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Thesis For The Degree Of Doctor Of Philosophy

# Topological finiteness and stability of hyperbolizable manifolds


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## Abstract

In this thesis we study the topology of closed hyperbolizable manifolds with bounded diameter and bounded volume entropy. We prove that their fundamental group contains free subgroups of large rank and generators of universally bounded length. This imply an entropy-cardinality inequality, and their topological finiteness. We also prove a systole-entropy inequality which imply, in dimension 3, their topological stability.

The philosophy behind these results is that we can think the volume entropy as an average, large scale version of the Ricci curvature. Hence, it allows us to prove analogues of classical results in Riemannian geometry, with a very strong topological hypothesis (being hyperbolizable), but with no assumptions at all on their curvature.

The quotes at the beginning of each chapter, arguably out of context, strive to recreate, for the reader, the author's unsettling, and emotionally tolling experience of working on a PhD in mathematics during a climate and ecological crisis. The appendix, which aims at explaining the work in accessible terms, is an attempt of resisting through community.

Keywords: volume entropy, systole, hyperbolizable manifold, 3-dimensional topology, hyperbolic geometry, Gromov hyperbolic space, climate crisis

# Indice

# Capitolo 1

# Introduction

*Il tuo respiro*
*misuralo a millenni,*
*vai con la foglia che spuntò*
*per prima sulla terra,*
*vai con gli uccelli che videro*
*un cielo che non è questo,*
*soffia sull'ultimo granello*
*del mondo, il tuo paese*
*è questa immensità.*

— Franco Arminio, Canti della gratitudine

One of the first theorems relating the topological complexity of a manifold with its curvature goes back to the $19^{th}$ century, when Bonnet proved that a complete manifold $M$ with sectional curvature bounded below by some positive constant is compact, and as a consequence has finite fundamental group.
This result can be generalised in various ways; for example Gromov showed that if the sectional curvature of $M$ is non-negative (resp. if the sectional curvature is bounded below by some negative constant, and the diameter is bounded), then the fundamental group of $M$ is finitely generated by a number of elements that only depends on the dimension (resp. on the dimension, the curvature, and the diameter)(cf. [Gro78], [Kar89]).
Furthermore one could ask if these results still hold when weaker invariants than the sectional curvature are constrained; for example Myers [Mye41] extended Bonnet's theorem to the case of Ricci curvature bounded below by a positive constant, while Kapovitch and Wilking [KW11] showed that there exists a function $\mathcal{C}(n, D)$ such that the fundamental group of a complete $n$-manifold $M$ with Ricci curvature $\geq -(n-1)$ and diameter $\leq D$ is generated by at most $\mathcal{C}(n, D)$ elements. On the other hand, a conjecture of Milnor [Mil68] asking whether the fundamental group of a complete manifold with non-negative Ricci curvature is finitely generated has been recently disproved [BNS23].
In dimension 3 Matveev's complexity and Gromov's simplicial volume are examples of other measures of topological complexity. Matveev's complexity is a combinatorial invariant of compact 3-manifolds that bounds from above the number of hyperbolic 3-manifolds [Mar04], in the sense that there are finitely many hyperbolic 3-manifolds $N$ with complexity $c(N) < n$, for any $n$. Similarly Gromov's simplicial volume bounds from above the number of atoroidal 3-manifolds, their connected sums, and $JSJ$-gluings [Gro82]. Its introduction was partly motivated by Cheeger's finiteness

theorem [Che70] for which there are finitely many closed Riemannian manifolds $M$ of dimension $n$ such that $|Sec(M)| \leq 1$, $Diam(M) < D$, and $Vol(M) > \epsilon$, for any fixed $n$, $D$, and $\epsilon$. Weakening the curvature assumptions of Cheeger's theorem, Zhu [Zhu93] showed that there are only finitely many homotopy types of Riemannian 3-manifolds satisfying a lower bound on the volume and on the Ricci curvature, and an upper bound on the diameter.
In the present work we will study how the topology of hyperbolizable manifolds is constrained by upper bounds of their volume entropy. A Riemannian manifold is *hyperbolizable* if it admits a complete metric locally isometric to $\mathbb{H}^n$. This condition is a qualitative form of negative curvature condition, because such manifolds are not hyperbolic, nor necessarily negatively curved; but in the compact case they have Gromov-hyperbolic universal cover, even though without any quantification of the $\delta$-hyperbolicity constant.
The *(volume) entropy* is an asymptotic, metric invariant: for a compact Riemannian manifold $X$ it is classically defined as

$$Ent(X, g) = \lim_{R \to \infty} \frac{\log(Vol(B_{\widetilde{X}}(R, \tilde{x})))}{R},$$

where $\widetilde{X}$ is the universal cover of $X$, and $B_{\widetilde{X}}(R, \tilde{x})$ is the open ball in $\widetilde{X}$ of radius $R$ centered at any point $\tilde{x}$.
This invariant is only significant for manifolds whose fundamental group has exponential growth: in the compact, non-positively curved case the volume entropy coincides with the topological entropy of the geodesic flow, and, as we shall see in Chapter 3, it measures the complexity of the fundamental group. However, unlike other well known measures which bound the complexity of $\pi_1$, such as the Gromov norm, or Matveev's complexity, the entropy does depend on the metric of $X$, and it is not even invariant by quasi-isometries.
An upper bound on the volume entropy can be thought as a weak, large scale replacement of a lower bound on the Ricci curvature (cf. [BCGS20], Comparison 3.9), by virtue of Bishop-Gromov's celebrated volume comparison theorem; in dimension 2 and 3 it can actually rigorously be interpreted as a space mean of the Ricci curvature on the unit tangent bundle [Kni94].
In particular, the volume entropy offers a synthetic theory of Ricci curvature at the asymptotic scale. Here by synthetic we mean that it is independent of the differential structure of the underlying space, just like CAT($k$)-spaces are synthetic analogues of manifolds with bounded sectional curvature, and injectivity radius bounded from below. Synthetic analogues of a geometric property are useful when one is interested in studying sequences of manifolds that satisfy that property, and which may converge to singular spaces. As noted in [CC97], the idea that there should be a synthetic theory of spaces whose Ricci curvature is bounded below in some generalized sense, goes back to Gromov, and it is now understood as the theory of $RCD$ spaces (see [Vil16]). For the purpose of our work, we will be satisfied by the observation that any relevant notion of spaces with a lower bound on their Ricci curvature should deal with metric measured spaces whose measure satisfy some doubling condition which is reminiscent of the Bishop-Gromov comparison theorem (cf. [Fuk87]). A notable example of this approach is the work of Breuillard, Green, and Tao [BGT12] on local groups satisfying a doubling condition; in particular they obtain a generalised Margulis lemma for metric spaces of bounded packing at a fixed scale, which is crucial in finding finiteness results and systolic estimates in the $\delta$-hyperbolic setting (cf. [BCGS20], [BCGS21], and [CS21]). In this perspective, an upper bound on the volume entropy gives only an asymptotic control of the

doubling.
Notice that entropy, being an asymptotic invariant, does not give any information on the *local* geometry of $(X, g)$: the metric $g$ can be easily perturbed locally while leaving the entropy unchanged. However, quite unexpectedly, hyperbolic metrics are characterised by minimizing this invariant (among all metrics with fixed volume, on hyperbolizable manifolds) (see [BCG95], and [CS19], [CS18], [BCGS21] for other interesting consequences of entropy bounds on manifolds).
Regarding the number of generators of the fundamental group of a manifold, one of the first results of this work will be the following bound on the cardinality of any generating set of the fundamental group of hyperbolizable manifolds.
Recall that the *minimal joint displacement* of a finite set of isometries $S$ acting on a metric space $X$ is defined as

$$L(S) \coloneqq \inf_{x \in X} \max_{s \in S} d(s.x, x)$$

Then we have the following:

**Entropy-Cardinality Inequality.** *There exists a function $\mathcal{S}(n, D, E)$ such that, for any complete, hyperbolizable, Riemannian $n$-manifold $X$ with entropy $\leq E$, every generating set $S$ of the fundamental group of $X$, with minimal joint displacement $L(S) \leq D$, contains at most $\mathcal{S}(n, D, E)$ elements.*

Notice that the bound holds for *any* generating set of *any* hyperbolizable manifold (compact or not). This is thanks to the fact that since the manifolds in consideration also support an hyperbolic metric, we are able to base our arguments purely on hyperbolic geometric, disregarding the geometry of $(X, g)$.
For manifolds of diameter $\leq D$, applying the Entropy-Cardinality inequality to the $3D$-short generating set[1]

$$S_{3D}(x) = \left\{ g \in \pi_1(X, x) \mid d_{\widetilde{X}}(g.\tilde{x}, \tilde{x}) < 3D \right\},$$

and using the well known fact that $\pi_1(X)$ admits a presentation on $S_{3D}$ with relations of length at most 3, together with Mostow's rigidity theorem, we immediately deduce the following corollary.

**Topological Finiteness.** *The class $\mathcal{H}_n(E, D)$ of closed, hyperbolizable, Riemannian $n$-manifolds with entropy $\leq E$ and diameter $\leq D$ is topologically finite, i.e. contains finitely many diffeomorphism classes.*

Actually, our argument allow us to bound (explicitly) the number of fundamental groups in the class $\mathcal{H}_n(E, D)$; the finiteness up to diffeomorphism then follows from Mostow's rigidity theorem.
It is interesting to compare this result with the finiteness theorems for Gromov-Hyperbolic spaces and groups of Besson et al. [BCGS21]. Among their results, they proved that the class of closed, aspherical, Riemannian manifolds of diameter $\leq D$, entropy $\leq E$, and with $\delta$-hyperbolic universal cover, has finitely many different topologies, whose number is bounded by a constant depending only on $\delta$, $D$, and $E$. The main difference between this theorem and our topological finiteness result is that we don't ask for the manifold to be $\delta$-hyperbolic, but only hyperbolizable. Namely, in our setting the universal covers $(\widetilde{X}, \tilde{g})$ are still Gromov-hyperbolic, but without any quantification of the hyperbolicity constant $\delta$.

[1] Notice that the definition of $S_{3D}$ does not depend explicitly on the choice of a lift $\tilde{x}$ of the base point $x$ because such a choice is already done when defining the action $\pi_1(X, x) \circlearrowleft \tilde{X}$

The idea of the proof of the Entropy-Cardinality inequality is basically the following: given any hyperbolizable $n$-manifold $(X, g)$ with $Ent(X) < E$ and a generating set $S$ of $G = \pi_1(X, x_0)$ with $L(S) < D$, we produce a subset $\widetilde{S} \subset S^N$ (for some universal constant $N$) generating a free subgroup $\mathbb{F}_n$ of $G$ of large rank $n$, compared to the cardinality of $S$.

**Free Subgroup Theorem.** *There exists a function $\mathcal{N}(n)$, depending only on the dimension, with the following property: let $S$ be any finite symmetric family of isometries of $\mathbb{H}^n$ generating a non-elementary discrete group; then, there exists a subset $\widetilde{S} \subset S^{\mathcal{N}(n)}$ which generates a free Schottky subgroup of rank $\geq \sqrt[4]{|S|}$.*

The proof of the free subgroup theorem is developed in Chapter 3. The main difficulty is that we do not have any geometric information on the set of hyperbolic isometries $S$ beside the fact that they generate non-elementary subgroup of $\text{Isom}(\mathbb{H}^n)$.
The construction of the free subgroup $\mathbb{F}_k \subset G$ of large rank compared to $|S|$ is inspired to similar constructions in [AL06] and [CS18].
In the former, the authors produce, from any subset $S$ of a *fixed*, *word hyperbolic* group $G$, a free subgroup of $G = \langle S \rangle$ of rank $n$ proportional to $|S|$; in the latter, a similar result is proved for groups acting *acylindrically* (possibly not discretely) on trees. However, the ration $n/|S|$ proved in [AL06] depends strongly on the group $G$: namely, it depends on the hyperbolicity constant of $G$ and on the cardinality of a large ball in $G$, which controls the small cancellation in the group.
In contrast, in our case the groups $G = \pi_1(X)$ under consideration, for $X$ belonging to the class $\mathcal{H}_n(E, D)$, will never be word-hyperbolic for the same hyperbolicity constant, whatever the generating set we choose, and we do not have any a priori estimate of the cardinality of balls in $G$.
Our approach draws a lot from the methods appearing in [CS18], but we adapt them in order to work when a lower bound on the translation length of isometries is not readily available as in the case of group actions on trees. To apply ideas from [AL06] and [CS18] to our context we introduce in Chapter 3 the notion of *$k$-acylindrical family of hyperbolic isometries* in $\mathbb{H}^n$ (which mimics the small cancellation condition in the geometric version of the small cancellation theory for groups acting on Gromov-hyperbolic spaces, as introduced by Dahmani, Guirardel, and Osin, cf. [DGO17] and [Cou16]) and then prove the $k$-acylindricity of families with translation length bounded away from $\delta_0 = \log(2)$ (the hyperbolicity constant of $\mathbb{H}^n$). We won't need the full power and generality of the geometric theory of small cancellation, since the proof of the Free Subgroup Theorem is obtained pretty elementary by producing large families of isometries in Schottky position in $\mathbb{H}^n$.
The finiteness theorem seems of particular interest in dimension 3, where hyperbolizable manifolds are one of the eight possible geometries. For the fundamental groups of non-geometric 3-manifolds Cerocchi and Sambusetti [CS18] already proved an entropy-cardinality inequality for $k$-acylindrical group actions on a tree, and deduced the topological finiteness of compact, non-geometric, Riemannian 3-manifolds with uniformly bounded entropy and diameter. This leads us to conjecture that the family of compact, orientable, Riemannian 3-manifolds, with fundamental group of exponential growth, and with uniformly bounded entropy and diameter, is topologically finite.
For closed, strictly negatively curved $n$-manifolds there exists a well known bound of the injectivity radius (which coincides with one half the systole, i.e. one half the length of the shortest non-contractible close geodesic) which is a consequence of the classical Marguils' lemma (see for example

[Gro78], [Hei76]): namely, if the curvature of $X$ satisfies $-k^2 \leq K(X) < 0$ and $Diam(X) \leq D$ then

$$inj(X) \geq i_0(n, k, D) > 0$$

A similar bound holds for closed, non-positively curved manifolds with entropy $\leq E$ and diameter $\leq D$, provided that the universal cover is $\delta$-hyperbolic, as proved in [BCGS20], which gives a bound of the injectivity radius in terms of $D, E$ and $\delta$ only.
In the second part of this work we prove analogous results on the systole of hyperbolizable manifolds and Seifert fibered 3-manifolds with hyperbolic base. In dimension 3, we use these estimates to prove another result regarding the differential stability of closed, hyperbolizable, Riemannian 3-manifolds. In particular we show that their topologies are *stable* with respect to the Gromov-Hausdorff convergence in a large class of Riemannian 3-manifolds, setting a conjecture proposed in [CS19], where the authors obtained an analogous result for non-geometric 3-manifolds. Namely we prove:

**Differential GH-Stability.** *There exists a function $\mathcal{E} = \mathcal{E}(D, E)$ with the following property: let $X$ be a closed, orientable, hyperbolizable, Riemannian 3-manifold, with diameter $Diam(X) < D$, and entropy $Ent(X) < E$; then, for any closed, orientable, Riemannian 3-manifold $Y$ with torsionless fundamental group, $Ent(Y) < E$ and $d_{GH}(X, Y) < \mathcal{E}$, it holds $\pi_1(Y) \cong \pi_1(X)$. In particular $Y$ is diffeomorphic to $X$.*

We can see this result as an analogue of Cheeger's and Colding's topological stability [CC97]; they showed that for any closed Riemannian $n$-manifold $X$ there exists an $\epsilon(X) > 0$ such that, if a closed Riemannian $n$-manifold $Y$ with Ricci curvature $\geq -(n-1)$ is at distance less than $\epsilon(X)$ to $X$, then $X$ and $Y$ are diffeomorphic.
Our result follows from the fact that, except for few special cases, we can distinguish two closed, orientable, Riemannian 3-manifolds by their fundamental groups, which are isomorphic whenever their Gromov-Hausdorff distance is small enough with respect to their systole. Hence the proof of this stability theorem basically relies on some systolic estimates for Seifert fibered, hyperbolizable, and non-geometric manifolds, with bounded entropy. The systolic estimate for non-geometric manifolds was proved in [CS19].
In Chapter 5 we prove the other two cases. First the following:

**Systolic Estimate for Hyperbolizable manifolds.** *There exists a function $\mathcal{S}_{hyp}(E, D) > 0$ such that, if $X$ is a closed, hyperbolizable, Riemannian manifold with $Diam(X) < D$ and $Ent(X) < E$, then $sys(X) \geq \mathcal{S}_{hyp}(E, D)$.*

The main idea behind this result is that if an element of the fundamental group $\pi_1(X) = G$ has small translation length, then $X$ has big entropy, so an upper bound to the entropy corresponds to a lower bound on the systole (i.e. the infimum of the translation lengths). In particular, by a quantitative version of the Tits' alternative for negatively curved manifolds proved in [DKL19] and [CS21], for any short loop $\gamma$ there is an associated free subgroup $\mathbb{F}_2 < G$ generated by a (controlled) power of $[\gamma]$ and some other element with bounded displacement, and it easy to see that the entropy of $\mathbb{F}_2$ (hence of $G$) is bounded below by a function of $\ell(\gamma)$ which tends to infinity as $\ell(\gamma) \to 0$.
In the case of Seifert fibered manifolds with hyperbolic base one needs to be a little more careful, as loops obtained as powers of the fibers can be arbitrarily small without affecting the entropy. This

is due to the fact that, while the volume entropy is useful to detect negative curvature, the abelian part of the fundamental group of a Seifert manifold comes from a flat factor of the universal cover. So we define a *fiber-free systole* for Seifert fibered manifolds with hyperbolic base, and we find a systolic estimate for this quantity when the entropy and the diameter are bounded from above.

**Systolic Estimate for Seifert Fibered manifolds.** *There exists a function $\mathcal{S}_{ff}(E, D) > 0$ such that, if $X$ is a closed, orientable, Riemannian Seifert 3-manifold with hyperbolic base, such that $Diam(X) < D$ and $Ent(X) < E$, then $sys_{ff}(X) \geq \mathcal{S}_{ff}(E, D)$.*

It is worth it to notice that all our results make no assumptions about the injectivity radius or the volume of the manifolds, as is usually required to have compactness, and in contrast to the usual assumptions made in the classic finiteness and stability theorems (see for example [Che70] and [CC97]). This is due to the fact that our results are limited to the very specific class of hyperbolizable manifolds; this condition automatically implies that other strong properties hold such as being aspherical, satisfying the Tits' alternative, or Mostow's rigidity. On the other hand the upper bound on the diameter is a normalization assumption necessary to use the entropy as a meaningful invariant, as entropy is a $\nu^{-n/2}$-homogeneous functional on the class of Riemannian $n$-manifolds.

# Capitolo 2

# Notations and preliminaries

*Victory will be achieved when those promoting the Kyoto treaty on the basis of extant science appear to be out of touch with reality* [...] *and there are no further inititives to thwart the threat of climate change.*

—Internal memo by the American Petroleum Institute, 1998

*Human activities, principally through emissions of greenhouse gases, have unequivocally caused global warming* [...] *Climate change is a threat to human well-being and planetary health.*

— $6^{th}$ Assessment Report, United Nation IPCC

In this chapter we will introduce the basic notations and definitions, and recall some useful results and work out some lemmas that will be needed in the forthcoming chapters. For our purpose, we will work with discrete group actions on the hyperbolic space $\mathbb{H}^n$, where estimates and inequalities are often explicit. Nonetheless many of the results and ideas still applies in the general context of discrete group actions on Gromov-hyperbolic spaces, where the same estimates, while qualitatively less precise, are usually more transparent. For this reason, in some cases, we will state and use results from the theory of discrete group actions on Gromov-hyperbolic spaces, in an effort to balance the generality of arguments with the convenience of proofs and exposition. We will first introduce hyperbolic and Gromov-hyperbolic spaces. Next we will discuss discrete group action on such spaces, and give a brief introduction on volume entropy. Afterwards we will recall some fundamental facts about the geometry and topology of closed 3-manifolds that will be used to prove the stability theorem in Chapter 6. Then we will introduce the notation of orbifold and some useful results regarding these spaces to study the structure of Seifert fibered manifolds. Lastly we will introduce the notion of $\delta$-covering, and the associated groups, that will be essential when studying the behaviours of fundamental groups of 3-manifold under Gromov-Hausdorff convergence.

## 2.1 Hyperbolic and Gromov-hyperbolic geometry

**Projections in hyperbolic space** Given two points in the hyperbolic space $x, y \in \mathbb{H}^n$ we denote $[x, y]$ the unique geodesic segment joining them. Let $\gamma : I \to [x, y]$ be a parametrization on the unit interval of the segment $[x, y]$ such that $\gamma(0) = x$ and $\gamma(1) = y$; given a point $z \in [x, y]$ we write $z = cy + (1 - c)x$ whenever $z = \gamma(c)$, i.e. $c = d(x, z)/d(x, y)$.
The following proposition holds in CAT(0) spaces, and in particular defines the *nearest-point*

*projection*, or *orthogonal projection*, $\pi_C$ to a convex subset $C$ in the hyperbolic space $\mathbb{H}^n$.

**Proposition 2.1.1** ([BH11], Proposition 2.4)**.** *Let $C$ be a convex, complete, subset of $\mathbb{H}^n$, then:*

1. *for every $x \in \mathbb{H}^n$, there exists a unique point $\pi_C(x) \in C$ such that $d(x, \pi_C(x)) = d(x, C)$,*
2. *if $x'$ belongs to the geodesic segment $[x, \pi_C(x)]$, then $\pi_C(x') = \pi_C(x)$,*
3. *given $x \notin C$ and $y \in C$, if $y \neq \pi(x)$ then $\sphericalangle_{\pi(x)}(x, y) \geq \pi/2$,*
4. *the map $x \to \pi(x)$ is a retraction of $\mathbb{H}^n$ into $C$ which does not increase distances, called the projection to $C$.*

We recall a quantitative version of the contraction property 4.

**Lemma 2.1.2** ([Kap10], Lemma 3.6)**.** *Suppose that $\gamma$ is a geodesic in $\mathbb{H}^n$ and $C \subset \mathbb{H}^n$ is a closed convex subset such that $d(\gamma, C) \geq h \geq 0.3$. Then the diameter of $\pi_C(\gamma)$ is at most $9\exp(-h)$.*

**Hyperbolic trigonometry** Given three distinct points $a, b, c \in \mathbb{H}^n$, we denote $\sphericalangle_a(b, c)$ the angle formed at $a$ by the geodesic segments $[a, b]$ and $[a, c]$, and $C_a^c(\alpha)$ the cone centered at $a$, of angle $\alpha$ in direction $c$,

$$C_a^b(\alpha) = \{p \in \mathbb{H}^n \mid \sphericalangle_a(p, c) \leq \alpha\}.$$

Let $\Delta(a, b, c)$ be the geodesic triangle with sides opposed to the vertices $a, b, c$, respectively of lengths $A, B, C$, and with corresponding angles $\alpha, \beta, \gamma$. Then, the following *hyperbolic cosine law* holds:

$$\cosh C = \cosh A \cosh B - \cos\gamma \sinh A \sinh B. \tag{2.1}$$

If $\Delta$ is a right triangle, with $\gamma = \pi/2$, then

$$\sin(\alpha) = \frac{\sinh A}{\sinh C}. \tag{2.2}$$

If $\Delta$ is a right triangle with $\gamma = \pi/2$ and $b \in \partial\mathbb{H}^2$, then $\alpha$ is called *angle of parallelism* and only depends on $C$, in particular it holds

$$\sin\alpha = \frac{1}{\cosh C}. \tag{2.3}$$

In the hyperbolic space the size of a triangle and its angles are interdependent. In particular sides are skewed inwards, making the triangles *thin* (cf. next paragraph). The following lemma quantifies precisely the relationship between the angle $\alpha$, and the distance of $a$ from its opposite side.

**Lemma 2.1.3** (Schweikart function)**.** *Let $\Delta \subset \mathbb{H}^n$ be a hyperbolic triangle of angle $\alpha$ in the vertex $a$ opposed to the side $[b, c]$, and denote $d = d(a, [b, c])$ we have*

$$\sin\left(\frac{\alpha}{2}\right) \leq \frac{1}{\cosh d}, \tag{2.4}$$

*where the equality holds when $[b, c]$ is a complete geodesic (i.e. two vertices are on the boundary).*

We call the function $\Delta(\alpha) = \cosh^{-1}\left(1/\sin(\alpha/2)\right)$ *Schweikart function* mindful of the Schweikart constant $\ln(1 + \sqrt{2}) = \Delta(\pi/2)$, so from 2.4 we get immediately

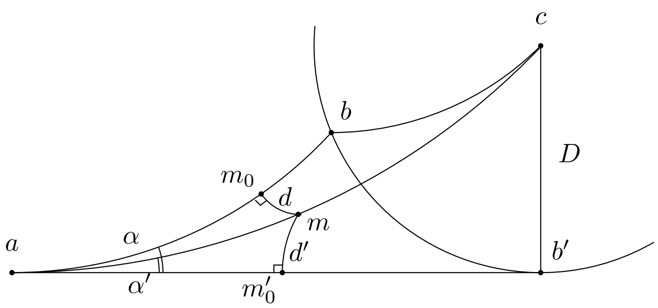
$$d \leq \Delta(\alpha).$$

*Dimostrazione.* Notice that it is sufficient to prove the statement in dimension 2. Consider a point $a \in \mathbb{H}^2$ and a geodesic $\gamma$ such that $d(a, \gamma) = d$. Any triangle that satisfy $d(a, A) = d$ is congruent to a triangle with vertices $a$ and two points $b, c \in \gamma$, and $\alpha = \sphericalangle_a(b, c) \leq \sphericalangle_a(\gamma^+, \gamma^-) = \alpha_d$, hence it is sufficient to prove $\sin\big(\alpha_d/2\big) \leq 1/\cosh d$.
This follows immediately from 2.3 noting that $\alpha_d = \sphericalangle_a(\pi_\gamma(a), \gamma^+ + \sphericalangle_a(\pi_\gamma(a), \gamma^-$ is the sum of two angles of parallelism. □

Another manifestation of the thinness of hyperbolic triangles is the fact that a point in a side is at a distance from one of the two other sides which is universally bound by a constant. Let $m$ be the midpoint of a side $[a, c]$ of a hyperbolic triangle $\Delta[a, b, c]$, and assume that $\ell([ac]) = D + L$, and $\ell([cb]) = D$. The following lemma quantifies precisely the distance between $m$ and its projection $m_0$ to $[a, b]$ when $L$ is very big.

**Lemma 2.1.4.** *Let $\Delta(a, b, c)$ be a geodesic triangle in a $\mathbb{H}^n$ space such that $\ell([ac]) = D + L$, and $\ell([cb]) = D$. Let $m \in [ac]$ be the middle point of $[ac]$, $m_0 \in [ab]$ its projection to $[ab]$, and denote $d = d(m, m_0)$. Then $\sinh d \leq (\frac{\sinh D}{2} \cosh \frac{D+L}{2})$.*

*Dimostrazione.* We may assume that $\Delta(a, b, c) \subset \mathbb{H}^2$. Since $L \geq 0$, the point $a$ lies outside the ball $B(D, c)$ of radius $D$ centered in $c$, so the continuous function $\sphericalangle_\cdot(a, c)$ defined on $\partial B(D, c)$ assumes all values in $[0, 2\pi]$. Then there exists a point $b'$ at distance $D$ from $c$ such that $\beta' = \sphericalangle_{b'}(a, c) = \frac{\pi}{2}$. Let us denote $\alpha'$ the angle at $a$ of the triangle $\Delta(a, b', c)$, $m_0'$ the projection of $m$ to $[a, b']$, and $d' = d(m, m_0')$. We claim that $\alpha \leq \alpha'$, from which follows, by 2.2, that $d \leq d'$.




In fact, since $\beta' = \pi/2$ and $\ell([c, b']) = D$, it follows that the ball $B(D, c)$ is contained in the cone $C_a^c(\alpha')$. In particular $b \in B(D, c)$ so $\alpha = \sphericalangle_a(b, c) \leq \alpha'$, as needed. Applying 2.2 to the triangles $\Delta(a, b', c)$ and $\Delta(a, m_0', m)$ we get

$$\sin \alpha' = \frac{\sinh d'}{\sinh \frac{D+L}{2}} = \frac{\sinh D}{\sinh D + L}.$$

The result then follow by monotonicity of the hyperbolic sine function and from the equality $\sinh(x)/\sinh(2x) = 1/2\cosh x$. □

**Gromov-hyperbolic spaces** Let $X$ be a geodesic metric space. Recall that for any geodesic triangle $\Delta = \Delta(x, y, z)$ there exists a unique comparison triangle in a tree (i.e. a tripod) $\overline{\Delta} = \overline{\Delta}(\bar{x}, \bar{y}, \bar{z})$ such that the lengths of $[\bar{x}, \bar{y}], [\bar{y}, \bar{z}], [\bar{z}, \bar{x}]$ equal the lengths of $[x, y], [y, z], [z, x]$ respectively. Then, there exists a unique map $f_{\bar{\Delta}} : \Delta(x, y, z) \to \overline{\Delta}$ that identifies isometrically the corresponding edges, and there are exactly three points $c_x \in [y, z], c_y \in [x, z], c_z \in [x, y]$ such that $f_{\bar{\Delta}}(c_x) = f_{\bar{\Delta}}(c_y) = f_{\bar{\Delta}}(c_z) = c$, where $c$ is the center of the tripod $\overline{\Delta}$.

A triangle $\Delta(x, y, z)$ is called $\delta$-thin if for every $u, v \in \Delta(x, y, z)$ such that $f_{\bar{\Delta}}(u) = f_{\bar{\Delta}}(v)$ it holds $d(u, v) \leq \delta$; in particular the mutual distances between $c_x, c_y$ and $c_z$ are at most $\delta$. A proper, geodesic metric space is said to be *Gromov $\delta$-hyperbolic*, or simply *$\delta$-hyperbolic*, if all its geodesic triangles are $\delta$-thin.
The *Gromov product* of $y$ and $z$ with respect to $x$ is defined as

$$(y, z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)),$$

and it holds $(y, z)_x = d(x, c_z) = d(x, c_y)$. As an immediate consequence of the $\delta$-thinness of triangles we have the following:

**Lemma 2.1.5.** *(Tripod approximation) Let $x, y, z \in X$ be three points, and $f_{\bar{\Delta}} : \Delta(x, y, z) \to \overline{\Delta}$ their tripod approximation, and $\{c_x, c_y, c_z\} = f_{\bar{\Delta}}^{-1}(c)$ as before. Then*

$$d(x, c_x) \leq \min\{d(x, c_z), d(x, c_y)\} + \delta.$$

In other words the Gromov product can be thought as the inverse of the angle at $x$ formed by the geodesic segments $[x, y]$ and $[x, z]$ (the smaller the angle, the bigger the product and vice versa), and it is approximately the length of the shortest segment joining $x$ to $[y, z]$.
**Remark 2.1.6.** The hyperbolic space $\mathbb{H}^n$ is $\delta$-hyperbolic, with optimal constant $\delta_0 = \log(1+\sqrt{2}) \approx 0.88$.

## 2.2 Discrete groups of isometries in negative curvature

**Classification of isometries** Let $g$ be an isometry of a metric space $X$. The *displacement function* $d_g$ of $g$ is defined as $d_g(x) = d(x, g.x)$, and the *translation length* of $g$ is defined as $\ell(g) = \inf_{x \in X} d_g(x)$. Note that

$$d_g(h.x) = d_{hgh^{-1}}(x), \tag{2.5}$$

and in particular the translation length is invariant by conjugation.
In $\mathbb{H}^n$ the distance is convex, in the sense that for any two geodesic segments $c_1$, $c_2$, the function $t \to d(c_1(t), c_2(t))$ is convex $X$. It follows that the displacement functions are convex, because for any geodesic $\gamma(t)$ we have $d(\gamma(t), g.\gamma(t))$ is convex.
Recall that if $g \in Isom(\mathbb{H}^3)$, then $g$ extends to a unique homeomorphism on $\partial\mathbb{H}^3 \cong S^2$ ([Mar22], Proposition 2.2.4). We denote by $Fix(g)$ the set of fixed points of the action of $g$ on the visual boundary $\partial\mathbb{H}^n$. For any $g, h \in Isom(\mathbb{H}^3)$ we clearly have $g.Fix(h) = Fix(ghg^{-1})$. Let $g \in Isom(\mathbb{H}^3) \setminus \{id\}$, then there are three cases ([Mar22], Proposition 2.2.5):

1. If $\ell(g) = 0$ and there is a point $x$ such that $d_g(x) = 0$, then $g$ is called *elliptic*, and $x$ is the unique fixed point;
2. If $\ell(g) = 0$ and $d_g(x) > 0$ for all $x \in \mathbb{H}^n$, then $g$ is called *parabolic*, and $g$ fixes a unique point $g^+$ on the visual boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$;
3. If $\ell(g) > 0$ then $g$ is called *loxodromic* (or *hyperbolic*), and $d_g = \ell(g)$ along a complete geodesic called the *axis* of $g$, and denoted $Ax(g)$. In this case the two endpoints $\gamma(+\infty)$ and $\gamma(-\infty)$ of $\gamma$ are the unique two fixed points for the action of $g$ on $\partial\mathbb{H}^n$, and we denote them $g^{\pm}$.

Note that, by (2.5) this classification is invariant under conjugation, in particular $Fix(ghg^{-1} = gFix(h)$ and $Ax(ghg^{-1}) = g.Ax(h)$.
If $X$ is a hyperbolic manifold, then $G = \pi_1(X, x_0)$ acts on freely and properly discontinuously on $\mathbb{H}^n$ by isometries through the monodromy action, once chosen a preimage $\tilde{x_0}$ of $x_0$ in $\widetilde{X} = \mathbb{H}^n$; so $G$ contains no elliptic elements. If $X$ is closed, then it is well known that $G$ does not contain parabolic elements either.
Using the fact that $\mathbb{H}^n$ is a Gromov-hyperbolic space for $\delta_0 = \log(1+\sqrt{2})$ we deduce the following facts on loxodromic isometries:

**Lemma 2.2.1.** *(cf. [BCGS20] Lemma* 7.11 *&* 8.21*) Let $g$ be any isometry of $\mathbb{H}^n$, and let $x$ be any point.*

*(i) If $m$ is the middle point of the geodesic segment $[x, g.x]$, then $d_g(m) \leq \ell(g) + 3\delta_0$;*
*(ii) moreover if $g$ is loxodromic with $\ell(g) > 3\delta_0$ then*

$$\frac{d_g(x) - \ell(g)}{2} \leq d(x, Ax(g)) \leq \frac{d_g(x) - \ell(g)}{2} + 3\delta_0.$$

**Lemma 2.2.2.** *([BCGS20] Lemma* 8.24*) Let $a$ and $b$ be two isometries of $\mathbb{H}^n$ such that*

$$d(a.x, b.x) \geq \max\{d_a(x), d_b(x)\} + 5\delta + \ell$$

*for some positive real value $\ell > 0$. Then there exists an isometry $s \in \{a, b, ab\}$ such that $\ell(s) \geq \ell$.*

**Non-elementary subgroups** Let $G$ be a discrete group of isometries of a Gromov-hyperbolic space $X$, and let $LG$ be the *limit set* of $G$ (i. e. the set of accumulation points of any orbit of the action of $G$ on $X$). The group $G$ is said to be *elementary* if $|LG| \leq 2$, and *non-elementary* otherwise.

**Proposition 2.2.3.** *([BCGS20], Proposition* 8.42*) Let $G$ be a discrete subgroup of isometries of $\mathbb{H}^n$:*

1. *if $g, h \in G$ are loxodromic isometries with a common fixed point at infinity, then they have the same pair of fixed points, and $Ax(g) = Ax(h)$;*
2. *if $G$ is virtually nilpotent, then its non elliptic elements are either all parabolic or all hyperbolic, and have the same set of fixed points at infinity. In particular, if $G$ is virtually nilpotent and has no elliptic elements, then it is elementary.*

**Remark 2.2.4.** If $G$ is the fundamental group of a closed hyperbolic 3-manifold, then $G$ acts geometrically on $\mathbb{H}^3$, so $LG \cong S^2$ ([KB02], Theorem 2.24). In particular $G$ is infinite and non-elementary.

**Minimal joint displacement** Let $G$ be a group, finitely generated by $S$, acting by isometries on a metric space $X$. The *minimal joint displacement* of $G$ with respect to $S$ is

$$L(S) := \inf_{x \in X} L_S(x) = \inf_{x \in X} \max_{s \in S} d(x, s.x).$$

From the definition immediately follows that $\ell(s) \leq L(S)$ for each $s \in S$.

**Lemma 2.2.5.** *Let $G$ be a discrete, non-elementary subgroup of loxodromic isometries of $\mathbb{H}^n$, and let be a $S$ a finite generating set of $G$ such that $\ell(s) = \ell$ for each $s \in S$. Then, there exists a unique point $x_0$ where the minimal joint displacement is realised, i.e. $L(S) = \max_{s\in S} d(x_0, s.x_0)$.*

*Dimostrazione.* We claim that $L_S$ is strictly convex. Since $L_S$ is defined as the point-wise maximum of a finite of functions, $\mathbb{H}^n$ is divided into regions $D_{ij}$ such that $L_{S|D_{ij}} = d_{s_i}$. Since the point-wise maximum of strictly convex functions is strictly convex, it is sufficient to show that on each of these regions $L_S$ is strictly convex. To see that, consider $x, y \in D_{ij}$ and notice that $d_{s_i}(tx + (1-t)y) \leq td_{s_i}(x) + (1-t)d_{s_i}(y)$ where the equality is achieved if and only if $[x, y] \in Ax(s_i)$, because each $d_s$ is strictly convex outside its axis. But if $[x, y] \in Ax(s_i)$, then we would have $\ell(s) = L_S(z) = \max_{s\in S} d_s(z)$ for any $z \in [x, y]$, and since $\ell(s)$ is the global minimum of each $d_s$, we find that $[x, y] \in Ax(s)$ for each $s \in S$, which is a contradiction by Lemma 2.2.3, $(i)$, because $G$ is non-elementary.
We now show that $L_S$ is coercive. Suppose by contradiction that $L_S(x_n) < C$ for some constant $C$ and some sequence $x_n \to \xi$ converging to a point at the boundary. Then by Lemma 2.2.1, $(ii)$, it holds $d(x_n, Ax(s)) < (C-\ell)/2 + 3\delta_0$, which means that $\xi \in Fix(s)$ for each $s \in S$, which is a contradiction by Lemma 2.2.3, $(i)$, because $G$ is non-elementary.
Since coercive, strictly convex functions have a unique minimum, this concludes the proof. □

We conclude this discussion by citing two estimates of the joint displacement of a finite set of isometries, which were proved in [BF21], Lemma 1.7 and [BCGS20], Theorem 4.17, in the general context of CAT(0) and Gromov-hyperbolic spaces respectively, and which will be useful in the following.
If $S$ is a finite, symmetric set of isometries we denote by $S^n$ the set of words of length at most $n$ in the letters $s \in S$.

**Lemma 2.2.6.** *Let $S$ be a finite, symmetric set of isometries of $\mathbb{H}^n$. Then:*

1. *the joint displacement of the powers of $S$ satisfy*

$$L(S^n) \geq \frac{\sqrt{n}}{2} L(S);$$

2. *moreover, if $L(S) \geq \frac{31}{2}\delta_0$, then there exists $s \in S^3$ such that $\ell(s) \geq \delta_0$.*

**The Margulis constant** Recall that a *Hadamard manifold* is a complete, simply-connected Riemannian manifold of non-positive sectional curvature. The celebrated Margulis' lemma says that the topology of a Hadamard manifold with bounded sectional curvature is simple at small scales, in the sense that the group generated by elements of small displacement has polynomial growth.

**Margulis Lemma 2.2.7** ([WB13], sec. 8.3)**.** *There exists numbers $I_0(n, K)$ and $\epsilon_0(n, K) > 0$ with the following properties: if $X$ is an $n$-dimensional Hadamard manifold of sectional curvature $-K^2 \leq K(X) \leq 0$, and $\Gamma < \text{Isom}(X)$ is a discrete subgroup of isometries acting on $X$, then, for any $x_0 \in X$, the group $\Gamma_\epsilon(x_0)$ generated by elements of $\Gamma$ that displace $x_0$ less than $\epsilon_0$, contains a nilpotent subgroup of finite index $I < I_0(n, K)$. The constant $\epsilon_0 = \epsilon_0(n, 1)$ is called Margulis constant.*

We will need the following estimates for the Margulis constant:

**Proposition 2.2.8.** *(cf. [Bel14], Proposition 5.2) there exists a constant $C > 0$ such that $\epsilon_0(n) \leq C/\sqrt{n}$. There are also lower bounds for $\epsilon_0(n)$ going to 0 as $n \to \infty$ (see [Kel04] for further references). In particular, in dimension 3 we have*

$$0.104 < \epsilon_0(3) < 1.8. \tag{2.6}$$

## 2.3 Entropy

Let $G$ be a group acting properly by isometries on a metric space $X$. For any $x \in X$ we consider the counting measure $\mu_{G,X}$ of the $G$-orbit of $x$: the *entropy* of $G$ acting on $X$ is defined as exponential growth rate of the measure of $R$-balls with respect to $\mu_{G,x}$, that is

$$Ent(G \curvearrowright X) = \limsup_{R\to\infty} \frac{\log(\mu_{G,X}(B_X(R,x))}{R},$$

where $B_X(R,x)$ is the ball of radius $R$ centered in $x$. This definition clearly does not depend on $x$, by the triangle inequality.

**Remark 2.3.1.** If the action of $G$ on $X$ is cocompact, then $Ent(G \curvearrowright X)$ is also independent from the choice of the $G$-invariant measure ([BCGS20], Proposition 3.3).

We will use this notion basically in two different contexts:

• Let $(G, S, \mathbf{w})$ be a group, a finite set of generators, and a weight function on $S$, $\mathbf{w} : S \to \mathbb{R}_{>0}$. The weight function determines a unique length metric $d_w$ on $G$ and on the Cayley graph $\mathcal{C}(G, S)$, for which an edge associated to a generator $s$ has length $\mathbf{w}(s)$: we will denote by $\mathcal{C}_{\mathbf{w}}(G, S)$ this metric space. If $L$ is a constant we will write $\mathcal{C}_L(G, S)$ and $d_L$ for the Cayley graph associated to the constant weight function $\mathbf{w} \equiv L$ and the corresponding distance. When $L = 1$, then this is just the usual Cayley graph $\mathcal{C}(G, S)$ whose distance we will denote by $d_S$ and $||g||_S := d(e, g)$ is the *word distance*, i.e. the length of the shortest word in the letters $S$ representing $g$. The *(algebraic) volume entropy* of $(G, S, \mathbf{w})$ is accordingly defined as the entropy of the action of $G$ on its weighted Cayley graph $\mathcal{C}_{\mathbf{w}}(G, S)$, which we will simply denote $Ent(G, S_w)$ (or $Ent(G, S_L)$, $Ent(G, S)$ in case $w \equiv L$ or $w \equiv 1$ respectively).

If $H$ is a subgroup (or a subsemigroup) of a finitely generated group $(G, S)$ with weight function $w$ we will similarly define

$$Ent(H, S_w) := \lim_{R\to\infty} \frac{\log\left(|B_H(R)|\right)}{R}$$

(resp. $Ent(H, S_L)$, $Ent(H, S)$ if $w \equiv L$ or $w \equiv 1$) where $B_H(R)$ is the $R$-ball in $H$ centered at the identity $e$ with respect to the distance $d_w$ of $G$.

A finitely generated group $G$ is said to be of *exponential growth* if $Ent(G, S) > 0$ for some (hence all) generating set $S$, and of *sub-exponential growth* otherwise.

We have the following estimate (which will be used later):

**Lemma 2.3.2** ([BCG03], Lemma 2.4)**.** *Let $\mathbb{F}^+(S)$ be the free semi-group generated by the generating set $S = \{a, b\}$, endowed with weight function $w$ such that $w(a) = A$ and $w(b) = B$. Then*

$$Ent(\mathbb{F}^+(S), S_w) \geq \frac{1}{2}\Big[\frac{1}{A}\log\big(1 + \frac{A}{B}\big) + \frac{1}{B}\log\big(1 + \frac{B}{A}\big)\Big].$$

*In particular, when $A < B$ then*

$$A > B \exp\big(-2B \cdot Ent(\mathbb{F}^+(S), S_w)\big).$$

• Let $M$ be a Riemannian manifold, and let $\widetilde{M}$ be the Riemannian universal cover. The *(Riemannian) volume entropy* of $M$ is defined as the exponential growth rate of the volume of balls in the universal cover (centered at any fixed point $\tilde{x}$, which coincides with the entropy of the Riemannian Galois action $G = \pi_1(X) \curvearrowright \widetilde{M} \curvearrowright \widetilde{M}$) (cf. for instance [Sam00], [BCGS20])

$$Ent(M) = \liminf_{R \to \infty} \frac{\log(Vol(N_R(p)))}{R}.$$

When $X$ is compact, by Remark 2.3.1 this equals the entropy of $\pi_1(X)$ acting by isometries on $(\widetilde{X}, \tilde{g})$.
The volume entropy of a Riemannian manifold $M$ can be thought as a very weak, asymptotic analogue of the Ricci curvature: see [BCGS20], Comparison 3.9, and [Kni94], Theorem 5.1 where $Ent(M)$ is interpreted, for negatively curved manifolds, as a space mean of curvatures of the unit tangent bundle of $M$. It bounds the complexity of the fundamental group of $M$ (cf. [CS18], and the entropy-cardinality inequality of Proposition 4.0.2), and in many cases yields, quite unexpectedly, lower bounds on the systole of $M$ (cf. for instance [Sam08], [BCG03], [CS19], [Cer12], [BCGS20], and Chapter 5)).
To simplify the notation we will simply write $Ent(M)$ when there will be no ambiguities about the metric $g$. Note that the metric $g$ and the choice of $x \in X$ induces a geometric distance $d_X(a, b)$ on the fundamental group $\pi_1(M, x)$, namely the distance $\tilde{d}(a.\tilde{x}, b.\tilde{X})$ on the universal cover $\widetilde{X}$ between the corresponding points of the orbit of a preimage $\tilde{X}$ of $x$, under the action of $\pi_1(X, x)$ on $\widetilde{X}$: then if $X$ is compact we have

$$Ent(X, g) = Ent(\pi_1(X, x) \curvearrowright \widetilde{M}) = Ent(G, ||\cdot||_X).$$

In particular, if the sectional curvature is negative, $Ent(M, g) > 0$.

## 2.4 Geometric and non-geometric 3-manifolds

Since we will deal exclusively with closed, orientable 3-manifolds, all results will be stated assuming these hypothesis. We refer to [Thu97], [AFW15], and [Sco83a] for a more detailed account of the subject in greater generality.
We recall that a closed 3-manifold $X$ is said to be *prime* if it cannot be decomposed non trivially as the connected sum of two manifolds, i.e. whenever $X = X_1 \# X_2$ then either $X_1$ or $X_2$ is diffeomorphic to the sphere $S^3$. A compact 3-manifold $X$ is called *irreducible* if every embedded 2-sphere in $X$ bounds a 3-ball in $X$ (and *reducible* otherwise). Every orientable, irreducible 3-manifold is prime; conversely, if $X$ is a closed, orientable 3-manifold which is prime, then either $X$ is irreducible or $X = S^1 \times S^2$.

**Proposition 2.4.1** (Prime Decomposition Theorem, [Mil62], Theorem 1)**.** *Let $X$ be a closed, oriented 3-manifold. Then $X$ is diffeomorphic to a connected sum $X_1 \# \dots \# X_s$ where all the components $X_i$ are oriented prime manifolds. Furthermore such a decomposition is unique, in the sense that,*

*if $X \cong X_1' \# \ldots \# X_r'$, then $r = s$ and (possibly after reordering) there exists orientation-preserving diffeomorphisms $X_i \to X_i'$.*
*In particular, if $X$ is not prime, the fundamental group of $X$ is isomorphic to the free product of the fundamental groups of its components, $\pi_1(X) = \pi_1(X_1) * \cdots * \pi_1(X_s)$.*

The following theorem by Stalling ([Sta56], Theorem $II.A.3$), which solves the Kneser's conjecture in the closed case, offers a partial converse to the Prime Decomposition Theorem.

**Proposition 2.4.2** (Kneser's Conjecture, [AFW15] Theorem 2.1)**.** *Let $X$ be a closed 3-manifold. If $\pi_1(X) = G_1 * \cdots * G_n$, then there exists closed 3-manifolds $X_1, \ldots, X_n$, such that $\pi_1(X_1) = G_i$ and $X = X_1 \# \ldots \# X_n$.*

Following [AFW15] and [Mar22] we say that a connected, embedded surface $S \subset X$ is *incompressible* if $\pi_1(S)$ is not trivial, and the induced map $\pi_1(S) \to \pi_1$ is injective. A closed manifold is called *atoroidal* if it is irreducible and contains no incompressible tori.
We also have the following decomposition for irreducible manifolds:

**Proposition 2.4.3** (JSJ-decomposition, [AFW15] Theorem 1.7)**.** *Let $X$ be a closed, orientable, irreducible 3-manifold. Then there exists a collection of disjointly embedded incompressible tori $S_1, \ldots, S_k$, such that each component of $X$ cut along $S1 \cup \cdots \cup S_k$ is Seifert fibered or atoroidal. Furthermore, any such collection with a minimal number of components is unique up to isotopy.*

We are especially interested in the interplay between geometry and topology, which exhibits very strong relations in dimension 3.
Recall that a 3-*dimensional geometry* is a smooth, simply connected 3-manifold $X$ which is equipped with a smooth, transitive action by diffeomorphisms of a Lie group $G$, with compact point stabilizers. One usually also add two technical conditions, which rule out redundant, or relatively "poor" geometries: the group of isometries is required to be maximal among Lie groups acting transitively on $X$ with compact point stabilizers; and $X$ is required to have a compact model. The Lie group $G$ is called the group of isometries of $X$. A *geometric structure* on a closed manifold $M$ is a diffeomorphism $M \to X/\pi$, where $\pi$ is a discrete subgroup of $G$ acting freely on $X$; in this case $M$ is said be a *geometric* 3*-manifold*, or to admit a $X$*-structure.*
In dimension 2 there exists three possible geometries, which are the classical spherical $S^2$, Euclidean $\mathbb{E}^2$, and hyperbolic $\mathbb{H}^2$. In [Thu97], Theorem 3.8.1, Thurston listed all possible 3-dimensional geometries, also known as *Thurston geometries*: $S^3$, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $Nil$, $Sol$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^2 \tilde{\times} \mathbb{R}$, and $\mathbb{H}^3$, where $\mathbb{H}^2 \widetilde{\times} \mathbb{R}$ is the twisted line bundle over $\mathbb{H}^2$, and is isomorphic to the universal cover of $SL(2, \mathbb{R})$, $Nil$ is the twisted line bundle over $\mathbb{R}^2$ and is isomorphic to the Heisenberg group, and $Sol$ is a $\mathbb{R}^2$-bundle over the line and is isomorphic to the connected component of the identity of the group of maps from 2-dimensional Minkowski space to itself that are either isometries or multiply the metric by $-1$.
We say that a closed manifold $M$ is *non-geometric* if it cannot be endowed with a complete metric which is locally isometric to one of the eight model geometries. A closed 3-manifold is called *hyperbolizable* if it can be endowed with a metric locally isometric to $\mathbb{H}^3$.
In contrast with the 2-dimensional case, a generic closed, irreducible 3-manifold does not admit a geometry. Nonetheless, Thurston's influential work on geometries, together with Hamilton's breakthrough on Ricci flow, culminated in Perelmann's proof of the Geometrization Conjecture,

which states that a closed, irreducible 3-manifold can be decomposed into geometric pieces.

**Proposition 2.4.4** (Geometric Decomposition Theorem, [AFW15] Theorem 1.19)**.** *Let $X$ be a closed, orientable, irreducible 3-manifold. Then there exists a collection of disjointly embedded incompressible surfaces $S_1, \dots, S_k$ which are either tori or Klein bottles, such that each component of $X$ cut along $S_1 \cup \dots \cup S_k$ is geometric. Furthermore, any such collection with a minimal number of components is unique up to isotopy.*

Hyperbolic manifolds have contractible universal cover, hence are aspherical, furthermore, by Preissman's theorem ([FdC13], Theorem 3.2) a closed hyperbolic manifold is atoroidal. The converse is known as Hyperbolisation Theorem:

**Proposition 2.4.5** (Hyperbolisation Theorem, [AFW15] Theorem 1.13)**.** *Let $X$ be a closed, orientable, aspherical, atoroidal 3-manifold. Then $X$ is hyperbolizable.*

In particular, for a closed, orientable, atoroidal 3-manifold $X$ the three following conditions are equivalent: $\pi_1(X)$ is torsion-less, $\pi_1(X)$ is infinite, $X$ is aspherical.
The fundamental groups of closed geometric manifolds are very much studied known, cf. (besides [Thu97]) the fundamental essay of Scott [Sco83a] and, for a complete overview, the book [AFW15]. In particular it is well known that if $M$ is a closed geometric 3-manifold, with fundamental group $G$, and if $M$ is modelled on $S^3$, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, or $Nil$, then $G$ is virtually nilpotent, hence of sub-exponential growth, while if $M$ is modelled on $Sol$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^2 \tilde{\times} \mathbb{R}$, or $\mathbb{H}^3$, then $G$ has exponential growth.
On the other hand, if $M$ is non-geometric then $G$ has always exponential growth, unless $M = \mathbb{P}^3\mathbb{R} \# M = \mathbb{P}^3\mathbb{R}$ (in which case $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_2$): in fact, either $X$ is not prime, and then $G$ is a non-trivial free product with $\pi_1(M) \neq \mathbb{Z}_2 * \mathbb{Z}_2$, hence it has exponential growth; or $X$ is prime and has a non-trivial $JSJ$-decomposition, then $G$ acts acylindrically on the Bass-Serre tree associated to the $JSJ$-decomposition (see [HW10], and [CS19] (Dichotomy, Section 4.1) $G$, in particular it contains in this case free subgroups of rank $\geq 2$.
A strong consequence of the Geometric Decomposition is that, except for a few cases, closed, orientable 3-manifolds are characterised by their fundamental group. We refer to Shalen, [DS02], Chapter 19, and to [AFW15], for more details. Here we just need the following well known fact, of which we give a brief proof for convenience.

**Proposition 2.4.6.** *Let $X$, and $Y$ be two closed, orientable 3-manifolds. If $X$ is atoroidal, and $Y$ is non-geometric, then $\pi_1(X)$ and $\pi_1(Y)$ cannot be isomorphic.*

*Dimostrazione.* Since $Y$ is non-geometric, then $\pi_1(Y)$ is infinite, so we can assume $\pi_1(X)$ to be infinite as well. In particular it means that $X$ is aspherical and hyperbolic by Proposition 2.4.5. Then we have two cases. If $Y$ is prime then it contains an embedded incompressible torus or Klein bottle. In the first case there is a subgroup $\mathbb{Z} \times \mathbb{Z} < \pi_1(Y)$. Since $X$ admits a metric of constant negative curvature, by Preissmann's theorem ([FdC13], Theorem 3.2), it cannot contain any abelian subgroup of rank $> 1$. In the second case there is a subgroup $\mathbb{Z} \times \mathbb{Z}_2 < \pi_1(Y)$, but $\pi_1(X)$ has no torsion. Thus $\pi_1(X)$ and $\pi_1(Y)$ cannot be isomorphic. On the other hand, if $Y$ is not prime then, by the Prime Decomposition Theorem 2.4.1 its fundamental group splits as a non-trivial free product of some groups $G_i$. Then, if we assume $\pi_1(X) \cong \pi_1(Y)$, by the Kneser's conjecture 2.4.2 $X$ splits as the connected sum of some manifolds $M_i$. In particular it contains some embedded incompressible sphere $S^2 \hookrightarrow X$, and we have $\pi_2(X) \neq 0$, which is a contradiction since $X$ is aspherical. In either cases $\pi_1(X)$ and $\pi_1(Y)$ cannot be isomorphic. □

All the following facts about the fundamental group of closed hyperbolic manifolds are well known, and we will give references or sketch brief proofs for the convenience of the reader.

**Proposition 2.4.7.** *Let $G$ be the fundamental group of a closed, hyperbolic 3-manifold $X$.*

*(i) $G$ is residually simple;*

*(ii) $G$ does not contain any non-trivial normal abelian subgroups;*

*(iii) $G$ as trivial center;*

*(iv) $G$ contains a non-abelian free subgroup on 2 generators;*

*Dimostrazione.* (i) is proven in [LR98], Corollary 1.3. From (i) and the fact that $G$ has no torsion (Proposition 2.7.7), it follows that $G$ does not contain any non-trivial normal abelian subgroups. In fact, assume by absurd that there is an element $g \in G$ generating an infinite cyclic normal subgroup. Then there is a surjective map $h : G \to S$ onto a simple group $S$, such that $h(g)^6 \neq 1$, but $\langle h(g) \rangle \trianglelefteq S$ would contain a non-trivial normal subgroup, which contradicts the hypothesis that $S$ is simple. That proves (ii). As an immediate corollary we get (iii). Finally (iv) is well known. For example, since $G$ it acts properly discontinuously by isometries on $\mathbb{H}^3$, it follows from Proposition 5.0.2. □

Let us conclude this section with two results about fundamental groups of aspherical manifolds. The next theorem, which is well known to experts, is a consequence of many deep results in topology and in the theory of 3- manifolds. In particular it involves the solution of the Borel conjecture for 3-manifolds, which relies on the Mostow–Prasad Rigidity Theorem and the Geometrization Theorem ([AFW15], Theorem 1.10 and Theorem 1.14), and work of Turaev ([Tur88], Theorem 1), of Waldhausen ([Wal68], Corollary 6.5), and Scott ([Sco83b], Theorem 3.1).

**Proposition 2.4.8.** *Two closed, orientable, aspherical 3-manifolds have isomorphic fundamental groups if and only if they are diffeomorphic.*

Here we sketch the proof for convenience of the reader.

*Dimostrazione.* The proof has three steps. Whitehead's theorem ([Whi78], Theorem $IV$.7.15) states that a map between CW-complexes is a homotopy equivalence if and only if it induces bijections on all homotopy groups. As a consequence two aspherical CW-complexes have isomorphic fundamental group if and only if they are homotopy equivalent (see [Lüc12], Theorem 2.1). Secondly, by the solution of the Borel conjecture in dimension 3, two closed, aspherical 3-manifolds are homotopy equivalent if and only if they are homeomorphic (see [AFW15] Theorem 2.1). Finally, by Moise's *Hauptvermutung* theorem ([Moi77], Theorem 32.2), and Munkres' work on smoothing PL structures ([Mun60], Theorem 6.5) two 3-manifolds are homeomorphic if and only if they are diffeomorphic. □

## 2.5 Orbifolds

In this section we introduce some basic definitions and results about orbifolds following [Thu02], Chapter 13. We will use these notions to describe the geometry of Seifert manifolds. Recall that an *orbifold* $O$ is a Hausdorff space $X_O$ together with an equivariant covering by finite quotients of open sets, closed by finite intersections. It means that there are a covering by open sets $\{U_i\}$, closed by finite intersection, and diffeomorphisms $\varphi_i : \tilde{U}_i/\Gamma_i \to U_i$, where $\tilde{U}_i \subset \mathbb{R}^n$ is an open set and $\Gamma_i$ is a finite group of diffeomorphisms of $\mathbb{R}^n$, such that for each inclusion $U_i \hookrightarrow U_j$ there are an injective homomorphism $\Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ such that the following diagram commutes

$$\begin{array}{ccc}
\tilde{U}_i & \xhookrightarrow{\tilde{\varphi}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\varphi_{ij}=\tilde{\varphi}_{ij}/\Gamma_i} & \tilde{U}_j/\Gamma_i \\
\uparrow & & \downarrow f_{ij} \\
 & & \tilde{U}_j/\Gamma_j \\
 & & \uparrow \\
U_i & \hookrightarrow & U_j
\end{array}$$

The points $x \in X_O$ that have a neighbourhood $U_i$ which is diffeomorphic to an open subset of $\mathbb{R}^n$ are called *regular* and form an open subset, its complement is denoted $S(O)$ and is called *singular locus.* In dimension 2, we call a *cone point* a critical point whose neighbourhood is a quotient by an open set by a finite group generated by a rotation. Note that a manifold is an orbifold with empty critical locus.

A *orbifold cover* of an orbifold $O$ is an orbifold $\tilde{O}$ together with a projection $p : O_{\tilde{X}} \to O_X$ between the underlying spaces such that every point $x \in O_X$ has an open neighbourhood $U \cong \tilde{U}/\Gamma$ (where $\tilde{U}$ is an open subset of $\mathbb{R}^n$) such that each component $v_i$ of $p^{-1}(U)$ is isomorphic to $\tilde{U}/\Gamma_i$ for some subgroup $\Gamma_i < \Gamma$, and the isomorphism must respect the projections, i.e. the following diagram must commute

$$\begin{array}{ccc}
 & \tilde{U} & \\
\swarrow & & \searrow \\
\tilde{U}/\Gamma_i \cong v_i & \xrightarrow{p} & U \cong \tilde{U}/\Gamma
\end{array}$$

Every orbifold has an *orbifold universal cover*, which is an orbifold cover that satisfy the usual universal property of universal covers. The *orbifold fundamental group* $\pi_1^{orb}(O)$ of an orbifold $O$ is the group of deck transformations of the orbifold universal cover.

An orbifold is *(very) good* if it is (finitely) covered by a manifold.

If $G$ is a group of real analytic diffeomorphisms of a real analytic manifold $X$, we say that an orbifold is *modelled on* $(G, X)$, or that it is a $(G, X)$-*orbifold*, if it has an equivariant covering of open sets $U_i \cong \tilde{U}_i/\Gamma_i$, where $\tilde{U}_i$ is an open subset of $X$ and $\Gamma_i$ is a finite subgroup of $G$. In particular the inclusion maps $U_i \subset U_j$ come from diffeomorphisms $\varphi_{ij} : \tilde{U}_i \to \tilde{U}_j$ in $G$. The following proposition states that $(G, X)$-orbifolds are good.

**Proposition 2.5.1** ([CHK00] Theorem 2.26)**.** *Let $O$ be an $(G, X)$-orbifold modelled on an analytic geometry $(G, X)$ where $G$ is a group of isometries of $X$. If $X_O$ is a complete metric space, then it is defined a developing map dev : $\tilde{O} \to X$, which is a covering map, and and holonomy representation $h : \pi_1^{orb}(O) \to G$; in particular $O$ is good. If $X$ is simply connected, then the $h$ is an isomorphism onto a discrete subgroup $\Gamma$ of $G$ which acts properly discontinuously on $X$, and $O$ is isometric to the quotient $X/\Gamma$.*

By Selberg's lemma, every finitely generated subgroup $\Gamma$ of $GL(n, \mathbb{C})$ has a torsion-free normal subgroup of finite index (see for example [Rat06] Ch. 7.6, Corollary 4). Hence if $G$ is a linear group it follows that every complete $(G, X)$-orbifold is very good. In particular we have the following.

**Corollary 2.5.2.** *(Geometric orbifolds are very good) If an orbifold is modelled on one of the three 2-dimensional geometries, or on one of the eight 3-dimensional Thurston geometries, then it is very good.*

## 2.6 Seifert manifolds

In this section we introduce some basic definitions and results about Seifert fibred manifolds following [Sco83a].

A *trivial fibred solid torus* is the product $S^1 \times D^2$ foliated by circles $S^1 \times \{x\}$. A *fibred solid torus* $T(q, p)$, where $q$ and $p$ are two coprime integers, is the mapping torus of a rotation of the disk of angle $2\pi q/p$. An (orientable) *Seifert fibred manifold* is a 3-manifold $M$ with a decomposition of $M$ into disjoint circles, called *fibres*, such that each circle has a neighbourhood in $M$ which is a union of fibres and is isomorphic to a fibred solid torus. If a fibre has a neighbourhood isomorphic to a trivial fibred solid torus, we call it *regular*, and *critical* otherwise. Given $y \in M$ will denote $F_y$ the fibre containing $y$.

Note that any non trivial fibred solid torus has exactly one singular fibre, corresponding to the point of $D^2$ fixed by the rotation. If $M$ is compact and orientable, the critical fibres are isolated, and the union of the regular fibres forms a fibre bundle.

**Remark 2.6.1.** In particular, all the regular fibres of $M$ are freely homotopic to each other, and any critical fibre has a power which is freely homotopic to a regular fibre.

Taking into account the critical fibres, $M$ can be seen as an $S^1$-orbifold bundle over an orbifold. By identifying each fibre with a point we obtain a surface $\Sigma$, called the *base* of $M$, which has a natural orbifold structure. To a critical fibre whose neighbourhood is isomorphic to $T(q, p)$ corresponds a cone point of angle $2\pi/p$ in $\Sigma$.

In particular, a Seifert manifold $M$ can be constructed by Dehn surgery performed on a finite number of fibres of trivial circle bundle over a surface. The Seifert-Van Kampen theorem then allows one to calculate the fundamental group of $M$.

**Proposition 2.6.2** ([JN83] Theorem 6.1)**.** *Let $M$ be a closed, orientable Seifert manifold whose underlying surface has genus $g$, and with $k$ critical fibres isomorphic to $\{T(q_i, p_i)\}_{i=1\dots k}$ for some coprime $p_i, q_i$. Then we have the following presentation of $\pi_1(M)$*

$$\pi_1(M) \cong \Big\langle a_i, b_i, c_j, f \Big| \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{k} c_j = f, c_j^{p_j} = f^{q_j}, [f, x] = 1 \Big\rangle,$$

*where $x$ represents any generator other than $f$.*

If we fix any point $p0 \in M$, the generators $f$ and $c_i$'s correspond to an element represented by a lasso whose head is a regular fiber and the $i^{th}$ critical fiber respectively, while the other generators correspond to the standard generators of the base.

The universal cover $\tilde{M}$ of $M$ has a foliation by circles or lines induced by the foliation of $M$, and its base $\tilde{\Sigma}$ is the orbifold universal cover of $\Sigma$.

The fundamental group $\pi_1(M)$ acts on $M$ and $\tilde{M}$ preserving the fibres, hence there is an induced action of $\pi_1(M)$ by deck transformations on $\tilde{\Sigma}$ which defines a map $p : \pi_1(M) \to \pi_1^{orb}(\Sigma)$. The kernel of this map consists of covering translations of $\tilde{M}$ which project to the identity map on $\tilde{\Sigma}$, and we have the following short exact sequence.

**Lemma 2.6.3** ([Sco83a] Lemma 3.2). *Let $M$ be a Seifert fibred manifold with base orbifold $\Sigma$. There is a short exact sequence*

$$1 \to F \to \pi_1(M) \to \pi_1^{orb}(\Sigma) \to 1, \tag{2.7}$$

*where $F$ denotes the cyclic subgroup of $\pi_1(M)$ generated by a regular fibre. The group $F$ is infinite except in the cases where $M$ is covered by $S^3$.*

It follows that the orbifold fundamental group of $\Sigma$ admits the following presentation

$$\pi_1^{orb}(\Sigma) = \big\langle a_i, b_i, c_j \big| \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{k} c_j = 1, c^{p_j} = 1 \big\rangle, \tag{2.8}$$

and it surjects to the surface group $\pi_1(\Sigma) = \langle a_i, b_i \big| \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$.
**Remark 2.6.4.** Since $\pi_1^{orb}(\Sigma)$ acts properly discontinuously, faithfully by isometries on $\mathbb{H}^2$, if $g \in \pi_1^{orb}(\Sigma)$ is a torsion element then it is elliptic, and has a unique fixed point. The fixed point correspond via the projection $p : \mathbb{H}^2 \to \Sigma$ precisely to the singular points of the orbifold; for every $x_i$ the exist a $c_i \in \pi_1^{orb}(\Sigma)$ that fixes some point $\tilde{x}_i$ in the preimage of $x_i$, and $p^{-1}(x_i)$ coincide with all conjugates of $c_i$. Hence $g \in Stab_G(h\tilde{x}_i) = hStab_G(\tilde{x}_i)h^{-1} = h\langle c_i \rangle h^{-1}$. In particular, it follows that an element $g \in \pi_1^{orb}(\Sigma)$ has torsion if and only if it is conjugated to some power of $c_j$ for some $j$.
We conclude this section with a remark on the Geometric Decomposition Theorem 2.4.4. If $X$ is a closed, orientable, irreducible manifold, every geometric piece in the decomposition of $X$ is either hyperbolic, *Sol*, or Seifert[1]. Every geometric Seifert piece fibres over a two dimensional orbifold that is modelled on $S^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$, and we call them *spherical*, *Euclidean*, and *hyperbolic Seifert*, respectively.

## 2.7 Fundamental groups under Hausdorff convergence

In this section we study the relations between the fundamental groups of manifolds that are close in the Gromov-Hausdorff distance with respect to their systole. Let us begin by recalling some basic facts about covering spaces and fundamental groups, further reference see [Hat02].
Let $M$ be a Riemannian manifold with base point $p_0 \in M$, and $\gamma : S^1 \to M$ a closed loop with $\gamma_0 = p_0$. We denote $[\gamma]$ and $[\gamma]_{p_0}$ the free homotopy class ans pointed homotopy class of $\gamma$ respectively. The obvious projection $\pi_1(M, p_0) \to C(S^1, M)$ of the fundamental group to the set of free homotopy classes defines a maps that is constant on the conjugacy classes of $\pi_1(M, p_0)$.
The fundamental group $\pi_1(M, p_0)$ is isomorphic to the group of deck transformations of the universal cover $\widetilde{M}$, so we can interpret it as an abstract group acting geometrically on $\widetilde{M}$, and we will drop the notation of the base point when not needed. Furthermore, to any subgroup $G \subset \pi_1(M)$ is associated a cover $\widetilde{M}_G = \widetilde{M}/G \xrightarrow{p} M$ whose fundamental group $\pi_1(\widetilde{M}_G)$ is isomorphic to $G$.

[1]the *Sol* pieces admits a further decomposition into Seifert manifolds called *JSJ-decomposition*

In general, if $M$ is a good[2] topological space, we can define a subgroup of the fundamental group, hence a cover of $M$, by specifying an open covering.
The theory of $\delta$-coverings is developed in the full generality of length spaces in [SW01], and [SW04], but for simplicity here we restrict ourselves to the case of compact Riemannian manifolds.
**Definition 2.7.1** (Lassos, $\delta$-coverings, and covering radius)**.** Let $(M, p_0)$ be a pointed Riemannian manifold. We call *lasso* a pointed loop $\gamma : S^1 \to (M, p_0)$ that can be decomposed as $\alpha^{-1} * \beta * \alpha$, where $\beta$ is a closed loop called the *head* of $\gamma$, and $\alpha$ is a path connecting $p_0$ to $\beta_0$.
If $\mathcal{U}$ is an open covering of $M$, we denote $\pi_1(M, p_0, \mathcal{U})$ the normal subgroup of $\pi_1(M, p_0)$ consisting of the normal closure of the set of pointed homotopy classes of loops that can be represented by a lasso pointed at $p_0$ and whose head is contained in some $U \in \mathcal{U}$.
If $\mathcal{U}_\delta$ is the open covering of $M$ consisting of open balls of diameter $\delta$,

$$\mathcal{U}_\delta = \{B_M(\delta, p) | p \in M\},$$

we simplify the notation and write $\pi_1^\delta(M, p_0)$ for $\pi_1(M, p_0, \mathcal{U}_\delta)$, and we define the *$\delta$-cover* $M^\delta \to M$ as the cover associated to the subgroup $\pi_1^\delta(M, p_0)$.
We denote $G(M, \delta) = \pi_1(M, p_0)/\pi_1^\delta(M, p_0)$ its group of deck transformations, and $\phi_{\delta,\tilde{p}} : \pi_1(M, p_0) \to G(M, \delta)$ the Galois projection, which depends on the choice of a point $\tilde{p}$ in $M^\delta$.
The *covering radius* of $\gamma$ is the infimum of the radii $R \geq 0$ such that $\gamma$ can be pushed freely homotopically inside an open ball of radius $R$, namely

$$cr(\gamma) = \inf\{R | \gamma' \subset B(R, p) \text{ for some } \gamma' \sim \gamma \text{ and } p \in M\}.$$

In other words, if $cr(\gamma) \leq R$ then, for all $\epsilon > 0$ there exists an open ball $B$ of radius $R + \epsilon$ for some $\epsilon > 0$, and a loop $\gamma'$ freely homotopic to $\gamma$ such that $\gamma' \subset B$. Clearly $\pi_1^\delta(M, p_0)$ is equal to the normal subgroup of $\pi_1(M, p_0)$ generate by loops of covering radius smaller than $\delta$.
One can think of $G(M, \delta)$ as roughly corresponding to long loops, of length at least $\delta$, in $\pi_1(M, p_0)$, as we shall now see.
**Definition 2.7.2** ($\delta$-translation length and systole)**.** The *translation length* of $g$ is defined as $\ell(g) = \min_{p \in M} d_{\widetilde{M}}(p, g.p)$, and it is also equal to the length of the shortest loop $\gamma$ in the free homotopy class of a representative of $g$.
The *systole* of $M$ is the shortest length of a non-contractible loop in $M$, namely $sys(M) = \min_{g \in \pi_1(M)} \ell(g)$.
Analogously, if $h \in G(M, \delta)$ we define the *$\delta$-length* of $h$ (resp. $g$), as $\ell(h, \delta) = \min_{p \in M^\delta} d_{M^\delta}(p, h.p)$ (resp. $\ell(h, \delta) = \min_{p \in M^\delta} d_{M^\delta}(p, \phi_{\delta,\tilde{p}}(g).p)$).
The following lemma makes the interpretation of $G(M, \delta)$ more precise.

**Proposition 2.7.3** ([SW01], Lemma 3.1)**.** *For any non-trivial $g \in G(M, \delta)$, we have $\ell(g, \delta) \geq \delta$.*

On the other hand some long loops in $M$ can indeed be trivial in $G(M, \delta)$, as clarified by the following proposition.

**Proposition 2.7.4.** *Let $(M, p_0)$ be a closed, pointed, Riemannian manifold, and let $\pi_1(M, p_0, \varepsilon)$ be*

[2] connected, locally path-connected, and semi-locally simply-connected

*the normal subgroup of* $\pi_1(M, p_0)$ *generated by all classes of loops* $\gamma$ *such that* $\ell(\gamma) < \varepsilon$. *Then*

$$\pi_1(M, p_0, 2\varepsilon) = \pi_1^{\varepsilon}(M, p_0).$$

*Dimostrazione.* It is easy to see that there is an inclusion

$$\pi_1(M, p_0, 2\varepsilon) \leq \pi_1^{\varepsilon}(M, p_0).$$

In fact, if $\gamma$ is a loop such that $\ell(\gamma) < 2\varepsilon$, let us denote $\gamma' \in [\gamma]$ the loop that minimise the length in the free homotopy class of $\gamma$, and let $p \in Im(\gamma')$ be a point in its image. Then $Im(\gamma') \subset B(\varepsilon, p)$, and we can choose a path $\alpha$ from $p_0$ to $p$ such that

$$[\gamma]_{p_0} = [\alpha^{-1} * \gamma' * \alpha]_{p_0} \in \pi_1^{\varepsilon}(M, p_0).$$

To show the inclusion in the other direction, assume that $\gamma = \alpha * \beta * \alpha^{-1}$ is a lasso whose head $\beta$ is contained in some open ball $B(\varepsilon, p)$. By compactness there is a $\delta > 0$ such that $\beta$ is contained in the ball $B(\varepsilon - \delta, p)$. If $s_i, s_j \in S^1$ let us denote $\beta_{ij} \coloneqq \beta_{|[s_i, s_{i+1}]}$ the restriction of $\beta$ to the interval $[s_i, s_j]$, and $r_i$ the geodesic segment joining $p$ to $\beta(s_i)$. There exists a family of points $\{s_i\}_{i=1\ldots k+1} \in S^1$ such that the length of $\beta_{ii+1}$ is $< 2\delta$ for each $i$, and $\beta(s_1) = \beta(s_{k+1}) = \beta_0$. Then by construction the loop $\bar{\beta}_i \coloneqq r_i * \beta_{ii+1} * r_j^{-1}$ is based at $p$ and has length $< 2\varepsilon$. Also by construction we have

$$[\beta]_{\beta_0} = [r_1^{-1} * \bar{\beta}_1 * \bar{\beta}_2 * \cdots * \bar{\beta}_k * r_1]_{\beta_0}.$$

In particular we have

$$[\gamma]_{p_0} = [\alpha * r_1^{-1} * \bar{\beta}_1 * \cdots * \bar{\beta}_k * r_1 * \alpha^{-1}]_{p_0}$$

which proves

$$\pi_1^{\varepsilon}(M, p_0) \leq \pi_1(M, p_0, 2\varepsilon).$$

□

As a corollary we have $G(M, \delta) = \pi_1(M, p_0)$ whenever $2\delta < sys(M)$.
The systole of a manifold gives then a scale at which we can describe the interactions between changes in the metric and changes in the topology.
Recall that the *Gromov-Hausdorff distance* $d_{GH}(X, Y)$ of two compact Riemannian manifolds $X$, $Y$ is defined as

$$d_{GH}(X, Y) = \inf\{d_H^Z(i(X), j(Y)) | X \overset{i}{\hookrightarrow} Z \overset{j}{\hookleftarrow} Y\}$$

where the infimum is taken among all compact metric spaces $Z$ and all isometric immersions $i$, $j$, and $d_H^Z(A, B)$ denotes the Hausdorff distance of two subsets $A, B$ of $Z$ (cf. [GLP81])
The following results are proven in [SW01], Theorem 2.1, Corollary 2.3, and Corollary 4.12, respectively.

**Proposition 2.7.5.** *Let* $X, Y$ *be two Riemannian manifolds. Then*

*(i) if* $d_{GH}(X, Y) < sys(X)/80 = \epsilon$, *then there is a surjective homomorphism,* $\phi : \pi_1(Y) \to \pi_1(X)$;

*(ii) if* $d_{GH}(X, Y) < \min\{sys(X), sys(Y)\}/80$, *then there is an isomorphism,* $\Phi : \pi_1(Y) \to \pi_1(X)$;

*(iii) if* $d_{GH}(X,Y) < \min(2\delta, sys(X) - 2\delta)/80$ *for some* $2\delta < sys(X)$*, then there is an isomorphism* $\Phi : G(Y, \delta) \to \pi_1(X)$*.*

We conclude this section with some generic results about the fundamental groups of aspherical manifolds and closed hyperbolic 3-manifolds.

**Proposition 2.7.6** ([Bro82] Proposition 8.1)**.** *The fundamental group of a closed aspherical* $n$*-manifold has cohomological dimension* $n$*.*

**Proposition 2.7.7** ( [Lüc12] Lemma 4.1)**.** *The fundamental group of an aspherical* $n$*-manifold is torsion-less.*

# Capitolo 3

# Free subgroups of large rank

*The era of global warming has ended;*
*The era of global boiling has arrived.*
*The air is unbreathable.*
*The heat is unbearable.*
*And the level of fossil fuel profits*
*and climate inaction is unacceptable.*
— António Guterres, secretary-general of the United Nations.

*In that moment I knew: something had to change*
*o I was going to die.*
Algo tenía que cambiar.
— Gloria Anzaldúa, *Borderlands/La frontiera. The new mestiza.*

In this chapter we prove that given a group $G$, generated by a set $S$, acting geometrically on $\mathbb{H}^n$ there is a free subgroup of rank proportional to $|S|$, whose generators have $S$-length bounded by a universal constant. The proof is inspired by an approach developed in [CS18] for groups acting on spaces admitting $k$-acylindrical splittings, and uses techniques similar to those described in [CS21] for $\delta$-hyperbolic groups. Notice however that the finitely generated groups $(G, S)$ acting on $\mathbb{H}^n$ that we are considering here neither are $\delta$-hyperbolic for some uniform value of $\delta$, nor, a priori, admit some $k$-acylindrical splitting in the sense of [CS18].

## 3.1 Acylindrical families and Schottky groups in $\mathbb{H}^n$

Recall that a group $G = \langle g_1, \dots, g_n \rangle$ acting on $\mathbb{H}^n$ is called *Schottky group of rank n* if for any $i = 1, \dots, n$ it is possible to find subsets $X_i \subset \mathbb{H}^n$, called *Dirichlet domains*, such that:

1. $X_i \cap X_j \neq \emptyset$ for all $i \neq j$,
2. $g_i^{\pm 1}(\mathbb{H}^n \setminus X_i) \subseteq X_i$ for all $i$.

**Remark 3.1.1.** By a classical ping-pong argument, it follows that $G$ is a free group on the generating set $S = \{g_i, \dots, g_n\}$. To show the ping pong argument, consider any point $p \in \mathbb{H}^n \setminus \bigcup_{i=1}^n X_i$, and suppose by absurd that a non-trivial reduced word $w = g_{i_1}^{n_1} \cdot \ldots \cdot g_{i_k}^{n_k}$ in the letters $g_1, \dots, g_n$ corresponds to the identity in $G$. Since the point $p$ does not belong to any Dirichlet domain $X_i$, it is sent into the domain $X_i$ by the first letter of $w$, namely $p_1 = g_{i_1}^{n_1}.p \in X_i$. The successive letters of $w$ keep sending the point from a domain to another, $p_j = g_{i_j}^{n_j} \in X_{i_j}$, so $w.p \in X_{i_k}$. Since $p$

is not contained in any domain it follows that $w.p \neq p$, so $w$ cannot be the identity, which is a contradiction.

**Definition 3.1.2.** A family of loxodromic isometries $S = \{a_1, \dots, a_n\} \subset \mathrm{Isom}(\mathbb{H}^n)$ with pairwise disjoint axes is said to be *$K$-acylindrical* if, for each $i \neq j$,

$$Diam(\pi_i(\alpha_j)) \leq K \max\{\ell_i, \ell_j\}, \tag{3.1}$$

where $\alpha_i$ denotes the axis of $a_i$, $\ell_i$ is its translation length, and $\pi_i$ is the orthogonal projection on $\alpha_i$.

The notion of $k$-acylindrical family is key to produce Schottky subgroups of large rank in a discrete isometry group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$, and is inspired by the condition of $k$-acilindricity of action on trees (i.e. $Diam(Fix(g)) \leq k$ for all elliptic $g$) introduced by Sela (cf. [Sel97]).

The condition (3.1) plays the role of a small cancellation condition for the group $\langle a_1, \dots, a_n \rangle$ in the geometric version of the small cancellation theory as introduced in [DGO17], where $Diam(\pi_i(\alpha_j))$ replaces the length of the "pieces", and the $\ell_i$'s would be the length of relators.

We will not need the general geometric theory of small cancellation: the only crucial geometric fact that we will use is the following:

**Proposition 3.1.3.** *There is a function $c(n) = 350/\epsilon_0(n)$, depending only on the dimension, with the following property: let $S = \{a_1, \dots, a_n\}$ be a set of isometries of $\mathbb{H}^n$ such that:*

1. *$\langle S \rangle$ is discrete;*
2. *$\langle a_i, a_j \rangle$ is non-elementary for each $i \neq j$;*
3. *$\ell_i = \ell > 3\delta_0 = 3 \log(1 + \sqrt{2})$ for each $i$.*

*Then $S$ is $c(n)$-acylindrical.*

This results is based on three fundamental ingredients: the hyperbolicity of $\mathbb{H}^n$, the convexity of the distance function, and the Margulis constant $\epsilon_0(n)$ appearing in the Margulis lemma.

Knowing that the translation lengths $\ell(a_i)$ are bounded from below by $3\delta_0$, it is immediate to deduce (3.1) from Lemma 2.1.2 when the axes $\alpha_i$ are sufficiently far away from each other. On the other hand, we need another argument to prove (3.1) when $d(\alpha_i, \alpha_j)$ is smaller than the constant 0.4 of Lemma 2.1.2.

Let $a$, $b$ be loxodromic isometries of $\mathbb{H}^n$ generating a discrete, non-elementary group, with axis $\alpha$ and $\beta$ respectively, and let $N_d(\beta)$ be the tubular neighbourhood of $\beta$ of radius $d$. For any $d \geq d(\alpha, \beta)$ the set $\alpha \cap N_d(\beta)$ is a nonempty, finite geodesic segment because the distance function from $\beta$ is convex when restricted to $\alpha$, and $\partial\alpha \cap \partial\beta = \emptyset$ (the group $\langle a, b \rangle$ being non-elementary).

The following lemma is an adaptation to our setting of a result originally due to Dey-Kapovich-Liu (see [DKL19]).

**Lemma 3.1.4.** *([CS21], Proposition 5.11) Let $a$, $b$ be loxodromic isometries of $\mathbb{H}^n$ generating a discrete, non-elementary group, and let $\alpha$ and $\beta$ be their respective axes. Suppose that $\ell(a) = \ell(b) = \ell > \epsilon_0/3$, and $d(\alpha, \beta) \leq \epsilon_0/74$.*

*Let $[z_-, z_+] = \alpha \cap N_d(\beta)$ be the set of points in $\alpha$ at distance less than $D = \epsilon_0/37$ from $\beta$. Then $Diam[z_-, z_+] < 5\ell$.*

*Proof of Proposition 3.1.3.* Given two isometries $a$, $b \in S$, let us denote $d$ the distance between their axes, $d = d(\alpha, \beta)$, and $\pi_\alpha$, $\pi_\beta$ the respective projections. We split the proof in three cases.

*Case* 1, $d > \frac{1}{2}$: In this case the estimate is straightforward from Lemma 2.1.2, which yields

$$Diam(\pi_\beta(\alpha)) < 9e^{-0.4} < 5\ell.$$

*Case* 2, $d \leq \frac{\epsilon_0}{74}$: Let $z_0 \in \alpha$ be such that $d(z_0, \beta) = d(\alpha, \beta) \leq \epsilon_0/74$, and $z_-$, $z_+$ the two points in $\alpha$ such that $d(z_\pm, \beta) = D = \epsilon_0/37$. By the estimates of the hyperbolicity constant and Margulis constants (cf. Proposition 2.2.8) for $\mathbb{H}^n$ we have $\ell > 3\delta_0 > \epsilon_0(n)/3$, so we can apply Lemma 3.1.4, so $d(z_-, z_+) < 5\ell$.
Assume that $\alpha$ is parametrised by arc length and $\alpha(0) = z_0$. The function $d_\beta(t) = d(\alpha(t), \beta)$ is convex, and satisfies $d_\beta(0) < d$ and $d_\beta(5\ell) > D$, so for any $t > 5\ell$ we have

$$d_\beta(t) > \frac{(D-d)}{5\ell} t \geq \frac{\epsilon_0(n)}{370\ell} t.$$

In particular $d_\beta(t) > 0.3$ as long as

$$t > \frac{165}{\epsilon_0(n)} \ell.$$

Hence, we can apply Lemma 2.1.2 to the two geodesic rays $C^+ = [\alpha(165/\epsilon_0(n)), \alpha^+]$ and $C^- = [\alpha^-, \alpha(-165\epsilon_0(n))]$, and we find that the diameter of the projection on $\beta$ of each of the two tails is less than

$$9e^{-0.3} < 9\frac{\ell}{3\delta_0} e^{-0.3} = \frac{9e^{-0.3}}{3\log(1+\sqrt{2})} \ell < 5\ell < \frac{10\ell}{\epsilon_0(n)},$$

where the last two inequalities come from the estimates of the $\delta$-hyperbolicity constant, and of the Margulis constant 2.2.8.
Then since $\alpha = C^- \cup [\alpha(-165/\epsilon_0(n)), \alpha(165/\epsilon_0(n))] \cup C^+$, we get

$$Diam(\pi_\beta(\alpha)) < \frac{350}{\epsilon_0(n)} \ell,$$

where the projection on $\beta$ of the segment $[\alpha(-165/\epsilon_0(n)), \alpha(165/\epsilon_0(n))]$ has diameter less than $330/\epsilon_0(n)$ by Proposition 2.1.1, 4.

*Case* 3, $\frac{\epsilon_0}{74} < d < \frac{1}{2}$: Denote $y_-$, $y_+$ the points on $\beta$ at distance $D = 1$ from $\alpha$, which project on $\alpha$ to points $x_-$, $x_+$ respectively. By construction $d(y_-, x_-) = D$, and since $x_- \neq x_+$, the geodesic segment $[x_-, y_+]$ has length $D + L$, with $L > 0$. We denote $m$ the middle point of $[x_-, y_+]$, and apply Lemma 2.1.4 to the triangles $\Delta(x_-, x_-+, y_+)$ and $\Delta(x_-, y_+, y_-)$, which yields

$$d \leq d(m, \alpha) + d(m, \beta) \leq 2\sinh^{-1}\Big(\frac{\sinh D}{2\cosh L}\Big). \tag{3.2}$$

So we obtain

$$Diam([x_-, x_+]) \leq L + 2D \leq 2\cosh^{-1}\Big(\frac{\sinh D}{2\sinh d/2}\Big) + D \leq$$
$$\leq 2\cosh^{-1}\Big(\frac{e}{4\sinh\big(\frac{\epsilon_0(n)}{148}\big)}\Big) + 1 \leq \frac{150}{\epsilon_0(n)}$$

as $D = 1$, $d > \epsilon_0/74$, and we assumed $\ell > 3\delta_0$. Using the assumption $\ell > 3\delta_0$, we can estimate $Diam([x_-, x_+]) < \frac{109}{\epsilon_0(n)} \ell$.

Furthermore, by Lemma 2.1.2 the diameter of the projection of each the two tails $[\beta^-, y_-]$ and $[y_+, \beta^+]$ is less than $9e^{-1}\ell/3\delta < 3\ell$. To conclude we have

$$Diam(\pi_\beta(\alpha)) < \frac{120}{\epsilon_0(n)}\ell.$$

□

**Proposition 3.1.5.** *There exists a function $N_0(K, d, \ell) > 0$ with the following property: let $S = \{a_1, \ldots, a_m\}$ be a $K$-acylindrical family of isometries of $\mathbb{H}^n$ with translation lengths $\geq \ell$, and let $x_0 \in \mathbb{H}^n$ be such that $d(x_0, Ax(a_i)) \leq d\ell$ for each $i$; then $\langle a_i^N, \ldots, a_m^N \rangle$ is a free Schottky group of rank $m$, for all $N \geq N_0(K, d, \ell)$.*

*Dimostrazione.* Let $\partial S' = \{\alpha_i^\pm\}$ denote the set of points at infinity of the isometries in $S$. Define

$$\Theta_{ij} := \min_{\epsilon, \epsilon' \in \{\pm 1\}} \sphericalangle_{x_0}(\alpha_i^\epsilon, \alpha_j^{\epsilon'}) \quad \text{and} \quad \Theta_0 = \min\{\Theta_{ij}\},$$

where the minimum is taken over all pairs of distinct points in $\partial S'$. We have $\Theta_0 > 0$, because by definition the isometries $a_i \in S$ generate pairwise non-elementary subgroups.

We now proceed to show that the angles $\Theta_{ij}$ satisfy the following uniform lower bound

$$\Theta_{ij} \geq \Theta\big((2d + K)\ell + \delta_0\big), \tag{3.3}$$

where the function on the right hand side is the Schweikart function defined in Lemma 2.1.3, and $\delta_0 = \log(1 + \sqrt{2})$ is the $\delta$-hyperbolicity constant of $\mathbb{H}^n$.

By hypothesis $d(x_0, Ax(a_i)) \leq d\ell$, so by Lemma 2.1.3 we have $\Theta_{ii} > \Theta(d\ell)$. Now, given two points $\alpha_i$, $\alpha_j$ fixed by different elements $a_i$, $a_j$, let us denote $\alpha_{ij} = [\alpha_i, \alpha_j]$ the geodesic segment joining them. The following argument is independent on the choice of an orientation for $Ax(a_i)$ and $Ax(a_j)$, so we will just assume $\alpha_i = \alpha_i^+$ and $\alpha_j = \alpha_j^+$ to simplify the notation. Let us denote $x_i = \pi_i(x_0)$ the projection of $x_0$ to $Ax(a_i)$, and $x_{ij} = \pi_j(x_i)$ and $z_{ij} = \pi_j(\alpha_i)$ the projection of $x_i$ and $\alpha_i$ to $Ax(a_j)$ respectively.

As $\sphericalangle_{z_{ij}}(\alpha_i, \alpha_j) = \frac{\pi}{2}$, we have $d(\alpha_{ij}, z_{ij}) = \Theta(\frac{\pi}{2}) = \delta_0$. Furthermore, since projections are contractions by Lemma 2.1.2, we have $d(x_j, x_{ij}) \leq d(x_0, x_j) \leq d\ell$, so we get

$$d(x_0, \alpha_{ij}) \leq d(x_0, x_j) + d(x_j, x_{ij}) + d(x_{ij}, z_{ij}) + d(z_{ij}, \alpha_{ij}) < (2d + K)\ell + \delta_0,$$

which proves (3.3).

We now proceed to show that for $N \geq N(K, d, \ell)$ the elements $\{a_1^N, \ldots, a_m^N\}$ generate a Schottky group of rank $m$. For each isometry $a_i$ define the domain

$$D_i = \{z \in \mathbb{H}^n \mid d(\pi_i(z), x_i) > L\},$$

where $\pi_i$ is the orthogonal projection on the axis $\alpha_i = Ax(a_i)$.

Every $D_i$ is the disjoint union of two connected components $D_i^+$ and $D_i^-$, which contains respectively $\alpha_i^+$ and $\alpha_i^-$. By Lemma 2.1.3, the connected component of $D_i^\pm$ containing $\alpha_i^\pm$ is contained in the cone

$$C_i^\pm = C_{x_0}^{\alpha_i^\pm}(\Theta(L_i^\pm)) = \big\{z \in \mathbb{H}^n \mid \sphericalangle_{x_0}(z, \alpha_i^\pm) \leq \Theta(L_i^\pm)\big\},$$

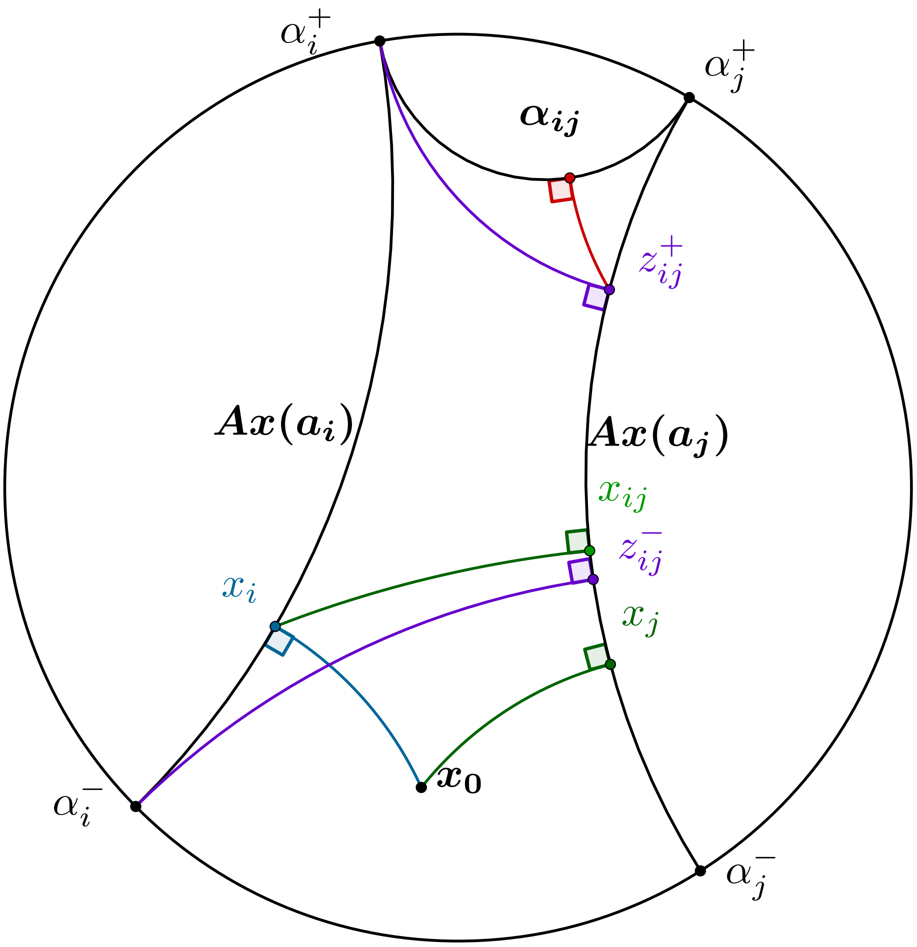



**Figura 3.1:** Estimate of the visual angle of $\alpha_{ij}$ from $x_0$

where $L_i^\pm = d(x_0, D_i^\pm)$. By construction we have $L_i^\pm > L$, so $\Theta(L_i^\pm) < \Theta(L)$.
As $\sphericalangle_{x_0}(\alpha_i^\pm, \alpha_j^\pm) > \Theta_0$ for each pair of different points $\alpha_i^\pm, \alpha_j^\pm$, it follows that, as soon as $\Theta(L) < \Theta_0/2$ (i.e. $L > \Theta^{-1}(\Theta_0/2)$), the cones $C_i^\pm$, and consequently the domains $D_i^\pm$, are all disjoint; it then suffice to take $L = N\ell$ such that $\Theta(L) < \Theta((2d+K)\ell + \delta_0)/2$, i.e.

$$N > \frac{1}{2} N_0(K, d, \ell) := \frac{\Theta^{-1}(\frac{1}{2}\Theta((2d+K)\ell + \delta_0)}{\ell}. \tag{3.4}$$

Then, for every isometry $a_i$ we have that $a_i^{\pm 2N}$ sends $\mathbb{H}^n \setminus D_i$ to $D_i$, because $\ell(a_i^{2N}) > 2L$, hence $\{a_1^{2N}, \ldots, a_m^{2N}\}$ generates a Schottky group of rank $m$. □

## 3.2 Free subgroup theorem in $\mathbb{H}^n$

In this section we prove the Free Subgroup Theorem:

**Free Subgroup Theorem 3.2.1.** *There exists a function $\mathcal{N}(n)$, depending only on the dimension, with the following property: let $S$ be any finite symmetric family of isometries of $\mathbb{H}^n$ generating a non-elementary discrete group; then, there exists a subset $\widetilde{S} \subset S^{\mathcal{N}(n)}$ which generates a free Schottky subgroup of rank $\geq \sqrt[4]{|S|}$.*

Notice that it is very easy to find free subgroups of $G = \langle S \rangle$ of arbitrary rank. In fact, by the Tits'

alternative a non-elementary subgroup of isometries of $\mathbb{H}^n$ contains a free non-abelian subgroup $\mathbb{F}_2 \supset \mathbb{F}_m$ for any $m$. The challenge (and interest) in this theorem is the condition $\widetilde{S} \subset S^{N(n)}$, i.e. the ability to find free subgroup of large rank and *bounded depth* with respect to $d_S$.
To prove the theorem we will use the theory of $k$-acylindrical families developed in the previous section, and the following proposition.

**Proposition 3.2.2.** *Let $G$ be a discrete, non-elementary group of loxodromic isometries of $\mathbb{H}^N$, and let $S$ be a finite symmetric generating set of $G$. Then there exists a set of isometries $\widetilde{S} \subset S^{H_0}$ and a point $x_0 \in \mathbb{H}^n$, such that*

*(i) $\widetilde{S}$ contains at least $\sqrt[4]{|S|}$ elements, all with different axes,*
*(ii) $\ell(g) = \ell_0 > 3\delta_0$ for each $g \in \widetilde{S}$,*
*(iii) $d(x_0, Ax(g)) \leq 73\ell(g)$ for each $g \in \widetilde{S}$.*

*In particular, $H_0 = H_0(n)$ is a function that only depends on the dimension, and it holds*

$$H_0(n) \leq 270\Big(\frac{31\delta_0}{\epsilon_0}\Big)^2 + 60,$$

*where $\delta_0 = \log(1+\sqrt{2})$ is the hyperbolicity constant, and $\epsilon_0 = \epsilon_0(n)$ is the Margulis constant in dimension $n$.*
*Notice that the group generated by $\widetilde{S}$ is still non-elementary.*

In order to prove this we will need some preliminary lemmas.
The first one allows us to construct a large $k$-acylindrical family of isometries with bounded depth and uniform translation length.

**Lemma 3.2.3.** *Let $G$ be a discrete, non-elementary group of isometries of $\mathbb{H}^n$, and let $S$ be a finite, symmetric generating set of $G$. Then there exists a subset $S_0$ of $G$ such that*

*(i) $S_0 \subset S^N$, with $N = 27n + 6$ for every $n \geq (31\delta_0/\epsilon_0(n))^2$,*
*(ii) $\ell(s) = \ell_0 \geq 3\delta_0$ for each $s \in S_0$,*
*(iii) $S_0$ contains at least $\sqrt{|S|}$ elements with different axes.*

*In particular $S_0$ still generates a non-elementary group.*

*Dimostrazione.* Since $G$ is non-elementary $L(S) \geq \epsilon_0$, otherwise, by the Margulis lemma there would exist $x_0 \in \mathbb{H}^n$ such that $d(x_0, s.x_0) < \epsilon_0$ for each $s \in S$, and $G = \langle S \rangle$ would be virtually nilpotent, hence elementary by Proposition 2.2.3, which is a contradiction.
By Lemma 2.2.6, for every $n \geq (31\delta_0/\epsilon_0)^2$ we have $L(S^n) \geq 31\delta_0/2$, so by Lemma 2.2.6 there exists an element $s \in S^{3n}$ such that $\ell(s) \geq \delta_0$. In particular $s$ is loxodromic. Now set $s' = s^3$, so that $\ell(s') \geq 3\delta_0$ and $||s'||_S \leq 9n$, and define the set

$$S' \coloneqq \{ss's^{-1} \mid s \in S\} \cup \{s'\}.$$

As the translation length is invariant by conjugation, we have $\ell(s) = \ell \geq 3\delta$ for each $s \in S'$. Notice that $\langle S' \rangle$ is non-elementary, otherwise it would mean that $s.Ax(s') = Ax(ss's^{-1}) = Ax(s')$ for each $s \in S$, so $G$ would be elementary since it would preserve $Ax(s')$.
Consider the relation on $S'$ defined by $s_1 \sim s_2$ if and only if $Ax(s_1) = Ax_(s_2)$.
If $|S'/\sim| \geq \sqrt{|S|}$, we pick a subset $S_0 \subset S'$ all of whose axes are different, and we are done. Otherwise $|S'/\sim| < \sqrt{|S|}$ and by the pigeonhole principle, there exists a class $C = \{s_1, \ldots, s_n\}$

with at least $\sqrt{|S|}$ elements. Furthermore, since $S'$ generates a non-elementary subgroup, there also exists some $s_0 \in S' \setminus C$. Then consider the set

$$S_0 := \{s_i s_0 s_i^{-1} \mid s_i \in C\}.$$

The elements of $S_0$ clearly satisfy ((i)) and ((ii)); we claim that they also have different axis. Actually, suppose by contradiction that there exists $s_i, s_j \in C$ such that $Ax(s_i s_0 s_i^{-1}) = Ax(s_j s_0 s_j^{-1})$. This is equivalent to saying that $s_j^{-1} s_i.Ax(s_0) = Ax(s_0)$.
But since $s_j$ and $s_i$ are two loxodromic isometries of a discrete group with the same axis, then $Ax(s_j^{-1}s_i) = Ax(s_i) = Ax(s_j)$, and since $s_j^{-1}s_i$ fixes $Ax(s_0)$ this would imply that $Ax(s_0) = Ax(s_j^{-1}s_i) = Ax(s_j)$ which contradicts the assumption that $s_0$ and $s_j$ belong to different $\sim$-classes. □

With the same notations as above, let $L_0 = L(S_0)$ be the minimal joint displacement of the subset $S_0$, and let $x_0 \in \mathbb{H}^n$ and $s_0 \in S_0$ be the point and element realising $L(S_0)$, i.e. $L_0 = \inf_x \max_s d_s(x) = d(x_0, s_0.x_0)$.
The following second preliminary lemma will be crucial to find a uniform bound on the distance of axis of elements in $S_0$ from $x_0$ as a function of their translation length.

**Lemma 3.2.4.** *With the notation as above, and the same assumptions of Lemma 3.2.3; there exists $s_0' \in S_0$ such that:*

1. $d(x_0, s_0'.x_0) \geq L_0/2$;
2. $(s_0.x_0, s_0'.x_0)_{x_0} < L_0/8$.

*Dimostrazione.* Let $x_0' \in [x_0, s_0.x_0]$ be the point along the geodesic from $x_0$ to $s_0.x_0$ at distance $L_0/8$ from $x_0$, and denote $S_0^* = \{s \in S_0 \mid d(x_0, s.x_0) \geq L_0/2\}$ the set of isometries in $S_0$ that displace $x_0$ by at least $L_0/2$. The set $S_0^* \setminus \{s_0\}$ is non empty. Otherwise, notice that $S_0^* \setminus \{s_0\}$ implies that $L_0 > \ell_0$, because $\ell(s) = \ell_0$ for all $s \in S_0$, and consider the point $z \in [x_0, s_0.x_0]$ at distance $L_0/4$ from $x_0$. Then $d_{s_0}(z) < L_0$ by convexity of the displacement function and by Lemma 2.2.1, $(i)$, furthermore for every $s \in S_0 \setminus \{s_0\}$ we also have $d_s(x_0) \leq 2d(x_0, z) + d(x_0, s.x_0) < L_0$, which is a contradiction.
Now, notice that if $s \in S_0 \setminus S_0^*$ then

$$d(x_0', g.x_0') \leq d(x_0, g.x_0) + 2d(x_0, x_0') < L_0/2 + 2L/4 = L_0. \tag{3.5}$$

On the other hand, for every $s \in S_0^*$ let $x_s$ be the point along the geodesic segment $[x_0, s.x_0]$ at distance $L_0/8$ from $x_0$, and let $m$ be the middle point of $[x_0, s.x_0]$. We can express $x_s$ as $x_s = cx_0 + (1-c)m$ where

$$c = \frac{d(x_s, m)}{d(x_0, m)} = 1 - \frac{L_0}{4d_s(x_0)} \in [\frac{1}{2}, \frac{3}{4}].$$

Recall that, by hypothesis $d_s(x_0) = d(x_0, s.x_0) \in [L/2, L]$ for all $s \in S_0^*$, furthermore by Lemma 2.2.1 we have

$$d(m, s.m) < \ell(s) + 3\delta_0 < L_0/16 + 3\delta_0 \leq L_0/2,$$

hence $d(m, s.m) < d(x_0, s.x_0)$. By Lemma 2.2.1 the displacement function is convex along $[x_0, s.x_0]$, so we have

$$\begin{aligned} d(x_s, s.x_s) &\leq c d_s(x_0) + (1-c) d(m, s.m) \leq \\ &\leq d_s(x_0) - \frac{L_0}{4} + \frac{L_0}{4 d_s(x_0)} d(m, s.m) \leq \frac{3L_0}{4} + \frac{1}{2}\big(\frac{L_0}{16} + 3\delta_0\big) < \\ &< \frac{25}{32} L_0 + \frac{3}{2}\delta_0. \end{aligned} \tag{3.6}$$

Now assume by contradiction that for all $s \in S_0^*$ we had $(s_0.x_0, s.x_0)_{x_0} \geq L/8$, then $x_0'$ and $x_s$ would be the preimage of the same point in the approximating tripod, and by Lemma 2.1.5 we would have $d(x_0', x_s) < \delta$: this would imply that for all $s \in S_0^*$ we would have

$$d(x_0', s.x_0') \leq d(x_s, s.x_s) + 2\delta < \frac{25}{32} L_0 + \frac{11}{2}\delta \leq \frac{28}{32} L_0,$$

by (3.6) and the fact that $3\delta_0 < L_0/16$. But then the joint displacement at $x_0'$ would be $\min_s d_s(x_0') < L_0$, a contradiction. So we proved that there is an isometry $s_0'$ such that $d(x_0, s_0'.x_0) \geq L/2$ and $(s_0.x_0, s_0'.x_0)_{x_0} < L/8$. □

*Proof of Proposition 3.2.2.* The proof is divided in two steps: firstly we find $g_0 \in S_0^2$ such that

(i) $d(x_0, s.x_0) \leq 16\ell(g_0)$ for each $s \in S'$,

(ii) $d(x_0, Ax(s_0)) \leq (17/2)\ell(s_0)$.

Secondly we use this element to construct a set $\widetilde{S}$ with the desired properties.

Let $S_0$ be the subset constructed in Lemma 3.2.3 satisfying ((i)), ((ii)), and ((iii)), in particular with $\ell(s) = \ell_0 \geq 3\delta_0$ for each $s \in S_0$. Let $x_0$ be the point where $L_0 := L(S_0)$ is realised, which exists and is unique because of Lemma 2.2.5. We have two cases.

*Case 1, $\ell_0 \geq L_0/16$:* This assumption means that all the elements of $S_0$ have axis relatively close to the point $x_0$. Then, pick any $s_0 \in S_0$ and we have

$$d(x_0, s_0.x_0) \leq 16\ell_0,$$

and by Lemma 2.2.1

$$d(x_0, Ax(s_0)) \leq \frac{L_0 - \ell_0}{2} + 3\delta_0 \leq \frac{15}{2}\ell_0 + 3\delta_0 \leq \frac{17}{2}\ell_0.$$

So, setting $g_0 = s_0$ properties ((i)) and ((ii)) are satisfied.

*Case 2, $3\delta_0 \leq \ell_0 < L_0/16$:* In this situation, a priori, the elements of $S_0$ can have axes arbitrarily far from the point $x_0$. Now consider the set of isometries of $S_0$ that displace $x_0$ at least $L_0/2$:

$$S_0^* := \{s \in S_0 \mid d(x_0 s.x_0) \geq L_0/2\}. \tag{3.7}$$

Let $s_0$ be an isometry that realise the minimal joint displacement at $x_0$, i.e. $L_0 = d_s(x_0)$. By Lemma 3.2.4, there is an isometry $s_0'$ such that

$$d(x_0, s_0'.x_0) \geq L_0/2$$

and

$$(s_0.x_0, s_0'.x_0)_{x_0} < L_0/8.$$

We now proceed to show that, by Lemma 2.2.2, the existence of such an isometry is sufficient to find an element that satisfy properties ((i)) and ((ii)).
In fact, for the hyperbolic triangle $\Delta(x_0, s_0.x_0, s_0'.x_0)$. we have

$$\begin{aligned} d(s_0.x_0, s_0'.x_0) &= d(x_0, s_0.x_0) + d(x_0, s_0'.x_0) - 2(s_0.x_0, s_0'.x_0)_{x_0} > \\ &> L_0 + L_0/2 - L_0/4 = \frac{5}{4}L_0, \end{aligned}$$

and since $\delta_0 < 48L_0$ we deduce that

$$d(s_0.x_0, s_0'.x_0) > L_0 + 5\delta_0 + \frac{L_0}{8} = \max\{d_{g_0'}(x_0), d_{g_0}(x_0)\} + 5\delta_0 + \frac{L_0}{8}.$$

Then, applying Lemma 2.2.2 with $\ell = L_0/8$ we know that there is an element $g_0 \in \{s_0, s_0'\}^2$ such that $\ell(g_0) > L_0/8 > 2\ell_0$, and in particular, so it satisfy properties ((i)) and ((ii)):

$$d_s(x_0) \leq 8\ell(g_0) \text{ for all } s \in S_0,$$

and, again by Lemma 2.2.1,

$$\begin{aligned} d(x_0, Ax(h)) &\leq \frac{d_{g_0}(x_0) - \ell(g_0)}{2} + 3\delta_0 \leq \frac{2L_0 - \ell(g_0)}{2} + 3\delta_0 \leq \\ &\leq \frac{16\ell(g_0) - \ell(g_0)}{2} + 3\delta_0 \leq \frac{15}{2}\ell(g_0) + 3\delta \leq \\ &\leq \frac{15}{2}\ell(g_0) + \ell_0 < 8\ell(g_0), \end{aligned}$$

which gives the result in case 2.
Consider the set of conjugates

$$S'' := \{sg_0s^{-1} \mid s \in S_0\} \cup \{g_0\}.$$

As in Lemma 3.2.3, $S''$ contains at least two elements with different axes, since the axes of elements $s \in S_0$ are all distinct.
We have two possibilities: either $S''$ contains at least $\sqrt{|S_0|}$ elements with distinct axes, and we set $\widetilde{S} := S''$, or there are at least $\sqrt{|S_0|}$ elements $g_1, \ldots, g_k \in S''$ with the same axis, and we define

$$\widetilde{S} := \{g_i g_0 g_j^{-1} \mid i = 1 \ldots k\}.$$

Notice that, as in the proof of Lemma 3.2.3, the elements in $\widetilde{S}$ have all distinct axes. Finally, since every element in $\widetilde{S}$ is of the form $ug_0u^{-1}$, with $u \in S_0^p$ for $p \leq 4$, we conclude by ((i)) and ((ii)) that, for all $\tilde{s} \in \widetilde{S}$ we have

$$\begin{aligned} d(x_0, Ax(\tilde{s})) &\leq d(x_0, u.x_0) + d(u.x_0, u.Ax(g_0)) \leq \\ &\leq 4 \cdot 16\ell(g_0) + \frac{17}{2}\ell(g_0) \leq 73\ell(g_0). \end{aligned}$$

□

*Proof of the Free Subgroup Theorem.* Let $\widetilde{S} = \{g_1, \ldots, g_k\} \subset S^{H_0}$ and $x_0 \in \mathbb{H}^n$ be the set of isometries and the point defined in Proposition 3.2.2. Recall that the following properties hold:

(i) $\widetilde{S}$ contains at least $\sqrt[4]{|S|}$ elements, all with different axes,

(ii) $\ell(g) = \ell_0 > 3\delta_0$ for each $g \in \widetilde{S}$,

(iii) $d(x_0, Ax(g)) \leq 73\ell(g)$ for each $g \in \widetilde{S}$.

Properties ((i)) and ((ii)) imply that $\widetilde{S}$ is a $c(n)$-acylindrical family, by Proposition 3.1.3, and by Proposition 3.1.5 properties ((ii)) and ((iii)) imply that $G = \langle g_1^N, \ldots, g_k^N \rangle$ is a free Schottky subgroup of rank $k$ for all $N > N_0(c(n), 73, 3\delta_0)$.
In particular $G$ is generated by elements of $S$-length less than

$$\mathcal{N}(n) = N_0(c(n), 73, 3\delta_0) \cdot H_0(n), \tag{3.8}$$

which only depends on the Margulis constant in dimension $n$. In particular, from the upper bound on $\epsilon_0(n)$ we can deduce a rough asymptotic estimate $\mathcal{N}(n) < C/\epsilon_0(n)^3$ for some constant $C < 10^6$. □

# Capitolo 4

# Topological finiteness of hyperbolizable manifolds

*Besides our own fate we worry about the whole earth that is facing the risk of turning to chaos. There is only one sky, and if it gets sick, everything will end. Maybe it won't happen now, but one day it might. Then it will be our children, their children, and their children's children who will die. That is why I want to pass on to white men these words of warning, which I draw from the great shamans. Through them, I want to make them understand that they should dream further and pay attention to the voice of the forest spirits.*

— Davi Kopenawa, a Yanõmamɨ

In this chapter we deduce an entropy-cardinality inequality for fundamental groups of compact hyperbolizable manifold from the existence of free subgroups of large rank, and we will use it to prove our finiteness theorem. The argument will be similar to the one used in [CS18], Theorem 6. Let us recall a classical result (see, for instance, [SS02]) that gives a canonical presentation of a manifold, with relations of length 3.

**Lemma 4.0.1** (Triangular presentations)**.** *Let $G$ act by homeomorphisms on a path-connected, simply connected topological space $X$, and let $U$ be a path-connected open set such that $G.U = X$. Let $S = \{s \in G \mid sU \cap U \neq \emptyset\}$ and $T = \{(s_1, s_2) \in S \times S \mid U \cap (s_1U) \cap (s_1s_2U) \neq \emptyset\}$; then $S$ generates $G$, and $G \cong F(\Sigma)/\langle\langle\Theta\rangle\rangle$ where $\Sigma$ is the set of symbols $\{x_s \mid s \in S\}$, and $\Theta$ is the set of words on $S \cup S^{-1}$ given by $\{x_{s_1}x_{s_2}x^{-1}_{s_1s_2} \mid (s1, s2) \in T\}$.*
*(Notice that if $(s1, s2) \in T$, then $s_1s_2 \in S$ so $x^{-1}_{s_1s_2}$ makes sense).*

In particular, if $X$ is a geodesic metric space with $Diam(X) \leq D$ and universal cover $\widetilde{X}$, for any $\tilde{x_0} =\in \widetilde{X}$ projecting on $x \in X$ the group $G = \pi_1(X, x_0)$ (acting discretely by isometries on $\widetilde{X}$) is generated by the set

$$S(x_0, L) := \{g \in \pi_1(X, x_0) \mid d(\tilde{x_0}, g.\tilde{x_0}) \leq L\}$$

for any $L > 2D$, with relations of length $\leq 3$, i.e. the group of relations is generated, as a normal subgroup of $\mathbb{F}(S(L))$, by relators of length at most 3. Notice that this subset only depend on the choice of $x_0 \in X$, and not on the choice of a lift of $x_0$. The subset $S(x_0, L)$ will be called an *$L$-short generating set of $X$*.

The following entropy-cardinality inequality gives an upper bound for the number of $L$-short generators of the $\pi_1$ of any complete, hyperbolizable, Riemannian $n$-manifolds, in terms of its entropy and diameter:

**Entropy-Cardinality Inequality 4.0.2.** *There exists a function $\mathcal{S}(n, L, E)$ with the following property: let $X$ be a complete, hyperbolizable, Riemannian $n$-manifold and $x_0 \in X$ with entropy $Ent(X) \leq E$, and let $S$ be a generating set for $\pi_1(X)$ such that $L(S) \leq L$, then*

$$|S| \leq \mathcal{S}(n, L, E) \coloneqq \exp\big(4\mathcal{N}(n)LE\big), \tag{4.1}$$

*where $\mathcal{N}$ is the function defined in Proposition 3.2.1.*

*Dimostrazione.* Let $\tilde{x}_0 \in \widetilde{X}$ such that $L_S(\tilde{x}_0) < L + \epsilon$. By Proposition 3.2.1 there is a free subgroup $\mathbb{F}_n < G = \pi_1(X)$ of rank $n > \sqrt[4]{|S|}$ generated by a subset $\widetilde{S} \subset S^{\mathcal{N}(n)}$. As every generator $s \in \widetilde{S}$ displaces $\tilde{x}_0$ less than $M = (L + \epsilon)\mathcal{N}(n)$, a straightforward computation shows that

$$\begin{aligned} Ent(X) = Ent(G \curvearrowright \widetilde{X}) \geq Ent(\mathbb{F}_n \curvearrowright \mathcal{C}_M(\mathbb{F}_n, \widetilde{S})) = \\ = \lim_{N \to \infty} \frac{\log(|B(NM)|)}{NM} = \frac{\log(n)}{M} \geq \frac{\log(\sqrt[4]{|S|})}{(L+\epsilon)\mathcal{N}(n)}, \end{aligned}$$

which gives the desired inequality for $\epsilon \to 0$. □

Now let us denote $\mathcal{H}_n(E, D)$ the set of closed, hyperbolizable, Riemannian $n$-manifolds $X$ such that $Ent(X) < E$ and $Diam(X) < D$.

**Theorem 4.0.3** (Topological finiteness)**.** *$\mathcal{H}_n(E, D)$ contains finitely many diffeomorphism classes.*

*Dimostrazione.* Let $X$ and $Y$ be two representative of some classes in $\mathcal{A}_3^{c,o}(E, D)$. In dimension 2 it is well known that if $X$ and $Y$ have the same fundamental group then they are diffeomorphic. In dimension $\geq 3$, since $X$ and $Y$ can be endowed with a hyperbolic metric, by Mostow's rigidity theorem if $X$ and $Y$ have the same fundamental group then they are diffeomorphic. So it is sufficient to show that there are a finite number of fundamental groups $G = \pi_1(X)$ for $X \in \mathcal{H}_n(E, D)$, up to isomorphisms. By Lemma 4.0.1 and Theorem 4.0.2, $\pi_1(X)$ admits a presentation with at most $e^{4\mathcal{N}(n)ED}$ generators, and relations of length at most 3. An easy combinatorial reasoning gives a rough upper bound to the number of such presentations, hence for the groups in $\{\pi_1(X) \,|\, X \in \mathcal{H}_n(E, D)\}$, namely

$$|\mathcal{H}_n(E, D)/\sim| < 2^{\mathcal{S}} \cdot 2^{\binom{\mathcal{S}}{3}}, \tag{4.2}$$

where $\mathcal{S} = \mathcal{S}(n, D, E)$ □

# Capitolo 5

# Systolic estimates of hyperbolizable and Seifert manifolds

*When there is a way and space to express silenced pain and on the other side full recognition, the perspective shifts and the very pain that is at the root of the desire for punishment sometimes disappears altogether (and sometimes not).*

— Giusy Palomba, La trama alternativa.

*You know, sometimes, when I think back to certain symphonies from Mahler, I am sure that we will make it, that we will stop the climate crisis and everything will be fine. I am sure that such beauty cannot just disappear.*

— F., a climate activist and mathematician.

In this chapter we prove two entropy systole inequalities for some classes of 3-manifolds analogous to the following inequality, which holds for non-geometric manifolds.

**Proposition 5.0.1** ([CS19] Theorem 1.1). *Let $X$ be a closed, orientable, non-geometric Riemannian 3-manifold, with torsion-less fundamental group, such that $Diam(X) < D$ and $Ent(X) < E$, then*

$$sys(X) \geq \mathcal{S}_{ng}(E, D) = \frac{1}{E} \log\left(1 + \frac{4}{e^{26ED} - 1}\right).$$

The following proposition is a special case of a result of Cavallucci and Sambusetti. They proved a quantitative version of Tits' alternative that holds on a very general class of metric spaces that have pinched negative curvature on a weak sense, and for such spaces it establishes the existence of free (semi)groups of isometries generated by elements of bounded length.

**Proposition 5.0.2** ([CS21] Theorem 1.1). *There exists a function $N_0 = N_0(n)$, depending only on the dimension, satisfying the following properties:*

*(i) for any couple of isometries $S = \{a, b\}$ of $\mathbb{H}^n$, where $a$ is non elliptic, such that the group $\langle a, b\rangle$ is discrete and non-elementary, there exists a word $w(a, b)$ in $a, b$ of length $||w||_S \leq N$ such that one of the semigroups $\langle a^N, w\rangle^+$, $\langle a^{-N}, w\rangle^+$ is free;*

*(ii) for any couple of isometries $S = \{a, b\}$ of $\mathbb{H}^n$ such that the group $\langle a, b\rangle$ is discrete, non-elementary and torsion-free, there exists a word $w(a, b)$ in $a, b$ of length $||w||_S \leq N$ such that the group $\langle a^N, w\rangle$ is free.*

**Remark 5.0.3.** The original proof of this theorem applies to $\delta$-hyperbolic, GCB-spaces[1] with a packing condition, and the constant $N_0$ depends on the (generalised) Margulis constant, the hyperbolicity constant, and the packing function of the space. For $\mathbb{H}^n$, the (optimal) hyperbolicity constant is known to be $\delta = \log(2)$ (cf. [NS16], Corollary 5.4), the Margulis constant is known to satisfy $0.104 < \epsilon_0(3) < 3\sqrt{(3)}/5$ (cf. [Kel04] for the lower bound, and [Bel14], Proposition 5.2, for the upper bound), and the packing function $\text{Pack}(R, r)$, denoting the maximal cardinality of an $r$-separated subset of a closed ball of radius $R$, can be bounded by the ratio

$$\text{Pack}(R, r) < \text{Vol}(B(R, x))/\text{Vol}(B(r, x)).$$

In particular, in dimension 3, $N_0(3) < \pi \exp(2 \cdot 10^{30})$.

## 5.1 Hyperbolizable manifolds

We will now prove the an entropy-systole inequality analogous to 5.0.1 for hyperbolizable manifolds.

**Theorem 5.1.1** (Entropy-systole inequality for hyperbolizable manifolds)**.** *Let $X$ be a closed, hyperbolizable, Riemannian manifold such that $Diam(X) < D$ and $Ent(X) < E$, then*

$$sys(X) \geq \mathcal{S}_{hyp}(E, D) = 3De^{-6N_0ED}, \tag{5.1}$$

*where $N_0$ is the number defined in Proposition 5.0.2.*

*Dimostrazione.* Let $\alpha$ be a non homotopically trivial loop in $X$ whose length realises $\ell(\alpha) = sys(X)$, and choose a point $x_0 \in \alpha$. Denote $a = [\alpha] \in \pi_1(X, x_0)$ and $S_{3D} = \{f \in \pi_1(X, x_0) | d(x_0, f.x_0) < 3D\}$ the set of $3D$-short generators of $\pi_1(X, x_0)$.
Since $X$ is hyperbolizable $\pi_1(X, x_0)$ admits a non-elementary, discrete, action by loxodromic isometries on $\mathbb{H}^n$. Then, since $\pi_1(X, x_0)$ is non elementary, there is an element $b \in S_{3D}$ such that $\langle a, b\rangle$ is non-elementary. Otherwise every generator of $g \in S_{3D}$ would have the same set of fixed points $Fix(b)$, and $\pi_1(X, x_0)$ would be elementary. Hence we can apply Proposition 5.0.2, $(ii)$, and there are a number $N_0$ and a word $w(a, b)$ of length $||w||_{\{a,b\}} \leq N_0$ such that $H :=< a^{N_0}, w >$ is isomorphic to the free group on two generators $\mathbb{F}^2 = \langle c_1, c_2\rangle$.
Clearly we have

$$E > Ent(X) = Ent(\pi_1(X, x_0), \tilde{X}) \geq Ent(H, \tilde{X}).$$

Now, since $||a||_X = sys(X)$ and $||b||_X < 3D$ we have $||a^{N_0}||_X \leq N_0 sys(X)$, $||w||_X \leq 3N_0 D$, so we can apply Lemma 2.3.2 and we get

$$Ent(H, \tilde{X}) > Ent(\mathbb{F}^2_+, ||\cdot||_X) > \frac{1}{6N_0 D} \log\big(\frac{3D}{sys(X)}\big).$$

In particular, since $N_0$ is independent on $X$, we have

$$sys(X) > \mathcal{S}_{hyp}(E, D) = 3De^{-6N_0ED}$$

[1] A GCB-space is a complete metric space together with a special family of geodesics called geodesic bicombing, as defined in [CS21].

which concludes the proof. □

## 5.2 Seifert manifolds with hyperbolic base

Let $X$ be a Seifert fibered manifold with hyperbolic base, and let

$$1 \to F \to \pi_1(M) \xrightarrow{p} \pi_1^{orb}(\Sigma) \to 1 \tag{5.2}$$

be the short exact sequence defined in 2.6.3. Recall that the kernel $F$ of $p$ is the subgroup generated by a regular fibre, and that, by Remark 2.6.1, if an element $g \in \pi_1(M)$ is represented by a loop freely homotopic to an exceptional fiber then it has a power contained in $F$. So we give the following definition.

**Definition 5.2.1.** The *fibre-free systole* of a closed, orientable, Seifert manifold of hyperbolic type $M$ is

$$sys_{ff}(M) := \min\{\ell(g), g \in \pi_1(M) | p(g)^n \neq 0 \text{ for all } n \in \mathbb{Z}\}$$

Since $X$ is a Seifert manifold with hyperbolic base, $sys_{ff}(X)$ is well defined. In fact $p$ is a surjective homomorphism onto $\pi_1^{orb}(\Sigma)$, which contains a surface group, so there is at least an element in $\pi_1(X)$ whose image in $\pi_1^{orb}(\Sigma)$ is non-trivial and torsion-less.

Notice that $p(g)$ has torsion if and only if all other conjugates $p(hgh^{-1})$ have torsion. Now, to a loop $\gamma : S^1 \to M$ there is associated a conjugacy class in $\pi_1(M)$, whose elements, by Remark 2.6.4, have torsion in $\pi_1^{orb}(\Sigma)$ if and only if they are conjugated to a power of $c_j$ for some $j$, or if they are a power of $f$. In either case, it follows that $\gamma$ is freely homotopic to a power of some fibre, either regular or not, i.e $[\gamma] = [F_y]^n$ for some $y \in M$ and $n \in \mathbb{Z}$.

**Remark 5.2.2.** To summarise we have shown that, if we denote $T$ (resp. $T_{p_0}$) the set of free homotopy classes (resp. pointed at $p_0$) of powers of fibers (both regular and exceptional), then $p(T_{p_0})$ equals the set of torsion elements in $\pi_1^{orb}(\Sigma)$, and that

$$sys_{ff}(M) = \min\{length(\gamma), \gamma : S^1 \to M | [\gamma] \notin T\}. \tag{5.3}$$

Note that, since we are considering manifolds with general metrics, the shortest representative of a fiber $F_{p_0}$ could be longer than a representative of some power $F_{p_0}^n$ [2], so excluding all $T$ and not just the fibers is necessary when taking the minimum.

**Theorem 5.2.3** (Entropy-systole inequality for hyperbolic Seifert manifolds)**.** *Let $X$ be a Riemannian Seifert 3-manifold with hyperbolic base, and such that $Diam(X) < D$ and $Ent(X) < E$, then*

$$sys_{ff}(X) \geq \mathcal{S}_{ff}(E, D) = \frac{3}{2} D e^{-6N_0 ED}, \tag{5.4}$$

*where $N_0$ is the number defined in Proposition 5.0.2.*

*Dimostrazione.* Let $\alpha$ be non homotopically trivial loop whose length realise $\ell(\alpha) < 2sys(X)$), and let $x_0 \in \alpha$ be a point in $\alpha$. Denote $\bar{a} = [\alpha] \in \pi_1(X, x_0)$, $a = p(\bar{a}) \in \pi_1 orb(\Sigma)$, $\bar{S}_{3D} = \{f \in$

[2]For example on a standard solid torus $D \times S^1$ one choose any simple loop $\alpha$ homotopic to $[\{x\} \times S^1]^n$ and deform the metric to make $\alpha$ shorter than any $\epsilon > 0$ without significantly change the length of $[\{x\} \times S^1]$ by dilating a small neighbourhood $N_\delta(\alpha)$ radially to $\alpha$ and contracting it along $\alpha$.

$\pi_1(X, x_0)|d(x_0, f.x_0) < 3D\}$ the set of $3D$-short generators of $\pi_1(X)$, and $S_{3D} = \{p(f), f \in \bar{S}_{3D}\}$. Note that since $p$ is surjective, $S_3D$ is a generating set for $\pi_1^{orb}(\Sigma)$.
Recall that since $X$ is Seifert hyperbolic, $\pi_1^{orb}(\Sigma)$ is a discrete, non-elementary, subgroup of isometries of $\mathbb{H}^3$ that acts properly discontinuously.
We claim that there is an element $b = p(\bar{b}) \in S_{3D}$ such that $\langle a, b\rangle$ is non-elementary. By definition, $a$ is a hyperbolic element. Suppose by absurd that $\langle a, b\rangle$ is elementary for all $b \in S_{3D}$. Then $b.Fix(a) = Fix(a)$ for all $b \in S_{3D}$, i.e. every generator of $\pi_1^{orb}(\Sigma)$ has the same set of fixed points. This implies that $\pi_1^{orb}(\Sigma)$ is elementary, which is a contradiction.
Let $b = p(\bar{b}) \in S_{3D}$ such that $\langle a, b\rangle \in \pi_1^{orb}(\Sigma)$ is non-elementary. Since $\pi_1^{orb}(\Sigma)$ is discrete, we can apply Proposition 5.0.2, $(i)$. So there are a number $N_0$ and a word $w(a, b)$ of length $||w||_{\{a,b\}} \leq N_0$ such that one of the semigroups $\langle a_0^N, w\rangle^+$, $\langle a^{N_0}, w\rangle^-$ is isomorphic to the free semigroup on two generators $\mathbb{F}_2^+$.
It implies that the semigroup $H := \langle \bar{a}^{N_0}, w(\bar{a}, \bar{b})\rangle < \pi_1(X)$ is also free. In fact, if there is a non-trivial relation $r(\bar{a}^{N_0}, w(\bar{a}, \bar{b})) = 1$ then there is also be a non trivial relation $p(r) = r(a^{N_0}, w(a, b)) = 1$ which is absurd. Then we have

$$E > Ent(X) = Ent(\pi_1(X, x_0), \tilde{X}) \geq Ent(H, \tilde{X}).$$

Now, since $||\bar{a}||_X < 2sys(X)$ and $||\bar{b}||_X < 3D$ we have $||\bar{a}^{N_0}||_X \leq 2N_0 sys(X)$, and $||w(\bar{a}, \bar{b})||_X \leq 3N_0 D$, so we can apply Lemma 2.3.2 and we get

$$Ent(H, \tilde{X}) > Ent(\mathbb{F}_+^2, ||\cdot||_X) > \frac{1}{6N_0 D} \log\big(\frac{3D}{2sys(X)}\big).$$

In particular, since $N_0$ is independent on $X$, we have

$$sys(X) > \mathcal{S}_{ff}(E, D) = \frac{3}{2} D e^{-6N_0 ED}.$$

□

# Capitolo 6

# Differential GH-stability of hyperbolizable 3-manifolds

*I do commit to rigorously learning how to gracefully collaborate, and step back when it's your turn with nothing to prove. I do commit to the work of going deep enough to find the necessary food that lights us up inside. I love you, and I have so much to learn. I love you and we are just now learning that it's possible, love on a scale we can survive.*

— Alexis Pauline Gumbs, Undrowned

**Theorem 6.0.1.** *There exists a function $\mathcal{E} = \mathcal{E}(D, E)$ with the following property: let $X$ be a closed, orientable, hyperbolizable, Riemannian 3-manifold, with diameter $Diam(X) < D$, and entropy $Ent(X) < E$; then, for any closed, orientable, Riemannian 3-manifold $Y$ with torsion-less fundamental group, $Ent(Y) < E$ and $d_{GH}(X, Y) < \mathcal{E}$, it holds $\pi_1(Y) \cong \pi_1(X)$. In particular $Y$ is diffeomorphic to $X$.*

*Dimostrazione.* First of all note that since $Ent(X) > 0$ the fundamental group $\pi_1(X)$ is infinite. Then, by Proposition 2.4.5, $X$ is hyperbolic and $\pi_1(X)$ admits a discrete, non-elementary action by isometries on $\mathbb{H}^3$. Then by Theorem 5.1.1 we have $sys(X) > \mathcal{S}_{hyp}(E, D)$. Let us and assume that

$$d_{GH}(X, Y) < \mathcal{S}_{hyp}(E, D)/80 \leq sys(X)/80. \tag{6.1}$$

By Proposition 2.7.5, ((i)) there is a surjective homomorphism $\pi_1(Y) \to \pi_1(X)$, and in particular we have $Ent(Y) > 0$. Thus we have four cases:

(i) $Y$ is non-geometric;
(ii) $Y$ can be modelled on $Sol$;
(iii) $Y$ can be modelled on $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \tilde{\times} \mathbb{R}$, i.e. is Seifert with hyperbolic base;
(iv) or $Y$ is hyperbolizable,

since the fundamental group of manifolds modelled on any other geometry is of sub-exponential growth. We shall deal with each of these cases separately. Let $D' = D + \mathcal{S}_{hyp}(E, D)/40$, then it follows from (6.1) that $Diam(Y) < D'$.

**Case** (i): If $Y$ is non-geometric, by Proposition 5.0.1 we have $sys(Y) \geq \mathcal{S}_{ng}(E, D')$. If we assume that

$$d_{GH}(X, Y) < \min\{\mathcal{S}_{hyp}(E, D), \mathcal{S}_{ng}(E, D')\}/80 \leq \min\{sys(X), sys(Y)\}/80,$$

then by Proposition 2.7.5, ((ii)) we have $\pi_1(X) \cong \pi_1(Y)$, which is a contradiction by Proposition 2.4.6.

**Case** (ii): If $Y$ can be modelled on $Sol$, then $\pi_1(Y)$ is solvable (see Bonahon, [DS02], Chapter 3, Section 2.3), but a solvable group is amenable ([CSC10], Theorem 4.6.3), and in particular it contains no free subgroups of rank $\geq 2$ ([CSC10], Corollary 4.5.2). This is a contradiction, since $\pi_1(X)$ contains a free subgroup of rank 2, by Proposition 2.4.7, ((iv)), and $\pi_1(Y)$ surjects on $\pi_1(X)$.

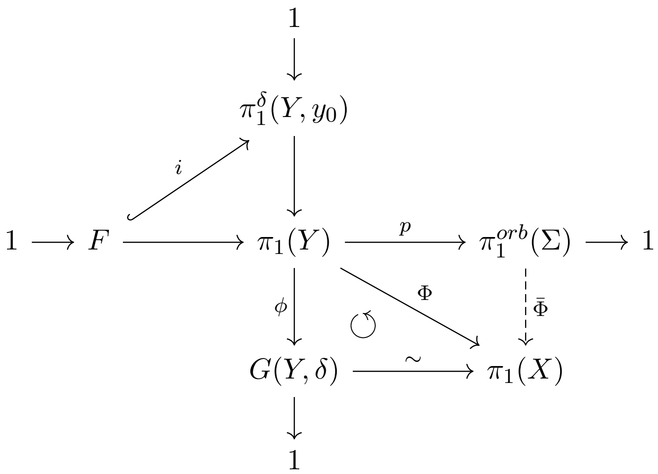
**Case** (iii): If $Y$ is a Seifert manifold with hyperbolic base, then we have the short exact sequence of Proposition 2.6.3

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(Y) \xrightarrow{\ p\ } \pi_1^{orb}(\Sigma) \longrightarrow 1,$$

where the base $\Sigma$ is a hyperbolic orbifold.
Let $T \subset C(S^1, Y)$ (resp. $T_{y_0} \subset \pi_1(Y, y_0)$) be the set of free homotopy classes (resp. pointed at $y_0$) of powers of fibers (both regular and exceptional) of $Y$ as in Remark 5.2.2. By Theorem 5.2.3, any loop $\gamma$ such that $[\gamma] \notin T$ has length greater than $\mathcal{S}_{ff}(E, D')$.
Let us set $2\delta := \mathcal{S}_{ff}(E, D')$, and assume that

$$d_{GH}(X,Y) < \min\{\mathcal{S}_{hyp}(E,D), 2\delta\}/80 \leq \min\{sys(X), sys_{ff}(Y)\}/80, \tag{6.2}$$

then by Proposition 2.7.5, ((iii)) there is an isomorphism $G(X,\delta) \cong \pi_1(X)$. Now recall that by definition $G(X,\delta) \cong \pi_1(Y)/\pi_1^{\delta}(Y, y_0)$, so we get the following diagram.




The inclusion $F \overset{i}{\hookrightarrow} \pi_1(Y,\mathcal{U}_\delta)$ is due to the fact that the kernel of $p$ is contained in the kernel of $\phi$. In fact, since $F$ is a central subgroup of $\pi_1(Y)$, and $\phi$ is surjective, $\phi(F)$ is contained in the center of $\pi_1(X)$, which is trivial by Proposition 2.4.7, ((iii)). It follows that $\Phi$ quotients to a surjective map $\bar{\Phi} : \pi_1^{orb}(\Sigma) \to \pi_1(X)$. If $g \in \pi_1^{orb}(\Sigma)$, for every $a, b \in p^{-1}(g)$ we have $ab^{-1} \in F < \ker(\Phi)$, hence $\bar{\Phi}$ is well defined, and it is clearly surjective.
Let $\overline{T} \triangleleft \pi_1^{orb}(\Sigma) = \langle\langle c_j \rangle\rangle$ be the normal subgroup generated by the torsion elements in $\pi_1^{orb}(\Sigma)$. We claim that $\overline{T}$ is the kernel of $\bar{\Phi}$.
Clearly $\overline{T} \leq \ker(\bar{\Phi})$ because $\pi_1(X)$ has no torsion. On the other hand, we have $\overline{T} \geq \ker(\bar{\Phi})$, as we shall see. By construction, $\bar{\Phi} = p \ker \Phi$, furthermore, since $p$ is surjective, we have $\overline{T} = p\langle\langle T_{y_0} \rangle\rangle$, so it suffice to show that $\langle\langle T_{y_0} \rangle\rangle \geq \ker(\Phi)$.

By Proposition 2.7.4 the kernel of $\Phi$ equals the normal subgroup of $\pi_1(Y, y_0)$ generated by elements shorter than $2\delta$, $\ker(\Phi) = \pi_1(Y, y_0, 2\delta)$, and since by definition any element of length $< 2\delta = \mathcal{S}_{ff}(E, D')$ must be contained in $T_{y_0}$, it follows $\langle\langle T_{y_0} \rangle\rangle \geq \ker(\Phi)$.
Then we have a contradiction. In fact, if $\overline{T} = \ker(\bar{\Phi})$ then

$$\pi_1(X) \cong \pi_1^{orb}(\Sigma)/\overline{T} \cong \pi_1(\Sigma),$$

which is impossible, because $\pi_1(X)$ and $\pi_1(\Sigma)$ are the fundamental groups of a closed, aspherical 3-manifold, and of a closed, aspherical surface respectively, which by Proposition 2.7.6 have different cohomological dimensions.

**Case** (iv): If $Y$ is hyperbolic, by Proposition 5.1.1 we have $sys(Y) \geq \mathcal{S}_{hyp}(E, D')$. If we assume that

$$d_{GH}(X, Y) < \min\{\mathcal{S}_{hyp}(E, D), \mathcal{S}_{hyp}(E, D')\}/80 \leq \min\{sys(X), sys(Y)\}/80,$$

then by Proposition 2.7.5, ((ii)) we have $\pi_1(X) \cong \pi_1(Y)$.

To conclude, let us define

$$\mathcal{E}(E, D) = \min\big\{\mathcal{S}_{hyp}(E, D), \mathcal{S}_{hyp}(E, D'), \mathcal{S}_{ff}(E, D'), \mathcal{S}_{ng}(E, D')\big\}/80.$$

We have shown that, if $d_{GH}(X, Y) < \mathcal{S}(E, D)$, then $Y$ must be hyperbolic, and $\pi_1(X) \cong \pi_1(Y)$. In particular, by Proposition 2.4.8, $X$ and $Y$ are diffeomorphic. $\square$

# Bibliografia


[AFW15] M. Aschenbrenner, S. Friedl, and H. Wilton. *3-manifold Groups*. EMS series of lectures in mathematics. European Mathematical Society, 2015.

[AL06] G. N. Arzhantseva and I. G. Lysenok. A lower bound on the growth of word hyperbolic groups. *Journal of the London Mathematical Society*, 73(1):109–125, 2006.

[BCG95] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geometric & Functional Analysis GAFA*, 5(5):731–799, 1995.

[BCG03] Gérard Besson, Gilles Courtois, and Sylvestre Gallot. Un lemme de margulis sans courbure et ses applications. *Prépublication de l'Institut Fourier*, 2003.

[BCGS20] Gérard Besson, Gilles Courtois, Sylvestre Gallot, and Andrea Sambusetti. Curvature-free margulis lemma for gromov-hyperbolic spaces. *arXiv preprint*, 2020.

[BCGS21] Gérard Besson, Gilles Courtois, Sylvestre Gallot, and Andrea Sambusetti. Finiteness theorems for gromov-hyperbolic spaces and groups. *arXiv preprint*, 2021.

[Bel14] Mikhail Belolipetsky. Hyperbolic orbifolds of small volume. *Proceedings of the international congress of mathematics*, pages 837–851, 2014.

[BF21] Emmanuel Breuillard and Koji Fujiwara. On the joint spectral radius for isometries of non-positively curved spaces and uniform growth. *Annales de l'Institut Fourier*, 71:317–391, 2021.

[BGT12] Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. *Publications mathématiques de l'IHÉS*, 2012.

[BH11] M.R. Bridson and A. Häfliger. *Metric Spaces of Non-Positive Curvature*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011.

[BNS23] Elia Bruè, Aaron Naber, and Daniele Semola. Fundamental groups and the milnor conjecture. *arXiv e-prints*, 2023.

[Bro82] K.S. Brown. *Cohomology of Groups*. Graduate Texts in Mathematics. Springer-Verlag Berlin and Heidelberg GmbH & Co. K, 1st edition edition, 1982.

[CC97] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *Journal of Differential Geometry*, 46(3):406 – 480, 1997.

[Cer12] Filippo Cerocchi. Margulis lemma, entropy and free products. *arXiv preprint*, 2012.

[Che70] Jeff Cheeger. Finiteness theorems for riemannian manifolds. *Am. J . Math.*, 92:61–75, 1970.

[CHK00] Daryl Cooper, Craig D. Hodgson, and Steven P. Kerckhoff. *Three-dimensional Orbifolds and Cone-Manifolds*. The Mathematical Society of Japan, Tokyo, 2000.

[Cou16] Rémi Coulon. Théorie de la petite simplification : une approche géométrique. *Asterisque*, 380:1–33, 2016.

[CS18] Filippo Cerocchi and Andrea Sambusetti. Entropy and finiteness of groups with acylindrical splittings. *arXive preprint*, 2018.

[CS19] Filippo Cerocchi and Andrea Sambusetti. Local topological rigidity of nongeometric 3–manifolds. *Geometry & Topology*, 23(6):2899–2927, 2019.

[CS21] Nicola Cavallucci and Andrea Sambusetti. Discrete groups of packed, non-positively curved, gromov hyperbolic metric spaces. *arXive preprint*, 2021.

[CSC10] T. Ceccherini-Silberstein and M. Coornaert. *Cellular Automata and Groups*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2010.

[DGO17] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Memoirs of the American Mathematical Society*, 245(1156), 2017.

[DKL19] Subhadip Dey, Michael Kapovich, and Beibei Liu. Ping-pong in hadamard manifolds. *Münster J. of Math.*, 12:453–471, 2019).

[DS02] R.J. Daverman and R.B. Sher. *Handbook of Geometric Topology*. Elsevier Science & Technology Books, 2002.

[FdC13] F. Flaherty and M.P. do Carmo. *Riemannian Geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, 2013.

[Fuk87] K. Fukaya. Collapsing of riemannian manifolds and eigenvalues of the laplace operator. *Invent. Math.*, 87:517–547, 1987.

[GLP81] M. Gromov, J. Lafontaine, and P. Pansu. *Structures métriques pour les variétés riemanniennes*. Textes mathématiques. Recherche. CEDIC/Fernand Nathan, 1981.

[Gro78] M. Gromov. Almost flat manifolds. *J. Diff. Geom.*, 13:231–241, 1978.

[Gro82] Michael Gromov. Volume and bounded cohomology. *Publications mathématiques de l'I.H.É.S.*, 56:5–99, 1982.

[Hat02] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[Hei76] E. Heintze. *Mannigfaltigkeiten negativer Krümmung*, volume 350. Bonn Univ., 1976. PhD thesis.

[HW10] P. Zalesskii H. Wilton. Profinite properties of graph manifolds. *Geom. Dedicata*, 147:29–45, 2010.

[JN83] M. Jankins and W.D. Neumann. *Lectures on Seifert Manifolds*. Brandeis lecture notes. Brandeis University, 1983.

[Kap10] Michael Kapovich. *Hyperbolic manifolds and discrete groups*, volume 53. Birkhäuser Boston, MA, 2010.

[Kar89] H. Karcher. Riemannian comparison constructions. *Studies in Global Geometry and Analysis*, 27, 1989. indirect reference.

[KB02] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. *In book: Combinatorial and geometric group theory*, pages 39,93, 2002.

[Kel04] Ruth Kellerhals. On the structure of hyperbolic manifolds. *Israel Journal of Mathematics*, 143:361–379, 2004.

[Kni94] Gerhard Knieper. Spherical means on compact riemannian manifolds of negative curvature. *Differential Geometry and its Applications*, 4:361,390, 1994.

[KW11] Vitali Kapovitch and Burkhard Wilking. Structure of fundamental groups of manifolds with ricci curvature bounded below. *arXiv e-prints*, 2011.

[LR98] D. D. Long and A. W. Reid. Simple quotients of hyperbolic 3-manifold groups. *Proceedings of the American Mathematical Society*, 12(3):877—-880, 1998.

[Lüc12] Wolfgang Lück. Aspherical manifolds. *Journal of the Manifold Atlas*, pages 1–17, 2012.

[Mar04] Bruno Martelli. Complexity of 3-manifolds. *arXiv preprint*, 2004.

[Mar22] Bruno Martelli. *An Introduction to Geometric Topology*. electronic version, 2022.

[Mil62] J. Milnor. A unique decomposition theorem for 3-manifolds. *American Journal of Mathematics*, 84(1):1–7, 1962.

[Mil68] John Milnor. A note on curvature and fundamental group. *J. Differential Geometry*, 2:1–7, 1968.

[Moi77] Edwin E. Moise. *Geometric topology in dimensions 2 and 3 / Edwin E. Moise*. Springer-Verlag New York, 1977.

[Mun60] James Munkres. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. *Annals of Mathematics*, 72(3):521–554, 1960.

[Mye41] Sumner B. Myers. Riemannian manifolds with positive mean curvature. *Duke Math. J.*, 8:401–404, 1941.

[NS16] Bogdan Nica and Jan Spakula. Strong hyperbolicity. *Groups Geom. Dyn.*, page 951–964, 2016.

[Rat06] J. Ratcliffe. *Foundations of Hyperbolic Manifolds.* Graduate Texts in Mathematics. Springer New York, 2006.

[Sam00] Andrea Sambusetti. On minimal entropy and stability. *Geometriae Dedicata*, 81:261–279, 2000.

[Sam08] Andrea Sambusetti. Asymptotic properties of coverings in negative curvature. *Geometry & Topology*, 12:617–637, 2008.

[Sco83a] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc,*, 15:401–487, 1983.

[Sco83b] Peter Scott. There are no fake seifert fibre spaces with infinite $\pi_1$. *Annals of Mathematics*, 117(1):35–70, 1983.

[Sel97] Z. Sela. Acylindrical accessibility for groups. *Inventiones mathematicae*, 129:527–565, 1997.

[SS02] J. Stilwell and J.P. Serre. *Trees.* Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2002.

[Sta56] J. Stalling. *Some topological proofs and extensions of Gruŝko's theorem.* Princeton University, 1956.

[SW01] Christina Sormani and Guofang Wei. Hausdorff convergence and universal covers. *Transactions of the American Mathematical Society*, 353(9):3585–3602, 2001.

[SW04] Christina Sormani and Guofang Wei. The covering spectrum of a compact length space. *J. Differential Geom.*, 67:35–77, May 2004.

[Thu97] William P. Thurston. *Three-dimensional geometry and topology.* Princeton mathematical series ; 35. lat. Princeton University Press, Princeton, N.J, 1997.

[Thu02] William P. Thurston. *The Geometry and Topology of Three-Manifolds.* Unpublished lecture notes, electronic version 1.1, 2002.

[Tur88] V.G. Turaev. Homeomorphisms of three-dimensional manifolds. *Mathematical Notes of the Academy of Sciences of the USSR*, 43:307–312, 1988.

[Vil16] Cédric Villani. Synthetic theory of ricci curvature bounds. *Jpn. J. Math.*, 11:219–263, 2016.

[Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Annals of Mathematics*, 87(1):56–88, 1968.

[WB13] Viktor Schroeder Werner Ballmann, Mikhael Gromov. *Manifolds of Nonpositive Curvature*, volume 61 of *Progress in Mathematics.* Birkhäuser Boston, MA, 2013.

[Whi78] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1978.

[Zhu93] Shun-Hui Zhu. A finiteness theorem for ricci curvature in dimension three. *J. Differential Geometry*, 37:711–727, 1993.

# Appendice A

# Questo lavoro è per tuttз

## Un racconto accessibile di spazi iperbolici e altre idee geometriche

*We are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics.*
— William P. Thurston, On proof and progress in mathematics

**Perché queste pagine?** Potrebbe sorprendere di trovare delle pagine dedicate a non specialistз sfogliando una tesi di dottorato in matematica, un manoscritto di ricerca tipicamente riservato ad espertз. Per me questa tesi rappresenta un rito di passaggio con il quale divento (anche) un matematico, e da qui sorge il bisogno di renderlo accessibile anche a tante persone che non sono esperte in materia.
Parlare di matematica a non espertз è sempre un'avventura interessante: a volte si incontrano persone curiose con molte domande, altre si incontra gente che ha molte cose bizzarre da dire a riguardo, ma più spesso mi è capitato di imbattermi in un pudico e timoroso distacco, articolato con frasi del tipo "per me è arabo", o "devi essere un genio". Queste risposte mettono una distanza fra me e l'altrə che rendono dolorosamente incomunicabile un aspetto della mia identità, il mio essere matematico.
Tuttavia non sono *solo* un matematico; vivo in una famiglia, in una città, in un pianeta, in tante comunità che si intersecano e che definiscono altri aspetti della mia identità. Per cui sono anche un figlio, nipote, fratello, un cittadino, e un attivista, che si interessano del mondo, si preoccupano del futuro, e cercano di capire e di prendersi cura di quello che hanno attorno. E anche parlare di questo mondo con dei matematici è stata in questi anni del dottorato un'esperienza complicata. Ho incontrato molta intelligenza, molta cultura, e grandi sensibilità, ma anche grandi frustrazioni, rassegnazione, e distrazione, che finiscono facilmente per relegare le conversazioni su temi umani o politici a incursioni occasionali, a parole altre, che non hanno ripercussioni concrete. Così mi sono accorto che la separazione fra queste due identità, quella matematica e quella, diciamo, mondana, è speculare; mi sembra nutrirsi di un distacco reciproco che finisce per dividere anche me.
Per cui, queste pagine in cui riassumo i punti salienti e risultati ottenuti in questi anni di ricerca sarebbero incomplete se non provassi almeno a mettere in fila le mie esperienze, e le immagini che

guidano la mia intuizione, per avvicinare questi due mondi.

**Iniziamo** Ricordo che quando avevo quattro o cinque anni mio papà passava molto tempo a parlarmi di matematica. Un pomeriggio mi aveva raccontato che moltiplicando due numeri negativi si otteneva un numero positivo, ed io proprio non riuscivo a capire il perché. Questo chiedere sempre perché? perché? che prima o poi si impossessa sempre dei bambini di non so quale età era il mio gioco preferito, ed è stata l'attitudine che mi ha guidato sia nella scoperta della matematica, che del mondo, con i suoi miracoli e le sue ingiustizie. "Perché i fili tra i pali della luce si curvano in basso?" "Perché c'è quel signore seduto fuori dalla chiesa?" "E perché?" Non la finivo più di ripetere queste domande lungo tutto il tragitto che mi portava da casa all'asilo.
E sempre a queste tratte mattutine è legata un'altra delle mie prime memorie matematiche: mamma sintonizzava la radio sui canali di musica classica, e mi faceva giocare a chiudere gli occhi e a immaginare le figure, le storie, i colori, che questa musica mi raccontava. Arrivavamo a scuola così, imparando a stare in relazione con la curiosità e la fantasia innate di ogni persona, ed è così che vorrei invitarvi in questo viaggio alla scoperta degli spazi iperbolici, delle varietà, e della geometria non euclidea.
Tuttз noi siamo familiari con i triangoli: tre lati, tre vertici, e tre angoli la cui somma è sempre 180°. Ma perché? Non potremmo immaginarci dei triangoli la cui somma degli angoli interni sia diversa da 180°?


**Figura A.1:** "Finché gli uomini massacreranno gli animali, si uccideranno a vicenda." — Pitagora

Ecco qua, con sei tratti di penna abbiamo attraversato 2000 anni di storia, da Euclide a Bolyai, e ci siamo immaginati dei triangoli diversi: tre lati, tre vertici, e tre angoli interni la cui somma può essere di più o di meno di 180°!
Si potrebbe obiettare che questi non sono triangoli, perché i loro lati non sono dritti, eppure...

**Linee dritte, linee curve** Che cos'è una linea dritta? Mi sono posto spesso questa domanda quando, soffocato dal desiderio di ottimizzare il mio percorso, cercavo di percorrere il tragitto più breve da piazzale Aldo Moro all'ingresso del Castelnuovo. Il problema di tagliare le diagonali interrotte da macchine parcheggiate, evitare le curve e le macchine in moto per raggiungere il dipartimento di matematica nel minor numero di passi non lasciava spazio a molto altro, finché non mi sono accorto che non *dovevo* per forza percorrere la strada più corta. Che magari in un giorno assolato era meglio fare il giro del cortile sotto l'ombra degli alberi che mi proteggevano dal sole piuttosto che attraversare un piazzale di asfalto infuocato.

Insomma, le distanze non sono sempre quelle che sembrano, e a volte vanno pesate con quello che ci costa attraversarle. Per cui se, seguendo l'intuizione di Archimede e Legendre, definiamo una linea retta come la linea più breve fra due punti, dobbiamo anche chiederci chi è che traccia questa linea, e dove si sta muovendo.
Ad esempio, per muoversi da un ramo all'altro una scimmia salterebbe, mentre una formica zompetterebbe fino alla prima biforcazione utile per poi tornare indietro lungo l'altro ramo, eppure entrambe seguirebbero la strada più breve per loro.
Una questione di punti di vista quindi, che in fisica chiameremmo sistemi di riferimento, ma anche di accessibilità dello spazio, o di forma dello spazio accessibile. Così se pensiamo a un bruco che cammina su una foglia, o a un albatros che vola attraverso gli oceani da un continente all'altro, ci accorgiamo che quei triangoli "storti" che abbiamo disegnato possono apparire dritti se li pensiamo dentro a spazi curvi, come una foglia o un globo.


**Figura A.2:** Triangoli inaspettati dentro i loro spazi modello

I triangoli che conosciamo da sempre sono quindi quelli che disegnamo sul piano, quelli che ci sembrano gonfi, quelli che disegnamo su una sfera. Lo spazio su quale invece disegnamo i triangoli fini si chiama spazio iperbolico.

**Disegnare una mappa** I lati di un triangolo sferico o iperbolico, quindi, appaiono curvi solo perché li abbiamo rappresentati sul piano, invece che su una superficie curva adatta, in cui sono a casa. Un po' come quando per rappresentare il globo terrestre usiamo il planisfero, e le le linee dritte, come di chi vola attraverso un oceano seguendo la rotta più breve, ci appaiono curve.
Un equivalente iperbolico del planisfero è il disco di Poincaré: una mappa tonda dello spazio iperbolico in cui le rette appaiono come archi di circonferenze che entrano ed escono dalla mappa facendo angoli di 90° con il bordo.

**Figura A.3:** Due rette sul piano iperbolico: a sinistra viste sul disco di Poincaré, a destra sul modello a sella.

Mentre faceva l'agrimensore per il regno di Hannover, Gauss dimostrò che per rappresentare sul piano una superficie curva bisogna per forza distorcere qualcosa. In effetti, anche senza accorgercene, questo è un fenomeno con il quale siamo familiari. Infatti la più comune mappa del pianeta, il planisfero di Mercatore, deforma le regioni che si trovano più vicine ai poli, ingrandendole notevolmente.

Questo artefatto visivo è dovuto al voler rappresentare sul piano una sfera in modo che le rotte delle navi appaiano dritte, che quindi rende questa mappa utile per la navigazione, ma che alimenta una percezione delle proporzioni fra diverse aree geografiche, in particolare fra nord e sud globali, decisamente distorte.

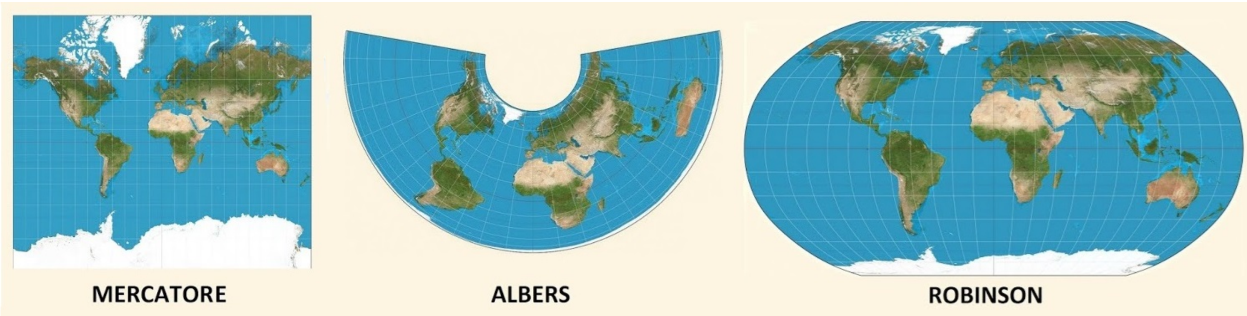



**Figura A.4:** Tre diverse proiezioni del globo su un piano con diverse distorsioni di forme, aree, e distanze.

Quali conseguenze può avere abituarsi a queste immagini su i nostri bias e pregiudizi coloniali ereditati dalla cultura in cui cresciamo? Quali alternative esistono?

La mappa di Poincaré dello spazio iperbolico non sfugge a queste distorsioni, ma presenta il problema opposto: più ci si allontana dal centro più le regioni appaiono piccole. Questo ha una conseguenza interessante. Il fatto di rappresentare la stessa figura sempre più piccola mano mano che la si allontana dal centro ci permette di rappresentare punti infinitamente lontani all'interno dei confini del disco di Poincaré. Immaginate di avere sempre lo stesso metro, ma di disegnarlo lungo 1 quando si trova al centro, lungo $\frac{1}{2}$ quando è spostato di un metro, $\frac{1}{4}$ quando è spostato di due metri e così via. Le distanze reali diventano sempre più grandi mano mano che il metro si allontana, ma vengono rappresentate con segni sempre più piccoli, creando una situazione, che sarebbe cara a Zenone, in cui una retta infinita si estende all'interno di un disco finito.

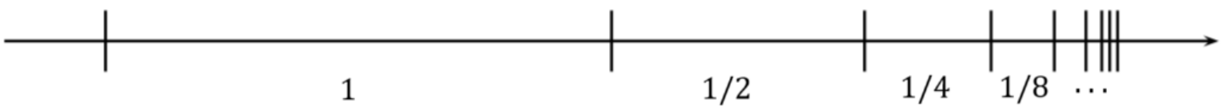



**Figura A.5:** "...quindi il movimento non esiste in quanto ciò che è in moto deve arrivare a metà strada prima di arrivare alla meta."— Zenone d'Elea

**Facciamo spazio**

*Respirare per liberare la mente,*
*lo yoga del fare spazio, dell'essere spazio*
*per accogliere il presente.*
*La più grande difficoltà*
*che incontro quando studio*
*quando affronto un vecchio problema*
*o mi confronto con chi la vede diversamente da me*
*è non presumere, non presupporre, non assumere*

*nulla che vada oltre*
*la nostra esperienza comune.*
*Respirare per fare spazio*
*per ricordarsi del fatto*
*che abbiamo tutti toccato la terra con mano,*
*vagito, respirato piangendo per una prima volta,*
*e poi poco altro, e poi quanto altro.*
*Le somiglianze che sfuggono allo sguardo*
*le somiglianze forzate, le analogie per un pelo,*
*le identità apparenti, la misura della diversità,*
*e la nostra natura di parenti.*
*Respirare per guardare il mondo con occhi innocenti,*
*fissi, alla scoperta di cosa accomuna una foglia,*
*al nostro annusare, alle profondità degli abissi.*

**Serve essere flessibili per esplorare spazi curvi?** Abbiamo visto che lo spazio iperbolico, ovvero quello in cui i triangoli appaiono magri e la somma dei loro angoli è più piccola di 180°, appare come una sella, una foglia, o un passo montano.

**Figura A.6:** Da sinistra a destra: Aiguille du Midi beau Chamonix, photo by Rosshe Witt, detail; Ficus Benjamina Leaves, photo by Forest and Kim Starr, detail

A differenza della sfera, o della sommità di una collina, in cui il terreno declina in ogni direzione, negli spazi iperbolici succede che, ovunque ci troviamo, ci sono delle direzioni in cui il terreno declina, ed altre in cui risale. Proprio come in un passo montano. E dico spazi, al plurale, perché ce ne sono tanti. Ecco un po' di esempi:
Gli spazi che curvano in questo modo, in modi diversi a seconda della direzione, si chiamano "curvi negativamente", mentre quelli che curvano nello stesso modo in ogni direzione, come le sfere, si chiamano "curvi positivamente", rifacendosi al fatto che moltiplicando due numeri con segni diversi si ottiene un numero negativo, e moltiplicando due numeri con lo stesso segno si ottiene un numero positivo.
Tutte queste forme dai nomi bizzarri hanno la proprietà di essere curvi negativamente; quindi i triangoli disegnati sulle loro superfici saranno sempre sottili.
Ma allora, se non è la loro forma, quando li guardiamo da fuori, che ci può aiutare a capire quando

**Figura A.7:** Da sinistra a destra: elicoide iperbolico, paraboloide iperbolico, pseudosfera.

uno spazio è iperbolico, come facciamo ad orientarci? Certamente il fatto che i triangoli che ci si disegnano sopra sono sottili è una caratteristica che li accomuna, ma ne possiamo trovare altre? In effetti si. Ed esplorarle si rivelerà essere un lavoro tutt'altro che inutile, che ci permetterà di capirli meglio, e di trovare molti nuovi esempi.

**Perdersi e ritrovarsi** La prima caratteristica che possiamo osservare è che in questi spazi ci si perde facilmente. Immaginatevi di essere un ragnetto in cima alla sua tela: per spostarvi a destra, al centro, o a sinistra, vi incamminerete agilmente lungo il filo che vi porta in quella direzione, nessun problema! Se invece foste un ragnetto iperbolico, sempre in cima alla sua tela fatta di triangoli, questa volta iperbolici, dovrete fare più attenzione: i fili partono tutti in direzioni simili, ed un piccolo errore nella direzione di partenza potrebbe farvi arrivare al posto sbagliato!

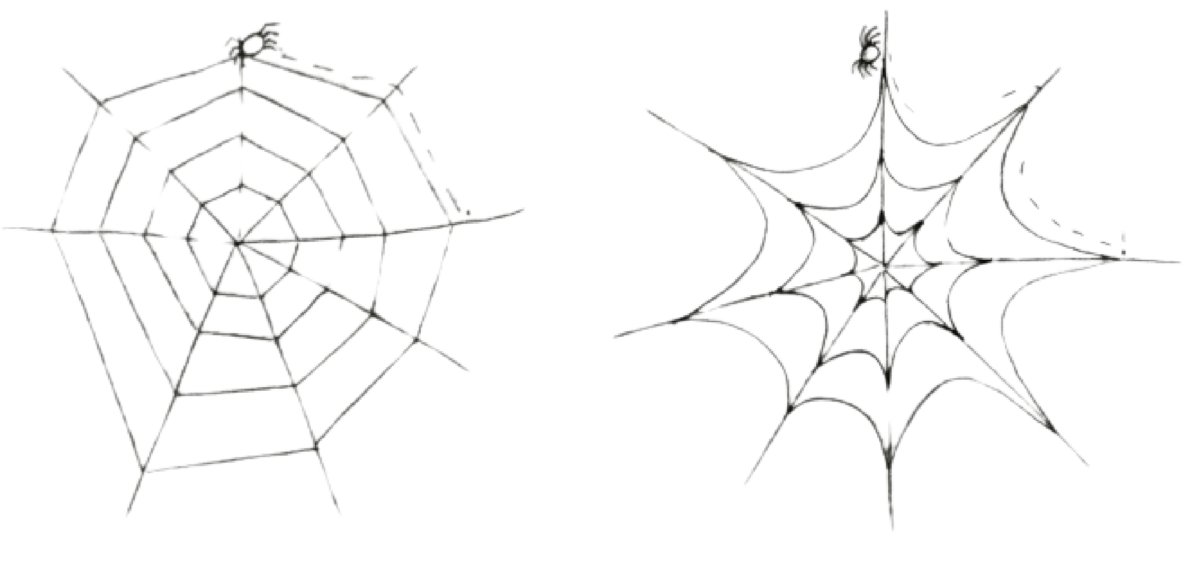


**Figura A.8:** A sinistra un ragno euclideo, a destra un ragno iperbolico

In un spazio iperbolico le rette che partono da un punto in direzioni diverse si allontanano molto rapidamente. Potrebbe non essere evidente a colpo d'occhio, perché i fili di seta sembrano allontanarsi dal centro in entrambe le ragnatele allo stesso modo. Eppure se partendo dal centro il ragnetto

euclideo arrivasse per sbaglio in alto, invece che al capo destro della tela, gli basterebbe fare due passi lungo i fili in senso orario per arrivare alla destinazione iniziale. Al ragnetto iperbolico le cose non vanno altrettanto bene: i fili si inarcano molto più vicini al centro, e fare gli stessi due passaggi potrebbe volerci lo stesso tempo che tornare al centro ed imboccare la strada giusta verso destra. Non ci sono scorciatoie. Se ci pensate questo è molto simile a quello che faceva qualche pagina fa la formica per passare da un ramo all'altro di un albero; in effetti gli alberi (ideali) sono proprio un esempio limite di spazio iperbolico!
Se partendo da un triangolo piano iniziate a risucchiarne i lati verso l'interno ottenete un triangolo iperbolico, e continuando ad asciugarlo fino all'osso vi ritroverete con un tripode: tre lati attaccati per il centro come in una "Y". Attaccando tanti tripodi fra di loro otteniamo un bell'albero!

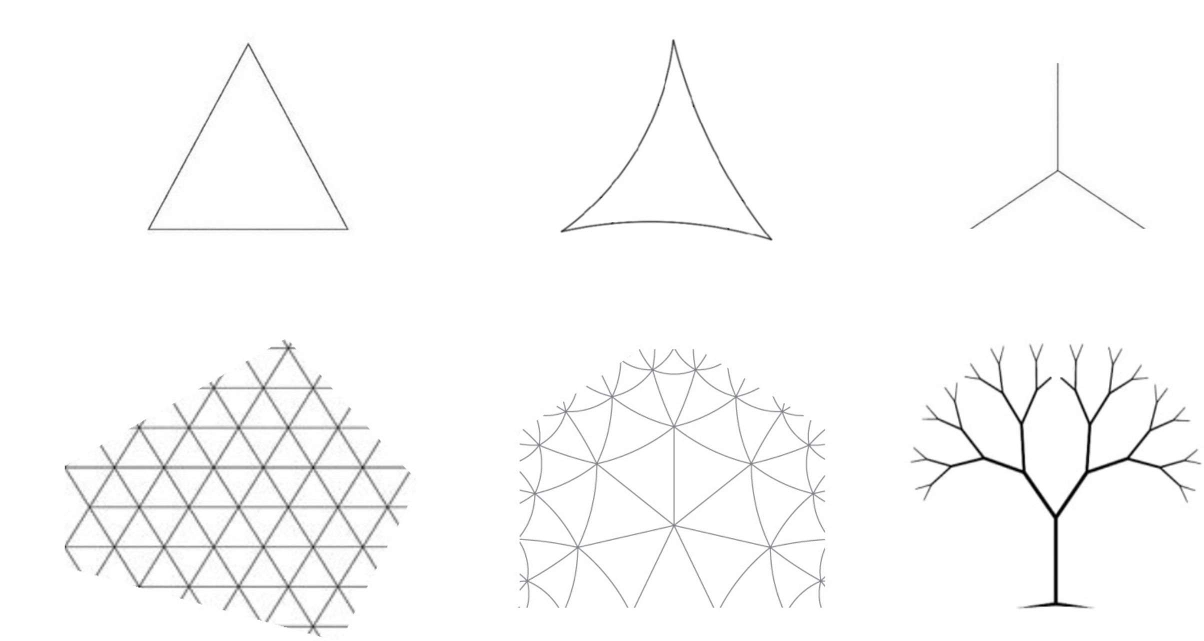


**Figura A.9:** Triangolo euclideo, triangolo iperbolico, e tripode, con i rispettivi tassellamenti.

E questo è molto interessante, perché gli alberi sono dappertutto! (Strano scrivere un'affermazione del genere, considerando che ogni anno perdiamo 10.000 $km^2$ di foresta amazzonica) Infatti, ogni volta che degli oggetti o degli individui sono legati fra di loro da delle relazioni d'ordine successive o gerarchiche (del tipo: viene prima questo, poi questi, e dopo questo quelli e così via), allora si possono rappresentare come degli alberi. Ad esempio: gli alberi genealogici, o quelli filogenetici che tracciano l'evoluzione delle specie. Ma con un po' di fantasia anche le reti di conoscenze fra le persone, o i network cerebrali con i quali percepiamo le immagini, i suoni, gli odori, sono legati da relazioni di questo genere. Per cui possiamo rappresentare e mappare tutte queste idee, relazioni, o oggetti, usando la geometria iperbolica, mettendole in connessione e capirle meglio grazie alle nostre conoscenze e intuizioni geometriche.

**Molto spazio, incredibilmente vicino** Ma sarà che in questi spazi a curvatura negativa ci si perde perché c'è troppo spazio dove potersi muovere? In effetti, per ritagliare un disco di un certo raggio (ovvero la distanza tra il centro e il bordo) ci servirà un certa quantità di stoffa, la cui area sappiamo calcolare fin dalle elementari.

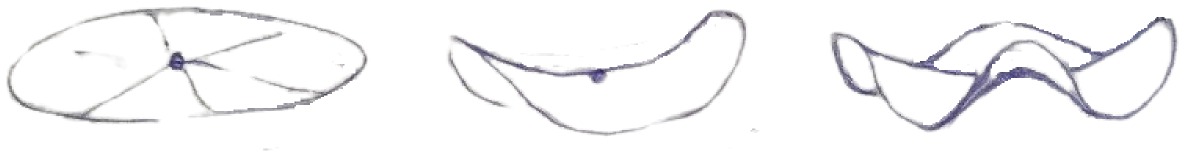


**Figura A.10:** Un disco, una sella, e una doppia sella. Le tre figure hanno lo stesso raggio, ma la lunghezza dei bordi, e la superficie, crescono all'aumentare delle creste.

Una sella con lo stesso raggio avrà una superficie più grande, perché invece di essere piatta oscilla e ci deve essere abbastanza tessuto per permetterle di farlo. Se accentuassimo la curvatura, aumentando il numero di creste e valli, o aumentando la loro profondità, l'area aumenterebbe ancora di più.
Il modo migliore per capire questa cosa è passare un pomeriggio a fare l'uncinetto: per fare un cappello si parte dal centro a fare giri di anellini, e mano mano che ci si sposta verso i bordi si devono aggiungere anelli per allargare il cappello. Se allontanandosi dal centro si aggiungono più anelli invece di un cappello viene un centrotavola, e aggiungendone ancora di più si ottengono queste forme sempre più arricciate che assomigliano ad alcuni coralli. In effetti queste forme appaiono spesso in natura, e non è un caso che moltissime piante hanno foglie curve negativamente in qualche misura. Infatti avere maggior superficie con distanze da percorrere più brevi permette di avere foglie ampie, quindi che prendono più sole, ma con i capillari dove scorre la linfa più corti, il che facilita il trasporto dei nutrienti.

**Figura A.11:** *Pod World – Plastic Fantastic*, part of the worldwide *Crochet Coral Reef* project by Margaret Wertheim and Christine Wertheim and the Institute For Figuring. (Featuring Jellyyarn coral by Kathleen Greco.) Photo courtesy $58^{th}$ Venice Biennale, by Francesco Galli.

**Stravolgersi e rivoltarsi**

*Parlo alla pelle su cui vivo*
*quel margine caro dove arriva lo stimolo,*
*il vento fresco, una carezza,*
*il brivido di adrenalina,*
*la tensione, il nervosismo, una paura.*
*Della pelle che si rivolta*
*quando lo sguardo si svolge*
*da dentro, verso dentro,*
*nella profondità enorme in cui si perde,*

*fioca luce, flebili forme.*
*La pelle che esplode quando mi lancio*
*fuori di me, non con lo sguardo, né col pensiero*
*ma me, intero, e divento nuvola, albero, straniero.*
*Ci sono mondi e mondi ad un fruscio di distanza*
*quando l'occhio balza fuori dal guscio*
*ed esce dalla stanza, non dall'uscio*
*ma dalla finestra sull'universo.*
*Dove il disco si fa sfera, io noi, e aperta la frontiera.*
*Non avanti, né indietro, né a destra né a sinistra.*
*Verso il sole. In altro.*

**Piccoli spazi sconfinati** Il modo intuitivo che abbiamo di pensare le tre dimensioni è in realtà abbastanza restrittivo. Infatti tutti gli spazi tridimensionali che ci vengono normalmente in mente sono abbastanza semplici. Un cubo, una stanza o una casa, un albero, una sfera... Cos'hanno in comune tutti questi spazi?
Certamente il fatto che li possiamo guardare da fuori: se volessimo, potremmo rappresentarli fedelmente, ad esempio farne una scultura, e tenerli *dentro* un altro spazio tridimensionale, la stanza dellə scultorə. E questa piccola ovvietà ha come conseguenza che ognuno di questi esempi ha un confine. Se ci immaginiamo un piccolo abitante di questi spazi (ad esempio: una termite per l'albero, un pesce per il mare, un punto astratto all'interno di un cubo, una mosca dentro un palloncino), ognunə di loro, muovendosi al suo interno, potrebbe finire per andare a sbattere contro una parete, contro un confine.
Eppure questo non deve succedere per forza. Con un po' di immaginazione possiamo riuscire a pensare a spazi senza confini, senza bordi, senza dover ricorrere all'universo e all'infinito. Per capire meglio facciamo un passo indietro e scendiamo di una dimensione.
Pensiamo a degli spazi bidimensionali, delle superfici. Queste possono essere il piano (illimitato e sconfinato), un foglio, o un quadrato o un disco (limitati e confinati), ma anche una sfera, o un toro, che è il nome matematico della superficie di una ciambella. Delle superfici senza bordi.
Una formica che cammini sulla superficie di una sfera o di un toro non incontrerà mai un dirupo. Non gli mancherà mai il terreno sotto i piedi.

**Sartoria, architettura, topologia** Una cosa da notare è che se disegnamo un disco o un quadrato, questi sono effettivamente un disco e un quadrato; mentre se disegnamo una sfera o un toro, questi vengono solo *rappresentati*. La sfera e il toro sono solo dei disegni; possono esistere effettivamente solo quando possiamo curvare la loro superficie al di fuori del foglio, in una terza dimensione. Tuttavia possiamo comunque provare a immaginarci questi spazi facendo ricorso a un po' di colla piuttosto che ad una dimensione in più. Una soluzione molto più terra terra, che sarà utile soprattutto quando faremo i conti con la quarta dimensione. Immaginatevi di prendere il toro e di tagliarlo con delle forbici per ottenere un tubo, e poi di tagliare il tubo per lungo fino a ottenere un rettangolo.
Questo rettangolo rappresenta tutta la superficie del toro, ma questa volta lo riusciamo a disegnare sul foglio e possiamo usarlo come mappa. Se un punto che si muove sul foglio attraversa il bordo in

alto della nostra mappa, sul toro di partenza starebbe semplicemente attraversando la prima linea tratteggiata lungo la quale abbiamo tagliato, per cui ricomparirebbe sulla mappa appena sopra il bordo inferiore. Allo stesso modo uscendo da destra si rientra a sinistra, proprio come in Pac-man, o Snake.

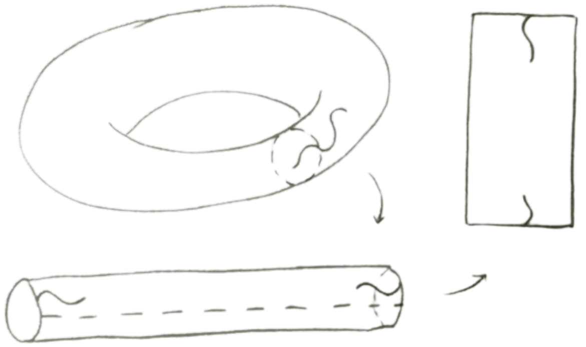


**Figura A.12:** Sartoria topologica: da toro a rettangolo. Un percorso che attraversa una linea di taglio appare agli estremi opposti della mappa.

Quello che ci ha permesso di rappresentare sul piano una superficie senza bordo che normalmente esisterebbe solo con una dimensione in più è stato quello di immaginare di incollare i lati opposti della mappa, di mettere in contatto i bordi opposti in modo che incontrare un confine non voglia dire andarci a sbattere ma attraversarlo per entrare da un'altro confine.

Questa idea ci permette di immaginare spazi tridimensionali completamente nuovi, senza confini, che non potremmo mai visualizzare con i nostri occhi, perché per realizzarli dovremmo curvarli in una quarta dimensione che non abbiamo mai visto.

Ad esempio possiamo immaginarci l'analogo tridimensionale del toro di cui abbiamo appena costruito la mappa. Al posto di un rettangolo abbiamo un cubo, potete pensare alla stanza in cui vi trovate adesso, del quale facciamo combaciare le pareti opposte. Come fossero dei portali magici, toccando la parete davanti a noi sentiremmo di poterla attraversare con la mano, quasi fosse la superficie di uno stagno, e le nostre dita spunterebbero dalla parete dietro di noi. Lo stesso succederebbe fra la parete alla nostra destra e quella alla nostra sinistra, e, arrampicandoci sulla scrivania per arrivare più in alto, fra il soffitto e il pavimento. Se le pareti fossero trasparenti potremmo voltarci per vedere la nostra mano spuntare dalla parete... e dietro di lei, noi, voltati di spalle per guardare la nostra mano spuntare dalla parete, e dietro di lei noi noi stessə, e così via a perdita d'occhio.

La stessa cosa succederebbe gettando lo sguardo in alto, e in basso. In ogni direzione la luce attraverserebbe le varie pareti per tornare a illuminare noi, quasi come se le pareti fossero tutte coperte da grandi specchi.

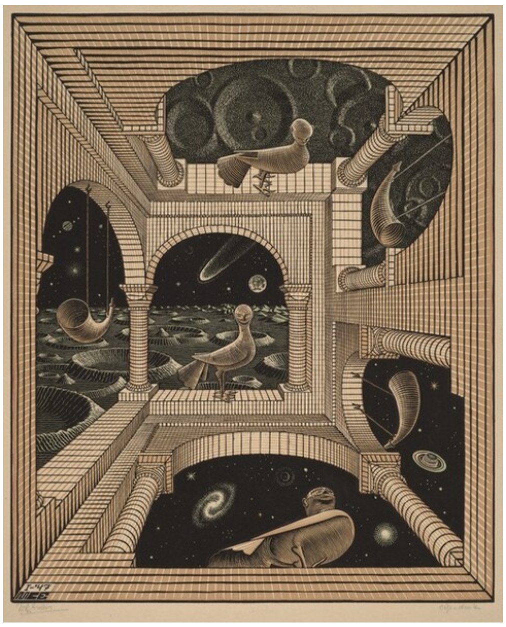


**Figura A.13:** Una varietà 3-dimensionale vista da dentro. *Another World II*, di M.C. Escher.

Tutti gli spazi con i quali ho lavorato e di cui parlo in questa tesi sono spazi (almeno) tridimensionali senza dei bordi, quindi simili al toro tridimensionale nel quale ci siamo appena rivisti. In questo caso però le architetture possono essere anche più complicate. Mi piace pensare questi spazi, che in matematica chiamiamo *varietà*, come dei castelli incantati, dove in ogni stanza corridoio e salone possiamo attraversare le pareti e finire in un altro punto del castello. Non c'è uscita, ma i percorsi e le avventure che si celano al suo interno sono infinite.

**Un paio di strumenti** Se vogliamo studiare questi spazi abbiamo bisogno di poter prendere qualche misura. In particolare a noi interessano due grandezze: il diametro, e l'entropia di volume, o complessità.

Il diametro è la più intuitiva fra le due, ed è la lunghezza del cammino che unisce i due punti dello spazio più distanti fra di loro. I due estremi di una diagonale di un quadrato, polo nord e polo sud, ecc.

Per capire cos'è l'entropia invece ci serve di guardare il castello da fuori. Immaginiamo di prendere una mappa, o una planimetria di questo luogo. Se fossimo in una stanza quadrata la nostra mappa sarebbe quadrata, e potremmo usare questa figura per piastrellare l'intero spazio piano.

Anche la mappa del nostro castello ha una forma che si incastra con se stessa come un unico pezzo che ripetendosi forma un grande puzzle. La differenza fra il nostro castello e i classici puzzle è che i nostri tasselli sono iperbolici, perché vogliamo che rappresentino fedelmente degli spazi curvi negativamente, e quindi dovremo posizionarli su uno spazio iperbolico. Possiamo farlo senza problemi, e visualizzare il risultato di questo tassellamento nella sua interezza usando il disco di Poincaré. Il risultato sono dei bellissimi motivi come quelli dei disegni di Escher.

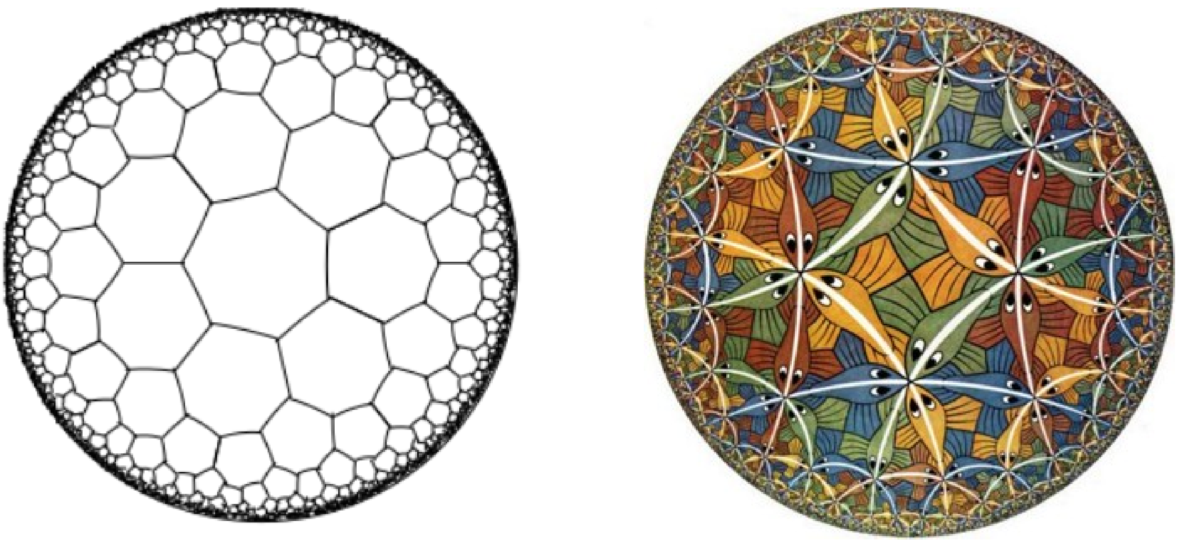


**Figura A.14:** In ordine: un tassellamento eptagonale del disco di Poincaré; *Circle Limit III*, di M.C.Escher.

Forme dei tasselli diverse danno luogo a motivi e simmetrie differenti, ma in tutti i disegni possiamo vedere che mano a mano che ci si allontana dal centro questi tasselli diventano molto numerosi molto in fretta. Sembrano moltiplicarsi molto più rapidamente di quanto facciano quelli quadrati nel piano.

Questo fenomeno è legato al fatto che, come abbiamo visto, negli spazi iperbolici c'è molto più spazio che in quelli piani; l'area dei cerchi, o il volume delle sfere, cresce molto rapidamente mano a mano che il raggio aumenta. E maggiore sarà la curvatura del nostro spazio, maggiore sarà il numero dei tasselli presenti alla stessa distanza, proprio come succedeva con gli anelli di lana nei modelli all'uncinetto.

La complessità dell'architettura del nostro castello può essere misurata come la velocità con cui cresce il numero dei tasselli mano mano che ci si allontana da quello iniziale. Quindi se la struttura della nostra varietà non è troppo complessa, il numero dei tasselli non cresce troppo rapidamente, e questo vuol dire che possiamo disegnarne le mappe su spazi che non sono troppo curvi.

**Forma e struttura** Fino ad ora abbiamo parlato di spazi astratti, della loro forma e delle loro geometrie, ma abbiamo usato, senza fare molta chiarezza sulle loro differenze, due concetti di "forma" molto diversi fra di loro.
All'inizio, cercando di capire quanti triangoli riuscivamo a immaginare e cosa ci serviva per andare oltre quelli che abbiamo conosciuto a scuola, abbiamo parlato di spazi curvi, che potevano essere sfere, tori, o selle. Poco dopo però abbiamo visto che la sella è solo un esempio di spazio curvo negativamente, e che ce ne possono essere di tante forme differenti. Inoltre abbiamo visto che possiamo rappresentare gli spazi più disparati tagliando e incollando i bordi delle loro mappe.
Per ora potrebbe non essere chiaro, ma come uno spazio si curva e come lo si ottiene incollando i bordi di una mappa sono due cose molto diverse. Sono due diversi modi di pensare alla forma di un oggetto, entrambi molto presenti in matematica, e studiare le loro interazioni nell'ambito di spazi iperbolici è stata buona parte del lavoro di questa tesi.
Quando studiamo la curvatura di uno spazio lo pensiamo come un oggetto rigido, come un vaso o una statua. La curvatura è un'informazione di tipo locale, molto circoscritta: lo tocchiamo in un punto e sappiamo come curva lì, ci spostiamo un po' e sappiamo come curva in quest'altro punto. Alla fine, come un cieco che ha sfiorato con le sue dita tutta la fredda superficie del David di Donatello, riusciamo a capire la forma di quello che abbiamo di fronte. Questo tipo di geometria si chiama geometria metrica, o Riemanniana, dal nome dell'allievo di Gauss che l'ha esplorata agli albori.
Quando studiamo la forma di uno spazio basandoci sul come incollare i bordi di una mappa invece siamo piuttosto vicini al fare il lavoro di unə sartə un po' inespertə. Non ci interessa tanto sapere come curva il tessuto nei singoli punti quanto, piuttosto, sapere le relazioni che intercorrono fra le varie parti del tessuto: quali parti sono vicine a quali altre? Dove si crea una manica? Dove c'è un'apertura? Non ci interessa davvero sapere che stile o misura avranno i pantaloni, quanto che siano dei pantaloni. Questo tipo di studio della geometria possiamo chiamarlo topologia.
Questa osservazione, che tutti i pantaloni hanno più o meno la stessa struttura, ci porta a chiederci che relazione ci sia fra questi due modi diversi di guardare alla geometria. Ad esempio il piano iperbolico e il piano euclideo hanno la stessa topologia, ma metriche differenti. In generale, che relazione c'è fra metrica e topologia?
Spesso, quando litighiamo con una persona che ci è cara non ci accorgiamo che al di là delle paure e delle incomprensioni stiamo dicendo la stessa cosa, e se solo fossimo capaci di cambiare punto di vista ci capiremmo molto meglio.
Allo stesso modo è possibile che due spazi appaiano molto simili quando guardiamo la loro forma dal punto di vista della metrica, ma molto diversi quando guardiamo la loro struttura, dal punto di vista topologico. E viceversa.
Per capirci; due spazi hanno una forma simile se possiamo più o meno sovrapporli e all'incirca li vediamo coincidere, se la loro figura è simile.
Invece due spazi hanno la stessa struttura se le istruzioni per montarli sono simili. Ad esempio se hanno lo stesso numero di gambe, di buchi, maniche, ecc). Quando dico struttura penso un po' all'architettura di un edificio al di là dell'arredamento, o all'anatomia di un animale al di là di pelo e piume.

**Figura A.15:** Un gabbiano reale zampegialle e una volpe volante malese in volo.

Vista così siamo molto familiari con questi concetti! Un gabbiano e un pipistrello hanno forme simili che gli permettono di volare, ma strutture molto diverse avendo l'anatomia di un mammifero l'uno e di un uccello l'altro. D'altra parte un pipistrello e una foca hanno forme decisamente diverse, ma condividono l'anatomia di un mammifero, hanno strutture analoghe. Vi vengono in mente altri esempi?

**Libertà, complessità, rigidità** In matematica ce ne sono molti! Prendiamo tre copie di uno stesso spazio, tipo il pipistrello.. Una di queste la possiamo deformare, stropicciare, curvare liberamente, fino ad ottenere uno spazio con la stessa struttura di quello di partenza ma con una forma che non c'entra niente, come la foca. D'altra parte possiamo prendere un'altra copia e non fargli quasi nulla, aggiungergli un occhiello, o una piccolissima manica: in questo modo otteniamo uno spazio che assomiglia moltissimo a quello di partenza dal punto di vista della metrica, ma la cui struttura è distinta, quindi con una topologia differente.
Sembrerebbe che non ci sia scampo, e che in ogni situazione ci sia la possibilità di fraintendere la forma o la struttura di uno spazio alla sua minima variazione. Eppure non è sempre questo il caso.

**Figura A.16:** Per quanto si provi a deformare la metrica di un toro per renderlo simile ad una sfera, la loro struttura sarà sempre diversa.

Nella mia tesi ho esplorato il forte legame che esiste fra forma e struttura quando gli spazi in considerazione sono iperbolici, e hanno un diametro e una complessità che non è troppo grande. In particolare uno dei risultati principali è stato dimostrare che se uno spazio iperbolico ed un altro spazio hanno forma è simile, e il loro diametro e la loro entropia non è troppo grande, allora la loro struttura è esattamente la stessa.
Questo risultato non è un'idea completamente nuova, e si posiziona all'interno di un ricco filone della

geometria in cui si studia come forma e struttura si parlano, e si influenzano a vicenda. In particolare ci dice che possiamo deformare la forma del secondo spazio in modo da renderlo iperbolico.
E quest'ultima è una bella sorpresa! Infatti non è sempre detto che uno spazio si possa deformare in modo da fargli avere la curvatura che più ci piace, e questa ostruzione è legata alla sua struttura. Ad esempio, una sfera è curva positivamente in ogni punto. Invece un toro ha dei punti curvi positivamente e altri curvi negativamente; ma ha una struttura diversa dalla sfera, e questo ci impedisce di deformarlo in modo da renderlo tondeggiante in ogni punto.

**Ma a cosa serve tutto questo?** Ecco una domanda che a un certo punto di questo racconto mi sento sempre fare. Penso che leggendo queste righe, fra i tanti esempi che abbiamo incontrato, si possano trovare mille applicazioni con la loro utilità, che certamente hanno stimolato il mio entusiasmo e la mia fantasia. Però non vorrei che fosse questa la mia risposta.
Negli anni '70, in concomitanza al suicidio di un suo collega e alla minaccia nucleare della guerra fredda, il grande geometra algebrico Alexander Grothendieck si chiedeva di fronte ad una platea che lo ascoltava al CERN se avremmo dovuto proseguire la ricerca scientifica. Amareggiato dalle ingerenze militari nella ricerca, e prendendo atto di un distacco sempre maggiore fra il lavoro della comunità scientifica e il bene comune, dubitava che studiare la matematica che amava fosse un buon modo di spendere il suo tempo.
In questi quattro anni di dottorato mi sono posto spesso la stessa domanda, e ho sperimentato in prima persona gli stessi dubbi. In questo tempo in cui le garanzie democratiche e i diritti umani stanno venendo sgretolati, in Italia come nel resto del mondo; con l'ENI, la compagnia petrolifera di stato, che finanzia per milioni la Sapienza, la mia università, con progetti che consolidano l'utilizzo dei combustibili fossili, mentre dovremmo disperatamente abbandonarli; sono sicuro che la ricerca scientifica non abbia bisogno di me.
Eppure in questi anni sono anche stato in contatto con il lato umano, emotivo, innamorato di chi vive la matematica con carta e penna in mano. Un lato solitamente ignorato dal grande pubblico, ma spesso dimenticato sotto strati di burocrazie, gerarchie, e paure, anche fra le mura del dipartimento. Ed ho capito che non c'è contraddizione nell'essere parte della comunità matematica e curarsi attivamente della società, mettere questa cura al primo posto. Penso che possiamo, e dovremmo, essere responsabili dei problemi più grandi anche come matematici. Pure perché abbiamo a disposizione strumenti e ricchezze molto difficili da trovare altrove.
Per cui se mi domando a cosa è servito tutto questo, e a cosa può servire, ripenso a tutti gli esempi e mi viene in mente che non sono importanti *quegli* esempi, quanto cosa li renda *degli* esempi. Cosa li accomuna. Capire questo è stato un viaggio alla scoperta di mondi nuovi, accessibili con un atto di fantasia e di ascolto. Forse il regalo più grande che la matematica mi ha fatto è stato insegnarmi, come una maestra, a cercare cosa accomuna le cose, le persone, gli animali, le piante; a guardare le connessioni fra di loro, e cosa accomuna loro a me. E' un dono prezioso in dei tempi che sembrano aver dimenticato la comunità, e che invece hanno tanto bisogno di empatia e collaborazione, fra esseri umani in guerra, e con le specie viventi e minerali che ci circondano, e insieme alle quali abitiamo questo pianeta.

*For any comment please write me at andrea.drago@uniroma1.it*