MEASURE-VALUED SOLUTIONS OF SCALAR HYPERBOLIC CONSERVATION LAWS, PART 1: EXISTENCE AND TIME EVOLUTION OF SINGULAR PARTS

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Abstract. We prove existence for a class of signed Radon measure-valued entropy solutions of the Cauchy problem for a first order scalar hyperbolic conservation law in one space dimension. The initial data of the problem is a finite superposition of Dirac masses, whereas the flux is Lipschitz continuous. Existence is proven by a constructive procedure which makes use of a suitable family of approximating problems. Relevant qualitative properties of such constructed solutions are pointed out.

1. Introduction

We study the Cauchy problem for the scalar conservation law:

$$(P) \qquad \begin{cases} u_t + [\varphi(u)]_x = 0 & \text{in } \mathbb{R}\times(0,T) =: S \\ u = u_0 & \text{in } \mathbb{R}\times\{0\}\,, \end{cases}$$

where u_0 is a finite signed Radon measure on $\mathbb{R}$ and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous (see assumption (A_2)). Specifically, we consider initial measures whose singular part is a finite superposition of Dirac masses:

$$(A_1) \qquad u_{0r} \in L^1(\mathbb{R})\,, \qquad u_{0s} = \sum_{i=1}^{P_+} p_{0i}\,\delta_{a_i} - \sum_{l=1}^{M_-} m_{0l}\,\delta_{b_l}$$

with $p_{0i} > 0, m_{0l} > 0,\ a_i \neq b_l\ (i = 1,\dots,P_+\,,\ l = 1,\dots,M_-\,; P_+, M_- \in \mathbb{N})$. Here $u_0 = u_{0ac} + u_{0s}$ is the Lebesgue decomposition of u_0 and u_{0r} denotes the density of u_{0ac} with respect to the Lebesgue measure. Observe that by (A_1) $\operatorname{supp} u_{0s}^+ = \{a_1,\cdots,a_{P_+}\}$, $\operatorname{supp} u_{0s}^- = \{b_1,\cdots,b_{M_-}\}$ and $\operatorname{supp} u_{0s}^+ \cap \operatorname{supp} u_{0s}^- = \varnothing$, $u_{0s}^\pm$ denoting the positive and the negative part of u_{0s}, respectively.

As for the function φ, we shall assume that

$$(A_2) \qquad \varphi(u) = \varphi_b(u) + C_0\,u_+\,, \qquad \varphi_b \in W^{1,\infty}(\mathbb{R})\,, \qquad C_0 \in \mathbb{R}$$

(hereafter $u_\pm := \max\{\pm u, 0\}$, $u \in \mathbb{R}$). Modelling motivations for the present study can be found in [**?**, **?**, **?**] (see also [**?**]).

In view of the lack of regularity of initial data, in the following we shall address problem (P) in the framework of the so called *Radon measure-valued solutions*, *i.e.* suitable weak solutions of (P) satisfying a specific entropic formulation, which takes into account the possible persistence of singular measures for positive times. It is worth observing that analogous notions of solutions have been also considered in the case of linear multi-dimensional transport equations with non-smooth coefficients ([**?**]), and in the Riemann problem for some physically relevant systems of conservation laws (*e.g.*, Keyfitz-Kranzer type systems). As for the latter, among the many contributions, we explicitly mention the concept of *delta-shock* solutions which arises to describe the appearance of delta functions supported on a shock (*e.g.*, see [**?**, **?**, **?**, **?**, **?**, **?**]).

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1.1. **The nonnegative case.** If u_0 is a finite nonnegative Radon measure on $\mathbb{R}$ and φ satisfies the following assumption:

$$(A_\varphi) \qquad \varphi \in \mathrm{Lip}([0,\infty))\,, \;\; \varphi(0)=0\,, \;\; \text{there exists } \lim_{u\to\infty}\frac{\varphi(u)}{u} =: C_0\,, \;\; \varphi(u)-C_0u \in L^\infty(0,\infty),$$

there exist (nonnegative) Radon measure-valued *entropy solutions* of problem (P) [?]. Existence is proven by studying convergence in a suitable topology of the sequence of solutions of the *approximating problems:*

$$\begin{cases} u_{nt} + [\varphi(u_n)]_x = 0 & \text{in } S \\ u_n = u_{0n} & \text{in } \mathbb{R}\times\{0\}\,, \end{cases}$$

where $\{u_{0n}\} \subseteq L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ is a convenient regularizing sequence of u_0. The solutions of (P) thus exhibited are called *constructed solutions*. Remarkably, in the proof it is not restrictive to assume that φ satisfies

$$(A'_\varphi) \qquad \varphi \in W^{1,\infty}(0,\infty)\,, \;\; \varphi(0)=0$$

instead of (A_φ). In fact, (A'_φ) implies (A_φ) with $C_0=0$. On the other hand, if (A_φ) holds and $u=u(x,t)$ is an entropy solution of (P), then $\tilde u = \tilde u(x,t) := u(x+C_0t,t)$ is an entropy solution of the problem

$$(\tilde P) \qquad \begin{cases} \tilde u_t + [\tilde\varphi(\tilde u)]_x = 0 & \text{in } S \\ \tilde u = \tilde u_0 & \text{in } \mathbb{R}\times\{0\} \end{cases}$$

with $\tilde u_0 = u_0$ and $\tilde\varphi(z) := \varphi(z) - C_0 z$ $(z\in\mathbb{R})$, thus $\tilde\varphi$ satisfies (A'_φ) (see [?, Remark 3.16]).

If $u_0 \ge 0$ and (A_φ) holds, it is a general feature of entropy solutions that the singular part $u_s(t)$ does not increase along the characteristic lines $x = C_0t + x_0$; moreover, if u_{0s} has a nonzero discrete part, there exists a positive time $\tau_0 \in (0,T]$ (only depending on u_{0s} and φ) until which $u_s(t) > 0$ (see [?, Proposition 3.8 and Theorem 3.18]). In particular, if (A'_φ) holds and

$$(1.1) \qquad u_{0r} \in L^1(\mathbb{R})\,, \quad u_{0s} = \sum_{i=1}^{P_+} p_{0i}\,\delta_{a_i}\,, \quad p_{0i} > 0\,,$$

- there exists $\tau \in (0,T]$ such that

$$(1.2) \qquad \operatorname{supp} u_s(t) = \operatorname{supp} u_{0s} = \{a_1,\cdots,a_{P_+}\} \quad \text{for any } t\in[0,\tau)\,;$$

- there exist nonincreasing functions $p_i : [0,T] \to [0,p_{0i}]$ such that $p_i(0) = p_{0i}$, and

$$(1.3) \qquad u_s(t) = \sum_{i=1}^{P_+} p_i(t)\,\delta_{a_i} \qquad (t\in[0,T])\,.$$

This suggests an existence proof for problem (P) different from that in [?], which can be outlined as follows. Consider the simple case where $u_{0s} = p_0\delta_a$, $p_0>0$, and φ satisfied (A'_φ). By the above remarks there exists a positive time τ until which the Dirac mass at a persists. Consider the *singular* Dirichlet initial-boundary value problems

$$(1.4) \qquad \begin{cases} u_t + [\varphi(u)]_x = 0 & \text{in } (-\infty,a)\times(0,\tau) \\ u = \infty & \text{in } \{a\}\times(0,\tau) \\ u = u_{0r} & \text{in } (-\infty,a)\times\{0\}\,, \end{cases} \qquad \begin{cases} u_t + [\varphi(u)]_x = 0 & \text{in } (a,\infty)\times(0,\tau) \\ u = \infty & \text{in } \{a\}\times(0,\tau) \\ u = u_{0r} & \text{in } (a,\infty)\times\{0\}\,. \end{cases}$$

The function u_r determined by solutions of (??) in $\mathbb{R}\times(0,\tau)$ is, by definition, the regular part of a Radon measure u whose singular part is $u_s(t) := p(t)\,\delta_a$ $(t\in(0,\tau))$, with p defined by the initial weight p_0 and the variation of mass at the point a. It can be proven that the measure u is an entropy solution of problem (P) in $(0,\tau)$. Moreover, it is the *unique* entropy solution of (P) satisfying a suitable integral form of the following condition:

$$(1.5)\qquad \begin{array}{l} \operatorname{sgn}_-(u_r(a^+,t)-k)[\varphi(u_r(a^+,t))-\varphi(k)] \le 0 \\ \operatorname{sgn}_-(u_r(a^-,t)-k)[\varphi(u_r(a^-,t))-\varphi(k)] \ge 0 \end{array} \qquad \text{for all } k\in[0,\infty)$$

(see (**??**) and [**?**]); hereafter we set $\operatorname{sgn}_\pm(u) := \pm\chi_{\{\pm u>0\}}(u)$, $\operatorname{sgn}(u) := \operatorname{sgn}_+(u)+\operatorname{sgn}_-(u)$, χ_E denoting the characteristic function of $E\subseteq\mathbb{R}$; $u\in\mathbb{R}$). Formally, condition (**??**) is equivalent to the *compatibility condition*

$$(1.6)\qquad \begin{array}{l} [\operatorname{sgn}(u_r(a^+,t)-k)-\operatorname{sgn}(b(t)-k)][\varphi(u_r(a^+,t))-\varphi(k)] \le 0 \\ [\operatorname{sgn}(u_r(a^-,t)-k)-\operatorname{sgn}(b(t)-k)][\varphi(u_r(a^-,t))-\varphi(k)] \ge 0 \end{array}$$

between the traces $u_r(a^\pm,t) := \lim_{x\to a^\pm} u_r(x,t)$ and the boundary data $b(t)=\infty$, for all k,t as above. It is known (see [**?**, **?**]) that the initial-boundary value problems in (**??**), with the boundary conditions "$u=\infty$" replaced by "$u=b$", are well posed if $b\in BV(0,T)$ and (**??**) holds (for the sake of completeness, we recall the weaker formulations of the boundary conditions for L^∞- or L^1-solutions in [**?**] and [**?**], respectively, as well as the results in [**?**, **?**, **?**] on the existence of strong traces).

To summarize, as long as the Dirac delta at $x=a$ survives, it behaves like a barrier which decouples the evolution of the regular part of the solution on either side of the singularity. As a consequence, the two Dirichlet conditions $u_r(a^\pm,t)=\infty$ at $x=a$ - namely, the compatibility condition (**??**) - are needed to prove uniqueness (in fact, it is known that the entropy inequalities are not enough to ensure uniqueness of measure-valued solutions of (P)).

The above considerations can be extended to any initial data of the form (**??**) and φ as in (A'_φ). The solution thus obtained turns out to belong to $C([0,T];\mathcal{M}^+(\mathbb{R}))$, thus the functions p_i in (**??**) are continuous in $[0,T]$ (see [**?**, Theorem 3.1]). We observe that the solutions constructed in [**?**] are known to satisfy the compatibility conditions (thus to coincide with those constructed in [**?**]) only under suitable conditions on φ (see [**?**, Proposition 3.17]).

If (A_φ) holds, for any nonnegative initial measure as in (**??**), analogous well-posedness results for (P) are obtained from those for $(\tilde{P})$. In this case, for the unique solution u of (P) (which satisfies a transformed form of the compatibility conditions (**??**) on the characteristic lines $\{(x,t)\in S \,|\, x=C_0t+a_i,\ t\in[0,T]\}$):
- there exists $\tau\in(0,T]$ such that

$$(1.7)\qquad \operatorname{supp} u_s(t) = \bigcup_{i=1}^{P_+}\{(x,t)\in S\,|\,x=C_0t+a_i,\ t\in[0,T]\} \quad \text{for any } t\in[0,\tau)\,;$$

- there holds

$$(1.8)\qquad u_s(t)=\sum_{i=1}^{P_+} p_i(t)\,\delta_{a_i+C_0t} \qquad (t\in[0,T])$$

with $p_i:[0,T]\to[0,p_{0i}]$, $p_i(0)=p_{0i}$ nonincreasing and continuous on $[0,T]$. Let us mention that now the map $t\mapsto\mathcal{T}_{-C_0t}\left(\sum_{i=1}^{P_+}p_i(t)\,\delta_{a_i}\right)$ belongs to $C((0,T];\mathcal{M}^+(\mathbb{R}))$, and is continuous at $t=0$ in the strong topology of $\mathcal{M}(\mathbb{R})$ if φ satisfies additional convexity assumptions (see [**?**, Proposition 3.20]). On the other hand, u_s is continuous on the whole interval $[0,T]$ in the weak* topology of $\mathcal{M}(\mathbb{R})$ (see [**?**, Proposition 3.5]).

1.2. **The signed case: novel features and outline of results.** As long as (A'_φ) holds, the above results can be generalized to signed measures satisfying (A_1) (see [**?**]).

As in the nonnegative case, the starting point is a monotonicity result: both the positive and the negative part, $u_s^\pm(t)$, of the singular part u_s of any entropy solution of (P) are nonincreasing with respect to t (for simplicity of notations, for singular Radon measures we prefer the symbols $u_s^\pm$ instead of $[u_s]_\pm$). Moreover, for any $i=1,\dots,P_+$, (respectively

$l = 1,\dots,M_-$) there exists $\tau_i \in (0,T]$ (respectively $\tau_l \in (0,T]$) such that $u_s(t)(\{a_i\}) > 0$ ($u_s(t)(\{b_l\}) > 0$, respectively). Therefore, there exists $\tau \in (0,T]$ such that for any $t \in [0,\tau)$

$$\text{supp}\, u_s^+(t) = \text{supp}\, u_{0s}^+ = \{a_1,\cdots,a_{P_+}\}\,, \quad \text{supp}\, u_s^-(t) = \text{supp}\, u_{0s}^- = \{b_1,\cdots,b_{M_-}\}\,. \tag{1.9}$$

Moreover, there exist nonincreasing functions $p_i : [0,T] \to [0,p_{0i}]$ with $p_i(0) = p_{0i}$, $m_l : [0,T] \to [0,m_{0l}]$ with $m_l(0) = m_{0l}$ ($i = 1,\dots,P_+, l = 1,\dots,M_-$), such that

$$u_s^+(t) = \sum_{i=1}^{P_+} p_i(t)\,\delta_{a_i}\,, \quad u_s^-(t) = \sum_{l=1}^{M_-} m_l(t)\,\delta_{b_l} \qquad (t \in [0,T])\,. \tag{1.10}$$

As in the nonnegative case there holds $u_s^\pm \in C([0,T];\mathcal{M}^+(\mathbb{R}))$ (see [?, Corollary 1]), thus all functions p_i and m_l in (??) are continuous in $[0,T]$.

Now an entropy solution u of (P) is said to satisfy the compatibility condition at a_i ($i = 1,\dots,P_+$) if

$$\pm\,\text{ess}\lim_{x\to a_i^\pm} \int_0^{\tau_i} \text{sgn}_-(u_r(x,t)-k)\left[\varphi(u_r(x,t)) - \varphi(k)\right]\beta(t)\,dt \le 0 \tag{1.11a}$$

for all $\beta \in C_c^1(0,\tau_i)$, $\beta \ge 0$ and $k \in \mathbb{R}$, respectively at b_l ($l = 1,\dots,M_-$) if

$$\pm\,\text{ess}\lim_{x\to b_l^\pm} \int_0^{\tau_l} \text{sgn}_+(u_r(x,t)-k)\left[\varphi(u_r(x,t)) - \varphi(k)\right]\beta(t)\,dt \le 0 \tag{1.11b}$$

for all $\beta \in C_c^1(0,\tau_l)$, $\beta \ge 0$ and $k \in \mathbb{R}$. A procedure which makes use of singular Dirichlet problems, thus generalizes that described in Subsection **??**, proves that the compatibility conditions identify a class of well-posedness for problem (P) (see [?, Theorem 3.5]).

The results in [?] make essential use of the fact that $\lim_{u\to\pm\infty} \frac{\varphi(u)}{u} = 0$. It is the purpose of the present paper to address problem (P), with u_0 as in (A_1), under the general assumption (A_2) of possibly unbounded fluxes. Observe that (A_2) is a special case of

$$\varphi(u) := \varphi_b(u) + C_+\,u_+ + C_-\,u_-\,, \qquad \varphi_b \in W^{1,\infty}(\mathbb{R})\,, \qquad C_\pm \in \mathbb{R}\,, \qquad (u \in \mathbb{R})\,. \tag{1.12}$$

Assuming (A_2) is not restrictive since, if (**??**) holds and $u = u(x,t)$ is an entropy solution of (P), then $\hat{u}(\hat{x},t) := u(\hat{x} - C_-t, t)$ is an entropy solution of the problem

$$(\hat{P}) \qquad \begin{cases} \hat{u}_t + [\hat{\varphi}(\hat{u})]_x = 0 & \text{in } S \\ \hat{u} = \hat{u}_0 & \text{in } \mathbb{R}\times\{0\} \end{cases}$$

with $\hat{u}_0 = u_0$ and $\hat{\varphi}(z) := \varphi(z) + C_-z (z \in \mathbb{R})$, thus $\tilde{\varphi}$ satisfies (A_2) with $C_0 = C_+ + C_-$.

In view of (A_2), now the Dirac masses with a positive weight are transported along the segments

$$P_i := \{(x,t)\,|\,x = a_i + C_0t,\ t \in [0,T]\} \qquad (i = 1,\dots,P_+)\,, \tag{1.13a}$$

while the same happens to the Dirac masses with a negative weight along the vertical segments

$$M_l := \{(x,t)\,|\,x = b_l,\ t \in [0,T]\} \qquad (l = 1,\dots,M_-)\,. \tag{1.13b}$$

If $C_0 = 0$, the situation is that already addressed in [?]. Instead, if $C_0 \ne 0$, two segments P_i and M_l possibly intersect at some point (x_{il},t_{il}) ($t_{il} \in (0,T)$) (thus the strip S is a finite union of triangles and possibly unbounded rectangles, rhombi and trapezoids). This is a major qualitative novelty, which gives rise to an intriguing dynamics of the singular part of entropy solutions of (P) (see Definition **??**).

To point out in a simple case the intricacies we are faced with, let $C_0 > 0$ and

$$u_{0s} = p_0\delta_a - m_0\delta_b \qquad (p_0, m_0 > 0;\ a < b)\,. \tag{1.14}$$

In this situation the parts $u_s^+(t)$ and $u_s^-(t)$ of any entropy solution u of (P) are transported for positive times along the segments $P := \{(x,t)\,|\,x = a + C_0t,\ t \in [0,T]\}$ and $M := \{(x,t)\,|\,x = b,\ t \in [0,T]\}$, respectively. As in the previous cases, the positive and the negative part of u_s are nonincreasing in time along P and M (see Proposition **??**). Therefore, there exist nonincreasing functions $p : [0,T] \to [0,p_0]$ and $m : [0,T] \to [0,m_0]$ such that $p(0) = p_0$, $m(0) = m_0$, and

$$u_s^+(t) = p(t)\,\delta_{a+C_0 t}\,, \quad u_s^-(t) = m(t)\,\delta_b \quad \text{for a.e. } t \in [0,T]$$

(see equalities (**??**) below). Since $C_0 > 0$, the segments P and M intersect at the point (b, t_0), with $t_0 := \frac{b-a}{C_0}$. There is no loss of generality in assuming $t_0 < T$.

It is natural to ask how the two Dirac masses (with "different signs") interact at the matching time t_0, if both survive until the time $t = t_0$ - an issue which obviously points at the problem of continuity in time of entropy solutions of (P). Proposition **??** below shows that both the absolutely continuous part and the singular part of an entropy solution u of (P) are continuous in time *in the whole interval* $[0,T]$ with respect to the weak* topology of $\mathcal{M}(\mathbb{R})$ (a preliminary continuity result of u in the same topology is given by Lemma **??**). As a consequence, we get the representation

$$u_s(t) = p(t^\pm)\,\delta_{a+C_0 t} - m(t^\pm)\,\delta_b \quad \text{for every } t \in (0,T)$$

(where $p(t^\pm) := \lim_{\tau\to t^\pm} p(\tau)$, $m(t^\pm) := \lim_{\tau\to t^\pm} m(\tau)$ exist by the monotonicity of p and m; the above equality is a particular case of (**??**)). Since $a + C_0 t_0 = b$, it follows that

$$u_s(t_0) \;=\; [p(t_0^-) - m(t_0^-)]\,\delta_b \;=\; [p(t_0^+) - m(t_0^+)]\,\delta_b\,. \tag{1.15}$$

It is important to stress that the weak* continuity at every point of $[0,T]$, ensured for u_s by Proposition **??**, *does not hold separately for* u_s^+ and u_s^-. Namely, continuity of p, m at $t = t_0$ need not hold - although the difference $w := p - m$ is continuous at t_0 , as shown by (**??**). Therefore, the entropic formulation alone does not determine the evolution of $u_s^\pm(t)$ after t_0, and additional information is needed.

This additional information is provided by a major feature of the solutions given by our existence proof (see Theorem **??**). Existence of entropy solutions to (P) is proven below by a constructive approach similar to that in [**?**], relying on a suitable family of approximating problems (see Subsection **??**). As in [**?**], the solutions thus obtained are called *constructed solutions*. An important qualitative property of theirs is the weak* continuity *from the right* of the positive and negative singular parts $t \mapsto u_s^\pm(t)$ at every point of $[0,T)$. Combined with the continuity of the difference $w := p - m$ at the intersection point (see (**??**)), this additional property allows to determine the behaviour for $t \in [t_0,T]$ of $u_s^\pm(t)$ for any constructed solution u of (P) with initial data as in (**??**) (similar results hold in the general case; see Lemma **??**). Different situations occur, depending on the sign of $w(t_0)$:
- if $w(t_0) > 0$, by (**??**) there holds $u_s(t_0) = w(t_0)\,\delta_b \in \mathcal{M}^+(\mathbb{R})$, thus $u_s(t_0) = u_s^+(t_0)$ and $u_s^-(t_0) = 0$. Then by the weak* continuity from the right and the nonincreasing character of u_s^- there holds $u_s^-(t) = 0$ for any $t \in [t_0,T]$ (see equalities (**??**));
- if $w(t_0) < 0$, we obtain similarly that $u_s^+(t) = 0$ for any $t \in [t_0,T]$;
- if $w(t_0) = 0$, then $u_s^\pm(t) = 0$ for any $t \in (t_0,T]$.

Finally, another important feature of constructed solutions is that they satisfy a more general version of the compatibility conditions (see Definition **??** and Theorem **??**). In the forthcoming paper [**?**] we shall prove that, as in [**?**, **?**], the compatibility conditions identify a class of uniqueness for problem (P). Therefore, well-posedness of problem (P) in the same class follows from the present constructive proof of existence. It is also worth observing that in [**?**, **?**], for bounded nonlinearities φ, existence of solutions satisfying the compatibility conditions has been proven by a different constructive approach (see Subsection **??**), not relying on regularization arguments of the initial measure. Thus, construction of solutions by the approximating problems in Subsection **??** is one of the major features of the present paper, and makes our construction consistent with respect to smoothing and regularization of initial data.

1.3. **Plan of the paper and notations.** The paper is organized as follows. After introducing our concepts of solution in Section **??**, the main results of the paper are presented in Section **??**. Sections **??**, **??** and **??** are devoted to proofs. Some general results used in the existence proofs are stated and proven in the Appendix.

Let us establish some notations. For all $u \in \mathbb{R}$ we set $u_\pm := \max\{\pm u, 0\}$, and for any $f : \mathbb{R} \mapsto \mathbb{R}$ $f_\pm(u) := [f(u)]_\pm$ $(u \in \mathbb{R})$, thus $f = f_+ - f_-$. We shall make use of the truncation $T_n(u) := \max\{-n, \min\{u, n\}\}$ $(n \in \mathbb{N}, u \in \mathbb{R})$. We denote by $|\cdot|$ the Lebesgue measure. A Borel set $E \subseteq \mathbb{R}$ such that $|E| = 0$ is called a null set, and "almost everywhere", or shortly "a.e.", means "up to null sets".

By $\mathcal{M}(\mathbb{R})$ (respectively $\mathcal{M}^+(\mathbb{R})$) we denote the space of finite signed (respectively, the cone of finite nonnegative) Radon measures on $\mathbb{R}$. The space $\mathcal{M}(\mathbb{R})$ is ordered by the inequality "$\le$" defined as follows: $\mu \le \nu$ if $\mu(E) \le \nu(E)$ for any Borel set $E \subseteq \mathbb{R}$ $(\mu, \nu \in \mathcal{M}(\mathbb{R}))$. For any $\mu \in \mathcal{M}(\mathbb{R})$ (i) μ_{ac} and μ_s denote the absolutely continuous and the singular part of μ with respect to the Lebesgue measure, thus $\mu = \mu_{ac} + \mu_s$, and $\mu_r \in L^1(\mathbb{R})$ is the density of μ_{ac}; (ii) μ^+ and μ^- are the positive and the negative part of μ, thus $\mu = \mu^+ - \mu^-$ (notice that $[\mu_s]^\pm = [\mu^\pm]_s =: \mu_s^\pm$); (iii) $|\mu|(\mathbb{R}) := \mu^+(\mathbb{R}) + \mu^-(\mathbb{R})$ is the total variation of μ. The space $\mathcal{M}(\mathbb{R})$ is a Banach space with norm $\|\mu\|_{\mathcal{M}(\mathbb{R})} := |\mu|(\mathbb{R})$. For any $\zeta \in C_c(\mathbb{R})$ the symbol $\langle \mu, \zeta \rangle$ denotes the duality between $\mu \in \mathcal{M}(\mathbb{R})$ and ζ. Similar remarks hold for the space $\mathcal{M}(S)$ of finite signed Radon measures on $S := \mathbb{R} \times (0, T)$.

For any Borel set $E \subseteq \mathbb{R}$, the restriction $\mu \llcorner E$ of $\mu \in \mathcal{M}(\mathbb{R})$ to E is defined by setting $(\mu \llcorner E)(A) := \mu(E \cap A)$ for every Borel set $A \subseteq \mathbb{R}$.

For every $a \in \mathbb{R}$ and $\mu \in \mathcal{M}(\mathbb{R})$, we denote by $\mathcal{T}_a \mu$ the *translated measure* of μ,

$$\langle \mathcal{T}_a \mu, \rho \rangle := \langle \mu, \rho(\cdot + a) \rangle \quad \text{for all } \rho \in C_c(\mathbb{R}). \tag{1.16}$$

2. Solution concepts and related notions

Let us recall that $u \in \mathcal{M}^+(S)$ belongs to the space $L^\infty_{w*}(0, T; \mathcal{M}^+(\mathbb{R}))$, if for a.e. $t \in (0, T)$ there exists $u(t) \in \mathcal{M}^+(\mathbb{R})$ such that:
(i) for every $\zeta \in C_c(\overline{S})$ the map $t \to \langle u(t), \zeta(\cdot, t) \rangle$ is Lebesgue measurable, and

$$\langle u, \zeta \rangle = \int_0^T \langle u(t), \zeta(\cdot, t) \rangle \, dt; \tag{2.1}$$

(ii) there exists a constant $C > 0$ such that $\operatorname{ess\,sup}_{t \in (0,T)} \|u(t)\|_{\mathcal{M}(\mathbb{R})} \le C$
(e.g., see [**?**, Chapter 4]). We set $\|u\|_{L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}))} := \operatorname{ess\,sup}_{t \in (0,T)} \|u(t)\|_{\mathcal{M}(\mathbb{R})}$.

If $u \in L^\infty_{w*}(0, T; \mathcal{M}^+(\mathbb{R}))$, it is easily seen that $u_{ac}, u_s \in L^\infty_{w*}(0, T; \mathcal{M}^+(\mathbb{R}))$ and $u_r \in L^\infty(0, T; L^1(\mathbb{R}))$. We say that a finite Radon measure $u \in \mathcal{M}(S)$ belongs to $L^\infty_{w*}(0, T; \mathcal{M}(\mathbb{R}))$ if both u_+ and u_- belong to $L^\infty_{w*}(0, T; \mathcal{M}^+(\mathbb{R}))$.

Our first concept of solution is given by the following definition.

Definition 2.1. Let $u_0 \in \mathcal{M}(\mathbb{R})$, and let (A_2) hold. By a *solution* of problem (P) we mean any $u \in L^\infty_{w*}(0, T; \mathcal{M}(\mathbb{R}))$ such that for any $\zeta \in C^1_c(\overline{S})$, $\zeta(\cdot, T) = 0$

$$\int_0^T \langle u(t), \zeta_t(\cdot, t) \rangle \, dt + \iint_S \varphi(u_r) \zeta_x \, dx dt + C_0 \int_0^T \langle u_s^+(t), \zeta_x(\cdot, t) \rangle \, dt = -\langle u_0, \zeta(\cdot, 0) \rangle. \tag{2.2}$$

Since $u_r \in L^\infty(0, T; L^1(\mathbb{R}))$ if $u \in L^\infty_{w*}(0, T; \mathcal{M}(\mathbb{R}))$, by assumption (A_2) there holds $\varphi(u_r) \in L^\infty(0, T; L^1(\mathbb{R}))$, thus equality (**??**) is well posed. From (**??**) by a proper choice of ζ and standard regularization results we get the following:

Lemma 2.1. *Let $u_0 \in \mathcal{M}(\mathbb{R})$, let (A_2) hold, and let u be a solution of problem (P). Then the map $t \mapsto u(t)$ has a representative defined for all $t \in [0, T]$ and continuous in $[0, T]$ with* *respect to the weak* topology of* $\mathcal{M}(\mathbb{R})$*. Moreover,* $u(0) = u_0$ *in* $\mathcal{M}(\mathbb{R})$*, and for any* $\tau \in [0,T]$ *and* $\rho \in C_c^1(\mathbb{R})$

$$\langle u(\tau),\rho\rangle = \langle u_0,\rho\rangle + \int_0^\tau\int_{\mathbb{R}} \varphi(u_r)\rho'(x)\,dxdt + C_0\int_0^\tau \langle u_s^+(t),\rho'\rangle\,dt. \tag{2.3}$$

Remark 2.1. In view of Lemma **??**, for any solution u of (P) the measure $u(t)$ is defined for all $t \in [0,T]$. In the following, for simplicity of notations, for every $t \in [0,T]$ we always set

$$u_s(t) := [u(t)]_s\,, \quad u_s^\pm(t) := [[u(t)]_s]^\pm\,,$$
$$|u_s(t)| := |[u(t)]_s|\,, \quad u_r(\cdot,t) := [u(t)]_r(\cdot)\,.$$

Definition 2.2. Let $u_0 \in \mathcal{M}(\mathbb{R})$, and let (A_2) hold. By an *entropy solution* of problem (P) we mean a solution u such that, for all $k \in \mathbb{R}$ and $\zeta \in C_c^1(\overline{S})$, $\zeta \geq 0$, $\zeta(\cdot,T) = 0$,

$$\begin{aligned}\iint_S \{|u_r-k|\zeta_t + \operatorname{sgn}(u_r-k)[\varphi(u_r)-\varphi(k)]\,\zeta_x\}\,dxdt\, + \\ + \int_0^T \langle |u_s(t)|,\zeta_t(\cdot,t)\rangle\,dt + C_0\int_0^T\langle u_s^+(t),\zeta_x(\cdot,t)\rangle\,dt \geq \\ \geq -\int_{\mathbb{R}} |u_{0r}-k|\,\zeta(x,0)\,dx - \langle |u_{0s}|,\zeta(\cdot,0)\rangle\,.\end{aligned} \tag{2.4}$$

Remark 2.2. Let u be an entropy solution of problem (P). Summing equality (**??**) to inequality (**??**) gives for all $k \in \mathbb{R}$ and $\zeta \in C_c^1(\overline{S})$, $\zeta \geq 0$, $\zeta(\cdot,T) = 0$,

$$\begin{aligned}\iint_S \{[u_r-k]_+\,\zeta_t + \operatorname{sgn}_+(u_r-k)[\varphi(u_r)-\varphi(k)]\,\zeta_x\}\,dxdt\, + \\ + \int_0^T \langle u_s^+(t),\zeta_t(\cdot,t)\rangle\,dt + C_0\int_0^T\langle u_s^+(t),\zeta_x(\cdot,t)\rangle\,dt \geq \\ \geq -\int_{\mathbb{R}} [u_{0r}-k]_+\,\zeta(x,0)\,dx - \langle u_{0s}^+,\zeta(\cdot,0)\rangle\,.\end{aligned} \tag{2.5a}$$

Similarly, subtracting (**??**) from (**??**) gives for all $k \in \mathbb{R}$ and ζ as above

$$\begin{aligned}\iint_S \{[u_r-k]_-\zeta_t + \operatorname{sgn}_-(u_r-k)[\varphi(u_r)-\varphi(k)]\,\zeta_x\}\,dxdt + \int_0^T \langle u_s^-(t),\zeta_t(\cdot,t)\rangle\,dt \geq \\ \geq -\int_{\mathbb{R}} [u_{0r}-k]_-\,\zeta(x,0)\,dx - \langle u_{0s}^-,\zeta(\cdot,0)\rangle\,.\end{aligned} \tag{2.5b}$$

3. Results

3.1. Monotonicity and support properties of the singular part of entropy solutions.

For general initial measures $u_0 \in \mathcal{M}(\mathbb{R})$, entropy solutions have the following monotonicity property:

Proposition 3.1. *Let* $u_0 \in \mathcal{M}(\mathbb{R})$*, let* (A_2) *hold, and let* u *be an entropy solution of problem* (P)*. Then*

$$u_s^+(t_2) \leq \mathcal{T}_{C_0(t_2-t_1)}u_s^+(t_1) \quad \text{for a.e. } 0 < t_1 \leq t_2 < T\,, \tag{3.1a}$$

$$u_s^+(t) \leq \mathcal{T}_{C_0 t}u_{0s}^+ \quad \text{for a.e. } t \in (0,T)\,, \tag{3.1b}$$

$$u_s^-(t_2) \leq u_s^-(t_1) \leq u_{0s}^- \quad \text{for a.e. } 0 \leq t_1 \leq t_2 \leq T\,. \tag{3.1c}$$

Let us address the case where u_{0s} is the sum of a finite number of Dirac masses - namely, assumption (A_1) holds. In view of Proposition **??**, the map $t \mapsto u_s^+(t)$ (respectively $t \mapsto u_s^-(t)$) is nonincreasing along the segment P_i (respectively M_l ; see (**??**)). More precisely, there exist nonincreasing functions $p_i : [0,T] \to [0,p_{0i}\,]$, $m_l : [0,T] \to [0,m_{0l}\,]$ and a null set $N \subseteq (0,T)$ such that

$$u_s^+(t) = \sum_{i=1}^{P_+} p_i(t)\,\delta_{a_i+C_0 t}\,, \quad u_s^-(t) = \sum_{l=1}^{M_-} m_l(t)\,\delta_{b_l} \quad \text{for any } t \in (0,T) \setminus N. \tag{3.2}$$

By the monotonicity of p_i and m_l, for all i, l there holds

$$p_i(t^+) \le p_i(t^-)\,, \quad m_l(t^+) \le m_l(t^-) \quad \text{for any } t \in (0,T)\,, \tag{3.3}$$

where $p_i(t^\pm) := \lim_{\tau\to t^\pm} p_i(\tau)$ and $m_l(t^\pm) := \lim_{\tau\to t^\pm} m_l(\tau)$. Without loss of generality, we may assume the functions p_i and m_l to be continuous at any point $t \in (0,T) \setminus N$.

To pursue our analysis we need the following proposition, which shows that both the absolutely continuous and the singular part of an entropy solution of (P) are weakly* time continuous in the whole interval $[0,T]$.

Proposition 3.2. *Let* (A_1)-(A_2) *hold, and let* u *be an entropy solution of problem* (P). *Then for any* $t_0 \in [0,T]$ *and* $\rho \in C_c(\mathbb{R})$ *there holds*

$$\operatorname*{ess\,lim}_{t\to t_0} \int_{\mathbb{R}} u_r(x,t)\,\rho(x)\,dx = \int_{\mathbb{R}} u_r(x,t_0)\,\rho(x)\,dx\,, \tag{3.4a}$$

$$\operatorname*{ess\,lim}_{t\to t_0} \langle u_s(t),\rho\rangle = \langle u_s(t_0),\rho\rangle\,. \tag{3.4b}$$

The following result follows at once from (**??**).

Proposition 3.3. *Let* (A_1)-(A_2) *hold, and let* u *be an entropy solution of problem* (P). *Then*

$$u_s(t) = \sum_{i=1}^{P_+} p_i(t^\pm)\,\delta_{a_i+C_0t} - \sum_{l=1}^{M_-} m_l(t^\pm)\,\delta_{b_l} \quad \textit{for all } t \in (0,T)\,, \tag{3.5a}$$

$$u_s(0) = u_{0s}\,, \qquad u_s(T) = \sum_{i=1}^{P_+} p_i(T^-)\,\delta_{a_i+C_0T} - \sum_{l=1}^{M_-} m_l(T^-)\,\delta_{b_l}\,. \tag{3.5b}$$

Remark 3.1. Let us point out that the weak* continuity of the map $t \mapsto u_s(t)$ in the whole interval $[0,T]$ (see Proposition **??**) need not hold separately for the maps $t \mapsto u_s^\pm(t)$. This indeed happens if $C_0 = 0$ (even in the strong topology of $\mathcal{M}(\mathbb{R})$; see [**?**, Corollary 1], [**?**, Proposition 3.20-(ii)]). However, as already observed, two segments P_i and M_l can intersect if $C_0 \ne 0$, in which case the continuity of the map $t \mapsto u_s^\pm(t)$ at the intersection point need not be true.

Corresponding remarks hold for the maps p_i, m_l in (**??**). By (**??**) and (**??**), what definitely applies for all $t \in (0,T)$ are the inequalities:

$$u_s^+(t) \le \sum_{i=1}^{P_+} p_i(t^+)\,\delta_{a_i+C_0t} \le \sum_{i=1}^{P_+} p_i(t^-)\,\delta_{a_i+C_0t}\,, \tag{3.6a}$$

$$u_s^-(t) \le \sum_{l=1}^{M_-} m_l(t^+)\,\delta_{b_l} \le \sum_{l=1}^{M_-} m_l(t^-)\,\delta_{b_l}\,, \tag{3.6b}$$

whereas for all $t \in [0,T]$ there holds (see (**??**)):

$$\operatorname{supp} u_s^+(t) \subseteq \bigcup_{i=1}^{P_+} \{(x,t)\,|\,x = a_i + C_0t\}\,, \qquad \operatorname{supp} u_s^-(t) \subseteq \bigcup_{l=1}^{M_-} \{(x,t)\,|\,x = b_l\}\,. \tag{3.6c}$$

3.2. **Existence.** As already pointed out, existence of entropy solutions of (P) is proven by a constructive approach analogous to that used in [**?**] for the nonnegative case. Consider the *approximating problems*

$$(P_n) \qquad \begin{cases} u_{nt} + [\varphi_n(u_n)]_x = 0 & \text{in } S \\ u_n = u_{0n} & \text{in } \mathbb{R}\times\{0\}\,, \end{cases}$$

where

$$\varphi_n(u) := \varphi_b(T_n(u)) + C_0\,u_+\,, \qquad T_n(u) := \max\{-n, \min\{u,n\}\} \qquad (n \in \mathbb{N}, u \in \mathbb{R})\,, \tag{3.7}$$

and $\{u_{0n}\} \subseteq BV(\mathbb{R})$ is defined by (**??**).

To prove existence the following assumption is used:

$$(A_3) \qquad \begin{cases} \text{for any } \bar{\xi} \in \mathbb{R} \text{ there exist } a, b \geq 0,\ a+b>0 \text{ such that} \\ \varphi \text{ is strictly convex or concave in } [\bar{\xi}-a, \bar{\xi}+b], \end{cases}$$

(see Section **??** and Theorem **??**; let us mention that (A_3) is a weaker form of assumption (C_2) used in [**?**, Theorem 3.7]). Our main existence result can be stated as follows.

Theorem 3.4. *Let assumptions (A_1)-(A_3) hold. Then there exists an entropy solution u of problem (P). Moreover, u is obtained as a limiting weak* point of the sequence $\{u_n\}$ of entropy solutions of the approximating problems (P_n), in the sense that*

$$u_n(\cdot,t) \overset{*}{\rightharpoonup} u(t) \ \text{ in } \mathcal{M}(\mathbb{R}) \quad \text{for all } t \in [0,T]. \tag{3.8}$$

Definition 3.1. Let assumptions (A_1)-(A_3) hold. By a *constructed entropy solution of* (P) we mean any entropy solution obtained as in the proof of Theorem **??**.

3.3. **Additional continuity properties of constructed entropy solutions.** The following theorem shows that for any *constructed* entropy solution u the mappings $t \mapsto u_s^{\pm}(t)$ are weakly* continuous from the right in $[0,T)$. As a consequence, we get the structural equalities (**??**) below, which hold *everywhere in* $[0,T)$ and improve on equalities (**??**).

Theorem 3.5. *Let assumptions (A_1)-(A_3) hold. Let u be a constructed entropy solution of problem (P). Then for any $t \in [0,T)$ there holds*

$$u_s^{\pm}(\tau) \overset{*}{\rightharpoonup} u_s^{\pm}(t) \quad \text{in } \mathcal{M}(\mathbb{R}) \ \text{as } \tau \to t^+, \tag{3.9a}$$

$$u_s^+(t) = \sum_{i=1}^{P_+} p_i(t^+)\, \delta_{a_i+C_0 t}, \qquad u_s^-(t) = \sum_{l=1}^{M_-} m_l(t^+)\, \delta_{b_l}. \tag{3.9b}$$

3.4. **Compatibility conditions.** Let us first state the following definition.

Definition 3.2. Let (A_1)-(A_2) hold, and let $\tau \in (0,T]$.
(i) Let $a_i + C_0\tau \in \operatorname{supp} u_s^+(\tau)$. An entropy solution of (P) satisfies the *compatibility conditions in* $[0,\tau]$ *at* a_i $(i=1,\dots,P_+)$ if for all $\beta \in C_c^1(0,\tau)$, $\beta \geq 0$, and $k \in \mathbb{R}$ there holds

$$\limsup_{\delta\to 0^+} \frac{1}{\delta}\Bigg\{ \int_0^\tau\!\!\int_{a_i+C_0t}^{a_i+C_0t+\delta} \operatorname{sgn}_-(u_r(x,t)-k)[\hat{\varphi}(u_r(x,t))-\hat{\varphi}(k)]\beta(t)\,dxdt \ - \tag{3.10a}$$
$$- \quad C_0 \int_0^\tau u_s^-(t)\big((a_i+C_0t, a_i+C_0t+\delta)\big)\,\beta(t)\,dt\Bigg\} \ \leq 0,$$

$$\liminf_{\delta\to 0^+} \frac{1}{\delta}\Bigg\{ \int_0^\tau\!\!\int_{a_i+C_0t-\delta}^{a_i+C_0t} \operatorname{sgn}_-(u_r(x,t)-k)[\hat{\varphi}(u_r(x,t))-\hat{\varphi}(k)]\beta(t)\,dxdt \ - \tag{3.10b}$$
$$- \quad C_0 \int_0^\tau u_s^-(t)\big((a_i+C_0t-\delta, a_i+C_0t)\big)\,\beta(t)\,dt\Bigg\} \ \geq 0,$$

where $\hat{\varphi}(u) := \varphi(u) - C_0\,u = \varphi_b(u) + C_0\,u_-$ $(u \in \mathbb{R})$.
(ii) Let $b_l \in \operatorname{supp} u_s^-(\tau)$. An entropy solution of (P) satisfies the *compatibility conditions in* $[0,\tau]$ *at* b_l $(l=1,\dots,M_-)$ if for all $\beta \in C_c^1(0,\tau)$, $\beta \geq 0$, and $k \in \mathbb{R}$ there holds

$$\limsup_{\delta\to 0^+} \frac{1}{\delta}\Bigg\{ \int_0^\tau\!\!\int_{b_l}^{b_l+\delta} \operatorname{sgn}_+(u_r(x,t)-k)[\varphi(u_r(x,t))-\varphi(k)]\beta(t)\,dxdt \ + \tag{3.11a}$$
$$+ \quad C_0 \int_0^\tau u_s^+(t)\big((b_l, b_l+\delta)\big)\,\beta(t)\,dt\Bigg\} \ \leq 0,$$

$$(3.11b)\qquad \liminf_{\delta\to 0^+}\ \frac{1}{\delta}\left\{\int_0^\tau\!\!\int_{b_l-\delta}^{b_l}\operatorname{sgn}_+(u_r(x,t)-k)[\varphi(u_r(x,t))-\varphi(k)]\beta(t)\,dxdt\,+\right.$$
$$\left.+\quad C_0\int_0^\tau u_s^+(t)\big((b_l-\delta,b_l)\big)\,\beta(t)\,dt\right\}\ \ge\ 0\,.$$

We can now point out another remarkable feature of constructed entropy solutions.

Theorem 3.6. *Let assumptions* (A_1)-(A_3) *hold. Then every constructed entropy solution of problem* (P) *satisfies the compatibility conditions.*

Remark 3.2. If $C_0=0$, inequalities (**??**) read

$$\pm\limsup_{\delta\to 0^+}\ \frac{1}{\delta}\left\{\int_0^\tau\!\!\int_{a_i}^{a_i\pm\delta}\operatorname{sgn}_-(u_r(x,t)-k)[\varphi(u_r(x,t))-\varphi(k)]\beta(t)\,dxdt\ \le\ 0\,,\right.$$

which corresponds to (**??**). Similarly, inequalities (**??**) when $C_0=0$ read

$$\pm\limsup_{\delta\to 0^+}\ \frac{1}{\delta}\left\{\int_0^\tau\!\!\int_{b_l}^{b_l\pm\delta}\operatorname{sgn}_+(u_r(x,t)-k)[\varphi(u_r(x,t))-\varphi(k)]\beta(t)\,dxdt\ \le\ 0\,,\right.$$

which corresponds to (**??**).

4. Monotonicity and support properties of the singular part: Proofs

Proof of Proposition **??**. We only prove inequality (**??**), the proof of (**??**)-(**??**) being similar. Let $t_2\in(0,T)$ be fixed. Choosing in (**??**) $\zeta(x,t)=\rho(x-C_0(t-t_2))\,h(t)$ with $\rho\in C_c^1(\mathbb{R})$, $\rho\ge 0$ and $h\in C^1([0,T])$, $h\ge 0$, $h(T)=0$, for any $k>0$ we get

$$(4.1)\qquad \int_0^T\langle u_s^+(t),\rho(\cdot-C_0(t-t_2))\rangle\ h'(t)\,dt\ +$$
$$+\quad\iint_S[u_r(x,t)-k]_+\,\rho(x-C_0(t-t_2))\,h'(t)\,dxdt\ +$$
$$+\quad\iint_S\operatorname{sgn}_+(u_r-k)[\varphi_b(u_r(x,t))-\varphi_b(k)]\,\rho'(x-C_0(t-t_2))\,h(t)\,dxdt\ \ge$$
$$\ge\quad -h(0)\left\{\int_{\mathbb{R}}[u_{0r}-k]_+\,\rho(x+C_0t_2)\,dx\ +\langle u_{0s}^+\,,\rho(\cdot+C_0t_2)\rangle\right\}\,.$$

Letting $k\to\infty$ in the above inequality, by the Dominated Convergence theorem, we obtain

$$(4.2)\qquad \int_0^T\langle u_s^+(t),\rho(\cdot-C_0(t-t_2))\rangle\ h'(t)\,dt\ \ge\ -h(0)\,\langle u_{0s}^+\,,\,\rho(\cdot+C_0t_2)\rangle\ .$$

Choosing in (**??**)

$$h(t):=\frac{1}{\delta}\left(t-t_1+\frac{\delta}{2}\right)\chi_{[t_1-\frac{\delta}{2},t_1+\frac{\delta}{2}]}(t)+\chi_{(t_1+\frac{\delta}{2},t_2-\frac{\delta}{2})}(t)+\frac{1}{\delta}\left(t_2+\frac{\delta}{2}-t\right)\chi_{[t_2-\frac{\delta}{2},t_2+\frac{\delta}{2}]}(t)\,,$$

with $\delta>0$ sufficiently small gives

$$(4.3)\qquad \frac{1}{\delta}\int_{t_2-\frac{\delta}{2}}^{t_2+\frac{\delta}{2}}\langle u_s^+(t),\rho(\cdot-C_0(t-t_2))\rangle\,dt\ \le\ \frac{1}{\delta}\int_{t_1-\frac{\delta}{2}}^{t_1+\frac{\delta}{2}}\langle u_s^+(t),\rho(\cdot-C_0(t-t_2))\rangle\,dt\,.$$

Since $u_s^+\in L^\infty_{w*}(0,T;\mathcal{M}^+(\mathbb{R}))$, there exists a null set $N\subseteq(0,T)$ such that for all $t_0\in(0,T)\smallsetminus N$ and $\zeta\in C_c(\overline{S})$ there holds

$$(4.4)\qquad \lim_{\delta\to 0^+}\frac{1}{\delta}\int_{t_0-\frac{\delta}{2}}^{t_0+\frac{\delta}{2}}\langle u_s^+(t),\zeta(\cdot,t)\rangle\,dt=\langle u_s^+(t_0),\zeta(\cdot,t_0)\rangle$$

(e.g., see the proof of [**?**, Lemma 3.1]). Then letting $\delta\to 0^+$ in (**??**) we get

$$\langle u_s^+(t_2),\rho\rangle\ \le\ \langle u_s^+(t_1),\rho(\cdot-C_0(t_1-t_2))\rangle$$

for any t_1, $t_2 \in (0,T) \setminus N$ and $\rho \in C^1_c(\mathbb{R})$, $\rho \geq 0$. Hence the result follows. □

Remark 4.1. Let u be an entropy solution of problem (P). Since $u_r \in L^\infty(0,T;L^1(\mathbb{R}))$ and $u_s^\pm \in L^\infty_{w*}(0,T;\mathcal{M}^+(\mathbb{R}))$, by standard separability arguments there exists a null set $N \subseteq (0,T)$ such that for all $t_0 \in (0,T) \setminus N$, $\zeta \in C_c(\overline{S})$ and $k \in \mathbb{R}$ there hold both (**??**) and

$$\lim_{\delta\to 0^+} \frac{1}{\delta} \int_{t_0-\frac{\delta}{2}}^{t_0+\frac{\delta}{2}} dt \int_{\mathbb{R}} [u_r - k]_\pm \zeta \, dx = \int_{\mathbb{R}} [u_r(x,t_0) - k]_\pm \zeta(x,t_0)\, dx\,. \tag{4.5}$$

Without loss of generality, we assume that inequalities (**??**) hold in $(0,T) \setminus N$.

Let $\tau \in (0,T) \setminus N$ be fixed, and choose in (**??**) $\zeta(x,t) = \rho(x - C_0(t-\tau))\, h(t)$ with $\rho \in C^1_c(\mathbb{R})$, $\rho \geq 0$ and

$$h(t) := \chi_{[0,\tau-\frac{\delta}{2})}(t) + \frac{1}{\delta}\left(\tau + \frac{\delta}{2} - t\right) \chi_{[\tau-\frac{\delta}{2},\tau+\frac{\delta}{2}]}(t) \tag{4.6}$$

with $\delta > 0$ sufficiently small. Then letting $\delta \to 0^+$ in (**??**) for any $k > 0$ we get

$$\begin{aligned} &\int_{\mathbb{R}} [u_r(x,\tau) - k]_+ \,\rho(x)\, dx + \langle u_s^+(\tau), \rho\rangle \leq \\ \leq\; &\int_{\mathbb{R}} [u_{0r} - k]_+ \,\rho(x + C_0\tau)\, dx + \langle u_{0s}^+, \rho(\cdot + C_0\tau)\rangle + \\ +\; &\int_0^\tau \int_{\mathbb{R}} \operatorname{sgn}_+(u_r - k)[\varphi_b(u_r) - \varphi_b(k)]\, \rho'(x - C_0(t-\tau))\, dxdt\,. \end{aligned} \tag{4.7}$$

Similarly, choosing in (**??**) $\zeta(x,t) = \rho(x)h(t)$ with h as in (**??**) and letting $\delta \to 0^+$ we obtain for any $k \in \mathbb{R}$

$$\begin{aligned} &\int_{\mathbb{R}} [u_r(x,\tau) - k]_- \,\rho(x)\, dx + \langle u_s^-(\tau), \rho\rangle \leq \\ \leq\; &\int_{\mathbb{R}} [u_{0r} - k]_- \,\rho(x)\, dx + \langle u_{0s}^-, \rho\rangle + \int_0^\tau \int_{\mathbb{R}} \operatorname{sgn}_-(u_r - k)[\varphi(u_r) - \varphi(k)]\, \rho'(x)\, dxdt\,. \end{aligned} \tag{4.8}$$

To prove Proposition **??** the following lemma is needed.

Lemma 4.1. *Let (A_1)-(A_2) hold, let u be an entropy solution of problem (P), and let $N \subseteq (0,T)$ be the null set in Remark* **??**. *Let $t_0 \in [0,T]$ and let $\{\tau_n\} \subseteq (0,T) \setminus N$ satisfy $\tau_n \to t_0$. Then there exist $f_{(\pm)} \in L^1(\mathbb{R})$, $f_{(\pm)} \geq 0$ a.e. in $\mathbb{R}$ such that, up to subsequences,*

$$[u_r(\cdot,\tau_n)]_\pm \overset{*}{\rightharpoonup} f_{(\pm)} \quad \text{in } \mathcal{M}(\mathbb{R})\,. \tag{4.9}$$

Proof. Observe preliminarily that, since φ_b is bounded in $\mathbb{R}$, there exist two sequences $\{\xi_q\} \subseteq \mathbb{R}$, $\{\xi'_q\} \subseteq \mathbb{R}$ such that $\lim_{q\to\infty} \xi_q = \lim_{q\to\infty} \xi'_q = \infty$, and

$$\sup_{z\geq \xi_q} [\varphi_b(z) - \varphi_b(\xi_q)] < \frac{1}{q}\,, \qquad \inf_{z\geq \xi'_q} [\varphi_b(z) - \varphi_b(\xi'_q)] > -\frac{1}{q} \qquad \text{for all } q \in \mathbb{N}\,. \tag{4.10}$$

Similarly, there exist $\{\tilde{\xi}_q\} \subseteq \mathbb{R}$, $\{\tilde{\xi}'_q\} \subseteq \mathbb{R}$ such that $\lim_{q\to\infty} \tilde{\xi}_q = \lim_{q\to\infty} \tilde{\xi}'_q = -\infty$, and

$$\inf_{z\leq \tilde{\xi}_q} [\varphi(z) - \varphi(\tilde{\xi}_q)] > -\frac{1}{q}\,, \qquad \sup_{z\leq \tilde{\xi}'_q} [\varphi(z) - \varphi(\tilde{\xi}'_q)] < \frac{1}{q} \qquad \text{for all } q \in \mathbb{N}\,. \tag{4.11}$$

We only prove (**??**) with " + " since the other case is similar, using (**??**) and (**??**) instead of (**??**) and (**??**). Since $[u_r]_+ \in L^\infty(0,T;L^1(\mathbb{R}))$, for all $k > 0$ there exist $\mu, \mu_k \in \mathcal{M}^+(\mathbb{R})$ such that, up to subsequences,

$$[u_r(\cdot,\tau_n)]_+ \overset{*}{\rightharpoonup} \mu\,, \qquad [u_r(\cdot,\tau_n) - k]_+ \overset{*}{\rightharpoonup} \mu_k \quad \text{in } \mathcal{M}(\mathbb{R})\,. \tag{4.12}$$

On the other hand, since

$$[u_r(\cdot,\tau_n)]_+ = \min\{k, [u_r(\cdot,\tau_n)]_+\} + [u_r(\cdot,\tau_n) - k]_+ \quad \text{for all } k > 0,$$

and $\{\min\{k,[u_r(\cdot,\tau_n)]_+\}\}$ is bounded in $L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$, letting $n\to\infty$ there holds

$$[\mu_k]_s = \mu_s \quad \text{in } \mathcal{M}(\mathbb{R}) \quad \text{for all } k>0\,. \tag{4.13}$$

Fix $\tau\in(0,T)\setminus N$, and recall that supp $u^+_{0s}=\{a_1,\cdots,a_{P_+}\}$ with $a_1<a_2<\cdots<a_{P_+}$. Let $i=1,\dots,P_+$. We set $z_i\equiv z_i(\tau):=a_i+C_0\tau$ and, for $i<P_+$,

$$I_0\equiv I_0(\tau):=(-\infty,z_1),\quad I_{P_+}\equiv I_{P_+}(\tau):=(z_{P_+},\infty),\quad I_i\equiv I_i(\tau):=(z_i,z_{i+1})$$
$$I_0(0):=(-\infty,a_1),\qquad I_{P_+}(0):=(a_{P_+},\infty),\qquad I_i(0):=(a_i,a_{i+1}).$$

By (**??**) there holds

$$u^+_s(\tau)\llcorner I_i(\tau) \;=\; u^+_{0s}\llcorner I_i(0) \;=\; 0 \quad \text{for } i=0,\cdots,P_+\,. \tag{4.14}$$

For all $i=1,\dots,P_+-1$, let $\alpha_{i,1}\in C^1_c([z_i,z_{i+1}))$, $\alpha_{i,2}\in C^1_c((z_i,z_{i+1}])$ be nonnegative, such that $\alpha_{i,1}+\alpha_{i,2}=1$ in $I_i\equiv I_i(\tau)$. Set also

$$\eta_{1,i,j}(x) \;:=\; j(x-z_i)\chi_{\left[z_i,z_i+\frac1j\right]}(x) \,+\, \chi_{\left(z_i+\frac1j,z_{i+1}\right]}(x)\,,$$

$$\eta_{2,i,j}(x) \;:=\; \chi_{\left[z_i,z_{i+1}-\frac1j\right]}(x) \,+\, j(z_{i+1}-x)\chi_{\left(z_{i+1}-\frac1j,z_{i+1}\right]}(x)$$

for any $j\in\mathbb{N}$ large enough. Observe that by (**??**) there holds

$$\langle u^+_s(\tau),\alpha_{i,1}\eta_{1,i,j}\rangle \;=\; \langle u^+_{0s},[\alpha_{i,1}\eta_{1,i,j}](\cdot+C_0\tau)\rangle \;=\; 0\,. \tag{4.15}$$

Let ξ_q be as in (**??**). Choosing $k=\xi_q$ and $\rho=\alpha_{i,1}\eta_{1,i,j}$ in (**??**), by (**??**) and (**??**) we get

$$\begin{aligned}
&\int_{\mathbb{R}}[u_r(x,\tau)-\xi_q]_+\,\alpha_{i,1}(x)\eta_{1,i,j}(x)\,dx \le \int_{\mathbb{R}}[u_{0r}-\xi_q]_+[\eta_{1,i,j}\alpha_{i,1}](x+C_0\tau)\,dx\,+\\
&+\int_0^\tau\!\!\int_{\mathbb{R}}\operatorname{sgn}_+(u_r-\xi_q)[\varphi_b(u_r)-\varphi_b(\xi_q)]\,[\eta_{1,i,j}\alpha'_{i,1}](x-C_0(t-\tau))\,dxdt\,+\\
&+\int_0^\tau\!\!\int_{\mathbb{R}}\operatorname{sgn}_+(u_r-\xi_q)[\varphi_b(u_r)-\varphi_b(\xi_q)]\,[\eta'_{1,i,j}\alpha_{i,1}](x-C_0(t-\tau))\,dxdt\,\le\\
&\le\int_{a_i}^{a_{i+1}}[u_{0r}-\xi_q]_+\,dx\,+\,2\|\varphi_b\|_\infty\|\alpha'_{i,1}\|_\infty\,|\{u_r>\xi_q\}|\,+\sup_{z\ge\xi_q}[\varphi_b(z)-\varphi_b(\xi_q)]\,\tau\,\|\eta'_{1,i,j}\|_1\,\le\\
&\le\int_{a_i}^{a_{i+1}}[u_{0r}-\xi_q]_+\,dx\,+\,2\|\varphi_b\|_\infty\|\alpha'_{i,1}\|_\infty\,|\{u_r>\xi_q\}|\,+\,\frac{\tau}{q}\,.
\end{aligned}$$

Letting $j\to\infty$ we obtain

$$\int_{z_i}^{z_{i+1}}[u_r(x,\tau)-\xi_q]_+\,\alpha_{i,1}(x)\,dx\le\int_{a_i}^{a_{i+1}}[u_{0r}-\xi_q]_+\,dx+2\|\varphi_b\|_\infty\|\alpha'_{i,1}\|_\infty\,|\{u_r>\xi_q\}|+\frac{\tau}{q}\,. \tag{4.16}$$

Similarly, choosing $k=\xi'_q$, $\rho=\alpha_{i,2}\eta_{2,i,j}$ in (**??**), and letting $j\to\infty$ gives

$$\int_{z_i}^{z_{i+1}}[u_r(x,\tau)-\xi'_q]_+\,\alpha_{i,2}(x)\,dx\le\int_{a_i}^{a_{i+1}}[u_{0r}-\xi'_q]_+\,dx+2\|\varphi_b\|_\infty\|\alpha'_{i,2}\|_\infty\,\left|\left\{u_r>\xi'_q\right\}\right|+\frac{\tau}{q}\,. \tag{4.17}$$

Since $\alpha_{i,1}+\alpha_{i,2}=1$ in I_i, from (**??**)-(**??**) for all $i=1,\dots,P_+-1$ we obtain

$$\begin{aligned}
\int_{I_i}[u_r(x,\tau)-k_q]_+\,dx\;\le\;&2\|\varphi_b\|_\infty(\|\alpha'_{i,1}\|_\infty\,|\{u_r>\xi_q\}|+\|\alpha'_{i,2}\|_\infty\,\left|\left\{u_r>\xi'_q\right\}\right|)+\frac{2\tau}{q}\,+\\
&+\int_{a_i}^{a_{i+1}}[u_{0r}-\xi_q]_+\,dx+\int_{a_i}^{a_{i+1}}[u_{0r}-\xi'_q]_+\,dx\,,
\end{aligned} \tag{4.18}$$

where $k_q:=\max\{\xi_q,\xi'_q\}$. Similarly, for all $\rho\in C^1_c(\mathbb{R})$, $0\le\rho\le1$ there holds

$$\begin{aligned}
&\int_{I_0}[u_r(x,\tau)-k_q]_+\,\rho(x)\,dx\le\int_{I_0}[u_r(x,\tau)-\xi'_q]_+\,\rho(x)\,dx\le\\
&\le\int_{-\infty}^{a_1}[u_{0r}-\xi'_q]_+\,dx\,+\,2\|\varphi_b\|_\infty\|\rho'\|_\infty\,\left|\left\{u_r>\xi'_q\right\}\right|\,+\,\frac{\tau}{q}\,,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}(4.20)\qquad &\int_{I_{P_+}} [u_r(x,\tau)-k_q]_+\,\rho(x)\,dx \le \int_{I_{P_+}} [u_r(x,\tau)-\xi_q]_+\,\rho(x)\,dx \le \\ &\le \int_{a_{P_+}}^{\infty} [u_{0r}-\xi_q]_+\,dx + 2\|\varphi_b\|_\infty \|\rho'\|_\infty\,|\{u_r>\xi_q\}| + \frac{\tau}{q}\,.\end{aligned}$$

By inequalities (**??**)-(**??**), for any $\rho\in C_c^1(\mathbb{R})$, $0\le\rho\le 1$ we obtain

$$\begin{aligned}(4.21)\qquad &\int_{\mathbb{R}} [u_r(x,\tau)-k_q]_+\,\rho\,dxdt \le \int_{-\infty}^{a_1} [u_{0r}-\xi_q']_+\,dx + \int_{a_{P_+}}^{\infty} [u_{0r}-\xi_q]_+\,dx + \\ &+ \sum_{i=1}^{P_+-1} \left\{ \int_{a_i}^{a_{i+1}} [u_{0r}-\xi_q]_+\,dx + \int_{a_i}^{a_{i+1}} [u_{0r}-\xi_q']_+\,dx \right\} + \\ &+ \sum_{i=1}^{P_+-1} \left\{ 2\|\varphi_b\|_\infty (\|\alpha_{i,1}'\|_\infty\,|\{u_r>\xi_q\}| + \|\alpha_{i,2}'\|_\infty\,\left|\left\{u_r>\xi_q'\right\}\right|) \right\} + \\ &+ 2\|\varphi_b\|_\infty \|\rho'\|_\infty \left(\left|\left\{u_r>\xi_q'\right\}\right| + |\{u_r>\xi_q\}| \right) + \frac{2\tau P_+}{q}\,.\end{aligned}$$

Choosing $\tau=\tau_n$ in the above estimate and letting $n\to\infty$, by (**??**)-(**??**) we get

$$\begin{aligned} &\langle\mu_s,\rho\rangle = \left\langle [\mu_{k_q}]_s\,,\,\rho\right\rangle \le \left\langle \mu_{k_q},\rho\right\rangle = \lim_{n\to\infty} \int_{\mathbb{R}} [u_r(x,\tau_n)-k_q]_+\,\rho\,dxdt \le \\ &\le \int_{-\infty}^{a_1} [u_{0r}-\xi_q']_+\,dx + \int_{a_{P_+}}^{\infty} [u_{0r}-\xi_q]_+\,dx + \\ &+ \sum_{i=1}^{P_+-1} \left\{ \int_{a_i}^{a_{i+1}} [u_{0r}-\xi_q]_+\,dx + \int_{a_i}^{a_{i+1}} [u_{0r}-\xi_q']_+\,dx \right\} + \\ &+ \sum_{i=1}^{P_+-1} \left\{ 2\|\varphi_b\|_\infty (\|\alpha_{i,1}'\|_\infty\,|\{u_r>\xi_q\}| + \|\alpha_{i,2}'\|_\infty\,\left|\left\{u_r>\xi_q'\right\}\right|) \right\} + \\ &+ 2\,\|\varphi_b\|_\infty \|\rho'\|_\infty \left(\left|\left\{u_r>\xi_q'\right\}\right| + |\{u_r>\xi_q\}| \right) + \frac{2t_0 P_+}{q}\,.\end{aligned}$$

Letting $q\to\infty$ in the above inequality gives $\langle\mu_s,\rho\rangle=0$ for any $\rho\in C_c^1(\mathbb{R})$, $0\le\rho\le 1$. Therefore, the limiting measure μ in (**??**) is absolutely continuous with respect to the Lebesgue measure, and from the first convergence in (**??**) we obtain (**??**) with "+". This completes the proof. □

Proof of Proposition **??**. Let $N\subseteq(0,T)$ be the null set in Remark **??**. Let $t_0\in[0,T]$ be fixed, and let $\{\tau_n\}\subseteq(0,T)\setminus N$ be any sequence such that $\tau_n\to t_0$. Since $u_s^\pm\in L^\infty_{w^*}(0,T;\mathcal{M}^+(\mathbb{R}))$, there exists $\nu_{(\pm)}\in\mathcal{M}^+(\mathbb{R})$ such that, up to subsequences, there holds

$$(4.22)\qquad u_s^\pm(\tau_n) \overset{*}{\rightharpoonup} \nu_{(\pm)} \quad \text{in } \mathcal{M}(\mathbb{R})\,,$$

whence

$$(4.23)\qquad u_s(\tau_n) \overset{*}{\rightharpoonup} \nu := \nu_{(+)}-\nu_{(-)} \quad \text{in } \mathcal{M}(\mathbb{R})\,.$$

In view of Lemma **??**, we also have that

$$(4.24)\qquad u_r(\cdot,\tau_n) \overset{*}{\rightharpoonup} f := f_{(+)}-f_{(-)} \quad \text{in } \mathcal{M}(\mathbb{R})\,,$$

with $f_{(\pm)}$ as in (**??**), thus $f\in L^1(\mathbb{R})$. Since $u(\tau_n)\overset{*}{\rightharpoonup} u(t_0)$ by Lemma **??**, from (**??**)-(**??**) it follows that

$$(4.25)\qquad u(t_0) = f+\nu \quad \text{in } \mathcal{M}(\mathbb{R})\,.$$

On the other hand, by inequalities (**??**) for any $n\in\mathbb{N}$ there holds

$$u_s^+(\tau_n) \le \mathcal{T}_{C_0\tau_n} u_{0s}^+\,, \qquad u_s^-(\tau_n) \le u_{0s}^- \quad \text{in } \mathcal{M}(\mathbb{R})\,.$$

Since $\mathcal{T}_{C_0\tau_n} u_{0s}^+ \overset{*}{\rightharpoonup} \mathcal{T}_{C_0 t_0} u_{0s}^+$ in $\mathcal{M}(\mathbb{R})$, letting $n\to\infty$ in the above inequalities and using (**??**) we obtain

$$\nu_{(+)} \le \mathcal{T}_{C_0 t_0} u_{0s}^+ , \qquad \nu_{(-)} \le u_{0s}^- \quad \text{in } \mathcal{M}(\mathbb{R}) .$$

Therefore, the measure ν in (**??**) is singular with respect to the Lebesgue measure. By the uniqueness of the Lebesgue decomposition, it follows from (**??**) that $\nu = u_s(t_0)$ in $\mathcal{M}(\mathbb{R})$, $f = [u(t_0)]_r(\cdot) = u_r(\cdot, t_0)$ a.e. in $\mathbb{R}$, and the convergences

$$u_r(\cdot,\tau_n) \overset{*}{\rightharpoonup} u_r(\cdot,t_0) , \qquad u_s(\tau_n) \overset{*}{\rightharpoonup} u_s(t_0) \quad \text{in } \mathcal{M}(\mathbb{R})$$

take place along the whole sequence $\{\tau_n\}$. By the arbitrariness of $\{\tau_n\}$ we get (**??**). □

5. Existence of entropy solutions: Proof

5.1. **The approximating problems.** Let φ_n be defined by (**??**). Observe that $\varphi_n \in \mathrm{Lip}\,(\mathbb{R})$, $\varphi_n(z) = \varphi(z)$ if $|z| \le n$, and $\varphi_n \to \varphi$ uniformly on the bounded subsets of $\mathbb{R}$. Let $u_0 \in \mathcal{M}(\mathbb{R})$ be any initial measure satisfying (A_1). For any $n \in \mathbb{N}$ set

$$I^+_{n,i} := \left(a_i - \frac{1}{2n^2}\, ,\, a_i + \frac{1}{2n^2}\right) \quad (i = 1,\dots,P_+), \qquad I_n := \left(\bigcup_{i=1}^{P_+} I^+_{n,i}\right) \cup \left(\bigcup_{l=1}^{M_-} I^-_{n,l}\right) .$$
$$I^-_{n,l} := \left(b_l - \frac{1}{2n^2}\, ,\, b_l + \frac{1}{2n^2}\right) \quad (l = 1,\dots,M_-),$$

Observe that for $n \in \mathbb{N}$ sufficiently large there holds $I^\pm_{n,j} \cap I^\pm_{n,k} = \varnothing$ for any $j \ne k$, and $I^+_{n,i} \cap I^-_{n,l} = \varnothing$ $(i = 1,\dots,P_+; l = 1,\dots,M_-)$.

Let p_{0i}, m_{0l} be as in (A_1). Let $\{\eta_n\}$ be a sequence of standard mollifiers. For $n \in \mathbb{N}$ we set

$$u_{0r,n} = (T_n(u_{0r}) * \eta_n)\chi_{[-n,n]\setminus I_n};\ \ u_{0s,n} = \sum_{i=1}^{P_+} p_{0i}\, n^2 \chi_{I^+_{n,i}} - \sum_{l=1}^{M_-} m_{0l}\, n^2 \chi_{I^-_{n,l}};\ \ u_{0n} = u_{0r,n} + u_{0s,n}. \tag{5.1}$$

Then $u_{0n} \in BV(\mathbb{R})$, there exists $M_0 > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u_{0n}\|_{L^1(\mathbb{R})} \le M_0 , \tag{5.2}$$

and there holds

$$f(u_{0n}) \overset{*}{\rightharpoonup} f(u_{0r}) + C_{f,+}\, u_{0s}^+ - C_{f,-}\, u_{0s}^- \quad \text{in } \mathcal{M}(\mathbb{R}) \tag{5.3}$$

for any $f \in C(\mathbb{R})$ such that

$$\lim_{\xi\to\pm\infty} \frac{f(\xi)}{\xi} =: C_{f,\pm} \in \mathbb{R} . \tag{5.4}$$

The last statement follows from the the proof of [**?**, Lemma 5.2], since $u_{0r,n} \to u_{0r}$ in $L^1(\mathbb{R})$ and a.e. in $\mathbb{R}$, and $[u_{0s,n}]_\pm \overset{*}{\rightharpoonup} u_{0s}^\pm$ in $\mathcal{M}(\mathbb{R})$).

For every $n \in \mathbb{N}$ there exists a unique entropy solution $u_n \in C([0,T]; L^1(\mathbb{R})) \cap L^\infty(S)$ of the Cauchy problem (P_n); moreover, there holds $u_n(\cdot,t) \in BV(\mathbb{R})$ for any $t \in [0,T]$ since $u_{0n} \in BV(\mathbb{R})$. Hereafter we shall identify $u_n(\cdot,t)$ with any of its representatives, which are defined pointwise in $\mathbb{R}$. Relying on the entropy inequalities:

$$\begin{aligned} &\int_{t_0}^{t_1}\!\!\int_{\mathbb{R}} \{[u_n(x,t)-k]_+\,\zeta_t + \mathrm{sgn}_+(u_n(x,t)-k)[\varphi_n(u_n(x,t)) - \varphi_n(k)]\,\zeta_x\}\, dxdt \ge \\ &\ge \int_{\mathbb{R}} [u_n(x,t_1)-k]_+\,\zeta(x,t_1)\, dx - \int_{\mathbb{R}} [u_n(x,t_0)-k]_+\,\zeta(x,t_0)\, dx , \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\int_{t_0}^{t_1}\!\!\int_{\mathbb{R}} \{[u_n(x,t)-k]_-\zeta_t + \mathrm{sgn}_-(u_n(x,t)-k)[\varphi_n(u_n(x,t)) - \varphi_n(k)]\,\zeta_x\}\, dxdt \ge \\ &\ge \int_{\mathbb{R}} [u_n(x,t_1)-k]_-\zeta(x,t_1)\, dx - \int_{\mathbb{R}} [u_n(x,t_0)-k]_-\zeta(x,t_0)\, dx , \end{aligned} \tag{5.6}$$

$$\int_{t_0}^{t_1}\int_{\mathbb{R}} \{|u_n - k|\zeta_t + \operatorname{sgn}(u_n(x,t)-k)[\varphi_n(u_n(x,t)) - \varphi_n(k)]\,\zeta_x\}\,dxdt \geq \tag{5.7}$$
$$\geq \int_{\mathbb{R}} |u_n(x,t_1)-k|\zeta(x,t_1)\,dx - \int_{\mathbb{R}} |u_n(x,t_0)-k|\zeta(x,t_0)\,dx$$

(which hold for any nonnegative $\zeta \in C^1(\overline{S})$, $k \in \mathbb{R}$, and $0 \le t_0 < t_1 \le T$), it can be proven that for all $n \in \mathbb{N}$ there holds

$$\|u_n\|_{L^\infty(0,T;L^1(\mathbb{R}))} \le \|u_{0n}\|_{L^1(\mathbb{R})} \le M_0\,. \tag{5.8}$$

Proposition 5.1. *Let u_n be the entropy solution of problem (P_n) with u_{0n} given by (**??**), and let $x_0 \in \mathbb{R}$ be fixed.*

(i) If $u_n(x_0^\pm, t_1) := \lim_{x\to x_0^\pm} u_n(x,t_1) > n$ for some $t_1 \in (0,T]$, there holds

$$u_n((x_0 + C_0(t_0 - t_1))^\pm, t_0) \ge n \quad \text{for all } t_0 \in [0,t_1)\,. \tag{5.9}$$

(ii) If $u_n(x_0^\pm, t_1) < -n$ for some $t_1 \in (0,T]$, there holds

$$u_n(x_0^\pm, t_0) \le -n \quad \text{for all } t_0 \in [0,t_1)\,. \tag{5.10}$$

Proof. Observe that the limits $u_n(x_0^\pm, t_1)$ exist, since $u_n(\cdot,t) \in BV(\mathbb{R})$ for all $t \in [0,T]$. To prove (i), we choose in (**??**) $k = n$ and $\zeta(x,t) = \rho(x - C_0(t-t_1))$ with $\rho \in C_c^1(\mathbb{R})$, $\rho \ge 0$. Since

$$\begin{aligned} &\operatorname{sgn}_+(u_n(x,t)-n)[\varphi_n(u_n(x,t)) - \varphi_n(n)] = \\ = \;&\operatorname{sgn}_+(u_n(x,t)-n)[\varphi_b(T_n(u_n(x,t))) - \varphi_b(n) + C_0([u_n(x,t)]_+ - n)] = \\ = \;&C_0 \operatorname{sgn}_+(u_n(x,t)-n)([u_n(x,t)]_+ - n) = C_0[u_n(x,t)-n]_+\,, \end{aligned}$$

with this choice the left-hand side of (**??**) vanishes. It follows that

$$\int_{\mathbb{R}} [u_n(x,t_1)-n]_+\,\rho(x)\,dx - \int_{\mathbb{R}} [u_n(x,t_0)-n]_+\,\rho(x - C_0(t_0-t_1))\,dx \le 0\,,$$

whence for all $0 \le t_0 < t_1 \le T$

$$\int_{\mathbb{R}} [u_n(x + C_0(t_0-t_1), t_0) - n]_+\,\rho(x)\,dx \ge \int_{\mathbb{R}} [u_n(x,t_1)-n]_+\,\rho(x)\,dx\,. \tag{5.11}$$

To prove (**??**), assume by contradiction that $u_n((x_0 + C_0(t_0-t_1))^+, t_0) < n$ for some $t_0 \in [0,t_1)$ (a similar argument holds for $u_n((x_0 + C_0(t_0-t_1))^-, t_0)$). Then there exists $\delta > 0$ (possibly depending on t_0) such that $[u_n(x + C_0(t_0-t_1), t_0) - n]_+ = 0$ for a.e. $x \in (x_0, x_0+\delta)$. Choosing ρ with $\operatorname{supp}\rho \subseteq (x_0, x_0+\delta)$ from (**??**) we get $\int_{x_0}^{x_0+\delta} [u_n(x,t_1)-n]_+\,\rho(x)\,dx \le 0$, a contradiction since $u_n(x_0^\pm, t_1) > n$. Hence the result follows in this case.

The proof of (ii) is similar, observing that

$$\begin{aligned} &\operatorname{sgn}_-(u_n(x,t)+n)[\varphi_n(u_n(x,t)) - \varphi_n(-n)] = \\ = \;&\operatorname{sgn}_-(u_n(x,t)+n)[\varphi_b(T_n(u_n(x,t))) - \varphi_b(-n) + C_0[u_n(x,t)]_+] = 0\,. \end{aligned}$$

By (**??**), with $k = -n$ and $\zeta(x,t) = \rho(x)$, ρ as above, we get for all $0 \le t_0 < t_1 \le T$

$$\int_{\mathbb{R}} [u_n(x,t_0)+n]_-\,\rho(x)\,dx \ge \int_{\mathbb{R}} [u_n(x,t_1)+n]_-\,\rho(x)\,dx\,,$$

whence the conclusion follows. □

5.2. **Letting $n \to \infty$ in the approximating problems.** In view of inequality (**??**), Theorem **??** can be applied to the sequence $\{u_n\}$ of entropy solutions of the approximating problems (see also Remark **??**). Hence there exist a Radon measure $u \in L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}))$, a subsequence of $\{u_n\}$ (not relabeled) and a Young measure $\nu \in \mathcal{Y}(S\times\mathbb{R})$ such that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{w*}(0,T;\mathcal{M}(\Omega)) \tag{5.12}$$

(see (**??**) and (**??**)), and $u_{b,(\pm)}(x,t) := \int_{\mathbb{R}} \xi_\pm\,d\nu_{(x,t)}(\xi)$ belong to $L^\infty(0,T;L^1(\mathbb{R}))$ (here $\{\nu_{(x,t)}\}$ $((x,t)\in S)$ denotes the disintegration of ν; e.g., see [**?**, Section 5.2]). Set

$$u_b := u_{b,(+)} - u_{b,(-)} \quad \text{a.e. in } S\,. \tag{5.13}$$

In view of (**??**) (with $f(\xi) = [\xi - k]_\pm$) and (**??**)-(**??**), it is easily seen that the results in Theorem **??** and Remark **??** (with $\Phi = \varphi$) hold along a suitable subsequence of $\{u_n\}$. Then we have the following theorem.

Theorem 5.2. *Let (A_1)-(A_2) hold. Let u, $\{u_n\}$, ν and u_b be as in* (**??**)-(**??**).
(i) There holds

$$u_b = u_r \quad \text{a.e. in } S. \tag{5.14}$$

(ii) There exists a subsequence of $\{u_n\}$ (not relabeled) such that in $L^\infty_{w^*}(0,T;\mathcal{M}(\mathbb{R}))$

$$[u_n]_\pm \overset{*}{\rightharpoonup} u_{b,(\pm)} + u_s^\pm\,, \tag{5.15}$$

$$f(u_n) \overset{*}{\rightharpoonup} f^* + C_{f,+}\, u_s^+ - C_{f,-}\, u_s^- \tag{5.16}$$

for any $f \in C(\mathbb{R})$ such that (**??**) *holds, with $f^* \in L^\infty(0,T;L^1_{\text{loc}}(\mathbb{R}))$ defined by*

$$f^*(x,t) = \int_{\mathbb{R}} f(\xi)\, d\nu_{(x,t)}(\xi) \quad \text{for a.e. } (x,t) \in S\,. \tag{5.17}$$

Our next task is to characterize the disintegration $\nu_{(x,t)}$ of the Young measure ν in Theorem **??**. To this purpose, both assumption (A_3) and a suitable parabolic approximation of the entropy solution u_n of (P_n) (for each $n \in \mathbb{N}$) are needed.

Arguing as in [**?**, Section 4] shows that u_n is a limiting point in $L^1(S)$ (and weakly* in $L^\infty(S)$) as $\epsilon \to 0^+$ of the family $\{u_n^\epsilon\}$ of solutions to the parabolic problems

$$(P_n^\epsilon) \qquad \begin{cases} \partial_t u_n^\epsilon + \partial_x[\varphi_n^\epsilon(u_n^\epsilon)] = \epsilon \partial_x^2 u_n^\epsilon & \text{in } S\,, \\ u_n^\epsilon = u_{0n}^\epsilon & \text{in } \mathbb{R} \times \{0\}\,. \end{cases}$$

Here

$$\varphi_n^\epsilon(\xi) = (\eta_\epsilon * \varphi_n)(\xi) \qquad (\xi \in \mathbb{R})$$

($\{\eta_\epsilon\}$ being a sequence of standard mollifiers), and $\{u_{0n}^\epsilon\} \subseteq C_c^\infty(\mathbb{R})$ satisfies

$$\|u_{0n}^\epsilon\|_{L^1(\mathbb{R})} \le \|u_{0n}\|_{L^1(\mathbb{R})} \le M_0\,, \qquad \|u_{0n}^\epsilon\|_{L^\infty(\mathbb{R})} \le \|u_{0n}\|_{L^\infty(\mathbb{R})}$$

$$u_{0n}^\epsilon \to u_{0n} \quad \text{in } L^1(\mathbb{R})\,, \qquad u_{0n}^\epsilon \overset{*}{\rightharpoonup} u_{0n} \quad \text{in } L^\infty(\mathbb{R})$$

as $\epsilon \to 0^+$. Relying on the above approximation, it can be checked that the disintegration $\nu_{(x,t)}$ satisfies equality (**??**) a.e. in S (the lengthy proof is modeled after the first part of that of [**?**, Proposition 5.8], thus we omit it.) Then by (**??**) and Theorem **??** below we get the following result.

Theorem 5.3. *Let assumptions (A_1)-(A_3) hold. Then there holds*

$$\nu_{(x,t)} = \delta_{u_r(x,t)} \quad \text{for a.e. } (x,t) \in S. \tag{5.18}$$

5.3. **Existence proof.** By (**??**)-(**??**) and (**??**), for any $\zeta \in C_c(\overline{S})$ and $0 \le t_1 < t_2 \le T$ there holds, up to subsequences,

$$\lim_{n\to\infty} \int_{t_1}^{t_2}\!\!\int_{\mathbb{R}} [u_n]_\pm\, \zeta\, dxdt = \int_{t_1}^{t_2}\!\!\int_{\mathbb{R}} [u_r]_\pm\, \zeta\, dxdt + \int_{t_1}^{t_2} \langle u_s^\pm(t), \zeta(\cdot,t)\rangle\, dt\,, \tag{5.19}$$

$$\begin{aligned} \lim_{n\to\infty} \int_{t_1}^{t_2}\!\!\int_{\mathbb{R}} f(u_n)\zeta\, dxdt &= \int_{t_1}^{t_2}\!\!\int_{\mathbb{R}} f(u_r)\, \zeta\, dxdt \,+ \\ &+ \; C_{f,+} \int_{t_0}^{t_1} \langle u_s^+(t), \zeta(\cdot,t)\rangle\, dt - C_{f,-} \int_{t_0}^{t_1} \langle u_s^-(t), \zeta(\cdot,t)\rangle\, dt \end{aligned} \tag{5.20}$$

for any $f \in C(\mathbb{R})$ such that (**??**) holds. Moreover,

$$\lim_{n\to\infty} \int_{t_1}^{t_2}\!\!\int_{\mathbb{R}} \varphi_n(u_n)\zeta\, dxdt = \int_{t_1}^{t_2}\!\!\int_{\mathbb{R}} \varphi(u_r)\, \zeta\, dxdt + C_0 \int_{t_1}^{t_2} \langle u_s^+(t), \zeta(\cdot,t)\rangle\, dt\,; \tag{5.21}$$

$$\lim_{n\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}}|u_n-k|\zeta\,dxdt=\int_{t_1}^{t_2}\int_{\mathbb{R}}|u_r-k|\zeta\,dxdt+\int_{t_0}^{t_1}\langle|u_s(t)|,\zeta(\cdot,t)\rangle\,dt\,, \tag{5.22}$$

$$\begin{aligned}\lim_{n\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}}\operatorname{sgn}(u_n(x,t)-k)[\varphi_n(u_n(x,t))-\varphi_n(k)]\,\zeta\,dxdt=\\ =\int_{t_1}^{t_2}\int_{\mathbb{R}}\operatorname{sgn}(u_r-k)[\varphi(u_r)-\varphi(k)]\,\zeta\,dxdt+C_0\int_{t_1}^{t_2}\langle u_s^+(t),\zeta(\cdot,t)\rangle\,dt\end{aligned} \tag{5.23}$$

for all $k\in\mathbb{R}$. In fact, the convergence in (??) follows from (??) with $f(s)=|s-k|$. As for (??) and (??), it suffices to choose $f(s)=\varphi(s)$, respectively $f(s)=\operatorname{sgn}(s-k)[\varphi(s)-\varphi(k)]$ in (??), since by (??) and (??) there holds

$$\iint_S|\varphi_n(u_n)-\varphi(u_n)|\,|\zeta|\,dxdt\le 2\|\zeta\|_\infty\|\varphi_b\|_\infty|\{|u_n|>n\}|\le\frac{2M_0\|\zeta\|_\infty\|\|\varphi_b\|_\infty}{n}\to 0\quad\text{as } n\to\infty\,,$$

$$\begin{aligned}&\left|\iint_S\operatorname{sgn}(u_n(x,t)-k)\,[\varphi_n(u_n(x,t))-\varphi_n(k)]-[\varphi(u_n)-\varphi(k)]\,\zeta\,dxdt\right|\le\\ \le\;&\iint_S\{|\varphi_n(u_n)-\varphi(u_n)|+|\varphi_n(k)-\varphi(k)|\}\,|\zeta|\,dxdt\to 0\quad\text{as } n\to\infty\,.\end{aligned}$$

Lemma 5.4. *Let (A_1)-(A_3) hold, and let u be the limiting measure in* (??). *Then u is a solution of problem (P). Moreover, for any sequence $\{u_n\}$ such that* (??)-(??) *are satisfied and for any $t\in(0,T]$ there holds*

$$u_n(\cdot,t)\overset{*}{\rightharpoonup}u(t)\quad\text{in }\mathcal{M}(\mathbb{R})\,. \tag{5.24}$$

In (??), at every $t\in(0,T]$ we have identified $u(t)$ with its continuous representative (with respect to the weak* topology of $\mathcal{M}(\mathbb{R})$), whose existence is ensured by Lemma **??**, since u is a solution of (P).

Proof. Choosing $f(s)=s$ in (??), we have

$$u_{0n}\overset{*}{\rightharpoonup}u_0\quad\text{in }\mathcal{M}(\mathbb{R})\,.$$

By the above convergence, (??) and (??), letting $n\to\infty$ in the weak formulation of problems (P_n) gives

$$\int_0^T\langle u(t),\zeta_t(\cdot,t)\rangle\,dt+\iint_S\varphi(u_r)\zeta_x\,dxdt+C_0\int_0^T\langle u_s^+(t),\zeta_x(\cdot,t)\rangle\,dt=-\langle u_0,\zeta(\cdot,0)\rangle \tag{5.25}$$

for any $\zeta\in C_c^1(\overline{S})$ such that $\zeta(\cdot,T)=0$. Thus u is a solution of (P) with initial data u_0. Let us address the convergence in (??). To this aim, fix any $t\in(0,T]$, and observe that the sequence $\{u_n(\cdot,t)\}$ is bounded in $L^1(\mathbb{R})$. Hence there exists $\mu^{(t)}\in\mathcal{M}(\mathbb{R})$ such that, up to a subsequences,

$$u_n(\cdot,t)\overset{*}{\rightharpoonup}\mu^{(t)}\quad\text{in }\mathcal{M}(\mathbb{R})\,. \tag{5.26}$$

On the other hand, for any $\rho\in C_c^1(\mathbb{R})$, taking the limit with respct to $n\to\infty$ in the equality

$$\int_{\mathbb{R}}u_n(x,t)\rho(x)\,dx=\int_{\mathbb{R}}u_{0n}(x)\rho(x)\,dx+\int_0^t\int_{\mathbb{R}}\varphi_n(u_n)\rho'(x)\,dxdt\,,$$

by (??) we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}}u_n(x,t)\rho(x)\,dx=\int_{t_1}^{t_2}\int_{\mathbb{R}}\varphi(u_r)\,\rho'(x)\,dxdt\,+\\ +C_0\int_{t_1}^{t_2}\langle u_s^+(t),\rho'\rangle\,dt+\langle u_0,\rho\rangle=\langle u(t),\rho\rangle\end{aligned} \tag{5.27}$$

(see (??)). From (??) and (??), it follows that $\mu^{(t)}=u(t)$ in $\mathcal{M}(\mathbb{R})$. □

Proof of Theorem **??**. In view of Lemma **??**, it is enough to check that the limiting measure u given by (**??**) satisfies the entropy inequality (**??**). Clearly, this follows by (**??**), (**??**) and the convergence

$$|u_{0n} - k| \overset{*}{\rightharpoonup} |u_{0r} - k| + |u_{0s}| \quad \text{in } \mathcal{M}(\mathbb{R}) \qquad (k \in \mathbb{R})$$

(see (**??**) with $f(s) = |s - k|$), letting $n \to \infty$ in (**??**). □

6. Qualitative properties of constructed entropy solutions: Proofs

6.1. **Continuity properties of the singular part.**

Lemma 6.1. *Let assumptions (A_1)-(A_3) hold. Let u be a constructed entropy solution of problem (P), and let $\{u_n\}$ be any sequence along which all convergences in the proof of Theorem* **??** *hold true. Then there exists a null set $N \subseteq (0,T)$ such that for every $\tau \in (0,T) \setminus N$ the following holds:*
(i) if $a_i + C_0\tau \in \operatorname{supp} u_s^+(\tau)$ $(i = 1, \dots, P_+)$, there exist $\{u_{n_j}\} \subseteq \{u_n\}$ and $\{\xi_j\} \subseteq \mathbb{R}$ such that

$$a_i + C_0\tau - \frac{1}{n_j^2} \le \xi_j \le a_i + C_0\tau + \frac{1}{n_j^2}\,, \qquad u_{n_j}(\xi_j^\pm, \tau) := \lim_{x \to \xi_j^\pm} u_{n_j}(x,\tau) > n_j\,; \tag{6.1a}$$

(ii) if $b_l \in \operatorname{supp} u_s^-(\tau)$ $(l = 1, \dots, M_-)$, there exist $\{u_{n_j}\} \subseteq \{u_n\}$ and $\{\xi_j\} \subseteq \mathbb{R}$ such that

$$b_l - \frac{1}{n_j^2} \le \xi_j \le b_l + \frac{1}{n_j^2}\,, \qquad u_{n_j}(\xi_j^\pm, \tau) < -n_j\,. \tag{6.1b}$$

Proof. We only prove claim (i). Since u is an entropy solution of (P), there exists a null set $N \subseteq (0,T)$ such that equalities (**??**) are satisfied for all $t \in (0,T) \setminus N$. Let $\tau \in (0,T) \setminus N$ be fixed, and observe that the limits in (**??**) exist since $u_{n_j}(\cdot,\tau) \in BV(\mathbb{R})$.

By the first equality in (**??**) there exists an interval $I_\delta := (a_i + C_0\tau - \delta, a_i + C_0\tau + \delta)$ $(\delta > 0)$ such that $u_s^+(\tau) \llcorner I_\delta = p_i(\tau)\,\delta_{\{a_i + C_0\tau\}}$. Set $J_n := (a_i + C_0\tau - \frac{1}{n^2}, a_i + C_0\tau + \frac{1}{n^2})$. We shall prove the following
Claim: There exist $\{u_{n_j}\} \subseteq \{u_n\}$ and $w \in L^1(I_\delta)$ such that

$$[u_{n_j}(\cdot,\tau)]_+ \chi_{I_\delta \setminus J_{n_j}} \overset{*}{\rightharpoonup} w \quad \text{in } \mathcal{M}(I_\delta)\,. \tag{6.2}$$

Part (i) follows from this Claim. In fact, we prove below that by (**??**) there holds

$$\lim_{j \to \infty} \frac{1}{n_j} \|[u_{n_j}(\cdot,\tau)]_+\|_{L^\infty(J_{n_j})} = \infty\,, \tag{6.3}$$

hence there exists $j_0 \in \mathbb{N}$ such that

$$\|u_{n_j}(\cdot,\tau)\|_{L^\infty(J_{n_j})} \ge 2n_j \quad \text{for all } j \ge j_0\,.$$

By the above inequality, for any $j \ge j_0$ there exists $\xi_j \in J_{n_j}$ such that $u_{n_j}(\xi_j^\pm, \tau) > n_j$. Hence (i) follows.

To prove (**??**) we argue by contradiction. Let there exist $M > 0$ and a subsequence (not relabeled for simplicity) such that

$$\|[u_{n_j}(\cdot,\tau)]_+\|_{L^\infty(J_{n_j})} \le M n_j \quad \text{for all } j \in \mathbb{N}\,,$$

thus for any $\rho \in C_c(I_\delta)$

$$\left| \int_{J_{n_j}} [u_{n_j}(x,\tau)]_+ \rho(x)\, dx \right| \le 2M n_j \frac{\|\rho\|_\infty}{n_j^2} \to 0 \quad \text{as } j \to \infty\,. \tag{6.4}$$

On the other hand, by (**??**) there holds

$$\int_{I_\delta \setminus J_{n_j}} [u_{n_j}(x,\tau)]_+ \rho(x)\, dx \to \int_{\mathbb{R}} w(x)\,\rho(x)\, dx\,. \tag{6.5}$$

From (**??**)-(**??**) we get $[u_{n_j}(\cdot,\tau)]_+ \overset{*}{\rightharpoonup} w$ in $\mathcal{M}(I_\delta)$. Since

$$[u_{n_j}(\cdot,\tau)]_+ = u_{n_j}(\cdot,\tau) + u^-_{n_j}(\cdot,\tau) \geq u_{n_j}(\cdot,\tau)\,,$$

the previous convergence and that in (**??**) give $u(\tau) \leq w$ in $\mathcal{M}(I_\delta)$. This implies that $u^+_s(\tau) \llcorner I_\delta = 0$, a contradiction.

Let us now prove the Claim. To this purpose, observe that the sequence $[u_n(\cdot,\tau)]_+ \chi_{I_\delta \setminus J_n}$ is bounded in $L^1(\mathbb{R})$, thus also in $L^1(I_\delta)$. Therefore there exist $\{u_{n_j}\} \subseteq \{u_n\}$ and $\mu \in \mathcal{M}^+(I_\delta)$ such that

$$[u_{n_j}(\cdot,\tau)]_+ \chi_{I_\delta \setminus J_{n_j}} \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(I_\delta)\,. \tag{6.6}$$

Moreover, for any $k > 0$ there exists $\mu_k \in \mathcal{M}^+(I_\delta)$ such that $[\mu_k]_s = \mu_s$ and, up to subsequences,

$$[u_{n_j}(\cdot,\tau) - k]_+ \chi_{I_\delta \setminus J_{n_j}} \overset{*}{\rightharpoonup} \mu_k \quad \text{in } \mathcal{M}(I_\delta). \tag{6.7}$$

Let $\{\xi_q\} \subseteq \mathbb{R}$ be any sequence as in (**??**), namely

$$\xi_q \to \infty\,, \qquad \sup_{\xi \geq \xi_q}[\varphi_b(\xi) - \varphi_b(\xi_q)] < \frac{1}{q} \quad \text{for all } q \in \mathbb{N}\,.$$

Set $n_j \equiv n$ for notational simplicity, and

$$\alpha_p(x) := p\left(x - a_i - C_0\tau - \frac{1}{n^2}\right)\chi_{[a_i+C_0\tau+\frac{1}{n^2},\, a_i+C_0\tau+\frac{1}{n^2}+\frac{1}{p}]}(x) + \chi_{(a_i+C_0\tau+\frac{1}{n^2}+\frac{1}{p},\infty)}(x) \qquad (p \in \mathbb{N})\,.$$

Let $q \in \mathbb{N}$ be fixed, let $n > \xi_q$ and let $\eta_\delta \in C^1_c(I_\delta)$, $0 \leq \eta_\delta \leq 1$, $\eta_\delta(a_i + C_0\tau) = 1$. Choosing in (**??**) $k = \xi_q$, $\zeta(x,t) = \alpha_p(x - C_0(t-\tau))\eta_\delta(x - C_0(t-\tau))$ and integrating on $(0,\tau)$ we get

$$\begin{aligned}
&\int_{\mathbb{R}}[u_n(x,\tau) - \xi_q]_+ \alpha_p(x)\eta_\delta(x)\,dx \leq \int_{\mathbb{R}}[u_{0n}(x) - \xi_q]_+(\alpha_p\eta_\delta)(x + C_0\tau)\,dx \;+ \\
&+ \int_0^\tau\!\!\int_{\mathbb{R}} \operatorname{sgn}_+(u_n(x,t) - \xi_q)[\varphi_b(T_n(u_n(x,t))) - \varphi_b(T_n(\xi_q))]\,[\alpha_p'\eta_\delta](x - C_0(t-\tau))\,dxdt \;+ \\
&+ \int_0^\tau\!\!\int_{\mathbb{R}} \operatorname{sgn}_+(u_n(x,t) - \xi_q)[\varphi_b(T_n(u_n(x,t))) - \varphi_b(T_n(\xi_q))]\,[\alpha_p\eta_\delta'](x - C_0(t-\tau))\,dxdt \;\leq \\
&\leq \int_{a_i+\frac{1}{n^2}}^{a_i+\delta}[u_{0n} - \xi_q]_+\,\eta_\delta(x + C_0\tau)dx + \frac{\tau}{q}\|\alpha_p'\|_1 + 2\|\varphi_b\|_\infty\|\eta_\delta'\|_\infty\,|\{u_n(x,t) > \xi_q\}|\;;
\end{aligned} \tag{6.8}$$

here we have used that

$$\sup_{\xi \geq \xi_q}[\varphi_b(T_n(\xi)) - \varphi_b(T_n(\xi_q))] = \sup_{\xi \geq \xi_q}[\varphi_b(T_n(\xi)) - \varphi_b(\xi_q)] \leq \sup_{\xi \geq \xi_q}[\varphi_b(\xi) - \varphi_b(\xi_q)] < \frac{1}{q}\,,$$

whence

$$\int_0^\tau\!\!\int_{\mathbb{R}} \operatorname{sgn}_+(u_n(x,t)-\xi_q)[\varphi_b(T_n(u_n(x,t)))-\varphi_b(T_n(\xi_q))]\,\underbrace{[\alpha_p'\eta_\delta]}_{\geq 0}(x-C_0(t-\tau))\,dxdt \;\leq\; \frac{\tau}{q}\|\alpha_p'\|_1\,.$$

Since $\|\alpha_p'\|_1 = 1$ and $0 \leq \eta_\delta \leq 1$, letting $p \to \infty$ in (**??**) gives

$$\begin{aligned}
&\int_{a_i+C_0\tau+\frac{1}{n^2}}^{a_i+C_0\tau+\delta}[u_n(x,t)(x,\tau) - \xi_q]_+\eta_\delta(x)\,dx \leq \int_{a_i+\frac{1}{n^2}}^{a_i+\delta}[u_{0n} - \xi_q]_+\,\eta_\delta(x + C_0\tau)dx + \frac{\tau}{q} \;+ \\
&+\; 2\|\varphi_b\|_\infty\|\eta_\delta'\|_\infty\,|\{u_n(x,t) > \xi_q\}| \leq \int_{a_i+\frac{1}{n^2}}^{a_i+\delta}[u_{0n} - \xi_q]_+\,dx + \frac{\tau}{q} + \frac{2M_0}{\xi_q}\|\varphi_b\|_\infty\|\eta_\delta'\|_\infty
\end{aligned} \tag{6.9}$$

(see also (**??**)). Arguing similarly, with $\{\xi_q'\}$ as in the second inequality of (**??**) we get

$$\int_{a_i+C_0\tau-\delta}^{a_i+C_0\tau-\frac{1}{n^2}}[u_n(x,\tau) - \xi_q']_+\eta_\delta(x)\,dx \leq \int_{a_i-\delta}^{a_i-\frac{1}{n^2}}[u_{0n} - \xi_q']_+\,dx + \frac{\tau}{q} + \frac{2M_0}{\xi_q'}\|\varphi_b\|_\infty\|\eta_\delta'\|_\infty\,. \tag{6.10}$$

Next, observe that in view of (A_1)-(**??**) we can fix $\delta>0$ so small that $u_{0s,n}=0$ in $(a_i-\delta,a_i+\delta)\setminus(a_i-\frac{1}{n^2},a_i+\frac{1}{n^2})$, whence (again by (**??**))

$$
\begin{aligned}
(6.11)\qquad &\lim_{n\to\infty}\left\{\int_{a_i+\frac{1}{n^2}}^{a_i+\delta}[u_{0n}-\xi_q]_+\,dx+\int_{a_i-\delta}^{a_i-\frac{1}{n^2}}[u_{0n}-\xi_q']_+\,dx\right\}=\\
=\ &\lim_{n\to\infty}\left\{\int_{a_i+\frac{1}{n^2}}^{a_i+\delta}[u_{0r,n}-\xi_q]_+\,dx+\int_{a_i-\delta}^{a_i-\frac{1}{n^2}}[u_{0r,n}-\xi_q']_+\,dx\right\}=\\
=\ &\int_{a_i}^{a_i+\delta}[u_{0r}-\xi_q]_+\,dx+\int_{a_i-\delta}^{a_i}[u_{0r}-\xi_q']_+\,dx\,.
\end{aligned}
$$

Set $k_q:=\max\{\xi_q,\xi_q'\}$. Since

$$
\begin{aligned}
&\int_{a_i+C_0\tau-\delta}^{a_i+C_0\tau-\frac{1}{n^2}}[u_n(x,\tau)-k_q]_+\eta_\delta(x)\,dx+\int_{a_i+C_0\tau+\frac{1}{n^2}}^{a_i+C_0\tau+\delta}[u_n(x,\tau)-k_q]_+\eta_\delta(x)\,dx\le\\
\le\ &\int_{a_i+C_0\tau-\delta}^{a_i+C_0\tau-\frac{1}{n^2}}[u_n(x,\tau)-\xi_q']_+\eta_\delta(x)\,dx+\int_{a_i+C_0\tau+\frac{1}{n^2}}^{a_i+C_0\tau+\delta}[u_n(x,\tau)-\xi_q]_+\eta_\delta(x)\,dx\,,
\end{aligned}
$$

summing up (**??**)-(**??**) and passing to the limit with respect to $n\to\infty$, by (**??**) (combined with the equality $\mu_s=\mu_k^s$ with $k=k_q$) and (**??**) we get

$$
\begin{aligned}
\langle\mu_s,\eta_\delta\rangle=\langle[\mu_{k_q}]_s,\eta_\delta\rangle\le\langle\mu_{k_q},\eta_\delta\rangle\le\int_{a_i-\delta}^{a_i}[u_{0r}-\xi_q']_+\,dx\,+\\
+\ \int_{a_i}^{a_i+\delta}[u_{0r}-\xi_q]_+\,dx+\frac{2\tau}{q}+2M_0\|\varphi_b\|_\infty\|\eta_\delta'\|_\infty\left(\frac{1}{\xi_q}+\frac{1}{\xi_q'}\right).
\end{aligned}
$$

Letting $q\to\infty$ in the above inequality, it follows that $\mu_s=0$. This proves the Claim. □

Lemma 6.2. *Let assumptions (A_1)-(A_3) hold, and let u be a constructed entropy solution of problem (P).*
(i) If $u_s^+(t_0)(\{\bar{x}\})=0$ for some $t_0\in[0,T)$, then there holds

$$(6.12)\qquad u_s^+(t)(\{\bar{x}+C_0(t-t_0)\})=0\quad \textit{for any } t\in(t_0,T).$$

(ii) If $u_s^-(t_0)(\{\bar{x}\})=0$ for some $t_0\in[0,T)$, then there holds

$$(6.13)\qquad u_s^-(t)(\{\bar{x}\})=0\quad \textit{for any } t\in(t_0,T).$$

Proof. Let us address only claim (i), the proof of (ii) being similar. To fix the ideas, assume that $C_0>0$, and let $t_0\in[0,T)$ be fixed.

If $u_s^+(t_0)(\{\bar{x}\})=0$, there holds $\bar{x}\ne a_i+C_0t_0$ for any $i=1,\dots,P_+$, thus $\bar{x}+C_0(t-t_0)\ne a_i+C_0t$ for any such i and $t\in(0,T)$. Then by (**??**) there holds

$$u_s^+(t)(\{\bar{x}+C_0(t-t_0)\})\le\sum_{i=1}^{P_+}p_i(t^+)\,\delta_{a_i+C_0t}(\{\bar{x}+C_0(t-t_0)\})=0\,,$$

and the conclusion follows in this case.

Now suppose that $\bar{x}=a_i+C_0t_0$ for some $i=1,\dots,P_+$. We argue by contradiction and prove the following

Claim. Let there exists $\tau_0\in(t_0,T)$ such that

$$(6.14)\qquad u_s^+(\tau_0)(\{\bar{x}+C_0(\tau_0-t_0)\})=u_s^+(\tau_0)(\{a_i+C_0\tau_0\})>0\,,$$

and let N be the null set in (**??**). Then for all $t\in(t_0,\tau_0)\setminus N$

$$(6.15\text{a})\qquad u_s^+(t)\llcorner\{a_i+C_0t\}=p_i(t)\,\delta_{a_i+C_0t}>0\,,$$

$$(6.15\text{b})\qquad a_i+C_0t_0\notin\operatorname{supp}u_s^-(t)\,.$$

Let us also observe for future reference that by (**??**) and (**??**) there holds

$$p_i(\tau_0^-)\,\delta_{a_i+C_0\tau_0} \;\geq\; p_i(\tau_0^+)\,\delta_{a_i+C_0\tau_0} \;\geq\; u_s^+(\tau_0) \llcorner \{a_i+C_0\tau_0\} > 0\,. \tag{6.16}$$

Using the *Claim* we can prove the result. Since $\operatorname{supp} u_s^-(t) \subseteq \{b_1,\dots,b_{M_-}\}$ $(t\in[0,T])$, by (**??**) there exists $\sigma>0$ such that

$$u_s^-(t) \llcorner (a_i+C_0t_0-\sigma, a_i+C_0t_0+\sigma) = 0 \quad \text{for a.e. } t \in (t_0,\tau_0)\,. \tag{6.17}$$

By (**??**) and (**??**), for any $\rho\in C_c(a_i+C_0t_0-\sigma, a_i+C_0t_0+\sigma)$, $\rho\geq 0$, and for a.e. $t\in(t_0,\tau_0)$ we get

$$\langle u_s(t)\,,\,\rho\rangle \;=\; \langle u_s^+(t)\,,\,\rho\rangle \;\geq\; p_i(\tau_0^-)\,\rho(a_i+C_0t)\,,$$

since the function p_i is nonincreasing in $[0,T]$. The by (**??**) we obtain

$$\langle u_s(t_0),\rho\rangle \;=\; \operatorname*{ess\,lim}_{t\to t_0^+} \langle u_s(t)\,,\,\rho\rangle \;\geq\; p_i(\tau_0^-)\,\rho(a_i+C_0t_0)\,.$$

By the arbitrariness of ρ, it follows that

$$u_s^+(t_0) \llcorner \{a_i+C_0t_0\} \;\geq\; p_i(\tau_0^-)\,\delta_{a_i+C_0t_0}\,. \tag{6.18}$$

Inequalities (**??**) and (**??**) contradict the assumption $u_s^+(t_0)(\{a_i+C_0t_0\})=0$, thus the conclusion follows.

It remains to prove the Claim. Inequality (**??**) follows from the first equality in (**??**) and (**??**) since p_i is noincreasing. To prove (**??**) we argue again by contradiction. Let there exists $\bar t\in(t_0,\tau_0)\smallsetminus N$ such that

$$a_i+C_0t_0 \;\in\; \operatorname{supp} u_s^-(\bar t)\,. \tag{6.19}$$

Since $C_0>0$, for any fixed $\tau\in(t_0,\bar t)\smallsetminus N$ there holds

$$a_i \;<\; a_i+C_0t_0 \;<\; a_i+C_0\tau\,. \tag{6.20}$$

Since $\tau\in(t_0,\bar t)\smallsetminus N \subseteq (t_0,\tau_0)\smallsetminus N$, from (**??**) with $t=\tau$ we obtain that $a_i+C_0\tau \in \operatorname{supp} u_s^+(\tau)$. Hence by Lemma **??**-(i) there exist $\{u_{n_j}\}\subseteq\{u_n\}$ and $\{\xi_j\}\subseteq\mathbb{R}$ such that

$$a_i+C_0\tau-\frac{1}{n_j^2} \;\leq\; \xi_j \;\leq\; a_i+C_0\tau+\frac{1}{n_j^2}\,, \qquad u_{n_j}(\xi_j^\pm,\tau) \;>\; n_j \tag{6.21}$$

(see (**??**)). On the other hand, by (**??**) and Lemma **??**-(ii) there exist a subsequence $\{u_{n_{j_k}}\}\subseteq\{u_{n_j}\}$ and $\{\bar\xi_k\}\subseteq\mathbb{R}$ such that

$$a_i+C_0t_0-\frac{1}{n_{j_k}^2} \;\leq\; \bar\xi_k \;\leq\; a_i+C_0t_0+\frac{1}{n_{j_k}^2}\,, \qquad u_{n_{j_k}}(\bar\xi_k^\pm,\bar t) < -n_{j_k}\,. \tag{6.22}$$

By Proposition **??**-(i) and the last inequality in (**??**) there holds

$$u_{n_j}((\xi_j+C_0(t-\tau))^\pm,t) \;\geq\; n_j \quad \text{for all } t\in[0,\tau)\,, \tag{6.23}$$

whereas by Proposition **??**-(ii) and the last inequality in (**??**),

$$u_{n_{j_k}}(\bar\xi_k^\pm,t) \leq -n_{j_k} \quad \text{for all } t\in[0,\bar t)\,. \tag{6.24}$$

Set $t_k := \tau+\frac{\bar\xi_k-\xi_{j_k}}{C_0}$. By (**??**)-(**??**), for any $k\in\mathbb{N}$ large enough there holds

$$t_k \;\geq\; t_0 \;-\; \frac{2}{C_0\,n_{j_k}^2} > 0\,, \qquad t_k \;\leq\; t_0 \;+\; \frac{2}{C_0\,n_{j_k}^2} \;<\; \tau\,.$$

To sum up, $0<t_k<\tau<\bar t$ and $\bar\xi_k=\xi_{j_k}+C_0(t_k-\tau)$ for sufficiently large $k\in\mathbb{N}$, whence by (**??**) and (**??**) for $t=t_k$ we obtain $-n_{j_k}\geq u_{n_{j_k}}(\bar\xi_k^\pm,t_k)\geq n_{j_k}$, a contradiction. □

Proof of Theorem **??**. Let $t_0 \in [0,T)$ be fixed. We distinguish two cases.
(i) Assume that $a_i + C_0 t_0 \neq b_l$ for $i = 1,\dots,P_+$ and $l = 1,\dots,M_-$. Then there exists $h > 0$ such that for any $t \in [t_0, t_0+h)$ there holds $a_i + C_0 t \neq b_l$ for all i,l, thus the measures $\sum_{i=1}^{P_+} p_i(t^\pm)\,\delta_{a_i+C_0 t}$ and $\sum_{l=1}^{M_-} m_l(t^\pm)\,\delta_{b_l}$ are mutually singular. Now (**??**) holds, since, by (**??**) for all $t \in [t_0, t_0+h)$ there holds

$$u_s^+(t) = \sum_{i=1}^{P_+} p_i(t^+)\delta_{a_i+C_0 t}\,, \qquad u_s^-(t) = \sum_{l=1}^{M_-} m_l(t^+)\delta_{b_l}\,. \tag{6.25}$$

We only prove (**??**) with " + ". Let $\{t_j\} \subseteq (t_0,T)$, $t_j \to t_0^+$. Let $\rho \in C_c(\mathbb{R})$. Since $\lim_{j\to\infty} p_i(t_j^+) = p_i(t_0^+)$, it follows from the first equalities in (**??**) and (**??**) that

$$\lim_{j\to\infty} \langle u_s^+(t_j), \rho\rangle = \lim_{j\to\infty}\left(\sum_{i=1}^{P_+} p_i(t_j^+)\,\rho(a_i + C_0 t_j)\right) = \sum_{i=1}^{P_+} p_i(t_0^+)\,\rho(a_i + C_0 t_0) = \langle u_s^+(t_0), \rho\rangle\,.$$

Hence the conclusion follows in this case.
(ii) Assume that $a_{i_0} + C_0 t_0 = b_{l_0}$ for some $i_0 \in \{1,\dots,P_+\}$, $l_0 \in \{1,\dots,M_-\}$. Since by assumption $a_{i_0} \neq b_{l_0}$, this is only possible if $t_0 > 0$. We address the case where moreover $a_i + C_0 t_0 \neq b_l$ for all pairs $(i,l) \neq (i_0,l_0)$, since the remaining case can be treated similarly.

Plainly, there exists $h > 0$ such that $a_i + C_0 t \neq b_l$ for all $t \in (t_0, t_0+h)$ and i,l. Hence equalities (**??**) hold true for all $t \in (t_0, t_0+h)$. To prove (**??**), we observe that, by (**??**),

$$\begin{aligned} u_s(t_0) &= \sum_{i=1, i\neq i_0}^{P_+} p_i(t_0^-)\,\delta_{a_i+C_0 t_0} - \sum_{l=1, l\neq l_0}^{M_-} m_l(t_0^-)\,\delta_{b_l} + [p_{i_0}(t_0^-) - m_{l_0}(t_0^-)]\,\delta_{b_{l_0}} = \\ &= \sum_{i=1, i\neq i_0}^{P_+} p_i(t_0^+)\,\delta_{a_i+C_0 t_0} - \sum_{l=1, l\neq l_0}^{M_-} m_l(t_0^+)\,\delta_{b_l} + [p_{i_0}(t_0^+) - m_{l_0}(t_0^+)]\,\delta_{b_{l_0}}\,. \end{aligned} \tag{6.26}$$

Assume that $p_{i_0}(t_0^-) \geq m_{l_0}(t_0^-)$ (if $p_{i_0}(t_0^-) < m_{l_0}(t_0^-)$ the proof is similar). By assumption the measures $\sum_{i=1,i\neq i_0}^{P_+} p_i(t_0^\pm)\,\delta_{a_i+C_0 t_0} + [p_{i_0}(t_0^\pm) - m_{l_0}(t_0^\pm)]\,\delta_{b_{l_0}}$ and $\sum_{l=1,l\neq l_0}^{M_-} m_l(t_0^\pm)\,\delta_{b_l}$ are mutually singular, thus by the first equality in (**??**) there holds

$$u_s^+(t_0) = \sum_{i=1, i\neq i_0}^{P_+} p_i(t_0^-)\,\delta_{a_i+C_0 t_0} + [p_{i_0}(t_0^-) - m_{l_0}(t_0^-)]\,\delta_{b_{l_0}}\,, \qquad u_s^-(t_0) = \sum_{l=1, l\neq l_0}^{M_-} m_l(t_0^-)\,\delta_{b_l}\,,$$

whence $u_s^-(t_0)(\{b_{l_0}\}) = 0$. As a consequence, by Lemma **??**-(ii) (see (**??**)) there holds $u_s^-(t)(\{b_{l_0}\}) = 0$ for a.e. $t \in (t_0,T)$, whence (see the second equality in (**??**))

$$m_{l_0}(t) = 0 \quad \text{for a.e. } t \in (t_0,T)\,.$$

This implies that $m_{l_0}(t_0^+) = 0$ and, since the function m_{l_0} is nonincreasing in $[0,T]$,

$$m_{l_0}(t^\pm) = 0 \quad \text{for all } t \in (t_0,T). \tag{6.27}$$

Since $m_{l_0}(t_0^+) = 0$, the second equality in (**??**) simply reads

$$u_s(t_0) = \sum_{i=1}^{P} p_i(t_0^+)\,\delta_{a_i+C_0 t_0} - \sum_{l=1, l\neq l_0}^{M} m_l(t_0^+)\,\delta_{b_l}\,,$$

whence

$$u_s^+(t_0) = \sum_{i=1}^{P_+} p_i(t_0^+)\,\delta_{a_i+C_0 t_0}\,, \qquad u_s^-(t_0) = \sum_{l=1, l\neq l_0}^{M_-} m_l(t_0^+)\,\delta_{b_l} = \sum_{l=1}^{M_-} m_l(t_0^+)\,\delta_{b_l}\,, \tag{6.28}$$

since by assumption the measures $\sum_{i=1}^{P_+} p_i(t_0^+)\,\delta_{a_i+C_0 t_0}$ and $\sum_{l=1,l\neq l_0}^{M_-} m_l(t_0^+)\,\delta_{b_l}$ are mutually singular and $m_{l_0}(t_0^+) = 0$. This proves (**??**). In addition, by (**??**) and (**??**),

$$u_s^+(t) = \sum_{i=1}^{P_+} p_i(t^+)\delta_{a_i+C_0 t}, \quad u_s^-(t) = \sum_{l=1, l\neq l_0}^{M_-} m_l(t^+)\delta_{b_l} \quad \text{for all } t \in (t_0, t_0+h). \tag{6.29}$$

Since $p_i(t^+) \to p_i(t_0^+)$ and $m_l(t^+) \to m_l(t_0^+)$ as $t \to t_0^+$, it follows from (**??**) and (**??**) that

$$\lim_{t\to t_0^+} \langle u_s^+(t), \rho\rangle = \lim_{t\to t_0^+} \sum_{i=1}^{P_+} p_i(t^+)\rho(a_i + C_0 t) = \sum_{i=1}^{P_+} p_i(t_0^+)\rho(a_i + C_0 t_0) = \langle u_s^+(t_0), \rho\rangle \ ,$$

$$\lim_{t\to t_0^+} \langle u_s^-(t), \rho\rangle = \lim_{t\to t_0^+} \sum_{l=1, l\neq l_0}^{M_-} m_l(t^+)\rho(b_l) = \sum_{l=1, l\neq l_0}^{M_-} m_l(t_0^+)\rho(b_l) = \langle u_s^-(t_0), \rho\rangle$$

for any $\rho \in C_c(\mathbb{R})$. This proves (**??**) and the result follows. □

6.2. **Compatibility conditions.** In the present subsection we prove Theorem **??**.

Proof of Theorem **??**. We only prove (**??**). By Lemma **??**-(i) and (**??**) there holds

$$a_i - \frac{1}{n_j^2} \le z_j \le a_i + \frac{1}{n_j^2}\,, \qquad u_{n_j}((z_j + C_0 t)^{\pm}, t) \ge n_j \quad \text{for all } t \in [0, \tau]\,, \tag{6.30}$$

where $z_j := \xi_j - C_0\tau$ (see (**??**)). Let $\delta > 0$ and $\rho \in C_c^1(a_i - \delta, a_i + \delta)$, $\rho \ge 0$. Let $j \in \mathbb{N}$ be so large that $z_j \in (a_i - \delta, a_i + \delta)$ (see (**??**)). Then the function

$$\alpha_m(y) := m(y - z_j)\rho(y)\chi_{[z_j, z_j + \frac{1}{m}]}(y) + \rho(y)\chi_{(z_j + \frac{1}{m}, \infty)}(y) \tag{6.31}$$

has compact support in $(a_i - \delta, a_i + \delta)$ for sufficiently large $m \in \mathbb{N}$. By standard regularization arguments we can choose in (**??**) $\zeta(x,t) = \alpha_m(x - C_0 t)\,\beta(t)$ with $\beta \in C_c^1(0,\tau)$, $\beta \ge 0$. Then

$$\begin{aligned} &\iint_S [u_{n_j}(x,t) - k]_- \left\{\alpha_m(x - C_0 t)\,\beta'(t) - C_0 \alpha_m'(x - C_0 t)\,\beta(t)\right\} dxdt \ + \\ &+ \ \iint_S \operatorname{sgn}_-(u_{n_j}(x,t) - k)\,[\varphi_{n_j}(u_{n_j}(x,t)) - \varphi_{n_j}(k)]\,\alpha_m'(x - C_0 t)\,\beta(t)\,dxdt \ge 0\,. \end{aligned} \tag{6.32}$$

Since for any $y, k \in \mathbb{R}$

$$\begin{aligned} & -C_0\,[y-k]_- + \operatorname{sgn}_-(y-k)[\varphi_{n_j}(y) - \varphi_{n_j}(k)] = \\ = \ & -C_0 \operatorname{sgn}_-(y-k)(y-k) + \operatorname{sgn}_-(y-k)[\varphi_b(T_{n_j}(y)) - \varphi_b(T_{n_j}(k)) + C_0(y_+ - k_+)] = \\ = \ & \operatorname{sgn}_-(y-k)[\varphi_b(T_{n_j}(y)) - \varphi_b(T_{n_j}(k)) + C_0(y_- - k_-)] =: g_{n_j,k}(y)\,, \end{aligned} \tag{6.33}$$

inequality (**??**) reads

$$\iint_S [u_{n_j}(x,t) - k]_-\,\alpha_m(x - C_0 t)\,\beta'(t)\,dxdt + \iint_S g_{n_j,k}(u_{n_j}(x,t))\,\alpha_m'(x - C_0 t)\,\beta(t)\,dxdt \ge 0\,. \tag{6.34}$$

Let us now send $m \to \infty$ in (**??**). To this purpose, observe that by (**??**) there holds

$$\begin{aligned} &\iint_S g_{n_j,k}(u_{n_j}(x,t))\,\alpha_m'(x - C_0 t)\,\beta(t)\,dxdt = \\ = \ & m\int_0^\tau \beta(t)\,dt \int_{z_j + C_0 t}^{z_j + C_0 t + 1/m} g_{n_j,k}(u_{n_j}(x,t))\rho(x - C_0 t)\,dx \ + \\ + \ & m\int_0^\tau \beta(t)\,dt \int_{z_j + C_0 t}^{z_j + C_0 t + 1/m} g_{n_j,k}(u_{n_j}(x,t))\,(x - C_0 t - z_j)\,\rho'(x - C_0 t)\,dx \ + \\ + \ & \int_0^\tau \beta(t)\,dt \int_{z_j + C_0 t + 1/m}^{\infty} g_{n_j,k}(u_{n_j}(x,t))\,\rho'(x - C_0 t)\,dx\,. \end{aligned} \tag{6.35}$$

Since $u_{n_j}(\cdot, t) \in BV(\mathbb{R})$ ($t \in (0,T)$), by the second inequality in (**??**) and the very definition of $g_{n_j,k}$ (see (**??**)), for any $n_j > k$ and for a.e. $t \in (0,\tau)$ there holds

$$\lim_{m\to\infty} m\int_{z_j + C_0 t}^{z_j + C_0 t + 1/m} g_{n_j,k}(u_{n_j}(x,t))\rho(x - C_0 t)\,dx = 0$$

and

$$\lim_{m\to\infty} m\int_{z_j + C_0 t}^{z_j + C_0 t + 1/m} g_{n_j,k}(u_{n_j}(x,t))\,(x - C_0 t - z_j)\,\rho'(x - C_0 t)\,dx = 0\,.$$

Since $u_{n_j} \in L^\infty(S)$, for any $j \in \mathbb{N}$ there exists $C_j > 0$ such that for all $m \in \mathbb{N}$ and $t \in [0,\tau]$

$$m \left| \int_{z_j+C_0t}^{z_j+C_0t+1/m} g_{n_j,k}(u_{n_j}(x,t))\rho(x-C_0t)\,dx \right| \le C_j\,,$$
$$m \left| \int_{z_j+C_0t}^{z_j+C_0t+1/m} g_{n_j,k}(u_{n_j}(x,t))\,(x-C_0t-z_j)\,\rho'(x-C_0t)\,dx \right| \le C_j\,.$$

Then by the Dominated Convergence Theorem there holds

$$\lim_{m\to\infty} m \int_0^\tau \beta(t)\,dt \int_{z_j+C_0t}^{z_j+C_0t+1/m} g_{n_j,k}(u_{n_j}(x,t))\rho(x-C_0t)\,dx = \tag{6.36}$$
$$= \lim_{m\to\infty} m \int_0^\tau \beta(t)\,dt \int_{z_j+C_0t}^{z_j+C_0t+1/m} g_{n_j,k}(u_{n_j}(x,t))\,(x-C_0t-z_j)\,\rho'(x-C_0t)\,dx = 0\,.$$

Moreover, it is easily seen that

$$\lim_{m\to\infty} \int_0^\tau \beta(t)\,dt \int_{z_j+C_0t+1/m}^{\infty} g_{n_j,k}(u_{n_j}(x,t))\,\rho'(x-C_0t)\,dx = \tag{6.37}$$
$$= \int_0^\tau \beta(t)\,dt \int_{z_j+C_0t}^{\infty} g_{n_j,k}(u_{n_j}(x,t))\,\rho'(x-C_0t)\,dx\,,$$

(6.38)
$$\lim_{m\to\infty} \iint_S [u_{n_j}(x,t)-k]_-\,\alpha_m(x-C_0t)\,\beta'(t)\,dxdt = \int_0^\tau\int_{z_j+C_0t}^{\infty} [u_{n_j}(x,t)-k]_-\,\rho(x-C_0t)\,\beta'(t)\,dxdt\,.$$

In view of (**??**)-(**??**), letting $m \to \infty$ in (**??**) we obtain

$$\int_0^\tau\int_{z_j+C_0t}^{\infty} [u_{n_j}(x,t)-k]_-\,\rho(x-C_0t)\,\beta'(t)\,dxdt \ge \tag{6.39}$$
$$\ge\ -\int_0^\tau\int_{z_j+C_0t}^{\infty} g_{n_j,k}(u_{n_j}(x,t))\,\rho'(x-C_0t)\,\beta(t)\,dxdt\,.$$

Since $\operatorname{supp}\rho \subseteq (a_i-\delta, a_i+\delta)$ and $z_j \in (a_i-\delta, a_i+\delta)$, from (**??**) we get

$$\int_0^\tau\int_{a_i+C_0t-\delta}^{a_i+C_0t+\delta} [u_{n_j}(x,t)-k]_-\,\rho(x-C_0t)\,|\beta'(t)|\,dxdt \ge \tag{6.40}$$
$$\ge\ -\int_0^\tau\int_{z_j+C_0t}^{\infty} g_{n_j,k}(u_{n_j}(x,t))\,\rho'(x-C_0t)\,\beta(t)\,dxdt\,.$$

The next step of the proof is sending $j \to \infty$ in (**??**). As for the left-hand side, using (**??**) with $f(y) = [y-k]_-$, we get

$$\lim_{j\to\infty} \int_0^\tau\int_{a_i+C_0t-\delta}^{a_i+C_0t+\delta} [u_{n_j}(x,t)-k]_-\,\rho(x-C_0t)\,|\beta'(t)|\,dxdt = \tag{6.41}$$
$$= \int_0^\tau |\beta'(t)|\,dt \int_{a_i+C_0t-\delta}^{a_i+C_0t+\delta} [u_r(x,t)-k]_-\,\rho(x-C_0t)\,dx +$$
$$+ \int_0^\tau \langle u_s^-(t)\,,\,\rho(\cdot-C_0t)\rangle\,|\beta'(t)|\,dt \le$$
$$\le \|\rho\|_{L^\infty(a_i-\delta,a_i+\delta)} \left\{ \int_0^\tau |\beta'(t)|\,dt \int_{a_i+C_0t-\delta}^{a_i+C_0t+\delta} [u_r(x,t)-k]_-\,dx + \right.$$
$$\left. + \int_0^\tau u_s^-(t)\big((a_i+C_0t-\delta, a_i+C_0t+\delta)\big)\,|\beta'(t)|\,dt \right\}\,.$$

To address the right-hand side of (**??**), set $g_k(y) := \operatorname{sgn}_-(y-k)[\hat\varphi(y)-\hat\varphi(k)]$ with $\hat\varphi(z) = \varphi_b(z)+C_0z_-$; $z \in \mathbb{R}$. Let $\rho \in C_c^1(a_i-\delta, a_i+\delta)$ satisfy: (i) $0 \le \rho \le 1$, (ii) $\rho' = 0$ in $[a_i-2\delta', a_i+2\delta']$ for some $\delta' \in \big(0, \frac{\delta}{2}\big)$, thus $\rho' \in M_0([a_i+2\delta', a_i+\delta])$. It follows that for sufficiently large j

$$\int_0^\tau\int_{z_j+C_0t}^{a_i+C_0t+2\delta'} g_k(u_{n_j})\,\rho'(x-C_0t)\beta(t)\,dxdt = 0\,,$$

since $z_j \in (a_i - \delta', a_i + \delta')$. Applying (**??**) with $f(y) = g_k(y)$ to $u_{n_j}(x + C_0 t, t)$, we obtain

$$
\begin{aligned}
(6.42)\qquad & \lim_{j\to\infty} \int_0^\tau \beta(t)\,dt \int_{z_j + C_0 t}^{\infty} g_k(u_{n_j}(x,t))\,\rho'(x - C_0 t)\,dx = \\
&= \lim_{j\to\infty} \int_0^\tau \beta(t)\,dt \int_{a_i + C_0 t + 2\delta'}^{a_i + C_0 t + \delta} g_k(u_{n_j}(x,t))\,\rho'(x - C_0 t)\,dx = \\
&= \int_0^\tau \beta(t)\,dt \int_{a_i + C_0 t + 2\delta'}^{a_i + C_0 t + \delta} g_k(u_r(x,t))\,\rho'(x - C_0 t)\,dx - \\
&- C_0 \int_0^\tau \langle u_s^-(t) \llcorner (a_i + C_0 t + 2\delta', a_i + C_0 t + \delta)\,,\, \rho'(\cdot - C_0 t)\rangle\,\beta(t)\,dt\,.
\end{aligned}
$$

Since

$$
g_{n_j,k}(u_{n_j}) - g_k(u_{n_j}) = \operatorname{sgn}_-(u_{n_j} - k)\big\{[\varphi_b(T_{n_j}(u_{n_j})) - \varphi_b(u_{n_j})] - [\varphi_b(T_{n_j}(k)) - \varphi_b(k)]\big\},
$$

it easily follows that

$$
\begin{aligned}
(6.43)\qquad & \limsup_{j\to\infty} \int_0^\tau \beta(t)\,dt \int_{z_j + C_0 t}^{\infty} |g_{n_j,k}(u_{n_j}(x,t)) - g_k(u_{n_j}(x,t))|\,|\rho'(x - C_0 t)|\,dx \le \\
&\le 2\|\varphi_b\|_\infty \|\beta\|_{L^\infty(0,T)} \|\rho'\|_{L^\infty(a_i - \delta, a_i + \delta)} \limsup_{j\to\infty} |\{(x,t) \in S : |u_{n_j}(x,t)| > n_j\}| = 0
\end{aligned}
$$

(in the last equality use that $\{u_{n_j}\}$ is bounded in $L^1(S)$). From (**??**) and (**??**) we obtain

$$
\begin{aligned}
(6.44)\qquad & \lim_{j\to\infty} \int_0^\tau \int_{z_j + C_0 t}^{\infty} g_{n_j,k}(u_{n_j}(x,t))\,\rho'(x - C_0 t)\,\beta(t)\,dxdt = \\
&= \int_0^\tau \beta(t) \int_{a_i + C_0 t + 2\delta'}^{a_i + C_0 t + \delta} g_k(u_r(x,t))\,\rho'(x - C_0 t)\,dxdt - \\
&- C_0 \int_0^\tau \langle u_s^-(t) \llcorner (a_i + C_0 t + 2\delta', a_i + C_0 t + \delta)\,,\, \rho'(\cdot - C_0 t)\rangle\,\beta(t)\,dt\,.
\end{aligned}
$$

In view of (**??**) and (**??**), sending $j \to \infty$ in (**??**) and recalling that $0 \le \rho \le 1$, we get

$$
\begin{aligned}
(6.45)\qquad & \int_0^\tau |\beta'(t)|\,dt \int_{a_i + C_0 t - \delta}^{a_i + C_0 t + \delta} [u_r(x,t) - k]_-\,dx + \\
&+ \int_0^\tau u_s^-(t)\big((a_i + C_0 t - \delta, a_i + C_0 t + \delta)\big)\,|\beta'(t)|\,dt \ge \\
&\ge -\int_0^\tau \beta(t) \int_{a_i + C_0 t + 2\delta'}^{a_i + C_0 t + \delta} g_k(u_r(x,t))\,\rho'(x - C_0 t)\,dxdt + \\
&+ C_0 \int_0^\tau \langle u_s^-(t) \llcorner (a_i + C_0 t + 2\delta', a_i + C_0 t + \delta)\,,\, \rho'(\cdot - C_0 t)\rangle\,\beta(t)\,dt\,.
\end{aligned}
$$

Let $q \in \mathbb{N}$, $q > \frac{4}{\delta}$. We choose in (**??**) $2\delta' = \frac{1}{q}$ and $\rho = \alpha_q \in C_c^1(a_i - \delta, a_i + \delta)$ defined by

$$
\begin{aligned}
(6.46)\qquad \alpha_q'(y) &= \tfrac{q}{\delta}\left(a_i + \tfrac{1}{q} - y\right)\chi_{\left[a_i + \frac{1}{q}, a_i + \frac{2}{q}\right]}(y) - \\
&- \tfrac{1}{\delta}\chi_{\left(a_i + \frac{2}{q}, a_i + \delta - \frac{2}{q}\right]}(y) + \tfrac{q}{\delta}\left(y - a_i - \delta + \tfrac{1}{q}\right)\chi_{\left(a_i + \delta - \frac{2}{q}, a_i + \delta - \frac{1}{q}\right]}(y)\,.
\end{aligned}
$$

Let $q \to \infty$. As for the right-hand side of (**??**), since $|\rho_q'(x)| \le \delta^{-1}$ and $\rho_q'(x) \to -\delta^{-1}\chi_{(a_i, a_i + \delta)}(x)$ for any $x \in [a_i, a_i + \delta]$ (see (**??**)), and since $u_r \in L^\infty(0,T; L^1(\mathbb{R}))$, $u_s \in L^\infty_{w*}(0,T; \mathcal{M}(\mathbb{R}))$, it follows from the Dominated Convergence Theorem that

$$
\begin{aligned}
(6.47)\qquad & \lim_{q\to\infty}\Bigg\{-\int_0^\tau \beta(t)\,dt\int_{a_i+C_0t+\frac{1}{q}}^{a_i+C_0t+\delta} g_k(u_r(x,t))\,\alpha_q'(x-C_0t)\,dx\ + \\
&+\quad C_0\int_0^\tau \left\langle u_s^-(t)\llcorner(a_i+C_0t+q^{-1},a_i+C_0t+\delta)\,,\,\alpha_q'(\cdot-C_0t)\right\rangle \beta(t)\,dt\Bigg\} = \\
&=\quad -\int_0^\tau \beta(t)\,dt\left(\lim_{q\to\infty}\int_{a_i+C_0t+\frac{1}{q}}^{a_i+C_0t+\delta} g_k(u_r(x,t))\,\alpha_q'(x-C_0t)\,dx\right) + \\
&+\quad C_0\int_0^\tau \lim_{q\to\infty}\left\langle u_s^-(t)\llcorner(a_i+C_0t+q^{-1},a_i+C_0t+\delta)\,,\,\alpha_q'(\cdot-C_0t)\right\rangle \beta(t)\,dt\ = \\
&=\quad \frac{1}{\delta}\int_0^\tau\!\!\int_{a_i+C_0t}^{a_i+C_0t+\delta} g_k(u_r(x,t))\,\beta(t)\,dxdt\ - \\
&-\quad \frac{C_0}{\delta}\int_0^\tau u_s^-(t)\big((a_i+C_0t,a_i+C_0t+\delta)\big)\,\beta(t)\,dt\,.
\end{aligned}
$$

By (**??**), it follows from (**??**) with $\rho=\alpha_q$ that

$$
\begin{aligned}
(6.48)\qquad & \frac{1}{\delta}\Bigg\{\int_0^\tau \beta(t)\,dt\int_{a_i+C_0t}^{a_i+C_0t+\delta} g_k(u_r(x,t))\,\beta(t)\,dx\ - \\
&-\quad C_0\int_0^\tau u_s^-(t)\big((a_i+C_0t,a_i+C_0t+\delta)\big)\,\beta(t)\,dt\Bigg\} \le \\
&\le\quad \int_0^\tau |\beta'(t)|\,dt\int_{a_i+C_0t-\delta}^{a_i+C_0t+\delta}[u_r-k]_-\,dx\ + \\
&+\quad \int_0^\tau u_s^-(t)\big((a_i+C_0t-\delta\,,\,a_i+C_0t+\delta)\big)\,|\beta'(t)|\,dt\,.
\end{aligned}
$$

Finally, we send $\delta\to0^+$ in the above inequality. Observe that for a.e. $t\in(0,\tau)$ there holds

$$
\lim_{\delta\to0^+} u_s^-(t)\big((a_i+C_0t-\delta\,,\,a_i+C_0t+\delta)\big)\ =\ u_s^-(t)\big(\{a_i+C_0t\}\big)\ =\ 0\,,
$$

since $\{a_i+C_0t\}\in\operatorname{supp}u_s^+(t)$ and $\operatorname{supp}u_s^+(t)\cap\operatorname{supp}u_s^-(t)\neq\varnothing$ only for finitely many t in $(0,\tau)$. Moreover,

$$
u_s^-(t)\big((a_i+C_0t\,,\,a_i+C_0t+\delta)\big)\ \le\ \operatorname{ess\,sup}_{t\in(0,T)}\|u_s(t)\|\,.
$$

Letting $\delta\to0^+$ in (**??**), (**??**) follows from the Dominated Convergence Theorem. □

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Appendix A

A.1. **Preliminaries.** In this Appendix we recall the proof of some results used in the existence proof. We refer the reader to [**?**] for a more general presentation of the underlying material.

Let $T>0$, $S:=\mathbb{R}^N\times(0,T)$, $M_0>0$ and $\{u_n\}\subseteq L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))$. Let

$$
(H_1)\qquad \sup_{n\in\mathbb{N}}\|u_n\|_{L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))}\ \le\ M_0\,.
$$

Recall (e.g., see [**?**, Proposition 4.4.16]) that a sequence $\{\mu_n\}\subseteq L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))$ weakly* converges to $\mu\in L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))$ (written $\mu_n\overset{*}{\rightharpoonup}\mu$ in $L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))$), if

$$
\int_0^T\langle\mu_n(t),\zeta(\cdot,t)\rangle\,dt\to\int_0^T\langle\mu(t),\zeta(\cdot,t)\rangle\,dt\quad\text{for any }\zeta\in L^1(0,T;C_0(\mathbb{R}^N))\,.
$$

Theorem A.1. *Let (H_1) hold. Then there exist a subsequence of $\{u_n\}$ (not relabeled), a Young measure $\nu \in \mathcal{Y}(S;\mathbb{R})$ and $\lambda_{(\pm)} \in L^\infty_{w^*}(0,T;\mathcal{M}^+(\mathbb{R}^N))$ such that*

$$u_n^\pm \overset{*}{\rightharpoonup} u_{b,(\pm)} + \lambda_{(\pm)} \quad \text{in } L^\infty_{w*}(0,T;\mathcal{M}^+(\mathbb{R}^N))\,, \tag{A.1}$$

where $u_{b,(\pm)} \in L^\infty(0,T;L^1(\mathbb{R}^N))$,

$$u_{b,(\pm)}(x,t) := \int_{\mathbb{R}} \xi_\pm \, d\nu_{(x,t)}(\xi) \quad \text{for a.e. } (x,t)\in S\,, \tag{A.2}$$

and $\{\nu_{(x,t)}\}$ denotes the disintegration of ν, defined for a.e. $(x,t)\in S$. If $f \in C(\mathbb{R})$ satisfies

$$\lim_{z\to\pm\infty} \frac{f(z)}{z} =: C_{f,\pm} \in \mathbb{R}\,, \tag{A.3}$$

then

$$f(u_{nr}) + C_{f,+}u^+_{ns} - C_{f,-}u^-_{ns} \overset{*}{\rightharpoonup} f^* + C_{f,+}\lambda_{(+)} - C_{f,-}\lambda_{(-)} \tag{A.4}$$

in $L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))$, where $f^* \in L^\infty(0,T;L^1(\mathbb{R}^N))$, $f^*(x,t) := \int_{\mathbb{R}} f(\xi)\,d\nu_{(x,t)}(\xi)$ for a.e. $(x,t)\in S$.

Proof. We split $u_n^\pm$ into the sum of their absolutely continuous parts with densities $[u_{nr}]_\pm$ and singular parts $u^\pm_{ns}$. Concerning the sequence $\{[u_{nr}]_\pm\}$, retracing the proof of [**?**, Lemmata A.1-A.2] shows that there exist Radon measures $\mu_{(\pm)} \in L^\infty_{w^*}(0,T;\mathcal{M}^+(\mathbb{R}^N))$ and a Young measure $\nu \in \mathcal{Y}(S;\mathbb{R})$ such that, up to subsequences,

$$[u_{nr}]_\pm \overset{*}{\rightharpoonup} u_{b,(\pm)} + \mu_{(\pm)} \quad \text{in } L^\infty_{w*}(0,T;\mathcal{M}^+(\mathbb{R}^N)) \tag{A.5}$$

with $u_{b,(\pm)} \in L^\infty(0,T;L^1(\mathbb{R}^N))$ given by (**??**), and

$$f(u_{nr}) \overset{*}{\rightharpoonup} f^* + C_{f,+}\mu_{(+)} - C_{f,-}\mu_{(-)} \quad \text{in } L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))\,, \tag{A.6}$$

with $f^* \in L^\infty(0,T;L^1(\mathbb{R}^N))$, $f^*(x,t) := \int_{\mathbb{R}} f(\xi)\,d\nu_{(x,t)}(\xi)$ for a.e. $(x,t)\in S$.

On the other hand, by (H_1) there also holds, up to subsequences,

$$u^\pm_{ns} \overset{*}{\rightharpoonup} \tau_{(\pm)} \quad \text{in } L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))\,,$$

for some $\tau_{(\pm)} \in L^\infty_{w*}(0,T;\mathcal{M}^+(\mathbb{R}^N))$. Setting $\lambda_{(\pm)} := \mu_{(\pm)} + \tau_{(\pm)}$ the conclusion follows. □

A.2. **Characterization of the measures $\lambda_{(\pm)}$.**

Theorem A.2. *Let $\{u_n\}$ and $\lambda_{(\pm)}$ be given by Theorem* **??**. *Let $u_0 \in \mathcal{M}(\mathbb{R}^N)$, $\{u_{0r,n}\} \subseteq L^1(\mathbb{R}^N)$, $\{u_{0s,n}\} \subseteq \mathcal{M}(\mathbb{R}^N)$, $\Phi \in C(\mathbb{R};\mathbb{R}^N)$ be such that for all $k>0$*

$$[u_{0r,n} \mp k]_\pm + u^\pm_{0s,n} \overset{*}{\rightharpoonup} [u_{0r} \mp k]_\pm + u^\pm_{0s} \quad \text{in } \mathcal{M}(\mathbb{R}^N) \text{ as } n\to\infty\,, \tag{A.7}$$

$$\lim_{\xi\to\pm\infty} \frac{\Phi(\xi)}{\xi} = M^\pm_\Phi \in \mathbb{R}^N. \tag{A.8}$$

Moreover, assume that for all $k\in\mathbb{R}$ and $\zeta \in C^1_c(\mathbb{R}^N\times[0,T))$, $\zeta\ge 0$, there holds

$$\begin{aligned}
&\int_0^T \langle u^\pm_{ns}(t), \zeta_t(\cdot,t)\rangle\, dt + \int_0^T\int_{\mathbb{R}^N} [u_{nr}\mp k]_\pm \zeta_t \, dxdt + \\
&+ \int_0^T\int_{\mathbb{R}^N} (\pm\chi_{\{\pm u_{nr}>k\}})\big[\Phi(u_{nr}) - \Phi(\pm k)\big]\cdot\nabla\zeta\, dxdt \ge \\
&\ge \int_{\mathbb{R}^N} [u_{0r,n}\mp k]_\pm \zeta(x,0)\,dx + \big\langle u^\pm_{0s,n}, \zeta(\cdot,0)\big\rangle + L^\pm(n,k,\zeta)
\end{aligned} \tag{A.9}$$

for some $L^\pm(n,k,\zeta)>0$ satisfying, for every ζ as above,

$$\lim_{k\to\infty}\limsup_{n\to\infty} L^\pm(n,k,\zeta) = 0. \tag{A.10}$$

Then for a.e. $\tau \in (0,T)$ *there holds*

$$\lambda_{(\pm)}(\tau) \le \mathcal{T}_{M_\Phi^\pm \tau}\, u_{0s}^\pm \quad \text{in } \mathcal{M}(\mathbb{R}^N)\,. \tag{A.11}$$

Proof. We only prove (**??**) with (+). By Theorem **??** (in particular, see (**??**)) and (**??**), letting $n \to \infty$ in (**??**) gives

$$\begin{aligned} &\int_0^T \left\langle \lambda_{(+)}(t), \zeta_t(\cdot,t)\right\rangle dt + \int_0^T\!\!\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}} [\xi-k]_+ d\nu_{(x,t)}(\xi)\right) \zeta_t\, dxdt + \\ &+ \int_0^T\!\!\int_{\mathbb{R}^N} \left(\int_{\{\xi>k\}} [\Phi(\xi)-\Phi(k)]\, d\nu_{(x,t)}(\xi)\right)\cdot \nabla\zeta\, dxdt + \int_0^T \left\langle \lambda_{(+)}, M_\phi^+ \cdot \nabla\zeta(\cdot,t)\right\rangle dt \ge \\ &\ge \int_{\mathbb{R}^N} [u_{0r}-k]_+ \zeta(x,0)\, dx + \langle u_{0s}^+, \zeta(\cdot,0)\rangle + \limsup_{n\to\infty} L^\pm(n,k,\zeta)\,, \end{aligned} \tag{A.12}$$

for every $\zeta \in C_c^1(\mathbb{R}^N \times [0,T))$, $\zeta \ge 0$, and $k > 0$. Observe that, since $u_{0r} \in L^1(\mathbb{R}^N)$,

$$[u_{0r}-k]_+ \to 0 \quad \text{in } L^1(\mathbb{R}^N) \quad \text{as } k \to \infty\,. \tag{A.13}$$

It is easily seen that for all $\xi \in \mathbb{R}$ there holds $[\xi-k]_+ \to 0$ as $k \to \infty$ and $[\xi-k]_+ \le \xi_+ \in L^1(\mathbb{R};\nu_{(x,t)})$ for a.e. $(x,t) \in S$. Therefore, by the Dominated Convergence Theorem, for a.e. $(x,t) \in S$

$$\int_{\mathbb{R}} [\xi-k]_+ d\nu_{(x,t)}(\xi) \to 0\,, \quad \int_{\mathbb{R}} [\xi-k]_+\, d\nu_{(x,t)}(\xi) \le \int_{\mathbb{R}} \xi_+\, d\nu_{(x,t)}(\xi) = u_{b,(+)}(x,t) \in L^1(S)$$

(see Theorem **??**). Again by the Dominated Convergence Theorem, it follows that

$$\int_{\mathbb{R}} [\xi-k]_+\, d\nu_{(x,t)}(\xi) \to 0 \quad \text{in } L^1(S) \quad \text{as } k \to \infty\,. \tag{A.14}$$

By similar arguments, it is easily checked that

$$\int_{\{\xi>k\}} [\Phi(\xi)-\Phi(k)]\, d\nu_{(x,t)}(\xi) \to 0 \quad \text{in } [L^1(S)]^N \quad \text{as } k \to \infty\,. \tag{A.15}$$

In view of (**??**) and (**??**)-(**??**), letting $k \to \infty$ in (**??**) gives

$$\int_0^T \left\langle \lambda_{(+)}(t), \zeta_t(\cdot,t)\right\rangle dt + \int_0^T \left\langle \lambda_{(+)}, M_\phi^+ \cdot \nabla\zeta(\cdot,t)\right\rangle dt \ge \langle u_{0s}^+, \zeta(\cdot,0)\rangle\,.$$

For any $\tau \in (0,T)$, $\rho \in C_c^1(\mathbb{R}^N)$, $\rho \ge 0$, and for any nonnegative $h \in C_c^1([0,T))$, choose $\zeta(x,t) = h(t)\,\rho(x - M_\Phi^+(t-\tau))$ in the above inequality. Then we obtain

$$\int_0^T \left\langle \lambda_{(+)}(t), \rho(\cdot - M_\Phi^+(t-\tau))\right\rangle h'(t)\, dt \ge h(0)\, \langle u_{0s}^+, \rho(\cdot + M_\Phi^+\tau)\rangle\,.$$

Arguing as in the last part of the proof of Proposition **??** the claim easily follows. □

Remark A.1. Let $\{u_n\}$ and Φ be as in Theorem **??**. Then by (**??**)

$$u_n \overset{*}{\rightharpoonup} u := u_b + \lambda \quad \text{in } L^\infty_{w^*}(0,T;\mathcal{M}(\mathbb{R}^N))\,, \tag{A.16}$$

where

$$u_b := u_{b,(+)} - u_{b,(-)} \in L^\infty(0,T;L^1(\mathbb{R}^N)) \quad \text{and} \quad \lambda := \lambda_{(+)} - \lambda_{(-)} \in L^\infty_{w^*}(0,T;\mathcal{M}(\mathbb{R}^N))\,. \tag{A.17}$$

First suppose that $M_\Phi^+ \ne 0$ or $M_\Phi^- \ne 0$. In this case we also assume that u_{0s} is as in (A_1) with a_i, $b_l \in \mathbb{R}^N$. Inequalities (**??**) imply that for a.e. $t \in (0,T)$ the nonnegative measures $\lambda_{(\pm)}(t)$ are singular with respect to the Lebesgue measure in $\mathbb{R}^N$. Moreover, there holds

$$\mathcal{T}_{M_\Phi^+\tau}(u_{0s}^+) = \sum_{i=1}^{P_+} p_{0i}\delta_{a_i+M_\Phi^+ t}\,, \quad \mathcal{T}_{M_\Phi^-\tau}(u_{0s}^-) = \sum_{l=1}^{M_-} m_{0l}\delta_{b_l+M_\Phi^- t}\,,$$

hence $\lambda_{(\pm)}(t)$ have disjoint supports for a.e. $t \in (0,T)$ such that $a_i + M_\Phi^+ t \neq b_l + M_\Phi^- t$ for all $i = 1,\cdots,P_+$ and $l = 1,\cdots,M_-$. Therefore, $\lambda_{(\pm)}(t)$ are mutually singular for a.e. $t \in (0,T)$. Since $u(t) = u_b(\cdot,t) + \lambda_{(+)}(t) - \lambda_{(-)}(t)$ in $\mathcal{M}(\mathbb{R}^N)$, we obtain that

(A.18a) $$\lambda_{(+)}(t) = u_s^+(t)\,, \quad \lambda_{(-)}(t) = u_s^-(t)\,, \quad \lambda(t) = u_s(t) \quad \text{for a.e. } t \in (0,T)\,,$$

(A.18b) $$u_s^\pm(t) \le \mathcal{T}_{M_\Phi^\pm \tau} u_{0s}^\pm \quad \text{in } \mathcal{M}(\mathbb{R}^N)\,,$$

(A.18c) $$u_b = u_r \quad \textit{a.e.} \text{ in } S = \mathbb{R}^N \times (0,T)\,.$$

Combining (**??**) with (**??**) gives for every $f \in C(\mathbb{R})$ satisfying (**??**):

(A.19) $$f(u_{nr}) \overset{*}{\rightharpoonup} f^* + C_{f,+}u_s^+ - C_{f,-}u_s^- \quad \text{in } L^\infty_{w*}(0,T;\mathcal{M}(\mathbb{R}^N))\,,$$

where $f^* \in L^\infty(0,T;L^1(\mathbb{R}^N))$, $f^*(x,t) := \int_{\mathbb{R}} f(\xi)\,d\nu_{(x,t)}(\xi)$ for a.e. $(x,t) \in S$.

If instead $M_\Phi^\pm = 0$, inequalities (**??**) read as

$$\lambda_{(\pm)}(\tau) \le u_{0s}^\pm \quad \text{in } \mathcal{M}(\mathbb{R}^N) \quad \text{for a.e. } \tau \in (0,T).$$

Arguing as above shows that (**??**)-(**??**) hold in this case, too.

A.3. **Characterization of the Young measure disintegration $\nu_{(x,t)}$ for $N = 1$.** In this subsection we assume that $N = 1$. For every $\phi \in \mathrm{Lip}(\mathbb{R})$ and $U \in C_c^2(\mathbb{R})$, set

(A.20) $$\Theta_U(\xi) := \int_c^\xi \phi'(s)U'(s)\,ds \qquad (c \in \mathbb{R})\,.$$

Proposition A.3. *Let $\nu \in \mathcal{Y}(S;\mathbb{R})$ be the Young measure given in Theorem* **??**. *Let there exist $\phi \in \mathrm{Lip}(\mathbb{R})$ such that for a.e. $(x,t) \in S$ and for every $U, V \in C_c^2(\mathbb{R})$ there holds*

(A.21) $$\int_{\mathbb{R}} \big[\Theta_U(\xi) - \Theta_U^*(x,t)\big]V(\xi)\,d\nu_{(x,t)}(\xi) = \int_{\mathbb{R}} \big[U(\xi) - U^*(x,t)\big]\Theta_V(\xi)\,d\nu_{(x,t)}(\xi)\,,$$

where

$$\Theta_U^*(x,t) = \int_{\mathbb{R}} \Theta_U(\xi)\,d\nu_{(x,t)}, \qquad U^*(x,t) = \int_{\mathbb{R}} U(\xi)\,d\nu_{(x,t)}(\xi)$$

and Θ_U, Θ_V are defined by (**??**). *Let u_b be the function in* (**??**). *Then*

(A.22) $$\phi(u_b(x,t)) = \int_{\mathbb{R}} \phi(\xi)\,d\nu_{(x,t)}(\xi) \quad \textit{for a.e. } (x,t) \in S.$$

Proof. Since for a.e. $(x,t) \in S$ the mapping $\xi \mapsto |\xi|$ belongs to $L^1(\mathbb{R};\nu_{(x,t)})$ and $\int_{\mathbb{R}} |\xi|\,d\nu_{(x,t)}(\xi) \le u_{b,+}(x,t) + u_{b,-}(x,t) \in L^1(S)$ (see (**??**)), it can be checked that equality (**??**) holds true for all $U, V \in W^{1,\infty}(\mathbb{R})$. Arguing as in [**?**, Proposition 5.8], (**??**) follows from (**??**). □

Theorem A.4. *Let the assumptions of Proposition* **??** *hold. Suppose that*

(A.23) $$\begin{cases} \textit{for every } \bar\xi \in \mathbb{R} \textit{ there exist } a,b \ge 0,\ a+b > 0 \textit{ such that} \\ \phi \textit{ is strictly convex or concave in } [\bar\xi - a, \bar\xi + b]\,. \end{cases}$$

Then there holds

(A.24) $$\nu_{(x,t)} = \delta_{u_b(x,t)} \quad \textit{for a.e. } (x,t) \in S.$$

Proof. Let $(x,t) \in S$ be such that (**??**) is satisfied for all $U, V \in W^{1,\infty}(\mathbb{R})$. Let $l_1 := u_b(x,t)$. Without loss of generality, we may assume that the map $\xi \mapsto |\xi|$ belongs to $L^1(\mathbb{R};\nu_{(x,t)})$.

In view of assumption (**??**), there exists $h > 0$ such that ϕ is strictly convex (or concave) in $[l_1, l_1 + h]$ or in $[l_1 - h, l_1]$. To fix the ideas, let ϕ be strictly convex in $[l_1, l_1 + h]$. For every $l_2 \in (l_1, l_1 + h)$ and $k \in \mathbb{N}$, let us consider the function

$$V_k(\xi) := k(\xi - l_1)\chi_{[l_1, l_1 + \frac{1}{k})}(\xi) + \chi_{[l_1 + \frac{1}{k}, l_2)}(\xi) + k\left(l_2 + \frac{1}{k} - \xi\right)\chi_{[l_2, l_2 + \frac{1}{k}]}(\xi)\,.$$

Then, for all $\xi \in \mathbb{R}$, in the limit as $k \to \infty$ we have

$$V_k(\xi) \to \chi_{(l_1,l_2]}(\xi)\,, \tag{A.25}$$

$$\Theta_{V_k}(\xi) = \int_{l_1}^{\xi} V_k'(s)\phi'(s)ds \to \phi_+'(l_1)\chi_{(l_1,l_2]}(\xi) + [\phi_+'(l_1) - \phi_+'(l_2)]\chi_{(l_2,\infty)}(\xi) \tag{A.26}$$

(observe that the right derivatives $\phi_+'(l_1)$, $\phi_+'(l_2) \in \mathbb{R}$, since $\phi \in \mathrm{Lip}(\mathbb{R})$). Moreover, there exists $\tilde{C} > 0$ such that

$$\|V_k\|_{L^\infty(\mathbb{R})} \le \tilde{C}\,, \qquad \|\Theta_{V_k}\|_{L^\infty(\mathbb{R})} \le \tilde{C}\,. \tag{A.27}$$

Choosing $U(\xi) = T_k(\xi) = \max\{-k, \min\{\xi, k\}\}$ and $V(\xi) = V_k(\xi)$ in (**??**), we get

$$\int_{\mathbb{R}} [\Theta_{T_k}(\xi) - \Theta_{T_k}^*(x,t)]V_k(\xi)d\nu_{(x,t)}(\xi) = \int_{\mathbb{R}} [T_k(\xi) - T_k^*(x,t)]\Theta_{V_k}(\xi)d\nu_{(x,t)}(\xi). \tag{A.28}$$

In order to take the limit as $k \to \infty$ in (**??**), observe that for all $\xi \in \mathbb{R}$ there holds

$$|T_k(\xi) - T_k^*(x,t)| \le |\xi| + \int_{\mathbb{R}} |\xi|\, d\nu_{(x,t)}(\xi) \in L^1(\mathbb{R}, \nu_{(x,t)})\,, \tag{A.29}$$

$$\begin{aligned} \left|\Theta_{T_k}^*(x,t) - \Theta_{T_k}(\xi)\right| &\le \int_{\mathbb{R}} \left| \int_0^{\xi} |\phi'(s)|\, ds \right| d\nu_{(x,t)} + \int_0^{\xi} |\phi'(s)|\, ds \le \\ &\le \|\phi'\|_\infty \left\{ \int_{\mathbb{R}} |\xi|\, d\nu_{(x,t)}(\xi) + |\xi| \right\} \in L^1(\mathbb{R}; \nu_{(x,t)})\,, \end{aligned} \tag{A.30}$$

and (see also (**??**) and (**??**))

$$T_k(\xi) - T_k^*(x,t) \to \xi - \int_{\mathbb{R}} \xi\, d\nu_{(x,t)}(\xi) = \xi - u_b(x,t) = \xi - l_1\,, \tag{A.31}$$

$$\Theta_{T_k}^*(x,t) - \Theta_{T_k}(\xi) \to \int_{\mathbb{R}} \phi(\xi)d\nu_{(x,t)}(\xi) - \phi(\xi) = \phi(u_b(x,t)) - \phi(\xi) = \phi(l_1) - \phi(\xi)\,. \tag{A.32}$$

By (**??**)-(**??**), (**??**)-(**??**) and the Dominated Convergence Theorem, we get

$$\begin{aligned} &\int_{\mathbb{R}} \left[\Theta_{T_k}(\xi) - \Theta_{T_k}^*(x,t)\right] V_k(\xi)\, d\nu_{(x,t)}(\xi) \to \int_{(l_1,l_2]} [\phi(\xi) - \phi(l_1)]\, d\nu_{(x,t)}(\xi)\,, \\ &\int_{\mathbb{R}} [T_k(\xi) - T_k^*(x,t)]\, \Theta_{V_k}(\xi)\, d\nu_{(x,t)}(\xi) \to \int_{(l_1,l_2]} \phi_+'(l_1)(\xi - l_1)\, d\nu_{(x,t)}(\xi) + \\ &\qquad\qquad + [\phi_+'(l_1) - \phi_+'(l_2)] \int_{(l_2,\infty)} (\xi - l_1)\, d\nu_{(x,t)}(\xi)\,. \end{aligned}$$

By the above convergences, letting $k \to \infty$ in (**??**) gives

$$\int_{(l_1,l_2]} [\phi(\xi) - \phi(l_1) - \phi_+'(l_1)(\xi - l_1)]d\nu_{(x,t)}(\xi) = [\phi_+'(l_1) - \phi_+'(l_2)] \int_{(l_2,\infty)} (\xi - l_1)d\nu_{(x,t)}(\xi). \tag{A.33}$$

Since ϕ is strictly convex in $[l_1, l_2]$, equality (**??**) follows from (**??**), arguing as in part (a) of the proof of [**?**, Proposition 5.9]. □

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References

[1] B. Andreianov & K. Sbihi, *Well-posedness of general boundary-value problems for scalar conservation laws,* Trans. Amer. Math. Soc. **367** (2015), 3763-3806.

[2] K. Ammar, P. Wittbold & J. Carrillo, *Scalar conservation laws with general boundary condition and continuous flux function,* J Differ Equ. **228** (2006), 111-139.

[3] J.M. Ball, *A version of the fundamental theorem for Young measures,* Partial Differential Equations and Continuum Models of Phase Transitions (Proceedings of an NSF-CNRS joint seminar; Springer, 1989).

[4] C. Bardos, A.Y. Le Roux & J.C. Nedelec, *First order quasilinear equations with boundary condition*, Comm. Partial Differential Equations **4** (1979), 1017-1034.

[5] M. Bertsch, F. Smarrazzo, A. Terracina & A. Tesei, *Radon measure-valued solutions of first order hyperbolic conservation laws,* Adv. in Nonlinear Anal. **9** (2020), 65-107.

[6] M. Bertsch, F. Smarrazzo, A. Terracina & A. Tesei, *A uniqueness criterion for measure-valued solutions of scalar hyperbolic conservation laws*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (2019), 137-168.

[7] M. Bertsch, F. Smarrazzo, A. Terracina & A. Tesei, *Signed Radon measure-valued solutions of flux saturated scalar conservation laws*, Discr. Cont. Dyn. Syst. **40** (2020), 3143-3169.

[8] M. Bertsch, F. Smarrazzo, A. Terracina & A. Tesei, *Measure-valued solutions of scalar hyperbolic conservation laws, Part 2: Uniqueness*, forthcoming.

[9] F. Demengel & D. Serre, *Nonvanishing singular parts of measure-valued solutions for scalar hyperbolic equations,* Comm. Part. Diff. Equ. **16** (1991), 221-254.

[10] V.G Danilov & V.M. Shelkovich, *Dynamics of propagation and interaction of δ-shock waves in conservation law systems,* J. Differential Equations **211** (2005), 333-381.

[11] A. Friedman, *Mathematics in Industrial Problems, Part 8*, IMA Volumes in Mathematics and its Applications **83** (Springer, 1997).

[12] K.T. Joseph, *A Riemann problem whose viscosity solutions contain δ-measures,* Asymptotic Anal. **7** (1993), 105-120.

[13] B.T. Hayes & P.G. LeFloch, *Measure solutions to a strictly hyperbolic system of conservation laws,* Nonlinearity **9** (1996), 1547-1563.

[14] D. Mitrovic & M. Nedeljkov, *Delta shock waves as a limit of shock waves,* J. Hyperbolic Differ. Equ. **4** (2007), 629-653.

[15] M. Nedeljkov, *Shadow waves: entropies and interactions for delta and singular shocks,* Arch. Rational Mech. Anal. **197** (2010), 489-537.

[16] F. Otto, *Initial-boundary value problem for a scalar conservation law,* Comptes Rendus Acad. Sci. **322** (1996), 729-734.

[17] E. Y. Panov, *Existence of strong traces for quasi-solutions of multidimensional conservation laws,* J. Hyperbolic Differ. Equ. **4** (2007), 729-770.

[18] F. Poupaud & M. Rascle, *Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients,* Comm. Part. Diff. Equ. **22** (1997), 337-358.

[19] M.M. Porzio, F. Smarrazzo & A. Tesei, *Radon measure-valued solutions for a class of quasilinear parabolic equations,* Arch. Rational Mech. Anal. **210** (2013), 713-772.

[20] M.M. Porzio, F. Smarrazzo & A. Tesei, *Noncoercive diffusion equations with Radon measures as initial data,* J. London Math. Soc. **105** (2022), 1823-1896.

[21] D.S. Ross, *Two new moving boundary problems for scalar conservation laws*, Comm. Pure Appl. Math **41** (1988), 725-737.

[22] D.S. Ross, *Ion etching: An application of the mathematical theory of hyperbolic conservation laws*, J. Electrochem. Soc. **135** (1988), 1235-1240.

[23] F. Smarrazzo & A. Tesei, *Measure Theory and Nonlinear Evolution Equations,* Studies in Mathematics **86** (De Gruyter, 2022).

[24] D.C. Tan, T. Zhang, T. Chang & Y.X Zheng *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, J. Differential Equations **112** (1994), 1-32.

[25] A. Terracina, *Comparison properties for scalar conservation laws with boundary conditions*, Nonlinear Anal. **28** (1997), 633-653.

[26] A. Vasseur, *Strong traces for solutions of multidimensional scalar conservation laws,* Arch. Ration. Mech. Anal. **160** (2001), 181-193.

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