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Irreversibility and response functions in the turbulent energy cascade

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Abstract. The statistical properties of turbulent flows are fundamentally different from those of systems at equilibrium due to the presence of an energy flux from the scales of injection to those where energy is dissipated by the viscous forces: a scenario dubbed “direct energy cascade”. Here, we aim at characterizing the non-equilibrium properties of turbulent cascades in a shell model of turbulence by studying an asymmetric time-correlation function and the relaxation behavior of an energy perturbation, measured at scales smaller or larger than the perturbed one. We shall contrast the behavior of these two observables in both non-equilibrium (forced and dissipated) and equilibrium (inviscid and unforced) cases. Finally, we shall show that equilibrium and non-equilibrium physics coexist in the same system, namely at scales larger and smaller, respectively, of the forcing scale.

1. Introduction

A key aspect of non equilibrium phenomena is the presence of currents induced by some external constraints, which entail the breaking of detailed balance and, consequently, of time-reversal symmetry [1]. A paradigmatic example of far-from-equilibrium system is found in turbulence: a current of energy flows from the large scales, where energy is injected, to the small scales, where it is dissipated by viscosity, with an average constant flux across the scales – the so called direct energy cascade scenario [2, 3]. The non-equilibrium nature of a physical system can be inferred with different approaches: for instance, suitable asymmetric time correlation functions can be constructed to reveal the breaking of time reversal symmetry [4, 5]. Also, looking at the time evolution of response functions, which describe how some system variables relax to their statistically steady state after a perturbation on the same or different degrees of freedom, can reveal aspects of the asymmetries between degrees of freedom induced by the presence of currents [6, 7, 8].

Here, we aim at studying the temporal properties of the turbulent energy cascade using the two indicators mentioned above: asymmetric time correlations and response functions. We employ a reduced model of turbulence, the Sabra shell model [9, 10], which retains only one scalar complex variable for each shell in wavevector space, whose radii grow in geometric progression. Despite its simplified nature, it is able to display all the most important statistical properties of turbulence.



We will compare how differently both functions behave when computed for the inviscid and non-forced model and for the viscous forced (i.e. turbulent) one, respectively a system at equilibrium (resembling the truncated Euler equation [11]) and one out-of-equilibrium. We will also examine a situation where both equilibrium and non-equilibrium properties are found, namely applying an intermediate-scale forcing.

2. Model

Shell models are finite-dimensional dynamical systems designed to reproduce the phenomenology of the turbulent energy cascade. The basic idea is to consider a discrete set of wavenumbers $k_n = k_0 2^{n-1}$, with $n = 1, \dots, N$ defining N shells in Fourier space. A single complex variable u_n is used to represent the velocity fluctuations across spatial scales $\sim 1/k_n$. Its evolution equation is formally analogous to the Navier-Stokes equation in Fourier space:

$$\dot{u}_n = i \left[k_{n+1} u_{n+2} u_{n+1}^* - \frac{1}{2} k_n u_{n+1} u_{n-1}^* + \frac{1}{2} k_{n-1} u_{n-1} u_{n-2} \right] - \nu k_n^2 u_n + f_n, \quad (1)$$

where ν is the viscosity, f_n the forcing on shell n and $*$ indicates complex conjugation. The non-linear terms in square parentheses, in the unforced and inviscid limit ($\nu = f_n = 0$) conserve the total energy E and helicity H :

$$E = \sum_{n=1}^N e_n \quad H = \sum_{n=1}^N (-1)^n k_n e_n \quad (2)$$

where $e_n = |u_n|^2/2$ is the kinetic energy of shell n .

3. Irreversibility via time correlations

It has been long known [4] that, for systems out of equilibrium, an asymmetry exists between time correlations among suitably chosen functions of the system state: $\langle f(t)g(t+\tau) \rangle = \langle g(t)f(t+\tau) \rangle$, where $\langle \cdot \rangle$ denotes a time average over t . This allows to detect and quantify the breaking of time-reversal symmetry by means of asymmetric time correlations.

Regarding our specific case, let us consider the energy of a given shell $e_n(t)$ and study the following (non-dimensional) asymmetric time correlation function:

$$\Psi_{e_n}(\tau) = \frac{\langle e_n^2(t) e_n(t+\tau) \rangle - \langle e_n(t) e_n^2(t+\tau) \rangle}{\langle e_n^3(t) \rangle}. \quad (3)$$

For systems far from statistical equilibrium one expects to see evident deviations from zero, with a nontrivial dependence on τ , whereas for the equilibrium case only small noisy fluctuations should be present. This is indeed observed in Figure 1, where we compare the functions (3) computed for values of n in the inertial range (dominated by energy transfer) of the turbulent model (left) with those computed in the inviscid unforced model (right). Again, they represent dynamical systems out of equilibrium and at equilibrium, respectively.

The peculiar behaviour observed in the non-equilibrium case can be understood by noticing that the functions Ψ_{e_n} are statistically equivalent to the following functions:

$$\Phi_{e_n}(\tau) = \frac{1}{3} \frac{\langle [e_n(t+\tau) - e_n(t)]^3 \rangle}{\langle e_n^3(t) \rangle} \quad (4)$$

whose physical meaning is more direct: they represent the third moment of a shell-energy variation after time τ . The trend shown by time correlations, namely a negative followed by a

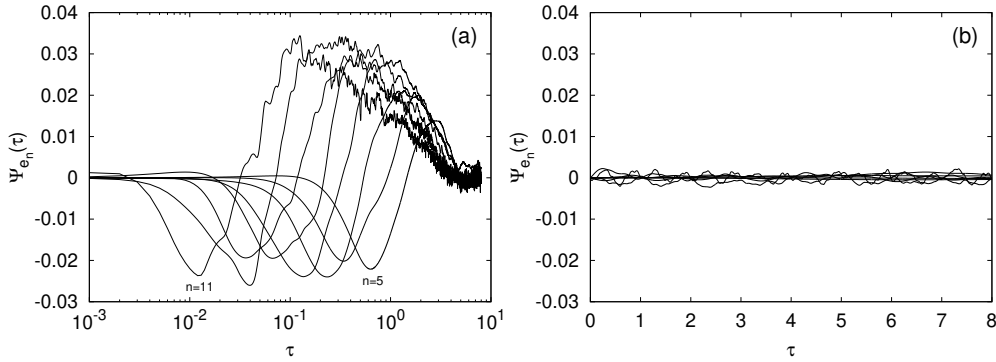


Figure 1. Asymmetric time correlation functions $\Psi_{e_n}(t)$ at varying n . (a) Turbulent case, shells n in the inertial range, from $n = 5$ (rightmost negative peak) to $n = 11$ (leftmost one). A qualitatively similar behavior is observed. (b) Inviscid case, from $n = 3$ (slowest fluctuations) to $n = 10$ (fastest fluctuations). Comparatively smaller and structureless fluctuations around zero are obtained.

positive peak, is thus compatible with the energy cascade since it predicts that energy is lost to faster (larger) shells and gained from slower (smaller) ones. Understanding the timescales of such correlation functions is not straightforward: in multiscale systems, the whole hierarchy of timescales needs to be taken into account [12]. Further discussions on the $\tau \rightarrow 0$ limit and the study of power statistics can be found in [13].

4. Energy cascade via response functions

An external perturbation acting on an individual degree of freedom will necessarily propagate to others if a dynamical coupling exists between them. If the interactions show some asymmetries one expects to observe a dissimilarity in the relaxation behaviour of different degrees of freedom towards a statistically stationary state. With this in mind, it is interesting to investigate such

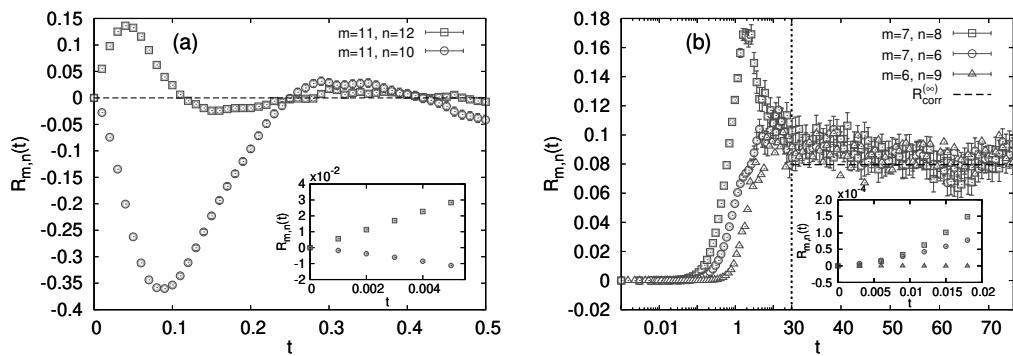


Figure 2. Non-diagonal energy RFs measured at the neighboring shells of the perturbed one. (a) Turbulent model. (b) Inviscid model. Logarithmic time axis up to vertical dotted line, then linear axis. Another RF has been added (triangles), with m further from n , for confirming the common asymptotic value. The dashed line is the asymptotic value taking into account the boundary corrections. The insets of both figures are enlargements of the initial-time range, showing respectively a non-zero and a zero first derivative at $t = 0$.

asymmetries by means of non-diagonal Response Functions (RFs) of the kind:

$$R_{m,n}(t) = \frac{\overline{\delta e_n(t)}}{\delta e_m(0)}, \quad m \neq n, \quad (5)$$

which indicates the average deviation of e_n from its unperturbed value measured after a delay t with respect to the initial perturbation on e_m [7]. With $\delta(\cdot)$ we denote this deviation, and the overbar is an average over a large number of realizations of such perturbation. The applied perturbation is of finite size, and such that it provides, at each realization, the same amount of extra energy.

Figure 2 shows the difference in the RFs measured in the turbulent non-equilibrium model (left) and for the inviscid equilibrium model (right) when m is close to n . The former plot shows a clear asymmetry in the sign of the RFs before the relaxation, whose meaning is that the perturbation tends, on average, to be transferred to the larger shell (smaller scale) while the smaller shell (larger scale) loses part of its energy, as we expect in presence of an energy cascade. In the latter plot the functions are all positive and reach a common non-zero asymptotic value, a situation that can be interpreted as the reaching of a new energy-equipartition state (with a boundary correction due to the finite- N truncation). The insets show the initial-time behaviour of the RFs, where non-zero initial derivatives appear in the turbulent case while they are zero in the inviscid one. This reveals the presence and absence, respectively, of an energy flux among different scales. For more results about forward ($m < n$) and backward ($m > n$) RFs, and a discussion about their initial time derivatives, see [13].

5. Intermediate-scale forcing: equilibrium and non-equilibrium

The turbulent cascade brings energy from the forcing scale down to the dissipative range, but what happens at scales larger than forcing, where there should not be a net energy flux? What is the statistical behaviour of such degrees of freedom? Numerical and experimental studies [14, 15] showed that such scales are described effectively by supposing absolute equilibrium, while a recent work [16] proved a deviation from equilibrium, since large-scale modes are actually in interaction with the scales where cascade is observed. Even though shell models do not display, by construction, direct interactions between scales which are far apart, it is nonetheless instructive to repeat the studies previously shown to a shell model with forcing at intermediate scales, and look at shells larger and smaller than the forced ones.

After having verified some preliminary signatures of equilibrium at scales larger than forcing, such as energy equipartition, zero net flux and skewness very close to zero, the computation of the asymmetric time correlations (3) and response functions (5) revealed that for scales smaller than forcing (in the “cascading” range) the result is very similar to Figure 1(a) and Figure 2(a), whereas the same functions computed at scales larger than forcing (in the supposedly equilibrium range) behave like the inviscid model shown in Figures 1(b) and 2(b). Additional figures can be found in [13].

Conclusion

In this work we used asymmetric time correlation functions and response functions to finite perturbations to ascertain the breaking of time reversal symmetry and asymmetries between the degrees of freedom in a simplified model for the energy cascade in turbulence, i.e. the Sabra shell model. We showed that by looking at a suitable time correlation of a single-shell energy one can clearly distinguish the case of a forced shell model from the case of an unforced inviscid model. In the latter case the asymmetric correlation vanishes (meaning time reversible dynamics) while in the former it is definitely different from zero and the behavior understandable in terms of the energy cascade scenario. Similarly, clear differences between equilibrium and non-equilibrium

physics are detectable using the energy response function, that is, perturbing the energy at given shell and looking at the time relaxation of the energy at a distant shell (either above or below the perturbed one). A net difference is observed in the forced case while looking at shells larger (scales smaller) or smaller (scales larger) than the perturbed one, as a consequence of the average energy current from large to small scales. The quantities studied in this work thus allow to observe the celebrated Richardson cascade scenario from the perspective of non-equilibrium Statistical Mechanics. Finally, we considered the case of a shell model forced at intermediate scales so to allow for scrutinizing the behavior of both quantities at scales larger or smaller than the forcing scale. The emerging picture seems to confirm the view that scales larger than the forcing display properties which can be ascribed to the equilibrium physics of the unforced-inviscid model. It is important to point out that our result on equilibrium at large scales cannot be directly transferred to Navier-Stokes turbulence, because direct interactions between large and small scales are claimed to be the main cause of departure from absolute equilibrium, but they are absent in shell models.

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