ for C_n:*

$$x_{ac}x_{bd} = p^{+}_{ac,bd}\, x_{ab}\, x_{cd} + p^{-}_{ac,bd}\, x_{ad}\, x_{bc} \ , \tag{1.4.8}$$

whenever $a, b, c, d, \overline{a}$ are in counter-clockwise order;

$$x_{ac}x_{a\bar{b}} = p^{+}_{ac,a\bar{b}}\, x_{ab}\, x_{a\bar{c}} + p^{-}_{ac,a\bar{b}}\, x^{2/r}_{a\bar{a}}\, x_{bc} \ , \tag{1.4.9}$$

whenever $a, b, c, \overline{a}$ *are in counter-clockwise order;*

$$x_{a\overline{a}} x_{b\overline{b}} = p^{+}_{a\overline{a},b\overline{b}} x^{r}_{ab} + p^{-}_{a\overline{a},b\overline{b}} x^{r}_{a\overline{b}} \ , \tag{1.4.10}$$

whenever $a, b, \overline{a}$ *are in counter-clockwise order. See Figure 1.16.*

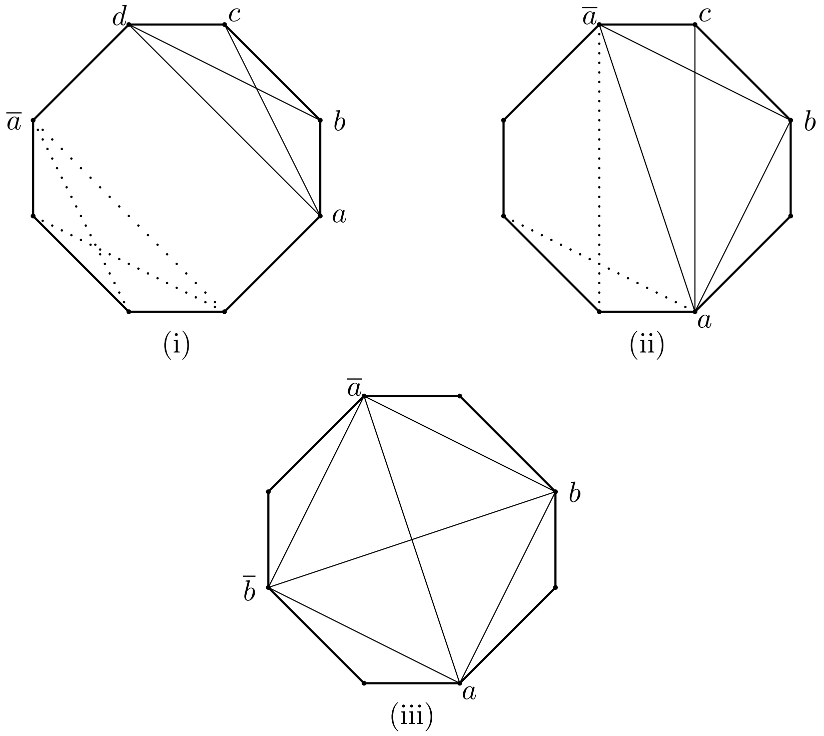


Figure 1.16: Exchanges in types B_n and C_n

Chapter 2

Cluster expansion formulas for cluster algebras of type B and C

In this chapter, mostly based on [17], we introduce cluster algebras of type B_n and C_n with principal coefficients in a given θ-invariant triangulation of $\mathbf{P}_{2n+2}$. Moreover, we define a simple operation on the diagonals of the polygon, the *restriction* (cf. Definition 2.1.1), which allow us to relate cluster variables of type B_n and C_n to cluster variables of type A_n. The main results of the chapter are the cluster expansion formulas for type B (cf. Theorem 2.2.2) and C (cf. Theorem 2.3.7). They are given in a combinatorial way, and the proofs rely on the geometric model for cluster algebras of type A, B and C discussed in Chapter 1.

2.1 Cluster algebras of type B and C with principal coefficients

Let n be a positive integer. Let $\mathbf{P}_{2n+2}$ be the regular polygon with $2n+2$ vertices, and let θ be the rotation of 180° of the polygon. Let $T = \{\tau_1, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$. It follows that T has $n-1$ pairs of centrally symmetric diagonals and exactly one diameter d, say $\tau_n = d$.

Assuming that d is oriented, we define cluster algebras of type B_n and C_n with principal coefficients in T.

Definition 2.1.1. Let $\mathcal{D}$ be a set of diagonals of $\mathbf{P}_{2n+2}$. We define the *restriction of* $\mathcal{D}$, and we denote it by $\mathrm{Res}(\mathcal{D})$, as the set of diagonals of $\mathbf{P}_{n+3}$ obtained from those of $\mathcal{D}$ by identifying all the vertices which lie on the right of d.

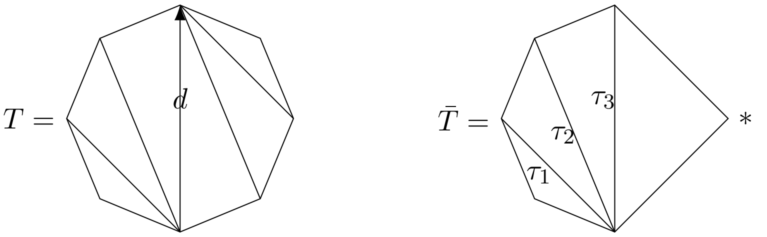
We use the label $*$ for the vertex of $\mathbf{P}_{n+3}$ which is obtained by identifying the vertices of $\mathbf{P}_{2n+2}$ which lie on the right of d.

Definition 2.1.2. Let $v \in \mathbb{Z}_{\geq 0}^{2n-1}$. We define the *restriction of* v, and we denote it by $\mathrm{Res}(v)$, as the vector of the first n coordinates of v.

Let $\bar{T} = \mathrm{Res}(T) = \{\tau_1, \dots, \tau_{n-1}, \tau_n = d\}$ be the triangulation of $\mathbf{P}_{n+3}$ which is obtained from T by identifying all the vertices of $\mathbf{P}_{2n+2}$ which lie on the right of d. Let $B(\bar{T}) = (\bar{b}_{ij})$ be the skew-symmetric $n \times n$ matrix associated with $\bar{T}$ (cf. 1.4.5). So $\bar{b}_{ij} = 1$ if and only if τ_i and τ_j are sides of a triangle of T, and τ_i is counterclockwise from τ_j. See Figure 2.1.

Let $D = \mathrm{diag}(1, \dots, 1, 2)$ be the $n \times n$ diagonal matrix with diagonal entries $(1, \dots, 1, 2)$. Since, by Remark 1.1.20, the symmetrizer is constant in the mutation class of a matrix, then $DB(\bar{T})$ is skew-symmetrizable of type B, and $B(\bar{T})D$ is skew-symmetrizable of type C, according to the convention of Section 1.4.

Example 2.1.3. Figure 2.1 illustrates how to compute the 3×3 skew-symmetric matrix $B(\bar{T})$ associated to the θ-invariant triangulation T of the octagon $\mathbf{P}_8$.



$$B(\bar{T}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 2.1: The matrix $B(\bar{T})$ associated with a θ-invariant triangulation of an octagon.

Let $D = \mathrm{diag}(1, 1, 2)$. Then the Cartan counterpart (cf. Definition 1.4.6) of $DB(\bar{T}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$ is the Cartan matrix of type B_3, while the one of $B(\bar{T})D = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$ is the Cartan matrix of type C_3. See Figure 1.8.

Definition 2.1.4. Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$. Then $\mathcal{A}_\bullet(T)^B := \mathcal{A}_\bullet(DB(\bar{T}))$ (cf. Definition 1.3.8) is the *cluster algebra of type B_n with principal coefficients in T*, and $\mathcal{A}_\bullet(T)^C := \mathcal{A}_\bullet(B(\bar{T})D)$ is the *cluster algebra of type C_n with principal coefficients in T*.

Remark 2.1.5. $\mathcal{A}_\bullet(T)^B$ (resp. $\mathcal{A}_\bullet(T)^C$), up to a change of coefficients, does not depend on T, but it depends only on n, since any two θ-invariant triangulations of $\mathbf{P}_{2n+2}$ can be obtained from each other by a sequence of flips of diameters and pairs of centrally symmetric flips.

2.2 Type B

Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d, and let $\bar{T} = \mathrm{Res}(T) = \{\tau_1, \dots, \tau_n = d\}$. Let $\mathcal{A}_\bullet^B(T)$ be the cluster algebra of type B_n with principal coefficients in T (cf. Definition 2.1.4), and let $\mathcal{A}_\bullet^A(\bar{T})$ be the cluster algebra of type A_n with principal coefficients in $\bar{T}$ (cf. Definition 1.4.15). For a diagonal γ of $\mathbf{P}_{n+3}$, let F_γ and $\mathbf{g}_\gamma$ denote the F-polynomial and the $\mathbf{g}$-vector respectively of the cluster variable $x_\gamma \in \mathcal{A}_\bullet^A(\bar{T})$. They have an explicit description, for example in terms of perfect matchings of the snake graph associated with γ. See Section 3.1 for details.

In this section, we present a formula for the expansion of an arbitrary cluster variable of $\mathcal{A}_\bullet^B(T)$ in terms of cluster variables of $\mathcal{A}_\bullet^A(\bar{T})$.

Definition 2.2.1. Let $[a,b] \not\subset T$ be an orbit of the action of θ on the diagonals of $\mathbf{P}_{2n+2}$. Let $D = \operatorname{diag}(1,\dots,1,2)$ be the $n \times n$ diagonal matrix with diagonal entries $(1,\dots,1,2)$, and let $\mathbf{e}_1,\dots,\mathbf{e}_n$ be the standard basis of $\mathbb{Z}^n$.

- If $\operatorname{Res}([a,b]) = \{\gamma\}$ (as in Figure 2.2),

$$F^B_{ab} := F_\gamma, \tag{2.2.1}$$

$$\mathbf{g}^B_{ab} := \begin{cases} D\mathbf{g}_\gamma & \text{if } \gamma \text{ does not cross } \tau_n = d; \\ D\mathbf{g}_\gamma + \mathbf{e}_n & \text{if } \gamma \text{ crosses } \tau_n = d, \end{cases} \tag{2.2.2}$$

- Otherwise, $\operatorname{Res}([a,b]) = \{\gamma_1,\gamma_2\}$ (as in Figure 2.3),

$$F^B_{ab} := F_{\gamma_1}F_{\gamma_2} - \mathbf{y}^{\mathbf{d}_{\gamma_1,\gamma_2}} F_{(a,\bar{b})}, \tag{2.2.3}$$

$$\mathbf{g}^B_{ab} := D(\mathbf{g}_{\gamma_1} + \mathbf{g}_{\gamma_2} + \mathbf{e}_n), \tag{2.2.4}$$

with the notation of Proposition 1.4.18. The definition is extended to any θ-orbit by letting $F^B_{ab} = 1$ and $\mathbf{g}^B_{ab} = \mathbf{e}_i$ if $[a,b] = \{\tau_i, \tau_{2n-i}\} \in T$, and $F^B_{ab} = 1$ and $\mathbf{g}^B_{ab} = \mathbf{0}$ if (a,b) is a boundary edge of $\mathbf{P}_{2n+2}$.

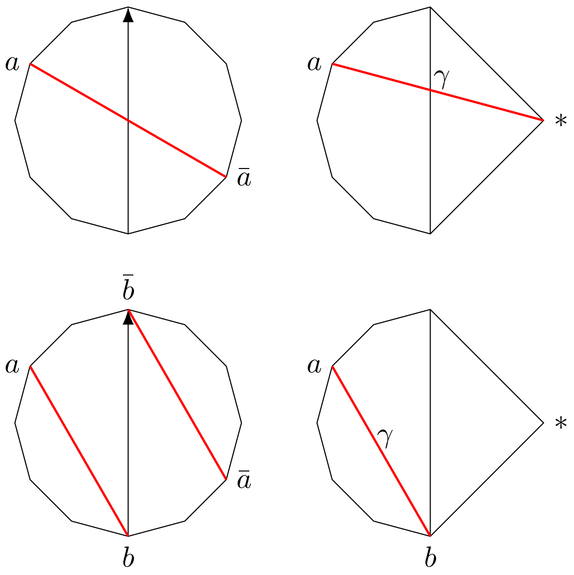


Figure 2.2: On the left, two θ-orbits $[a,\bar{a}]$ and $[a,b]$. On the right, their restrictions.

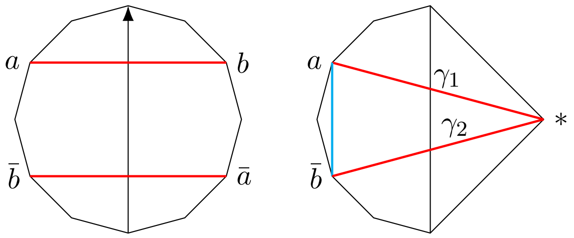


Figure 2.3: On the left, a θ-orbit $[a,b]$. On the right, its restriction in red, and the diagonal $(a,\bar{b})$ in blue.

Theorem 2.2.2. *Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d, and let $\mathcal{A} = \mathcal{A}_\bullet(T)^B$ be the cluster algebra of type B_n with principal coefficients in T. Let $[a,b]$ be an orbit of the action of θ on the diagonals of the polygon, and x_{ab} the cluster variable of $\mathcal{A}$ which corresponds to $[a,b]$. Let F_{ab} and $\mathbf{g}_{ab}$ denote the F-polynomial and the $\mathbf{g}$-vector of x_{ab}, respectively. Then $F_{ab} = F_{ab}^B$ and $\mathbf{g}_{ab} = \mathbf{g}_{ab}^B$.*

Remark 2.2.3. Since, for a diagonal γ of $\mathbf{P}_{n+3}$, F_γ and $\mathbf{g}_\gamma$ have an explicit description in terms of perfect matchings of the snake graph associated with γ (see Section 3.1), Theorem 2.2.2 also allows us to get the expansion of cluster variables of type B_n in terms of the cluster variables of the initial seed.

Example 2.2.4. By Theorem 2.2.2, the F-polynomial of the cluster variable of type B_3 which corresponds to the θ-orbit $[a,b]$ of $\mathbf{P}_8$ in Figure 2.4 is

$$\begin{aligned} F_{ab} &= F_{\gamma_1}F_{\gamma_2} - y_1y_2y_3 = (y_3+1)(y_1y_2y_3 + y_1y_3 + y_1 + y_3 + 1) - y_1y_2y_3 \\ &= y_1y_2y_3^2 + y_1y_3^2 + 2y_1y_3 + y_3^2 + y_1 + 2y_3 + 1, \end{aligned}$$

and the $\mathbf{g}$-vector is

$$\mathbf{g}_{ab} = D(\mathbf{g}_{\gamma_1} + \mathbf{g}_{\gamma_2} + \mathbf{e}_3) = D(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) = D(\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}) = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}.$$

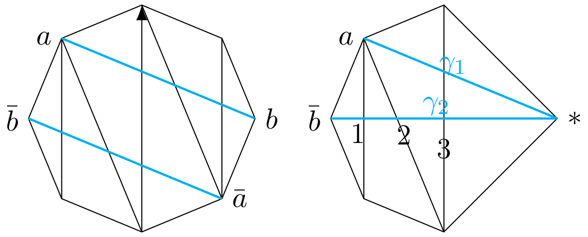


Figure 2.4: A θ-orbit $[a,b]$ in a triangulated octagon and its restriction.

For the computation of $F_{\gamma_1}, F_{\gamma_2}, \mathbf{g}_{\gamma_1}, \mathbf{g}_{\gamma_2}$ we refer to Example 1.4.19.

In order to present the proof of Theorem 2.2.2, we first need some lemmas.

Lemma 2.2.5. *If each diagonal of $[a,b]$ crosses only one diagonal of T, then $F_{ab} = F_{ab}^B$ and $\mathbf{g}_{ab} = \mathbf{g}_{ab}^B$.*

Proof. Let $T = \{\tau_1, \ldots, \tau_{2n-1}\}$. Since each diagonal of $[a,b]$ crosses only one diagonal of T, either $[a,b]$ is a pair of diagonals which do not cross d or $[a,b] = \{a,\bar{a}\}$ is the diagonal which crosses only d. Therefore, $\mathrm{Res}([a,b]) = \{\gamma_j\}$, where γ_j is the diagonal of $\mathbf{P}_{n+3}$ which crosses only τ_j. Let $DB(\bar{T}) = (b_{ij})$ and $B(\bar{T}) = (\bar{b}_{ij})$. We have

$$x_{ab}x_j = y_j \prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b_{ij}<0} x_i^{-b_{ij}}, \tag{2.2.5}$$

and

$$x_{\gamma_j}x_j = y_j \prod_{\bar{b}_{ij}>0} x_i^{\bar{b}_{ij}} + \prod_{\bar{b}_{ij}<0} x_i^{-\bar{b}_{ij}}. \tag{2.2.6}$$

So

$$F_{ab} = y_j + 1 = F_{\gamma_j} = F^B_{ab}. \tag{2.2.7}$$

If $k \neq n$,

$$(\mathbf{g}_{ab})_k = \left(\mathbf{deg}\left(\frac{\prod_{b_{ij}<0} x_i^{-b_{ij}}}{x_j}\right)\right)_k = \left(\mathbf{deg}\left(\frac{\prod_{\bar{b}_{ij}<0} x_i^{-\bar{b}_{ij}}}{x_j}\right)\right)_k = (\mathbf{g}_{\gamma_j})_k = (\mathbf{g}^B_{ab})_k. \tag{2.2.8}$$

If $k = n$ and $j \neq n$,

$$(\mathbf{g}_{ab})_n = \left(\mathbf{deg}\left(\frac{\prod_{b_{ij}<0} x_i^{-b_{ij}}}{x_j}\right)\right)_n = 2\left(\mathbf{deg}\left(\frac{\prod_{\bar{b}_{ij}<0} x_i^{-\bar{b}_{ij}}}{x_j}\right)\right)_n = 2(\mathbf{g}_{\gamma_j})_n = (\mathbf{g}^B_{ab})_n. \tag{2.2.9}$$

Finally, if $k = n$ and $j = n$,

$$(\mathbf{g}_{ab})_n = \left(\mathbf{deg}\left(\frac{1}{x_n}\right)\right)_n = -1 = (\mathbf{g}_{\gamma_n})_n = 2(\mathbf{g}_{\gamma_n})_n + 1 = (\mathbf{g}^B_{ab})_n. \tag{2.2.10}$$

□

Lemma 2.2.6. *Let B be a skew-symmetric $n \times n$ matrix, and let I be the $n \times n$ identity matrix. Let $D = diag(1, \ldots, 1, 2)$ be $n \times n$ diagonal matrix with diagonal entries $(1, \ldots, 1, 2)$. Let $\mu_{i_1} \cdots \mu_{i_k}(\begin{bmatrix} B \\ I \end{bmatrix}) = \begin{bmatrix} B' \\ C \end{bmatrix}$, and let $\mu_{i_1} \cdots \mu_{i_k}(\begin{bmatrix} DB \\ I \end{bmatrix}) = \begin{bmatrix} DB' \\ C' \end{bmatrix}$, for any $1 \leq i_1 < \cdots < i_k \leq n$. Then,*

i) if $i_j \neq n$ for every $j = 1, \ldots, k$, $C = C'$;

ii) if $i_k = n$, the columns $(C')^1, \ldots, (C')^{i_1-1}$ of C' are equal to $DC^1, \ldots, DC^{i_1-1}$.

Proof. *i)* holds since μ_{i_j} does not consider the n-th row for every $j = 1, \ldots, k$.

We prove *ii)* by induction on k. If $k = 1$, $(C')^n = -\mathbf{e}_n$, and for $j \neq n$

$$(C')^j = \begin{cases} \mathbf{e}_j & \text{if } b_{nj} \leq 0 \\ \mathbf{e}_j + 2\mathbf{e}_n & \text{otherwise} \end{cases} = DC^j.$$

Assume $k > 1$. By inductive hypothesis, the columns $(C')^1, \ldots, (C')^{i_1-1}$ of C' are equal to $DC^1, \ldots, DC^{i_1-1}$. Then, we mutate at $i_1 - 1$. If $\mu_{i_1-1}\mu_{i_1} \cdots \mu_{i_k}(\begin{bmatrix} \mathrm{B} \\ I \end{bmatrix}) = \begin{bmatrix} B'' \\ C'' \end{bmatrix}$, and $\mu_{i_1-1}\mu_{i_1} \cdots \mu_{i_k}(\begin{bmatrix} DB \\ I \end{bmatrix}) = \begin{bmatrix} DB'' \\ C''' \end{bmatrix}$, we have that $(C''')^j = D(C'')^j$ for every $j = 1, \ldots, i_1 - 2$. □

Lemma 2.2.7 ([46], Lemma 4.3)**.** *Let $\bar{T} = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of $\mathbf{P}_{n+3}$. Let $\gamma \not\subset \bar{T}$ be a diagonal on which we fixed an orientation such that γ is going from s to t. Let $s =$*

$p_0, p_1, ..., p_d, p_{d+1} = t$ be the intersection points of γ and $\bar{T}$ in order of occurrence on γ, and let $i_1, i_2, \dots, i_d$ be such that p_k lies on τ_{i_k}, for $k = 1, \dots, d$. Then $x_\gamma \in \mu_{i_1} \dots \mu_{i_d}((x_{\tau_1}, \dots, x_{\tau_n}))$.

Proof of Theorem 2.2.2. We prove the theorem by induction on the number k of intersections between each diagonal of $[a, b]$ and $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$.

If $k = 0$, the theorem holds by Definition 2.2.1. If $k = 1$, the theorem holds by Lemma 2.2.5. Assume $k > 1$. Let $\bar{T} = \operatorname{Res}(T) = \{\tau_1, \dots, \tau_n = d\}$, and let $\mathbf{x}_{\bar{T}} = \{x_{\tau_1}, \dots, x_{\tau_n}\} = \{x_1, \dots, x_n\}$. There are three cases to consider.

1) Let $[a, b] = \{(a, b), (\bar{b}, \bar{a})\}$ be such that $\operatorname{Res}([a, b]) = \{(a, b)\}$. Let $a = p_0, p_1, \dots, p_k$, $p_{k+1} = b$ be the intersection points of (a, b) and $\bar{T}$ in order of occurrence on (a, b), and let $i_1, i_2, \dots, i_k$ be such that p_j lies on the diagonal $\tau_{i_j} \in \bar{T}$, for $j = 1, \dots, k$. Let $[c, d] = \{\tau_{i_1}, \tau_{i_{2n-i_1}}\}$.

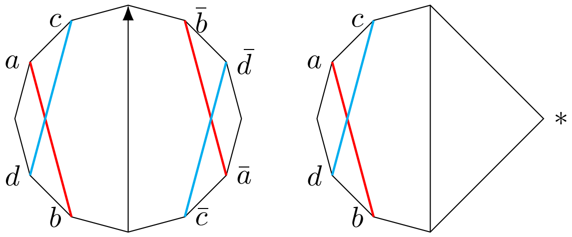


Figure 2.5: On the left, the two θ-orbits $[a, b]$ and $[c, d]$. On the right, their restrictions.

Then, by Lemma 2.2.7, $x_{(a,b)} \in \mu_{i_1} \cdots \mu_{i_k}(\mathbf{x}_{\bar{T}})$. Therefore, the c-vector corresponding to the exchange between $[a, b]$ and $[c, d]$ is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_k}(\begin{bmatrix} DB(\bar{T}) \\ I \end{bmatrix})$. Since $i_j \neq n$ for each j, by Lemma 2.2.6 i), this is equal to the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_k}(\begin{bmatrix} B(\bar{T}) \\ I \end{bmatrix})$, which is given by Proposition 1.4.18. Therefore, we have the following exchange relation

$$x_{i_1} x_{ab} = \mathbf{y}^{\mathbf{d}_{ac,bd}} x_{ad} x_{bc} + \mathbf{y}^{\mathbf{d}_{ad,bc}} x_{ac} x_{bd}. \tag{2.2.11}$$

Since (c, d) is the first diagonal of T that is crossed by (a, b), (a, c) and (a, d) must be either boundary edges or diagonals of $\bar{T}$. It follows from 2.2.11 that

$$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,bd}} F_{bc} + \mathbf{y}^{\mathbf{d}_{ad,bc}} F_{bd}. \tag{2.2.12}$$

Since each diagonal of T which crosses (b, c) (resp. (b, d)) also crosses (a, b), the number of intersections between (b, c) (resp. (b, d)) and T is strictly lower than the number of crossings between (a, b) and T. By inductive hypothesis and Proposition 1.4.18,

$$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,bd}} F_{(b,c)} + \mathbf{y}^{\mathbf{d}_{ad,bc}} F_{(b,d)} = F_{(a,b)} = F^B_{ab}. \tag{2.2.13}$$

It also follows from 2.2.11 that

$$\mathbf{e}_{i_1} + \mathbf{g}_{ab} = \begin{cases} \mathbf{g}_{ad} + \mathbf{g}_{bc} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,bd}} = 1 \\ \mathbf{g}_{ac} + \mathbf{g}_{bd} & \text{otherwise.} \end{cases} \tag{2.2.14}$$

By inductive hypothesis and Proposition 1.4.18,

$$\mathbf{g}_{ab} = \begin{cases} -\mathbf{e}_{i_1} + D\mathbf{g}_{(a,d)} + D\mathbf{g}_{(b,c)} = D(-\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + \mathbf{g}_{(b,c)}) & \text{if } \mathbf{y}^{\mathbf{d}_{ac,bd}} = 1 \\ -\mathbf{e}_{i_1} + D\mathbf{g}_{(a,c)} + D\mathbf{g}_{(b,d)} = D(-\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + \mathbf{g}_{(b,d)}) & \text{otherwise.} \end{cases} = D\mathbf{g}_{(a,b)} = \mathbf{g}^B_{ab}.$$

2) Let $[a,b] = [a,\bar{a}]$ be a diameter. So $\mathrm{Res}([a,\bar{a}]) = \{(a,*)\}$. Let $* = p_0, p_1, \dots, p_s, p_{s+1} = a$ be the intersection points of $(a,*)$ and $\bar{T}$ in order of occurrence on $(*,a)$, $s \leq k$, and let $i_1, i_2, \dots, i_s$ be such that p_j lies on the diagonal $\tau_{i_j} \in \bar{T}$, for $j = 1, \dots, s$. Thus $i_1 = n$. Let $[b,\bar{b}] = \{\tau_n\} = \{d\}$.

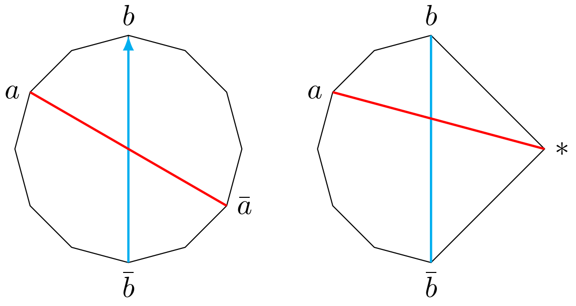


Figure 2.6: On the left, the two ρ-orbits $[a,\bar{a}]$ and $[b,\bar{b}]$. On the right, their restrictions.

Then, by Lemma 2.2.7, $x_{(a,*)} \in \mu_{i_1} \cdots \mu_{i_s}(\mathbf{x}_{\bar{T}})$. Therefore, the c-vector corresponding to the exchange between $[a,\bar{a}]$ and $[b,\bar{b}]$ is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} DB(\bar{T}) \\ I \end{bmatrix})$. Since $i_j \neq n$ for each j, by Lemma 2.2.6 i), this is equal to the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} B(\bar{T}) \\ I \end{bmatrix})$, which is given by Proposition 1.4.18. Therefore, we have the following exchange relation

$$x_n x_{a\bar{a}} = \mathbf{y}^{\mathbf{d}_{ab,\bar{b}*}} x_{a\bar{b}} + \mathbf{y}^{\mathbf{d}_{b*,a\bar{b}}} x_{ab}. \tag{2.2.15}$$

It follows from 2.2.15 that

$$F_{a\bar{a}} = \mathbf{y}^{\mathbf{d}_{ab,\bar{b}*}} F_{a\bar{b}} + \mathbf{y}^{\mathbf{d}_{b*,a\bar{b}}} F_{ab}. \tag{2.2.16}$$

By inductive hypothesis and Proposition 1.4.18,

$$F_{a\bar{a}} = \mathbf{y}^{\mathbf{d}_{ab,\bar{b}*}} F_{(a,\bar{b})} + \mathbf{y}^{\mathbf{d}_{b*,a\bar{b}}} F_{(a,b)} = F_{(a,*)} = F^B_{a\bar{a}}. \tag{2.2.17}$$

It also follows from 2.2.15 that

$$\mathbf{e}_n + \mathbf{g}_{a\bar{a}} = \begin{cases} \mathbf{g}_{a\bar{b}} & \text{if } \mathbf{y}^{\mathbf{d}_{ab,\bar{b}*}} = 1 \\ \mathbf{g}_{ab} & \text{otherwise.} \end{cases} \tag{2.2.18}$$

By inductive hypothesis and Proposition 1.4.18,

$$\mathbf{g}_{a\bar{a}} = \begin{cases} -\mathbf{e}_n + D\mathbf{g}_{(a,\bar{b})} = D(-\mathbf{e}_n + \mathbf{g}_{(a,\bar{b})}) + \mathbf{e}_n & \text{if } \mathbf{y}^{\mathbf{d}_{ab,\bar{b}*}} = 1 \\ -\mathbf{e}_n + D\mathbf{g}_{(a,b)} = D(-\mathbf{e}_n + \mathbf{g}_{(a,b)}) + \mathbf{e}_n & \text{otherwise.} \end{cases} = D\mathbf{g}_{(a,*)} + \mathbf{e}_n = \mathbf{g}^B_{a\bar{a}}.$$

3) Let $[a,b] = \{(a,b),(\bar{a},\bar{b})\}$ be a pair of diagonals which cross d, so $\mathrm{Res}([a,b]) = \{(a,*),(\bar{b},*)\}$. Let $a = p_0, p_1, \dots, p_s, p_{s+1} = *$ be the intersection points of $(a,*)$ and $\bar{T}$ in order of occurrence on $(a,*)$, $s \leq k$, and let $i_1, i_2, \dots, i_s$ be such that p_j lies on the diagonal $\tau_{i_j} \in \bar{T}$, for $j = 1, \dots, s$. So $i_s = n$. Let $[c,d] = \{\tau_{i_1}, \tau_{i_{2n-i_1}}\}$. Assume that $(c,d) = \tau_{i_1}$ intersects $(a,*)$ (otherwise we consider $(\bar{b},*)$ instead of $(a,*)$).

Then, by Lemma 2.2.7, $x_{(a,*)} \in \mu_{i_1} \cdots \mu_{i_s}(\mathbf{x}_{\bar{T}})$. Therefore, the c-vector corresponding to the exchange between $[a,b]$ and $[c,d]$ is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} DB(\bar{T}) \\ I \end{bmatrix})$. Since $i_s = n$, by Lemma 2.2.6 ii), this is equal to DC^{i_1}, where C^{i_1} is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} B(\bar{T}) \\ I \end{bmatrix})$, which is given by Proposition 1.4.18.

Now, we have two cases to consider:

a) c is not an endpoint of τ_n;

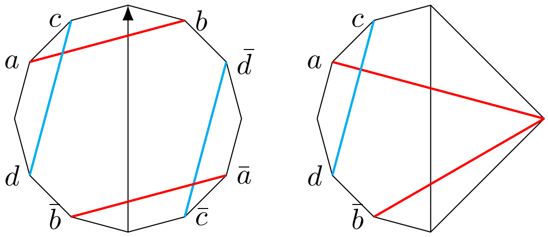


Figure 2.7: On the left, the two θ-orbits $[a,b]$ and $[c,d]$. On the right, their restrictions.

b) c is an endpoint of τ_n.

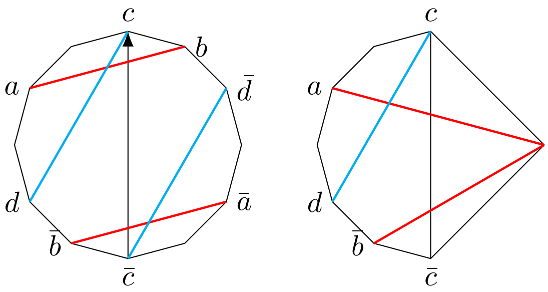


Figure 2.8: On the left, the two ρ-orbits $[a,b]$ and $[c,d]$. On the right, their restrictions.

In case a), we have the following exchange relation:

$$x_{i_1} x_{ab} = \mathbf{y}^{D\mathbf{d}_{ac,d*}} x_{ad} x_{bc} + \mathbf{y}^{D\mathbf{d}_{ad,c*}} x_{ac} x_{bd} = \mathbf{y}^{\mathbf{d}_{ac,d*}} x_{ad} x_{bc} + \mathbf{y}^{\mathbf{d}_{ad,c*}} x_{ac} x_{bd}, \tag{2.2.19}$$

where the last equality is due to the fact that the n-th coordinate of $\mathbf{d}_{ab,c*}$ and $\mathbf{d}_{a*,bc}$ must be 0, since L_n cannot cross both (a,c) and $(d,*)$, nor both (a,d) and $(c,*)$. It follows from 2.2.19 that

$$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,d*}} F_{bc} + \mathbf{y}^{\mathbf{d}_{ad,c*}} F_{bd}, \tag{2.2.20}$$

where we have used that $F_{ad} = F_{ac} = 1$, since $[a,d]$ and $[a,c]$ must be either boundary edges or pairs of diagonals of T.

By inductive hypothesis and Proposition 1.4.18,

$$\begin{aligned} F_{ab} &= \mathbf{y}^{\mathbf{d}_{ac,d*}}(F_{(\bar{b},*)}F_{(c,*)} - \mathbf{y}^{\mathbf{d}_{\bar{b}*,c*}}F_{(c,\bar{b})}) + \mathbf{y}^{\mathbf{d}_{ad,c*}}(F_{(\bar{b},*)}F_{(d,*)} - \mathbf{y}^{\mathbf{d}_{\bar{b}*,d*}}F_{(d,\bar{b})}) \\ &= F_{(\bar{b},*)}(\mathbf{y}^{\mathbf{d}_{ac,d*}}F_{(c,*)} + \mathbf{y}^{\mathbf{d}_{ad,c*}}F_{(d,*)}) - \mathbf{y}^{\mathbf{d}_{a*,\bar{b}*}}(\mathbf{y}^{\mathbf{d}_{ac,d\bar{b}}}F_{(c,\bar{b})} + \mathbf{y}^{\mathbf{d}_{ad,c\bar{b}}}F_{(d,\bar{b})}) \\ &= F_{(\bar{b},*)}F_{(a,*)} - \mathbf{y}^{\mathbf{d}_{a*,\bar{b}*}}F_{(a,\bar{b})} = F^B_{ab}. \end{aligned}$$

It also follows from 2.2.19 that

$$\mathbf{e}_{i_1} + \mathbf{g}_{ab} = \begin{cases} \mathbf{g}_{ad} + \mathbf{g}_{bc} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ \mathbf{g}_{ac} + \mathbf{g}_{bd} & \text{otherwise.} \end{cases} \tag{2.2.21}$$

By inductive hypothesis and Proposition 1.4.18,

$$\begin{aligned} \mathbf{g}_{ab} &= \begin{cases} -\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + D(\mathbf{g}_{(\bar{b},*)} + \mathbf{g}_{(c,*)} + \mathbf{e}_n) & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ -\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + D(\mathbf{g}_{(\bar{b},*)} + \mathbf{g}_{(d,*)} + \mathbf{e}_n) & \text{otherwise;} \end{cases} \\ &= \begin{cases} D(-\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + \mathbf{g}_{(\bar{b},*)} + \mathbf{g}_{(c,*)} + \mathbf{e}_n) & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ D(-\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + \mathbf{g}_{(\bar{b},*)} + \mathbf{g}_{(d,*)} + \mathbf{e}_n) & \text{otherwise.} \end{cases} \\ &= D(\mathbf{g}_{(a,*)} + \mathbf{g}_{(\bar{b},*)} + \mathbf{e}_n) = \mathbf{g}^B_{ab}. \end{aligned}$$

On the other hand, in case b), we have the following exchange relation:

$$x_{i_1}x_{ab} = \mathbf{y}^{D\mathbf{d}_{ac,d*}}x_{ad}x_{bc} + \mathbf{y}^{D\mathbf{d}_{ad,c*}}x_{ac}x_{bd} = \mathbf{y}^{D\mathbf{d}_{ac,d*}}x_{ad}x_{bc} + \mathbf{y}^{\mathbf{d}_{ad,c*}}x_{ac}x_{bd}, \tag{2.2.22}$$

where the last equality is due to the fact that the n-th coordinate of $\mathbf{d}_{ad,c*}$ must be 0, since L_n cannot cross both (a,d) and $(c,*)$.

It follows from 2.2.22 that

$$F_{ab} = \mathbf{y}^{D\mathbf{d}_{ac,d*}} F_{bc} + \mathbf{y}^{\mathbf{d}_{ad,c*}} F_{bd}, \tag{2.2.23}$$

where we have used that $F_{ad} = F_{ac} = 1$, since $[a,d]$ and $[a,c]$ must be either boundary edges or pairs of diagonals of T.

By inductive hypothesis and repeated applications of Proposition 1.4.18,

$$\begin{aligned} F_{ab} &= \mathbf{y}^{D\mathbf{d}_{ac,d*}} F_{(\bar{b},\bar{c})} + \mathbf{y}^{\mathbf{d}_{ad,c*}} (F_{(\bar{b},*)} F_{(d,*)} - \mathbf{y}^{\mathbf{d}_{\bar{b}*,d*}} F_{(d,\bar{b})}) \\ &= F_{(a,*)} F_{(\bar{b},*)} - \mathbf{y}^{\mathbf{d}_{a*,\bar{b}*}} F_{(a,\bar{b})}. \end{aligned}$$

It also follows from 2.2.22 that

$$\mathbf{e}_{i_1} + \mathbf{g}_{ab} = \begin{cases} \mathbf{g}_{ad} + \mathbf{g}_{bc} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ \mathbf{g}_{ac} + \mathbf{g}_{bd} & \text{otherwise.} \end{cases} \tag{2.2.24}$$

By inductive hypothesis and Proposition 1.4.18,

$$\begin{aligned} \mathbf{g}_{ab} &= \begin{cases} -\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + D\mathbf{g}_{(\bar{b},\bar{c})} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ -\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + D(\mathbf{g}_{(\bar{b},*)} + \mathbf{g}_{(d,*)} + \mathbf{e}_n) & \text{otherwise;} \end{cases} \\ &= \begin{cases} D(-\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + \mathbf{g}_{(\bar{b},\bar{c})}) & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ D(-\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + \mathbf{g}_{(\bar{b},*)} + \mathbf{g}_{(d,*)} + \mathbf{e}_n) & \text{otherwise.} \end{cases} \\ &= D(\mathbf{g}_{(a,*)} + \mathbf{g}_{(\bar{b},*)} + \mathbf{e}_n) = \mathbf{g}^B_{ab}. \end{aligned}$$

□

2.3 Type C

Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d, and let $\bar{T} = \operatorname{Res}(T) = \{\tau_1, \dots, \tau_n = d\}$. Let $\mathcal{A}^C_\bullet(T)$ be the cluster algebra of type C_n with principal coefficients in T (cf. Definition 2.1.4), and let $\mathcal{A}^A_\bullet(\bar{T})$ be the cluster algebra of type A_n with principal coefficients in $\bar{T}$ (cf. Definition 1.4.15). For a diagonal γ of $\mathbf{P}_{n+3}$, let F_γ and $\mathbf{g}_\gamma$ denote the F-polynomial and the $\mathbf{g}$-vector respectively of the cluster variable $x_\gamma \in \mathcal{A}^A_\bullet(\bar{T})$. They have an explicit description, for example in terms of perfect matchings of the snake graph associated with γ. See Section 3.1 for details.

In this section, we present a formula for the expansion of an arbitrary cluster variable of $\mathcal{A}^C_\bullet(T)$ in terms of cluster variables of $\mathcal{A}^A_\bullet(\bar{T})$.

Definition 2.3.1. Let $[a, b]$ be a θ-orbit of $\mathbf{P}_{2n+2}$. We define the *rotated restriction of* $[a, b]$, and we denote it by $\widetilde{\operatorname{Res}}([a, b])$, as follows.

- If $[a, b] = [a, \bar{a}]$ is a diameter, so $\operatorname{Res}([a, \bar{a}]) = \{\gamma\}$, then $\widetilde{\operatorname{Res}}([a, \bar{a}]) := \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, where $\tilde{\gamma}_1 = \gamma$ and $\tilde{\gamma}_2$, if it exists, is the diagonal of $\mathbf{P}_{n+3}$ which intersects the same diagonals of T as γ but d. If there is no such diagonal, $\widetilde{\operatorname{Res}}([a, \bar{a}]) := \{\tilde{\gamma}_1\}$. A possible situation is represented in Figure 2.9.

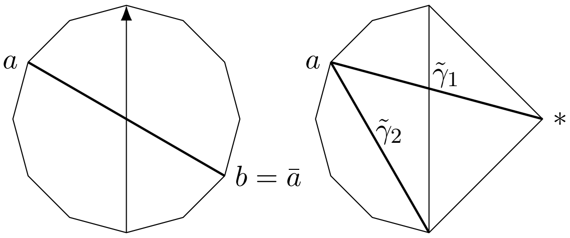


Figure 2.9: On the left, a diameter $[a, \bar{a}]$. On the right, its rotated restriction.

- If $[a, b]$ is a pair of diagonals which do not cross d, then $\widetilde{\mathrm{Res}}([a, b]) := \mathrm{Res}([a, b])$. A possible situation is represented in Figure 2.10.

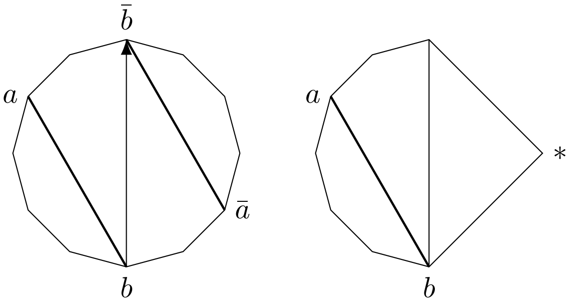


Figure 2.10: On the left, a θ-orbit $[a, b]$. On the right, its rotated restriction.

- If $[a, b]$ is a pair of diagonals which cross d, then $\mathrm{Res}([a, b]) = \{\gamma_1, \gamma_2\}$, where γ_1 and γ_2 are two diagonals of $\mathbf{P}_{n+3}$ that share the right endpoint, and such that γ_2 is obtained from γ_1 by rotating counterclockwise (resp. clockwise) its left endpoint if τ_{n-1} is counterclockwise (resp. clockwise) from τ_n. We define $\widetilde{\mathrm{Res}}([a, b]) := \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, where $\tilde{\gamma}_1 = \gamma_1$ and $\tilde{\gamma}_2$, if it exists, is the diagonal of $\mathbf{P}_{n+3}$ which intersects the same diagonals of T as γ_2 but the diameter. If there is no such diagonal, $\widetilde{\mathrm{Res}}([a, b]) := \{\tilde{\gamma}_1\}$. A possible situation is represented in Figure 2.11.

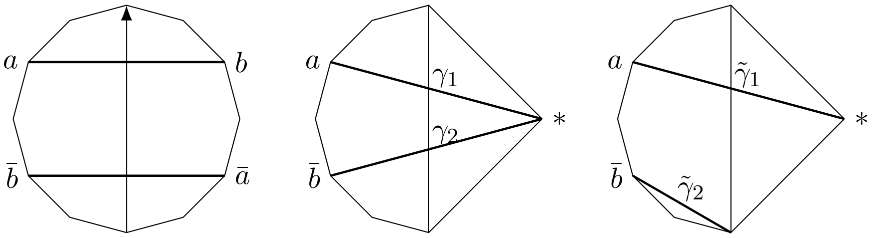


Figure 2.11: From left to right, a θ-orbit $[a, b]$, its restriction and its rotated restriction.

Definition 2.3.2. Let $v \in \mathbb{Z}_{\geq 0}^{2n-1}$. We define the *rotated restriction of* v, and we denote it by $\widetilde{\mathrm{Res}}(v)$, as the vector of the first n coordinates of v, with the n-th one divided by 2.

Definition 2.3.3. Let $[a, b] \not\subset T$ be a θ-orbit of $\mathbf{P}_{2n+2}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{Z}^n$.

- If $\widetilde{\mathrm{Res}}([a, b]) = \{\tilde{\gamma}\}$,

$$F^C_{ab} := F_{\tilde{\gamma}}, \tag{2.3.1}$$

$$\mathbf{g}^C_{ab} := \begin{cases} \mathbf{g}_{\tilde\gamma} + \mathbf{e}_i & \text{if } \tau_i \text{ and } \tau_n = d \text{ are two different sides of a triangle of } T, \\ & \tau_i \text{ is clockwise from } \tau_n, \text{ and } \tilde\gamma \text{ crosses } \tau_n = d; \\ \mathbf{g}_{\tilde\gamma} & \text{otherwise.} \end{cases} \tag{2.3.2}$$

- If $(a,b) = (a,\bar a)$ is a diameter, $\widetilde{\mathrm{Res}}([a,\bar a]) = \{\tilde\gamma_1, \tilde\gamma_2\}$, and there are uniquely determined two θ-orbits $[a,\bar c]$ and $[a,\bar b]$, such that $\widetilde{\mathrm{Res}}([a,\bar c]) = \{\tilde\gamma_1\}$ and $\widetilde{\mathrm{Res}}([a,\bar b]) = \{\tilde\gamma_2\}$. A possible situation is represented in Figure 2.12.

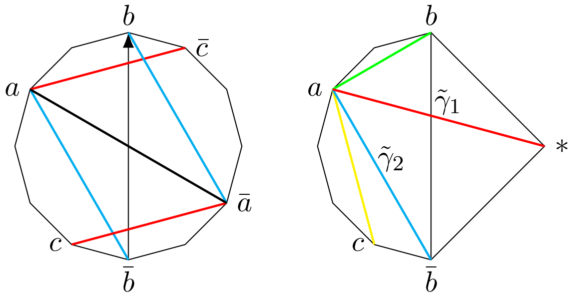


Figure 2.12: On the left, the θ-orbits $[a,\bar a]$, $[a,\bar c]$, $[a,\bar b]$. On the right, their rotated restrictions, and the diagonals (a,b) and (a,c).

Then

$$F^C_{a\bar a} := F_{\tilde\gamma_1} F_{\tilde\gamma_2} - \mathbf{y}^{\widetilde{\mathrm{Res}}(\mathbf{d}_{a*,c\bar b} + \mathbf{d}_{a\bar b,b*})} F_{(a,b)} F_{(a,c)}, \tag{2.3.3}$$

$$\mathbf{g}^C_{a\bar a} := \begin{cases} \mathbf{g}_{\tilde\gamma_1} + \mathbf{g}_{\tilde\gamma_2} + \mathbf{e}_i - \mathbf{g}_{(\bar b,c)} & \text{if } \tau_i \text{ and } \tau_n \text{ are two different sides of a triangle of } T, \\ & \text{and } \tau_i \text{ is clockwise from } \tau_n; \\ \mathbf{g}_{\tilde\gamma_1} + \mathbf{g}_{\tilde\gamma_2} & \text{otherwise.} \end{cases} \tag{2.3.4}$$

- If $[a,b]$ is a pair of diagonals which cross d, and $\widetilde{\mathrm{Res}}([a,b]) = \{\tilde\gamma_1, \tilde\gamma_2\}$, there are uniquely determined two θ-orbits $[a,d]$ and $[b,c]$, such that $\widetilde{\mathrm{Res}}([a,d]) = \{\tilde\gamma_1\}$ and $\widetilde{\mathrm{Res}}([b,c]) = \{\tilde\gamma_2\}$. A possible situation is represented in Figure 2.13.

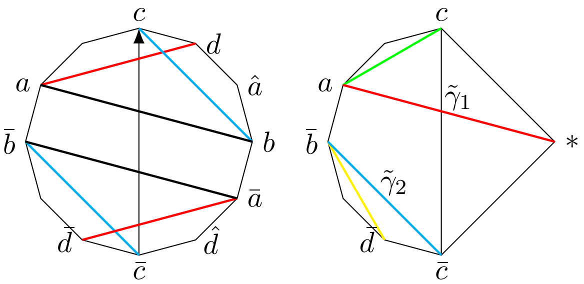


Figure 2.13: On the left, the θ-orbits $[a,b]$, $[a,d]$, $[b,c]$. On the right, their rotated restrictions, and the diagonals (a,c) and $(\bar b,\bar d)$.

Then

$$F^C_{ab} := F_{\tilde\gamma_1} F_{\tilde\gamma_2} - \mathbf{y}^{\widetilde{\mathrm{Res}}(\mathbf{d}_{\bar b*,\bar d\bar c} + \mathbf{d}_{a\bar c,c*})} F_{(a,c)} F_{(\bar b,\bar d)}, \tag{2.3.5}$$

$$\mathbf{g}^C_{ab} := \begin{cases} \mathbf{g}_{\tilde\gamma_1} + \mathbf{g}_{\tilde\gamma_2} + \mathbf{e}_i - \mathbf{g}_{(\bar c,\bar d)} & \text{if } \tau_i \text{ and } \tau_n \text{ are two different sides of a triangle of } T, \\ & \text{and } \tau_i \text{ is clockwise from } \tau_n; \\ \mathbf{g}_{\tilde\gamma_1} + \mathbf{g}_{\tilde\gamma_2} & \text{otherwise.} \end{cases} \tag{2.3.6}$$

The definition is extended to any θ-orbit by letting $F^C_{ab} = 1$ and $\mathbf{g}^C_{ab} = \mathbf{e}_i$ if $[a,b] = \{\tau_i, \tau_{2n-i}\} \in T$, and $F^C_{ab} = 1$ and $\mathbf{g}^C_{ab} = \mathbf{0}$ if (a,b) is a boundary edge of $\mathbf{P}_{2n+2}$.

Remark 2.3.4. $(\bar b, c)$ in 2.3.4 and $(\bar c, \bar d)$) in 2.3.6 are either diagonals of $\bar T$ or boundary edges, since $\widetilde{\mathrm{Res}}([a,\bar c]) = \{\tilde\gamma_1\}$ and $\widetilde{\mathrm{Res}}([a,d]) = \{\tilde\gamma_1\}$ respectively. Remember that by convention $x_{(a,b)} = 1$ if (a,b) is a boundary edge, and so in that case $\mathbf{g}_{(a,b)} = \mathbf{0}$.

Remark 2.3.5. We note that $F^C_{a\bar a}$ (resp. F^C_{ab} for $[a,b]$ pair of diagonals which cross d) are well-defined polynomial in $y_1, \ldots, y_n$, since if L_n crosses $(a,*)$ and $(c,\bar b)$ (resp. $(\bar b,*)$ and $(\bar d,\bar c)$), then it also crosses $(a,\bar b)$ and $(b,*)$ (resp. $(a,\bar c)$ and $(c,*)$).

Remark 2.3.6. We observe that $\widetilde{\mathrm{Res}}(\mathbf{d}_{a*,c\bar b} + \mathbf{d}_{a\bar b,b*})$ is either equal to $\mathbf{d}_{a*,c\bar b}$ or equal to $\mathbf{d}_{a\bar b,b*}$. In fact, if L_i and L_j, $i,j \neq n$, are the elementary lamination of two diagonals τ_i and τ_j of T such that L_i crosses both $(a,*)$ and $(c,\bar b)$, and L_j crosses both $(a,\bar b)$ and $(b,*)$, then L_i crosses L_j, so τ_i crosses τ_j. Moreover, if L_n crosses both $(a,*)$ and $(c,\bar b)$, it also crosses $(a,\bar b)$ and $(b,*)$. Similarly, $\widetilde{\mathrm{Res}}(\mathbf{d}_{\bar b*,\bar d\bar c} + \mathbf{d}_{a\bar c,c*})$ is either equal to $\mathbf{d}_{\bar b*,\bar d\bar c}$ or equal to $\mathbf{d}_{a\bar c,c*}$.

Theorem 2.3.7. *Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d, and let $\mathcal{A} = \mathcal{A}_\bullet(T)^C$ be the cluster algebra of type C_n with principal coefficients in T. Let $[a,b]$ be an orbit of the action of θ on the diagonals of the polygon, and x_{ab} the cluster variable of $\mathcal{A}$ which corresponds to $[a,b]$. Let F_{ab} and $\mathbf{g}_{ab}$ denote the F-polynomial and the $\mathbf{g}$-vector of x_{ab}, respectively. Then $F_{ab} = F^C_{ab}$ and $\mathbf{g}_{ab} = \mathbf{g}^C_{ab}$.*

Remark 2.3.8. Since, for a diagonal γ of $\mathbf{P}_{n+3}$, F_γ and $\mathbf{g}_\gamma$ have an explicit description in terms of perfect matchings of the snake graph associated with γ (see Section 3.1), Theorem 2.3.7 also allows us to get the expansion of cluster variables of type C_n in terms of the cluster variables of the initial seed.

Example 2.3.9. By Theorem 2.3.7, the F-polynomial of the cluster variable of type C_3 which corresponds to the θ-orbit $[a,b]$ of $\mathbf{P}_8$ in Figure 2.14 is

$$F_{ab} = F_{\tilde\gamma_1} F_{\tilde\gamma_2} - y_3 F_{(a,c)} = (y_3 y_2 + y_3 + 1)(y_1 + 1) - y_3(y_2 + 1) = y_1 y_2 y_3 + y_1 y_3 + y_1 + 1,$$

and the $\mathbf{g}$-vector is

$$\mathbf{g}_{ab} = \mathbf{g}_{\tilde\gamma_1} + \mathbf{g}_{\tilde\gamma_2} + \mathbf{e}_2 - \mathbf{e}_2 = \mathbf{g}_{\tilde\gamma_1} + \mathbf{g}_{\tilde\gamma_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

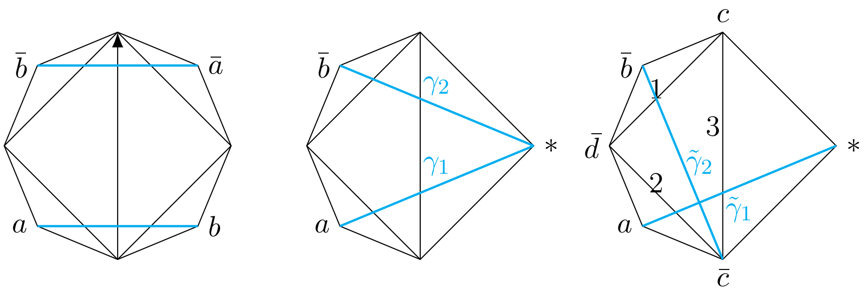


Figure 2.14: A θ-orbit $[a, b]$ in a triangulated octagon, its restriction and its rotated restriction.

In order to present the proof of Theorem 2.3.7, we first need some lemmas.

Lemma 2.3.10. *If each diagonal of* $[a, b]$ *crosses only one diagonal of* T*, then* $F_{ab} = F^C_{ab}$ *and* $\mathbf{g}_{ab} = \mathbf{g}^C_{ab}$.

Proof. With the notation of the proof of Lemma 2.2.5, $\tilde{\text{Res}}([a, b]) = \text{Res}([a, b]) = \{\gamma_j\}$, where γ_j is the diagonal of $\mathbf{P}_{n+3}$ which crosses only τ_j. Let $B(\bar{T})D = (b_{ij})$ and $B(\bar{T}) = (\bar{b}_{ij})$. We have

$$x_{ab}x_j = y_j \prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b_{ij}<0} x_i^{-b_{ij}}, \tag{2.3.7}$$

and

$$x_{\gamma_j}x_j = y_j \prod_{\bar{b}_{ij}>0} x_i^{\bar{b}_{ij}} + \prod_{\bar{b}_{ij}<0} x_i^{-\bar{b}_{ij}}. \tag{2.3.8}$$

So

$$F_{ab} = y_j + 1 = F_{\gamma_j} = F^C_{ab}. \tag{2.3.9}$$

If $j = n$ and k is such that τ_k and τ_n are both sides of a triangle of T, and τ_k is clockwise from τ_n, then $b_{kn} = -2$, while $\bar{b}_{kn} = -1$. So

$$(\mathbf{g}_{ab})_k = \left(\deg\left(\frac{\prod_{b_{in}<0} x_i^{-b_{in}}}{x_n}\right)\right)_k = \left(\deg\left(\frac{\prod_{\bar{b}_{in}<0} x_i^{-\bar{b}_{in}}}{x_n}\right)\right)_k + 1 = (\mathbf{g}_{\gamma_n})_k + 1 = (\mathbf{g}^C_{ab})_k. \tag{2.3.10}$$

Otherwise,

$$(\mathbf{g}_{ab})_k = \left(\deg\left(\frac{\prod_{b_{in}<0} x_i^{-b_{in}}}{x_n}\right)\right)_k = \left(\deg\left(\frac{\prod_{\bar{b}_{in}<0} x_i^{-\bar{b}_{in}}}{x_n}\right)\right)_k = (\mathbf{g}_{\gamma_n})_k = (\mathbf{g}^C_{ab})_k. \tag{2.3.11}$$

□

Lemma 2.3.11. *Let* B *be a skew-symmetric* $n \times n$ *matrix, and let* I *be the* $n \times n$ *identity matrix. Let* $D = diag(1, \ldots, 1, 2)$ *be* $n \times n$ *diagonal matrix with diagonal entries* $(1, \ldots, 1, 2)$.

i) Let $\mu_{i_1} \cdots \mu_{i_k}(\begin{bmatrix} B \\ I \end{bmatrix}) = \begin{bmatrix} B' \\ C \end{bmatrix}$*, and let* $\mu_{i_1} \cdots \mu_{i_k}(\begin{bmatrix} BD \\ I \end{bmatrix}) = \begin{bmatrix} B'D \\ C' \end{bmatrix}$*, for any* $1 \leq i_1 < \cdots < i_k \leq n$*. Then,* $C^k = (C')^k$ *for any* $k \neq n$.

ii) Let $\mu_{i_k}\cdots\mu_{i_1}(\begin{bmatrix} B \\ I \end{bmatrix}) = \begin{bmatrix} B' \\ C \end{bmatrix}$*, and let* $\mu_{i_k}\cdots\mu_{i_1}(\begin{bmatrix} BD \\ I \end{bmatrix}) = \begin{bmatrix} B'D \\ C' \end{bmatrix}$*, for any* $1 \leq i_1 < \cdots < i_k < n$*. Then* $((C')^n)_i = \begin{cases} 2(C^n)_i & \text{if } i \neq n, \\ (C^n)_n & \text{if } i = n. \end{cases}$

Proof. B and BD differ only in the n-th column, and the n-th column of BD is equal to the n-th one of B multiplied by 2. $i)$ follows from the fact the 2 can appear in the bottom part of the matrix only in the n-th column, since we mutate at n only eventually once at the beginning. In $ii)$, we start mutating from the left. So in the bottom part of the n-th column, other than the last coordinate, only the entries corresponding to $i_1, \ldots, i_k$ can be non-zero. For each j, $\mu_{i_j}\cdots\mu_{i_1}(BD) = \mu_{i_j}\cdots\mu_{i_1}(B)D$, since the symmetrizer is constant in the mutation class of B (cf. Remark 1.1.20), i.e. $\mu_{i_j}\cdots\mu_{i_1}(BD)$ is equal to $\mu_{i_j}\cdots\mu_{i_1}(B)$ with the n-th column multiplied by 2. So, for any $i \neq n$, $((C')^n)_i \neq 0$ if and only if $(C^n)_i \neq 0$, and $((C')^n)_i = 2(C^n)_i$. Finally, $((C')^n)_n$ doesn't change after mutations, as well as $(C^n)_n$, so $((C')^n)_n = 1 = (C^n)_n$ ☐

Proof of Theorem 2.3.7. We prove the theorem by induction on the number k of intersections between each diagonal of $[a,b]$ and $T = \{\tau_1, \ldots, \tau_n = d, \ldots, \tau_{2n-1}\}$.

If $k = 0$, the theorem holds by Definition 2.3.3. If $k = 1$, the theorem holds by Lemma 2.3.10. Assume $k > 1$. Let $\bar{T} = \mathrm{Res}(T) = \{\tau_1, \ldots, \tau_n = d\}$, and let $\mathbf{x}_{\bar{T}} = \{x_{\tau_1}, \ldots, x_{\tau_n}\} = \{x_1, \ldots, x_n\}$. There are three cases to consider.

1) Let $[a,b] = \{(a,b), (\bar{b},\bar{a})\}$ be such that $\widetilde{\mathrm{Res}}([a,b]) = \{(a,b)\}$. Let $a = p_0, p_1, \ldots, p_k$, $p_{k+1} = b$ be the intersection points of (a,b) and $\bar{T}$ in order of occurrence on (a,b), and let $i_1, i_2, \ldots, i_k$ be such that p_j lies on the diagonal $\tau_{i_j} \in \bar{T}$, for $j = 1, \ldots, k$. Let $[c,d] = \{\tau_{i_1}, \tau_{i_{2n-i_1}}\}$.

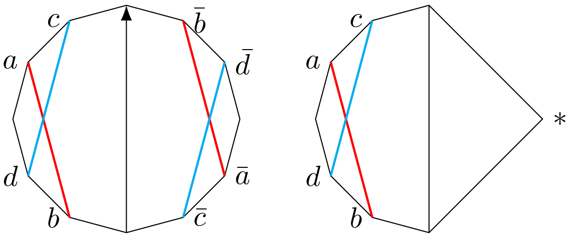


Figure 2.15: On the left, the two θ-orbits $[a,b]$ and $[c,d]$. On the right, their rotated restrictions.

Then, by Lemma 2.2.7, $x_{(a,b)} \in \mu_{i_1}\cdots\mu_{i_k}(\mathbf{x}_{\bar{T}})$. Therefore, the c-vector corresponding to the exchange between $[a,b]$ and $[c,d]$ is the bottom part of the i_1-th column of $\mu_{i_2}\cdots\mu_{i_k}(\begin{bmatrix} B(\bar{T})D \\ I \end{bmatrix})$. Since $i_1 \neq n$, by Lemma 2.3.11 i), this is equal to the bottom part of the i_1-th column of $\mu_{i_2}\cdots\mu_{i_k}(\begin{bmatrix} B(\bar{T}) \\ I \end{bmatrix})$, which is given by Proposition 1.4.18. Therefore, we have the following exchange relation

$$x_{i_1} x_{ab} = \mathbf{y}^{\mathbf{d}_{ac,bd}} x_{ad} x_{bc} + \mathbf{y}^{\mathbf{d}_{ad,bc}} x_{ac} x_{bd}. \tag{2.3.12}$$

Since (a,c) and (a,d) must be either boundary edges or diagonals of $\bar{T}$, it follows from 2.2.11

that

$$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,bd}} F_{bc} + \mathbf{y}^{\mathbf{d}_{ad,bc}} F_{bd}. \tag{2.3.13}$$

By inductive hypothesis and Proposition 1.4.18,

$$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,bd}} F_{(b,c)} + \mathbf{y}^{\mathbf{d}_{ad,bc}} F_{(b,d)} = F_{(a,b)} = F^{C}_{ab}. \tag{2.3.14}$$

It also follows from 2.3.12 that

$$\mathbf{e}_{i_1} + \mathbf{g}_{ab} = \begin{cases} \mathbf{g}_{ad} + \mathbf{g}_{bc} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,bd}} = 1 \\ \mathbf{g}_{ac} + \mathbf{g}_{bd} & \text{otherwise.} \end{cases} \tag{2.3.15}$$

By inductive hypothesis and Proposition 1.4.18,

$$\mathbf{g}_{ab} = \begin{cases} -\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + \mathbf{g}_{(b,c)} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,bd}} = 1 \\ -\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + \mathbf{g}_{(b,d)} & \text{otherwise.} \end{cases} = \mathbf{g}_{(a,b)} = \mathbf{g}^{C}_{ab}.$$

2) Let $[a, \bar{a}]$ be a diameter. So $\widetilde{\mathrm{Res}}([a, \bar{a}]) = \{(a, *), (a, \bar{b})\}$. Let $* = p_0, p_1, \dots, p_s$, $p_{s+1} = a$ be the intersection points of $(a, *)$ and $\bar{T}$ in order of occurrence on $(*, a)$, $s \leq k$, and let $i_1, i_2, \dots, i_s$ be such that p_j lies on the diagonal $\tau_{i_j} \in \bar{T}$, for $j = 1, \dots, s$. Thus $i_1 = n$. Let $[b, \bar{b}] = \{\tau_n\} = \{d\}$. We have two cases to consider:

   i) there is no $i \in \{1, \dots, n\}$ such that τ_i and τ_n are both sides of a triangle of T, and τ_i is clockwise from τ_n;

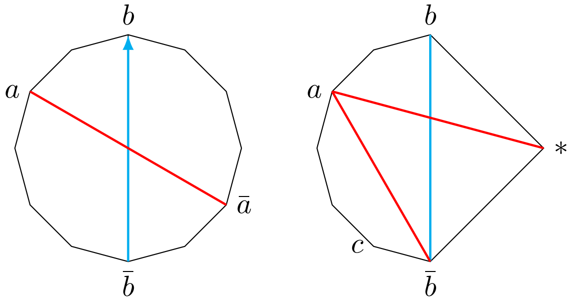


Figure 2.16: On the left, the two ρ-orbits $[a, \bar{a}]$ and $[b, \bar{b}]$. On the right, their rotated restrictions.

   ii) there exists $i \in \{1, \dots, n\}$ such that τ_i is clockwise from τ_n.

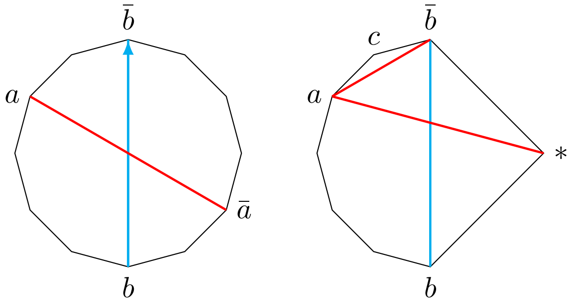


Figure 2.17: On the left, the two ρ-orbits $[a, \bar{a}]$ and $[b, \bar{b}]$. On the right, their rotated restrictions.

First we prove i). By Lemma 2.2.7, $x_{(a,*)} \in \mu_{i_1} \cdots \mu_{i_s}(\mathbf{x}_{\bar{T}})$. Therefore, the c-vector corresponding to the exchange between $[a, \bar{a}]$ and $[b, \bar{b}]$ is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} B(\bar{T})D \\ I \end{bmatrix})$. By Lemma 2.3.11 ii), this is equal to the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} B(\bar{T}) \\ I \end{bmatrix})$, which is given by Proposition 1.4.18, with all coordinates multiplied by two except the n-th one. If $v \in \mathbb{Z}_{\geq 0}$, we indicate by $\bar{v}$ the vector whose coordinates are multiplied by two but the n-th one. Therefore, we have the following exchange relation

$$x_n x_{a\bar{a}} = \mathbf{y}^{\bar{\mathbf{d}}_{ab,\bar{b}*}} x_{a\bar{b}}^2 + \mathbf{y}^{\bar{\mathbf{d}}_{b*,a\bar{b}}} x_{ab}^2. \tag{2.3.16}$$

We note that $\mathbf{y}^{\bar{\mathbf{d}}_{ab,\bar{b}*}} = 1$, since it cannot exist i such that L_i intersects both (a, b) and $(\bar{b}, *)$. It follows from 2.3.16 that

$$F_{a\bar{a}} = F_{a\bar{b}}^2 + \mathbf{y}^{\bar{\mathbf{d}}_{b*,a\bar{b}}} F_{ab}^2. \tag{2.3.17}$$

By inductive hypothesis and repeated applications of Proposition 1.4.18,

$$F_{a\bar{a}} = F_{(a,\bar{b})}^2 + \mathbf{y}^{\bar{\mathbf{d}}_{b*,a\bar{b}}} F_{(a,b)}^2 = F_{(a,*)} F_{(a,\bar{b})} - \mathbf{y}^{\tilde{\mathrm{Res}}(\mathbf{d}_{a*,c\bar{b}} + \mathbf{d}_{a\bar{b},b*})} F_{(a,b)} F_{(a,c)} = F_{a\bar{a}}^C. \tag{2.3.18}$$

It also follows from 2.3.16 that

$$\mathbf{e}_n + \mathbf{g}_{a\bar{a}} = 2\mathbf{g}_{a\bar{b}}. \tag{2.3.19}$$

By inductive hypothesis and Proposition 1.4.18,

$$\mathbf{g}_{a\bar{a}} = -\mathbf{e}_n + 2\mathbf{g}_{(a,\bar{b})} = \mathbf{g}_{(a,*)} + \mathbf{g}_{(a,\bar{b})} = \mathbf{g}_{a\bar{a}}^C.$$

We now prove ii). As in i), we have the exchange relation

$$x_n x_{a\bar{a}} = \mathbf{y}^{\bar{\mathbf{d}}_{ab,\bar{b}*}} x_{a\bar{b}}^2 + \mathbf{y}^{\bar{\mathbf{d}}_{b*,a\bar{b}}} x_{ab}^2. \tag{2.3.20}$$

In this case $\mathbf{y}^{\bar{\mathbf{d}}_{b*,a\bar{b}}} = 1$, since it cannot exist i such that L_i intersects both $(b, *)$ and $(a, \bar{b})$. It follows from 2.3.20 that

$$F_{a\bar{a}} = \mathbf{y}^{\bar{\mathbf{d}}_{ab,\bar{b}*}} F_{a\bar{b}}^2 + F_{ab}^2. \tag{2.3.21}$$

By inductive hypothesis and repeated applications of Proposition 1.4.18,

$$F_{a\bar{a}} = \mathbf{y}^{\bar{\mathbf{d}}_{ab,\bar{b}*}} F_{(a,\bar{b})}^2 + F_{(a,b)}^2 = F_{(a,*)} F_{(a,\bar{b})} - \mathbf{y}^{\tilde{\mathrm{Res}}(\mathbf{d}_{a*,c\bar{b}} + \mathbf{d}_{a\bar{b},b*})} F_{(a,b)} F_{(a,c)} = F_{a\bar{a}}^C. \tag{2.3.22}$$

It also follows from 2.3.20 that

$$\mathbf{e}_n + \mathbf{g}_{a\bar{a}} = 2\mathbf{g}_{ab}. \tag{2.3.23}$$

By inductive hypothesis and Proposition 1.4.18,

$$\mathbf{g}_{a\bar{a}} = -\mathbf{e}_n + 2\mathbf{g}_{(a,b)} = \mathbf{g}_{(a,*)} + \mathbf{g}_{(a,\bar{b})} + \mathbf{e}_i - \mathbf{g}_{(\bar{b},c)} = \mathbf{g}_{a\bar{a}}^C.$$

3) Let $[a,b] = \{(a,b),(\bar{b},\bar{a}\}$ be such that $\widetilde{\mathrm{Res}}([a,b]) = \{(a,*),(\bar{b},\bar{e})\}$.

Let $a = p_0, p_1, \dots, p_s, p_{s+1} = *$ be the intersection points of $(a,*)$ and $\bar{T}$ in order of occurrence on $(a,*)$, and let $i_1, i_2, \dots, i_s$ be such that p_j lies on the diagonal $\tau_{i_j} \in \bar{T}$, for $j = 1, \dots, s$. So $i_s = n$. Let $[c,d] = \{\tau_{i_1}, \tau_{i_{2n-i_1}}\}$. Assume that $(c,d) = \tau_{i_1}$ intersects $(a,*)$ (otherwise we consider $(\bar{b},\bar{e})$ instead of $(a,*)$).

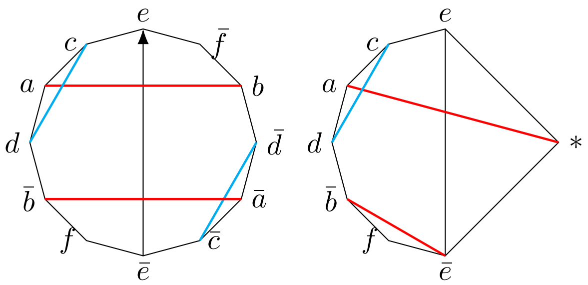


Figure 2.18: On the left, the two θ-orbits $[a,b]$ and $[c,d]$. On the right, their rotated restrictions.

Then, by Lemma 2.2.7, $x_{(a,*)} \in \mu_{i_1} \cdots \mu_{i_s}(\mathbf{x}_{\bar{T}})$. Therefore, the c-vector corresponding to the exchange between $[a,b]$ and $[c,d]$ is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} B(\bar{T})D \\ I \end{bmatrix})$. By Lemma 2.3.11 i), this is equal to C^{i_1}, where C^{i_1} is the bottom part of the i_1-th column of $\mu_{i_2} \cdots \mu_{i_s}(\begin{bmatrix} B(\bar{T}) \\ I \end{bmatrix})$, which is given by Proposition 1.4.18.

We have the following exchange relation:

$$x_{i_1} x_{ab} = \mathbf{y}^{\mathbf{d}_{ac,d*}} x_{ad} x_{bc} + \mathbf{y}^{\mathbf{d}_{ad,c*}} x_{ac} x_{bd}. \tag{2.3.24}$$

It follows from 2.3.24 that

$$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,d*}} F_{bc} + \mathbf{y}^{\mathbf{d}_{ad,c*}} F_{bd}, \tag{2.3.25}$$

where we have used that $F_{ad} = F_{ac} = 1$, since $[a,d]$ and $[a,c]$ must be either boundary edges or pairs of diagonals of T.

By inductive hypothesis and repeated applications of Proposition 1.4.18,

$F_{ab} = \mathbf{y}^{\mathbf{d}_{ac,d*}}(F_{(c,*)}F_{(\bar{b},\bar{e})} - \mathbf{y}^{\widetilde{\mathrm{Res}}(\mathbf{d}_{\bar{b}*,f\bar{e}} + \mathbf{d}_{c\bar{e},e*})} F_{(c,e)} F_{(\bar{b},f)}) + \mathbf{y}^{\mathbf{d}_{ad,c*}}(F_{(d,*)}F_{(\bar{b},\bar{e})}$
$- \mathbf{y}^{\widetilde{\mathrm{Res}}(\mathbf{d}_{\bar{b}*,f\bar{e}} + \mathbf{d}_{d\bar{e},e*})} F_{(d,e)} F_{(\bar{b},f)}) = F_{(a,*)}F_{(\bar{b},\bar{e})} - \mathbf{y}^{\widetilde{\mathrm{Res}}(\mathbf{d}_{\bar{b}*,f\bar{e}} + \mathbf{d}_{a\bar{e},e*})} F_{(a,e)} F_{(\bar{b},f)} = F^C_{ab}.$

It also follows from 2.3.24 that

$$\mathbf{e}_{i_1} + \mathbf{g}_{ab} = \begin{cases} \mathbf{g}_{ad} + \mathbf{g}_{bc} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ \mathbf{g}_{ac} + \mathbf{g}_{bd} & \text{otherwise.} \end{cases} \tag{2.3.26}$$

By inductive hypothesis and Proposition 1.4.18,

$$\begin{aligned}\mathbf{g}_{ab} &= \begin{cases} -\mathbf{e}_{i_1} + \mathbf{g}_{(a,d)} + \mathbf{g}_{(c,*)}\big(+\mathbf{e}_i - \mathbf{g}_{(\bar{e},f)}\big) + \mathbf{g}_{(\bar{b},\bar{e})} & \text{if } \mathbf{y}^{\mathbf{d}_{ac,d*}} = 1 \\ -\mathbf{e}_{i_1} + \mathbf{g}_{(a,c)} + \mathbf{g}_{(d,*)}\big(+\mathbf{e}_i - \mathbf{g}_{(\bar{e},f)}\big) + \mathbf{g}_{(\bar{b},\bar{e})} & \text{otherwise;} \end{cases} \\ &= \mathbf{g}_{(a,*)}\big(+\mathbf{e}_i - \mathbf{g}_{(\bar{e},f)}\big) + \mathbf{g}_{(\bar{b},\bar{e})} = \mathbf{g}^{C}_{ab}.\end{aligned}$$

□

Chapter 3

Snake graphs and perfect matching Laurent polynomials from polygons

In this chapter, mostly based on [16], we associate a modified snake graph $\mathcal{G}_{ab}$ to each θ-orbit $[a, b]$, and prove that the perfect matching Laurent polynomial of $\mathcal{G}_{ab}$ is precisely the cluster variable which corresponds to $[a, b]$. The shape of $\mathcal{G}_{ab}$ has been inspired by the work of Musiker for cluster algebras of type B and C [39], which this chapter extends to every seed.

3.1 Labeled snake graphs from polygons

In this section, we recall from [9–11] the combinatorial construction of the snake graph associated to a diagonal of a polygon, and the main results about these objects that we will need in the next section.

Let $\mathbf{P}_{n+3}$ be the regular polygon with $n+3$ vertices. Let $\bar{T} = \{\tau_1, \dots, \tau_n\}$ be a triangulation of $\mathbf{P}_{n+3}$. Let γ be a diagonal which is not in $\bar{T}$. After choosing an orientation on γ, such that s is its starting point, and t its endpoint, let

$$s = p_0, p_1, p_2, \dots, p_{d+1} = t$$

the points of intersection of γ and $\bar{T}$ in order, with $p_j \in \tau_{i_j}$. Let Δ_{j-1} and Δ_j be the two triangles in $\bar{T}$ on either side of τ_{i_j}. Let G_j be the graph with 4 vertices and 5 edges, having the shape of a square with a diagonal, such that there is a bijection between the edges of G_j and the 5 diagonals in the two triangles Δ_{j-1} and Δ_j, which preserves the signed adjacency of the diagonals up to sign, and such that the diagonal in G_j corresponds to the diagonal τ_{i_j}. Thus G_j, which is called *tile*, is given by the quadrilateral in $\bar{T}$ whose diagonal is τ_{i_j}.

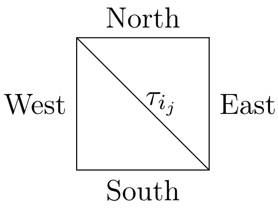


Figure 3.1: A tile G_j.

Given a planar embedding $\tilde{G}_j$ of G_j the relative orientation $\text{Rel}(\tilde{G}_j, \bar{T})$ of $\tilde{G}_j$ with respect to $\bar{T}$ is +1 (resp. -1) if its triangles agree (resp. disagree) in orientation with those of $\bar{T}$.

Diagonals τ_{i_j} and $\tau_{i_{j+1}}$ form two edges of the triangle Δ_j in $\bar{T}$. To the third edge in this triangle is given label $\tau_{[j]}$. Tiles $G_1, \dots, G_d$ in order from 1 to d are glued together in the following way: G_{j+1} is glued to $\tilde{G}_j$, along the edge labeled $\tau_{[j]}$, choosing a planar embedding $\tilde{G}_{j+1}$ for G_{j+1} such that $\text{rel}(\tilde{G}_{j+1}, \bar{T}) \neq \text{rel}(\tilde{G}_j, \bar{T})$, as in Figure 3.2. The resulting graph embedded in the plane is denoted by $\mathcal{G}^{\Delta}_{\gamma}$.

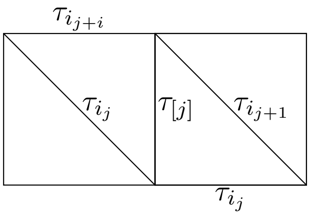


Figure 3.2: Gluing tiles $\tilde{G}_j$ and $\tilde{G}_{j+1}$ along the edge labeled $\tau_{[j]}$.

Definition 3.1.1. The *snake graph* $\mathcal{G}_\gamma$ *associated to* γ is obtained from $\mathcal{G}^{\Delta}_{\gamma}$ by removing the diagonal in each tile.

Example 3.1.2. In the example in Figure 3.3, we compute the snake graph $\mathcal{G}_\gamma$ associated to a diagonal γ in a triangulated polygon. The diagonal γ crosses three diagonals of the triangulation, hence the snake graph $\mathcal{G}_\gamma$ has three tiles.

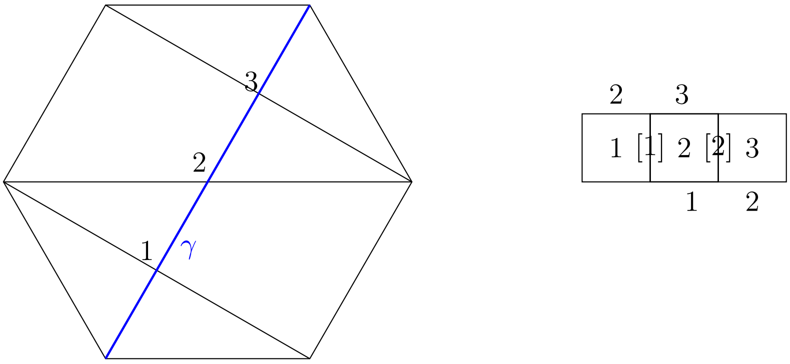


Figure 3.3: On the left, a diagonal γ in a triangulated hexagon; on the right, its snake graph $\mathcal{G}_\gamma$.

Definition 3.1.3. A *perfect matching* of a graph $\mathcal{G}$ is a subset P of the edges of $\mathcal{G}$ such that each vertex of $\mathcal{G}$ is incident to exactly one edge of P. The set of all perfect matchings of $\mathcal{G}$ is denoted by $Match(\mathcal{G})$.

Example 3.1.4. The edges in red form two different perfect matchings of the snake graph with three tiles in Figure 3.4.

Figure 3.4: Two different perfect matchings of a snake graph with three tiles.

Definition 3.1.5. Let γ be a diagonal. $\mathcal{G}_\gamma$ has precisely two perfect matchings which contain only boundary edges. If $\mathrm{Rel}(\tilde{G}_1, \bar{T}) = +1$ (resp. -1), e_1 and e_2 are defined to be the two edges of $\mathcal{G}_\gamma^\Delta$ which lie in the counterclockwise (resp. clockwise) direction from the diagonal of $\tilde{G}_1$. Then $P_- = P_-(\mathcal{G}_\gamma^\Delta)$ is the unique matching which contains only boundary edges and does not contain edges e_1 or e_2. P_- is called the *minimal matching*. $P_+ = P_+(\mathcal{G}_\gamma^\Delta)$, the *maximal matching*, is the other matching with only boundary edges.

Let $P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$ denote the symmetric difference of a perfect matching P of the snake graph $\mathcal{G}_\gamma$ with the minimal matching P_-. By [40, Theorem 5.1], $P_- \ominus P$ is the set of boundary edges of a subgraph $\mathcal{G}_P$ of $\mathcal{G}_\gamma$, and $\mathcal{G}_P$ is a union of tiles

$$\mathcal{G}_P = \bigcup_{i \in I} G_i.$$

Definition 3.1.6. Let P be a perfect matching of $\mathcal{G}_\gamma$. The *height monomial* of P is

$$y(P) = \prod_{i \in I} y_i.$$

Thus $y(P)$ is the product of all y_i for which the tile G_i lies inside $P_- \ominus P$.

Lemma 3.1.7. *Let $\tilde{I} = \{i \mid P_- \cap P$ does not contain any edges of $G_i\}$. Then $\tilde{I} = I$.*

Proof. If e is an edge of G_i such that $e \in P_- \cap P$, then $P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$ does not contain e, so it cannot contain G_i. Vice versa, if $P_- \cap P$ does not contain any edges of G_i, then $P_- \ominus P$ must contain G_i, since P_- and P are perfect matchings of $\mathcal{G}_\gamma$ which do not have any edges in common at the level of G_i, so their union must have all the edges of G_i. ☐

Remark 3.1.8. It follows from Lemma 3.1.7, that $y(P)$ is the product of all y_i such that $P_- \cap P$ does not contain any edges of G_i.

Definition 3.1.9. If γ is a diagonal of $\mathbf{P}_{n+3}$ and $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ is the sequence of diagonals of $\bar{T}$ which γ crosses, the *crossing monomial* of γ with respect to $\bar{T}$ is

$$\mathrm{cross}(\bar{T}, \gamma) = \prod_{j=1}^{d} x_{\tau_{i_j}}.$$

Definition 3.1.10. If $\mathcal{G}_\gamma$ is a snake graph and the edges of a perfect matching P of $\mathcal{G}_\gamma$ are labeled $\tau_{j_1}, \ldots, \tau_{j_r}$, then the *weight* $x(P)$ of P is the monomial $x_{\tau_{j_1}} \ldots x_{\tau_{j_r}}$.

Definition 3.1.11. Let $\bar{T}$ be a triangulation of $\mathbf{P}_{n+3}$. Let γ be a diagonal which is not in $\bar{T}$, and let $\mathcal{G}_\gamma$ be its snake graph. The *perfect matching Laurent polynomial of* $\mathcal{G}_\gamma$ is

$$x_{\mathcal{G}_\gamma} = \frac{1}{\mathrm{cross}(\bar{T}, \gamma)} \sum_{P \in Match(\mathcal{G}_\gamma)} x(P) y(P). \tag{3.1.1}$$

Moreover, the *perfect matching polynomial of* $\mathcal{G}_\gamma$ is the polynomial $F_{\mathcal{G}_\gamma}$ obtained from $x_{\mathcal{G}_\gamma}$ by specializing all the x_{τ_i} to 1, i.e.

$$F_{\mathcal{G}_\gamma} = \sum_{P \in Match(\mathcal{G}_\gamma)} y(P), \tag{3.1.2}$$

and the **g**-*vector of* $\mathcal{G}_\gamma$ is the multi-degree of the Laurent monomial $\dfrac{x(P_-(\mathcal{G}_\gamma^\Delta))}{\text{cross}(\gamma,\bar{T})}$ with respect to the multi-grading given by $\mathbf{deg}(x_{\tau_i}) = \mathbf{e}_i$, i.e.

$$\mathbf{g}_{\mathcal{G}_\gamma} = \mathbf{deg}\Big(\frac{x(P_-(\mathcal{G}_\gamma^\Delta))}{\text{cross}(\gamma,\bar{T})}\Big) = \sum_{\tau_i\in P_-(\mathcal{G}_\gamma^\Delta)} \mathbf{e}_i - \sum_{j=i}^{d} \mathbf{e}_{i_j}, \tag{3.1.3}$$

where $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ is the sequence of diagonals of $\bar{T}$ which γ crosses. The definitions of $x_{\mathcal{G}_\gamma}$, $F_{\mathcal{G}_\gamma}$ and $\mathbf{g}_{\mathcal{G}_\gamma}$ are extended to any diagonal of $\mathbf{P}_{n+3}$ by letting $x_{\mathcal{G}_\gamma} = x_i$, $F_{\mathcal{G}_\gamma} = 1$ and $\mathbf{g}_{\mathcal{G}_\gamma} = \mathbf{e}_i$ if $\gamma = \tau_i \in \bar{T}$, while $x_\gamma = 1$, $F_{\mathcal{G}_\gamma} = 1$ and $\mathbf{g}_{\mathcal{G}_\gamma} = \mathbf{0}$ if γ is a boundary edge of $\mathbf{P}_{n+3}$.

Example 3.1.12. We compute the perfect matching Laurent polynomial of the snake graph $\mathcal{G}_\gamma$ associated to the diagonal γ of Example 3.1.2. The graph $\mathcal{G}_\gamma$ admits exactly 5 perfect matchings (drawn in red), and they form a linear poset in which $P_-(\mathcal{G}_\gamma^\Delta)$ is the unique minimal element and $P_+(\mathcal{G}_\gamma^\Delta)$ is the unique maximal element.

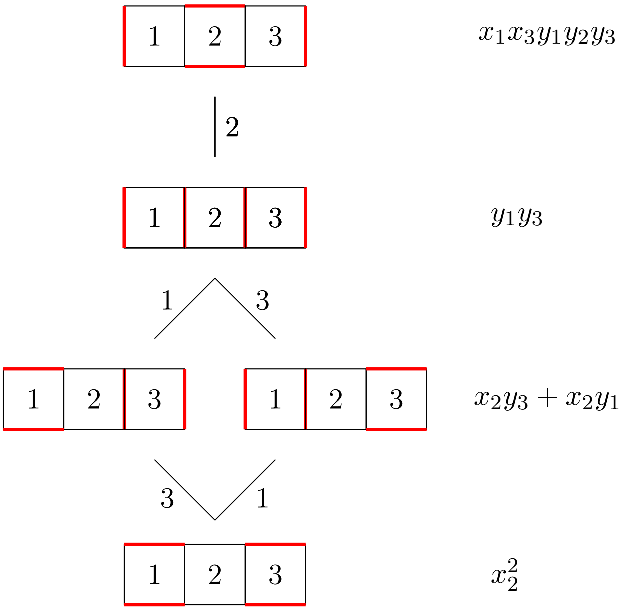


Figure 3.5: The poset of perfect matchings of $\mathcal{G}_\gamma$, and the corresponding monomials.

Therefore, we have

$$x_{\mathcal{G}_\gamma} = \frac{x_1x_3y_1y_2y_3 + y_1y_3 + x_2y_1 + x_2y_3 + x_2^2}{x_1x_2x_3}.$$

Hence, $F_{\mathcal{G}_\gamma} = y_1y_2y_3 + y_1y_3 + y_1 + y_3 + 1$, and $\mathbf{g}_{\mathcal{G}_\gamma} = \begin{pmatrix}0\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$.

The following theorem is a particular case of the much more general result of [40, Theorem 5.1].

Theorem 3.1.13. *Let* $\bar{T} = \{\tau_1, \ldots, \tau_n\}$ *be a triangulation of* $\mathbf{P}_{n+3}$*, and let* $\mathcal{A} = \mathcal{A}_\bullet^A(\bar{T})$ *be the cluster algebra of type* A_n *with principal coefficients in* $\bar{T}$ *(cf. Definition 1.4.15). Let* γ *be a diagonal* *of* $\mathbf{P}_{n+3}$*, and let* x_γ *be the cluster variable of* $\mathcal{A}$ *which corresponds to* γ*. Then* $x_\gamma = x_{\mathcal{G}_\gamma}$*, and* $F_{\mathcal{G}_\gamma}$ *and* $\mathbf{g}_{\mathcal{G}_\gamma}$ *are its* F*-polynomial and its* $\mathbf{g}$*-vector, respectively.*

3.2 Modified snake graphs from θ-orbits

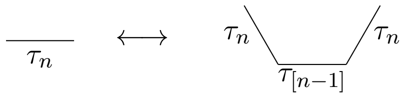
Let n be a positive integer. Let $\mathbf{P}_{2n+2}$ be the regular polygon with $2n+2$ vertices, and let θ be the rotation of 180° of the polygon. Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$, and let $\mathcal{A}^B_\bullet(T)$ and $\mathcal{A}^C_\bullet(T)$ be the cluster algebras with principal coefficients in T of type B_n and C_n, respectively (cf. Definition 2.1.4).

In this section we associate a labeled modified snake graph $\mathcal{G}_{ab}$ to each θ-orbit $[a, b]$, and prove that the perfect matching polynomial $F_{\mathcal{G}_{ab}}$ (resp. the $\mathbf{g}$-vector $\mathbf{g}_{\mathcal{G}_{ab}}$) of $\mathcal{G}_{ab}$ is equal to the F-polynomial (resp. $\mathbf{g}$-vector) of the cluster variable of $\mathcal{A}^B_\bullet(T)$, and $\mathcal{A}^C_\bullet(T)$ respectively, which corresponds to $[a, b]$. The definition of $\mathcal{G}_{ab}$ has been inspired by the work of Musiker [39] for type B and C cluster algebras.

3.2.1 Type B

Definition 3.2.1. Let $\bar{T} = \{\tau_1, \dots, \tau_n\}$ be a triangulation of $\mathbf{P}_{n+3}$, such that τ_n is an edge of a triangle of $\bar{T}$ whose other two edges are boundary edges. Let γ be a diagonal of $\mathbf{P}_{n+3}$ which is not in $\bar{T}$. We define the *labeled modified snake graph* $\hat{\mathcal{G}}_\gamma$ associated to γ as the usual labeled snake graph $\mathcal{G}_\gamma$ of Definition 3.1.1 with these two modifications:

- the edge with label τ_n in the tile G_{n-1} is replaced by three new edges in order to have $\hat{G}_{n-1}$ homeomorphic to a hexagon in the following way:



- if l is the label of an edge e of G_n, and e is an internal edge of $\mathcal{G}_\gamma$, then l is also the label of the edge of $\hat{G}_n$ opposite to e.

Remark 3.2.2. In $\hat{\mathcal{G}}_\gamma$, unlike $\mathcal{G}_\gamma$, $\tau_{[n-1]}$ can also be the label of an external edge. This is the edge along which we will glue labeled modified snake graphs of diagonals to construct labeled modified snake graphs associated to θ-orbits. See Definition 3.2.4.

Example 3.2.3. In the example for $n = 3$ in Figure 3.21, we compute the snake graphs $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$ associated to a diagonal γ in a triangulated polygon. The diagonal γ crosses three diagonals of the triangulation $\bar{T}$, hence $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$ have three tiles.

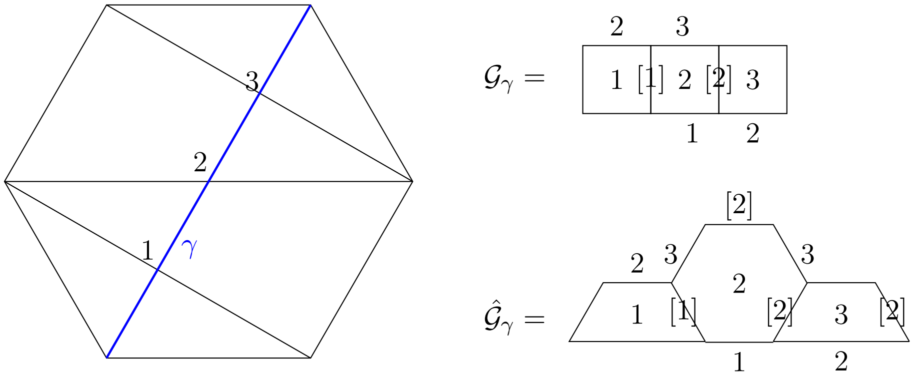


Figure 3.6: On the left, a diagonal γ in a triangulated hexagon; on the right, its snake graphs $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$.

Definition 3.2.4. Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ such that τ_n and τ_{n-1} are edges of a triangle of T whose third edge is a boundary edge, and $\tau_n = d$ is oriented. Let $[a, b]$ be a θ-orbit which is not in T. We associate to $[a, b]$ the labeled modified snake graph $\mathcal{G}_{ab}$ defined in the following way:

- if $\mathrm{Res}([a, b]) = \{\gamma\}$, then $\mathcal{G}_{ab} := \hat{\mathcal{G}}_\gamma$;
- if $\mathrm{Res}([a, b]) = \{\gamma_1, \gamma_2\}$, with γ_1 counterclockwise (resp. clockwise) from γ_2 if τ_{n-1} is counterclockwise (resp. clockwise) from τ_n, then $\mathcal{G}_{ab}$ is obtained by gluing the tile with label n of $\hat{\mathcal{G}}_{\gamma_2}$ to the tile with label $n-1$ of $\hat{\mathcal{G}}_{\gamma_1}$ along $\tau_{[n-1]}$, following the gluing rule recalled in Section 3.1. If both $\hat{\mathcal{G}}_{\gamma_1}$ and $\hat{\mathcal{G}}_{\gamma_2}$ contain a tile with label $n-1$, we add an edge from the top right vertex of the tile of $\hat{\mathcal{G}}_{\gamma_1}$ with label n to the top left vertex of the tile of $\hat{\mathcal{G}}_{\gamma_2}$ with label $n-1$, as in Figure 3.7.

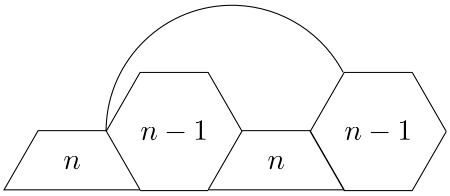


Figure 3.7

Example 3.2.5. We compute the labeled modified snake graph $\mathcal{G}_{ab}$ of the θ-orbit $[a, b]$ in Figure 3.8.

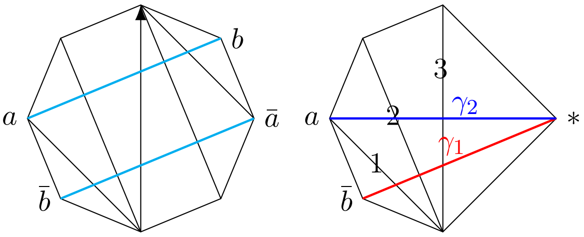
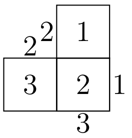


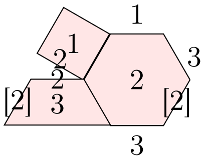
Figure 3.8: A θ-orbit $[a, b]$ in a triangulated octagon and its restriction.

First, we compute $\hat{\mathcal{G}}_{\gamma_1}$ (in red) and $\hat{\mathcal{G}}_{\gamma_2}$ (in blue) from $\mathcal{G}_{\gamma_1}$ and $\mathcal{G}_{\gamma_2}$, according to Definition 3.2.1:

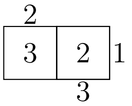
$\mathcal{G}_{\gamma_1} =$



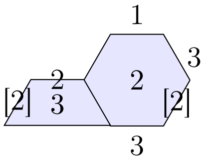
$\hat{\mathcal{G}}_{\gamma_1} =$



$\mathcal{G}_{\gamma_2} =$

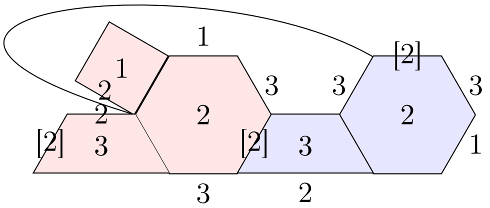


$\hat{\mathcal{G}}_{\gamma_2} =$



Then, according to Definition 3.2.4, we glue them together, and we add an edge from the top right vertex of the tile of $\hat{\mathcal{G}}_{\gamma_1}$ with label 3 to the top left vertex of the tile of $\hat{\mathcal{G}}_{\gamma_2}$ with label 2. We get:

$\mathcal{G}_{ab} =$



If $\mathcal{G}_{ab}$ contains an additional edge as in Figure 3.7, we extend the definition of height monomial $y(P)$ of a perfect matching P of $\mathcal{G}_{ab}$ using Remark 3.1.8.

Definition 3.2.6. Let P be a perfect matching of $\mathcal{G}_{ab}$. The *height monomial* of P is

$$y(P) = \prod_i y_i,$$

where the product is over all i for which $P_- \cap P$ does not contain any edge of G_i with label different from τ_n.

For a θ-orbit $[a, b]$ of $\mathbf{P}_{2n+2}$ (resp. a diagonal γ of $\mathbf{P}_{n+3}$) the minimal matching $P_-(\mathcal{G}_{ab})$ (resp. $P_-(\hat{\mathcal{G}}_\gamma)$) is defined as in Definition 3.1.5.

Definition 3.2.7. Let $[a, b]$ be a θ-orbit which is not in T, and $\tau_{i_1}, \ldots, \tau_{i_d}$ be the sequence of diagonals of $\bar{T} = \mathrm{Res}(T)$ crossed by the diagonals of $\mathrm{Res}([a, b])$. Then the *perfect matching polynomial of* $\mathcal{G}_{ab}$ is

$$F_{\mathcal{G}_{ab}} = \sum_P y(P),$$

where the sum is over all perfect matchings P of $\mathcal{G}_{ab}$, and the $\mathbf{g}$-*vector* is

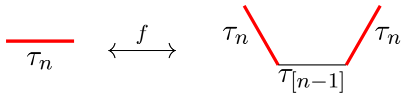
$$\mathbf{g}_{\mathcal{G}_{ab}} = \sum_{\tau_i \in P_-(\mathcal{G}_{ab})} \mathbf{e}_i - \sum_{j=i}^{d} e_{i_j}.$$

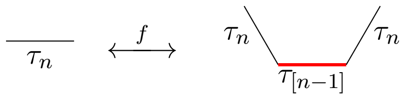
The definition is extended to any θ-orbit by letting $F_{\mathcal{G}_{ab}} = 1$ and $\mathbf{g}_{\mathcal{G}_{ab}} = \mathbf{e}_i$ if $[a, b] = \{\tau_i, \tau_{2n-i}\} \in T$, and $F_{\mathcal{G}_{ab}} = 1$ and $\mathbf{g}_{\mathcal{G}_{ab}} = \mathbf{0}$ if (a, b) is a boundary edge of $\mathbf{P}_{2n+2}$.

Remark 3.2.8. In the definition of $\mathbf{g}_{\mathcal{G}_{ab}}$ we do not consider the edges of $P_-(\mathcal{G}_{ab})$ with a label of the form $\tau_{[i]}$.

Lemma 3.2.9. *Let $\bar{T} = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of $\mathbf{P}_{n+3}$, such that τ_n is an edge of a triangle of $\bar{T}$ whose other two edges are boundary edges. Let γ be a diagonal of $\mathbf{P}_{n+3}$ which is not in $\bar{T}$. Then $F_{\hat{\mathcal{G}}_\gamma} = F_{\mathcal{G}_\gamma}$.*

Proof. The operation $f : Match(\mathcal{G}_\gamma) \to Match(\hat{\mathcal{G}}_\gamma)$ defined as follows:





is a poset preserving isomorphism between the set of perfect matchings of $\mathcal{G}_\gamma$ and the set of perfect matchings of $\hat{\mathcal{G}}_\gamma$. Moreover, $y(P) = y(f(P))$, for any $P \in Match(\mathcal{G}_\gamma)$. Theorefore, $F_{\hat{\mathcal{G}}_\gamma} = F_{\mathcal{G}_\gamma}$. □

Example 3.2.10. We illustrate the proof of Lemma 3.2.9 for the diagonal γ of Example 3.2.3.

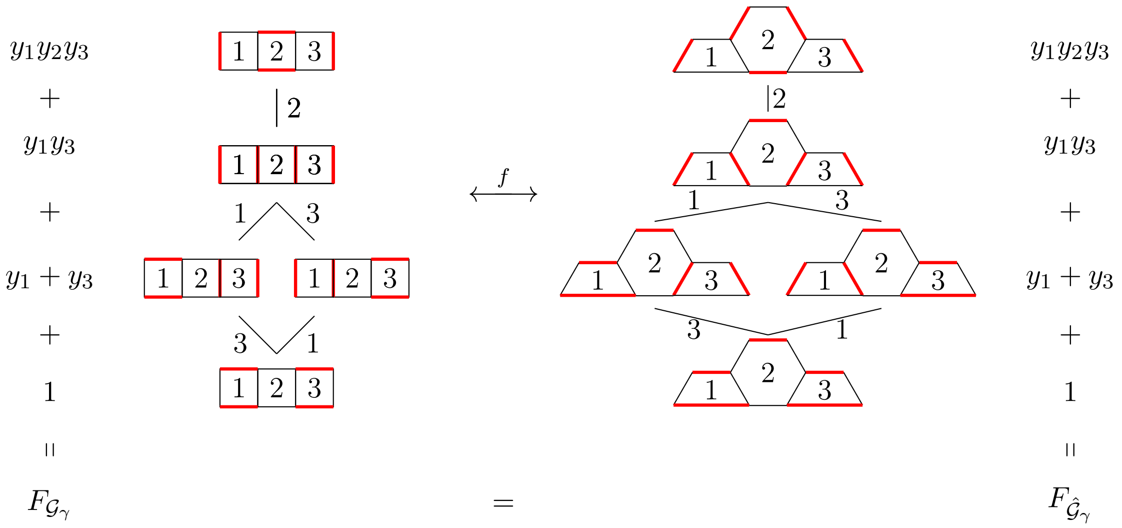


Figure 3.9: The posets of perfect matchings of $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$, and the corresponding monomials which give $F_{\mathcal{G}_\gamma} = F_{\hat{\mathcal{G}}_\gamma}$.

Example 3.2.11. We compute the perfect matching polynomial $F_{\mathcal{G}_{ab}}$ and the $\mathbf{g}$-vector $\mathbf{g}_{\mathcal{G}_{ab}}$ of the labeled modified snake graph $\mathcal{G}_{ab}$ of Example 3.2.5. The set of all perfect matchings of $\mathcal{G}_{ab}$, with the corresponding monomials, is the following:

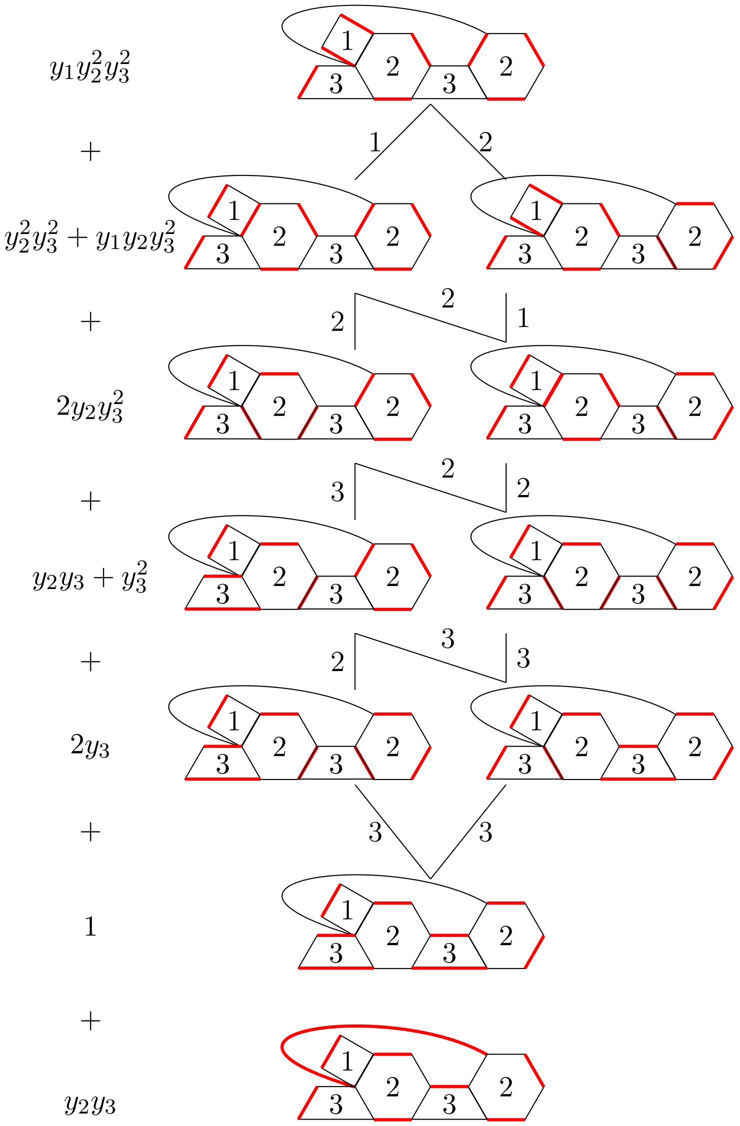


Figure 3.10: The perfect matchings of $\mathcal{G}_{ab}$, and the corresponding monomials.

Therefore, we have

$$F_{\mathcal{G}_{ab}} = y_1y_2^2y_3^2 + y_2^2y_3^2 + y_1y_2y_3^2 + 2y_2y_3^2 + 2y_2y_3 + y_3^2 + 2y_3 + 1,$$

and

$$\mathbf{g}_{\mathcal{G}_{ab}} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Lemma 3.2.12. *Let $T = \{\tau_1, \ldots, \tau_n = d, \ldots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ such that τ_n and τ_{n-1} are edges of a triangle of T whose third edge is a boundary edge, and $\tau_n = d$ is oriented. For any θ-orbit $[a, b]$ of $\mathbf{P}_{2n+2}$, $F_{\mathcal{G}_{ab}} = F_{ab}^B$ (cf. Definition 2.2.1).*

Proof. If $\mathrm{Res}([a,b]) = \{\gamma\}$, the statement holds since $F_{\mathcal{G}_{ab}} = F_{\hat{\mathcal{G}}_\gamma} = F_{\mathcal{G}_\gamma}$ (cf. Lemma 3.2.9). Otherwise, if $\mathrm{Res}([a,b]) = \{\gamma_1, \gamma_2\}$, we have two cases to consider.

i) One of γ_1 and γ_2, say γ_2, intersects only τ_n. So $\mathcal{G}_{ab}$ will be of the following form:

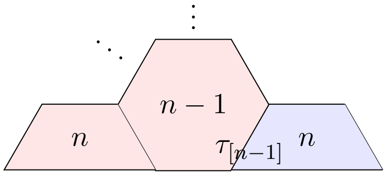


Figure 3.11

where the red part represents $\hat{\mathcal{G}}_{\gamma_1}$, and the blue part represents $\hat{\mathcal{G}}_{\gamma_2}$. We have that

$$F_{\mathcal{G}_{ab}} = F_{\hat{\mathcal{G}}_{\gamma_1}} F_{\hat{\mathcal{G}}_{\gamma_2}} - R = F_{\mathcal{G}_{\gamma_1}} F_{\mathcal{G}_{\gamma_2}} - R, \tag{3.2.1}$$

where R is the sum of the monomials which correspond to gluing of perfect matchings of $\hat{\mathcal{G}}_{\gamma_1}$ (in red) and perfect matchings of $\hat{\mathcal{G}}_{\gamma_2}$ (in blue) which are not perfect matchings of $\mathcal{G}_{ab}$. They are all of the form

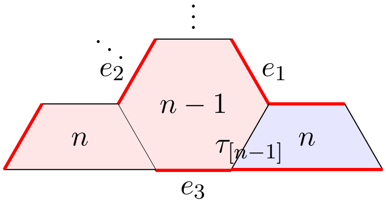


Figure 3.12

Therefore, we have to describe all perfect matchings of $\hat{\mathcal{G}}_{\gamma_1}$ which do not contain the edge with label $\tau_{[n-1]}$, along which we glue $\hat{G}_n$ of $\hat{\mathcal{G}}_{\gamma_2}$ and $\hat{G}_{n-1}$ of $\hat{\mathcal{G}}_{\gamma_1}$. A similar question for $\hat{\mathcal{G}}_{\gamma_2}$ is trivial, as it is just one tile.

We consider the type A exchange relation corresponding to the crossing of diagonals τ_{n-1} and $\tilde{\gamma}_1$, which is the diagonal of $\mathbf{P}_{n+3}$ which intersects the same diagonals of $\bar{T}$ as γ_1 but τ_n. We have two cases to consider.

1) $\tau_{[n-1]}$ is not in $P_-(\hat{\mathcal{G}}_{\gamma_1})$. So the red edges e_1, e_2, e_3 of $\hat{G}_{n-1}$ in Figure 3.12 are in $P_-(\hat{\mathcal{G}}_{\gamma_1})$. Let f_1, f_2, f_3 be the edges of the tile of $\hat{\mathcal{G}}_{\gamma_1}$ with label $n-1$ which are not in the set $\{e_1, e_2, e_3\}$. Then $P = P_-(\hat{\mathcal{G}}_{\gamma_1}) \setminus \{e_1, e_2, e_3\} \cup \{f_1, f_2, f_3\}$ is a perfect matching of $\hat{\mathcal{G}}_{\gamma_1}$ such that $h(P) = y_{n-1}$. Therefore, y_{n-1} is a summand of $F_{\mathcal{G}_{\gamma_1}}$. It follows that τ_{n-1} has to be counterclockwise from τ_n. In fact, we will see in Chapter 5 that $F_{\mathcal{G}_{\gamma_1}} = F_{L_{\gamma_1}}$, where $F_{L_{\gamma_1}}$ is the F-polynomial of the representation L_{γ_1} of the quiver $Q(\bar{T})$ associated to $\bar{T}$ (cf. Definition 1.1.10). If τ_{n-1} was clockwise from τ_n, then there would be an arrow $n-1 \longrightarrow n$ in $Q(\bar{T})$, and so y_{n-1} would not be a summand of $F_{L_{\gamma_1}}$, which contradicts what we have just proved. This argument will be used several other times in the proof. We will not repeat it.

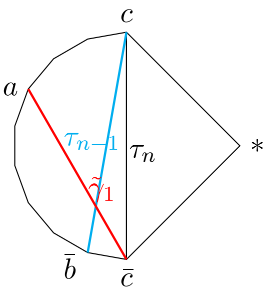


Figure 3.13: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_1$ and τ_{n-1}.

We have that

$$F_{\tilde{\gamma}_1} = \mathbf{y}^{\mathbf{d}_{a\bar{b},c\bar{c}}} F_{(a,c)} + F_{(a,\bar{b})}. \tag{3.2.2}$$

Since e_1, e_2, e_3 are in the minimal perfect matching of $\hat{\mathcal{G}}_{\gamma_1}$, and so of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$, the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain them in the right hand side of 3.2.2 is $F_{(a,\bar{b})}$. Therefore,

$$R = y_n F_{(a,\bar{b})} = \mathbf{y}^{\mathbf{d}_{a*,\bar{b}*}} F_{(a,\bar{b})}, \tag{3.2.3}$$

So, we obtain

$$F_{\mathcal{G}_{ab}} = F_{\gamma_1} F_{\gamma_2} - \mathbf{y}^{\mathbf{d}_{\gamma_1,\gamma_2}} F_{(a,\bar{b})} = F^B_{ab}.$$

2) $\tau_{[n-1]}$ is in $P_-(\hat{\mathcal{G}}_{\gamma_1})$. So the edges e_1, e_2, e_3 in Figure 3.12 are not in $P_-(\hat{\mathcal{G}}_{\gamma_1})$. It follows that y_{n-1} is not a summand of $F_{\mathcal{G}_{\gamma_1}}$. Therefore, τ_{n-1} has to be clockwise from τ_n.

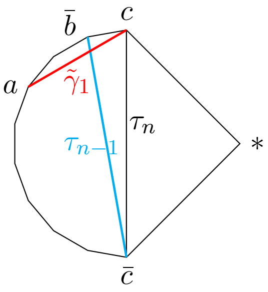


Figure 3.14: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_1$ and τ_{n-1}.

We have that

$$F_{\tilde{\gamma}_1} = \mathbf{y}^{\mathbf{d}_{a\bar{c},\bar{b}c}} F_{(a,\bar{b})} + F_{(a,\bar{c})}. \tag{3.2.4}$$

Since e_1, e_2, e_3 are not in the minimal perfect matching of $\hat{\mathcal{G}}_{\gamma_1}$, and so of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$, the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain them in the right hand side of 3.2.4 is $\mathbf{y}^{\mathbf{d}_{a\bar{c},\bar{b}c}} F_{(a,\bar{b})}$. Therefore,

$$R = y_n \mathbf{y}^{\mathbf{d}_{a\bar{c},\bar{b}c}} F_{(a,\bar{b})} = \mathbf{y}^{\mathbf{d}_{a*,\bar{b}*}} F_{(a,\bar{b})}. \tag{3.2.5}$$

So, we obtain

$$F_{\mathcal{G}_{ab}} = F_{\gamma_1} F_{\gamma_2} - \mathbf{y}^{\mathbf{d}_{\gamma_1,\gamma_2}} F_{(a,\bar{b})} = F^B_{ab}.$$

ii) Both γ_1 and γ_2 intersect τ_{n-1}. So $\mathcal{G}_{ab}$ will be of the following form:

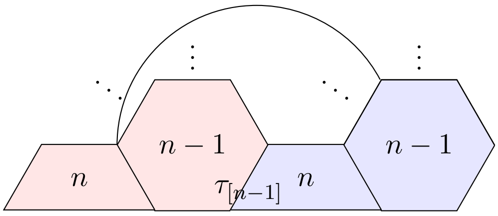


Figure 3.15

where the red part represents $\hat{\mathcal{G}}_{\gamma_1}$, and the blue part represents $\hat{\mathcal{G}}_{\gamma_2}$. We have that

$$F_{\mathcal{G}_{ab}} = F_{\hat{\mathcal{G}}_{\gamma_1}} F_{\hat{\mathcal{G}}_{\gamma_2}} - R + S = F_{\mathcal{G}_{\gamma_1}} F_{\mathcal{G}_{\gamma_2}} - R + S, \tag{3.2.6}$$

where R is the sum of the monomials which correspond to gluing of perfect matchings of $\hat{\mathcal{G}}_{\gamma_1}$ (in red) and perfect matchings of $\hat{\mathcal{G}}_{\gamma_2}$ (in blue) which are not perfect matchings of $\mathcal{G}_{ab}$, and so they are of the form

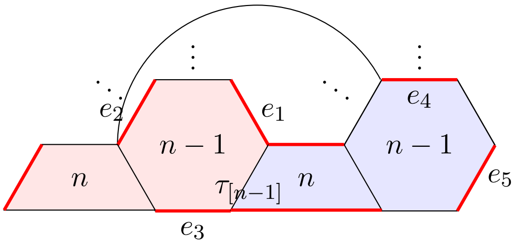


Figure 3.16

while S is the sum of the monomials which correspond to perfect matchings of $\mathcal{G}_{ab}$ which contain the additional edge, and so they are of the form

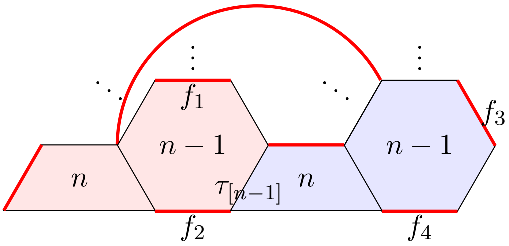


Figure 3.17

Therefore, first we have to describe all perfect matchings of $\hat{\mathcal{G}}_{\gamma_1}$ and $\hat{\mathcal{G}}_{\gamma_2}$ which do not contain the edge with label $\tau_{[n-1]}$, along which we glue $\hat{G}_n$ of $\hat{\mathcal{G}}_{\gamma_2}$ and $\hat{G}_{n-1}$ of $\hat{\mathcal{G}}_{\gamma_1}$.

We consider the type A exchange relation corresponding to the crossing of diagonals τ_{n-1} and $\tilde{\gamma}_1$ (resp. $\tilde{\gamma}_2$), which is the diagonal of $\mathbf{P}_{n+3}$ which intersects the same diagonals of $\bar{T}$ as γ_1 (resp. γ_2) but τ_n. We have two cases to consider.

1) $\tau_{[n-1]}$ is not in $P_-(\hat{\mathcal{G}}_{\gamma_1})$. So the red edges e_1, e_2, e_3 of the tile of $\hat{\mathcal{G}}_{\gamma_1}$ with label $n-1$ in Figure 3.16 are in $P_-(\hat{\mathcal{G}}_{\gamma_1})$. It follows that y_{n-1} is a summand of $F_{\mathcal{G}_{\gamma_1}}$. Therefore, τ_{n-1} has to be counterclockwise from τ_n.

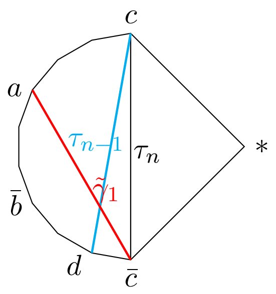


Figure 3.18: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_1$ and τ_{n-1}.

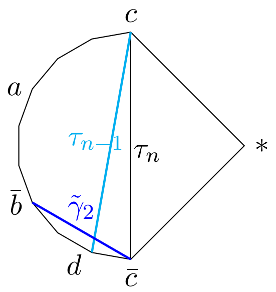


Figure 3.19: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_2$ and τ_{n-1}.

We have

$$F_{\tilde{\gamma}_1} = \mathbf{y}^{\mathbf{d}_{ad,c\bar{c}}} F_{(a,c)} + F_{(a,d)}, \tag{3.2.7}$$

and

$$F_{\tilde{\gamma}_2} = \mathbf{y}^{\mathbf{d}_{\bar{b}d,c\bar{c}}} F_{(\bar{b},c)} + F_{(\bar{b},d)}. \tag{3.2.8}$$

Since e_1, e_2, e_3 are in $P_-(\hat{\mathcal{G}}_{\gamma_1})$, and so in $P_-(\hat{\mathcal{G}}_{\tilde{\gamma}_1})$, the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain them in the right hand side of 3.2.7 is $F_{(a,d)}$. Since e_1, e_2, e_3 are in $P_-(\hat{\mathcal{G}}_{\gamma_1})$, it also follows that the red edges e_4, e_5 of the tile of $\hat{\mathcal{G}}_{\gamma_2}$ with label $n-1$ in Figure 3.16 are not in $P_-(\hat{\mathcal{G}}_{\gamma_2})$. So the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ which contain e_4, e_5 in the right hand side of 3.2.8 is $\mathbf{y}^{\mathbf{d}_{\bar{b}d,c\bar{c}}} F_{(\bar{b},c)}$. Therefore,

$$R = y_n \mathbf{y}^{\mathbf{d}_{\bar{b}d,c\bar{c}}} F_{(a,d)} F_{(\bar{b},c)}. \tag{3.2.9}$$

On the other hand, let f_1, f_2 be the red edges of the tile of $\hat{\mathcal{G}}_{\gamma_1}$ with label $n-1$ in Figure 3.17. Then $f_2 = e_3$ has label τ_n, while f_1 has a different label. Since e_1, e_2, e_3 are in $P_-(\hat{\mathcal{G}}_{\gamma_1})$, it follows that f_1 is not in $P_-(\hat{\mathcal{G}}_{\gamma_1})$, and so it is not in $P_-(\hat{\mathcal{G}}_{\tilde{\gamma}_1})$. So, if P is a perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain f_1, then $h(P)$ is a multiple of y_{n-1}. Therefore, the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain f_1 in the right hand side of 3.2.7 is $\mathbf{y}^{\mathbf{d}_{ad,c\bar{c}}} F_{(a,c)}$. Moreover, since e_1, e_2, e_3

are in $P_-(\hat{\mathcal{G}}_{\gamma_1})$, it also follows that f_3, f_4 are in $P_-(\hat{\mathcal{G}}_{\gamma_2})$, and so in $P_-(\hat{\mathcal{G}}_{\tilde{\gamma}_2})$. Then the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ which contain f_3, f_4 in the right hand side of 3.2.8 is $F_{(\bar{b},d)}$. Therefore,

$$S = y_n \mathbf{y}^{\mathbf{d}_{ad,c\bar{c}}} F_{(a,c)} F_{(\bar{b},d)}. \tag{3.2.10}$$

Finally, we consider the exchange relation corresponding to the crossing of (a,d) and $(\bar{b},c)$.

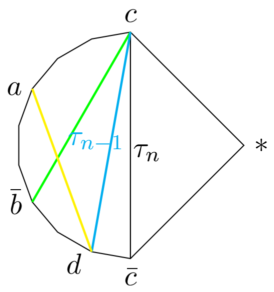


Figure 3.20: Type A exchange relation corresponding to the crossing of (a,d) and $(\bar{b},c)$.

We have that

$$F_{(a,d)} F_{(\bar{b},c)} = \mathbf{y}^{\mathbf{d}_{ac,\bar{b}d}} F_{(a,\bar{b})} + \mathbf{y}^{\mathbf{d}_{a\bar{b},cd}} F_{(a,c)} F_{(\bar{b},d)}. \tag{3.2.11}$$

Therefore,

$$-R + S = -y_n \mathbf{y}^{\mathbf{d}_{\bar{b}d,c\bar{c}}} \mathbf{y}^{\mathbf{d}_{ac,\bar{b}d}} F_{(a,\bar{b})} = -\mathbf{y}^{a*,\bar{b}*} F_{(a,\bar{b})}. \tag{3.2.12}$$

So, we obtain

$$F_{\mathcal{G}_{ab}} = F_{\gamma_1} F_{\gamma_2} - \mathbf{y}^{\mathbf{d}_{\gamma_1,\gamma_2}} F_{(a,\bar{b})} = F^B_{ab}.$$

2) The case in which $\tau_{[n-1]}$ is in the minimal perfect matching of $\hat{\mathcal{G}}_{\gamma_1}$ is analogous, exchanging the roles of γ_1 and γ_2.

□

Lemma 3.2.13. *Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ such that τ_n and τ_{n-1} are edges of a triangle of T whose third edge is a boundary edge, and $\tau_n = d$ is oriented. For any θ-orbit $[a,b]$ of $\mathbf{P}_{2n+2}$, $\mathbf{g}_{\mathcal{G}_{ab}} = g^B_{ab}$ (cf. Definition 2.2.1).*

Proof. If $\mathrm{Res}([a,b]) = \{\gamma\}$, by construction, an edge with label n is in $\mathcal{G}_\gamma$ if and only if two edges with label n are in $\hat{\mathcal{G}}_\gamma$. Therefore,

- If γ does not cross τ_n, $\mathbf{g}_{\mathcal{G}_{ab}} = \mathbf{g}_{\hat{\mathcal{G}}_\gamma} = D\mathbf{g}_{\mathcal{G}_\gamma}$.
- Otherwise, if γ crosses τ_n, then $\mathbf{g}_{\mathcal{G}_{ab}} = \mathbf{g}_{\hat{\mathcal{G}}_\gamma} = D\mathbf{g}_{\mathcal{G}_\gamma} + e_n$, since in $D\mathbf{g}_{\mathcal{G}_\gamma}$ we have subtracted e_n twice, so we have to add it once.

If $\mathrm{Res}([a,b]) = \{\gamma_1, \gamma_2\}$, the statement follows since the minimal matching of $\mathcal{G}_{ab}$ is the gluing of the minimal matchings of $\hat{\mathcal{G}}_{\gamma_1}$ and $\hat{\mathcal{G}}_{\gamma_2}$. □

Theorem 3.2.14. *Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ such that $\tau_n = d$ and τ_{n-1} are edges of a triangle of T whose third edge is a boundary edge, and $\tau_n = d$ is oriented. Let $\mathcal{A} = \mathcal{A}_\bullet(T)^B$ be the cluster algebra of type B_n with principal coefficients in T (cf. Definition 2.1.4). Let $[a, b]$ be an orbit of the action of θ on the diagonals of the polygon, and x_{ab} the cluster variable of $\mathcal{A}$ which corresponds to $[a, b]$. Let F_{ab} and $\mathbf{g}_{ab}$ be the F-polynomial and the $\mathbf{g}$-vector of x_{ab}, respectively. Then $F_{ab} = F_{\mathcal{G}_{ab}}$ and $\mathbf{g}_{ab} = \mathbf{g}_{\mathcal{G}_{ab}}$.*

Proof. The result follows directly from Theorem 2.2.2, Lemma 3.2.12 and Lemma 3.2.13. □

Remark 3.2.15. Theorem 3.2.14 extends the result of [39] for cluster algebras of type B to every seed whose cluster corresponds to a θ-invariant triangulation $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ of $\mathbf{P}_{2n+2}$, such that $\tau_n = d$ and τ_{n-1} are edges of a triangle of T whose third edge is a boundary edge.

Example 3.2.16. Let $[a, b]$ be the θ-orbit in the triangulated octagon in Figure 3.8. It follows from Theorem 3.2.14 that the Laurent polynomial $F_{\mathcal{G}_{ab}}$, and the integer vector $\mathbf{g}_{\mathcal{G}_{ab}}$, computed in Example 3.2.11, are the F-polynomial, and the $\mathbf{g}$-vector respectively, of $x_{ab} \in \mathcal{A}_\bullet(T)^B$, where T is the θ-invariant triangulation of the octagon in Figure 3.8.

3.2.2 Type C

Definition 3.2.17. Let $\bar{T} = \{\tau_1, \dots, \tau_n\}$ be a triangulation of $\mathbf{P}_{n+3}$, such that τ_n is an edge of a triangle of $\bar{T}$ whose other two edges are boundary edges. Let γ be a diagonal of $\mathbf{P}_{n+3}$ which is not in $\bar{T}$. We define the *labeled modified snake graph* $\hat{\mathcal{G}}_\gamma$ associated to γ as the usual labeled snake graph $\mathcal{G}_\gamma$ of Definition 3.1.1 with the following additional labels on the tile $\hat{G}_n$ with label n: if l is a label of an edge e of G_n, l is also a label of the edge of $\hat{G}_n$ opposite to e.

Remark 3.2.18. A cluster algebra of type C_n can also be realized as a disk with one orbifold point of weight $\frac{1}{2}$, and $n+1$ boundary marked points [23]. In [12], Canakci and Tumarkin introduce snake and band graphs associated to curves in a triangulated orbifold with orbifold points of weight $\frac{1}{2}$, including type C. The tile $\hat{G}_n$ of $\hat{\mathcal{G}}_\gamma$ is the same as the tile they associate to the pending arc, i.e. the arc of the triangulation of the orbifold connecting a boundary point to the orbifold point.

Example 3.2.19. In the example for $n = 3$ in Figure 3.21, we compute the snake graphs $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$ associated to a diagonal γ in a triangulated polygon. The diagonal γ crosses three diagonals of the triangulation $\bar{T}$, hence $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$ have three tiles.

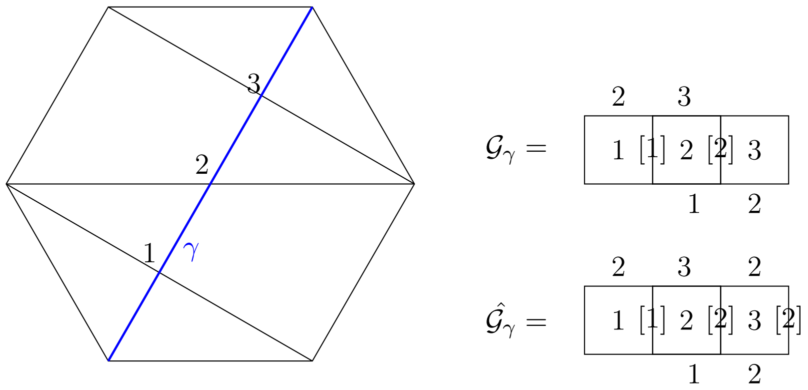


Figure 3.21: On the left, a diagonal γ in a triangulated hexagon; on the right, its snake graphs $\mathcal{G}_\gamma$ and $\hat{\mathcal{G}}_\gamma$.

Definition 3.2.20. Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$. Let $[a,b]$ be a θ-orbit which is not in T. We associate to $[a,b]$ the labeled modified snake graph $\mathcal{G}_{ab}$ defined in the following way:

- if $\widetilde{\mathrm{Res}}([a,b]) = \{\tilde{\gamma}\}$, then $\mathcal{G}_{ab} := \hat{\mathcal{G}}_{\tilde{\gamma}}$;
- if $\widetilde{\mathrm{Res}}([a,b]) = \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, then $\mathcal{G}_{ab}$ is obtained by gluing $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ and $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ along their common exterior edge, following the gluing rule recalled in Section 3.1.

Remark 3.2.21. In the case where $\widetilde{\mathrm{Res}}([a,b]) = \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, with the notation of Definition 2.3.3, the edge along which we glue $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ and $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ is $(\bar{b}, c)$ if $[a,b] = [a,\bar{a}]$ is a diameter, while it is $(\bar{c}, \bar{d})$ if $[a,b]$ is a pair of diagonals which cross d.

Remark 3.2.22. Let $[a,b]$ be a θ-orbit such that $\mathrm{Res}([a,b]) = \{\gamma_1, \gamma_2\}$. Then $\mathcal{G}_{ab}$ is obtained by superimposing $\mathcal{G}_{\gamma_1}$ and $\mathcal{G}_{\gamma_2}$ over their tile G_n with label n, in the only way such that G_n has different relative orientation with respect to $\bar{T} = \mathrm{Res}(T)$ in $\mathcal{G}_{\gamma_1}$ and $\mathcal{G}_{\gamma_2}$.

We define the perfect matching polynomial $F_{\mathcal{G}_{ab}}$ and the **g**-vector $\mathbf{g}_{\mathcal{G}_{ab}}$ of $\mathcal{G}_{ab}$ as in Definition 3.2.7, where for the height monomial we can use Definition 3.1.6 since, unlike type B, we do not have an additional edge.

Example 3.2.23. We compute the labeled modified snake graph $\mathcal{G}_{ab}$ of the θ-orbit $[a,b]$ in Figure 3.22.

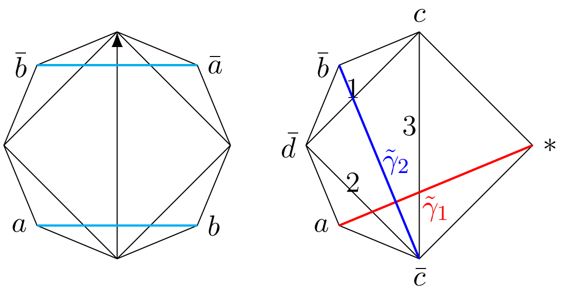


Figure 3.22: A θ-orbit $[a,b]$ in a triangulated octagon and its rotated restriction.

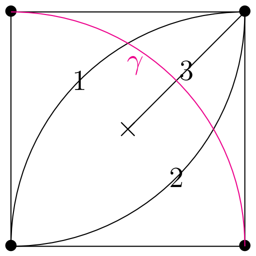
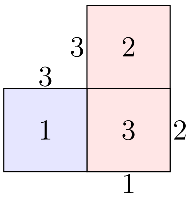


Figure 3.23: A triangulated orbifold with one orbifold point of weight $\frac{1}{2}$.

$\mathcal{G}_{ab} =$



Remark 3.2.24. $\mathcal{G}_{ab}$ is the snake graph associated in [12] to the arc γ in the triangulated orbifold in Figure 3.23.

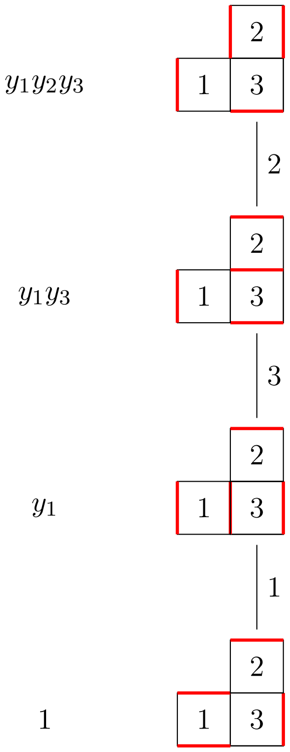


Figure 3.24: The poset of perfect matchings of $\mathcal{G}_{ab}$, and the corresponding monomials.

Therefore, we have

$$F_{\mathcal{G}_{ab}} = y_1y_2y_3 + y_1y_3 + y_1 + 1,$$

and

$$\mathbf{g}_{\mathcal{G}_{ab}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Lemma 3.2.25. *Let $T = \{\tau_1, \dots, \tau_n = d, \dots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d. For any θ-orbit $[a,b]$ of $\mathbf{P}_{2n+2}$, $F_{\mathcal{G}_{ab}} = F^C_{ab}$ (cf. Definition 2.3.3).*

Proof. If $\widetilde{\operatorname{Res}}([a,b]) = \{\tilde{\gamma}\}$, the statement holds since $F_{\mathcal{G}_{ab}} = F_{\hat{\mathcal{G}}_{\tilde{\gamma}}} = F_{\mathcal{G}_{\tilde{\gamma}}}$. Otherwise, if $\widetilde{\operatorname{Res}}([a,b]) = \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, then

$$F_{\mathcal{G}_{ab}} = F_{\hat{\mathcal{G}}_{\tilde{\gamma}_1}} F_{\hat{\mathcal{G}}_{\tilde{\gamma}_2}} - R = F_{\mathcal{G}_{\tilde{\gamma}_1}} F_{\mathcal{G}_{\tilde{\gamma}_2}} - R, \tag{3.2.13}$$

where R is the sum of the monomials which correspond to gluing of perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ (in red in Figure 3.25) and perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ (in blue in Figure 3.25) which are not perfect matchings of $\mathcal{G}_{ab}$. They are all of the form

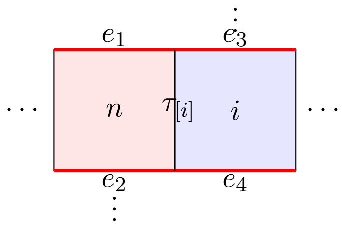


Figure 3.25

Therefore, we have to describe all perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ and of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ which do not contain the edge with label $\tau_{[i]}$, along which we glue $\hat{G}_n$ of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ and $\hat{G}_i$ of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$.

We prove the statement in the case where $[a,b] = [a,\bar{a}]$ is a diameter. If $[a,b]$ is a pair of diagonals which cross d, the proof is completely analogous. We consider the type A exchange relation corresponding to the crossing of diagonals $\tilde{\gamma}_1$ and τ_n.

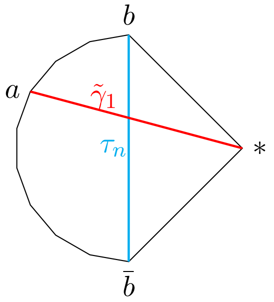


Figure 3.26: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_1$ and τ_n.

We have that

$$F_{\tilde{\gamma}_1} = \mathbf{y}^{\mathbf{d}_{a\bar{b},b*}} F_{(a,b)} + F_{(a,\bar{b})}. \tag{3.2.14}$$

At this point, we have two cases to consider.

1) $\tau_{[i]}$ is not in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$. So the red edges e_1, e_2 of $\hat{G}_n$ in Figure 3.25 are in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$. So the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain them in the right hand side of 3.2.14 is $F_{(a,\bar{b})}$. Moreover, the fact that e_1, e_2 are in the minimal matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ means that the monomial y_n is a summand of $F_{\mathcal{G}_{\tilde{\gamma}_1}}$. Therefore, τ_i has to be clockwise from τ_n. To prove the last two sentences we use the argument stated explicitly in the proof of Lemma 3.2.12 point i-1).

   We consider the type A exchange relation corresponding to the crossing of diagonals $\tilde{\gamma}_2$ and τ_i.

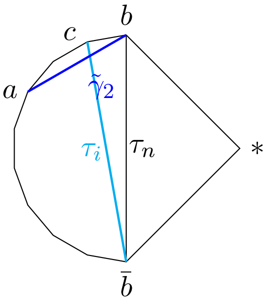


Figure 3.27: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_2$ and τ_i.

   We have that

$$F_{\tilde{\gamma}_2} = \mathbf{y}^{\mathbf{d}_{bc,a\bar{b}}} F_{(a,c)} + F_{(a,\bar{b})}. \tag{3.2.15}$$

   Since e_1, e_2 are in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$, the red edges e_3, e_4 of $\hat{G}_i$ in Figure 3.25 cannot be in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$. So the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ which contain e_3, e_4 in the right hand side of 3.2.15 is $\mathbf{y}^{\mathbf{d}_{bc,a\bar{b}}} F_{(a,c)}$. Therefore,

$$R = \mathbf{y}^{\mathbf{d}_{bc,a\bar{b}}} F_{(a,\bar{b})} F_{(a,c)} = \mathbf{y}^{\mathbf{d}_{bc,a*}} F_{(a,\bar{b})} F_{(a,c)}. \tag{3.2.16}$$

   So, we obtain (cf. Remark 2.3.6)

$$F_{\mathcal{G}_{ab}} = F_{\tilde{\gamma}_1} F_{\tilde{\gamma}_2} - \mathbf{y}^{\mathbf{d}_{bc,a*}} F_{(a,\bar{b})} F_{(a,c)} = F^{C}_{ab}.$$

2) $\tau_{[i]}$ is in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$. So e_1, e_2 are not in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$. So the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ which contain e_1, e_2 in the right hand side of 3.2.14 is $\mathbf{y}^{\mathbf{d}_{a\bar{b},b*}} F_{(a,b)}$. Moreover, the fact that e_1, e_2 are not in the minimal matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ means that the monomial y_n is not a summand of $F_{\mathcal{G}_{\tilde{\gamma}_1}}$. Therefore, τ_i has to be counterclockwise from τ_n.

   We consider the type A exchange relation corresponding to the crossing of diagonals $\tilde{\gamma}_2$ and τ_i.

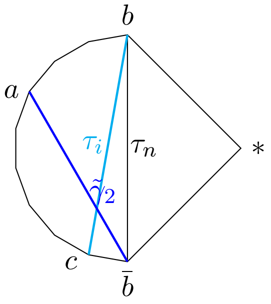


Figure 3.28: Type A exchange relation corresponding to the crossing of $\tilde{\gamma}_2$ and τ_i.

We have that

$$F_{\tilde{\gamma}_2} = \mathbf{y}^{\mathbf{d}_{ac,b\bar{b}}} F_{(a,b)} + F_{(a,c)}. \tag{3.2.17}$$

Since e_1, e_2 are not in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$, e_3, e_4 must be in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$. So the sum of the monomials which correspond to the perfect matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$ which contain e_3, e_4 in the right hand side of 3.2.17 is $F_{(a,c)}$. Therefore,

$$R = \mathbf{y}^{\mathbf{d}_{a\bar{b},b*}} F_{(a,b)} F_{(a,c)}. \tag{3.2.18}$$

So, we obtain (cf. Remark 2.3.6)

$$F_{\mathcal{G}_{ab}} = F_{\tilde{\gamma}_1} F_{\tilde{\gamma}_2} - \mathbf{y}^{\mathbf{d}_{a\bar{b},b*}} F_{(a,b)} F_{(a,c)} = F^C_{ab}.$$

☐

Lemma 3.2.26. *Let $T = \{\tau_1, \ldots, \tau_n = d, \ldots, \tau_{2n-1}\}$ be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d. For any θ-orbit $[a, b]$ of $\mathbf{P}_{2n+2}$, $\mathbf{g}_{\mathcal{G}_{ab}} = g^C_{ab}$ (cf. Definition 2.3.3).*

Proof. If $\widetilde{\mathrm{Res}}([a, b]) = \{\tilde{\gamma}\}$, we have two cases to consider.

- If τ_i and τ_n are two different sides of a triangle of T, τ_i is clockwise from τ_n, and $\tilde{\gamma}$ crosses τ_n, then the edge of $\hat{G}_n$ with label i and its opposite must be in the minimal perfect matching $P_-(\mathcal{G}_{ab})$ of $\mathcal{G}_{ab}$. Since they both have label i in $\hat{G}_n$, it follows that $\mathbf{g}_{\mathcal{G}_{ab}} = \mathbf{g}_{\mathcal{G}_{\tilde{\gamma}}} + \mathbf{e}_i = \mathbf{g}_{\tilde{\gamma}} + \mathbf{e}_i$.
- Otherwise, $\mathbf{g}_{\mathcal{G}_{ab}} = \mathbf{g}_{\mathcal{G}_{\tilde{\gamma}}} = \mathbf{g}_{\tilde{\gamma}}$.

If $\widetilde{\mathrm{Res}}([a, b]) = \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, the statement follows since the minimal matching of $\mathcal{G}_{ab}$ is the gluing of the minimal matchings of $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ and $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$. If τ_i and τ_n are two different sides of a triangle of T and τ_i is clockwise from τ_n, we have to subtract the vector of the canonical basis of $\mathbb{R}^n$ which corresponds to the edge e of the triangle containing τ_n along which we glue $\hat{\mathcal{G}}_{\tilde{\gamma}_1}$ and $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$, i.e. $\mathbf{g}_{(\bar{c},\bar{d})}$ (resp. $\mathbf{g}_{(\bar{b},c)}$ if $[a, b] = (a, \bar{a})$ is a diameter) with the notation of Definition 2.3.3 (cf. Remark 3.2.21). We have to do this since e is in the minimal perfect matching of $\hat{\mathcal{G}}_{\tilde{\gamma}_2}$, but it is not in $P_-(\mathcal{G}_{ab})$ since it becomes an interior edge of $\mathcal{G}_{ab}$ after gluing. ☐

Theorem 3.2.27. *Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d, and let $\mathcal{A} = \mathcal{A}_\bullet(T)^C$ be the cluster algebra of type C_n with principal coefficients in T (cf. Definition*

2.1.4). Let $[a, b]$ be an orbit of the action of θ on the diagonals of the polygon, and x_{ab} the cluster variable of $\mathcal{A}$ which corresponds to $[a, b]$. Let F_{ab} and $\mathbf{g}_{ab}$ be the F-polynomial and the $\mathbf{g}$-vector of x_{ab}, respectively. Then $F_{ab} = F_{\mathcal{G}_{ab}}$ and $\mathbf{g}_{ab} = \mathbf{g}_{\mathcal{G}_{ab}}$.

Proof. The result follows directly from Theorem 2.3.7, Lemma 3.2.25 and Lemma 3.2.26. ☐

Remark 3.2.28. Theorem 3.2.27 extends the result of [39] for cluster algebras of type C to every seed.

Example 3.2.29. Let $[a, b]$ be the θ-orbit in the triangulated octagon in Figure 3.22. It follows from Theorem 3.2.27 that the Laurent polynomial $F_{\mathcal{G}_{ab}}$, and the integer vector $\mathbf{g}_{\mathcal{G}_{ab}}$, computed in Example 3.2.23, are the F-polynomial, and the $\mathbf{g}$-vector respectively, of $x_{ab} \in \mathcal{A}_{\bullet}(T)^C$, where T is the θ-invariant triangulation of the octagon in Figure 3.22.

Chapter 4

Quiver representations

This chapter is devoted to a recollection on quiver representations, and it gives the background necessary for the last chapter. We will review the notions of projective and injective modules (Section 4.1.1), quivers and path algebras (Section 4.2), representations of bound quivers (Section 4.3.1), representations given by diagonals of a polygon (Section 4.3.2), quiver Grassmannians (Section 4.3.3), F-polynomials and $\mathbf{g}$-vectors of quiver representations (Section 4.3.4), Auslander-Reiten translation and Auslander-Reiten formulas (Section 4.4). For the exposition we follow mainly [1, 47]. We assume that the reader is familiar with the basic notions of category theory. Otherwise one can find, for example in [1, Appendix A], a reminder on the subject.

In this chapter K denotes an algebrically closed field.

4.1 Modules over an algebra

Definition 4.1.1. Let A be a K-algebra with identity element denoted by 1. A *right A-module* (or a right module over A) is a pair $(L, \cdot)$, where L is a K-vector space and $\cdot : L \times A \to L$, $(m, a) \mapsto ma$, is a binary operation satisfying the following conditions:

i) $(x + y)a = xa + ya$;

ii) $x(a + b) = xa + xb$;

iii) $x(ab) = (xab)$;

iv) $x1 = x$;

v) $(x\lambda)a = x(a\lambda) = (xa)\lambda$

for all $x, y \in L$, $a, b \in A$ and $\lambda \in K$. A K-subspace L' of L is said to be an *A-submodule* of L if $la \in L'$ for all $l \in L', a \in A$.

The definition of a left A-module is analogous. Throughout, we will write L or L_A instead of $(L, \cdot)$. Moreover, we will write A_A and ${}_AA$ to denote the algebra A viewed as a right or left A-module, respectively.

Let A be a K-algebra. Let L and L' be right A-modules. A K-linear map $h : L \to L'$ is said to be an *A-module homomorphism* (or simply an A-homomorphism) if $h(la) = h(l)a$ for any $l \in L$ and $a \in A$. An A-module homomorphism is said to be a *monomorphism* (resp. an *epimorphism*) if it is injective (resp. surjective). A bijective A-module homomorphism is called an *isomorphism.* Two right A-modules L and L' are said to be *isomorphic* if there exists an A-module isomorphism $h : L \to L'$. We denote by $\operatorname{Mod} A$ the abelian category of right A-modules, i.e. the category whose objects are right A-modules, the morphisms are A-module homomorphisms, and the composition of morphisms is the usual composition of maps.

A right module L is *finite dimensional* if the dimension $\dim_K L$ of the underlying K-vector space of L is finite, while it is *finitely generated* if there exists a finite subset $\{l_1, \dots, l_s\}$ of elements of L such that for any element $l \in L$ there exist $a_1, \dots, a_s \in A$ such that $l = l_1a_1 + \cdots + l_sa_s$. We denote by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ whose objects are the finitely generated modules. It follows that, if A is a finite dimensional K-algebra, then all modules in $\operatorname{mod} A$ are finite dimensional.

4.1.1 Projective and injective modules

Definition 4.1.2. A right A-module P is *projective* if, for any epimorphism $h : M \to N$, the induced map $\operatorname{Hom}_A(P, h) : \operatorname{Hom}_A(P, M) \to \operatorname{Hom}_A(P, N)$ is surjective, i.e., for any epimorphism $h : M \to N$ and any $f \in \operatorname{Hom}_A(P, N)$, there is an $f' \in \operatorname{Hom}_A(P, M)$ such that the following diagram is commutative

$$\begin{tikzcd} & P \arrow[dl, "f'"'] \arrow[d, "f"] & \\ M \arrow[r, "h"] & N \arrow[r] & 0 \end{tikzcd}$$

The full subcategory of $\operatorname{mod} A$ whose objects are the projective modules is denoted by $\operatorname{proj} A$.

Definition 4.1.3. A right A-module I is injective if, for any monomorphism $u : L \to M$, the induced map $\operatorname{Hom}_A(u, I) : \operatorname{Hom}_A(M, I) \to \operatorname{Hom}_A(L, I)$ is surjective, i.e., for any monomorphism $u : L \to M$ and any $g \in \operatorname{Hom}_A(L, I)$, there is a $g' \in \operatorname{Hom}_A(M, I)$ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "u"] \arrow[d, "g"'] & M \arrow[dl, "g'"] \\ & I & \end{tikzcd}$$

The full subcategory of $\operatorname{mod} A$ whose objects are the injective modules is denoted by $\operatorname{inj} A$.

The functor $D(-) = \operatorname{Hom}_K(-, K)$, known as *standard K-duality*, defines two dualities of categories

$$\operatorname{mod} A \xrightarrow{D} \operatorname{mod} A^{\mathrm{op}} \xrightarrow{D} \operatorname{mod} A$$

such that there are natural equivalences of functors $D \circ D \cong 1_{\operatorname{mod} A}$ and $D \circ D \cong 1_{\operatorname{mod} A^{\mathrm{op}}}$, where A^{op} is the opposite algebra of A or, in other words, $\operatorname{mod} A^{\mathrm{op}}$ is the category of finitely generated left A-modules. This allows us to study the injective modules in $\operatorname{mod} A$ by means of the projective modules in $\operatorname{mod} A^{\mathrm{op}}$.

Definition 4.1.4. The *Nakayama functor* of $\operatorname{mod} A$ is defined to be the endofunctor

$$\nu = D \circ \operatorname{Hom}_A(-, A) : \operatorname{mod} A \to \operatorname{mod} A.$$

Lemma 4.1.5. *The Nakayama functor ν is right exact.*

Proof. See [1], III.2.9. ☐

Proposition 4.1.6. *The restriction of the Nakayama functor $\nu : \operatorname{mod} A \to \operatorname{mod} A$ to $\operatorname{proj} A$ induces an equivalence between $\operatorname{proj} A$ and $\operatorname{inj} A$. The quasi-inverse of this restriction is given by $\nu^{-1} : \operatorname{Hom}_A(D({}_A A), -) : \operatorname{inj} A \to \operatorname{proj} A$.*

Proof. See [1], III.2.10. ☐

Let A be a finite dimensional K-algebra. Then A can be viewed as a bimodule over itself. Let $\{e_1, \dots, e_n\}$ be a *complete set of primitive orthogonal idempotents* of A, so $e_1, \dots, e_n$ are orthogonal idempotents, i.e. $e_i e_j = e_j e_i = \delta_{i,j}$, e_i cannot be written as a sum $e_i = e'_i + e''_i$, where e'_i and e''_i are non-zero orthogonal idempotents, and $1 = e_1 + \dots + e_n$. Therefore, $A = e_1 A \oplus \dots \oplus e_n A$ is a decomposition of A into indecomposable submodules. We have the following result:

Theorem 4.1.7. *i) Every indecomposable projective right A-module is isomorphic to one of the modules*

$$P(1) = e_1 A, P(2) = e_2 A, \dots, P(n) = e_n A.$$

2) Every indecomposable injective right A-module is isomorphic to one of the modules

$$I(1) = D(Ae_1), I(2) = D(Ae_2), \dots, I(n) = D(Ae_n).$$

Proof. See [1], I.5.17. ☐

Definition 4.1.8. A *projective resolution* of a right A-module L is a complex

$$P_\bullet : \dots \to P_m \xrightarrow{p_m} P_{m-1} \to \dots \to P_1 \xrightarrow{p_1} P_0 \to 0$$

of projective A-modules together with an epimorphism $p_0 : P_0 \xrightarrow{p_0} L$ of right A-modules such that the sequence

$$\dots \to P_m \xrightarrow{p_m} P_{m-1} \to \dots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0 \tag{4.1.1}$$

is exact.

Definition 4.1.9. Dually, an *injective resolution* of L is a complex

$$I_\bullet : 0 \to I_0 \xrightarrow{i_1} I_1 \to \dots \to I_m \xrightarrow{i_{m+1}} I_{m+1} \to \cdots$$

of injective A-modules together with a monomorphism $i_0 : L \to I_0$ of right A-modules such that the sequence

$$0 \to L \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \to \dots \to I_m \xrightarrow{i_{m+1}} I_{m+1} \to \cdots \tag{4.1.2}$$

is exact.

Definition 4.1.10. An A-submodule X of L is *superfluous* if for every submodule Y of L the equality $X+Y=L$ implies $Y=L$. An A-epimorphism $h: L \to M$ in $\operatorname{mod} A$ is *minimal* if $\ker(h)$ is superfluous in L. An epimorphism $p: P \to L$ in $\operatorname{mod} A$ is called a *projective cover* of L if P is a projective module and p is a minimal epimorphism.

The notions dual to minimal epimorphism and to projective cover are defined as follows.

Definition 4.1.11. An A-module monomorphism $u: L \to M$ in $\operatorname{mod} A$ is *minimal* if every non-zero submodule X of M has a non-zero intersection with $\operatorname{im}(u)$. A monomorphism $i: L \to I$ in $\operatorname{mod} A$ is called an *injective envelope* of L if I is an injective module and i is a minimal monomorphism.

Definition 4.1.12. An exact sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0$$

in $\operatorname{mod} A$ is called a *minimal projective presentation* of the A-module L if the A-module homomorphisms $P_0 \xrightarrow{p_0} L$ and $P_1 \xrightarrow{p_1} \ker(p_0)$ are projective covers.

An exact sequence as in 4.1.1 in $\operatorname{mod} A$ is called a *minimal projective resolution* of L if $p_j : P_j \to \operatorname{im}(p_j)$ is a projective cover for all $j \geq 1$ and $P_0 \xrightarrow{p_0} L$ is a projective cover.

Theorem 4.1.13. *Let A be a finite dimensional K-algebra. Any module L in* $\operatorname{mod} A$ *admits a minimal projective presentation and a minimal projective resolution in* $\operatorname{mod} A$.

Proof. See [1], I.5.10. ☐

Definition 4.1.14. Dually, an exact sequence

$$0 \to L \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1$$

in $\operatorname{mod} A$ is called a *minimal injective presentation* of the A-module L if the A-module homomorphisms $L \xrightarrow{i_0} I_0$ and $\operatorname{im}(i_1) \hookrightarrow I_1$ are injective envelops.

An exact sequence as in 4.1.2 in $\operatorname{mod} A$ is called a *minimal projective resolution* of L if $\operatorname{im}(i_m) \to I_m$ is an injective envelope for all $m \geq 1$ and $L \xrightarrow{i_0} I_0$ is an injective envelope.

Theorem 4.1.15. *Let A be a finite dimensional K-algebra. Any module L in* $\operatorname{mod} A$ *admits a minimal injective presentation and a minimal injective resolution in* $\operatorname{mod} A$.

Proof. See [1], I.5.16. ☐

Definition 4.1.16. Let A be a finite dimensional K-algebra. The *projective dimension* of a right A-module M is the non-negative integer $\operatorname{pd} L = m$ such that there exists a projective resolution

$$0 \to P_m \xrightarrow{p_m} P_{m-1} \to \cdots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0$$

of L of length m and L has no projective resolutions of length $m-1$, if such a number m exists. If L admits no projective resolutions of finite length, the projective dimension $\operatorname{pd} L$ of L is defined to be infinity. Dually, the *injective dimension* of L is the non-negative integer $\operatorname{id} L = m$ such that there exists an injective resolution of L of length m and L has no injective resolutions of length $m-1$, if such a number m exists. If L admits no injective resolutions of finite length, $\operatorname{id} L$ of L is defined to be infinity.

Definition 4.1.17. Let A be a finite dimensional K-algebra. The *global dimension* of A is the non-negative integer $\mathrm{gl.dim}\, A = \max\{\mathrm{pd}\, L \mid L \in \mathrm{mod}\, A\}$.

One can show that the global dimension of A is also equal to $\max\{\mathrm{id}\, L \mid L \in \mathrm{mod}\, A\}$. See [1, A.4] for details.

Definition 4.1.18. Let A be a finite dimensional K-algebra. A is said to be *hereditary* if the global dimension of A is at most 1.

Given two A-modules L and N, we take a projective resolution $P_\bullet$ of N and construct the induced cochain complex

$$\begin{aligned}\mathrm{Hom}_A(P_\bullet, L) : 0 \to \mathrm{Hom}_A(P_0, L) &\xrightarrow{\mathrm{Hom}_A(p_1, L)} \mathrm{Hom}_A(P_1, L) \to \cdots \\ \cdots \to \mathrm{Hom}_A(P_m, L) &\xrightarrow{\mathrm{Hom}_A(p_{m+1}, L)} \mathrm{Hom}_A(P_{m+1}, L) \to \cdots\end{aligned}$$

of K-vector spaces.

Definition 4.1.19. The *i-th extension group* $\mathrm{Ext}^i_A(N, L)$, $i \geq 1$, is defined to be the i-th cohomology group of this complex, i.e.,

$$\mathrm{Ext}^i_A(N, L) = \ker(\mathrm{Hom}_A(p_{m+1}, L))/\mathrm{im}(\mathrm{Hom}_A(p_m, L)).$$

One can show that this definition does not depend on the choice of the projective resolution; see [1, A.4].

Notation 4.1.20. We will denote by $[N, L]^i$ the dimension of the vector space $\mathrm{Ext}^i_A(N, L)$.

Remark 4.1.21. If A is hereditary, $\mathrm{Ext}^i_A(N, L) = 0$ for any $i \geq 2$, and for any A-modules L and N.

The Ext^1-groups provide very interesting information. Now will see that the vector space $\mathrm{Ext}^1(N, L)$ is isomorphic to the vector space of extensions of N by L.

Definition 4.1.22. An *extension* ξ of N by L is a short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$. Two extensions ξ and ξ' are called *equivalent* if there is a commutative diagram:

$$\begin{tikzcd}
\xi : 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "="] & M \arrow[r, "g"] \arrow[d, "\cong"] & N \arrow[r] \arrow[d, "="] & 0 \\
\xi' : 0 \arrow[r] & L \arrow[r, "f'"] & M' \arrow[r, "g'"] & N \arrow[r] & 0
\end{tikzcd}$$

An extension is *split*, or *trivial*, if the short exact sequence is *split*, i.e., if the extension is equivalent to the short exact sequence

$$0 \to L \to L \oplus N \to N \to 0.$$

It can be easily seen that this happens if and only if f is a section (equivalently, g is a retraction).

Given two extensions ξ and ξ' of N by L, one can define their sum $\xi + \xi'$; see [1, A.5] for details. The set of equivalence classes $\mathcal{E}_A(N, L)$ of extensions of N by L together with the sum of extensions is an abelian group, and the class of the split extension is the zero element of that group. We have the following result:

Theorem 4.1.23. *For any pair of A-modules L and N, there is a functorial isomorphism*

$$\chi : \mathcal{E}_A(N, L) \to \operatorname{Ext}^1_A(N, L).$$

Proof. See [1], A.5.9. □

4.2 Quivers and path algebras

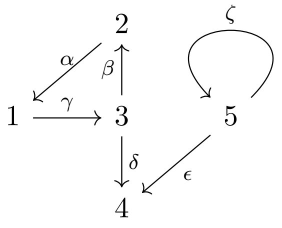
We recall the definition of quiver given in Section 1.1.

Definition 4.2.1. A (finite) *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two (finite) sets: Q_0 (whose elements are called *vertices*) and Q_1 (whose elements are called *arrows*), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$, respectively.

An arrow $\alpha \in Q_1$ of source $i = s(\alpha)$ and target $j = t(\alpha)$ is usually denoted by $\alpha : i \to j$. A quiver $Q = (Q_0, Q_1, s, t)$ is usually denoted briefly by $Q = (Q_0, Q_1)$, or even simply by Q.

Thus, in this section, in contrast to Definition 1.1.1 given in Section 1.1, we do not have any restrictions neither on the number of arrows between two points, nor on the existence of loops or oriented 2-cycles.

Example 4.2.2. The following quiver is given by $Q_0 = \{1, 2, 3, 4, 5\}$, $Q_1 = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$, $s(\alpha) = 2$, $s(\beta) = 3$, $s(\gamma) = 1$, $s(\delta) = 3$, $s(\epsilon) = 5$, $s(\zeta) = 5$, and $t(\alpha) = 1$, $t(\beta) = 2$, $t(\gamma) = 3$, $t(\delta) = 4$, $t(\epsilon) = 4$, $t(\zeta) = 5$.



Definition 4.2.3. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $i, j \in Q_0$. A *path* of length $l \geq 1$ with source i and target j (or, more briefly, from i to j) is a sequence

$$(i \mid \alpha_1, \alpha_2, \ldots, \alpha_l \mid j),$$

where $\alpha_k \in Q_1$ for all $1 \leq k \leq l$, and $s(\alpha_1) = i$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k < l$, and finally $t(\alpha_l) = b$. Such a path is denoted briefly by $\alpha_1 \alpha_2 \cdots \alpha_l$, and may be visualised as follows:

$$i = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \to \cdots \xrightarrow{\alpha_l} a_l = j.$$

It is also associated with each point $i \in Q_0$ a path of length $l = 0$, called the *trivial path at* i, and denoted by ϵ_i.

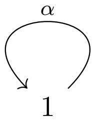
Thus the paths of lengths 0 and 1 are in bijective correspondence with the elements of Q_0 and Q_1, respectively. A path of length $l \geq 1$ is called an *oriented cycle* whenever its source and target coincide. A cycle of length 1 is called a *loop*. A quiver is called *acyclic* if it contains no oriented cycles.

Definition 4.2.4. Let Q be a quiver. The *path algebra* KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all paths $(i \mid \alpha_1, \dots, \alpha_l \mid j)$ of length $l \geq 0$ in Q and such that the product of two basis vectors $(a \mid \alpha_1, \dots, \alpha_l \mid b)$ and $(c \mid \beta_1, \dots, \beta_k \mid d)$ of KQ is defined by

$$(a \mid \alpha_1, \dots, \alpha_l \mid b)(c \mid \beta_1, \dots, \beta_k \mid d) = \delta_{bc}(a \mid \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_k \mid d),$$

where δ_{bc} denotes the Kronecker delta. In other words, the product of two paths $\alpha_1 \cdots \alpha_l$ and $\beta_1 \cdots \beta_k$ is equal to zero if $t(\alpha_l) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_k$ if $t(\alpha_l) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of KQ by distributivity.

Example 4.2.5. Let Q be the quiver



consisting of a single point and a single loop. The defining basis of the path algebra KQ is $\{\epsilon_1, \alpha, \alpha^2, \dots, \alpha^l, \dots\}$, and the multiplication of basis vectors is given by

$$\begin{aligned} &\epsilon_1 \alpha^l = \alpha^l \epsilon_1 = \alpha^l && \text{for all } l \geq 0, \text{ and} \\ &\alpha^l \alpha^k = \alpha^{l+k} && \text{for all } l, k \geq 0, \end{aligned}$$

where $\alpha^0 = \epsilon_1$. Thus KQ is isomorphic to the polynomial algebra $K[t]$ in one indeterminate t, the isomorphism being induced by the K-linear map such that

$$\epsilon_1 \mapsto 1 \quad \text{and} \quad \alpha \mapsto t.$$

Example 4.2.6. Let Q be the quiver

$$1 \xleftarrow{\alpha} 2 \ .$$

The path algebra KQ has as its defining basis the set $\{\epsilon_1, \epsilon_2, \alpha\}$. Clearly, KQ is isomorphic to the 2×2 lower triangular matrix algebra

$$\mathbb{T}_2(K) = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in K \right\},$$

where the isomorphism is induced by the K-linear map such that

$$\epsilon_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Definition 4.2.7. Let Q be a finite and connected quiver. The two-sided ideal of the path algebra KQ generated (as an ideal) by the arrows of Q is called the *arrow ideal* of KQ, and is denoted by R_Q.

Note that there is a direct sum decomposition

$$R_Q = KQ_1 \oplus KQ_2 \oplus \cdots \oplus KQ_l \oplus \cdots$$

of the K-vector space R_Q, where KQ_l is the subspace of KQ generated by the set Q_l of all paths of length l. In particular, the underlying K-vector space of R_Q is generated by all paths in Q of length $l \geq 1$. This implies that, for each $l \geq 1$,

$$R_Q^l = \bigoplus_{m \geq l} KQ_m,$$

and therefore R_Q^l is the ideal of KQ generated, as a K-vector space, by the set of all paths of length $\geq l$.

Theorem 4.2.8. *Let Q be a finite, connected, and acyclic quiver. The path algebra KQ is a basic and connected associative finite dimensional K-algebra with identity $1 = \sum_{i \in Q_0} \epsilon_i$, having the arrow ideal as radical, and the set $\{\epsilon_i \mid i \in Q_0\}$ as a complete set of primitive orthogonal idempotents.*

Proof. See [1], II.1.11. ☐

4.2.1 Admissible ideals and quotients of the path algebra

Let Q be a finite quiver. By Theorem 4.2.8, the path algebra KQ of Q is an associative algebra with an identity, and it is finite dimensional if and only if Q is acyclic. The aim of this section is to study the finite dimensional quotients of a not necessarily finite dimensional path algebra. We will see that they correspond to certain ideals which are called admissible.

Definition 4.2.9. Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra KQ. A two-sided ideal I of KQ is said to be *admissible* if there exists $m \geq 2$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

If I is an admissible ideal of KQ, the pair (Q, I) is said to be a *bound quiver.* The quotient algebra KQ/I is said to be the *algebra of the bound quiver* (Q, I), or simply a *bound quiver algebra.*

Remark 4.2.10. The condition $R_Q^m \subseteq I$ implies that the admissible ideal I contains all paths of length greater or equal to m, which guarantees that the bound quiver algebra is finite dimensional.

If the quiver Q does not contain any oriented cycles, then there always exists m such that R_Q^m is a subset of I, it suffices to take m greater than the length of the longest path in Q. Thus, if Q has no oriented cycles, an ideal is admissible if and only if it is contained in R_Q^2.

The condition $I \subseteq R_Q^2$ guarantees that we do not cut any arrows when we take the quotient, so that the bound quiver algebra is connected.

Examples 4.2.11. i) For any finite quiver Q and any $m \geq 2$, the ideal R_Q^m is admissible.

ii) The zero ideal is admissible in KQ if and only if Q is acyclic. Indeed, the zero ideal is admissible if and only if there exists $m \geq 2$ such that $R_Q^m = 0$, i.e. any product of m arrows in KQ is zero. This is the case if and only if Q is acyclic.

iii) Let Q be the quiver

$$\begin{tikzcd}
 & 2 \arrow[dl, "\beta"'] \\
1 & 4 \arrow[l, "\lambda"] \arrow[u, "\alpha"'] \arrow[d, "\gamma"] \\
 & 3 \arrow[ul, "\delta"]
\end{tikzcd}$$

The ideal $I_1 = \langle \alpha\beta - \gamma\delta \rangle$ of the K-algebra KQ is admissible, but $I_2 = \langle \alpha\beta - \lambda \rangle$ is not; indeed, $\alpha\beta - \lambda \notin R_Q^2$.

iv) Let Q be the quiver

$$\begin{tikzcd}
1 & 2 \arrow[l, shift right, "\beta"'] \arrow[l, shift left, "\gamma"] & 3 \arrow[l, "\alpha"']
\end{tikzcd}$$

Each of the ideals $I_1 = \langle \alpha\beta \rangle$ and $I_2 = \langle \alpha\beta - \alpha\gamma \rangle$ is clearly admissible. The bound quiver algebras kQ/I_1 and kQ/I_2 are isomorphic under the isomorphism $kQ/I_1 \to kQ/I_2$ induces by the correspondence $\epsilon_i \mapsto \epsilon_i$ for $i = 1, 2, 3$, $\alpha \mapsto \alpha$, $\beta \mapsto \beta - \gamma$, and $\gamma \mapsto \gamma$.

Lemma 4.2.12. *Let Q be a finite quiver. Every admissible ideal I of KQ is finitely generated.*

Proof. See [1], II.2.8. ☐

Definition 4.2.13. Let Q be a quiver. A *relation* in Q with coefficients in K is a K-linear combination of paths of length at least two having the same source and target. Thus, a relation ρ is an element of KQ such that

$$\rho = \sum_{i=1}^{m} \lambda_i w_i,$$

where the λ_i are scalars (not all zero) and the w_i are paths in Q of length at least 2 such that, if $i \neq j$, then the source and the target of w_i coincide with those of w_j.

If $(\rho_j)_{j \in J}$ is a set of relations for a quiver Q such that the ideal they generate $\langle \rho_j \mid j \in J \rangle$ is admissible, we say that the quiver Q is bound by the relations $(\rho_j)_{j \in J}$, or by the relations $\rho_j = 0$ for all $j \in J$.

Example 4.2.14. The quiver Q of Example 4.2.11 iii) is bound by the relation $\alpha\beta = \gamma\delta$.

Remark 4.2.15. For every admissible ideal I, there exists a set of relations that generate I. In fact, let I be an admissible ideal. Then I is generated by some elements $\sigma_1, \sigma_2, \ldots, \sigma_s$ (cf. Lemma 4.2.12). For every pair of vertices x, y, the element $e_x \sigma_i e_y$ is a linear combination of paths from x to y, hence a relation. Since $\sigma_i = \sum_{x,y} e_x \sigma_i e_y$, we have that the ideal I is also generated by the set of relations $\{e_x \sigma_i e_y \mid i = 1, 2, \ldots, s; x, y \in Q_0\}$.

Theorem 4.2.16. *Let Q be a finite connected quiver, R_Q be the arrow ideal of KQ, and I be an admissible ideal of KQ. The bound quiver algebra KQ/I is a basic and connected finite dimensional algebra with an identity, having R_Q/I as radical, and $\{\epsilon_i \mid i \in Q_0\}$ as complete set of primitive orthogonal idempotents.*

Proof. See [1], II.2.12. ☐

We recall that, for a K-algebra A, $\operatorname{rad} A$ denotes the *Jacobson radical* of A, i.e., the intersection of all the maximal right ideals in A.

Definition 4.2.17. Let A be a basic and connected finite dimensional K-algebra, and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. The *quiver of A*, also known as the *Gabriel quiver of A*, denoted by Q_A, is defined as follows:

i) the vertices of Q_A are the idempotents $e_1, \dots, e_n$;

ii) the number of arrows from e_i to e_j is the dimension of the K-vector space $e_i(\operatorname{rad} A/\operatorname{rad}^2 A)e_j$.

Remark 4.2.18. If $A = KQ/I$ for a finite quiver Q and an admissible ideal I, then $Q_A = Q$.

The following important result shows that every basic and connected finite dimensional K-algebra is actually a bound quiver algebra:

Theorem 4.2.19. *Let A be a basic and connected finite dimensional K-algebra. There exists an admissible ideal I of KQ_A such that $A \cong KQ_A/I$.*

Proof. See [1], II.3.7. ☐

Theorem 4.2.20. *Let $A \cong KQ_A/I$ be a basic and connected finite dimensional K-algebra. Then A is hereditary if and only if Q_A is acyclic and $I = 0$.*

Proof. See [1], VII.1.7. ☐

4.3 Representations and modules

As we saw in Section 4.2, quivers provide a convenient way to visualise finite dimensional algebras. In this section we will see how quivers may be used to visualise modules.

Definition 4.3.1. Let Q be a finite quiver. A *K-linear representation L of Q*, or more briefly a *representation of Q*, or simply a *Q-representation*, is defined by the following data:

i) to each point i in Q_0 is associated a K-vector space V_i;

ii) to each arrow $\alpha : i \to j$ in Q_1 is associated a K-linear map $\phi_\alpha : V_i \to V_j$.

Such a representation is denoted as $L = (V_i, \phi_\alpha)_{i\in Q_0, \alpha\in Q_1}$, or simply $L = (V_i, \phi_\alpha)$. It is called *finite dimensional* if each vector space V_i is finite dimensional.

Definition 4.3.2. Let $L = (V_i, \phi_\alpha)$ be a representation of Q, and let $|Q_0| = n$. The *dimension vector of L* is defined to be the vector

$$\mathbf{dim}\, L = \begin{bmatrix} \dim_K V_1 \\ \vdots \\ \dim_K V_n \end{bmatrix} \in \mathbb{Z}^n_{\geq 0}.$$

Example 4.3.3. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dl] \arrow[d] & \\ 1 & 3 \arrow[d] & 5 \arrow[dl] \\ & 4 & \end{tikzcd}$$

. A representation L of Q of dimension vector $(1,1,1,0,0)^T$ is given by

$$\begin{tikzcd} & k \arrow[dl, "1"'] \arrow[d, "1"] & \\ k & k \arrow[d, "0"] & 0 \arrow[dl, "0"'] \\ & 0 & \end{tikzcd}$$

Notation 4.3.4. Sometimes we will use indices of vertices with a non-zero dimensional vector space to indicate representations. For instance, for L of the previous example the shorthand is $\begin{smallmatrix} 2 \\ 13 \end{smallmatrix}$. With this notation we mean that $V_1 = V_2 = V_3 = k$, and there are two arrows going downward from 2, one going to 1 and one going to 3, which both carry the identity map.

Definition 4.3.5. Let $L = (V_i, \phi_\alpha)$ and $L' = (V'_i, \phi'_\alpha)$ be two representations of Q. A *morphism* of representations $f : L \to L'$ is a family $f = (f_i)_{i \in Q_0}$ of K-linear maps $(f_i : L_i \to L'_i)_{i \in Q_0}$ that are compatible with the structure maps ϕ_α and ϕ'_α, i.e., for each arrow $\alpha : i \to j$, we have $\phi'_\alpha f_i = f_j \phi_\alpha$ or, equivalently, the following square is commutative:

$$\begin{tikzcd} V_i \arrow[r, "\phi_\alpha"] \arrow[d, "f_i"] & V_j \arrow[d, "f_j"] \\ V'_i \arrow[r, "\phi'_\alpha"] & V'_j \end{tikzcd}$$

A morphism $f = (f_i) : L \to L'$ is an *isomorphism* if each f_i is bijective. The class of all representations that are isomorphic to a given representation L is called the *isoclass* of L.

Let $f : L \to L'$ and $g : L' \to L''$ be two morphisms of representations of Q, where $f = (f_i)_{i \in Q_0}$ and $g = (g_i)_{i \in Q_0}$. Their composition is defined to be the family $g \circ f = (g_i \circ f_i)_{i \in Q_0}$. Then $g \circ f$ is easily seen to be a morphism from L to L''.

This defines a category Rep(Q) of K-linear representations of Q. We denote by rep(Q) the full subcategory of Rep(Q) consisting of the finite dimensional representations of Q.

Example 4.3.6. Let Q be the Kronecker quiver $1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$.

A representation L of Q is given by

$$\begin{tikzcd}
K^2 & K \arrow[l, shift right, "\begin{bmatrix}1\\0\end{bmatrix}"'] \arrow[l, shift left, "\begin{bmatrix}0\\1\end{bmatrix}"]
\end{tikzcd}$$

Another representation L' is given by

$$\begin{tikzcd}
K^2 & K^2 \arrow[l, shift right, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"'] \arrow[l, shift left, "\begin{bmatrix}0&0\\1&0\end{bmatrix}"]
\end{tikzcd}$$

Both are finite dimensional. We have a morphism $L \to L'$ defined by

$$\begin{tikzcd}
K^2 \arrow[d, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"'] & K \arrow[l, shift right, "\begin{bmatrix}1\\0\end{bmatrix}"'] \arrow[l, shift left, "\begin{bmatrix}0\\1\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"] \\
K^2 & K^2 \arrow[l, shift right, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"'] \arrow[l, shift left, "\begin{bmatrix}0&0\\1&0\end{bmatrix}"]
\end{tikzcd}$$

Indeed, $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$.

Lemma 4.3.7. *Let Q be a finite quiver. Then* $\mathrm{Rep}_K(Q)$ *and* $\mathrm{rep}_K(Q)$ *are abelian K-categories.*

Proof. See [1], III.1.3. ☐

Definition 4.3.8. A representation $L \in \mathrm{rep}(Q)$ is called *indecomposable* if $L \neq 0$ and L cannot be written as a direct sum of two non-zero representations, that is, whenever $L \cong M \oplus N$ with $M, N \in \mathrm{rep}(Q)$, then $M = 0$ or $N = 0$.

Definition 4.3.9. A quiver Q is said to be of *finite representation type* if the number of isoclasses of indecomposable representations of Q is finite.

Quivers of finite representation type are listed by the following theorem which is one of the most important results of the theory:

Theorem 4.3.10 (Gabriel's Theorem)**.** *Let Q be a connected quiver. Then*

1. *Q is of finite representation type if and only if Q is any orientation of a Dynkin diagram of type A, D or E (cf. Figure 1.9).*
2. *If Q is any orientation of a Dynkin diagram Γ of type A, D or E, then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots of the root system corresponding to Γ.*

4.3.1 Representations of bound quivers

Definition 4.3.11. Let Q be a finite quiver and $L = (V_i, \phi_\alpha)$ be a representation of Q. For any non-trivial path $w = \alpha_1\alpha_2 \cdots \alpha_l$ from i to j in Q, the *evaluation of L on the path w* is the K-linear map from V_i to V_j defined by

$$\phi_w = \phi_{\alpha_l} \circ \phi_{\alpha_{l-1}} \circ \cdots \circ \phi_{\alpha_2} \circ \phi_{\alpha_1}.$$

The definition of evaluation extends to K-linear combinations of paths with a common source and a common target; thus let

$$\rho = \sum_{i=1}^{m} \lambda_i w_i$$

be such a combination, where $\lambda_i \in K$ and w_i is a path in Q, for each i, then

$$\phi_\rho = \sum_{i=1}^{m} \lambda_i \phi_{w_i}.$$

This allows us to define a notion of representation of a bound quiver.

Definition 4.3.12. Let Q be a finite quiver and I be an admissible ideal of KQ. A representation $L = (V_i, \phi_\alpha)$ of Q is said to be *bound by I*, or to *satisfy the relations in I*, if

$$\phi_\rho = 0, \qquad \text{for all relations} \qquad \rho \in I.$$

If L is a representation of Q bound by I, we say simple that L is a *representation of* (Q, I), or a (Q, I)*-representation.*

We denote by $\mathrm{Rep}(Q, I)$ (resp. by $\mathrm{rep}(Q, I)$) the full subcategory of $\mathrm{Rep}(Q)$ (resp. of $\mathrm{rep}(Q)$) consisting of the representations of (Q, I) (resp. of the finite dimensional representations of (Q, I)).

Example 4.3.13. Let Q be the quiver of Example 4.2.11 iii). We consider the Q-representations L given by

$$\begin{tikzcd}
 & K \arrow[dl, "\begin{bmatrix}1\\0\end{bmatrix}"'] \\
K^2 & 0 \arrow[l] \arrow[u] \arrow[d] \\
 & K \arrow[ul, "\begin{bmatrix}0\\1\end{bmatrix}"]
\end{tikzcd}$$

and L' given by

$$\begin{tikzcd}
 & 0 \arrow[dl] \\
K & K \arrow[l, "1"'] \arrow[u] \arrow[d, "1"] \\
 & K \arrow[ul, "1"]
\end{tikzcd}$$

With the notation of Example 4.2.11 iii), we have that L is a representation of (Q, I_1), while L' is not.

Theorem 4.3.14. *Let $A = KQ/I$, where Q is a finite connected quiver and I is an admissible ideal of KQ. There exists a K-linear equivalence of categories*

$$F : \operatorname{Mod} A \xrightarrow{\simeq} \operatorname{Rep}(Q, I)$$

that restricts to an equivalence of categories $F : \operatorname{mod} A \xrightarrow{\simeq} \operatorname{rep}(Q, I)$.

4.3.2 Representations given by diagonals of a polygon

Let $\bar{T} = \{\tau_1, \dots, \tau_n\}$ be a triangulation of $\mathbf{P}_{n+3}$, and let $Q(\bar{T})$ be the quiver associated to $\bar{T}$ (cf. Definition 1.1.10), so that there is an arrow from the vertex j to the vertex i if and only if τ_i and τ_j are sides of a triangle of $\bar{T}$, and τ_i is counterclockwise from τ_j. Let I be the admissible ideal generated by the set of relations given by all paths $i \to j \to k$ such that there exists an arrow $k \to i$.

Definition 4.3.15. Let $\bar{T}, Q(\bar{T})$ and I as above. Then $(Q(\bar{T}), I)$ is called a *cluster-tilted bound quiver of type A_n*.

Remark 4.3.16. If each triangle of $\bar{T}$ has at least one edge on the boundary of the polygon, then $I = 0$, and $Q(\bar{T})$ is an orientation of a Dynkin diagram of type A_n. Furthermore, all quivers of type A_n arise in this way.

Since $\bar{T}$ is a triangulation of the polygon, any other diagonal γ which is not already in $\bar{T}$ will cut through a certain number of diagonals in $\bar{T}$; in fact, any such diagonal γ is uniquely determined by the set of diagonals in $\bar{T}$ that γ crosses.

Definition 4.3.17. To such a diagonal γ, it is associated a representation $L_\gamma = (V_i, \phi_\alpha)$ of $(Q(\bar{T}), I)$ defined as follows:

$$V_i = \begin{cases} k \text{ if } \gamma \text{ crosses the diagonal } i; \\ 0 \text{ otherwise;} \end{cases}$$

and $\phi_\alpha = 1$ whenever $V_{s(\alpha)} = V_{t(\alpha)} = k$, and $\phi_\alpha = 0$ otherwise.

Example 4.3.18. The triangulation of the octagon in Figure 4.1 gives rise to the quiver $Q(\bar{T})$ of type A_5. In this case $I = 0$, since each triangle of the triangulation has at least one side on the boundary of the polygon.

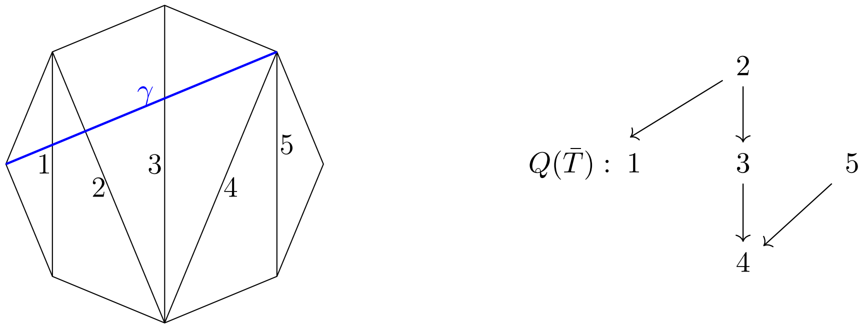
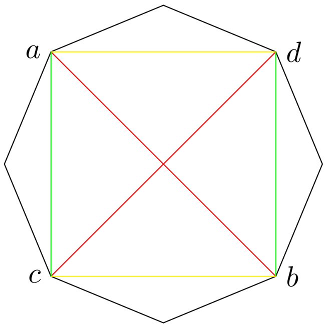


Figure 4.1: On the left, a triangulation $\bar{T}$ of the octagon, on the right the associated quiver $Q(\bar{T})$.

The diagonal γ crosses the diagonals 1,2,3 of $\bar{T}$, so the corresponding $Q(\bar{T})$-representation L_γ is given by

$$\begin{tikzcd}
& k \arrow[ld, "1"'] \arrow[d, "1"] & \\
k & k \arrow[d, "0"] & 0 \arrow[ld, "0"'] \\
& 0 &
\end{tikzcd}$$

The map $\gamma \mapsto L_\gamma$ is a bijection from the set of diagonals that are not in $\bar{T}$ and the set of isoclasses of indecomposable representations of $(Q(\bar{T}), I)$. Moreover, let (a, b) and (c, d) be the diagonals which connect vertices a and b, and c and d respectively. If (a, b) and (c, d) cross, then exactly one of the groups $\mathrm{Ext}^1(L_{(a,b)}, L_{(c,d)})$ and $\mathrm{Ext}^1(L_{(c,d)}, L_{(a,b)})$ is non-zero; otherwise they are both zero. Furthermore, if (a, b) and (c, d) cross, then the dimension of the non-trivial group is 1, and the middle term of the non-trivial extension is given by $L_{(a,d)} \oplus L_{(b,c)}$ (resp. $L_{(a,c)} \oplus L_{(b,d)}$) if there exists i such that the elementary lamination L_i associated to τ_i (cf. Definition 1.4.16) crosses both (a, d) and (b, c) (resp. (a, c) and (b, d)). For more details see [8].



4.3.3 Quiver Grassmannian

In this section we introduce a family of projective varieties which generalize the Grassmannian of vector spaces to the context of quiver representations. For the exposition we follow mainly [44].

Let Q be a finite quiver, I be an admissible ideal, and $L = (V_i, \phi_\alpha)$ be a representation of (Q, I).

Definition 4.3.19. A *subrepresentation* of L is a tuple $(W_i)_{i \in Q_0}$ such that

- each W_i is a subspace of V_i;
- for each arrow α in Q_1, $\phi_\alpha(W_{s(\alpha)}) \subset W_{t(\alpha)}$.

In this case, $L' = (W_i, \phi_{\alpha_{|W_{s(\alpha)}}})$ is a representation of (Q, I), and the canonical inclusion into L is a morphism of representations.

Grassmannians of vector spaces are projective varieties whose points parametrize subvector spaces of a given dimension. Quiver Granssmannians generalize this notion: they are projective varieties whose points parametrize subrepresentations of a given dimension vector.

Definition 4.3.20. Let $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector. The *quiver Grassmannian of L with dimension vector* $\mathbf{e}$ is the subset $\mathrm{Gr}_{\mathbf{e}}(L)$ of $\prod_{i\in Q_0} \mathrm{Gr}_{e_i}(V_i)$ of all points $(W_i)_{i\in Q_0}$ defining a subrepresentation of V.

The quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(L)$ is a Zariski-closed subset of $\prod_{i\in Q_0} \mathrm{Gr}_{e_i}(V_i)$, hence it is a projective variety.

Examples 4.3.21. i) If the quiver Q has only one vertex and no arrows, then representations of Q are just vector spaces, and their quiver Grassmannians are just usual Grassmannians.

ii) Let Q be the Kronecker quiver
$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd} .$$

Consider the representation L of Q given by

$$\begin{tikzcd} K^2 \arrow[r, shift left, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}"'] & K^2 \end{tikzcd} .$$

There are six dimension vectors for which the quiver Grassmannian of L is non-empty. The table below lists those dimension vectors and gives a variety isomorphic to the corresponding quiver Grassmannian.

$\mathbf{e}$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,2)$	$(2,2)$
$\mathrm{Gr}_{\mathbf{e}}(L)$	point	$\mathbb{P}^1$	point	point	$\mathbb{P}^1$	point

4.3.4 F-polynomials and g-vectors of quiver representations

In this section we give the definition of F-polynomial and $\mathbf{g}$-vector of a representation of a bound quiver (Q, I). Roughly, the F-polynomial can be seen as a generating function for counting subrepresentations of a given representation. This theory originates from [7], although F-polynomials and $\mathbf{g}$-vectors of quiver representations appeared later in [20]. For the exposition we follow mainly [44].

In this section, the base field K is the field $\mathbb{C}$ of complex numbers.

Definition 4.3.22. Let L be a representation of (Q, I). Its *F-polynomial* is

$$F_L(\mathbf{y}) := \sum_{\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}} \chi\big(\mathrm{Gr}_{\mathbf{e}}(L)\big)\mathbf{y}^{\mathbf{e}},$$

where

- $\mathbf{y}$ is the tuple of variables $(y_i \mid i \in Q_0)$;
- $\mathbf{y}^{\mathbf{e}} = \prod_{i \in Q_0} y_i^{e_i}$;
- $\mathrm{Gr}_{\mathbf{e}}(L)$ is the quiver Grassmannian of L with dimension vector $\mathbf{e}$ (cf. Definition 4.3.20);
- χ is the Euler-Poincaré characteristic.

It is easy to see that the F-polynomial of a representation L only depends on the isomorphism class of L.

Proposition 4.3.23. *Let L and M be two representations of (Q, I). Then $F_{L \oplus M} = F_L F_M$.*

Proof. See [7], [21]. □

Examples 4.3.24. i) Let Q be the quiver with one vertex and no arrows. Its path algebra is simply $\mathbb{C}$, and representations of Q are just vector spaces. Let $L = V$ be a d-dimensional vector space. Then

$$F_L(y) = \sum_{i=0}^{d} \binom{d}{i} y^i.$$

This can be seen by observing that, for $d = 1$, the F-polynomial is $1 + y$, and then by applying Proposition 4.3.23. As a corollary, we get a nice proof of the known fact that the Euler-Poincaré characteristic of the usual Grassmannian $\mathrm{Gr}_i(\mathbb{C}^d)$ is equal to $\binom{d}{i}$.

ii) Let Q and L be as in Example 4.3.21 ii). Then $F_L(y_1, y_2) = 1 + 2y_2 + y_2^2 + y_1y_2 + 2y_1y_2^2 + y_1^2y_2^2$.

iii) If L and L' are isomorphic, then $F_L = F_{L'}$. The converse is false: consider the Kronecker quiver $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd}$. Then the representations

$$\begin{tikzcd} L : \mathbb{C} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \mathbb{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L' : \mathbb{C} \arrow[r, shift left, "1"] \arrow[r, shift right, "0"'] & \mathbb{C} \end{tikzcd}$$

are not isomorphic, but their F-polynomials are both equal to $1 + y_2 + y_1y_2$.

iv) If F_L is an irreducible polynomial, then L is indecomposable. The converse is false: consider the quiver $\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$. Then the representation

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, shift left, "{\begin{bmatrix} 1 & 1 \end{bmatrix}}"] & \mathbb{C} \arrow[l, shift left, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] \end{tikzcd}$$

is indecomposable, but its F-polynomial is $1 + y_1 + y_1y_2 + y_1^2y_2 = (1 + y_1y_2)(1 + y_1)$.

v) An F-polynomial may have negative coefficients. An example for a quiver with two vertices and four arrows is given in [20, Example 3.6].

Definition 4.3.25. Let L be a representation of a bound quiver (Q, I) viewed as $A = KQ/I$-module. Let

$$0 \to L \to I_0 \to I_1$$

be a minimal injective presentation of L in $\operatorname{mod} A$, where $I_0 = \bigoplus_{i \in Q_0} I(i)^{a_i}$ and $I_1 = \bigoplus_{i \in Q_0} I(i)^{b_i}$ (cf. Theorem 4.1.7). Then the $\mathbf{g}$-vector of L is the vector $\mathbf{g}_L \in \mathbb{Z}^{Q_0}$ whose i-th coordinate is given by

$$(\mathbf{g}_L)_i := b_i - a_i.$$

Example 4.3.26. Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[d, "\beta"'] & 3 \arrow[l, "\alpha"'] \\
2 \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

and let $I = \langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle$. Then $I(1) = \begin{smallmatrix}3\\1\end{smallmatrix}$, $I(2) = \begin{smallmatrix}1\\2\end{smallmatrix}$, $I(3) = \begin{smallmatrix}2\\3\end{smallmatrix}$. Let L be the representation of (Q, I) given by

$$\begin{tikzcd}
\mathbb{C} \arrow[d] & 0 \arrow[l] \\
0 \arrow[ur] &
\end{tikzcd}$$

Then a minimal injective presentation of L is

$$0 \to L \to I(1) \to I(3).$$

Therefore, $\mathbf{g}_L = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Definition 4.3.27. Let $B = B(Q)$ be the exchange matrix of a quiver Q (cf. Definition 1.1.12), and let $n = |Q_0|$. Let L be a representation of (Q, I). The *cluster character of* L, also known as *Caldero-Chapoton map*, is the Laurent polynomial

$$CC(L) = \sum_{\mathbf{e} \in \mathbb{Z}^n_{\geq 0}} \chi(\mathrm{Gr}_{\mathbf{e}}(L)) \mathbf{x}^{B\mathbf{e} + \mathbf{g}_L} \mathbf{y}^{\mathbf{e}} \in \mathbb{Z}[y_1, \dots, y_n][x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Remark 4.3.28. One can obtain F_L by specializing $CC(L)$ at $x_1 = \cdots = x_n = 1$. On the other hand, $\mathbf{g}$ is the multi-degree of $CC(L)$ with respect to the $\mathbb{Z}^n$-grading in $\mathbb{Z}[y_1, \dots, y_n][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ given by $\deg(x_i) = \mathbf{e_i}$ and $\deg(y_j) = -\mathbf{b}_j$, where $\mathbf{e_i}$ is the i-th vector of the standard basis of $\mathbb{Z}^n$, and $\mathbf{b}_j$ is the j-th column of B.

4.4 Auslander–Reiten theory

As we saw in the previous sections, quivers provide a convenient way to visualise finite dimensional algebras and their modules. However, to actually compute the indecomposable modules and the homomorphisms between them, we need other tools. Particularly useful in this context are the notions of irreducible morphisms and almost split sequences.

In this section A is a finite dimensional K-algebra. In order to define almost split sequences, we first need to introduce the concept of almost split morphisms.

Definition 4.4.1. Let L, M be modules in $\operatorname{mod} A$. A morphism $f : L \to M$ is called *left minimal almost split* if

i) f is not a section, i.e., there is no morphism $h : M \to L$ such that $h \circ f = 1_L$;

ii) for each morphism $u : L \to U$ in $\operatorname{mod} A$ which is not a section, there exists a morphism $u' : M \to U$ such that $u' \circ f = u$;

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[dl, "u'"] \\ U & \end{tikzcd}$$

iii) if $h : M \to M$ is such that $h \circ f = f$, then h is an automorphism of M.

Similarly, a morphism $g : M \to N$ is called *right minimal almost split* if

i) g is not a retraction, i.e., there is no morphism $h : N \to M$ such that $g \circ h = 1_N$;

ii) for each morphism $v : V \to N$ in $\operatorname{mod} A$ which is not a retraction, there exists a morphism $v' : V \to M$ such that $g \circ v' = v$;

$$\begin{tikzcd} & V \arrow[dl, "v'"'] \arrow[d, "v"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

iii) if $h : M \to M$ is such that $g \circ h = g$, then h is an automorphism of M.

Definition 4.4.2. A short exact sequence in $\operatorname{mod} A$

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called an *almost split sequence* (or Auslander-Reiten sequence) if f is a left minimal almost split morphism, and g is a right minimal almost split morphism.

Remark 4.4.3. An almost split sequence is not split since, by definition, f is not a section and g is not a retraction.

Definition 4.4.4. A morphism $f : X \to Y$ in $\operatorname{mod} A$ is called *irreducible* if

i) f is not a section;

ii) f is not a retraction;

iii) whenever $f = g \circ h$ for some morphisms $h : X \to Z$ and $g : Z \to Y$, then either h is a section or g is a retraction.

Lemma 4.4.5. *A short exact sequence in* $\operatorname{mod} A$

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is an almost split sequence if and only if L *and* N *are indecomposable and* f *and* g *are irreducible morphisms.*

Proof. See [1], IV.1.13. ☐

Definition 4.4.6. Let $\mathcal{C}$ be an additive K-category. The *Jacobson radical* $\operatorname{rad}_{\mathcal{C}}$ is defined to be the class of all morphisms such that for any pair of objects X, Y in $\mathcal{C}$, $\operatorname{rad}_{\mathcal{C}}(X,Y)$ is equal to

$$\big\{ f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \mid (1_X - h \circ f) \text{ is an isomorphism for all } h \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \big\}.$$

Remark 4.4.7. Note that if $f \in \operatorname{rad}_{\mathcal{C}}(X,Y)$, then f is not an isomorphism, since otherwise $1_X - f^{-1} \circ f = 0$.

Now we will characterize irreducible morphisms in terms of the Jacobson radical of $\operatorname{mod} A$. For simplicity, we will write rad instead of $\operatorname{rad}_{\operatorname{mod} A}$.

Let $\operatorname{rad}^2(X,Y)$ be the span of all morphisms $f : X \to Y$ for which we have a factorization $f = g \circ h$, with $g \in \operatorname{rad}(Z,Y)$ and $h \in \operatorname{rad}(X,Z)$, for some module Z. Clearly $\operatorname{rad}^2(X,Y) \subset \operatorname{rad}(X,Y)$.

Lemma 4.4.8. *Let* X*,* Y *be indecomposable* A*-modules. Then* $f : X \to Y$ *is irreducible if and only if* $f \in \operatorname{rad}(X,Y) \setminus \operatorname{rad}^2(X,Y)$.

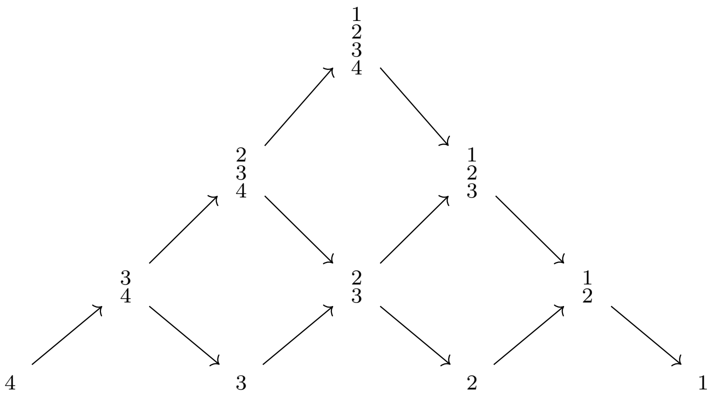
Proof. See [47], Lemma 7.8. ☐

We are now able to give the definition of Auslander-Reiten quiver.

Definition 4.4.9. Let A be a finite dimensional algebra. The *Auslander-Reiten quiver of* A is the quiver whose vertices are the isomorphism classes $[X]$ of indecomposable A-modules X; and if X and Y are indecomposable A-modules, then the number of arrows from $[X]$ to $[Y]$ is equal to the dimension of the K-vector space $\operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$.

Remark 4.4.10. If follows from Lemma 4.4.8 that there exists an arrow $[X] \to [Y]$ if and if there exists an irreducible morphism $X \to Y$.

Example 4.4.11. Let Q be the quiver $1 \to 2 \to 3 \to 4$. The Auslander-Reiten quiver of KQ is:



4.4.1 Auslander–Reiten Translation

Let L be an A-module. The Auslander–Reiten translation τL of L is defined as follows: start with a minimal projective presentation of L

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0.$$

Thus $P_0 \xrightarrow{p_0} L$ is a projective cover and $P_1 \xrightarrow{p_1} \ker(p_0)$ is a projective cover. Applying the Nakayama functor yields an exact sequence

$$0 \to \ker(\nu p_1) \to \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu L \to 0.$$

Definition 4.4.12. The *Auslander–Reiten translate* of L is $\tau L := \ker(\nu p_1)$.

Example 4.4.13. Let Q be the quiver $1 \to 2 \to 3 \to 4$ of Example 4.4.11. Let $L = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$. A minimal projective resolution of L is given by

$$0 \to P(4) = {\scriptstyle 4} \to P(1) = \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \to L \to 0.$$

Applying the Nakayama functor we get the exact sequence

$$0 \to \begin{smallmatrix}2\\3\\4\end{smallmatrix} \to I(4) = \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \to I(1) = {\scriptstyle 1} \to \nu L \to 0.$$

Therefore, we have that $\tau L = \begin{smallmatrix}2\\3\\4\end{smallmatrix}$.

Dually, start with a minimal injective presentation

$$0 \to L \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1$$

and apply ν^{-1} to get an exact sequence

$$0 \to \nu^{-1} L \xrightarrow{\nu^{-1} i_0} \nu^{-1} I_0 \xrightarrow{\nu^{-1} i_1} \nu^{-1} I_1 \to \operatorname{coker}(\nu^{-1} i_1) \to 0.$$

Definition 4.4.14. The *inverse Auslander–Reiten translate* of L is $\tau^{-1} L = \operatorname{coker}(\nu^{-1} i_1)$.

Remark 4.4.15. If L is projective, then $\tau L = 0$, since in the minimal projective presentation above $L = P_0$ and $P_1 = 0$. Dually, if L is injective, then $\tau^{-1} L = 0$, since in the minimal injective presentation above $L = I_0$ and $I_1 = 0$.

The following result, due to Auslander and Reiten, is the main existence theorem for almost split sequences:

Theorem 4.4.16. *i) For any indecomposable non-projective A-module N, there exists an almost split sequence $0 \to \tau N \to E \to N$ in $\operatorname{mod} A$.*

ii) For any indecomposable non-injective A-module L, there exists an almost split sequence $0 \to L \to F \to \tau^{-1} L \to 0$ in $\operatorname{mod} A$.

Proof. See [1], IV.3.1. □

Now we will state the Auslander–Reiten formulas. This is a fundamental result which, on the one hand, is of conceptual nature, since it describes a relation between short exact sequences and morphisms in the module category, on the other hand, it provides a powerful computational tool, since it allows us to calculate Ext^1 in terms of morphisms.

Definition 4.4.17. Let $P(X,Y)$ be the set of all morphisms $f \in \mathrm{Hom}(X,Y)$ such that f factors through a projective A-module, and define

$$\underline{\mathrm{Hom}}(X,Y) := \mathrm{Hom}(X,Y)/P(X,Y).$$

Dually, let $I(X,Y)$ be the set of all morphisms $f \in \mathrm{Hom}(X,Y)$ such that f factors through an injective A-module, and define

$$\overline{\mathrm{Hom}}(X,Y) := \mathrm{Hom}(X,Y)/I(X,Y).$$

Theorem 4.4.18 (Auslander–Reiten formulas)**.** *Let X, Y be A-modules. Then there are isomorphisms*

$$\mathrm{Ext}^1(X,Y) \cong D\underline{\mathrm{Hom}}(\tau^{-1}Y,X) \cong D\overline{\mathrm{Hom}}(Y,\tau X)$$

that are functorial in both variables, that is, the following functors are isomorphic:

$$\begin{gathered}\mathrm{Ext}^1(-,Y) \cong D\underline{\mathrm{Hom}}(\tau^{-1}Y,-) \cong D\overline{\mathrm{Hom}}(Y,\tau -)\\ \mathrm{Ext}^1(X,-) \cong D\underline{\mathrm{Hom}}(\tau^{-1}-,X) \cong D\overline{\mathrm{Hom}}(-,\tau X)\end{gathered}$$

Proof. See [1], IV.2.13. □

Corollary 4.4.19. *Let X, Y be A-modules. Then*

i) If $\mathrm{pd}\, X \leq 1$ and Y is arbitrary, then there exists a K-linear isomorphism

$$\mathrm{Ext}^1(X,Y) \cong D\mathrm{Hom}(Y,\tau X).$$

ii) If $\mathrm{id}\, Y \leq 1$ and X is arbitrary, then there exists a K-linear isomorphism

$$\mathrm{Ext}^1(X,Y) \cong D\mathrm{Hom}(\tau^{-1}Y,X).$$

Corollary 4.4.20. *Let X, Y be non-projective indecomposable A-modules such that $\mathrm{pd}\, X \leq 1$ and $\mathrm{id}\, \tau Y \leq 1$. Then*

$$\mathrm{Hom}(\tau Y,\tau X) \cong \mathrm{Hom}(Y,X).$$

Chapter 5

Categorification of cluster algebras of type B and C through symmetric quivers

In this last chapter, mostly based on [17], we present a categorification of cluster algebras of type B and C through symmetric quivers and their representations. First, in Section 5.1, we give an introduction to symmetric representation theory. Then, in Section 5.4, we explain how each cluster variable of type B_n and C_n corresponds to a symmetric indecomposable representation of a cluster tilted bound symmetric quiver of type A_{2n-1}. Finally, in Section 5.5, we show a categorical interpretation of the formula relating cluster variables of type B to cluster variables of type A (cf. Theorem 2.2.2 of Chapter 2), in the case of quivers without oriented cycles. This result (cf. Theorem 5.5.15) relies on the cluster multiplication formula of Cerulli Irelli, Esposito, Franzen, Reineke [14] for acyclic quivers. We also present a conjectural extension of this formula to the case of bound quivers, and provide several examples supporting our conjecture.

We work over the field $K = \mathbb{C}$ of complex numbers.

5.1 Symmetric quivers and their representations

The representation theory of symmetric quivers has been developed by Derksen and Weyman in [19], as well as Boos and Cerulli Irelli in [5].

Definition 5.1.1. A *symmetric quiver* is a pair (Q, σ), where Q is a finite quiver and σ is an involution of Q_0 and of Q_1 which reverses the orientation of arrows.

Example 5.1.2. Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $Q' = 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$ be two quivers of type A_3. Then Q is symmetric, with the involution σ given by $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(\alpha) = \beta$, while Q' is not symmetric, i.e., it cannot be endowed with the structure of a symmetric quiver.

Definition 5.1.3. Let (Q, σ) be a symmetric quiver. Let $I \subset kQ$ be an admissible ideal such that $\sigma(I) = I$. Then (Q, I, σ) is called a *bound symmetric quiver*, and the pair $(A = kQ/I, \sigma)$ is called a *symmetric quiver algebra*.

Remark 5.1.4. If $(A = KQ/I, \sigma)$ is a symmetric quiver algebra, then A is isomorphic via σ to its opposite $A^{\mathrm{op}} = KQ^{\mathrm{op}}/\sigma(I)$, where Q^{op} is the opposite quiver of Q, i.e., the quiver obtained from Q by reversing the direction of all arrows.

Definition 5.1.5. A *symmetric representation* of a bound symmetric quiver (Q, I, σ) is a triple $(V_i, \phi_\alpha, \langle \cdot, \cdot \rangle)$, where (V_i, ϕ_α) is a representation of (Q, I), $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric or skew-symmetric scalar product on $V = \bigoplus_{i \in Q_0} V_i$ such that its restriction to $V_i \times V_j$ is 0 if $j \neq \sigma(i)$, and $\langle \phi_\alpha(v), w \rangle + \langle v, \phi_{\sigma(\alpha)}(w) \rangle = 0$, for every $\alpha : i \to j \in Q_1$, $v \in V_i$, $w \in V_{\sigma(j)}$. If $\langle \cdot, \cdot \rangle$ is symmetric (resp. skew-symmetric), $(V_i, \phi_\alpha, \langle \cdot, \cdot \rangle)$ is called *orthogonal* (resp *symplectic*).

Remark 5.1.6. If $\mathbf{d}$ is the dimension vector of a symmetric representation $(V_i, \phi_\alpha, \langle \cdot, \cdot \rangle)$ of a bound symmetric quiver (Q, I, σ), then $d_i = d_{\sigma(i)}$. If the dimension vector $\mathbf{d}$ of a (Q, I)-representation has this property, we say that it is *symmetric.*

Definition 5.1.7. If $(V_i, \phi_\alpha, \langle \cdot, \cdot \rangle)$ and $(V_i', \phi_\alpha', \langle \cdot, \cdot \rangle')$ are two orthogonal (resp. two symplectic) representations of a bound symmetric quiver Q, then their direct sum is given by $(V_i \oplus V_i', \phi_\alpha \oplus \phi_\alpha', \langle \cdot, \cdot \rangle + \langle \cdot, \cdot \rangle')$. A symmetric representation is called *indecomposable* if it is non-trivial, and it is not isomorphic to the direct sum of two non-trivial symmetric representations.

Remark 5.1.8. Let Q be a symmetric quiver. The full subcategory of $\mathrm{Rep}(Q)$ generated by orthogonal (resp. symplectic) representations of Q is an additive category which is not abelian.

Example 5.1.9. Let $Q : \ 1 \longleftarrow 2 \longleftarrow 3$. Let $f : \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ be the morphism between orthogonal representations of Q given by:

$$\begin{tikzcd}
K \arrow[d, "1"] & K^2 \arrow[l, "{\begin{bmatrix}1 0\end{bmatrix}}"'] \arrow[d, "{\begin{bmatrix}1 0\end{bmatrix}}"] & K \arrow[l, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] \arrow[d, "0"] \\
K & K \arrow[l, "1"'] & K \arrow[l, "1"']
\end{tikzcd}$$

We have that $\ker(f) = \begin{smallmatrix}3\\2\end{smallmatrix}$, which is not orthogonal.

Definition 5.1.10. Let $L = (V_i, \phi_\alpha)$ be a representation of a bound symmetric quiver Q. The *twisted dual* of L is the Q-representation $\nabla L = (\nabla V_i, \nabla \phi_\alpha)$, where $\nabla V_i = V^*_{\sigma(i)}$ and $\nabla \phi_\alpha = -\phi^*_{\sigma(\alpha)}$ ($*$ denotes the linear dual).

Remark 5.1.11. The twisted dual is a contravariant exact endofunctor on $\mathrm{Rep}(Q)$. Moreover, if L is symmetric, the scalar product $\langle \cdot, \cdot \rangle$ induces an isomorphism from $V = \bigoplus_{i \in Q_0} V_i$ to $\nabla V = \bigoplus_{i \in Q_0} \nabla V_i$.

Proposition 5.1.12. *The functor ∇ fulfills*

$$\nabla \tau = \tau^- \nabla,$$

where τ is the Auslander-Reiten translation (cf. Definition 4.4.12).

Proof. See [19], Proposition 3.4. □

The following result shows that every indecomposable symmetric representation is uniquely determined by the ∇-orbit of an ordinary indecomposable representation:

Lemma 5.1.13. *Let M be an indecomposable symmetric representation of a bound symmetric quiver Q. Then, one and only one of the following three cases can occur:*

- *(I) M is indecomposable as a Q-representation; in this case, M is called of type (I), for "indecomposable";*
- *(S) there exists an indecomposable Q-representation L such that $M = L \oplus \nabla L$ and $L \ncong \nabla L$; in this case, M is called of type (S), for "split";*
- *(R) there exists an indecomposable Q-representation L such that $M = L \oplus \nabla L$ and $L \cong \nabla L$; in this case, M is called of type (R) for "ramified".*

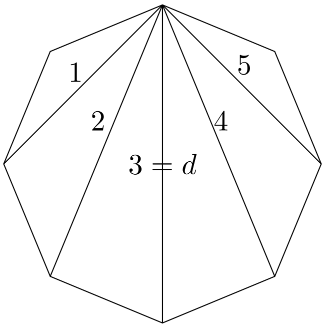
Proof. See [5], Lemma 2.10. $\square$

5.2 ρ-orbits as orthogonal and symplectic representations

Let $\mathbf{P}_{2n+2}$ be the regular polygon with $2n+2$ vertices. Let d be a diameter of $\mathbf{P}_{2n+2}$. Let ρ denote the reflection of the polygon along d. It induces an action on the diagonals of $\mathbf{P}_{2n+2}$. If T' is a ρ-invariant triangulation of $\mathbf{P}_{2n+2}$, then $(Q(T'), \sigma_\rho)$ (cf. Definition 1.1.10) is a cluster-tilted bound symmetric quiver of type A_{2n-1} (cf. Definition 4.3.15), with involution σ_ρ induced by ρ.

Example 5.2.1. Let ρ be the reflection of the octagon along the diameter d in Figure 5.1. Let σ_ρ be the involution of $Q(T')$ defined by $\sigma_\rho(1) = \rho(1) = 5$, $\sigma_\rho(2) = \rho(2) = 4$, $\sigma_\rho(3) = \rho(3) = 3$, and $\sigma_\rho(\alpha) = \delta$, $\sigma_\rho(\beta) = \gamma$. Then $(Q(T'), \sigma_\rho)$ is a symmetric quiver of type A_5.

$T' =$



$$Q(T') : 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4 \xleftarrow{\delta} 5$$

Figure 5.1: A ρ-invariant triangulation of $\mathbf{P}_8$ and the associated quiver.

Moreover, if $[a, b]^\rho = \{\alpha_1, \alpha_2\}$ is a ρ-orbit, and α_1 corresponds to the indecomposable representation of $Q(T')$ L_{α_1} (cf. Definition 4.3.17), then α_2 corresponds to $L_{\alpha_2} = \nabla L_{\alpha_1}$. In fact, if we denote by $\mathbf{d}_{\alpha_i}$ the vector of indices of diagonals of T' crossed by α_i, i.e. the dimension vector of L_{α_i}, we have that both $\mathbf{d}_{\alpha_1}$ and $\mathbf{d}_{\alpha_2}$ are not symmetric, while $\mathbf{d}_{\alpha_1} + \mathbf{d}_{\alpha_2}$ is. It follows from Lemma 5.1.13 that $L_{\alpha_1} \oplus L_{\alpha_2}$ is symmetric indecomposable of type S, so $L_{\alpha_2} = \nabla L_{\alpha_1}$.

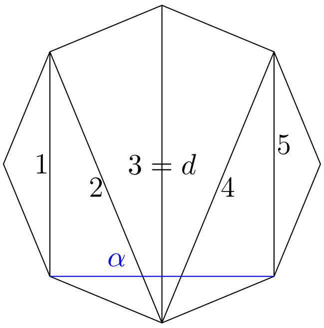
On the other hand, if $[a, b]^\rho = \{\alpha\}$, then α corresponds to the ∇-invariant indecomposable representation of $Q(T')$ L_α, since $\mathbf{d}_\alpha$ is symmetric.

Let $T' = \{\tau_1, \dots, \tau_{2n-1}\}$ be a ρ-invariant triangulation of $\mathbf{P}_{2n+2}$. Then it has $n-1$ ρ-invariant pairs of diagonals not orthogonal to d, and exactly one ρ-invariant diagonal τ_n. We have two cases to consider.

$\tau_n = d$ In this case $Q(T')$ has a fixed vertex n and no fixed arrows. Therefore, every ρ-invariant diagonal α which is not in T' crosses τ_n. So L_α is orthogonal indecomposable of type I, while $L_\alpha \oplus L_\alpha$ is symplectic indecomposable of type R, since in the latter case the non-zero vector space at vertex n of the quiver must be a symplectic space, so it must have dimension 2.

Example 5.2.2.

$T' =$



$Q(T')$:

$$\begin{tikzcd}
 & 2 \arrow[dl] \arrow[d] & \\
1 & 3 \arrow[d] & 5 \arrow[dl] \\
 & 4 &
\end{tikzcd}$$

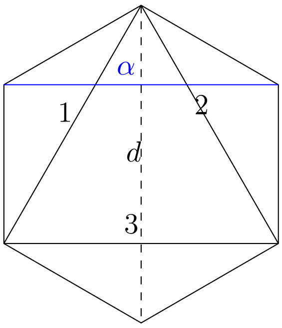
$\tau_n \neq d$ In this case $Q(T')$ has a fixed vertex n and a fixed arrow $\beta : i \to j$. Therefore, every ρ-invariant diagonal α which is not in T' crosses i and j, while it cannot cross τ_n. Let $\{v\}$ be a basis of the 1-dimensional vector space of L_α at vertex i and let $\{w\}$ be a basis of the 1-dimensional vector space of L_α at vertex j. If $(L_\alpha, \langle\cdot,\cdot\rangle)$ is a symmetric representation of $Q(T')$, then by definition

$$\langle w, v\rangle = \langle f_\beta(v), v\rangle = -\langle v, f_{\sigma_\rho(\beta)}(v)\rangle = -\langle v, f_\beta(v)\rangle = -\langle v, w\rangle. \tag{5.2.1}$$

Since $\langle\cdot,\cdot\rangle$ is a non-degenerate scalar product, it must be skew-symmetric. It follows from Lemma 5.1.13 that L_α is symplectic indecomposable of type I, while $L_\alpha \oplus L_\alpha$ is orthogonal indecomposable of type R.

Example 5.2.3.

$T' =$



$Q(T')$:

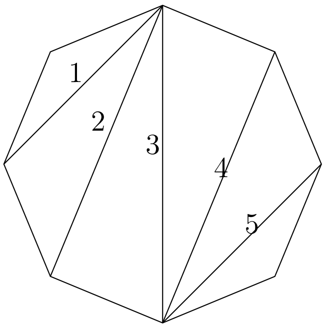
$$\begin{tikzcd}
 & 2 \arrow[dl] \\
1 \arrow[r] & 3 \arrow[u]
\end{tikzcd}$$

5.3 From θ-orbits to ρ-orbits

Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d. Then the quiver $Q(T)$ associated to T (cf. Definition 1.1.10) is not symmetric.

Example 5.3.1. Let T be θ-invariant triangulation of the octagon in Figure 5.2. Then the quiver $Q(T)$ is not symmetric.

$T =$



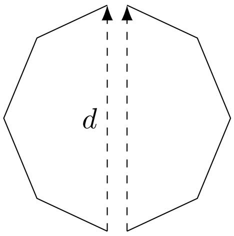
$$Q(T) : 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5$$

Figure 5.2: A θ-invariant triangulation of $\mathbf{P}_8$ and the associated quiver.

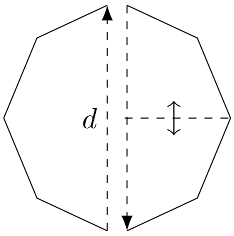
In order to get a symmetric quiver, we define an involution on the polygon depending on d, that we call F_d:

Definition 5.3.2. F_d is the operation on $\mathbf{P}_{2n+2}$ which consists of the following three steps in order:

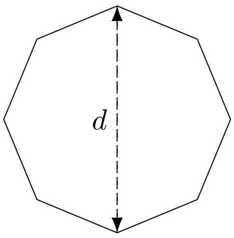
1) cut the polygon along d;



2) reflect the right part with respect to the axis of symmetry of d;



3) glue again the right part along d.



Remark 5.3.3. F_d induces an action on isotopy classes of diagonals of the polygon.

Lemma 5.3.4. *Under the bijection F_d, θ-orbits correspond to ρ-orbits. In particular, diameters correspond to ρ-invariant diagonals, while pairs of centrally symmetric diagonals correspond to ρ-invariant pairs of diagonals which are not orthogonal to d.*

Proof. Let $[a, b]$ be a θ-orbit. We have three cases to consider:

i) (a, b) is a diameter (illustrated in Figure 5.3);

ii) $[a, b]$ is a pair of centrally symmetric diagonals which cross d (illustrated in Figure 5.4);

iii) $[a, b]$ is a pair of centrally symmetric diagonals which do not cross d (illustrated in Figure 5.5).

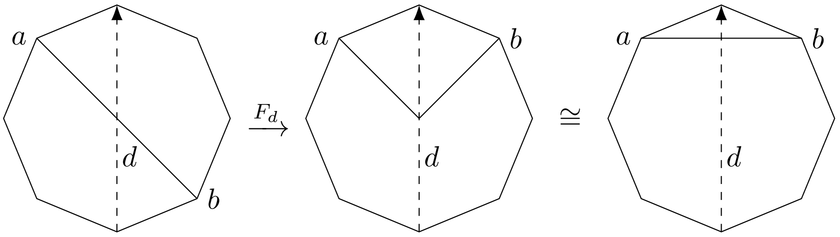


Figure 5.3: The action of F_d on the diameter (a, b).

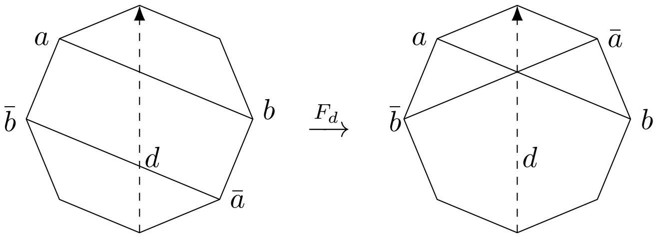


Figure 5.4: The action of F_d on the θ-orbit $[a, b]$ whose diagonals cross d.

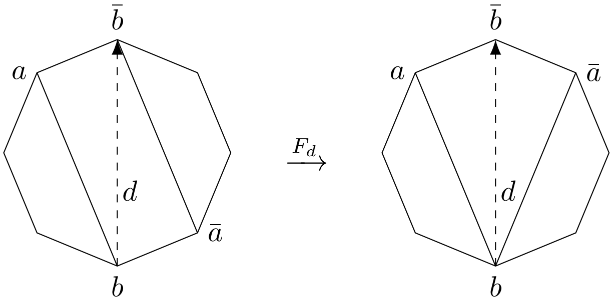


Figure 5.5: The action of F_d on the θ-orbit $[a, b]$ whose diagonals do not cross d.

□

Remark 5.3.5. Let T' be the element in the isotopy class of $F_d(T)$ which is also a triangulation. It follows from Lemma 5.3.4 that T' is a ρ-invariant triangulation of $\mathbf{P}_{2n+2}$ which contains the diameter d. Then $Q(T')$ is a cluster-tilted bound symmetric quiver of type A_{2n-1} with a fixed vertex corresponding to d and no fixed arrows (cf. Section 5.2).

5.4 Categorification of cluster algebras of type B and C

In Section 1.4.2, we have seen that clusters of cluster algebras of type B_n and C_n are in bijection with triangulations of $\mathbf{P}_{2n+2}$ invariant under the action of the 180°-rotation θ of the polygon. Furthermore, cluster variables correspond to the orbits of the action induced by θ on the diagonals of $\mathbf{P}_{2n+2}$. In this section we will explain how each cluster variable of type B_n and C_n corresponds to a symmetric indecomposable representation of a cluster tilted bound symmetric quiver of type A_{2n-1} (cf. Definition 4.3.15 and Section 5.1).

Let T be a θ-invariant triangulation of $\mathbf{P}_{2n+2}$ with oriented diameter d. In Section 5.3 we have seen that, if T' is the element in the isotopy class of $F_d(T)$ which is also a triangulation, then T' is a ρ-invariant triangulation of $\mathbf{P}_{2n+2}$ which contains the diameter d, and $Q(T')$ (cf. Definition 1.1.10) is a cluster-tilted bound symmetric quiver of type A_{2n-1} with a fixed vertex corresponding to d and no fixed arrows (cf. Remark 5.3.5).

Now let $\mathcal{A} = \mathcal{A}_\bullet(T)^B$ be the cluster algebra of type B_n with principal coefficients in T (cf. Definition 2.1.4), and let $[a, b]$ be a θ-orbit which is not in T, and x_{ab} the cluster variable of $\mathcal{A}$ which corresponds to it. If $F_d([a, b]) = \{\alpha\}$ consists of only one ρ-invariant diagonal, then x_{ab} corresponds to the orthogonal indecomposable $Q(T')$-representation L_α of type I (cf. Section 5.2). Otherwise, $F_d([a, b]) = \{\alpha_1, \alpha_2\}$. In this case, x_{ab} corresponds to $L_{\alpha_1} \oplus L_{\alpha_2} = L_{\alpha_1} \oplus \nabla L_{\alpha_1}$ which is an orthogonal indecomposable $Q(T')$-representation of type S (cf. Section 5.2).

Moreover, the restriction on θ-orbits (cf. Definition 2.1.1) corresponds to an operation on orthogonal indecomposable $Q(T')$-representations defined in the following way:

Definition 5.4.1. Let $M = (V_i, \phi_\alpha, \langle \cdot, \cdot \rangle)$ be an orthogonal indecomposable $Q(T')$- representation. Then the *restriction* of M is $\mathrm{Res}(M) = (\mathrm{Res}(V)_i, \mathrm{Res}(\phi)_\alpha)$, where $\mathrm{Res}(V)_i = V_i$ if $i \leq n$, and $\mathrm{Res}(V)_i = 0$ otherwise; while $\mathrm{Res}(\phi)_\alpha = \phi_\alpha$ if $\alpha : i \to j$, with $i, j \leq n$, and $\mathrm{Res}(\phi)_\alpha = 0$ otherwise. In other words, if $[a, b]$ is the θ-orbit which corresponds to M, and $\mathrm{Res}([a, b]) = \{\gamma_1, \gamma_2\}$ (resp. $\mathrm{Res}([a, b]) = \{\gamma\}$), then $\mathrm{Res}(M) = L_{\gamma_1} \oplus L_{\gamma_2}$ (resp. $\mathrm{Res}(M) = L_\gamma$).

Remark 5.4.2. Note that $\mathrm{Res}(M)$ is no longer orthogonal. Moreover, $\mathrm{Res}(M)$ is a representation of the quiver associated to the triangulation of $\mathbf{P}_{n+3}$ obtained from T' by identifying the vertices which lie on the right of d, i.e. $\bar{T} = \mathrm{Res}(T') = \mathrm{Res}(T)$ (the part of T on the left of d is equal to the one of T' on the left of d).

On the other hand, let $\mathcal{A} = \mathcal{A}_\bullet(T)^C$ be the cluster algebra of type C with principal coefficients in T (cf. Definition 2.1.4). Let $[a, b]$ be a θ-orbit, and let x_{ab} be the cluster variable of $\mathcal{A}$ which corresponds to $[a, b]$. If $F_d([a, b]) = \{\alpha\}$ consists of only one ρ-invariant diagonal, then x_{ab} corresponds to the symplectic indecomposable $Q(T')$-representation $L_\alpha \oplus L_\alpha$ of type R (cf. Section 5.2). Otherwise, $F_d([a, b]) = \{\alpha_1, \alpha_2\}$. As before, in this case x_{ab} corresponds to the symplectic indecomposable $Q(T')$-representation $L_{\alpha_1} \oplus L_{\alpha_2} = L_{\alpha_1} \oplus \nabla L_{\alpha_1}$ of type S.

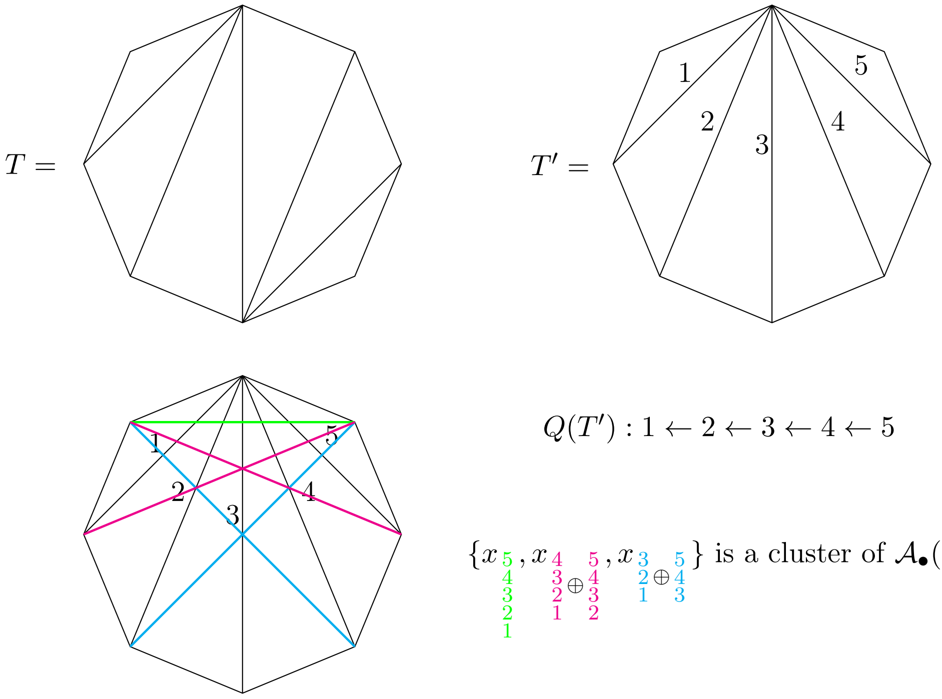
Moreover, the rotated restriction on θ-orbits (cf. Definition 2.3.1) corresponds to the operation on symplectic $Q(T')$-representations defined in the following way:

Definition 5.4.3. Let M be an indecomposable symplectic representation of $Q(T')$, and let $[a, b]$ be the θ-orbit that corresponds to M. If $\widetilde{\mathrm{Res}}([a, b]) = \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ (resp. $\widetilde{\mathrm{Res}}([a, b]) = \{\tilde{\gamma}\}$), then

$\widetilde{\operatorname{Res}}(M) = L_{\tilde{\gamma}_1} \oplus L_{\tilde{\gamma}_2}$ (resp. $\widetilde{\operatorname{Res}}(M) = L_{\tilde{\gamma}}$).

Remark 5.4.4. Note that $\widetilde{\operatorname{Res}}(M)$ is no longer symplectic. Moreover, as for $\operatorname{Res}(M)$, $\widetilde{\operatorname{Res}}(M)$ is a representation of the quiver associated to the triangulation $\bar{T} = \operatorname{Res}(T') = \operatorname{Res}(T)$ of $\mathbf{P}_{n+3}$.

Example 5.4.5.



$Q(T') : 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$

$\{x_{\substack{5\\4\\3\\2\\1}}, x_{\substack{4\\3\\2\\1} \oplus \substack{5\\4\\3\\2}}, x_{\substack{3\\2\\1} \oplus \substack{5\\4\\3}}\}$ is a cluster of $\mathcal{A}_{\bullet}(T)^B$, while

$\{x_{\substack{5\\4\\3\\2\\1} \oplus \substack{5\\4\\3\\2\\1}}, x_{\substack{4\\3\\2\\1} \oplus \substack{5\\4\\3\\2}}, x_{\substack{3\\2\\1} \oplus \substack{5\\4\\3}}\}$ is a cluster of $\mathcal{A}_{\bullet}(T)^C$.

Figure 5.6: An example of cluster for a cluster algebra of type B_3 and C_3.

Finally, on the one hand, Theorem 2.2.2 and Theorem 2.3.7 give two formulas (the former for type B_n and the latter for type C_n) to express each cluster variable associated to a θ-orbit in terms of cluster variables of type A_n, on the other hand Theorem 3.2.14 and Theorem 3.2.27 (the former for type B_n and the latter for type C_n) give its cluster expansion in the initial cluster variables. It follows from the above correspondence that, given a cluster-tilted bound symmetric quiver Q of type A_{2n-1} with no fixed arrows, they allow us to express the type B_n (resp. type C_n) cluster variable that corresponds to an orthogonal (resp. symplectic) indecomposable representation of Q, on the one hand in terms of (ordinary) representations of $Q(\bar{T})$, where $\bar{T} = \operatorname{Res}(T')$, and T' is the triangulation of $\mathbf{P}_{2n+2}$ such that $Q = Q(T')$, on the other hand in terms of the initial cluster variables. In other words, we get a Caldero-Chapoton like map (see [7]) from the categories of orthogonal and symplectic representations of cluster tilted bound symmetric quivers of type A_{2n-1} (with no fixed arrows) to cluster algebras of type B_n and C_n.

Remark 5.4.6. The techniques presented in this section could be used to produce a categorification of other classes of skew-symmetrizable cluster algebras through the representation theory of symmetric quivers. For example, they could provide an alternative categorification of non-skew-symmetric cluster algebras associated by Felikson, Shapiro and Tumarkin [23] to surfaces with marked points and order-2 orbifold points. These algebras have been categorified in the work of Geuenich and Labardini-Fragoso [35, 36] by species with potential.

5.5 Categorical interpretation of the cluster expansion formula in the acyclic case

In this section, we show a categorical interpretation of the formula relating cluster variables of type B to cluster variables of type A (cf. Theorem 2.2.2 of Chapter 2), in the case of quivers without oriented cycles. The result we are going to present (cf. Theorem 5.5.15) relies on the cluster multiplication formula of Cerulli Irelli, Esposito, Franzen, Reineke [14] for acyclic quivers. We will also see a conjectural extension of this formula to the case of bound quivers, and provide some examples supporting our conjecture.

5.5.1 Cluster multiplication formula for acyclic quivers

In the following Q is an acyclic quiver. Let

$$\xi : 0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} S \to 0$$

be a short exact sequence in $\operatorname{rep}(Q)$. This induces the map

$$\Psi^\xi : \mathrm{Gr}_{\mathbf{e}}(Y) \to \coprod_{\mathbf{f}+\mathbf{g}=\mathbf{e}} \mathrm{Gr}_{\mathbf{f}}(X) \times \mathrm{Gr}_{\mathbf{g}}(S) : \; N \mapsto (\iota^{-1}N, \pi(N))$$

between quiver Grassmannians (cf. Definition 4.3.20).

By taking the preimage $\mathcal{S}^\xi_{\mathbf{f},\mathbf{g}} = (\Psi^\xi)^{-1}(\mathrm{Gr}_{\mathbf{f}}(X) \times \mathrm{Gr}_{\mathbf{g}}(S))$ of each piece, we get the algebraic map

$$\Psi^\xi_{\mathbf{f},\mathbf{g}} : \mathcal{S}^\xi_{\mathbf{f},\mathbf{g}} \to \mathrm{Gr}_{\mathbf{f}}(X) \times \mathrm{Gr}_{\mathbf{g}}(S) : \; N \mapsto (\iota^{-1}N, \pi(N)).$$

Remark 5.5.1. If ξ is split, the map Ψ^ξ is crearly surjective for any $\mathbf{e}$.

Definition 5.5.2. An element $\xi \in \mathrm{Ext}^1(S, X)$ is *generating* if $\mathrm{Ext}^1(S, X) = \mathbb{C}\xi$.

In other words $\xi \in \mathrm{Ext}^1(S, X)$ is generating if either $[S, X]^1 = 0$ and $\xi = 0$, or $[S, X]^1 = 1$ and $\xi \neq 0$ (cf. 4.1.20).

Let X, S be representations of Q such that $[S, X]^1 = 1$. In [14], Lemma 27, it is shown that the following subrepresentations of X and S, respectively, are well defined:

$$X_S := \max\{N \subset X \,|\, [S, X/N]^1 = 1\} \subset X, \quad S^X := \min\{N \subset S \,|\, [N, X]^1 = 1\} \subseteq S.$$

In other words, the subrepresentation X_S is the maximal subrepresentation of X such that the "push-out" sequence

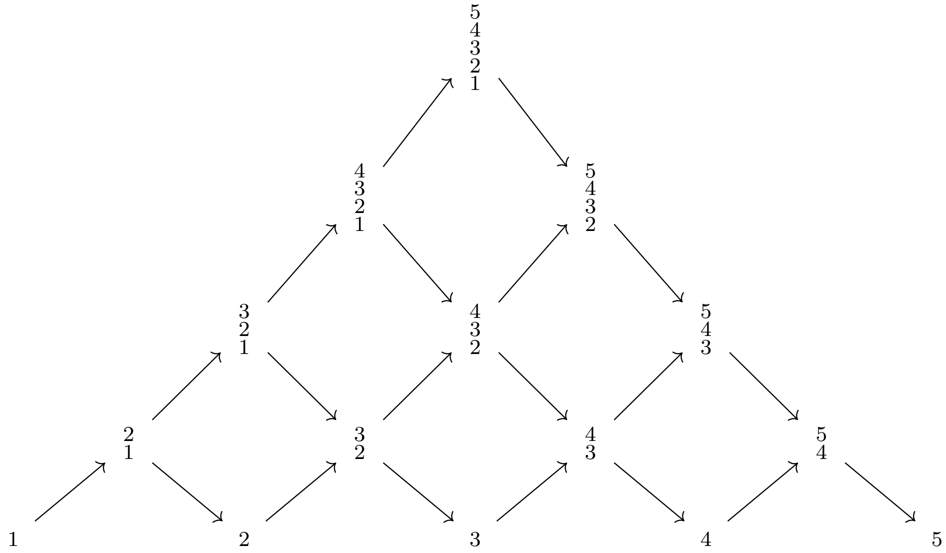
$$\begin{tikzcd}
\xi : 0 \arrow[r] & X \arrow[r] \arrow[d, two heads, "p"] & Y \arrow[r] \arrow[d, two heads] & S \arrow[r] \arrow[d, "="] & 0 \\
p_*\xi : 0 \arrow[r] & X/X_S \arrow[r] & \bar{Y} \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

does not split. Dually, the subrepresentation $S^X \subseteq S$ is the minimal subrepresentation such that the “pull-back” sequence

$$\begin{tikzcd}
i^*\xi : 0 \arrow[r] & X \arrow[r] \arrow[d, "="] & \tilde{Y} \arrow[r] \arrow[d, hook] & S^X \arrow[r] \arrow[d, hook, "i"] & 0 \\
\xi : 0 \arrow[r] & X \arrow[r] & \bar{Y} \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

does not split. If ξ is almost split (cf. Definition 4.4.2), then this description implies that $S^X = S$ and $X_S = 0$.

Example 5.5.3. Let Q be the quiver $1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$. The Auslander-Reiten quiver of KQ is:



We consider the short exact sequence

$$\xi : 0 \to X = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \to Y = \begin{smallmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \, {\scriptstyle 3} \to S = \begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix} \to 0.$$

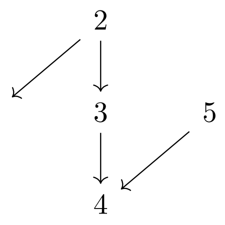
Then $X_S = {\scriptstyle 1}$ and $S^X = \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$.

Definition 5.5.4. A generating extension $\xi : 0 \to X \to Y \to S \to 0$ is called a *generalized almost split* sequence if $S^X = S$ and $X_S = 0$.

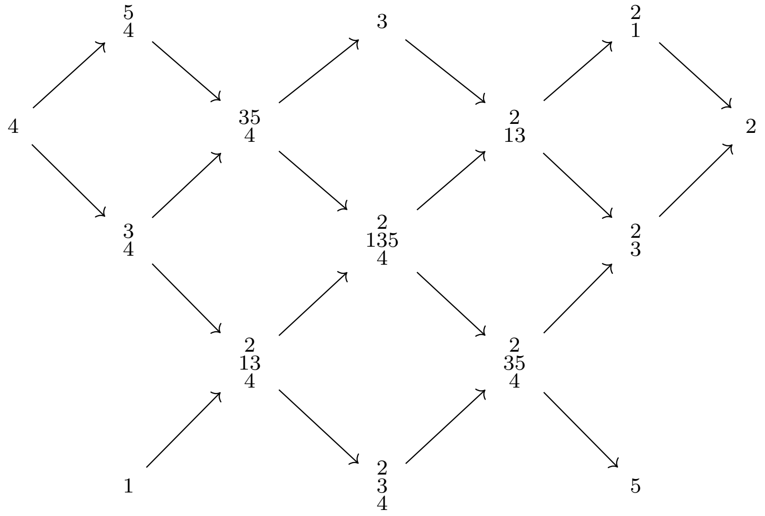
Remark 5.5.5. An almost split sequence is generalized almost split.

Example 5.5.6. We exhibit an example of generalized almost split sequence which is not almost split for a type A quiver. This is a counterexample for [14, Example 30] which asserts that, for a quiver of type A, generalized almost split sequences are almost split.

Let Q be the quiver



The Auslander-Reiten quiver of KQ is:



We consider the short exact sequence

$$\xi : 0 \to \substack{3\\4} \to 3 \oplus \substack{2\\3\\4} \to \substack{2\\3} \to 0.$$

We have that ξ is generalized almost split, but not almost split.

If $\xi \in \operatorname{Ext}^1(S, X)$ is generating and non-split, then, by the Auslander-Reiten formulas (cf. Theorem 4.4.18 and Corollary 4.4.19), $[X, \tau S] = [\tau^{-1}X, S] = 1$. Let $f : X \to \tau S$ and $g : \tau^{-1}X \to S$ be two non-zero maps, then

Lemma 5.5.7.

$$X_S = \ker(f) \quad \textit{and} \quad S^X = \operatorname{im}(g). \tag{5.5.1}$$

Proof. See [14], Lemma 31. □

It turns out that a pair $(N_1, N_2) \in \mathrm{Gr}_{\mathbf{f}}(X) \times \mathrm{Gr}_{\mathbf{g}}(S)$ is *not* in the image of $\Psi^{\xi}_{\mathbf{f},\mathbf{g}}$ if and only if $N_1 \subseteq X_S$ and $N_2 \supseteq S^X$. We hence have the following result:

Theorem 5.5.8. *Let $0 \neq \xi \in \operatorname{Ext}^1(S, X)$ be a generating extension. Then*

$$\operatorname{im}(\Psi^{\xi}_{\mathbf{f},\mathbf{g}}) = (\mathrm{Gr}_{\mathbf{f}}(X) \times \mathrm{Gr}_{\mathbf{g}}(S)) \setminus \big(\mathrm{Gr}_{\mathbf{f}}(X_S) \times \mathrm{Gr}_{\mathbf{g}-\mathbf{dim}\, S^X}(S/S^X)\big).$$

Proof. See [14], Theorem 32. □

Corollary 5.5.9. *Let $0 \neq \xi \in \operatorname{Ext}^1(S, X)$ be a generating extension. Then, for $(\mathbf{f}, \mathbf{g}) \neq (\mathbf{0}, \mathbf{dim}\, S^X)$, the map $\Psi^{\xi}_{\mathbf{f},\mathbf{g}}$ is surjective if and only if ξ is generalized almost split.*

Definition 5.5.10. A representation L of Q is called *rigid* if $\operatorname{Ext}^1(L, L) = 0$.

Let X, S be representations of Q such that $[S, X]^1 = 1$. Then there exists an exact sequence $0 \to X/X_S \to \tau S^X \to I \to 0$, where I is either injective or zero [14, Lemma 31]. Let $|Q_0| = n$, let $I = I(1)^{f_1} \oplus I(2)^{f_2} \oplus \cdots \oplus I(n)^{f_n}$ be the indecomposable decomposition of I (cf. Theorem 4.1.7), and let $\mathbf{f} = (f_1, \cdots, f_n)$. Let $B = B(Q)$ be the exchange matrix of Q (cf. Definition 1.1.12), and let CC be the cluster character (cf. Definition 4.3.27). We have the following multiplication formula for cluster characters:

Theorem 5.5.11. *Let $\xi \in \operatorname{Ext}^1(S, X)$ be a generating extension with middle term Y. Then*

$$CC(X)CC(S) = CC(Y) + \mathbf{y}^{\boldsymbol{dim} S^X} CC(X_S \oplus S/S^X)\mathbf{x}^{\mathbf{f}}, \tag{5.5.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Moreover, if $\operatorname{Ext}^1(X, S) = 0$, and both X and S are rigid and indecomposable, then formula 5.5.10 is an exchange relation between the cluster variables $CC(X)$ and $CC(S)$ for the cluster algebra $\mathcal{A}_{\bullet}(B)$ with principal coefficients at the initial seed $(\mathbf{x}, \mathbf{y}, B)$ (cf. Definition 1.3.8).

Proof. See [14], Theorem 67. □

Remark 5.5.12. The multiplication formula of 5.5.10 can be interpreted as a categorification of the exchange relations in the cluster algebra associated with Q.

Remark 5.5.13. Let Q be a symmetric quiver, and let L be a representation of Q such that $[\nabla L, L]^1 = 1$. By definition, $L_{\nabla L} = \ker(L \to \tau \nabla L)$, and $\nabla L^L = \operatorname{im}(\tau^{-1} L \to \nabla L)$. So we have that $\nabla(L_{\nabla L}) = \operatorname{coker}(\tau^{-1} L \to \nabla L) = \nabla L / \nabla L^L$, where we have used the fact that $\nabla \tau = \tau^{-1} \nabla$ (cf. Proposition 5.1.12). Therefore, $L_{\nabla L} \oplus \nabla L / \nabla L^L$ is a symmetric representation of Q.

Example 5.5.14. Let Q, ξ, X, S as in Example 5.5.3. Then it follows from Theorem 5.5.11 that

$$CC(X)CC(S) = CC\left(\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right)CC\left(\begin{smallmatrix}5\\4\\3\end{smallmatrix}\right) = CC\left(\begin{smallmatrix}5\\4\\3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}\right) + y_3 y_4 CC\left(\begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}5\end{smallmatrix}\right) \tag{5.5.3}$$

In this case $\mathbf{f}$ is zero in 5.5.10, since $\tau S^X = X/X_S$. Since both X and S are rigid and indecomposable, 5.5.3 is an exchange relation for the cluster algebra $\mathcal{A}_{\bullet}(B)$ of type A_5, where $B = B(Q)$.

5.5.2 Categorical interpretation of Theorem 2.2.2 in the acyclic case

In this section, Q is a symmetric quiver of type A_{2n-1}, i.e., a symmetric orientation of the Dynkin diagram of type A_{2n-1}.

Observe that, if L is a representation of Q, then

$$CC(L) = \sum_{\{\mathbf{e} = \mathbf{dim} L' \in \mathbb{Z}^n \mid L' \subseteq L\}} \mathbf{y}^{\mathbf{e}} \mathbf{x}^{B\mathbf{e} + \mathbf{g}_L}, \tag{5.5.4}$$

where $B = B(Q)$ (cf. Definition 1.1.12), since $Gr_{\mathbf{e}}(M)$ is either empty or a point.

Let T' be the triangulation of $\mathbf{P}_{2n+2}$ such that $Q = Q(T')$ (cf. Definition 1.1.10 and Remark 4.3.16). Since Q has a fixed vertex n and no fixed arrows, then T' contains a diameter $d = \tau_n$, and if ρ is the reflection along d, T' is ρ-invariant (cf. Section 5.2). Let $[a, b] = \{(a, b), (\bar{b}, \bar{a})\}$ be a θ-orbit such that each diagonal of $[a, b]$ crosses d, so $\operatorname{Res}([a, b]) = \{(a, *), (\bar{b}, *)\}$, and let $(a, \bar{a}), (\bar{b}, b)$ be the diameters starting in a and $\bar{b}$ respectively, so that $\operatorname{Res}([a, \bar{a}]) = \{(a, *)\}$ and $\operatorname{Res}([b, \bar{b}]) = \{(\bar{b}, *)\}$, see Figure 5.7 (the restriction is with respect to d). Therefore $[a, b]$ corresponds via F_d (cf. Definition 5.3.2) to $L_{(a, \rho(\bar{b}))} \oplus \nabla L_{(a, \rho(\bar{b}))}$, with $[\nabla L_{(a, \rho(\bar{b}))}, L_{(a, \rho(\bar{b}))}]^1 = 1$ because $(a, \rho(\bar{b}))$ and $(\bar{b}, \rho(a))$ cross (cf. Section 4.3.2). Since the elementary lamination L_n associated to $\tau_n = d$ crosses both $(a, \rho(a))$ and

$(\bar{b}, \rho(\bar{b}))$, then there exists a non-degenerate square in the Auslander-Reiten quiver of Q from $L_{(a,\rho(\bar{b}))}$ to $\nabla L_{(a,\rho(\bar{b}))} = L_{(\bar{b},\rho(a))}$, whose middle vertices are the ∇-invariant Q-representations $L_{(a,\rho(a))}$ and $L_{(\bar{b},\rho(\bar{b}))}$. In other words, there is the non-split short exact sequence

$$0 \to L_{(a,\rho(\bar{b}))} \to L_{(a,\rho(a))} \oplus L_{(\bar{b},\rho(\bar{b}))} \to \nabla L_{(a,\rho(\bar{b}))} \to 0. \tag{5.5.5}$$

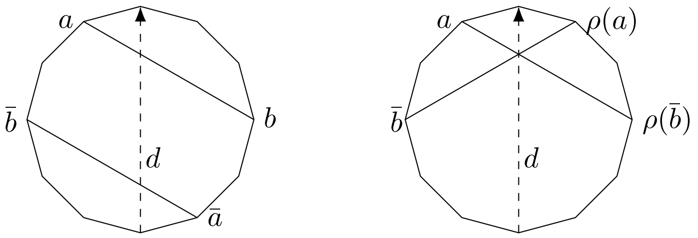


Figure 5.7: The action of F_d on the θ-orbit $[a, b]$ whose diagonals cross d.

By Theorem 5.5.11, we have that

$$F_{L_{(a,\rho(\bar{b}))} \oplus \nabla L_{(a,\rho(\bar{b}))}} = F_{L_{(a,\rho(a))} \oplus L_{(\bar{b},\rho(\bar{b}))}} + \mathbf{y}^{\mathbf{dim} \nabla L_{(a,\rho(\bar{b}))}^{L_{(a,\rho(\bar{b}))}}} F_{(L_{(a,\rho(\bar{b}))})_{\nabla L_{(a,\rho(\bar{b}))}} \oplus \nabla L_{(a,\rho(\bar{b}))} / \nabla L_{(a,\rho(\bar{b}))}^{L_{(a,\rho(\bar{b}))}}}.$$

On the other hand, by Proposition 1.4.18,

$$F_{L_{(a,\rho(\bar{b}))} \oplus \nabla L_{(a,\rho(\bar{b}))}} = F_{L_{(a,\rho(a))} \oplus L_{(\bar{b},\rho(\bar{b}))}} + \mathbf{y}^{\mathbf{d}_{a\rho(a),\bar{b}\rho(\bar{b})}} F_{L_{(a,\bar{b})} \oplus \nabla L_{(a,\bar{b})}}.$$

Thus

$$\mathbf{dim} \nabla L_{(a,\rho(\bar{b}))}^{L_{(a,\rho(\bar{b}))}} = \mathbf{d}_{a\rho(a),\bar{b}\rho(\bar{b})},$$

and

$$(L_{(a,\rho(\bar{b}))})_{\nabla L_{(a,\rho(\bar{b}))}} \oplus \nabla L_{(a,\rho(\bar{b}))} / \nabla L_{(a,\rho(\bar{b}))}^{L_{(a,\rho(\bar{b}))}} = L_{(a,\bar{b})} \oplus \nabla L_{(a,\bar{b})}.$$

Let $\mathcal{A}_{\bullet}^{B}(T)$ be the cluster algebra of type B_n with principal coefficients in the θ-invariant triangulation T of $\mathbf{P}_{2n+2}$ in the isotopy class of $F_d(T')$ (cf. Definition 2.1.4). Let M be an orthogonal indecomposable representation of Q. We denote by F_M and $\mathbf{g}_M$ the F-polynomial and the $\mathbf{g}$-vector respectively of the cluster variable of $\mathcal{A}_{\bullet}^{B}(T)$ that corresponds to M (cf. Section 5.4). On the other hand, $F_{\mathrm{Res}(M)}$ and $\mathbf{g}_{\mathrm{Res}(M)}$ are the F-polynomial and the $\mathbf{g}$-vector respectively of the Q-representation $\mathrm{Res}(M)$ (cf. Definitions 4.3.22, 4.3.25 and 5.4.1). Then, it follows from the above discussion that Theorem 2.2.2 can be reformulated as:

Theorem 5.5.15. *Let M be an orthogonal indecomposable Q-representation. If $Res(M) = (V_i, \phi_\alpha)$ is indecomposable as Q-representation, then*

$$F_M = F_{Res(M)}, \tag{5.5.6}$$

and

$$\mathbf{g}_M = \begin{cases} D\mathbf{g}_{Res(M)} & \textit{if } \boldsymbol{dim} V_n = 0; \\ D\mathbf{g}_{Res(M)} + \mathbf{e}_n & \textit{if } \boldsymbol{dim} V_n \neq 0. \end{cases} \tag{5.5.7}$$

Otherwise, $M = L \oplus \nabla L$ with dim $Ext^1(\nabla L, L) = 1$, and there exists a non-split short exact sequence

$$0 \to L \to G_1 \oplus G_2 \to \nabla L \to 0,$$

where G_1 and G_2 are orthogonal indecomposable Q-representations of type I. Then

$$F_M = F_{Res(M)} - \mathbf{y}^{Res(\boldsymbol{dim} \nabla L^L)} F_{Res(L_{\nabla L} \oplus \nabla L / \nabla L^L)}, \tag{5.5.8}$$

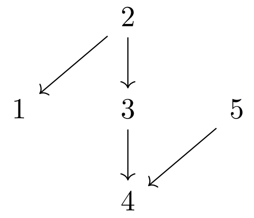
and

$$\mathbf{g}_M = D(\mathbf{g}_{Res(M)} + \mathbf{e}_n). \tag{5.5.9}$$

Remark 5.5.16. Observe that on the right hand sides of 5.5.6, 5.5.7, 5.5.8, 5.5.9 we have only F-polynomials and $\mathbf{g}$-vectors of ordinary type A quiver representations.

Remark 5.5.17. By Remark 5.5.13, $L_{\nabla L} \oplus \nabla L / \nabla L^L$ is an orthogonal indecomposable representation of Q.

Example 5.5.18. Let Q be the quiver



of Example 4.3.18 and Example 5.5.6. We compute the F-polynomial and the $\mathbf{g}$-vector of Example 2.2.4 using Theorem 5.5.15. Let $M = L \oplus \nabla L = \begin{smallmatrix} 35 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 13 \end{smallmatrix}$ be an orthogonal indecomposable Q-representation. We have the short exact sequence

$$0 \to \begin{smallmatrix} 35 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 135 \\ 4 \end{smallmatrix} \oplus 3 \to \begin{smallmatrix} 2 \\ 13 \end{smallmatrix} \to 0.$$

Since the sequence is almost split, $L_{\nabla L} = 0$ and $\nabla L^L = \nabla L$. Therefore

$$F_M = F_{\mathrm{Res}(\begin{smallmatrix} 35 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 13 \end{smallmatrix})} - \mathbf{y}^{\mathrm{Res}(\mathbf{dim} \begin{smallmatrix} 2 \\ 13 \end{smallmatrix})} = F_3 F_{\begin{smallmatrix} 2 \\ 13 \end{smallmatrix}} - y_1 y_2 y_3 = y_1 y_2 y_3^2 + y_1 y_3^2 + 2 y_1 y_3 + y_3^2 + y_1 + 2 y_3 + 1.$$

On the other hand, the $\mathbf{g}$-vector is

$$\mathbf{g}_M = D(\mathbf{g}_{\mathbf{g}_{\mathrm{Res}(M)}} + \mathbf{e}_3) = D(\mathbf{g}_{3 \oplus \begin{smallmatrix} 2 \\ 13 \end{smallmatrix}} + \mathbf{e}_3) = D(\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) = D(\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}) = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}.$$

5.5.3 Conjecture in the cyclic case

The definitions of X_S and S^X, and the proof of Theorem 5.5.11 heavily rely on the fact that, if Q has no oriented cycles, the algebra $A = KQ$ is hereditary (cf. Theorem 4.2.20), and therefore $\mathrm{Ext}^i_A(N, L) = 0$ for any $i \geq 2$, and for any A-modules L and N.

We would like to extend the multiplication formula of 5.5.10 to finite dimensional non-hereditary algebras. This would allow us to have a categorical interpretation of Theorem 2.2.2 even in the case where $Q(T')$ has oriented cycles, i.e., when T' has at least one triangle with no edges on the boundary of the polygon.

In this section, we present a conjectural extension of the formula of Cerulli Irelli, Esposito, Franzen, Reineke to the case of bound quivers, and provide some examples supporting our conjecture.

Let (Q, I) be a bound quiver. Let X, S be representations of (Q, I) such that $[S, X]^1 = 1$. The idea is to define X_S and S^X directly as

$$X_S := \ker(f) \subset X; \qquad\qquad S^X := \mathrm{im}(g) \subseteq S,$$

where $f : X \to \tau S$ is the only (up to scalar) non-zero morphism from X to τS which does not factor through an injective KQ/I-module (cf. Theorem 4.4.18), and $g : \tau^{-1}X \to S$ is the only (up to scalar) non-zero morphism from $\tau^{-1}X$ to S which does not factor through a projective KQ/I-module (cf. Theorem 4.4.18). We have the following conjecture on the product of the F-polynomials of X and S (cf. Definition 4.3.22):

Conjecture 5.5.19. *Let (Q, I) be a bound quiver. Let X, S be representations of (Q, I) such that $[S, X]^1 = 1$. Let $\xi \in \mathrm{Ext}^1(S, X)$ be a generating extension with middle term Y. Then*

$$F_X F_S = F_Y + \mathbf{y}^{\boldsymbol{dim} S^X} F_M, \tag{5.5.10}$$

where

- *$M = X_S \oplus S/S^X$ if $\mathrm{Ext}^1(S^X, X/X_S) \neq 0$;*
- *otherwise, if $\mathrm{Ext}^1(S^X, X/X_S) = 0$, M is the unique non-trivial extension between S/S^X and X_S.*

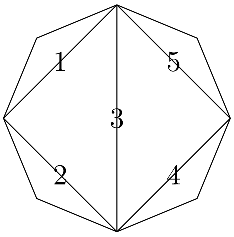
Remark 5.5.20. If the algebra KQ/I is hereditary, i.e., Q has no oriented cycles and $I = 0$, $\mathrm{Ext}^1(S^X, X/X_S)$ is always non-zero (see [14], Lemma 31), so we recover the specialization of the formula of Theorem 5.5.11 at $x_1 = \cdots = x_n = 1$, where $n = |Q_0|$.

Example 5.5.21. Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"'] & & 3 \arrow[ll, "\gamma"'] \arrow[dr, "\zeta"'] & & 5 \arrow[ll, "\delta"'] \\
& 2 \arrow[ur, "\beta"'] & & 4 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

and let $I = \langle \alpha\beta, \beta\gamma, \gamma\alpha, \zeta\epsilon, \epsilon\delta, \delta\zeta \rangle$.

(Q, I) is a cluster tilted bound symmetric quiver of type A_5, and it comes from the following triangulation of the octagon $\mathbf{P}_8$:



The Auslander-Reiten quiver of KQ/I is the following, where one has to identify the two representations labeled $\begin{smallmatrix}1\\2\end{smallmatrix}$ and the two representations labeled $\begin{smallmatrix}4\\5\end{smallmatrix}$, so it has the shape of a Moebius strip:

$$\begin{tikzcd}
\begin{smallmatrix}4\\5\end{smallmatrix} \arrow[dr] & & & & \begin{smallmatrix}5\\3\\1\end{smallmatrix} \arrow[dr] & & & & \begin{smallmatrix}1\\2\end{smallmatrix} \\
& 4 \arrow[dr] & & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}5\\3\end{smallmatrix} \arrow[dr] & & 2 \arrow[ur] & \\
& & \begin{smallmatrix}3\\14\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}25\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \\
& 1 \arrow[ur] & & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] & & 5 \arrow[dr] & \\
\begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & & & & \begin{smallmatrix}2\\3\\4\end{smallmatrix} \arrow[ur] & & & & \begin{smallmatrix}4\\5\end{smallmatrix}
\end{tikzcd}$$

We consider the short exact sequence

$$\xi : 0 \to X = \begin{smallmatrix}3\\14\end{smallmatrix} \to Y = \begin{smallmatrix}5\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix} \to S = \begin{smallmatrix}25\\3\end{smallmatrix} \to 0$$

Then $X_S = \ker(X \to \tau S) = \ker(\begin{smallmatrix}3\\14\end{smallmatrix} \to \begin{smallmatrix}3\end{smallmatrix}) = \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}4\end{smallmatrix}$, and $S^X = \mathrm{im}(\tau^{-1}X \to S) = \mathrm{im}(\begin{smallmatrix}3\end{smallmatrix} \to \begin{smallmatrix}25\\3\end{smallmatrix}) = \begin{smallmatrix}3\end{smallmatrix}$. In this case, $\mathrm{Ext}^1(S^X, X/X_S) = \mathrm{Ext}^1(\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}) = 0$, and in fact

$$F_{\begin{smallmatrix}3\\14\end{smallmatrix}} F_{\begin{smallmatrix}25\\3\end{smallmatrix}} = F_{\begin{smallmatrix}5\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix}} + y_3 F_{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\end{smallmatrix}},$$

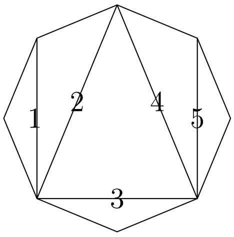
where $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\end{smallmatrix}$ is the only non-trivial extension between $S/S^X = \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}5\end{smallmatrix}$ and $X_S = \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}4\end{smallmatrix}$. Indeed, there is the non-split short exact sequence

$$0 \to \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}5\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\end{smallmatrix} \to \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}4\end{smallmatrix} \to 0.$$

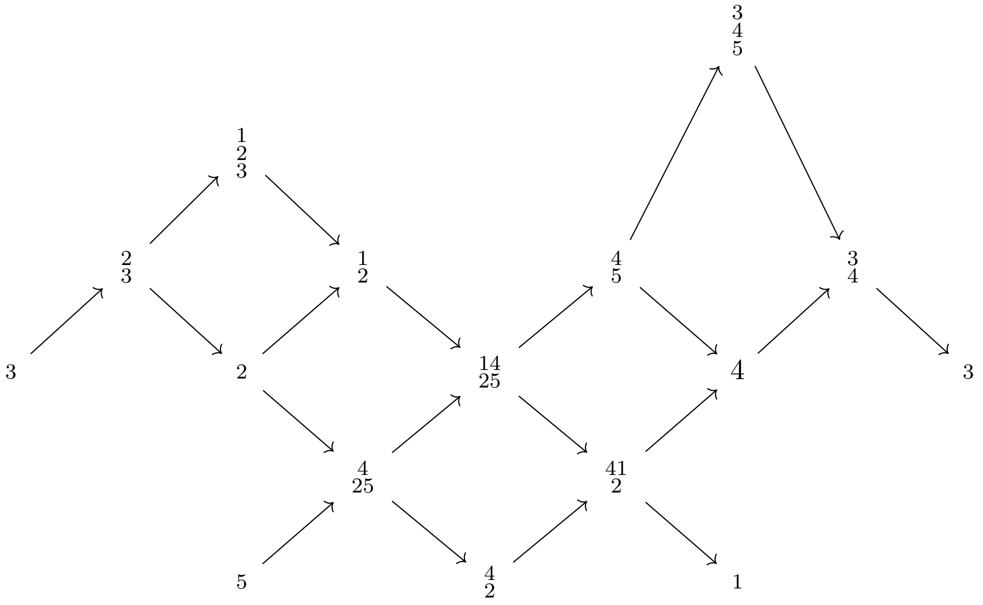
Example 5.5.22. Let Q be the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"'] & & 4 \arrow[ll, "\delta"'] \arrow[dr, "\epsilon"] & \\
1 \arrow[ur, "\alpha"] & & 3 \arrow[ur, "\gamma"'] & & 5
\end{tikzcd}$$

and let $I = \langle \beta\gamma, \gamma\delta, \delta\beta \rangle$. (Q, I) is a cluster tilted bound symmetric quiver of type A_5, and it comes from the following triangulation of the octagon $\mathbf{P}_8$:



The Auslander-Reiten quiver of KQ/I is the following, where one has to identify the two representations labeled 3:



We consider the short exact sequence

$$\xi : 0 \to X = \begin{smallmatrix}2\\3\end{smallmatrix} \to Y = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\25\end{smallmatrix} \to S = \begin{smallmatrix}14\\25\end{smallmatrix} \to 0.$$

Then $X_S = \ker(X \to \tau S) = \ker(\begin{smallmatrix}2\\3\end{smallmatrix} \to 2) = 3$, and $S^X = \operatorname{im}(\tau^{-1}X \to S) = \operatorname{im}(\begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}14\\25\end{smallmatrix}) = \begin{smallmatrix}1\\2\end{smallmatrix}$. In this case, $\operatorname{Ext}^1(S^X, X/X_S) = \operatorname{Ext}^1(\begin{smallmatrix}1\\2\end{smallmatrix}, 2) = 0$, and in fact

$$F_{\begin{smallmatrix}2\\3\end{smallmatrix}} F_{\begin{smallmatrix}14\\25\end{smallmatrix}} = F_{\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\25\end{smallmatrix}} + y_1 y_2 F_{\begin{smallmatrix}3\\4\\5\end{smallmatrix}},$$

where $\begin{smallmatrix}3\\4\\5\end{smallmatrix}$ is the only non-trivial extension between $S/S^X = \begin{smallmatrix}4\\5\end{smallmatrix}$ and $X_S = 3$. Indeed, there is a non-split short exact sequence

$$0 \to \begin{smallmatrix}4\\5\end{smallmatrix} \to \begin{smallmatrix}3\\4\\5\end{smallmatrix} \to 3 \to 0.$$

We also consider the short exact sequence

$$\xi' : 0 \to X = \begin{smallmatrix}2\\3\end{smallmatrix} \to Y = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 2 \to S = \begin{smallmatrix}1\\2\end{smallmatrix} \to 0.$$

Then $X_S = 0$ and $S^X = S$, since ξ' is almost split. In this case, $\operatorname{Ext}^1(S^X, X/X_S) = \operatorname{Ext}^1(S, X) \neq 0$, and $F_{X_S \oplus S/S^X} = 1$. In fact, we have

$$F_{\begin{smallmatrix}2\\3\end{smallmatrix}}F_{\begin{smallmatrix}1\\2\end{smallmatrix}} = F_{\begin{smallmatrix}1\\2\oplus 2\\3\end{smallmatrix}} + y_1y_2.$$

Example 5.5.23. Let Q be the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, "\beta"'] & \\
1 & & 3 \arrow[ul, "\alpha"'] \arrow[ll, "\gamma"']
\end{tikzcd}$$

and let $I = \langle \alpha\beta \rangle$. The Auslander-Reiten quiver of KQ/I is the following, where one has to identify the two representations labeled 2:

$$\begin{tikzcd}
& \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}32\\21\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \\
& \begin{smallmatrix}3\\21\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}32\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] & & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[ur] & & 2
\end{tikzcd}$$

We consider the short exact sequence

$$\xi : 0 \to X = \begin{smallmatrix}2\\1\end{smallmatrix} \to Y = \begin{smallmatrix}32\\1\end{smallmatrix} \to S = 3 \to 0.$$

In this case, $\mathrm{Hom}(X, \tau S) = \mathrm{Hom}(\begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}32\\21\end{smallmatrix})$ is of dimension 2, since there is the irreducible injective morphism f that corresponds to the arrow in the Auslander-Reiten quiver, and there is also the morphism f' with kernel 1 and image 2 corresponding to the lower left 2 in $\begin{smallmatrix}32\\21\end{smallmatrix}$. The morphism f' is given in the Auslander-Reiten quiver as the composition of 5 arrows, starting at $\begin{smallmatrix}2\\1\end{smallmatrix}$ going downward until 2 and then, using the identification in the Auslander-Reiten quiver, going upward until $\begin{smallmatrix}32\\21\end{smallmatrix}$. Therefore, f' factors through the injective module $I(1) = \begin{smallmatrix}32\\1\end{smallmatrix}$. Then $X_S = \ker(f) = 0$. On the other hand, $S^X = \mathrm{im}(\tau^{-1}X \to S) = \mathrm{im}(\begin{smallmatrix}3\\2\end{smallmatrix} \to 3) = 3$.

We have that $\mathrm{Ext}^1(S^X, X/X_S) = \mathrm{Ext}^1(3, \begin{smallmatrix}2\\1\end{smallmatrix}) \neq 0$, and $F_{X_S \oplus S/S^X} = 1$. In fact, we can compute that

$$F_{\begin{smallmatrix}2\\1\end{smallmatrix}}F_3 = F_{\begin{smallmatrix}32\\1\end{smallmatrix}} + y_3.$$

Bibliography

[1] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[2] E. Banaian and E. Kelley. Snake graphs from triangulated orbifolds. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 138, 50, 2020.

[3] V. Bazier-Matte, A. Chan, and Wright K. Marked non-orientable surfaces and cluster categories via symmetric representations. *arXiv:2211.15863*, 2023.

[4] I. N. Bernšteĭn, I. M. Gelfand, and V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[5] M. Boos and G. Cerulli Irelli. On degenerations and extensions of symplectic and orthogonal quiver representations. *arXiv:2106.08666*, 2021.

[6] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.

[7] P. Caldero and F. Chapoton. Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3):595–616, 2006.

[8] P. Caldero, F. Chapoton, and R. Schiffler. Quivers with relations arising from clusters (A_n case). *Trans. Amer. Math. Soc.*, 358(3):1347–1364, 2006.

[9] I. Canakci and R. Schiffler. Snake graph calculus and cluster algebras from surfaces. *J. Algebra*, 382:240–281, 2013.

[10] I. Canakci and R. Schiffler. Snake graph calculus and cluster algebras from surfaces II: self-crossing snake graphs. *Math. Z.*, 281(1-2):55–102, 2015.

[11] I. Canakci and R. Schiffler. Snake graph calculus and cluster algebras from surfaces III: Band graphs and snake rings. *Int. Math. Res. Not. IMRN*, (4):1145–1226, 2019.

[12] I. Canakci and P. Tumarkin. Bases for cluster algebras from orbifolds with one marked point. *Algebr. Comb.*, 2(3):355–365, 2019.

[13] R. Carter. *Lie algebras of finite and affine type*, volume 96 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2005.

[14] G. Cerulli Irelli, F. Esposito, H. Franzen, and M. Reineke. Cell decompositions and algebraicity of cohomology for quiver Grassmannians. *Adv. Math.*, 379:Paper No. 107544, 47, 2021.

[15] G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, and P.-G. Plamondon. Linear independence of cluster monomials for skew-symmetric cluster algebras. *Compos. Math.*, 149(10):1753–1764, 2013.

[16] A. Ciliberti. Cluster expansion formulas and perfect matchings for type b and c. *arXiv:2405.14915*, 2024.

[17] Azzurra Ciliberti. A categorification of cluster algebras of type b and c through symmetric quivers. *Journal of Algebra*, 664:1–41, 2025.

[18] L. Demonet. Categorification of skew-symmetrizable cluster algebras. *Algebr. Represent. Theory*, 14(6):1087–1162, 2011.

[19] H. Derksen and J. Weyman. Generalized quivers associated to reductive groups. *Colloq. Math.*, 94(2):151–173, 2002.

[20] H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations II: applications to cluster algebras. *J. Amer. Math. Soc.*, 23(3):749–790, 2010.

[21] S. Dominguez and C. Geiss. A Caldero-Chapoton formula for generalized cluster categories. *J. Algebra*, 399:887–893, 2014.

[22] G. Dupont. An approach to non-simply laced cluster algebras. *J. Algebra*, 320(4):1626–1661, 2008.

[23] A. Felikson, M. Shapiro, and P. Tumarkin. Cluster algebras and triangulated orbifolds. *Adv. Math.*, 231(5):2953–3002, 2012.

[24] A. Felikson, M. Shapiro, and P. Tumarkin. Cluster algebras of finite mutation type via unfoldings. *Int. Math. Res. Not. IMRN*, (8):1768–1804, 2012.

[25] A. Felikson and P. Tumarkin. Bases for cluster algebras from orbifolds. *Adv. Math.*, 318:191–232, 2017.

[26] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[27] S. Fomin and D. Thurston. Cluster algebras and triangulated surfaces Part II: Lambda lengths. *Mem. Amer. Math. Soc.*, 255(1223):v+97, 2018.

[28] S. Fomin, L. Williams, and A. Zelevinsky. Introduction to cluster algebras. chapters 1-3, 2021.

[29] S. Fomin, L. Williams, and A. Zelevinsky. Introduction to cluster algebras. chapters 4-5, 2021.

[30] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[31] S. Fomin and A. Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[32] S. Fomin and A. Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, 143(1):112–164, 2007.

[33] C. Geiss, B. Leclerc, and J. Schröer. Generic bases for cluster algebras and the Chamber ansatz. *J. Amer. Math. Soc.*, 25(1):21–76, 2012.

[34] C. Geiss, B. Leclerc, and J. Schröer. Quivers with relations for symmetrizable Cartan matrices I: Foundations. *Invent. Math.*, 209(1):61–158, 2017.

[35] J. Geuenich and D. Labardini-Fragoso. Species with potential arising from surfaces with orbifold points of order 2, part I: one choice of weights. *Math. Z.*, 286(3-4):1065–1143, 2017.

[36] J. Geuenich and D. Labardini-Fragoso. Species with potential arising from surfaces with orbifold points of order 2, Part II: Arbitrary weights. *Int. Math. Res. Not. IMRN*, (12):3649–3752, 2020.

[37] B. Keller. The periodicity conjecture for pairs of Dynkin diagrams. *Ann. of Math. (2)*, 177(1):111–170, 2013.

[38] D. Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc. (3)*, 98(3):797–839, 2009.

[39] G. Musiker. A graph theoretic expansion formula for cluster algebras of classical type. *Ann. Comb.*, 15(1):147–184, 2011.

[40] G. Musiker and R. Schiffler. Cluster expansion formulas and perfect matchings. *J. Algebraic Combin.*, 32(2):187–209, 2010.

[41] G. Musiker, R. Schiffler, and L. Williams. Positivity for cluster algebras from surfaces. *Adv. Math.*, 227(6):2241–2308, 2011.

[42] T. Nakanishi and S. Stella. Diagrammatic description of c-vectors and d-vectors of cluster algebras of finite type. *Electron. J. Combin.*, 21(1):Paper 1.3, 107, 2014.

[43] T. Nakanishi and A. Zelevinsky. On tropical dualities in cluster algebras. In *Algebraic groups and quantum groups*, volume 565 of *Contemp. Math.*, pages 217–226. Amer. Math. Soc., Providence, RI, 2012.

[44] P.-G. Plamondon. Cluster characters. *arXiv:1610.07546*, 2017.

[45] N. Reading. Dominance phenomena: mutation, scattering and cluster algebras. *Trans. Amer. Math. Soc.*, 376(2):773–835, 2023.

[46] R. Schiffler. On cluster algebras arising from unpunctured surfaces. II. *Adv. Math.*, 223(6):1885–1923, 2010.

[47] R. Schiffler. *Quiver representations.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham, 2014.