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Modern aspects of convexity and the interplay between geometry and analysis

by

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Introduction

One does not simply sum up years of research and training in one manuscript. Especially if the research and the training concerns one of the most prolific, yet mysterious, branches of modern Mathematics (at least in the short experience of the author).

The word "Convexity" in the title is not supposed to be exhaustive, nor precise. On the contrary, we tried to keep an indecisiveness which reflects the multitude of aspects in which one can encounter this topic. Strictly speaking, a convex set in a space (whenever is possible to define geodesic lines) is characterized by the following property: It contains all the shortest geodesic segments between every couple of points belonging to the set. Still, it is good practice to introduce an operative definition of convexity, not in what it is, but on terms of what it does (even though informally). We (the author) like to think that an object is convex when, through some kind of dark magic, it is still possible to perform differential calculus with it: Seeking maxima and minima, studying extremal properties, investigating geometric phenomena that usually concerns only differential geometry. The list goes on, and so does the amount of applications of this field.

The first question historically connected with convexity is probably the Isoperimetric Problem: That is, which curve in the plane minimizes its perimeter, with the constraint of enclosing a fixed area. This problem, as well as the answer (the circle!), was already well known by Hellenic mathematicians. If we move from the plane to the three dimensional space, solving the same problem explains why a soap bubble is round. These are simple instances, but, for example, contact surfaces between soap bubbles and rigid structures to encase them are nowadays a fertile source of questions and examples stemming new research. In general, studying the surfaces (as well as other geometric characteristics) of classes of objects is a meaningful and interesting task, and we try to follow in this tradition.

Since modern problems require modern solutions, it is a matter of fact that a Mathematician (as every other scientist) cannot rely anymore only on the tools provided by its native field. New instruments are realized on the border line between different areas, and Convexity is certainly a branch that understood this lesson a long time ago. We focus in particular (as the title suggests) on the interplay between

Geometry and Analysis. As it is fundamental and fruitful in the Physics of small particles to understand the double nature of these objects as waves and physical matter, when treating a convex object one must never forget its double nature of geometric and at the same time analytic entity (algebraic and combinatorial aspects arise too, the specialists will pardon us for the narrow treatment we can provide). Such double nature is at the core of our work.

This thesis builds on many of the ingredients that lead to the solution of problems like the Isoperimetric one. For example, one of the classic approaches is study how a surface behaves under appropriate perturbations, and then use this information to determine whether or not an adequate solution has been found. This is known as *Variational Approach*, and explains why it is interesting and important to evaluate, in practice, these variations. Such task, in the specific instance of the world of convex functions, is the aim of Chapter 1. Already from this chapter, it is possible to understand the meaning of the second part of the title. Indeed, our strategy does not follow the usual route of employing hardcore calculations in order to study complex problems related to functions (as one does). Without renouncing to the hardcore part, we study this problem from a geometric point of view, interpreting the geometric nature of functions.

In Chapter 2, we dig deeper into the connection with geometry, entering the world of the *Theory of Valuations*. This topic saw the light of the world more than a century ago, and builds on the following question: What are the properties that make something a measure? Indeed, it is meaningful to understand the intrinsic nature of objects like volume, or surface area, if one wants to tackle problems concerning these and other quantities. Without entering on the technicalities of what is the suitable definition of a measure (but we do not mean the classical notion here), this field anticipated on many regards what is nowadays known as *Geometric Measure Theory*, and it is with it deeply intertwined. One of the most interesting aspects is that the tools provided by this theory are specifically aimed to study and classify specific functionals (called valuations) starting from a bunch of properties. In this chapter we provide an overview of the topic, with a particular emphasis on the modern developments concerning valuations on spaces of functions. In particular, spaces of convex functions. Again, we will show how Geometry and Analysis play together, spacing from geometric constructions to instruments of functional analysis. The main idea, throughout this treatment, is to show the similarities between this modern theory and the instruments which built the classical one.

This work closes with Chapter 3, where we talk about *Symmetrizations*. As the name suggests, the idea is to work on objects making them more symmetric, with one catch: Some geometric properties must be preserved in the process. This kind of instrument allowed one of the first formal solutions of the Isoperimetric Problem, and has since then been a fruitful source of proofs for many of its variations and extensions. For example, symmetrization techniques can be employed to study functionals of the type appearing in Chapters 1 and 2. Nonetheless, we focus on another aspect of this field. As one does in the Theory of Valuations, in the recent years there has been a successful attempt to study symmetrizations starting only from some fundamental properties. Surprisingly, these properties are sufficient to identify many fundamental behaviors and classify them, shedding a light on an instrument which is as old as mysterious. We do not use the latter adjective lightly. The reader will find in the last section of this chapter many examples showing some pathological behaviors that we have not been able to explain yet, even though they can be easily formulated.

Regarding the structure of this work, the experienced reader might read the three chapters in the order they prefer. Chapter 1 starts with a series of preliminaries and notations that will be kept during the whole manuscript. The other two chapters are provided with a section introducing the further necessary background, and can be read independently, provided that one goes through said initial preliminaries contained in Chapter 1. The main content of all three is made of original works of the author and collaborators, which we hope can provide a fresh perspective on this field. We tried to keep the exposition as self contained as possible. Where this was not feasible, we have provided suitable and extensive references.

List of symbols

$\lvert \cdot \rvert$	Euclidean norm
$\lVert\cdot\rVert_\infty$	sup-norm
$[\mathbb{A}]_{n-i}$	$(n-i)$-th elementary symmetric function of the eigenvalues of a symmetric matrix $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^n$
λ_n	n-dimensional Lebesgue measure
B^n	Euclidean unit ball in $\mathbb{R}^n$
$B^n_r(x)$	Euclidean ball with center at x and radius r in $\mathbb{R}^n$
$\mathbb{S}^{n-1}$	unit sphere in $\mathbb{R}^n$
$\mathbb{S}^{n-1}_-$	open lower half-sphere in $\mathbb{R}^n$
λ_n	n-dimensional Lebesgue measure
$\mathcal{H}^n$	n-dimensional Hausdorff measure
$x \cdot y$	standard scalar product
∂	topological boundary
pr_H	projection from $\mathbb{R}^{n+1}$ to a copy of $\mathbb{R}^n$ identified as a fixed hyperplane H
$\lVert\cdot\rVert_p$	L^p norm
$\wedge$	pointwise maximum
$\vee$	pointwise minimum
$\mathscr{K}^n$	non-empty compact convex subsets of $\mathbb{R}^n$
$\mathscr{K}^n_n$	elements of $\mathscr{K}^n$ with nonempty interior
$\mathcal{C}^n$	compact subsets of $\mathbb{R}^n$
$\mathcal{P}^n$	polytopes of $\mathbb{R}^n$
$\mathcal{C}^n_n$	elements of $\mathcal{C}^n$ with nonempty interior
h_K	support function of the convex set K
I_K	indicator function of the convex set K
N_K	generalized Gauss map of K
$\mathrm{Nor}K$	normal bundle of K
τ_K	reverse spherical image of K
$S_{n-1}(K,\cdot)$	surface area measure of the convex body K
$\Theta_i(K,\cdot)$	i-th support measure of K
$\mathrm{epi}(u)$	epigraph of the function u
$\mathrm{dom}(u)$	domain of the function u
D^2	Hessian matrix of the function u
∂u	subgradient of the function u
u^*	Fenchel-Legendre transform of the function u
$u \square v$	infimal convolution of the functions u and v
$t \cdot u$	epi-multiplication of the function u by a factor t
∂K_-	lower boundary of the convex body K
$\lfloor K \rfloor$	function representing the lower boundary of the convex body K
$\lceil K \rceil$	function representing the upper boundary of the convex body K
g	gnomonic projection from $\mathbb{S}^n_-$ to $\mathbb{R}^n$

$\mathrm{Conv}(\mathbb{R}^n)$	convex and lower semi-continuous convex functions
$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$	super-coercive functions in $\mathrm{Conv}(\mathbb{R}^n)$
$\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$	finite-valued convex functions
$\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$	functions in $\mathrm{Conv}(\mathbb{R}^n)$ with compact domain
ρ_ζ	recession function of the function ζ
$C_{rec}(\mathbb{R}^n)$	continuous functions with bounded and continuous recession function
$[f]$	Wulff shape of the function f
$F_t K$	Wulff shape of the function $h_K + tf$
$\mathrm{supp}\ Z$	support of the valuation Z
S_H	Schwarz symmetrization with respect to the subspace H
M_H	Minkowski symmetrization with respect to the subspace H
F_H	Fiber symmetrization with respect to the subspace H
$\diamondsuit_H$	H-symmetrization

If in doubt, Meriadoc, always follow your nose.

Gandalf the Grey

Chapter 1

A general point of view

1.1 Preliminaries

The ambient space where we work is the Euclidean space $\mathbb{R}^n, n \in \mathbb{N}$ (at times, we switch the point of view to $\mathbb{R}^{n+1}$). We start by summarizing some of the main results from the classical theory of convex bodies and convex functions.

Definition 1.1. A set $K \subset \mathbb{R}^n$ is *convex* if for every $x, y \in K, t \in [0,1]$,

$$(1-t)x + ty \in K.$$

Of particular interest is the family of non-empty compact convex sets of $\mathbb{R}^n$, denoted by $\mathscr{K}^n$. The theory concerning these sets is nowadays well established. For an exhaustive exposition, see, for example, the books of Gruber [Gru07], Hadwiger [Had57], Hug and Weil [HW20], and Schneider [Sch14]. The latter, in particular, is the main source for these preliminaries.

1.1.1 Convex bodies

Topological properties of $\mathscr{K}^n$. We denote by $\mathcal{C}^n$ the family of compact subsets of $\mathbb{R}^n$. Clearly, $\mathscr{K}^n$ is a subfamily of $\mathcal{C}^n$. Two further subfamilies we consider are

$$\mathcal{C}^n_n \coloneqq \{K \in \mathcal{C}^n \,:\, K \text{ has non-empty interior}\},$$

and $\mathscr{K}^n_n := \mathscr{K}^n \cap \mathcal{C}^n_n$. The elements in the latter family are called *convex bodies*.

On $\mathcal{C}^n$, we consider the topology of the Hausdorff metric. The corresponding distance, for $K, L \in \mathcal{C}^n$, is given by

$$d_{\mathcal{H}}(K, L) \coloneqq \max\{\sup_{x\in L} d(x, K), \sup_{y \in K} d(y, L)\},$$

where $d(x, K) \coloneqq \inf_{z\in K} |x - z|$. We summarize in the following statement the main properties of $\mathcal{C}^n$ when endowed with this metric (see [Sch14, Theorem 1.8.3-1.8.7]).

Theorem 1.2 (Blaschke's selection Theorem)**.** *The space $\mathcal{C}^n$ endowed with the Hausdorff metric is a complete metric space.*
The subspace $\mathscr{K}^n$ is closed in $\mathcal{C}^n$ and therefore is a complete metric space as well.

In these spaces, every bounded subset is compact, and thus every bounded sequence admits a converging subsequence.

When the boundary ∂K of a set $K \in \mathscr{K}^n_n$ is of class C^2_+, that is, the principal curvatures of ∂K as a manifold are strictly positive, we say that K is C^2_+. We have the following useful fact (for example, see [Sch14, Theorem 2.7.1]).

Proposition 1.3. *The set of convex bodies of $\mathbb{R}^n$ of class C^2_+ is dense in $\mathscr{K}^n$ with respect to the Hausdorff metric.*

Minkowski addition.

Definition 1.4. Given two sets $A, B \subset \mathbb{R}^n$, their *Minkowski sum* is the set

$$A + B := \{x + y : x \in A, y \in B\}.$$

The corresponding operation is called *Minkowski addition.*

This operation is closed in $\mathcal{C}^n$ and $\mathscr{K}^n$. The same is not true for measurable sets. See Sierpinski [Sie20].

For a set $A \subset \mathbb{R}^n$ we can define its *convex hull*

$$\mathrm{conv}(A) := \bigcap_{K \subset \mathbb{R}^n \text{ convex},\ A \subset K} K.$$

Even though we will not use it, an important subclass of $\mathscr{K}^n$ is the family $\mathcal{P}^n$ of *polytopes*, that is, the subsets of $\mathbb{R}^n$ obtained as convex hulls of finitely many points. The *diameter* is defined as

$$\mathrm{diam}(A) := \sup_{x,y \in A} |x - y|.$$

An interesting property of Minkowski addition is the following regularizing effect. This result can be found, for example, in [Sch14, Theorem 3.1.6].

Theorem 1.5 (Shapley, Folkman, and Starr)**.** *Let $A_1, \dots, A_k \in \mathcal{C}^n$, and suppose that $x \in conv(A_1 + \cdots + A_k)$. Then there exists a point $a \in A_1 + \cdots + A_k$ such that*

$$|x - a| \leq \sqrt{n} \max_{1 \leq i \leq k} diam(A_i),$$

hence

$$d_{\mathcal{H}} \left(\sum_{i=1}^{k} A_i, conv \left(\sum_{i=1}^{k} A_i \right) \right) \leq \sqrt{n} \max_{1 \leq i \leq k} diam(A_i).$$

Theorem 1.5 will be crucial in Section 3.2.

Denote by V_n the *volume* corresponding to the standard n-dimensional Lebesgue measure. A wide portion of the literature in convex geometry is concerned with estimates of the volume of Minkowski sums. A milestone on this topic is the *Brunn-Minkowski inequality* which reads as follows.

Theorem 1.6 (Brunn-Minkowski inequality)**.** *Let $A, B \subset \mathbb{R}^n$ be bounded measurable sets such that $A + B$ is measurable. Then*

$$V_n(A+B)^{1/n} \geq V_n(A)^{1/n} + V_n(B)^{1/n}. \tag{1.1.1}$$

If $V_n(A)V_n(B) > 0$, equality holds if and only if A and B are homothetic convex sets up to removing negligible subsets. If $A, B \in \mathscr{K}^n$, equality can be alternatively achieved if both A and B lie on parallel hyperplanes.

For a detailed survey on this inequality, its connections to convex geometry, and its generalizations, see the survey from Gardner [Gar02].

Support functions. To every $K \in \mathscr{K}^n$ we can associate a function to $\mathbb{R}^n$, called *support function*, defined as

$$h_K(x) \coloneqq \sup\{x \cdot y : y \in K\}.$$

This function is positively homogeneous of degree 1, that is, $h_K(tx) = th_K(x)$ for every $t \geq 0, x \in \mathbb{R}^n$ and $K \in \mathscr{K}^n$. Thus, its restriction on $\mathbb{S}^{n-1}$ determines it. Moreover, for every $x, y \in \mathbb{R}^n$

$$h_K(x+y) \leq h_K(x) + h_K(y),$$

and thus support functions are *sublinear*. This property completely characterizes them (see [Sch14, Theorem 1.7.1]).

Theorem 1.7. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there exists a unique convex set $K \in \mathscr{K}^n$ with $h_K = f$.*

An explicit connection between support functions and convex sets is the following: For $\xi \in \mathbb{S}^{n-1}$, $h_K(\xi)$ gives the signed distance from the origin of the (unique) hyperplane tangent to K and orthogonal to ξ, such that ξ is an outer normal vector of K at the contact point. The existence and uniqueness of supporting hyperplanes for convex sets (in general Banach spaces!) is a crucial topic in functional analysis. See, for example, the book from Brezis [Bre11].

Support functions behave nicely with respect to Minkowski addition. Indeed, for every $K, L \in \mathscr{K}^n$, one has

$$h_K + h_L = h_{K+L}.$$

Mixed volumes. The interaction between volume and Minkowski addition holds further consequences, as the following theorem shows ([Sch14, Theorem 5.1.7])

Theorem 1.8. *There is a non-negative symmetric function $V \colon (\mathscr{K}^n)^n \to \mathbb{R}$, called mixed volume, such that for $m \in \mathbb{N}$,*

$$V_n(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1,\ldots,i_n=1}^{m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, \ldots, K_{i_n}) \tag{1.1.2}$$

for arbitrary convex compact sets $K_1, \ldots, K_m$ and $t_1, \ldots, t_m \geq 0$.

Consider for $K \in \mathscr{K}^n$ and $t \geq 0$ the particular case $K + tB^n$, known as *parallel set* of K. Then (1.1.2) is known as *Steiner Formula*, and states that

$$V_n(K + tB^n) = \sum_{i=0}^{n} t^{n-i} \kappa_{n-i} V_i(K), \tag{1.1.3}$$

where κ_{n-i} is the volume of the unit ball in $\mathbb{R}^{n-i}$. The functionals $V_i : \mathscr{K}^n \to \mathbb{R}$ are called *intrinsic volumes*. Notice that V_n corresponds to the volume itself. Other notable cases are V_0, which corresponds to the Euler characteristic, $2\kappa_{n-1}/n\kappa_n V_1$, known as *mean width*, and $2V_{n-1}$, which is the *surface area*. The latter can be defined a priori by the limit

$$\lim_{t \to 0^+} \frac{V_n(K + tB^n) - V_n(K)}{t},$$

and corresponds to the $(n-1)$-dimensional Hausdorff measure of the boundary of K, $\mathcal{H}^{n-1}(\partial K)$.

Surface area and boundary structure. Let $K \in \mathscr{K}^n$ and consider its boundary ∂K. When K is a C^2_+ body, the *Gauss map*

$$N_K : \partial K \to \mathbb{S}^{n-1}$$

that for each $x \in \partial K$ gives the unit normal vector to ∂K at x is well defined and bijective. When the convex set K is clear from the context, we will omit it and write N instead of N_K.

For every $K \in \mathscr{K}^n$, the map N_K is defined $\mathcal{H}^{n-1}$-almost everywhere on ∂K. In the points x where this is not single-valued, we consider $N_K(x)$ as the *unit normal cone* of K at x, or *generalized Gauss map*. This is the set of all the unit vectors ξ such that K has a tangent hyperplane at x with outer normal ξ. Conversely, we can define the *reverse spherical image* of K

$$\tau_K : \mathbb{S}^{n-1} \to \partial K,$$

which pairs every vector $\xi \in \mathbb{S}^{n-1}$ to the set of points $x \in \partial K$ such that $\xi^\perp + x$ is a supporting plane of K at x with ξ as outer normal vector. Then, the *surface area measure* of K is defined as

$$S_{n-1}(K, B) = \mathcal{H}^{n-1}(\tau_K(B)) \tag{1.1.4}$$

for every Borel set $B \subset \mathbb{S}^{n-1}$.

When K is of class C^2_+, the measure $S_{n-1}(K, \cdot)$ is absolutely continuous with respect to the Hausdorff measure on $\mathbb{S}^{n-1}$ and its density at $\xi \in \mathbb{S}^{n-1}$ is the product of the radii of curvature at the point of $x \in \partial K$ such that $N_K(x) = \xi$.

The surface area measures are finite Borel measures on $\mathbb{S}^{n-1}$, and they are weakly continuous (see [Sch14, Section 4.2]), meaning that if a sequence of convex compact sets $K_m \in \mathscr{K}^n$ converges to $K \in \mathscr{K}^n$, then, for every $f \in C(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} f(\xi)\, dS_{n-1}(K_j, \xi) \to \int_{\mathbb{S}^{n-1}} f(\xi)\, dS_{n-1}(K, \xi).$$

The determination of a convex set from its surface area measure is a problem known as *Minkowski problem*. Its solution is classical (see, for example, [Sch14, Theorem 8.2.2]), and reads as follows.

Theorem 1.9. *Let σ be a Borel measure on the sphere $\mathbb{S}^{n-1}$ with the properties*

$$\int_{\mathbb{S}^{n-1}} \xi \, d\sigma(\xi) = 0$$

and $\sigma(s) < \sigma(\mathbb{S}^{n-1})$ for each great subsphere s of $\mathbb{S}^{n-1}$. Then there is a convex body $K \in \mathscr{K}_n^n$ for which $S_{n-1}(K, \cdot) = \sigma$.

1.1.2 Convex functions

Definition 1.10. A function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if its epigraph

$$\operatorname{epi}(u) := \{(x,t) \in \mathbb{R}^{n+1} : t \geq u(x)\}$$

is a convex subset of $\mathbb{R}^{n+1}$. Equivalently, if $u \not\equiv +\infty$ and $u \not\equiv -\infty$, it satisfies the condition

$$u((1-t)x + ty) \leq (1-t)u(x) + tu(y)$$

for every $x, y \in \mathbb{R}^n, t \in [0,1]$.

Examples of convex functions are support functions.

The natural space to consider in this setting is

$$\operatorname{Conv}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \text{ s.t. } u \text{ is convex, lower semi-continuous, } u \not\equiv +\infty\}.$$

The *domain* of a convex function is the set

$$\operatorname{dom}(u) := \{x \in \mathbb{R}^n : u(x) < +\infty\}.$$

Notice that a convex function is always (and only) continuous in its domain. Later, we introduce many subspaces of $\operatorname{Conv}(\mathbb{R}^n)$ in order to obtain specialized results.

Topological properties of $\operatorname{Conv}(\mathbb{R}^n)$**.** On the space $\operatorname{Conv}(\mathbb{R}^n)$ we consider the topology of *epi-convergence*, characterized as follows: A sequence of functions $u_j \in \operatorname{Conv}(\mathbb{R}^n)$ *epi-converges* to $u \in \operatorname{Conv}(\mathbb{R}^n)$ if for every $x \in \mathbb{R}^n$, the following conditions are satisfied.

- For every sequence of points $x_j \in \mathbb{R}^n$ converging to $x \in \mathbb{R}^n$, $u(x) \leq \liminf_{j\to\infty} u_j(x_j)$.
- There exists a sequence (x_j) converging to x, such that $u(x) = \lim_{j\to\infty} u_j(x_j)$.

We are particularly interested in the subfamily

$$\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \{u \in \operatorname{Conv}(\mathbb{R}^n) : \lim_{|x|\to\infty} \frac{u(x)}{|x|} = +\infty\}$$

of *convex super-coercive functions.* Here, a more intuitive characterization is given by convergence of level sets. For $u \in \operatorname{Conv}(\mathbb{R}^n), t \in \mathbb{R} \cup \{+\infty\}$ consider the *sublevel set*

$$\{u \leq t\} := \{x \in \mathbb{R}^n : u(x) \leq t\}.$$

For a sequence of functions $u_j \in \operatorname{Conv}(\mathbb{R}^n)$ we use the convention $\{u_j \leq t\} \to \emptyset$ if there exists $j_0 \in \mathbb{N}$ such that $\{u_j \leq t\} = \emptyset$ for every $j \geq j_0$.

Lemma 1.11 ([CLM20b, Lemma 10])**.** *A sequence of functions* $(u_j) \subset \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *epi-converges to* $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *if and only if* $\{u_k \le t\} \to \{u \le t\}$ *in the Hausdorff metric for every* $t \in \mathbb{R}$ *with* $t \neq \min_{x\in\mathbb{R}^n} u(x)$*.*

We provide a further lemma concerning the level sets of coercive convex functions. This fact can be considered folklore, but since we could not find a suitable source, we provide a proof for the convenience of the reader.

Lemma 1.12. *For every coercive* $u \in \mathrm{Conv}(\mathbb{R}^n)$ *the family of level sets* $\{u \le t\}$ *is continuous in* $t \in \mathbb{R}$ *for every* $t \neq \min_{x\in\mathbb{R}^n} u(x)$ *with respect to the Hausdorff metric.*

Proof. First, notice that the hypothesis of coercivity is necessary in order for the statement to make sense. Indeed, it is a classical fact that a convex function is coercive if and only if all its level sets are compact.

If $t < \min_{x\in\mathbb{R}^n} u(x)$ then $\{u \le t\} = \emptyset$ for every such t, and there is nothing to prove. Suppose instead that $t > \min_{x\in\mathbb{R}^n} u(x)$ and the family $\{u \le t\}$ is not continuous at some t_0. Then, we can find a sequence $t_m, m \in \mathbb{N}$, converging to t_0 such that $\{u \le t_m\}$ does not converge to $\{u \le t_0\}$ with respect to the Hausdorff metric. Therefore, there exists some $\alpha > 0$ fixed and independent of m such that we have at least one of the two following scenarios: We can find a sequence of points $x_m \in \{u \le t_m\}$ such that $d(x_m, \{u \le t_0\}) > \alpha$, or there exists $x_0 \in \{u \le t_0\}$ such that $d(x_0, \{u \le t_m\}) > \alpha$ for every $m \in \mathbb{N}$.

Suppose the latter case is true. Choose $t > \min_{x\in\mathbb{R}^n} u(x)$. Then since u is convex we have that for every $y \in \{u \le t\}$ the segment between (y, t) and (x_0, t_0) is completely included in $\mathrm{epi}(u)$. But then this segment crosses all the level sets between t and t_0, that is $d(x_0, \{u \le t\}) \to 0$ as $t \to t_0$. As t can be arbitrarily chosen, we have a contradiction and therefore this scenario is not possible.

Consider then the remaining case. By definition, $\{u \le t_m\} \subseteq \{u \le \sup_{m\in\mathbb{N}} t_m\}$ for every $m \in \mathbb{N}$. Thus,

$$(x_m, t_m) \in [\min_{x\in\mathbb{R}^n} u(x), \max_{m\in\mathbb{N}} t_m] \times \{u \le \max_{m\in\mathbb{N}} t_m\}$$

for every $m \in \mathbb{N}$, and this set is compact. Therefore, we can find a subsequence of (x_m, t_m) converging to some (x, t_0) and since $\mathrm{epi}(u)$ is closed, $(x, t_0) \in \mathrm{epi}(u)$. In particular, $x \in \{u \le t_0\}$. By construction, $d(x_m, \{u \le t_0\}) > \alpha$ for every $m \in \mathbb{N}$, and by the continuity of the Euclidean distance $d(x, \{u \le t_0\}) > \alpha$, which is a contradiction. The proof is therefore concluded. □

The Fenchel-Legendre transform. On $\mathrm{Conv}(\mathbb{R}^n)$, we consider the following transform, known as *Fenchel-Legendre transform.* For $u \in \mathrm{Conv}(\mathbb{R}^n)$, it is defined as

$$u^*(x) := \sup_{y\in\mathbb{R}^n} \{x \cdot y - u(y)\}.$$

Note that since it is the supremum of affine (and thus convex) functions, u^* is a convex function. Moreover, for a non-convex function $f : \mathbb{R}^n \to \mathbb{R}$, it still makes sense to define f^* as above if we admit the trivial case $f \equiv +\infty$. Still, when $u \in \mathrm{Conv}(\mathbb{R}^n)$ one has the important fact

$$(u^*)^* = u.$$

When $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, the supremum is now a maximum, and if $\bar{y}$ is the point where the maximum is achieved, i.e. $u^*(x) = x \cdot \bar{y} - u(\bar{y})$, a quick calculation shows that $x = \nabla u(\bar{y})$. This behavior can be generalized without smoothness assumptions: What this transform does, in practice, is creating a correspondence between the *subgradient*

$$\partial u(x) \coloneqq \{p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot x \text{ for every } y \in \mathbb{R}^n\}$$

and the points p of the domain of u^* such that $x \in \partial u^*(p)$, and vice-versa. By Theorem [Sch14, Theorem 1.5.3], if u is convex then it is lipschitz on compact subsets of the interior of $\mathrm{dom}(u)$, and by Rademacher's theorem (see, for example [Mag12, Section 7.3]) the gradient of u exists almost everywhere in $\mathrm{dom}(u)$. That is, for almost every $x \in \mathrm{dom}(u)$ we have that $\partial u(x) = \nabla u(x)$.

For our purposes, note that if we consider the family of *real-valued convex functions*

$$\mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \coloneqq \{u \in \mathrm{Conv}(\mathbb{R}^n) : u < +\infty\},$$

one has that

$$\begin{aligned} \mathcal{L} \colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) &\to \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \\ u &\mapsto u^* \end{aligned} \tag{1.1.5}$$

is a homeomorphism (see, for example, [RW98, Theorem 11.8]). In analogy with Lemma 1.11, the epi-convergence on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ can be characterized as follows (see, for example, [RW98, Theorem 7.17]).

Lemma 1.13. *A sequence $(u_j) \subset \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ if and only if $u_j \to u$ uniformly on compact subsets of $\mathbb{R}^n$.*

In the families $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, the transform of $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ behaves as the support function of the epigraph of u (see (1.1.13) later). The opposite is true as well with suitable precautions.

In analogy with the Minkowski addition, on $\mathrm{Conv}(\mathbb{R}^n)$, we can consider two operations: Pointwise addition and *infimal convolution.* For $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ the latter is defined as

$$u \square v(x) \coloneqq \inf\{u(y) + v(z) : z + y = x\},$$

and $u \square v \in \mathrm{Conv}(\mathbb{R}^n)$. For $u \in \mathrm{Conv}(\mathbb{R}^n), t \geq 0$, instead of the dilation, we have two corresponding notions: The classical scalar multiplication and the epi-multiplication

$$(t \cdot u)(x) \coloneqq tu\left(\frac{x}{t}\right)$$

for $t > 0$, $0 \cdot u = I_{\{0\}}$. It is easy to prove the relation

$$((t \cdot u) \square (s \cdot v))^* = tu^* + sv^* \tag{1.1.6}$$

for $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ and $s, t \geq 0$. Notice that for $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ and $t > 0$, $\mathrm{epi}(u \square v)$ is the Minkowski sum $\mathrm{epi}(u) + \mathrm{epi}(v)$, while $\mathrm{epi}(t \cdot v)$ is the dilation by a factor t of $\mathrm{epi}(v)$.

Another important relationship is the one between support functions and indicator functions of convex compact sets. Indeed, if $K \in \mathscr{K}^n$ and K contains the origin, a quick calculation shows that

$$I_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed $(I_K)^* = h_K$.

1.1.3 The space $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$

In this section, we present some tools and introduce the main strategy for many results in this work. In particular, we need to switch often the point of view between $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$. To do so, we consider on $\mathbb{R}^{n+1}$ the standard basis $\{e_1, \ldots, e_{n+1}\}$, and we identify $\mathbb{R}^n$ as the subspace with basis $\{e_1, \ldots, e_n\}$. When referring to a hyperplane H in $\mathbb{R}^{n+1}$, when not differently stated, we use the notation $H := e_{n+1}^{\perp} \equiv \mathbb{R}^n$.

Our main results concern the space

$$\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) := \{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : u \text{ has compact domain}\},$$

which is a subset of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We study these spaces with the topology of *epi-convergence* introduced earlier. The results and notions exposed in this subsection are from the author and Knoerr [KU23], where the family $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ was introduced as a tool to infer geometric properties of convex functions through the properties of corresponding convex bodies.

These functions can be obtained from convex compact sets in $\mathbb{R}^{n+1}$ using the following construction: To every $K \in \mathscr{K}^{n+1}$ we associate the function $\lfloor K \rfloor : \mathbb{R}^n \to (-\infty, +\infty]$ defined by

$$\lfloor K \rfloor(x) := \inf\{t \in \mathbb{R} : (x,t) \in K\}. \tag{1.1.7}$$

In addition, $\lfloor K \rfloor(x) = +\infty$ if and only if $(x,t) \notin K$ for all $t \in \mathbb{R}$. Analogously, for every $x \in \mathbb{R}^n$ and $K \in \mathscr{K}^{n+1}$ we can define the concave function

$$\lceil K \rceil(x) := \sup\{t \in \mathbb{R} : (x,t) \in K\}.$$

In this case $\lceil K \rceil(x) = -\infty$ if $(x,t) \notin K$ for all $t \in \mathbb{R}$. The following results are proved only for the map $\lfloor \cdot \rfloor$ for the sake of brevity, but they also hold for the map $\lceil \cdot \rceil$ since for $K \in \mathscr{K}^n$ one has $\lceil K \rceil(x) = -\lfloor R_H K \rfloor(x) + c$, where $R_H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the reflection with respect to H and c is a suitable constant.

Lemma 1.14 ([KU23, Lemma 3.1]). *For every $K \in \mathcal{K}^{n+1}$, $\lfloor K \rfloor \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$.*

Proof. Note first that $\lfloor K \rfloor$ is bounded from below by

$$\inf\{t \in \mathbb{R} : (x,t) \in K \text{ for some } x \in \mathbb{R}^n\},$$

which is finite due to the compactness of K. In particular, $\lfloor K \rfloor(x) \in (-\infty, +\infty]$ for every $x \in \mathbb{R}^n$.

Assume that $x \in \mathbb{R}^n$ satisfies $\lfloor K \rfloor(x) < +\infty$. As K is compact, this implies that the infimum in (1.1.7) is attained, so $(x, \lfloor K \rfloor(x)) \in K$ in this case. Therefore,

$$\operatorname{dom}\lfloor K \rfloor = \operatorname{pr}_H(K)$$

where $\operatorname{pr}_H : \mathbb{R}^{n+1} \to H \cong \mathbb{R}^n$ denotes the orthogonal projection. In particular, $\operatorname{dom}\lfloor K \rfloor$ is non-empty, so $\lfloor K \rfloor$ is proper.

Let us show that $\lfloor K \rfloor$ is lower semi-continuous. If $x \in \operatorname{dom}(\lfloor K \rfloor)$ and (x_j) is a sequence in $\operatorname{dom}(\lfloor K \rfloor)$ converging to x, then $(x_j, \lfloor K \rfloor(x_j)) \in K$ for all $j \in \mathbb{N}$. In particular, this sequence is bounded in $\mathbb{R}^{n+1}$, so $t := \liminf_{j\to\infty} \lfloor K \rfloor(x_j)$ exists and is finite. Thus, (x, t) is a limit point of the sequence $(x_j, \lfloor K \rfloor(x_j))$ and therefore belongs to K. In particular,

$$\lfloor K \rfloor(x) \le t = \liminf_{j\to\infty} \lfloor K \rfloor(x_j).$$

On the other hand, $x \in \mathbb{R}^n \setminus \operatorname{dom}(\lfloor K \rfloor)$ implies that $\lfloor K \rfloor$ is equal to $+\infty$ on a neighborhood of x, as the domain is closed. Thus $\lfloor K \rfloor$ is lower semi-continuous outside its domain too. It is easy to see that $\lfloor K \rfloor$ is convex. In total, we thus obtain $\lfloor K \rfloor \in \operatorname{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ for all $K \in \mathcal{K}^{n+1}$. ☐

Lemma 1.15 ([KU23, Corollary 3.2]). *The inclusion* $\operatorname{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \subset \operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *is dense.*

Proof. For $u \in \operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, set $u_j := \lfloor \operatorname{epi}(u) \cap (B_j^n(0) \times [-j, j]) \rfloor$. Then $u_j \in \operatorname{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ for all $j \in \mathbb{N}$ large enough. As u has compact sublevel sets, given $t \in \mathbb{R}$ we have

$$\{u_j \le t\} = \{u \le t\} \quad \text{for all } j \in \mathbb{N} \text{ large enough.}$$

Lemma 1.11 thus implies that (u_j) converges to u in $\operatorname{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, which shows the claim. ☐

Lemma 1.16 ([KU23, Lemma 3.3]). *The map* $\lfloor \cdot \rfloor : \mathcal{K}^{n+1} \to \operatorname{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ *is continuous.*

Proof. Consider a sequence $(K_j) \subset \mathscr{K}^{n+1}$ such that $K_j \to K \in \mathscr{K}^{n+1}$. Then

$$\tilde{K}_j := K_j + [0, e_{n+1}]$$

converges to $\tilde{K} := K + [0, e_{n+1}]$. We may thus choose $R > 0$ such that $\tilde{K}_j, \tilde{K} \subset B_R^n(0) \times [-R, R]$ for all $j \in \mathbb{N}$.

As $\lfloor \tilde{K}_j \rfloor = \lfloor K_j \rfloor$, $\lfloor \tilde{K} \rfloor = \lfloor K \rfloor$ for all $j \in \mathbb{N}$, we obtain

$$\{\lfloor K_j \rfloor \le t\} = \operatorname{pr}_H(\tilde{K}_j \cap (B_R^n(0) \times [-(R+1), t])).$$

and a similar formula holds for the sublevel sets of $\lfloor K \rfloor$. Note that the sets $\tilde{K}$ and $(B_R^n(0) \times [-(R+1), t]))$ can not be separated by a hyperplane for $t > \min_{(x,s)\in\tilde{K}} s = \min_{x\in\mathbb{R}^n} \lfloor K \rfloor(x)$. For $t > \min_{x\in\mathbb{R}^n} \lfloor K \rfloor(x)$, [Sch14, Theorem 1.8.10] thus implies $\tilde{K}_j \cap (B_R^n(0) \times [-(R+1), t]) \ne \emptyset$ for every $j \in \mathbb{N}$ sufficiently large and

$$\tilde{K}_j \cap (B_R^n(0) \times [-(R+1), t]) \to \tilde{K} \cap (B_R^n(0) \times [-(R+1), t])$$

for $j \to \infty$. Applying the natural projection onto H to both sides, we obtain for $t > \min_{x \in \mathbb{R}^n} \lfloor K \rfloor(x)$

$$\{\lfloor K_j \rfloor \leq t\} \to \{\lfloor K \rfloor \leq t\}.$$

On the other hand, $t < \min_{x \in \mathbb{R}^n} \lfloor K \rfloor(x)$ implies that $\{\lfloor K \rfloor \leq t\} = \emptyset$. Therefore $\{\lfloor K_j \rfloor \leq t\} = \emptyset$ for almost all $j \in \mathbb{N}$, as we may otherwise find a sequence $x_{j_k} \in \mathbb{R}^n$ with

$$(x_{j_k}, \lfloor K_{j_k} \rfloor(x_{j_k})) \in \tilde{K}_{j_k} \cap (B_R^n(0) \times [-(R+1), t]),$$

from which we can construct a limit point $(x, t_0) \in \tilde{K} \cap (B_R^n(0) \times [-(R+1), t])$.

Lemma 1.11 thus implies that $\lfloor K_j \rfloor \to \lfloor K \rfloor$ in the topology of epi-convergence. As $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ inherits the topology from $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, this shows that $\lfloor \cdot \rfloor$ is continuous. □

Conversely, we may associate to any $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ a convex set in the following way: Consider, for $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, its epigraph $\mathrm{epi}(u)$. We set $M_u := \max_{x \in \mathrm{dom}(u)} u(x)$, which is finite since the domain of u is compact and u is convex, and define

$$K^u := \mathrm{epi}(u - M_u) \cap R_H(\mathrm{epi}(u - M_u)) + M_u e_{n+1}, \tag{1.1.8}$$

where R_H is the reflection with respect to H. This is a compact and convex set, so $K^u \in \mathscr{K}^{n+1}$. We thus obtain a map

$$\begin{aligned} \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) &\to \mathscr{K}^{n+1} \\ u &\mapsto K^u. \end{aligned} \tag{1.1.9}$$

By construction $u = \lfloor K^u \rfloor$ for $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. In particular, we have the following.

Lemma 1.17. *The map $\lfloor \cdot \rfloor : \mathcal{K}^{n+1} \to \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ is surjective.*

Consider the lower half-sphere $\mathbb{S}^n_- := \{X \in \mathbb{S}^n : X \cdot e_{n+1} < 0\}$. We define the *lower boundary* of K by

$$\partial K_- := \{X \in \partial K : \text{some unit normal to } K \text{ in } X \text{ belongs to } \mathbb{S}^n_-\}.$$

Notice that the graph of $\lfloor K \rfloor$ coincides with the closure of ∂K_-.

If K is C^2_+, the map that associates to $X \in \partial K$ its unique outer normal unit vector establishes a diffeomorphism between ∂K_- and $\mathbb{S}^n_-$. More generally, if $K \in \mathcal{K}^{n+1}$ is a convex set such that there exists an open subset $U \subset \mathbb{R}^{n+1}$ with the property that $U \cap \partial K$ is the graph of a convex function of class C^2_+, then $N_K : \partial K \cap U \to \mathbb{S}^n$ is well defined and establishes a diffeomorphism onto an open subset of $\mathbb{S}^n$. In this case, we can relate integrals over $\partial K \cap U$ to integrals with respect to the surface area of K by

$$\int_{\mathbb{S}^n \cap N_K(\partial K \cap U)} \eta(N)\, dS_n(K, N) = \int_{\partial K \cap U} \eta(N_K(X))\, d\mathcal{H}^n(X), \tag{1.1.10}$$

where $\eta : \mathbb{S}^n \cap N_K(\partial K \cap U) \to$ is a bounded Borel measurable function, compare [Sch14, (2.61)].

On the other hand, if $u \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$, then the closure of $\partial K^u_- \subset \mathbb{R}^{n+1}$ is the graph of u, and we can parameterize it using the map

$$f_u : \mathrm{dom}(u) \to \mathbb{R}^{n+1}$$
$$x \mapsto (x, u(x)).$$

If $\gamma : \mathbb{R}^{n+1} \to \mathbb{R}$ is bounded and Borel measurable, then we obtain using the area formula [Mag12, Theorem 8.1]

$$\int_{\partial K^u_-} \gamma(X)\, d\mathcal{H}(X) = \int_{\mathrm{dom}(u)} \gamma\left((x, u(x))\right) \sqrt{1 + |\nabla u(x)|^2}\, dx, \tag{1.1.11}$$

where $\sqrt{1 + |\nabla u(x)|^2}$ is the approximate Jacobian of f_u.

If $u \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$ is differentiable in $x \in \mathrm{dom}(u)$, then $\frac{(\nabla u(x), -1)}{\sqrt{1+|\nabla u(x)|^2}}$ is the unique unit outer normal to $\mathrm{epi}(u)$ in $(x, u(x))$. Since u is convex, it is differentiable almost everywhere, and thus the unit normal vectors to the epigraph are defined almost everywhere. We have the following.

Lemma 1.18 ([KU23, Corollary 3.7]). *For every* $u \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$ *and* $\eta \in C(\mathbb{S}^n_-)$

$$\int_{\mathbb{S}^n_-} \eta(N)\, dS_n(K^u, N) = \int_{\mathrm{dom}(u)} \eta\left(\frac{(\nabla u(x), -1)}{\sqrt{1 + |\nabla u(x)|^2}}\right) \sqrt{1 + |\nabla u(x)|^2}\, dx. \tag{1.1.12}$$

Proof. If $K \in \mathcal{K}^{n+1}$ is C^2_+, then $u := \lfloor K \rfloor$ satisfies the equation by direct change of variable. If $u \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$ is an arbitrary function, then we may approximate K^u in the Hausdorff metric by a sequence (K_j) of C^2_+ bodies. As the surface area measure is weakly continuous [Sch14, Theorem 4.2.1], we obtain

$$\begin{aligned}\int_{\mathbb{S}^n_-} \eta(N)\, dS_n(K^u, N) &= \int_{\mathbb{S}^n} \eta(N)\, dS_n(K^u, N) = \lim_{j\to\infty} \int_{\mathbb{S}^n} \eta(N)\, dS_n(K_j, N) \\ &= \lim_{j\to\infty} \int_{\mathbb{S}^n_-} \eta(N)\, dS_n(K_j, N) \\ &= \lim_{j\to\infty} \int_{\mathrm{dom}(\lfloor K_j \rfloor)} \eta\left(\frac{(\nabla \lfloor K_j \rfloor(x), -1)}{\sqrt{1 + |\nabla \lfloor K_j \rfloor(x)|^2}}\right) \sqrt{1 + |\nabla \lfloor K_j \rfloor(x)|^2}\, dx.\end{aligned}$$

On the other hand, the map $u \mapsto \int_{\mathrm{dom}(u)} \eta\left(\frac{(\nabla u(x), -1)}{\sqrt{1+|\nabla u(x)|^2}}\right) \sqrt{1 + |\nabla u(x)|^2}\, dx$ is continuous with respect to epi-convergence by [CLM20b, Proposition 20]. As $\lfloor \cdot \rfloor$ is continuous by Lemma 1.16, we thus obtain

$$\begin{aligned}\int_{\mathbb{S}^n_-} \eta(N)\, dS_n(K^u, N) &= \lim_{j\to\infty} \int_{\mathrm{dom}(\lfloor K_j \rfloor)} \eta\left(\frac{(\nabla \lfloor K_j \rfloor(x), -1)}{\sqrt{1 + |\nabla \lfloor K_j \rfloor(x)|^2}}\right) \sqrt{1 + |\nabla \lfloor K_j \rfloor(x)|^2}\, dx \\ &= \int_{\mathrm{dom}(\lfloor K^u \rfloor)} \eta\left(\frac{(\nabla \lfloor K^u \rfloor(x), -1)}{\sqrt{1 + |\nabla \lfloor K^u \rfloor(x)|^2}}\right) \sqrt{1 + |\nabla \lfloor K^u \rfloor(x)|^2}\, dx.\end{aligned}$$

As $\lfloor K^u \rfloor = u$, the claim follows. $\square$

Consider the Fenchel-Legendre transform of the function $\lfloor K \rfloor$ obtained by $K \in \mathcal{K}^{n+1}$. Via explicit calculations we infer

$$\begin{aligned} h_K(y,-1) &= \sup\{X \cdot (y,-1) \colon X \in K\} = \sup\{X \cdot (y,-1) \colon X \in \partial K\} \\ &= \sup\{(x, \lfloor K \rfloor(x)) \cdot (y,-1) \colon x \in \mathrm{dom}(\lfloor K \rfloor)\} \\ &= \sup\{x \cdot y - \lfloor K \rfloor(x) \colon x \in \mathrm{dom}(\lfloor K \rfloor)\} \\ &= \sup\{x \cdot y - \lfloor K \rfloor(x) \colon x \in \mathbb{R}^n\} = \lfloor K \rfloor^*(y). \end{aligned} \tag{1.1.13}$$

When $K = K^u$ for some $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, (1.1.13) takes the form

$$h_{K^u}(y,-1) = u^*(y) \tag{1.1.14}$$

which will be very useful in the following pages. The map

$$K \mapsto h_K(\cdot,-1)$$

was already considered, for example, by Knoerr in [Kno21] to create a correspondence between $\mathscr{K}^{n+1}$ and $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$. Equation (1.1.13) shows that the point of view presented here and the one in [Kno21] are dual. We conclude with a remark on the integrability of specific functions, which will be useful later.

Corollary 1.19 ([KU23, Corollary 3.8]). *If* $u, v \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, *then* $x \mapsto v^*(\nabla u(x))$ *is integrable on* $\mathrm{dom}(u)$.

Proof. By Lemma 1.18,

$$\begin{aligned} \int_{\mathrm{dom}(u)} v^*(\nabla u(x))\, dx &= \int_{\partial K^u_-} \frac{v^*(\nabla u(x))}{\sqrt{1+|\nabla u(x)|^2}}\, d\mathcal{H}^n((x,u(x))) \\ &= \int_{\partial K^u_-} \frac{h^*_{K^v}(\nabla u(x))}{\sqrt{1+|\nabla u(x)|^2}}\, d\mathcal{H}^n((x,u(x))) \\ &= \int_{\mathbb{S}^n_-} h_{K^v}(N) dS_n(K^u,N) \le V(K^u[n-1],K^v). \end{aligned}$$

Since the surface area measure of a convex set is finite, the claim follows. □

1.2 First variations for measures of epigraphs

One of the trending topics in the last years concerning convexity is *marginals* of measures. The idea is pretty simple, and it is the one we are used to from the first courses in probability. In particular, we are interested in the following cases. Consider over $\mathbb{R}^{n+1}$ a measure μ such that

$$d\mu(z,x) = d\omega(z) d\eta(x), z \in \mathbb{R}, x \in \mathbb{R}^n$$

where ω and η are positive Borel measures on $\mathbb{R}$ and $\mathbb{R}^n$ respectively. The space of interest is always some subset of the family of functions $\mathrm{Conv}(\mathbb{R}^n)$, and we want to evaluate the measure of the epigraph of u through μ, that is

$$\mu(\mathrm{epi}(u)) = \int_{\mathrm{epi}(u)} d\mu(z,x) = \int_{\mathrm{dom}(u)} \int_{u(x)}^{+\infty} d\omega(z) d\eta(x).$$

Ignoring, for now, the various summability assumptions, if we define $\Phi(t) = \omega([t, +\infty))$ we can write

$$\mu(\mathrm{epi}(u)) = \int_{\mathrm{dom}(u)} \Phi(u(x))\, d\eta(x).$$

We focus on the case where there exist $\phi \in C(\mathbb{R})$ and $\psi \in C(\mathbb{R}^n)$ such that $d\omega(z) = \phi(z)dz$ and $d\eta(x) = \psi(x)dx$.

For simplicity, we work in the family $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. For $u, v \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, we want to evaluate (when it exists) the first variation

$$\lim_{t\to 0^+} \frac{\mu(\mathrm{epi}(u \square (t \cdot v))) - \mu(\mathrm{epi}(u))}{t}, \tag{1.2.1}$$

where $u \square v$ is the infimal convolution of u and v, while $t \cdot v$ is the epi-multiplication. Colesanti and Fragalà [CF13, Theorem 4.6] were the first to investigate this topic, and building from that and the successive literature, in Theorem 1.32 we prove that (1.2.1) exists and is equal to

$$\begin{aligned} &\int_{\mathrm{dom}(u)} v^*(\nabla u(x))\phi(u(x))\psi(x)\, dx \\ &+ \int_{\partial\mathrm{dom}(u)} h_{\mathrm{dom}(v)}(N(y))\Phi(u(y))\psi(x)\, d\mathcal{H}^{n-1}(y). \end{aligned} \tag{1.2.2}$$

Here $N(y)$ is the outer unit normal vector at $y \in \partial\mathrm{dom}(u)$, which is well-defined $\mathcal{H}^{n-1}$-almost everywhere since $\mathrm{dom}(u)$ is convex. In fact, we will prove a formula contemplating a wider class of deformations, which we introduce in the next section. Doing so, at the end of this chapter we show a variant of [Rot22a, Theorem 1.5] (which for the convenience of the reader is reported later as Theorem 1.35).

1.2.1 Wulff shapes of convex functions

The concept of Wulff shape, introduced more than a century ago by Wulff [Wul01], is nowadays a well-established scientific tool, especially in the study of the shapes of crystals. Significant developments have been obtained throughout the 20th century from the mathematical perspective. See, for example, the work of Fonseca [Fon91].

Definition 1.20. Consider a function $f : \mathbb{S}^{n-1} \to \mathbb{R}$. Its *Wulff shape* is the set

$$[f] \coloneqq \bigcap_{N\in\mathbb{S}^{n-1}} H_N^-(f(N)),$$

where $H_N^-(t) \coloneqq \{x \in \mathbb{R}^n : x \cdot N \leq t\}$ is the negative closed half-space with outer normal N and distance t from the origin. Equivalently, $[f]$ is the unique maximal (with respect to inclusion) convex set satisfying the condition

$$h_{[f]}(\xi) \leq f(\xi) \text{ for every } \xi \in \mathbb{S}^{n-1}. \tag{1.2.3}$$

Notice that if $f > c > 0$ then clearly $[f]$ is non-empty. In general, if $\ell_y(x) \coloneqq y \cdot x, y \in \mathbb{R}^n$ and $f - \ell_y > c > 0$, then $[f]$ is non-empty and y is in the interior of $[f]$. In particular, if $[f]$ is non-empty,

$$[f + \ell_y] = [f] + y, \tag{1.2.4}$$

for every $y \in \mathbb{R}^n$ (notice that $\ell_y = h_{\{y\}}$). Indeed, by (1.2.3), for every $y \in \mathbb{R}^n$

$$h_{[f]+y} = h_{[f]} + \ell_y \leq f + \ell_y.$$

If, by contradiction, $h_{[f]+y}$ was not maximal for $f + \ell_y$, neither would be $h_{[f]}$ for f, which would contradict (1.2.3), proving (1.2.4).

We now try to extend the same concept to convex functions. To our knowledge, this is the first time that this approach has been followed, but many of the ideas we present are scattered around the literature. See, for example, [CEK15, Rot22a]. We work with $u \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$ and $\zeta \in C(\mathbb{R}^n)$ such that its *recession function*

$$\rho_\zeta(N) := \lim_{t \to +\infty} \frac{\zeta(tN)}{t}, \quad N \in \mathbb{S}^{n-1}$$

exists, is finite, and depends continuously on N. With this in mind, we define the family of functions

$$C_{rec}(\mathbb{R}^n) \coloneqq \{\zeta \in C(\mathbb{R}^n) : \rho_\zeta \text{ exists and is finite and continuous}\}.$$

Consider now the convex compact set K^u associated to u as in (1.1.8), and the function $\bar{\zeta}$ on $\mathbb{S}^n_-$ obtained as

$$\bar{\zeta}(N) \coloneqq \frac{\zeta\left(g(N)\right)}{\sqrt{1 + |g(N)|^2}}, \tag{1.2.5}$$

where $\zeta \in C_{rec}(\mathbb{R}^n)$. Here $g : \mathbb{S}^n_- \to \mathbb{R}^n$ is the gnomonic projection, which is given by

$$N \mapsto \frac{N - (e_{n+1} \cdot N)e_{n+1}}{e_{n+1} \cdot N}. \tag{1.2.6}$$

We will make use of the extension of $\bar{\zeta}$ on the whole $\mathbb{S}^n$ obtained by reflection on H, that is, if $N = (N_1, \ldots, N_{n+1}) \in \mathbb{S}^n_-$, $\bar{\zeta}(N) = \bar{\zeta}((N_1, \ldots, N_n, -N_{n+1}))$. When $N_{n+1} = 0$, $\bar{\zeta}$ is extended by continuity (which is finite since $\zeta \in C_{rec}(\mathbb{R}^n)$). Note that with the identification $\mathbb{S}^n \cap H \equiv \mathbb{S}^{n-1}$ the continuous extension of the function $\bar{\zeta}$ to $\mathbb{S}^{n-1}$ is equal to the recession function of ζ. To make the notation lighter, we refer to $\bar{\zeta}$ both for the transform (1.2.5) and the extension.

Consider on $\mathbb{S}^n$ the function

$$h_{u,t}(N) \coloneqq h_{K^u}(N) + t\bar{\zeta}(N)$$

and its classical Wulff shape $[h_{u,t}]$, which we denote by $K^{u,t}$. The function $h_{u,t}$ can be extended to $\mathbb{R}^{n+1}$ considering its positively homogeneous extension $h_{u,t}(X) = |X| h_{u,t}(X/|X|)$ for $X \in \mathbb{R}^{n+1}, |X| \neq 0$.

Notice that the Fenchel-Legendre transform behaves similarly to the Wulff shape. Indeed by (1.1.13), for every $v \in \mathrm{Conv}(\mathbb{R}^n)$ (since the concept of support function can be considered for unbounded sets too)

$$\begin{aligned} \mathrm{epi}(v) &= \bigcap_{N \in \mathbb{S}^n_-} H^-_N\left(h_{\mathrm{epi}(v)}(N)\right) = \bigcap_{N \in \mathbb{S}^n_-} H^-_N\left(\frac{h_{\mathrm{epi}(v)}((g(N), -1))}{\sqrt{1 + |g(N)|^2}}\right) \\ &= \bigcap_{x \in \mathbb{R}^n} H^-_{g^{-1}(x)}\left(\frac{h_{\mathrm{epi}(v)}((x, -1))}{\sqrt{1 + |x|^2}}\right) = \bigcap_{x \in \mathbb{R}^n} H^-_{g^{-1}(x)}\left(\frac{v^*(x)}{\sqrt{1 + |x|^2}}\right). \end{aligned} \tag{1.2.7}$$

In general, if $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is proper (thus f^* is proper by [RW98, Theorem 11.1]), by the definition of the Fenchel-Legendre transform the epigraph of f^* is the intersection of the epigraphs of the affine functions $x\mapsto x\cdot y-f(x), y\in\mathbb{R}^n$ (the supremum of a family of functions corresponds to the intersection of their epigraphs). These epigraphs are delimited by the affine hyperplanes with unit normal vectors $(x,-1)/\sqrt{1+|x|^2}$ and distance $f(x)/\sqrt{1+|x|^2}$ from the origin (if $f(x)=+\infty$, the epigraph of the corresponding hyperplane is trivially equivalent to $\mathbb{R}^{n+1}$). Then, equivalently

$$\operatorname{epi}(f^*)=\bigcap_{x\in\mathbb{R}^n}H^-_{g^{-1}(x)}\left(\frac{f(x)}{\sqrt{1+|x|^2}}\right). \tag{1.2.8}$$

This relation can be tied with the definition of Wulff shape as follows.

Lemma 1.21. *Let $u\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, $\zeta\in C_{rec}(\mathbb{R}^n)$, and $t\in\mathbb{R}$. Then*

$$\bigcap_{N\in\mathbb{S}^n_-}H^-_N(h_{u,t}(N))=\operatorname{epi}((u^*+t\zeta)^*).$$

Proof. Notice that using (1.1.14),

$$\begin{aligned}\bigcap_{N\in\mathbb{S}^n_-}H^-_N(h_{u,t}(N))&=\bigcap_{N\in\mathbb{S}^n_-}H^-_N(h_{K^u}(N)+t\bar{\zeta}(N))\\&=\bigcap_{x\in\mathbb{R}^n}H^-_N\left(\frac{h_{K^u}(x,-1)}{\sqrt{1+|x|^2}}+t\frac{\zeta(g^{-1}(x))}{\sqrt{1+|x|^2}}\right)\\&=\bigcap_{x\in\mathbb{R}^n}H^-_{g^{-1}(x)}\left(\frac{(u^*+t\zeta)(x)}{\sqrt{1+|x|^2}}\right).\end{aligned}$$

Thanks to (1.2.8), the last line equals to $\operatorname{epi}((u^*+t\zeta)^*)$, concluding the proof.

□

Suppose now that $\zeta=v^*, v\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. By the homeomorphism (1.1.5), $v\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ if and only if v^* is a Lipschitz convex function. In particular, this implies $v^*\in C_{rec}(\mathbb{R}^n)$ and convex. In this case $\overline{v^*}$, i.e. the function on $\mathbb{S}^n$ corresponding to v^* via (1.2.5), coincides with h_{K^v} by (1.1.14). Thus, for $t\geq 0$,

$$h_{u,t}=h_{K^u}+th_{K^v}=h_{K^u+tK^v},$$

that is, the Wulff shape simply gives a Minkowski addition and $[h_{u,t}]=K^u+tK^v$. By (1.2.7)

$$\operatorname{epi}((u^*+tv^*)^*)=\bigcap_{x\in\mathbb{R}^n}H^-_{g^{-1}(x)}\left(\frac{(u^*+tv^*)^*(x)}{\sqrt{1+|x|^2}}\right)=\bigcap_{N\in\mathbb{S}^n_-}H^-_N\left(h_{K^u+tK^v}(N)\right),$$

and thus

$$\lfloor K^{u,t}\rfloor=\lfloor K^u+tK^v\rfloor=(u^*+tv^*)^*=u\square(t\cdot v). \tag{1.2.9}$$

Unfortunately, it is not always the case that for a generic $\zeta\in C_{rec}(\mathbb{R}^n)$ one has

$$\lfloor K^{u,t}\rfloor=(u^*+t\zeta)^*.$$

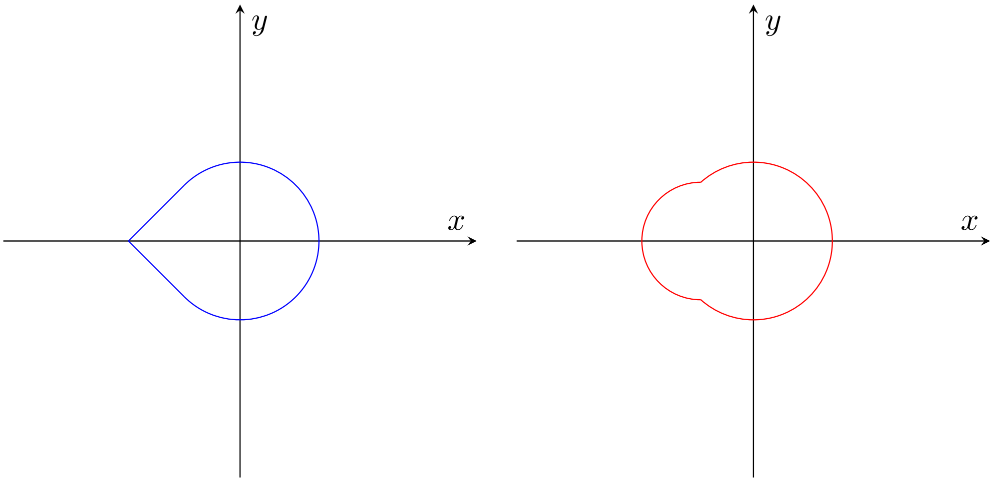

Figure 1.1: On the left, the body K^u. On the right, its support function h_{K^u}.

Indeed the envelope

$$\bigcap_{N\in\mathbb{S}^n_-} H^-_N(h_{K^{u,t}}(N))$$

might be such that its projection onto $e^\perp_{n+1}$ is not the same as $\mathrm{dom}((u^*+t\zeta)^*)$, as we show in the following example.

Example 1.22. *Consider the function* $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R})$ *given by*

$$u(x) = \begin{cases} -x-\sqrt{2} & -\sqrt{2}\le x\le -\sqrt{2}/2, \\ -\sqrt{1-|x|^2} & -\sqrt{2}/2 \le x \le 1, \\ +\infty & \textit{otherwise.} \end{cases}$$

Then $K^u \in \mathscr{K}^2$ *(on the left in Figure 1.1) is such that its support function (on the right in Figure 1.1) is, in polar coordinates for* $\theta\in[-\pi,\pi]$,

$$h_{K^u}(\theta) = \begin{cases} 1 & -\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi, \\ \frac{\sqrt{2}}{\sqrt{1+(\tan\theta)^2}} & \textit{otherwise.} \end{cases}$$

Notice that K^u *can be determined by considering only the half-spaces corresponding to* $\theta\in[-3\pi/4, 3\pi/4]$. *Indeed, if we define*

$$\tilde{K} := \bigcap_{\theta\in[-3\pi/4,3\pi/4]} H^-_{(\cos\theta,\sin\theta)}(h_{K^u}(\theta)),$$

one has that

$$\tilde{K} \subseteq H^-_{(\cos\theta,\sin\theta)}(h_{K^u}(\theta))$$

for every $\theta \notin [-3\pi/4, 3\pi/4]$. *Therefore,* $\tilde{K}=K^u$.

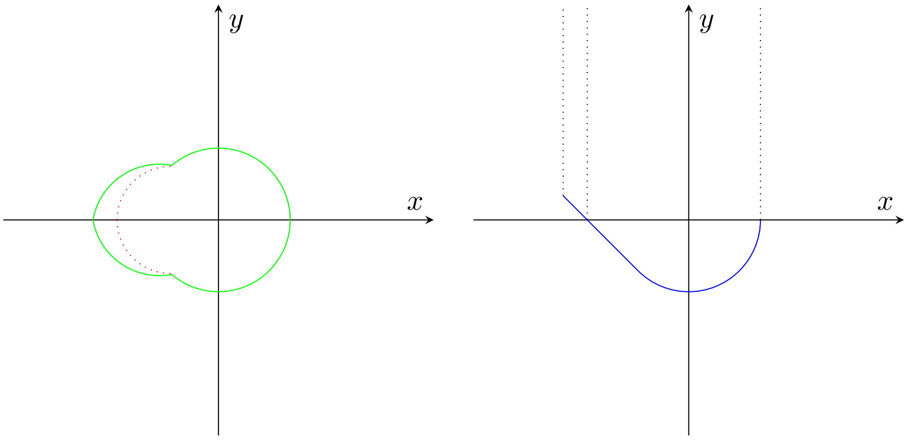

Figure 1.2: On the left, the plot of $h_{K^u} + t\bar{\zeta}$ (in red, the perturbed part). On the right, the epigraph of $(u^* + t\zeta)$.

Consider $\xi \in C(\mathbb{S}^1)$ *such that it is symmetric with respect to the horizontal axis,* $\zeta(\theta) \equiv 0$ *for* $\theta \in [-3\pi/4, 3\pi/4]$*, and is strictly positive otherwise. Then, by construction, for every* $t \geq 0$

$$K^u \subset H^-_{(\cos\theta,\sin\theta)}(h_{K^u}(\theta) + t\xi(\theta)),$$

and therefore

$$K^{u,t} = [h_{K^u} + t\bar{\zeta}] = K^u \tag{1.2.10}$$

for every $t \geq 0$.

Fix $x_0 < -\sqrt{2}$*, and for* $t \in [0,1]$ *consider the intersection of the line* $x = tx_0$ *with the line* $y = -x - \sqrt{2}$*, which is the tangent to* K^u *with outer unit normal vector* $(-\sqrt{2}/2, -\sqrt{2}/2)$*. We denote this intersection by the point* (x_t, y_t)*. We now choose* $\bar{\zeta}$ *such that the lines determining the half-spaces* $H^-_{(\cos\theta,\sin\theta)}(h_{K^u}(\theta) + t\xi(\theta)), \theta \notin [-3\pi/4, 3\pi/4]$*, intersect* $x = tx_0$ *in* (x_t, y_t)*, while the remaining ones are unchanged. By explicit calculations, this corresponds to the choice (see Figure 1.2 on the left)*

$$\bar{\zeta}(\theta) = \begin{cases} 0 & -\frac{3}{4}\pi \leq \theta \leq \frac{3}{4}\pi, \\ \frac{|(\sqrt{2}+x_0)(|\cot\theta|-1)|}{\sqrt{1+(\cot\theta)^2}} & \textit{otherwise.} \end{cases}$$

If we consider the envelope

$$\bigcap_{\theta\in(-\pi,0)} H^-_N(h_{K^u}(\theta) + t\bar{\zeta}(\theta)),$$

by construction we obtain an epigraph whose corresponding function agrees with u *in* $[-\sqrt{2}, 1]$*, with an additional part corresponding to the epigraph of* $y = -x - \sqrt{2} + I_{[x_0,-\sqrt{2}]}(x)$ *(see Figure 1.2 on the right). Considering*

$$\zeta(x) = \bar{\zeta}(g^{-1}(x))\sqrt{1 + (g^{-1}(x))^2},$$

by Lemma 1.21 and (1.2.10) *we just proved that, in this example,*

$$\lfloor K^{u,t} \rfloor \neq (u^* + t\zeta)^*.$$

Nonetheless, considering the relation highlighted in Lemma 1.21 and in order to provide a wider treatment, we introduce the following definition.

Definition 1.23. Fix $\zeta \in C_{rec}(\mathbb{R}^n)$ and consider $t \in \mathbb{R}$. Then, for $u \in \mathrm{Conv}(\mathbb{R}^n)$,

$$u_t(x) := (u^* + t\zeta)^*(x)$$

is the *functional Wulff shape* of u at t with respect to ζ.

In order to work with a well-behaved family of Wulff shapes, from now on we require the following property.

Definition 1.24. We say that the functional Wulff shape u_t has the property (**P**) in an interval I if for every $t \in I$

$$u_t = \lfloor K^{u,t} \rfloor. \tag{$\mathbf{P}$}$$

This is always satisfied, as we mentioned before, if $\zeta = v^*, v \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$ and $t \geq 0$. Other examples can be obtained by small non-convex perturbations of the previous case. Another sufficient condition is the following.

Lemma 1.25. *Consider $u \in \mathrm{Conv_{cd}}(\mathbb{R}^n)$ and $\zeta \in C_{rec}(\mathbb{R}^n)$. If for every t in an interval I we have that $u^* + t\zeta$ is convex, then (**P**) is satisfied for u, ζ, and $t \in I$.*

Proof. If $u + t\zeta$ is convex, then since $\zeta \in C_{rec}(\mathbb{R}^n)$ we have $u + t\zeta \in \mathrm{Conv}(\mathbb{R}^n)$, and therefore there exist bodies $K_t \in \mathscr{K}^n$ such that $h_{K_t}(x, -1) = u(x) + t\zeta(x)$ for every $t \in I$. Then, by (1.1.13) $u_t = \lfloor K_t \rfloor$. Since

$$K_t = [h_{K_t}] = [h_{K^u} + t\bar{\zeta}] = K^{u,t},$$

we infer $u_t = \lfloor K^{u,t} \rfloor$, which is precisely the required property. ☐

Let us provide a practical example (communicated to us by Mussnig [Mus]) where ζ is non-convex, but $u^* + t\zeta$ is.

Example 1.26. *Fix $c > 0$ and consider the function $u(x) = c|x| + I_{B^n}(x)$. Its Fenchel-Legendre transform is given by*

$$u^*(x) = \begin{cases} 0 & |x| \leq c, \\ |x| - c & |x| \geq c. \end{cases}$$

Now, take $\zeta \in C_{rec}(\mathbb{R}^n)$ defined as

$$\zeta(x) = \begin{cases} |x| - c & |x| \leq c, \\ 0 & |x| \geq c, \end{cases}$$

and consider the functional Wulff shape of u at time t with respect to ζ, taking $t \in [0, 1]$. Notice that if we take the function $v(x) = (|x| - c) + I_{B_{c(0)}}(x)$, then

$$u^* + t\zeta = u^* \wedge tv,$$

and by the properties of the Fenchel-Legendre transform

$$u_t = (u^* + t\zeta)^* = (u^* \wedge tv)^* = u \vee (t \cdot v^*).$$

Therefore,

$$u_t(x) = \begin{cases} ct & |x| \le t, \\ c|x| & t \le |x| \le 1, \\ +\infty & |x| > 1. \end{cases}$$

In practice, the perturbation cuts the cone which is the graph of u orthogonally with respect to its axis, creating a plateau. As t increases, so do the height and width of the plateau.

When (**P**) is satisfied, we have the following consequence, which will be fundamental in the proof of our main result.

Lemma 1.27. *Let $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, and $\zeta \in C_{rec}(\mathbb{R}^n)$ such that they satisfy* (**P**) *for $t \in [0, \varepsilon], \varepsilon > 0$. Consider the segment $\ell_\tau = \{s e_{n+1} \colon s \in [0, \tau]\}, \tau > 0$. Then*

$$[h_{K^u + \ell_\tau} + t\bar{\zeta}] = [h_{K^u} + t\bar{\zeta}] + \ell_\tau.$$

Proof. Consider $K_{\tau,t} \coloneqq [h_{K^u+\ell_\tau} + t\bar{\zeta}], K^{u,t} = [h_{K^u} + t\bar{\zeta}]$. Notice that $h_{\ell_\tau}(N) = 0$ for every $N \in \mathbb{S}^n$ such that $N \cdot e_{n+1} \le 0$ and therefore in these directions

$$h_{K^u+\ell_\tau}(N) + t\bar{\zeta}(N) = h_{K^u}(N) + h_{\ell_\tau}(N) + t\bar{\zeta}(N) = h_{K^u}(N) + t\bar{\zeta}(N).$$

In particular, by property (**P**) this is sufficient to grant that

$$\lfloor [h_{K^u+\ell_\tau} + t\bar{\zeta}] \rfloor = \lfloor [h_{K^u} + t\bar{\zeta}] \rfloor,$$

and as a consequence we obtain

$$\lfloor K_{\tau,t} \rfloor = (u^* + t\zeta)^* = u_t.$$

Notice that, by construction, $K^{u,t}$ is symmetric with respect to $e_{n+1}^\perp$ up to a suitable vertical translation. With the notation $\tilde{u}_t = \lceil K^{u,t} \rceil$, this implies that u_t and $-\tilde{u}_t$ are equal up to a constant. Now, if $e_{n+1} \cdot N \ge 0$, we have

$$\begin{aligned} &h_{K^u+\ell_\tau}(N) + t\bar{\zeta}(N) = h_{K^u}(N) + h_{\ell_\tau}(N) + t\bar{\zeta}(N) = \\ &h_{K^u}(N) + \tau e_{n+1} \cdot N + t\bar{\zeta}(N) = h_{K^u}(N) + h_{\{\tau e_{n+1}\}}(N) + t\bar{\zeta}(N). \end{aligned}$$

By (1.2.4),

$$[h_{K^u} + h_{\{\tau e_{n+1}\}} + t\bar{\zeta}] = [h_{K^u} + t\bar{\zeta}] + \tau e_{n+1} = K^{u,t} + \tau e_{n+1}.$$

Therefore,

$$\lceil [h_{K^u+\ell_\tau} + t\bar{\zeta}] \rceil = \lceil [h_{K^u} + h_{\{\tau e_{n+1}\}} + t\bar{\zeta}] \rceil = \lceil [h_{K^u} + t\bar{\zeta}] \rceil + \tau = \tilde{u}_t + \tau.$$

By definition,

$$\begin{aligned}K_{\tau,t} =& \left(\bigcap_{N\cdot e_{n+1}\leq 0} H_N^-(h_{K^u+\ell_\tau}(N)+t\bar{\zeta}(N))\right)\\ &\cap \left(\bigcap_{N\cdot e_{n+1}\geq 0} H_N^-(h_{K^u+\ell_\tau}(N)+t\bar{\zeta}(N))\right)\\ =&\operatorname{epi}(u_t)\cap\operatorname{epi}(\tilde{u}_t+\tau)=K^{u,t}+\ell_\tau,\end{aligned}$$

concluding the proof. □

The core ideas for the proof of our main result are encoded in the following properties of Wulff shapes proved by Willson [Wil80]. Consider $K\in\mathscr{K}^n, t\geq 0$ and $f\in C(\mathbb{S}^{n-1})$; we use the notation F_tK for the Wulff shape of h_K+tf, that is

$$F_tK=[h_K+tf]. \tag{1.2.11}$$

Theorems 5.1 and 5.6 from [Wil80] read as follows.

Theorem 1.28. *If $K_m\to K$ in $\mathscr{K}^n$, $t_m\to t_0$ in $\mathbb{R}$ and $F_{t_0}K$ has non-empty interior, then $F_{t_m}K_m$ has non-empty interior for m large and $F_{t_m}K_m\to F_{t_0}K$ in $\mathscr{K}^n$.*

In particular, Theorem 1.28 implies that F_tK is continuous in t.

Theorem 1.29. *Let s and t be nonnegative real numbers. Let $K\in\mathscr{K}^n_n, f\in C(\mathbb{S}^{n-1})$ and assume F_tK has non-empty interior. Then*

$$F_sF_tK=F_{s+t}K$$

In the functional notation, Theorem 1.29 reads as

$$[[h_K+tf]+sf]=[h_K+tf+sf].$$

This will be very useful later to obtain differentiability in t for the measure of some F_tK.

A consequence of Theorem 1.28 and Lemma 1.21 is the following.

Corollary 1.30. *Consider $\zeta\in C_{rec}(\mathbb{R}^n)$. If a sequence of functions $u_m\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ epi-converges to $u\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, $t_m\to t_0$, $\operatorname{epi}(u_{t_0})$ has non-empty interior, and (**P**) is satisfied for m sufficiently large and for t sufficiently close to t_0, then $v_m:=(u_m^*+t_m\zeta)^*$ has full-dimensional domain for m large and v_m epi-converges to u_{t_0}.*

Proof. Since $\operatorname{epi}(u_{t_0})$ has non-empty interior in $\mathbb{R}^{n+1}$, then $\operatorname{dom}(u_{t_0})$, which coincides with the projection of $\operatorname{epi}(u_{t_0})$ on $\mathbb{R}^n$, has non-empty interior on $\mathbb{R}^n$.

If u_{t_0} is constant on its domain, say $u_{t_0}\equiv c\in\mathbb{R}$ on $\operatorname{dom}(u_{t_0})$, then $K^{u_{t_0}}$ is contained in the hyperplane $\{x\in\mathbb{R}^{n+1}\colon x\cdot e_{n+1}=c\}$ and $K^{u_{t_0}}$ has empty interior. Assume that u_{t_0} is not constant on its domain. In this case $K^{u_{t_0}}$ has non-empty interior.

Let us use the notation introduced in (1.2.11), with $f = \bar{\zeta}$. Since $K^{u_m} \to K^u$, by the continuity of the map

$$\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \ni v \mapsto K^v \in \mathscr{K}^{n+1},$$

and $t_m \to t_0$ then $F_{t_m}(K^{u_m}) \to F_{t_0}(K^u)$, by Theorem 1.28. Moreover, the domain of $\lfloor F_{t_m} K_m \rfloor$ has non-empty interior for m sufficiently large. Since property (**P**) is satisfied for t sufficiently close to t_0 and m sufficiently large, for these values of m we have $\lfloor F_{t_m} K_m \rfloor = (u_m^* + t_m\zeta)^* = v_m$ and by Lemma 1.16 $\lfloor F_{t_0} K \rfloor = u_{t_0}$, completing the proof in this case.

It remains to deal with the case when u_{t_0} is constant on its domain. In this case we can prove, arguing as above, that

$$F_{t_m}(K^{u_m} + \ell) \to F_{t_0}(K^u + \ell),$$

where $\ell_1 = \{se_{n+1} \colon s \in [0,1]\}$. This fact and property (**P**) imply that

$$(u_m^* + t_m\zeta)^* = \lfloor F_{t_m} K^{u_m} \rfloor = \lfloor F_{t_m}(K^{u_m} + \ell) \rfloor$$

converges to

$$u_{t_0} = \lfloor F_{t_0} K^u \rfloor = \lfloor F_{t_0}(K^u + \ell) \rfloor.$$

□

1.2.2 Measure-theoretic Brunn-Minkowski theory

The measure-theoretic Brunn-Minkowski theory is a relatively recent development in the world of convex geometry. See, for example, Livshyts [Liv19], Alonso-Gutierrez, Hernandez, Roysdon, Yepes Nicolàs, and Zvavitch [AGHCR$^+$21], Rotem [Rot22b], and Kryvonos and Langharst [KL22] and the references therein. Consider a measure μ on $\mathbb{R}^{n+1}$ such that is has continuous density Ψ with respect to the Lebesgue measure (milder hypotheses can be considered, but continuity will suffice for an exhaustive picture). Then it is possible to generalize the notion of surface area measure in (1.1.4) considering its weighted version

$$S_{\mu,K}(B) = \int_{\tau_K(B)} \Psi(X)\, d\mathcal{H}^n(X)$$

for every $K \in \mathscr{K}^n$ and Borel set $B \subset \mathbb{S}^n$.

Many achievements, which can be found in the works listed above and the references therein, have been accomplished starting from this notion. We are mainly interested in Lemma 2.7 from [KL22], which generalizes Aleksandrov's Lemma [Sch14, Lemma 7.5.3].

Lemma 1.31. *Let μ be a Borel measure on $\mathbb{R}^{n+1}$ with continuous density Ψ with respect to the Lebesgue measure. Then for $f \in C(\mathbb{S}^n)$ and $K \in \mathscr{K}^{n+1}_{n+1}$, we have*

$$\lim_{t\to 0^+} \frac{\mu([h_K + tf]) - \mu(K)}{t} = \int_{\mathbb{S}^n} f(N)\, dS_{\mu,K}(N) = \int_{\partial K} f(N_K(X))\Psi(X)\, d\mathcal{H}^n(X). \tag{1.2.12}$$

Lemma 1.31 was originally formulated asking for the origin to be in the interior of K. This is not necessary, since the boundary structure of a convex compact set is invariant under translations, and thus both the set and the measure can be suitably translated so that the Lemma holds in the form we proposed. Indeed, suppose that the origin is not contained in K. For every point Y in the interior of K, $h_K - \ell_Y > 0$, where $\ell_Y(X) = Y \cdot X, X \in \mathbb{R}^{n+1}$, and $K - Y$ has the origin in its interior. Then, by (1.2.4), we can consider the Wulff shape $[h_K - \ell_Y + tf]$ to infer by Lemma 1.31

$$\lim_{t\to 0^+} \frac{\mu([h_K - \ell_Y + tf]) - \mu(K - Y)}{t} = \int_{\partial(K-Y)} f(N_{K-Y}(X))\Psi(X)\, d\mathcal{H}^n(X).$$

Consider $\tilde{\Psi}(\cdot) = \Psi(\cdot + Y)$ and the measure $\tilde{\mu}$ which has $\tilde{\Psi}$ as density. By Lemma 1.31, the translation invariance of the Hausdorff measure, and since $N_{K-Y}(X) = N_K(X+Y)$,

$$\begin{aligned} \lim_{t\to 0^+} \frac{\mu([h_K + tf]) - \mu(K)}{t} &= \lim_{t\to 0^+} \frac{\tilde{\mu}([h_K - \ell_Y + tf]) - \tilde{\mu}(K - Y)}{t} = \\ \int_{\partial(K-Y)} f(N_{K-Y}(X))\tilde{\Psi}(X)\, d\mathcal{H}^n(X) &= \int_{\partial K} f(N_K(Z))\tilde{\Psi}(Z - Y)\, d\mathcal{H}^n(Z) \\ &= \int_{\partial K} f(N_K(Z))\Psi(Z)\, d\mathcal{H}^n(Z), \end{aligned}$$

proving that in Lemma 1.31 we do not need the origin to belong to the interior of K.

1.2.3 Proof of the variational formula

We are finally ready to prove the main result of this chapter. It reads as follows. We recall that if $\nu \in \mathbb{S}^{n-1} \subset H \subset \mathbb{R}^{n+1}$, then $\bar{\zeta}((\nu, 0)) = \rho_\zeta(\nu)$.

Theorem 1.32. *Let $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, and $\zeta \in C_{rec}(\mathbb{R}^n)$ such that they satisfy ($\mathbf{P}$) for every $t \geq 0$ sufficiently small. Consider, moreover, a measure μ on $\mathbb{R}^{n+1}$ such that $d\mu(z, x) = \phi(z)\psi(x)dz\, dx$ with positive functions $\phi \in C(\mathbb{R}) \cap L^1([a, +\infty))$ for some $a \in \mathbb{R}$ and $\psi \in C(\mathbb{R}^n)$ such that $\phi(z) \to 0$ as $z \to +\infty$. Then, if $0 < \mu(\mathrm{epi}(u)) < \infty$,*

$$\mu(u, \zeta) := \lim_{t\to 0^+} \frac{\mu(\mathrm{epi}((u^* + t\zeta)^*)) - \mu(\mathrm{epi}(u))}{t}$$

$$= \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\phi(u(x))\psi(x)\, dx + \int_{\partial \mathrm{dom}(u)} \rho_\zeta(N_{\mathrm{dom}(u)}(y))\Phi(u(y))\psi(x)\, d\mathcal{H}^{n-1}(y),$$

exists and is finite, where $\Phi(t) = \int_t^{+\infty} \phi(z)dz$.

In particular, if $\zeta = v^*, v \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, we recover (1.2.2).

As anticipated, our strategy is to work between convex sets and convex functions. In order to perform this passage formally, we introduce the family

$$\mathscr{K}_+^{n+1} := \{K \subset \mathbb{R}^{n+1} : K \text{ is convex, closed, with nonempty interior, and } \mathrm{pr}_H(K) \in \mathscr{K}_n^n\}.$$

Notice that this family includes precisely $\mathscr{K}_{n+1}^{n+1}$ and the epigraphs of the functions in $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ with n-dimensional domain, where $\mathbb{R}^n \equiv H = e_{n+1}^\perp$. It is good practice, even though we will not use this in the proofs, to present a topology for this space. Let $K, L \in \mathscr{K}_+^{n+1}$ and let μ be a measure on $\mathbb{R}^{n+1}$. The *μ-symmetric-difference* between K and L is

$$d_\mu(K,L) \coloneqq \mu(K\Delta L) = \int_{\mathbb{R}^{n+1}} \chi_{K\Delta L}(x)\, d\mu(x),$$

where for a measurable set A we denote by χ_A is characteristic function. In particular, if we use the Gaussian measure γ_{n+1} on $\mathbb{R}^{n+1}$, defined by its density

$$d\gamma_{n+1}(x) = \frac{1}{\sqrt{2\pi}^{n+1}} e^{-|x|^2/2}\, dx,$$

then $d_{\gamma_{n+1}}$ defines a metric on $\mathscr{K}_+^{n+1}$. This follows from the convexity of the involved sets and the sets being of dimension $n+1$. Notice that $d_{\gamma_{n+1}}(K,L) \in [0,1]$ for every $K, L \in \mathscr{K}_+^{n+1}$. This kind of metrics were studied in a wider generality by Li and Mussnig [LM22]. Here we provide a proof that $d_{\gamma_{n+1}}$ induces an appropriate metric on $\mathscr{K}_+^{n+1}$, that is, we can approximate functions in $\mathrm{Conv_{cd}}(\mathbb{R}^n)$ by convex bodies in $\mathscr{K}_{n+1}^{n+1}$.

Lemma 1.33. *Let γ_{n+1} be the Gaussian measure on $\mathbb{R}^{n+1}$. The function $d_{\gamma_{n+1}}$: $\mathscr{K}_+^{n+1} \times \mathscr{K}_+^{n+1} \to [0,1]$ is a distance. Moreover, its restrictions to $\mathscr{K}_{n+1}^{n+1}$ and the family of epigraphs of functions in $\mathrm{Conv_{cd}}(\mathbb{R}^n)$ induce the topology of the Hausdorff metric and epi-convergence, respectively.*

Proof. The equivalence of epi-convergence and convergence with respect to $d_{\gamma_{n+1}}$ can be proved analogously to Theorem 1.2 in [LM22].

To prove that $d_{\gamma_{n+1}}$ induces the Hausdorff metric on $\mathscr{K}_{n+1}^{n+1}$, we use the first part of the proof as follows: by Lemma 1.11, the Hausdorff convergence of a sequence $\mathscr{K}_m \in \mathscr{K}_{n+1}^{n+1}$ to some $\mathscr{K} \in \mathscr{K}_{n+1}^{n+1}$ is equivalent to the epi-convergence of I_{K_m} to I_K as functions on $\mathbb{R}^{n+1}$. But as we just proved, this is equivalent to the convergence of $\mathrm{epi}(I_{K_m})$ to $\mathrm{epi}(I_K)$ with respect to $d_{\gamma_{n+2}}$. Direct calculations show that

$$d_{\gamma_{n+1}}(\mathrm{epi}(I_{K_m}), \mathrm{epi}(I_K)) = C d_{\gamma_{n+1}}(K_m, K)$$

for some absolute constant $C > 0$, concluding the proof. □

For the convenience of the reader, we recall the following classical result (see, for example, [Rud76, Theorem 7.17]).

Lemma 1.34. *Suppose $f_m : [a,b] \to \mathbb{R}, m \in \mathbb{N}$ is a sequence of functions, differentiable on $[a,b]$ and such that $f_m(x_0)$ converges for some $x_0 \in [a,b]$. If the derivatives f'_m converge uniformly on $[a,b]$, then f_m converges uniformly on $[a,b]$, to a function f, and*

$$f'(x) = \lim_{m\to\infty} f'_m(x)$$

for every $x \in [a,b]$.

In the following proof it is convenient to consider

$$\lim_{t\to 0^+} \frac{\mu(\mathrm{epi}((u^* + t\zeta)^*)) - \mu(\mathrm{epi}(u))}{t}$$

as the right derivative of $\mu(\mathrm{epi}((u^*+t\zeta)^*)$ at $t=0$. With these instruments at hand, we can start the proof of the main result of this chapter.

Proof of Theorem 1.32. Let us first sketch the outline of the proof. We start by appropriately re-writing the variational formula (1.2.12) with $K=K^v$ for some $v\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, $\Psi(x,z)=\phi(z)\psi(x)$, and $f=\bar\zeta$ as defined in (1.2.5). For $\tilde v=\lceil K^v\rceil$, direct computations using (1.1.11) show

$$\begin{aligned}
&\lim_{t\to0^+}\frac{\mu([h_{K^v}+t\bar\zeta])-\mu(K^v)}{t}=\int_{\partial K^v}\bar\zeta(N_{K^v}((x,z)))\phi(z)\psi(x)\,d\mathcal{H}^n((x,z))\\
&=\int_{\mathrm{dom}(v)}\zeta(\nabla v(x))\phi(v(x))\psi(x)\,dx\\
&+\int_{\mathrm{dom}(v)}\zeta(\nabla v(x))\phi(\tilde v(x))\psi(x)\,dx\\
&+\int_{\partial\mathrm{dom}(v)}\bar\zeta((N_{\mathrm{dom}(v)}(x),0))\left(\int_{v(x)}^{\tilde v(x)}\phi(s)\,ds\right)\psi(x)\,d\mathcal{H}^{n-1}(x).
\end{aligned}\tag{1.2.13}$$

In particular, notice that $\bar\zeta$ restricted to the equator is by definition ρ_ζ. We have used the following fact: Since v and $\tilde v$ are one a suitable reflection of the other, $\nabla v(x)=-\nabla\tilde v(x)$ for every x where the gradient exists. For v, the normal to the epigraph expressed through the gradient is $(\nabla v,-1)/\sqrt{1+|\nabla v|^2}$, while for $\tilde v$ we have $(-\nabla\tilde v,1)/\sqrt{1+|\nabla\tilde v|^2}$. Moreover, by construction $\bar\zeta\,(N_{K^v}(x,z))=\bar\zeta\,(R_H(N_{K^v}(x,z)))$. Since

$$\bar\zeta\,(R_H(N_{K^v}(x,z)))=\bar\zeta\left(\frac{-\nabla\tilde v(x),1}{\sqrt{1+|\nabla\tilde v(x)|^2}}\right)=\bar\zeta\left(\frac{\nabla v(x),1}{\sqrt{1+|\nabla v(x)|^2}}\right)=\zeta(\nabla v(x)),$$

we have the integral on the third line of (1.2.13).

From here, for a fixed $u\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ the idea is to approximate $\mathrm{epi}(u_t)$ with a suitable sequence of convex bodies, in order to obtain our claim as a limit of the integrals of the form (1.2.13). Then, we conclude checking the hypotheses required to apply Lemma 1.34. The proof is structured in three steps, after introducing the sequence approximating $\mathrm{epi}(u_t)$.

Fix $u\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ such that $0<\mu(\mathrm{epi}(u))$. Clearly, $\dim(\mathrm{epi}(u))=n+1$. Consider now the segment $\ell_m=\{se_{n+1}\colon s\in[0,m]\}$ for $m\in\mathbb{N}$. We remark that $K^u+\ell_m\to\mathrm{epi}(u)$ in the symmetric-difference distance $d_{\gamma_{n+1}}$ as $m\to\infty$. Consider the sequence of Wulff shapes $K_{m,t}:=F_t(K^u+\ell_m)=[h_{K^u+\ell_m}+t\bar\zeta]$. Notice that $K_{m,0}$ is monotonic with respect to inclusion, and thus the sequence $\mu(K_{m,0})$ is increasing. By the properties of the sequence $K_{m,0}$ and the measure μ it follows from the monotone convergence theorem that

$$\lim_{m\to\infty}\mu(K_{m,0})=\mu(\mathrm{epi}(u_0))=\mu(\mathrm{epi}(u)).$$

Moreover, notice that the sequence of sets $K_{m,t}$ converges to $\mathrm{epi}(u_t)$ in the symmetric-difference distance topology monotonically. In particular since (**P**) is satisfied by u and ζ, Lemma 1.27 implies that $K_{m,t}=K^{u,t}+\ell_m$, where $K^{u,t}=F_t(K^u)=[h_{K^u}+t\bar\zeta]$.

Step 1 (Limit as $t\to0^+$). Define now $u_m:=\lfloor K^u+\ell_m\rfloor$ and $\tilde u_m=\lceil K^u+\ell_m\rceil$. Notice that $u_m=u$ and $\tilde u_m=\tilde u+m$, where $\tilde u=\lceil K^u\rceil$. Then, replacing K^v with

$K^u + \ell_m$ in (1.2.13), we infer

$$\lim_{t\to 0^+} \frac{\mu(K_{m,t}) - \mu(K^u + \ell_m)}{t} = \int_{\partial(K^u+\ell_m)} f(N_{K^u+\ell_m}((x,z)))\phi(z)\psi(x)\, d\mathcal{H}^n((x,z))$$
$$= \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\phi(u(x))\psi(x)\, dx$$
$$+ \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\phi(\tilde u(x)+m)\psi(x)\, dx$$
$$+ \int_{\partial \mathrm{dom}(u)} \rho_\zeta((N_{\mathrm{dom}(u)}(x)) \left(\int_{u(x)}^{\tilde u(x)+m} \phi(s)\, ds \right) \psi(x)\, d\mathcal{H}^{n-1}(x). \tag{1.2.14}$$

Notice that for m fixed the integrals in the last three lines of (1.2.14) are all finite, since they are just a suitable decomposition of the right-hand side of the first line. We claim that the right-hand side of (1.2.14) converges to

$$\int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\phi(u(x))\psi(x)\, dx + \int_{\partial \mathrm{dom}(u)} \rho_\zeta(N_{\mathrm{dom}(u)}(y))\Phi(u(y))\psi(x)\, d\mathcal{H}^{n-1}(y)$$

as $m \to \infty$. The first integral is determined by the graph of u and thus is fixed by the sequence we are considering. Concerning the second one, since $\tilde u_m = \tilde u + m$ is the parametrization of the upper part of $\partial(K^u + \ell_m)$, we infer $\lim_{m\to\infty} \tilde u_m(x) = +\infty$ for every $x \in \mathrm{dom}(u)$. Therefore $\lim_{m\to\infty} \phi(\tilde u_m(x)) = 0$ for every $x \in \mathrm{dom}(u)$ since ϕ converges to 0 by hypothesis. Then,

$$\left| \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\phi(\tilde u(x)+m)\psi(x)\, dx \right| \le$$
$$\max_{x\in \mathrm{dom}(u)} \phi(\tilde u(x)+m) \int_{\mathrm{dom}(u)} \left|\zeta(\nabla u(x))\psi(x)\right|\, dx.$$

Since the integral on the right-hand side of the inequality is finite (it is part of the weighted surface area measure of K^u when in Lemma 1.31 we consider $\Psi(x,z) = \psi(x)$) and

$$\max_{x\in \mathrm{dom}(u)} \phi(\tilde u(x)+m) \le \sup_{y\in[\inf \tilde u+m, +\infty]} \phi(y) \to 0$$

as $m \to \infty$, the integral on the third line of (1.2.14) converges to 0.

We conclude this step observing that

$$\int_{\partial \mathrm{dom}(u)} \rho_\zeta(N(x)) \left(\int_{u(x)}^{\tilde u_m(x)} \phi(s)\, ds \right) \psi(x)\, d\mathcal{H}^{n-1}(x)$$

converges to

$$\int_{\partial \mathrm{dom}(u)} \rho_\zeta(N(x))\Phi(u(x))\psi(x)\, d\mathcal{H}^{n-1}(x)$$

as $m \to \infty$. Indeed $\tilde u_m(x) \to \infty$ as $m \to \infty$, thus

$$\int_{u(x)}^{\tilde u_m(x)} \phi(s)\, ds \to \int_{u(x)}^{+\infty} \phi(s)\, ds = \Phi(u(x))$$

increasingly, and the limit is finite by hypothesis. Then since

$$\left|\rho_\zeta(N(x))\left(\int_{u(x)}^{\tilde{u}_m(x)}\phi(s)\,ds\right)\psi(x)\right| \leq |\rho_\zeta(N(x))\psi(x)|\,\Phi(u(x)),$$

the desired convergence is granted by the dominated convergence theorem.

Step 2 (The derivative exists). We now prove that $\mu(K_{m,t})$ is differentiable with respect to t for each t in $[0,\varepsilon]$ for every $m \in \mathbb{N}$ and ε suitably small, and its derivative converges uniformly as $m \to \infty$ on $[0,\varepsilon]$ to

$$\int_{\mathrm{dom}(u_t)} \zeta(\nabla u_t(x))\phi(u_t(x))\psi(x)\,dx + \int_{\partial\mathrm{dom}(u_t)} \rho_\zeta(N(x))\Phi(u_t(x))\psi(x)\,d\mathcal{H}^{n-1}(x).$$

For m fixed, the differentiability of $\mu(K_{m,t})$ follows from Theorem 1.29. Indeed this theorem implies

$$\frac{\mu(F_{t+t_0}(K^u+\ell_m)) - \mu(F_{t_0}(K^u+\ell_m))}{t} = \frac{\mu(F_tF_{t_0}(K^u+\ell_m)) - \mu(F_{t_0}(K^u+\ell_m))}{t},$$

and as long as K_{m,t_0} has non-empty interior we can apply Lemma 1.31 and have, for $t_0 \in [0,\varepsilon]$,

$$\frac{d\mu(K_{m,t})}{dt}\Big|_{t=t_0^+} = \int_{\mathbb{S}^n} \bar{\zeta}(N)\,dS_{\mu,K_{m,t_0}}(N). \tag{1.2.15}$$

Notice that $K^u+\ell_m$ has non-empty interior for every m since $K^u+\ell_1 \subset K^u+\ell_m$. Thus, we can choose ε such that (1.2.15) remains true for every m and for every $t_0 \in [0,\varepsilon]$. Since $K_{m,t}$ is continuous in t by Theorem 1.28 and $dS_{\mu,K_{m,t}}$ is weakly continuous (see Livshyts [Liv19, Proposition A.3]), the right derivative of $\mu(K_{m,t})$ is continuous in t on $[0,\varepsilon]$. Since if the right derivative of a function is continuous then the function itself is differentiable (see Bruckner [Bru94, Theorem 1.3, p. 40]), the function $\mu(K_{m,t})$ is differentiable in $(0,\varepsilon)$, as desired.

Step 3 (Uniform convergence). To prove the uniform convergence of the derivatives (1.2.15), we start by repeating the procedure for the limits of (1.2.14) for $t \in [0,\varepsilon]$, $\varepsilon > 0$ as chosen in the previous step. Consider the decomposition in (1.2.14) applied to $K_{m,t}$ for a general $t \in [0,\varepsilon]$. The first integral is independent of m. Indeed $\lfloor K_{m,t} \rfloor = u_t$ for every m since $K_{m,t} = K^{u,t} + \ell_m$. Furthermore, $\mathrm{dom}(u_{m,t}) = \mathrm{dom}(u_t)$ for every m. For the second integral, notice that since for $t \to 0^+$ one has $\lceil K_{m,t} \rceil =: \tilde{u}_{m,t} \to \tilde{u}_m = \tilde{u} + m$, we can find a sequence of values $M_m \to +\infty$ as $m \to \infty$ such that for every $t \in [0,\varepsilon]$ we have $M_m \leq \min_{\mathrm{dom}(u_t)} \tilde{u}_{m,t}$. Moreover, notice that for every $x \in \mathrm{dom}(u_t)$ we have $\nabla u_t(x) = -\nabla \tilde{u}_{m,t}(x)$ since $\tilde{u}_{m,t}$ is the reflection of u_t up to a constant. Then, for the second integral in (1.2.14) we have

$$\begin{aligned}
&\left|\int_{\mathrm{dom}(u_t)} \zeta(\nabla u_t(x))\phi(\tilde{u}_{m,t}(x))\psi(x)\,dx\right| \\
&\le \left(\sup_{t\in[M_m,+\infty]}\phi(t)\right)\int_{\mathrm{dom}(u_t)}|\zeta(\nabla u_t(x))\psi(x)|\,dx \\
&= \left(\sup_{t\in[M_m,+\infty]}\phi(t)\right)\int_{\mathrm{dom}(u_t)}|\zeta(\nabla u_t(x))\psi(x)|\,dx \\
&= \left(\sup_{t\in[M_m,+\infty]}\phi(t)\right)\int_{\partial K_-^{u,t}}|\bar{\zeta}(g^{-1}(\nabla u_t(x)))|\psi(x)\,d\mathcal{H}^n((x,z)) \\
&\le \left(\sup_{t\in[M_m,+\infty]}\phi(t)\right)\max_{N\in\mathbb{S}^n}|\bar{\zeta}(N)|\max_{t\in[0,\varepsilon]}\left(\int_{\partial K^{u,t}}\psi(x)\,d\mathcal{H}^n((x,z))\right),
\end{aligned}$$

where g was defined in (1.2.6), and therefore the first line converges to 0 independently of t. In the last inequality we have used that $\phi \ge 0$ and $\partial K_-^{u,t} \subset \partial K^{u,t}$. Notice that the maximum in t is bounded since $K^{u,t}$ is continuous in t. Finally, for the last integral, the convergence is granted again by the dominated convergence theorem and making use of the fact that $\mathrm{dom}(u_t) = \mathrm{dom}(u_{m,t})$. Indeed, we get

$$\begin{aligned}
&\Bigg|\int_{\partial\mathrm{dom}(u_t)} \rho_\zeta(N_{\mathrm{dom}(u_t)}(x))\Phi(u_t(x))\psi(x)\,d\mathcal{H}^{n-1}(x) - \\
&\int_{\partial\mathrm{dom}(u_{m,t})} \rho_\zeta(N_{\mathrm{dom}(u_{m,t})}(x))\left(\int_{u_t(x)}^{\tilde{u}_{m,t}}\phi(s)\,ds\right)\psi(x)\,d\mathcal{H}^{n-1}(x)\Bigg| \\
&= \left|\int_{\partial\mathrm{dom}(u_t)} \rho_\zeta(N(x))\left(\int_{\tilde{u}_{m,t}(x)}^{+\infty}\phi(s)\,ds\right)\psi(x)\,d\mathcal{H}^{n-1}(x)\right| \\
&\le \max_{\xi\in\mathbb{S}^{n-1}}|\rho_\zeta(\xi)|\max_{t\in[0,\varepsilon]}\left|\int_{\partial\mathrm{dom}(u_t)}\psi(x)\,d\mathcal{H}^{n-1}(x)\right|\left(\int_{C+m/2}^{+\infty}\phi(s)\,ds\right),
\end{aligned}$$

where $C = \max_{t\in[0,\varepsilon]}\max_{x\in\mathrm{dom}(u_t)} u_t(x)$. Notice that the maximum on $[0,\varepsilon]$ is finite since the integral is continuous in t, as $\mathrm{dom}(u_t)$ is the projection on $e_{n+1}^\perp$ of $K^{u,t}$, which is continuous in t. Then, as $m \to \infty$ the last integral converges to 0 uniformly on t.

Conclusion. Having concluded all the steps, we can now safely apply Lemma 1.34 with $f_m(t) = \mu(K_{m,t})$, concluding the proof. □

1.2.4 An application: Moment measures

Consider the case $\phi(z) = e^{-z}$, $\psi \equiv 1$. This point of view has been initially investigated by Colesanti and Fragalà [CF13] and Cordero-Erausquin and Klartag [CEK15]. The two interpretations stem from independent perspectives. The former aims to generalize some classical concepts from the Brunn-Minkowski theory. The latter originates from the interest in the KLS conjecture and solutions of differential equations in Kähler-Einstein manifolds; see, for example, Klartag [Kla14]. The first approach required C^2 regularity on the interior of the domain, while the second was restricted to *essentially continuous convex functions* (see [CEK15, Definition 2]).

Recently Rotem, in [Rot22a, Rot22c], significantly improved these results, dropping all regularity assumptions.

Theorem 1.35 (Rotem)**.** *Let* $u, v \in \mathrm{Conv}(\mathbb{R}^n)$*. If* $0 < \int_{\mathbb{R}^n} e^{-u(x)}\,dx < +\infty$*, then*

$$\mu(u, v^*) = \int_{\mathrm{dom}(u)} v^*(\nabla u(x)) e^{-u(x)}\,dx + \int_{\partial \mathrm{dom}(u)} h_{\mathrm{dom}(v)}(N(y)) e^{-u(y)}\,d\mathcal{H}^{n-1}(y).$$

Theorem 1.32 immediately implies the following variant of Theorem 1.35.

Corollary 1.36. *For every* $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ *and* $\zeta \in C_{rec}(\mathbb{R}^n)$ *satisfying* (**P**) *for* $t \geq 0$ *sufficiently small, and* μ *the measure in* $\mathbb{R}^{n+1}$ *such that* $d\mu(x, z) = e^{-z} dz dx$*, the first variation* $\mu(u, \zeta)$ *exists and is finite. Moreover, it has the same form as in Theorem 1.32.*

The truth is too simple: One must always get there by a complicated route.

Aurore Dupin (a.k.a. George Sand)

Chapter 2

Valuations: Old and new

The theory of valuations has been for decades a powerful tool and a fascinating subject, connecting analysis, algebra, and geometry. Consider a family $\mathcal{E}$ of subsets of $\mathbb{R}^n$ and a commutative semigroup $\mathcal{S}$. Our focus will be in particular on real-valued valuations, that is, $\mathcal{S} = (\mathbb{R}, +)$.

Definition 2.1. A functional $Y : \mathcal{E} \to \mathcal{S}$ is a valuation if

$$Y(A \cup B) + Y(A \cap B) = Y(A) + Y(B)$$

for every $A, B \in \mathcal{E}$ such that $A \cup B, A \cap B \in \mathcal{E}$.

These instruments were introduced by Dehn to solve Hilbert's third problem a few months after it was stated. In fact, it was the first problem from Hilbert's list to be solved [1]. There the choice for the family of sets was $\mathcal{E} = \mathcal{P}^n$, the polytopes of $\mathbb{R}^n$. More on this can be found, for example, in [Sah79]. It developed as an accessible and fruitful tool, especially in integral geometry (see [SW08]).

Consider now a real-valued valuation Y. Suppose that on $\mathcal{E}$ we have a binary associative operation (let us denote it by the symbol "+"), and that $\mathcal{E}$ is closed under multiplication for positive real numbers. That is, $\mathcal{E}$ is a *cone*. Our case studies are $\mathscr{K}^n$ with the Minkowski addition and the dilation, and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and its subfamilies with infimal convolution and epi-multiplication. For $A_1, \dots, A_m \in \mathcal{E}$ and $t_1, \dots, t_m \in [0, +\infty)$ we can consider the *polarization* of Y

$$Y(t_1 A_1 + \dots + t_m A_m).$$

One of the central concepts in the theory of valuations is that, under appropriate assumptions, this polarization exhibits a polynomial behavior in the coefficients $t_1, \dots, t_m$. We have seen an example of this in Theorem 1.8. In particular, a crucial role is taken by the coefficients of these polynomials.

We start by giving a summary of the main results and techniques in this field. The reader can find in [Sch14, Chapter 6] an exhaustive introduction to the topic.

[1] In a recent historical research, Ciesielska and Ciesielski [CC18] have traced back a proof of this result by Ludwik Antoni Birkenmajer. It was written for a mathematical competition in 1882.

2.1 Preliminaries

2.1.1 Valuations on $\mathscr{K}^n$

The main direction in which the theory is developed is for the choice $\mathcal{E} = \mathscr{K}^n$. The aim is to classify and characterize valuations that satisfy suitable properties. Let us define some of those playing a role in this chapter. Let $Y : \mathscr{K}^n \to \mathbb{R}$ be a valuation.

1. *Continuity*: We say that Y is *continuous* if for every sequence $K_m \to K$ in the Hausdorff metric, then $Y(K_m) \to Y(K)$.

2. *G-Invariance*: Let G be a group acting on $\mathscr{K}^n$. We say that Y is *G-invariant* if $Y(gK) = Y(K)$ for every $g \in G$ and $K \in \mathscr{K}^n$. The main choices are the group of translations and the group of rotations.

3. *a-Homogeneity*: If there exists $a \in \mathbb{R}$ such that $Y(tK) = t^a Y(K)$ for every $t \geq 0, K \in \mathscr{K}^n$, we say that Y is *a-homogeneous*.

We have introduced in Chapter 1 the concept of intrinsic volumes, and it is quite easy to check that, for example, the functionals V_i for $i = 0, \dots, n-1, n$ are valuations. Moreover, they are invariant under *rigid motions* (that is, the group generated by rotations and translations), continuous, and i-homogeneous. A bit more work shows that they are for every i. Something stronger is true for intrinsic volumes, as the content of the celebrated *Hadwiger's characterization theorem* (see [Had57]) shows.

Theorem 2.2. *A functional $Y : \mathscr{K}^n \to \mathbb{R}$ is a continuous, translation, and rotation invariant valuation if and only if there are constants $c_0, \dots, c_n \in \mathbb{R}^n$ such that*

$$Y(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

for every $K \in \mathscr{K}^n$.

The main applications of this result consist of integral formulas in stochastic geometry. Consider, for example, the group of rotations $SO(n)$ and the corresponding Haar probability measure σ on this group. For an hyperplane $H \subset \mathbb{R}^n$ consider the orthogonal projection $\mathrm{pr}_H : \mathbb{R}^n \to H$. Then it is easy to prove that, for a fixed hyperplane H, the functional on $\mathscr{K}^n$

$$K \mapsto \int_{SO(n)} V_{n-1}(\mathrm{pr}_{\theta H} K)\, d\sigma(\theta)$$

is a valuation, and is $(n-1)$-homogeneous, continuous and invariant under rigid motions. Thus, up to a multiplicative constant, this functional is the surface area. This kind of formula can be generalized to projections on lower-dimensional subspaces, and it is known as *Cauchy-Kubota formula* (see, for example, [HW20, Theorem 5.6]).

What happens if we remove the hypothesis of invariance under $SO(n)$? Again, even though less explicitly, it is possible to describe the space of these valuations, which possess a graded algebra structure. It is known as McMullen's homogeneous decomposition theorem [McM77] and reads as follows.

Theorem 2.3. *Let $Y : \mathcal{K}^n \to \mathbb{R}$ be a continuous, translation invariant valuation. Then there are continuous, translation invariant valuations $Y_0, \dots, Y_n$ on $\mathcal{K}^n$ such that Y_i is homogeneous of degree i, for every $i = 0, 1, \dots, n$, and*

$$Y(K) = Y_0(K) + \cdots + Y_n(K),$$

for every $K \in \mathcal{K}^n$.

If we consider

$$\mathrm{Val}(\mathbb{R}^n) := \{Y : \mathcal{K}^n \to \mathbb{R} : Y \text{ is a continuous and translation invariant valuation}\}$$

and its subfamilies

$$\mathrm{Val}^i(\mathbb{R}^n) := \{Y \in \mathrm{Val}(\mathbb{R}^n) : Y \text{ is } i\text{-homogeneous}\},$$

for $0 \le i \le n$, then Theorem 2.3 reads as

$$\mathrm{Val}(\mathbb{R}^n) = \bigoplus_{i=0}^{n} \mathrm{Val}^i(\mathbb{R}^n).$$

In other words, continuous and translation invariant valuations form a graded algebra, where the degree is set by homogeneity.

These functionals can be described more explicitly in four cases, namely $i = 0, 1, n-1, n$. For $i = 0$, Y_0 is proportional to the Euler characteristic. For $i = n$, Hadwiger [Had57, p. 79] proved the following.

Theorem 2.4. *Let $Y : \mathcal{P}^n \to \mathbb{R}$ be a translation invariant valuation on $\mathcal{P}^n$. If Y is homogeneous of degree n, then $Y = cV_n$ with a real constant c. If Y is continuous, the result is extended by continuity to $\mathcal{K}^n$.*

The case $i = n-1$ was treated by McMullen [McM80].

Theorem 2.5. *For a functional $Y : \mathcal{K}^n \to \mathbb{R}$, $Y \in \mathrm{Val}^{n-1}(\mathbb{R}^n)$ if and only if there exists a continuous function $\eta : \mathbb{S}^{n-1} \to \mathbb{R}$ such that*

$$Y(K) = \int_{\mathbb{S}^{n-1}} \eta(\nu)\, dS_{n-1}(K, \nu) \tag{2.1.1}$$

for every $K \in \mathcal{K}^n$. The function η is uniquely determined up to the addition of the restriction of a linear function to $\mathbb{S}^{n-1}$.

Finally, for $i = 1$, Goodey and Weil [GW84] gave the following characterization.

Theorem 2.6. *For a functional $Y : \mathcal{K}^n \to \mathbb{R}$, $Y \in \mathrm{Val}^1(\mathbb{R}^n)$ if and only if there are two sequences of convex compact sets $(L_j), (W_j)$ in $\mathcal{K}^n$ such that*

$$Y(K) = \lim_{j\to\infty} [V(K, L_j, \dots, L_j) - V(K, W_j, \dots, W_j)] \tag{2.1.2}$$

holds uniformly on compact subsets of $\mathcal{K}^n$. That is, for $R > 0$ fixed, for every $K \subset B_R(0)$ the limit in (2.1.2) converges depending only on R.

Later, we prove functional generalizations of Theorems 2.6 and 2.5. To conclude the picture, McMullen conjectured that all translation invariant and continuous valuations are approximated by combinations of mixed volumes. This was positively solved by Alesker [Ale01] through his remarkable *Irreducibility theorem.*

2.1.2 Parallel sets and support measures.

We now present some instruments for the local study of the geometry of convex sets. These instruments play a vital role in the theory of valuations, and we will use them to recover functional versions of valuations. In Section 1.1, we introduced the concept of parallel set and the Steiner formula (1.1.3) describing intrinsic volumes. This section aims to show the local behavior of these objects. In this chapter, when referring to measurable sets, we always refer to Borel measurable sets. For the exposition, we follow [Sch14, Chapter 4].

Consider $K \in \mathscr{K}^n, t > 0$. The Minkowski sum $K_t = K + tB^n$ can be considered as the set of points $x \in \mathbb{R}^n$ such that $0 \leq d(x, K) \leq t$. It is then natural to consider, for a subset $\beta \subset K$, the set of points $x \in \mathbb{R}^n$ for which $d(K, x) \leq t$ and for which the *nearest point* $p(x, K) \coloneqq \operatorname{argmin}_{y \in K} d(x, y)$ (also known as *metric projection*) is in β. Notice that from the uniqueness of the projection on closed convex sets (see, for example, [Bre11, Theorem 5.2]), $p(x, K)$ is unique for every $x \in \mathbb{R}^n$. Alternatively, we can consider $\beta \subset \mathbb{S}^{n-1}$ and ask for the set of all $x \in \mathbb{R}^n$ such that $0 < d(K, x) \leq t$ and the unit vector $u(K, x)$ from $p(K, x)$ pointing towards x is in β.

Let us now be more precise. Consider $\Sigma = \mathbb{R}^n \times \mathbb{S}^{n-1}$. For a fixed $K \in \mathscr{K}^n$, a pair $(x, \xi) \in \Sigma$ is called a *support element* of K if $x \in \partial K$ and ξ is an outer unit normal vector of K at x. Note that for every $x \in \mathbb{R}^n$, the pair $(p(K, x), u(K, x))$ is a support element of K. We define the *normal bundle* of K as the set

$$\mathrm{Nor} K = \{(x, \xi) \in \Sigma \,:\, (x, \xi) \text{ is a support element of } K\}.$$

Consider, moreover, the map

$$\begin{aligned} f_t \,:\, K_t \setminus K \to & \qquad \Sigma \\ & x \mapsto (p(K, x), u(K, x)), \end{aligned}$$

which is continuous and measurable. Thus we can consider on Σ the image measure $\mu_t(K, \cdot)$ under f_t of the Lebesgue measure on $\mathbb{R}^n$. The map $\mu_t(\cdot, \cdot)$ is a valuation on the semigroup (with the operation of addition) of the Borel measures on Σ (where we consider the product topology between the classical ones on $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ respectively), and for each Borel set $\beta \subset \Sigma$, $\mu_t(\cdot, \beta)$ is measurable (considering the Borel sigma algebra on $\mathscr{K}^n$) by [Sch14, Theorems 4.1.2 and 4.1.3].

A useful feature of $\mu_t(K, \cdot)$ is that it admits a polynomial expansion analogous to (1.1.3). The main properties of this measure are summarized by the following statement (see [Sch14, Theorem 4.2.1]).

Theorem 2.7. *For every convex set $K \in \mathscr{K}^n$ there exist finite positive measures $\Theta_i(K, \cdot), 0 \leq i \leq n-1$, on the Borel sigma-algebra of Σ such that for every $t > 0$ the measure $\mu_t(K, \cdot)$ satisfies the polynomial expansion*

$$\mu_t(K, \beta) = \sum_{i=0}^{n-1} t^{n-i} \binom{n}{i} \Theta_i(K, \beta)$$

for every Borel set $\beta \subset \Sigma$.

The mapping $K \mapsto \Theta_i(K, \cdot)$ is weakly continuous and is a measure-valued valuation. Moreover, for each Borel set $\beta \subset \Sigma$, the function $\Theta_i(\cdot, \beta)$ is measurable. The coefficients Θ_i are called support measures.

We have the following immediate consequence, which gives us a large class of continuous valuations on $\mathscr{K}^n$.

Corollary 2.8. *Consider* $f \in C(\Sigma)$. *Then, for every* $0 \leq i \leq n-1$*, the functional* $Z : \mathscr{K}^n \to \mathbb{R}$ *defined by*

$$K \mapsto \int_{\Sigma} f(x,\xi)\, d\Theta_i\,(K,(x,\xi)) \tag{2.1.3}$$

is a continuous valuation.

Proof. The statement follows at once from Theorem 2.7: Indeed since f is continuous, the weak continuity of the support measures implies the continuity of Z. The valuation property for Z descends from the same property of the measures Θ_i. □

Note that by [Sch14, Lemma 4.2.2] if K is a convex body with boundary of class C^2 (strict convexity is not required), (2.1.3) becomes

$$\int_{\mathbb{R}^n} f(x, u(K,x))\, dC_i(K,x),$$

while if K is strictly convex (∂K contains no segments) it reads as

$$\int_{\mathbb{S}^{n-1}} f(p(K,\xi),\xi)\, dS_i(K,\xi), \tag{2.1.4}$$

where the measures C_i are S_i are the marginals of the support measure Θ_i, and are respectively known as i-th *curvature measure* and i-th *surface area measure.* Notice that S_{n-1} corresponds to (1.1.4).

2.1.3 Valuations on convex functions

The theory we briefly presented in $\mathscr{K}^n$ has a functional counterpart. This process started with the works of Ludwig [Lud11a,Lud11b,Lud12] and Tsang [Tsa10a,Tsa10b], with a focus on L^p and Sobolev spaces. The idea is pretty simple: We still consider unions and intersections of sets, but since these sets are epigraphs, these operations can be replaced by pointwise minimum and maximum of functions, respectively.

Definition 2.9. If $\mathcal{E}$ is a family of real-valued functions, a functional $Z : \mathcal{E} \to \mathbb{R}$ is a (real-valued) valuation if

$$Z(u \wedge w) + Z(u \vee w) = Z(u) + Z(w)$$

for every $u, w \in \mathcal{E}$ such that $u \wedge w, u \vee w \in \mathcal{E}$, where $\vee$ and $\wedge$ are the pointwise minimum and maximum, respectively.

More recently, Colesanti, Ludwig, and Mussnig wrote a series of papers [CLM17, CLM20a, CLM20b, CLM20c, CLM21, CLM22a, CLM22b] focusing on convex functions. In this section, we survey the main advances in this direction, upon which we build the remainder of this chapter. A different approach was followed by Knoerr [Kno21, Kno22, Kno], who studied this topic from the points of view of functional analysis and differential geometry.

As in Section 2.1.1, the main results concern the classification and characterization of these functionals under certain conditions. The family where this process appeared to be more successful is $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. The topology we consider on this space is again the one induced by epi-convergence. The reason behind this fact is that, as proved in [CLM20b], the only meaningful valuation on the family $\mathrm{Conv}(\mathbb{R}^n)$ satisfying suitable hypothesis is the constant one, which motivates the choice of a smaller (even though still dense) family of functions. As mentioned in the preliminaries, $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is equivalent to the other family $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ through duality. Indeed, by the homeomorphism (1.1.5), $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ if and only if $u^* \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$. We remark that the results obtained by this theory are not just functional versions of the classical theory, but they properly generalize those since $\mathscr{K}^n$ can be embedded in $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ via the map

$$K \mapsto I_K.$$

The properties we require for these functionals are exactly the same ones we require in $\mathscr{K}^n$, but now we can formulate them from a functional point of view. Let $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ be a valuation.

1. *Epi-continuity*: We say that Z is *epi-continuous* if for every sequence u_m converging to u under epi-convergence, then $Z(u_m) \to Z(u)$.

2. *G-invariance*: Let G be a group acting on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We say that Z is G-invariant if $Z(gu) = Z(u)$ for every $g \in G$ and $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$. We consider rotations of the domain, that is $Z(u \circ \theta^{-1}) = Z(u)$ for every $\theta \in SO(n)$, and *epi-translations*, that is $Z(u \circ \tau + t) = Z(u)$ for every translation τ on $\mathbb{R}^n$ and $t \in \mathbb{R}$.

3. *Epi-homogeneity*: If there exists $a \in \mathbb{R}$ such that $Z(t \cdot u) = t^a Z(u)$ for every $t \geq 0$ and $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, we say that Z is *epi-homogeneous* of degree a.

By the Fenchel-Legendre transform, all these properties have a dual expression in $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$. We discuss this in Section 2.2.2.

As a consequence of these properties, the functional

$$\begin{aligned} Y \colon \mathscr{K}^n &\to \mathbb{R} \\ K &\mapsto Z(I_K) \end{aligned}$$

is a valuation that inherits respectively continuity with respect to the Hausdorff metric, G-invariance (when G-acts on $\mathbb{R}^n$ only), and homogeneity, respectively.

But what do these valuations on convex functions look like? For $\mathscr{K}^n$, we mentioned in Section 2.1.1 that mixed volumes are examples of translation invariant and continuous valuations, and one gains rotation invariance in the particular case of intrinsic volumes. In general, Corollary 2.8 describes a wide class of continuous valuations. A similar process was followed in [CLM20a], where *Hessian measures* were used, instead of support measures.

To define Hessian measures, we first need to introduce an alternative notion of parallel set: For $u \in \mathrm{Conv}(\mathbb{R}^n)$, $t > 0$, and $\beta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$, consider the set

$$P_t(u, \beta) := \{x + ty \,:\, (x, y) \in \beta, y \in \partial u(x)\}.$$

Here the subgradient substitutes the notion of normal cone, and instead of the normal bundle, for $u \in \mathrm{Conv}(\mathbb{R}^n)$ we consider the graph of the subgradient

$$\Gamma_u := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, y \in \partial u(x)\}.$$

If we evaluate the n-dimensional Hausdorff measure of $P_t(u,\beta)$, we obtain a Steiner-type formula as proved in [CLM20a, Theorem 7.1] (note the analogy with Theorem 2.7).

Theorem 2.10. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t > 0$, there are non-negative Borel measures $\Xi_i(u,\cdot), 0 \le i \le n$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that*

$$\mathcal{H}^n(P_t(u,\beta)) = \sum_{i=0}^{n} \binom{n}{i} t^i \Xi_{n-i}(u,\beta)$$

for every $\beta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$ and $t > 0$. We call these measures Hessian measures.

By [CLM20a, Theorem 7.3], Hessian measures are weakly-continuous with respect to epi-convergence. Moreover, if $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, for $\beta \in \mathcal{B}(\mathbb{R}^n)$ they take the form

$$\Xi_i(u, \beta \times \mathbb{R}^n) = \int_\beta [\det D^2 u(x)]_{n-i}\, dx,$$

where for a diagonalizable matrix $\mathbb{A}$, $[\mathbb{A}]_{n-i}$ is the $(n-i)$-th elementary symmetric function of the eigenvalues of $\mathbb{A}$, and D^2 is the Hessian matrix.

For the family $\mathrm{Conv}(\mathbb{R}^n)$, Colesanti, Ludwig, and Mussnig in [CLM20a, Theorem 1.1] proved the following result.

Theorem 2.11. *Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support with respect to the second and third variables. For every $1 \le i \le n$, the functional $F_{i,\zeta} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, defined by*

$$F_{i,\zeta}(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(u(x), x, y)\, d\Xi_i(u, (x,y)),$$

is a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$. If $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then

$$F_{i,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x)) [D^2 u(x)]_i\, dx.$$

The proof of Theorem 2.11 uses the crucial fact that Hessian measures are measure-valued valuations on $\mathrm{Conv}(\mathbb{R}^n)$, as proved in [CLM20a, Theorem 9.2]. With a better idea of what to expect, we can start requiring something else on top of epi-continuity. Consider, for example, epi-translation invariance. As we mentioned earlier, $\mathrm{Conv}(\mathbb{R}^n)$ is "too big" to obtain meaningful valuation under invariance hypotheses, and, from now on, the results we present have $\mathrm{Conv_{sc}}(\mathbb{R}^n)$ (or its subfamily $\mathrm{Conv_{cd}}(\mathbb{R}^n)$) as ambient space. In [CLM20b, Theorem 1] the following extension of Theorem 2.3 was proved. This result was also obtained by different methods in [Kno21, Theorem 1.1].

Theorem 2.12. *If $Z : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ is a continuous and epi-translation invariant valuation, then there are continuous and epi-translation invariant valuations $Z_0, \dots, Z_n : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that Z_i is epi-homogeneous of degree i and*

$$Z = Z_0 + \cdots + Z_n.$$

By Theorem 2.11, some examples of continuous, epi-translation invariant, and epi-homogeneous of degree i valuations have the form

$$Z_{n-i,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(\nabla u(x))[D^2u(x)]_{n-i}\,dx$$

for $\zeta \in C_c(\mathbb{R}^n)$ if $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C^2_+(\mathbb{R}^n)$.

As for Theorem 2.3, Theorem 2.12 can be expressed in the language of graded algebras. Consider the family of valuations

$$\mathrm{VConv}_{\mathrm{sc}}(\mathbb{R}^n) := \{Z\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R} \,:\, Z \text{ is a continuous and translation invariant valuation}\}.$$

As before, we can then consider splitting this family into its homogeneous components, that is, for $0 \le i \le n$,

$$\mathrm{VConv}^i_{\mathrm{sc}}(\mathbb{R}^n) := \{Z \in \mathrm{VConv}_{\mathrm{sc}}(\mathbb{R}^n) \,:\, Z \text{ is epi-homogeneous of degree } i\},$$

and therefore Theorem 2.12 reads as

$$\mathrm{VConv}_{\mathrm{sc}}(\mathbb{R}^n) = \bigoplus_{i=0}^{n} \mathrm{VConv}^i_{\mathrm{sc}}(\mathbb{R}^n).$$

The extension of Theorem 2.2 is instead more delicate and requires the introduction of specific families of functions. We briefly report the following result for completeness. Let $C_b((0,\infty))$ be the set of continuous functions on $(0,\infty)$ with bounded support. For $0 \le j \le n-1$, let

$$D^n_j := \left\{\zeta \in C_b((0,\infty)) \,:\, \lim_{s\to 0^+} s^{n-j}\zeta(s) = 0,\ \lim_{s\to 0^+} \int_s^\infty t^{n-j-1}\zeta(t)dt \text{ exists and is finite}\right\}.$$

Here we use the notation $D^n_n = C_c([0,\infty))$. The characterization of continuous, epi-translation invariant, and rotation invariant valuations [CLM20c, Theorems 1.2 and 1.3] reads as follows. The first part regards the well-posedness of the integral form of the valuations, while the second one extends Theorem 2.2.

Theorem 2.13. *For $i \in \{0,\dots,n\}$ and $\zeta \in D^n_i$, there exists a unique, continuous, epi-translation and rotation invariant valuation $V_{i,\zeta} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ such that*

$$V_{i,\zeta}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[D^2u(x)]_{n-i}\,dx$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap C^2_+(\mathbb{R}^n)$.
Consider now a functional $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$. Then, Z is a continuous, epi-translation and rotation invariant valuation if and only if there exist functions $\zeta_i \in D^n_i, i = 0,\dots,n$, such that

$$Z = V_{0,\zeta_0} + \cdots + V_{n,\zeta_n}.$$

Other descriptions and further discussions on these functionals can be found in [CLM21, CLM22a, CLM22b].

As for continuous and translation invariant valuations on $\mathcal{K}^n$ with a fixed degree of homogeneity, the investigation can be brought further in some cases. In particular, continuous and epi-translation invariant valuations with homogeneity of degree $i = n$ or $i = 1$ have more detailed descriptions. The case $i = n$ was investigated for the first time in [CLM20b, Theorem 2].

Theorem 2.14. *For a functional $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, $Z \in \mathrm{VConv}^n_{\mathrm{sc}}(\mathbb{R}^n)$ if and only if there exists $\zeta \in C_c(\mathbb{R}^n)$ such that*

$$Z(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\, dx$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

As we show in Section 2.2.1, not only does this result acts as a functional version of Theorem 2.5, it can be deduced from the latter result in $\mathcal{K}^{n+1}$. This was proved in [KU23, Section 4].

For $i = 1$, we have the counterpart of Theorem 2.6. This result is in [KU23, Theorem 1.5].

Theorem 2.15. *Every $Z \in VConv^1_{sc}(\mathbb{R}^n)$ can be approximated uniformly on compact subsets of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ by a sequence (Z_j) of valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ with the following properties:*

1. *Z_j is a continuous, epi-translation invariant valuation for each $j \in \mathbb{N}$.*
2. *For every $j \in \mathbb{N}$ there exist two functions $\ell_j, w_j \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ such that*

$$Z_j(u) = \int_{\mathrm{dom}(\ell_j)} u^*(\nabla \ell_j(x))\, dx - \int_{\mathrm{dom}(w_j)} u^*(\nabla w_j(x))\, dx$$

   for all $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$.

In particular, for every compact set $C \subset \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $Z_j(u) \to Z(u)$ uniformly for every $u \in C$. That is, the convergence behavior depends only on C.

Identifying compact subsets in $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is not as easy as in $\mathcal{K}^n$. A characterization of relatively compact subsets of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is given by [Kno21, Proposition 2.4], and reads as follows.

Proposition 2.16. *A subset $U \subset \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is relatively compact if and only if it is bounded on compact subsets of $\mathbb{R}^n$, that is, for any compact subset $A \subset \mathbb{R}^n$ there exists a constant $c(A) > 0$ such that*

$$\sup_{x \in A} u^*(x) \leq c(A)$$

for every $u \in U$.

In the following sections, we present one of our contributions to this field. In particular, our approach is to exploit the relation between $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ and $\mathcal{K}^{n+1}$, as shown in Section 1.1.3. We investigate the relation between the valuations on these two spaces and use them to prove Theorems 2.14 and 2.15.

2.2 Inducing valuations from $\mathscr{K}^{n+1}$ to $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$

The content of this section is from [KU23]. We will use the map $\lfloor\cdot\rfloor : \mathscr{K}^{n+1} \to \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ to interpret valuations on $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ as valuations on convex sets in $\mathbb{R}^{n+1}$. This is precisely the content of the following Lemma.

Lemma 2.17. *If $K, L \in \mathscr{K}^{n+1}$ are such that $K \cup L \in \mathscr{K}^{n+1}$, then*

$$\lfloor K\cap L\rfloor = \lfloor K\rfloor \vee \lfloor L\rfloor, \qquad\qquad \lfloor K\cup L\rfloor = \lfloor K\rfloor \wedge \lfloor L\rfloor.$$

Proof. By definition

$$\begin{aligned}
\lfloor K\cap L\rfloor(x) =& \inf\{t\in\mathbb{R} : (x,t)\in K\cap L\}\\
\geq& \inf\{t\in\mathbb{R} : (x,t)\in K\} \vee \inf\{t\in\mathbb{R} : (x,t)\in K\cap L\}\\
=& \lfloor K\rfloor(x) \vee \lfloor L\rfloor(x),\\
\lfloor K\cup L\rfloor(x) =& \inf\{t\in\mathbb{R} : (x,t)\in K\cup L\}\\
\leq& \inf\{t\in\mathbb{R} : (x,t)\in K\} \wedge \inf\{t\in\mathbb{R} : (x,t)\in K\cap L\}\\
=& \lfloor K\rfloor(x) \wedge \lfloor L\rfloor(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{dom}(\lfloor K\cap L\rfloor) &= \mathrm{dom}(\lfloor K\rfloor) \cap \mathrm{dom}(\lfloor L\rfloor),\\
\mathrm{dom}(\lfloor K\cup L\rfloor) &= \mathrm{dom}(\lfloor K\rfloor) \cup \mathrm{dom}(\lfloor L\rfloor),
\end{aligned}$$

as the domains are just the image of the corresponding convex set under the natural projection onto $H \cong \mathbb{R}^n$. In particular, both sides of each of the inequalities are finite if and only if one of the two sides is finite. We thus only have to consider points belonging to the corresponding domains. Assume that $\lfloor K\cap L\rfloor(x) < +\infty$. As $\lfloor K\rfloor(x) \vee \lfloor L\rfloor(x) \leq \lfloor K\cap L\rfloor(x) < +\infty$,

$$\begin{aligned}
\{(x,t)\in\mathbb{R}^{n+1} : t\in[\lfloor K\rfloor(x), \lfloor K\cap L\rfloor(x)]\} \subset K,\\
\{(x,t)\in\mathbb{R}^{n+1} : t\in[\lfloor L\rfloor(x), \lfloor K\cap L\rfloor(x)]\} \subset L
\end{aligned}$$

by convexity, as the points corresponding to the boundary points belong to these sets. Thus $(x, \lfloor K\rfloor(x) \vee \lfloor L\rfloor(x)) \in K\cap L$, which implies

$$\lfloor K\cap L\rfloor(x) \leq \lfloor K\rfloor(x) \vee \lfloor L\rfloor(x).$$

Now assume that $\lfloor K\cup L\rfloor(x) < +\infty$. Then $(x, \lfloor K\cup L\rfloor(x)) \in K\cup L$. Without loss of generality, we may assume that $(x, \lfloor K\cup L\rfloor(x)) \in K$. Then

$$\lfloor K\rfloor(x) \wedge \lfloor L\rfloor(x) \leq \lfloor K\rfloor(x) \leq \lfloor K\cup L\rfloor(x)$$

by the definition of $\lfloor K\rfloor(x)$. □

With Lemma 2.17 at hand, we can now formally show how a valuation on $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ induces a valuation on $\mathscr{K}^{n+1}$.

Theorem 2.18. *For* $Z : \mathrm{Conv_{cd}}(\mathbb{R}^n) \to \mathbb{R}$ *consider* $Y : \mathcal{K}^{n+1} \to \mathbb{R}$ *defined by*

$$Y(K) = Z(\lfloor K \rfloor).$$

Then Y *has the following properties:*

1. *If* Z *is a valuation, then so is* Y.
2. *If* Z *is continuous, then* Y *is continuous with respect to the Hausdorff metric.*
3. *If* Z *is epi-translation invariant, then* Y *is translation invariant, that is*
$$Y(K+X) = Y(K) \quad \text{for all } K \in \mathcal{K}^{n+1}, X \in \mathbb{R}^{n+1}.$$
4. *If* Z *is epi-homogeneous of degree* j*, then* Y *is* j*-homogeneous, that is,*
$$Y(tK) = t^j Y(K) \quad \text{for all } K \in \mathcal{K}^{n+1}, t > 0.$$

Proof. 1. If $K, L \in \mathcal{K}^{n+1}$ satisfy $K \cup L \in \mathcal{K}^{n+1}$, then

$$\lfloor K \cap L \rfloor = \lfloor K \rfloor \vee \lfloor L \rfloor, \qquad \lfloor K \cup L \rfloor = \lfloor K \rfloor \wedge \lfloor L \rfloor.$$

by Lemma 2.17. Thus

$$\begin{aligned} Y(K \cup L) + Y(K \cap L) =& Z(\lfloor K \cup L \rfloor) + Z(\lfloor K \cap L \rfloor) \\ =& Z(\lfloor K \rfloor \wedge \lfloor L \rfloor) + Z(\lfloor K \rfloor \vee \lfloor L \rfloor) \\ =& Z(\lfloor K \rfloor) + Z(\lfloor L \rfloor) = Y(K) + Y(L). \end{aligned}$$

2. If Z is continuous, then $Y = Z \circ \lfloor \cdot \rfloor$ is continuous due to the continuity of $\lfloor \cdot \rfloor$, compare Lemma 1.16.

3. For $X = (v, c) \in \mathbb{R}^n \times \mathbb{R}$ and $K \in \mathcal{K}^{n+1}$, the definition of $\lfloor K \rfloor$ implies for $x \in \mathbb{R}^n$

$$\begin{aligned} \lfloor K + X \rfloor(x) =& \inf\{s \in \mathbb{R} : (x, s) \in K + X\} = \inf\{s \in \mathbb{R} : (x - v, s - c) \in K\} \\ =& \inf\{s + c : s \in \mathbb{R}, (x - v, s) \in K + X\} \\ =& \lfloor K \rfloor(x - v) + c. \end{aligned}$$

If Z is epi-translation invariant, we obtain

$$Y(K + X) = Z(\lfloor K + X \rfloor) = Z(\lfloor K \rfloor(\cdot - v) + c) = Z(\lfloor K \rfloor) = Y(K).$$

Thus Y is translation invariant.

4. For $t > 0$ we calculate for $x \in \mathbb{R}^n$

$$\begin{aligned} \lfloor tK \rfloor(x) =& \inf\{s \in \mathbb{R} : (x, s) \in tK\} = \inf\left\{s \in \mathbb{R} : \left(\frac{x}{t}, \frac{s}{t}\right) \in K\right\} \\ =& \inf\left\{ts : s \in \mathbb{R}, \left(\frac{x}{t}, s\right) \in K\right\} \\ =& t\lfloor K \rfloor\left(\frac{x}{t}\right). \end{aligned}$$

Thus $\lfloor tK \rfloor = t \cdot \lfloor K \rfloor$, which implies

$$Y(tK) = Z(\lfloor tK \rfloor) = Z(t \cdot \lfloor K \rfloor) = t^j Z(\lfloor K \rfloor) = t^j Y(K)$$

if Z is epi-homogeneous of degree j.

□

2.2.1 New proof of Theorem 2.14

We will deduce the representation formula established in Theorem 2.14 from McMullen's Theorem 2.5. More precisely, we will show that the same representation holds for continuous, epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ that are epi-homogeneous of degree n. As $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \subset \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is dense, we will prove in Corollary 2.20 that this establishes the representation formula for the corresponding space of valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ by continuity.

Theorem 2.19. *Let $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \to \mathbb{R}$ be a valuation that is continuous, epi-translation invariant, and epi-homogeneous of degree n. Then there exists a unique function $\zeta \in C_c(\mathbb{R}^n)$ such that*

$$Z(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\, dx \tag{2.2.1}$$

for every $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$.

Proof. Given a functional Z with the properties stated above, $Y(K) := Z(\lfloor K \rfloor)$ defines a valuation on $\mathscr{K}^{n+1}$ which is continuous, translation invariant, and n-homogeneous by Theorem 2.18. By McMullen's Theorem 2.5 there exists $\eta \in C(\mathbb{S}^n)$ such that

$$Y(K) = \int_{\mathbb{S}^n} \eta(N)\, dS_n(K, N)$$

for every $K \in \mathscr{K}^{n+1}$. If we define $\tilde{\eta}(N) := [\eta(N) + \eta(R_H N)]/2$, then the valuation

$$\tilde{Y}(K) := \int_{\mathbb{S}^n} \tilde{\eta}(N)\, dS_n(K, N)$$

thus satisfies

$$Z(u) = Y(K^u) = \tilde{Y}(K^u).$$

We will work with $\tilde{Y}$ and the function $\tilde{\eta} \in C(\mathbb{S}^n)$.

For $K \in \mathscr{K}^n$ a convex set in H and $\ell > 0$, consider the cylinder $C(K, \ell) = K \times [0, \ell] \in \mathscr{K}^{n+1}$. Then by definition $I_K = \lfloor C(K, \ell) \rfloor$, and therefore we infer

$$\begin{aligned} Z(I_K) =& \tilde{Y}(C(K,\ell)) = 2\tilde{\eta}(-e_{n+1}) V_n(K) + \ell \int_{\mathbb{S}^n \cap H} \tilde{\eta}(N)\, dS_n(C(K,\ell), N) \\ =& 2\tilde{\eta}(-e_{n+1}) V_n(K) + \ell \int_{\mathbb{S}^{n-1}} \tilde{\eta}(\nu)\, dS_{n-1}(K, \nu), \end{aligned}$$

where we identify $\mathbb{S}^{n-1}$ and $\mathbb{S}^n \cap H$. As the left-hand side of this equation is independent of $\ell > 0$ (see Figure 2.1), we infer that

$$\int_{\mathbb{S}^{n-1}} \tilde{\eta}(\nu)\, dS_{n-1}(K, \nu) = 0 \tag{2.2.2}$$

for every $K \in \mathscr{K}^n$. We may consider the left-hand side of (2.2.2) as a valuation on convex sets in $\mathscr{K}^n$ that is continuous, translation invariant, and $(n-1)$-homogeneous. As it vanishes identically, McMullen's Theorem 2.5 implies that $\tilde{\eta}|_{\mathbb{S}^n \cap H}$ is the restriction of a linear function to $\mathbb{S}^n \cap H$.

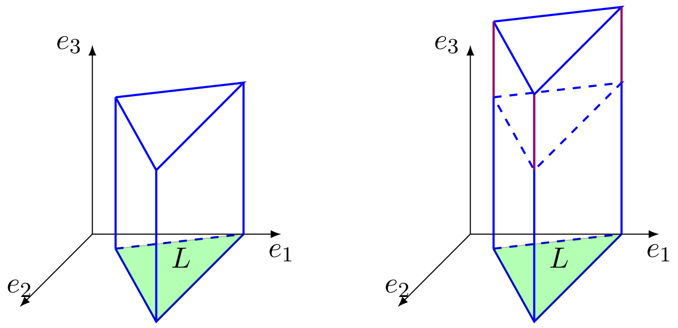

Figure 2.1: If we make the cylinder taller, it does not affect the portion of the boundary perceived by the valuation.

In particular, there exists a linear function $l : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\tilde{\eta} + l \equiv 0$ on the equator $\mathbb{S}^n \cap H$, and we set $\hat{\eta} = \tilde{\eta} + \frac{1}{2}[l + l \circ R_H]$. Then $\hat{\eta}$ vanishes on the equator $\mathbb{S}^n \cap H$. Using Theorem 1.18 and the fact that linear functions belong to the kernel of the surface area measure, we thus obtain for $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$

$$\begin{aligned} Z(u) =& \tilde{Y}(K^u) = \int_{\mathbb{S}^n} \hat{\eta}(N)\, dS_n(K^u, N) = 2 \int_{\mathbb{S}^n_-} \hat{\eta}(N)\, dS_n(K^u, N) \\ =& \int_{\mathrm{dom}(u)} 2\hat{\eta} \left(\frac{(\nabla u(x), -1)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \sqrt{1 + |\nabla u(x)|^y}\, dx, \end{aligned}$$

which for $\zeta(y) := 2\hat{\eta}\left(\frac{(y,-1)}{\sqrt{1+|y|^2}}\right)\sqrt{1+|y|^2}$ gives the desired representation in equation (2.2.1). Here we used that $\hat{\eta}$ vanishes on the equator $\mathbb{S}^n \cap H$ and is symmetric with respect to H.

To prove that ζ has compact support, one can use the same argument given by Colesanti, Ludwig, and Mussnig in the proof of [CLM20b, Proposition 27], which we include for completeness. Suppose by contradiction that the support is not compact. Then we can find a sequence $y_j \in \mathbb{R}^n$ such that $|y_j| \to \infty$, $\zeta(y_j) \neq 0$ for every $j \in \mathbb{N}$ and

$$\lim_{j \to \infty} \frac{y_j}{|y_j|} = \nu.$$

Consider the sets

$$B_j := \{x \in y_j^\perp : |x| \le 1\}, \quad B_\infty := \{x \in \nu^\perp : |x| \le 1\}$$

and define the cylinders

$$C_j := \left\{ x + t\frac{y_j}{|y_j|} : x \in B_j,\ t \in \left[0, \frac{1}{|\zeta(y_j)|}\right] \right\}.$$

For $y \in \mathbb{R}^n$ let l_y denote the linear function $x \mapsto x \cdot y$. Consider the sequence

$$u_j = l_{y_j} + I_{C_j}$$

in $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. By construction, $\min_{x\in\mathrm{dom}(u_j)} u_j(x) = 0$. For $t > 0$, the sublevel sets are given by

$$\{u_j \le t\} = \left\{ x + s\frac{y_j}{|y_j|} : x \in B_j,\ s \in \left[0, \min\left\{\frac{t}{|y_j|}, \frac{1}{|\zeta(y_j)|}\right\}\right]\right\},$$

so $\{u_j \le t\} \to B_\infty$ in this case. Obviously, the sublevel sets are empty for $t < 0$. Lemma 1.11 thus implies that the sequence (u_j) converges to I_{B_∞}.

Now note that $S_n(K^{u_j})$ is concentrated on $(H \cap \mathbb{S}^n) \cup \left\{\frac{(y_j,-1)}{\sqrt{1+|y_j|^2}}, \frac{(y_j,1)}{\sqrt{1+|y_j|^2}}\right\}$, so

$$\begin{aligned}
Z(u_j) &= \int_{\mathbb{S}^n} \hat\eta(N)\, dS_n(K^{u_j}, N)\\
&= \left[\hat\eta\left(\frac{(y_j,-1)}{\sqrt{1+|y_j|^2}}\right) + \hat\eta\left(\frac{(y_j,1)}{\sqrt{1+|y_j|^2}}\right)\right]\sqrt{1+|y_j|^2}\mathrm{vol}_n(C_j)\\
&=\zeta(y_j)\mathrm{vol}_n(C_j) = \kappa_{n-1},
\end{aligned}$$

because $\hat\eta$ is symmetric with respect to H and vanishes on $\mathbb{S}^n \cap H$. By continuity we obtain

$$Z(I_{B_\infty}) = \lim_{j\to\infty} Z(u_j) = \kappa_{n-1}.$$

On the other hand, Z is n-homogeneous and B_∞ is a convex set of dimension $n-1$, so $Z(I_{B_\infty}) = 0$, which is a contradiction. Thus ζ has compact support.

Therefore, obtain

$$Z(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\, dx,$$

where ζ has compact support.

Finally, let us show how one can use McMullen's Theorem 2.5 to see that ζ is uniquely determined by the valuation Z. Let us thus assume that $\zeta, \zeta' \in C_c(\mathbb{R}^n)$ are such that for all $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$

$$Z(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\, dx = \int_{\mathrm{dom}(u)} \zeta'(\nabla u(x))\, dx.$$

Consider the functions $\eta, \eta' \in C(\mathbb{S}^n_-)$ given for $(y, -\sqrt{1-|y|^2}) \in \mathbb{S}^n_-$, $y \in \{y \in \mathbb{R}^n : |y| < 1\}$, by

$$\begin{aligned}
\eta\left(y, -\sqrt{1-|y|^2}\right) &:= \zeta\left(\frac{y}{\sqrt{1-|y|^2}}\right)\sqrt{1-|y|^2},\\
\eta'\left(y, -\sqrt{1-|y|^2}\right) &:= \zeta'\left(\frac{y}{\sqrt{1-|y|^2}}\right)\sqrt{1-|y|^2}.
\end{aligned}$$

As the support of these functions is strictly contained in $\mathbb{S}^n_-$, we extend them trivially to $\mathbb{S}^n$. Using Theorem 1.18, we obtain

$$Y(K) = Z(\lfloor K \rfloor) = \int_{\mathbb{S}^n} \eta\, dS_n(K) = \int_{\mathbb{S}^n} \eta'\, dS_n(K)$$

for all $K \in \mathscr{K}^{n+1}$. By McMullen's Theorem 2.5, η and η' thus differ by the restriction of a linear function to $\mathbb{S}^n$. However, they are both equal to 0 on the complement of $\mathbb{S}^n_-$, so $\eta - \eta'$ vanishes on an open subset. As this difference is linear, it thus has to vanish identically. In particular, $\eta = \eta'$. □

Let us add the following observation.

Corollary 2.20. *Let $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous, epi-translation invariant valuation that is epi-homogeneous of degree n. Then Z extends uniquely to a continuous valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.*

Proof. By Theorem 2.19, any such valuation $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \to \mathbb{R}$ is given by

$$Z(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\, dx \quad \text{for } u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$$

for some $\zeta \in C_c(\mathbb{R}^n)$. The right-hand side of this equation defines a continuous valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ by [CLM20b, Proposition 20], which yields the desired continuous extension. As $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \subset \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is dense, this extension is unique. □

2.2.2 Proof of Theorem 2.15

For the proof of Theorem 2.15, we will switch to the dual setting: For any functional $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, we may define a functional $Z^* : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ by

$$Z^*(u) := Z(u^*) \quad \text{for } u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}),$$

where u^* denotes the Fenchel-Legendre transform. In $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, by the duality induced by the Fenchel-Legendre transform, we replace epi-translation with *dual epi-translation*, i.e., for a linear functional $L : \mathbb{R}^n \to \mathbb{R}^n$ and $c \in \mathbb{R}$, the map

$$\begin{aligned} \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) &\to \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \\ v &\mapsto v + L + c \end{aligned}$$

is the dual epi-translation corresponding to the affine function $L+c$. Again by duality, we consider the following properties (see also the discussion in [CLM20a, Section 3.1]).

- Z is a valuation if and only if Z^* is a valuation.
- Z is continuous if and only if Z^* is continuous.
- Z is epi-translation invariant if and only if Z^* is dually epi-translation invariant, that is,
$$Z^*(u) = Z^*(u + L + c)$$
for every linear functional $L : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$.
- Z is epi-homogeneous of degree i if and only if Z^* is i-homogeneous in the classical sense, that is,
$$Z^*(tu) = t^i Z^*(u) \quad \text{for all } u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}), t \geq 0.$$

Now assume that Z is epi-homogeneous of degree 1. It is a general fact that 1-homogeneous valuations are *additive*, that is

$$Z^*(u+v) = Z^*(u) + Z^*(v) \quad \text{for all } u, v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}). \tag{2.2.3}$$

This can be directly proved by studying the polarization of the valuation. See [CLM20b, Corollary 24] for a proof.

Goodey-Weil distributions. Before proving Theorem 2.15, let us briefly introduce some preliminary notions concerning the theory of distributions. As reference, we use the book by Rudin [Rud91]. Consider the space of infinitely differentiable functions with compact support on $\mathbb{R}^n$, denoted by $C_c^\infty(\mathbb{R}^n)$. On this space, we consider the usual topology (see [Rud91, Definition 6.3] for the details).

A continuous linear functional on $C_c^\infty(\mathbb{R}^n)$ is called a *distribution*, and the space of distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$. A characterization for distributions is the following (see, for example, [Rud91, Theorem 6.8]): a linear functional T on $C_c^\infty(\mathbb{R}^n)$ is continuous if and only if, for every compact subset $A \subset \mathbb{R}^n$ there exists a constant $c(A)$ and $k \in \mathbb{N}$ such that for every $\phi \in C_c^\infty(\mathbb{R}^n)$ with support contained in A,

$$|T(f)| \leq c(A) \|f\|_{C^k},$$

where

$$\|f\|_{C^k} := \sup\{|\partial^\alpha f(x)| \colon x \in \mathbb{R}^n, |\alpha| \leq k\}. \tag{2.2.4}$$

Here $\alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \in \mathbb{N}, 1 \leq i \leq n$, is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} f.$$

When a distribution T has compact support, that is, there exists a compact set $A \subset \mathbb{R}^n$ such that

$$T(f) = T(h)$$

for every $f, h \in C^\infty(\mathbb{R}^n)$ such that $f(x) = h(x)$ for every x in a neighborhood of A, we can consider T as a functional on $C^\infty(\mathbb{R}^n)$ instead of $C_c^\infty(\mathbb{R}^n)$ (see [Rud91, Theorem 6.24]). Indeed, we can consider a function $h \in C_c^\infty(\mathbb{R}^n)$ such that $h \equiv 1$ on A, and for $T \in \mathcal{D}'(\mathbb{R}^n)$ supported on A we can define

$$T(f) \coloneqq T(hf)$$

for every $f \in C^\infty(\mathbb{R}^n)$. By the definition of support, this extension is independent of the choice of h.

In [Kno21], property (2.2.3) was used to lift dually epi-translation invariant valuations to *distributions* on $\mathbb{R}^n$.

Theorem 2.21 ([Kno21] Theorem 2)**.** *For every* 1*-homogeneous, dually epi-translation invariant, continuous valuation* $Z : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ *there exists a unique* $\mathrm{GW}(Z) \in \mathcal{D}'(\mathbb{R}^n)$ *with compact support which satisfies*

$$\mathrm{GW}(Z)[u] = Z(u) \quad \textit{for all } u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n).$$

A similar construction is possible for homogeneous valuations of arbitrary degree of homogeneity. It is based on ideas of Goodey and Weil [GW84] for translation invariant valuations on convex sets. For the convenience of the reader we report the sketch of the proof in the 1-homogeneous case since it is very instructive and gives a powerful insight on the machinery at hand.

Sketch of the proof of Theorem 2.21. The proof consists of two main parts: Existence and compactness of the support.

Existence. The main idea is to define $\mathrm{GW}(Z)$ by evaluating Z on differences of convex functions. By [Kno21, Lemma 5.1], if we consider the space $C_b^2(\mathbb{R}^n)$ of twice differentiable functions such that

$$\|\phi\|_{C_b^2} \coloneqq \|\phi\|_\infty + \|\nabla\phi\|_\infty + \sup_{x\in\mathbb{R}^n,\xi\in\mathbb{S}^{n-1}} (D^2\phi(x)\xi)\cdot\xi$$

is bounded, for every $\phi \in C_b^2(\mathbb{R}^n)$ there exist $f, h \in \mathrm{Conv}(\mathbb{R}^n,\mathbb{R}) \cap C^\infty(R^n)$ such that $f - h = \phi$ and for every compact set $A \subset \mathbb{R}^n$ the restrictions of f and h to A are bounded by $c(A)\|\phi\|_{C_b^2}$, where $c(A) \coloneqq \sup_{x\in A} |x|^2/2 + 1$.

Now, if Z is a 1-homogeneous, dually epi-translation invariant, continuous valuation, for $\phi \in C_b^2(\mathbb{R}^n)$ and f, h as above we can define the functional

$$\mathrm{GW}(Z)[\phi] \coloneqq Z(f) - Z(h).$$

By (2.2.3), this definition is independent on the choice of f and h, and the functional is unique.

Consider the set F of convex functions bounded by $c(A)$ for every compact set A in the sense of Proposition 2.16. Then, by that proposition (since F is closed), F is a compact subset of $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$. Note that the functions $\bar{f} \coloneqq f/\|\phi\|_{C_b^2}$ and $\bar{h} \coloneqq h/\|\phi\|_{C_b^2}$ are in F. Then,

$$|\mathrm{GW}(Z)[\phi]| \leq |Z(h) - Z(f)| \leq |Z(\bar{h}) - Z(\bar{f})|\|\phi\|_{C_b^2} \leq 2\sup_{v\in F}|Z(v)|\|\phi\|_{C_b^2}. \quad (2.2.5)$$

The space of twice differentiable continuous functions with compact support $C_c^2(\mathbb{R}^n)$ is contained in the space of differences of elements of $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ (for a proof, see, for example, [Kno21, Lemma 5.1]). Therefore, we can consider $\mathrm{GW}(Z)$ as a functional defined on this space. Since $C_c^\infty(\mathbb{R}^n) \subset C_c^2(\mathbb{R}^n)$ and the norm $\|\phi\|_{C_b^2}$ in the last term of the inequality (2.2.5) can be majorized by the norms (2.2.4) for every $k \geq 2$, we have that $\mathrm{GW}(Z) \in \mathcal{D}'(\mathbb{R}^n)$.

Compactness of the support. We will argue by contradiction and assume the support of Z is not compact. Then we can find a sequence $(x_j, \phi_j) \in \mathbb{R}^n \times C_c^\infty(\mathbb{R}^n)$ with the following properties:

1. $\lim_{j\to\infty} |x_j| = \infty$,
2. $\mathrm{supp}\ \phi_j \cap \mathrm{supp}\ \phi_i = \emptyset$ for every $i \neq j$,
3. $\mathrm{supp}\ \phi_j \subset (\mathbb{R}^n \setminus B_{|x_j|+1}(0))$ for all $j \in \mathbb{N}$,

4. $\mathrm{GW}(Z)[\phi_j] = 1$.

We may further assume that $(|x_j|)_j$ is strictly increasing, by passing to a suitable subsequence.

Consider $\bar{\phi} = \sum_{j=1}^{\infty} \phi_j$. Since the supports of the functions ϕ_j are disjoint, ϕ is locally finite and in $C^\infty(\mathbb{R}^n)$. By [Kno21, Lemma 5.6] since

$$\sup_{x\in B_m(0),|\xi|=1} (D^2\phi_j(x)\xi)\cdot\xi \le \sup_{x\in B_m(0),|\xi|=1} (D^2\bar{\phi}(x)\xi)\cdot\xi$$

for every $m \in \mathbb{N}$, there exists $h \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ such that for each $j \in \mathbb{N}$ one has $h + \phi_j \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$. Thus for all $j \in \mathbb{N}$.

$$1 = \mathrm{GW}(Z)[\phi_j] = Z(h + \phi_j) - Z(h).$$

Note that $h + \phi_j$ converges pointwise to h. In particular the convergence is uniform on every compact subset of $\mathbb{R}^n$ since for every compact set A there exists $\bar{j}$ such that $h + \phi_j = h$ on A for every $j \ge \bar{j}$. Therefore, by Lemma 1.13 $h + \phi_j$ epi-converges to h, and the continuity of Z implies

$$1 = \lim_{j\to\infty} \mathrm{GW}(Z)[\phi_j] = 0,$$

which is the desired contradiction. Thus the support of $\mathrm{GW}(Z)$ is compact.

It remains to see that $\mathrm{GW}(Z)[u] = Z(u)$ for $u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$. Take $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ on $B_1(0)$, $\operatorname{supp}\phi \subset B_2(0)$ and set $\phi_j(x) := \phi\left(\frac{x}{j}\right)$. Then

$$\begin{aligned} D^2(\phi_j u)[x] &= uD^2\phi_j(x) + \nabla\phi_j(x)\cdot\nabla u(x)^T + \nabla u(x)\cdot\nabla\phi_j^T(x) + \phi_j(x)D^2u(x) \\ &= \frac{1}{j^2}u(x)D^2\phi\left(\frac{x}{j}\right) + \frac{1}{j}\nabla\phi\left(\frac{x}{j}\right)\cdot\nabla u(x)^T + \nabla u\cdot\frac{1}{j}\nabla\phi\left(\frac{x}{j}\right)^T + \phi\left(\frac{x}{j}\right)D^2u(x) \end{aligned}$$

so

$$\sup_{x\in\mathbb{R}^n} |D^2(\phi_j u)[x]| \le \left(\frac{1}{j^2}\sup_{|x|\le 2j}|u(x)| + \frac{2}{j}\sup_{|x|\le 2j}|\nabla u(x)| + \sup_{x\le 2j}|D^2u(x)|\right)\|\phi\|_{C_b^2(\mathbb{R}^n)}.$$

Therefore, we can apply [Kno21, Lemma 5.6] to find $h \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ such that $h + \phi_j u \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ for $j \in \mathbb{N}$. As $\mathrm{GW}(Z)$ has compact support,

$$\mathrm{GW}(Z)[u] = \lim_{j\to\infty} \mathrm{GW}(Z)[(\phi_j u)] = \lim_{j\to\infty} Z(\phi_j u) = \lim_{j\to\infty} (Z(h + \phi_j u) - Z(h)).$$

But $h + \phi_j u$ converges uniformly on compact sets to $h + u$ and thus epi-converges. Thus the continuity and linearity of Z imply

$$\mathrm{GW}(Z)[u] = Z(h + u) - Z(u) = Z(u) + Z(h) - Z(h) = Z(u),$$

which yields the desired formula. □

For a 1-homogeneous, dually epi-translation invariant, continuous valuation Z on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, we define the support $\mathrm{supp}\, Z := \mathrm{supp}\, \mathrm{GW}(Z)$. Then this is a compact subset of $\mathbb{R}^n$ which has the property that

$$Z(u) = Z(v) \quad \text{for all } u, v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \text{ s.t. } u \equiv v \text{ on a neighborhood of } \mathrm{supp}\, Z,$$

compare [Kno21, Proposition 6.3].

If T is a distribution with compact support and $\phi \in C^\infty(\mathbb{R}^n)$, we define their *convolution* (see [Rud91, Definition 6.34]) as

$$T * \phi(x) = T(\phi(x - \cdot)),$$

which is a function in $C^\infty(\mathbb{R}^n)$. For $\phi \in C^\infty(\mathbb{R}^n)$ consider its reflection $\check{\phi}(x) = \phi(-x)$. If T is a distribution, then $\check{T}$ is again a distribution characterized by

$$T * \check{\phi} = \widecheck{\check{T} * \phi}.$$

Lemma 2.22. *If T is a distribution with compact support, and $\phi \in C_c^\infty(\mathbb{R}^n), \psi \in C^\infty(\mathbb{R}^n)$, then*

$$T(\psi * \phi) = \int_{\mathbb{R}^n} \psi(x) T(\phi(\cdot - x))\, dx = \int_{\mathbb{R}^n} \phi(x) T(\psi(\cdot - x))\, dx.$$

Proof. Notice that, in general,

$$T(\zeta) = [T * \check{\zeta}](0)$$

for every $\zeta \in C^\infty(\mathbb{R}^n)$. Moreover, since T has compact support, by [Rud91, Theorems 6.35 and 6.37] the convolution and its standard properties are well defined even though $\psi \notin C_c^\infty(\mathbb{R}^n)$. Then, by the commutativity of the convolution and swapping the roles of ϕ and ψ, we infer

$$\begin{gathered} T(\psi * \phi) = [T * (\widecheck{\psi * \phi})](0) = [\check{T} * (\psi * \phi)](0) = \\ [\check{T} * \psi * \phi](0) = [\psi * \check{T} * \phi](0) = [\psi * (\check{T} * \phi)](0) = \\ [\psi * (\widecheck{T * \check{\phi}})](0) = \int_{\mathbb{R}^n} \psi(x) T(\phi(\cdot - x))\, dx. \end{gathered}$$

The second equality in the statement follows swapping the roles of ϕ and ψ in the same calculations. □

Proof of theorem 2.15. The following Lemma can also be deduced from [Kno, Theorem 1] in combination with Lemma 5.3 of the same article. The proof we give here is self-contained and does not rely on the machinery developed in [Kno]. It uses a standard approximation argument, which we include for the convenience of the reader.

Lemma 2.23. *Let $Z : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ be a valuation that is continuous, dually epi-translation invariant, and homogeneous of degree 1. If $\mathrm{supp}\, Z \subset B_R(0)$, then there exists a sequence (ψ_j) in $C_c^\infty(\mathbb{R}^n)$ such that*

1. $\operatorname{supp} \psi_j \subset B_{R+1}(0)$ *for all* $j \in \mathbb{N}$,
2. $\int_{\mathbb{R}^n} \psi_j(x)\, dx = \int_{\mathbb{R}^n} x_i \psi_j(x)\, dx = 0$ *for all* $1 \le i \le n$ *for all* $j \in \mathbb{N}$,

and such that the valuations Z_j *given by*

$$Z_j(v) := \int_{\mathbb{R}^n} v(x)\psi_j(x)\, dx$$

are continuous on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, *dually epi-translation invariant and converge uniformly to* Z *on compact subsets of* $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$.

Proof. First, notice that the continuity of the Z_j is immediate by construction, by Lemma 1.13. Let $T := \mathrm{GW}(Z)$ denote the Goodey-Weil distribution of Z. Fix a non-negative function $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$, $\operatorname{supp} \phi \subset B_1(0)$ and consider the sequence $\phi_j := j^n \phi(j\cdot)$, $j \in \mathbb{N}$. Then, the functions $\phi_{T,j}(x) := T(\phi_j(\cdot - x)) = T * \check{\phi}_j$ define a sequence in $C_c^\infty(\mathbb{R}^n)$. Every $\phi_{T,j}$ is supported in $B_{R+1}(0)$ for every $j \in \mathbb{N}$ (since the support of the convolution is included in the Minkowski sum of the supports; see, for example, [Rud91, Theorem 6.37]). To these functions we associate the distributions

$$T_j(\psi) := \int_{\mathbb{R}^n} \psi(x)\phi_{T,j}(x)\, dx \quad \text{for } \psi \in C_c^\infty(\mathbb{R}^n).$$

Notice that, equivalently,

$$T_j(\psi) = T(\psi * \phi_j),$$

by Lemma 2.22.

If l is an affine function, then the usual convolution of functions

$$[l * \phi_j](y) = \int_{\mathbb{R}^n} l(y - x)\phi_j(x)\, dx$$

is affine as well. Indeed, if $l(x) = x \cdot b + c$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$,

$$[l * \phi_j](y) = \int_{\mathbb{R}^n} [(y - x) \cdot b + c]\phi_j(x)\, dx = y \cdot b \int_{\mathbb{R}^n} \phi_j(x)\, dx + \int_{\mathbb{R}^n} (c - x \cdot b)\phi_j(x)\, dx.$$

By Lemma 2.22 with $\psi = l$ and $\phi = \phi_j$, we obtain

$$\int_{\mathbb{R}^n} l(x)\phi_{T,j}(x)\, dx = T(l * \phi_j).$$

Since l is affine, by the definition of Goodey-Weil distribution, and noticing that Z vanishes on affine functions since it is homogeneous and dually epi-translation invariant,

$$\int_{\mathbb{R}^n} l(x)\phi_{T,j}(x)\, dx = Z(l * \phi_j) = 0.$$

Indeed,

$$\int_{\mathbb{R}^n} l(x)\phi_{T,j}(x)\, dx = \int_{\mathbb{R}^n} (x \cdot b)\phi_{T,j}(x)\, dx + c \int_{\mathbb{R}^n} \phi_{T,j}(x)\, dx,$$

and $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are arbitrary. Therefore, we have that condition 2 in the statement, with $\phi_j = \psi_{T,j}$, holds.

Now, we define $Z_j : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ by

$$Z_j(v) := \int_{\mathbb{R}^n} v(x)\phi_{T,j}(x)\,dx.$$

For every $u, v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ such that $u \vee v, u \wedge v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, we have $u(x) + v(x) = (u \vee v)(x) + (u \wedge v)(x)$, and therefore since Z_j is linear

$$Z_j(u \vee v) + Z_j(u \wedge v) = Z_j(u \vee v + u \wedge v) = Z_j(u + v) = Z_j(u) + Z_j(v)$$

for every j. Thus Z_j is a valuation. Since epi-convergence in $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ is equivalent to uniform convergence on compact sets of $\mathbb{R}^n$ and since the support of $\phi_{T,j}$ is compact, Z_j is continuous. Property 2 of the statement implies that Z_j is dually epi-translation invariant. It is straightforward to check that $\mathrm{GW}(Z_j) = T_j$. It remains to check that (Z_j) converges to Z uniformly on compact subsets. To see this, notice that for $v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$ the function

$$[v * \phi_j](y) = \int_{\mathbb{R}^n} v(y - x)\phi_j(x)\,dx,$$

is convex as ϕ_j is non-negative. Moreover, by Lemma 2.22

$$T(v * \phi_j) = \int_{\mathbb{R}^n} T(v(\cdot - x))\phi_j(x)\,dx.$$

Then, for any $v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$

$$\begin{aligned} Z_j(v) =& \mathrm{GW}(Z_j)[v] = T_j(v) = \int_{\mathbb{R}^n} T(v(\cdot - x))\phi_j(x)\,dx \\ =& \int_{\mathbb{R}^n} Z(v(\cdot - x))\phi_j(x)\,dx. \end{aligned}$$

We want to prove that the previous identity remains valid for every $v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$. On $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, the topology induced by epi-convergence coincides with the topology of uniform convergence on compact subsets of $\mathbb{R}^n$, see [RW98, Theorem 7.17]. Using this fact, it is easy to see that the map

$$\begin{aligned} \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R}^n &\to \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \\ (v, x) &\mapsto v(\cdot - x) \end{aligned}$$

is jointly continuous. In particular,

$$v \mapsto \int_{\mathbb{R}^n} Z(v(\cdot - x))\phi_j(x)\,dx$$

defines a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$. By continuity, we thus obtain

$$Z_j(v) = \int_{\mathbb{R}^n} Z(v(\cdot - x))\phi_j(x)\,dx \quad \text{for all } v \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}).$$

Let $\epsilon > 0$ be given. Our previous discussion implies

$$\begin{aligned} |Z_j(v) - Z(v)| &= \left| \int_{\mathbb{R}^n} Z(v(\cdot - x))\phi_j(x)\,dx - \int_{\mathbb{R}^n} Z(v)\phi_j(x)\,dx \right| \\ &\leq \int_{\mathbb{R}^n} \left| Z(v(\cdot - x)) - Z(v) \right| \phi_j(x)\,dx. \end{aligned}$$

As the map

$$\begin{aligned}\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}^n &\to \mathrm{Conv}(\mathbb{R}^n,\mathbb{R})\\ (v,x)&\mapsto v(\cdot-x)\end{aligned}$$

is continuous, it is uniformly continuous on compact subsets. Given a compact subset $C\subset\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$, we can thus find $\delta>0$ such that

$$|Z(v(\cdot-x))-Z(v)|\le\epsilon\quad\text{for all } v\in C \text{ and all } x\in\mathbb{R}^n \text{ with } |x|<\delta.$$

As ϕ is supported on $B_1(0)$, supp $\phi_j\subset B_\delta(0)$ for all $j\ge\frac{1}{\delta}$, so

$$|Z_j(v)-Z(v)|\le\int_{\mathbb{R}^n}|Z(v(\cdot-x))-Z(v)|\,j^n\phi(jx)\,dx\le\epsilon$$

for all $v\in K$ and $j\ge\frac{1}{\delta}$. Thus (Z_j) converges uniformly to Z on the compact subset $C\subset\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$, which concludes the proof. □

Proof of Theorem 2.15. Let $Z:\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\to\mathbb{R}$ be a continuous, epi-translation invariant valuation that is epi-homogeneous of degree 1. Applying Theorem 2.21 to Z^*, we deduce that it has compact support. Thus, we can assume that supp $Z^*\subset B_R(0)$ for $R>0$ large enough. By Lemma 2.23 applied to Z^*, there exists a sequence $\phi_j\in C_c^\infty(\mathbb{R}^n)$ with supp $\phi_j\subset B_{R+1}(0)$ such that

$$Q_j(v):=\int_{\mathbb{R}^n}v(x)\phi_j(x)\,dx$$

defines a sequence of continuous, dually epi-translation invariant, and homogeneous valuations of degree 1 that converge uniformly to Z^* on compact subsets of $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$. If we consider the sequence of valuations Z_j on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ defined by

$$Z_j(u):=Q_j(u)=\int_{\mathbb{R}^n}u^*(x)\phi_j(x)\,dx,$$

its elements are continuous, epi-translation invariant, and epi-homogeneous of degree 1. Moreover, they converge uniformly to Z on compact subsets of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ since the Fenchel-Legendre transform establishes a homeomorphism between $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ (see [RW98, Theorem 11.8], here (1.1.5)), so the preimage of any compact subset of $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ under this map is compact.

It remains to see that Z_j has the desired representation on $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. Consider the function $b=\lfloor B^{n+1}(0)\rfloor\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, that is,

$$b(x)=\begin{cases}-\sqrt{1-|x|^2} & |x|\le 1,\\ +\infty & |x|>1.\end{cases}$$

Then $b^*(x)=\sqrt{1+|x|^2}-1$. From a direct calculation one infers that $\det D^2b^*(x)=(1+|x|^2)^{-(n/2+1)}$, and using (1.1.11) we can thus write for $u\in\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$

$$\begin{aligned}Z_j(u)&=\int_{\mathbb{R}^n}u^*(x)\phi_j(x)\,dx=\int_{\mathbb{R}^n}u^*(x)\phi_j(x)(1+|x|^2)^{n/2+1}\det D^2b^*(x)\,dx\\ &=\int_{\mathrm{dom}(b)}u^*(\nabla b(y))\phi_j(\nabla b(y))(1+|\nabla b(y)|^2)^{n/2+1}\,dy=\int_{\mathbb{S}^n_-}\frac{u^*(g(N))}{\sqrt{1+|g(N)|^2}}f_j(N)dN,\end{aligned}$$

where $f_j(N) := \phi_j(g(N))(1+|g(N)|^2)^{n/2+1}$ is a function whose support is strictly contained in the lower half sphere (in particular, f_j vanishes in a neighborhood of the equator), and the change of variable from the first to the second line is $x \mapsto \nabla b(y)$. We trivially extend f_j to a smooth function on $\mathbb{S}^n$.

By equation (1.1.14), we thus obtain the representation

$$Z_j(u) = \int_{\mathbb{S}^n} \frac{h_{K^u}((g(N),-1))}{1+|g(N)|^2} f_j(N)dN =$$
$$\int_{\mathbb{S}^n} h_{K^u}\left(\frac{(g(N),-1))}{1+|g(N)|^2}\right) f_j(N)dN = \int_{\mathbb{S}^n} h_{K^u}(N) f_j(N)dN.$$

In fact, $Z_j(u) = \int_{\mathbb{S}^n} h_K(N)f_j(N)dN$ for any $K \in \mathscr{K}^{n+1}$ with $h_K = h_{K^u}$ on $\mathbb{S}^n_-$. As Z_j is epi-translation invariant and epi-homogeneous of degree 1, we thus obtain, for every $z \in \mathbb{R}^n, c \in \mathbb{R}$,

$$0 = Z_j\left(0 \cdot (I_{\{0\}})\right) = Z_j\left(I_{\{0\}}\right) = Z_j\left(I_{\{z\}} + c\right) = \int_{\mathbb{S}^n} h_{\{(z,c)\}}(N) f_j(N)dN.$$

In the last equality we have used that $I_{\{z\}} + c = \lfloor\{(z,c)\}\rfloor$ since $\{(z,c)\} \in \mathscr{K}^{n+1}$. As $h_{\{(z,c)\}}(N) = (z,c)^T \cdot N$, we conclude that

$$\int_{\mathbb{S}^n} N f_j(N)dN = 0.$$

The non-negative measure

$$\mu_j(B) := \int_B (1 + \|f_j\|_\infty + f_j)dN \quad \text{for a Borel subset } B \subset \mathbb{S}^n$$

is thus not concentrated on a great sphere and satisfies

$$\int_{\mathbb{S}^n} N\, d\mu_j(N) = 0.$$

By Minkowski's existence theorem (Theorem 1.9 earlier), there thus exists a convex set $L_j \in \mathscr{K}^{n+1}$ such that $\mu_j = S_n(L_j)$. In particular,

$$Z_j(u) = \int_{\mathbb{S}^n} h_{K^u} f_j(N)dN = \int_{\mathbb{S}^n} h_{K^u}\, dS_n(L_j) - \int_{\mathbb{S}^n} h_{K^u}\, dS_n(\sqrt[n]{1+\|f_j\|_\infty} B_1(0)).$$

Here we have used that the surface area measure on $\mathbb{R}^{n+1}$ is n-homogeneous and that $S_n(B_1(0))$ is the spherical Lebesgue measure. Set $W_j := \sqrt[n]{1+\|f_j\|_\infty} B_1(0)$. By construction, $S_n(L_j)$ and $S_n(W_j)$ are absolutely continuous with respect to the spherical Lebesgue measure, and their densities only differ on the support of f_j, that is, on a compact subset contained in the lower half sphere. Therefore

$$Z_j(u) = \int_{\mathbb{S}^n_-} h_{K^u}\, dS_n(L_j) - \int_{\mathbb{S}^n_-} h_{K^u}\, dS_n(W_j).$$

Set $\ell_j = \lfloor L_j \rfloor, w_j = \lfloor W_j \rfloor$. By construction, the measures $S_n(L_j)$ and $S_n(K^{\ell_j})$ agree on $\mathbb{S}^n_-$, and the same is true for W_j and K^{w_j}. We now apply Lemma 1.18 with

$\eta = h_{K^u}, u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, and using that support functions are 1-homogeneous, we thus obtain

$$\begin{aligned} Z_j(u) &= \int_{\mathbb{S}^n_-} h_{K^u}(N)\, dS_n(K^{\ell_j}, N) - \int_{\mathbb{S}^n_-} h_{K^u}(N)\, dS_n(K^{w_j}, N) \\ &= \int_{\mathrm{dom}(\ell_j)} h_{K^u}\left(\frac{(\nabla \ell_j(x), -1)}{\sqrt{1+|\nabla \ell_j(x)|^2}}\right) \sqrt{1+|\nabla \ell_j(x)|^2}\, dx \\ &\quad - \int_{\mathrm{dom}(w_j)} h_{K^u}\left(\frac{(\nabla w_j(x), -1)}{\sqrt{1+|\nabla w_j(x)|^2}}\right) \sqrt{1+|\nabla w_j(x)|^2}\, dx. \end{aligned}$$

Finally, by (1.1.14)

$$Z_j(u) = \int_{\mathrm{dom}(\ell_j)} u^*(\nabla \ell_j(x))\, dx - \int_{\mathrm{dom}(w_j)} u^*(\nabla w_j(x))\, dx,$$

and thus Z_j has the desired representation. In particular, Z can be approximated uniformly on compact subsets of $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ by valuations of this type. □

I open at the close.

The golden snitch

Chapter 3

Symmetrization processes

Tracing back the origin of a mathematical concept is always a difficult task. Leaving these technicalities to the historians, one of the first attempts at Convex Geometry goes back to the work of Steiner on the isoperimetric inequality, that for convex bodies of $\mathbb{R}^n$ states

$$\left(\frac{V_{n-1}(K)}{V_{n-1}(B^n)}\right)^{1/(n-1)} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{1/n},$$

with $K \in \mathscr{K}_n^n$ and B^n the unit ball in $\mathbb{R}^n$. In particular, around 1836, he introduced a technique nowadays known as *Steiner symmetrization*. In this chapter, λ_n is the n-dimensional Lebesgue measure.

Definition 3.1. Consider $K \in \mathscr{K}_n^n$ and a hyperplane H of $\mathbb{R}^n$. The *Steiner symmetral* of K is the set

$$S_H K := \bigcup_{x \in H} \ell_x,$$

where ℓ_x is the segment orthogonal to H with length $\lambda_1(K \cap (H^\perp + x))$ and centered at x.

This operation clearly preserves the volume, by the Cavalieri principle, but also has another relevant property: The surface area decreases under this transformation. From here, the idea is that an iteration of symmetrizations with different hyperplanes could transform $K \in \mathscr{K}_n^n$ into a ball, which is invariant under this symmetrization for every choice of H. The latter fact shows that balls are minimizers in the isoperimetric inequality. Even though Steiner's original proof lacked the topological details, that is, the lower semi-continuity for the surface area, this proof was completed later by Blaschke [Bla56]. An exhaustive treatment can be found, for example, in [Gru07, Chapter 9]. This instrument has been since then a huge success, especially thanks to the pioneering work of Pólya and Szegö [PS51], with many applications in the world of PDEs and Mathematical Physics. Even nowadays, many geometric inequalities for variational functionals are still proved through these methods, called *rearrangements* in the functional setting.

Our focus is on a different direction. As mentioned earlier, one of the crucial properties of Steiner symmetrization is the following (see, for example, [Sch14, Theorem 10.3.2]).

Theorem 3.2. *If $K \in \mathscr{K}_n^n$, then there exists a sequence of hyperplanes (H_m) such that the iterated symmetrals*

$$S_{H_m} \cdots S_{H_1} K$$

converge to a ball in the Hausdorff metric.

This result can be refined, for example, through results like Theorem 3.4, proving that these sequences can be chosen independently on K. In $\mathbb{R}^2$, Steiner achieved this by considering two lines forming an angle that is an irrational multiple of π. Alternating these two subspaces in the sequence guarantees the convergence to a disk.

In the next chapter, we present a general definition of the concept of symmetrization, bringing many examples and case studies. What are the main properties that characterize a "well-behaved" symmetrization with respect to these limit processes? And how pathological can the counterexamples that arise be, even for the harmless Steiner symmetrization? These are the questions that we try to answer in this chapter. Building on the works of Bianchi, Gardner, and Gronchi [BGG17, BGG22a], who introduced this perspective on the topic, in this chapter we exhibit original results from [Uli21, Uli23], where we investigated these and other questions.

3.1 Preliminaries

Steiner symmetrization is just an example of how one can improve the symmetry of an object while trying at the same time to preserve some of its intrinsic properties. Recently, Bianchi, Gardner, and Gronchi in [BGG17, BGG22a] introduced a language and several tools for the study of mappings that can be more widely referred to as symmetrizations. The domain of these maps is usually a family $\mathcal{E}$ of subsets of $\mathbb{R}^n$. For us, the relevant families are $\mathcal{C}^n, \mathcal{C}_n^n, \mathscr{K}^n$, and $\mathscr{K}_n^n$. We note that, in this section, the letter H is now used to refer to generic subspaces.

Definition 3.3. Given a family $\mathcal{E}$ of sets and a subspace H of $\mathbb{R}^n$, an *H-symmetrization* is a map

$$\diamondsuit_H : \mathcal{E} \to \mathcal{E}_H$$

with $\mathcal{E}_H = \{E \in \mathcal{E} \mid R_H E = E\}$, where R_H is the *reflection* with respect to H

$$x \mapsto x - 2\text{proj}_{H^\perp}\{x\}.$$

A *symmetrization* is a map that satisfies this property for every subspace H.

For example, Steiner symmetrization is an H-symmetrization for every hyperplane H, but in Definition 3.3, we consider lower-dimensional subspaces too. Clearly, the information given in this definition is not enough to provide meaningful results, and one needs further properties.

1. (*Monotonicity*): For every $K, L \in \mathcal{E}$ if $K \subseteq L$, then $\diamondsuit_H K \subseteq \diamondsuit_H L$.

2. (*Idempotence*): For every $K \in \mathcal{E}$, we have $\diamondsuit_H K = \diamondsuit_H \diamondsuit_H K$.

3. (*$H^\perp$-translation invariance for H-symmetric sets*): If $K \in \mathcal{E}$ and $R_H K = K$, then for every $x \in H^\perp$ we have $\diamondsuit_H (K + x) = \diamondsuit_H K$.

4. (*Invariance for H-symmetric sets*): If $K \in \mathcal{E}$ and $R_H K = K$, then $\diamondsuit_H K = K$.

5. (*F-invariance*): There exist a function $F : \mathcal{E} \to \mathbb{R}$ such that $F(K) = F(\diamondsuit_H K)$ for every $K \in \mathcal{E}$.

Here are some relevant examples of symmetrizations, which are the focus of this chapter.

Schwarz Symmetrization Let $K \in \mathcal{C}^n$, for a fixed $H \in \mathcal{G}(n, i), 1 \leq i \leq n-1$. The *Schwarz symmetral* of K is the set

$$S_H K = \bigcup_{x \in H} B(x, r_x),$$

where r_x is such that $\lambda_{n-i}(K \cap (H^\perp + x)) = \lambda_{n-i}(B(x, r_x))$ if $\lambda_{n-i}(K \cap (H^\perp + x)) > 0$. If the measure of the section at $x \in H$ is zero, but the section is non-empty, we replace it with x. Otherwise, we replace this section with the empty set. From Fubini's Theorem, it follows that this symmetrization preserves the volume, thus satisfying Property 5 for $F(\cdot) = \lambda_n(\cdot)$. When $i = n-1$ this is the Steiner symmetrization from Definition 3.1, and, in general, it decreases intrinsic volumes (see [Had57, Satz XI, p. 260] or [Sch14, Theorem 10.4.1]). Both in $\mathcal{C}^n$ and $\mathcal{K}^n$ Schwarz symmetrization satisfies Properties 1, 2, 5, while 3, 4 hold only for convex sets in the case $i = n-1$.

Minkowski Symmetrization Let $K \in \mathcal{C}^n$ and let $H \in \mathcal{G}(n, i), 1 \leq i \leq n$. The *Minkowski symmetral* of K is the set

$$M_H K = \frac{K + R_H K}{2}.$$

It preserves the mean width $V_1(K)$ when K is convex, thus Property 5 holds for $F = V_1$. From the Brunn-Minkowski inequality (1.1.1) it follows that

$$\lambda_n(M_H K) \geq \lambda_n(K).$$

It may be useful to consider the *central Minkowski symmetral*, i.e.,

$$\triangle K = \frac{K - K}{2},$$

which is origin symmetric. If K lies in an affine subspace $H + x, x \in H^\perp$, then we write $\triangle_x K$ for the *central Minkowski symmetral of K in $H + x$*, i.e.

$$\triangle_x K = \frac{K + R_{H^\perp} K}{2}.$$

In $\mathcal{K}^n$ Minkowski symmetrization satisfies all the listed properties, but in $\mathcal{C}^n$ only Property 1 holds.

Fiber Symmetrization Let $K \in \mathcal{C}^n$ and let $H \in \mathcal{G}(n,i)$, $1 \leq i \leq n$. Then the *fiber symmetral* of K is the set

$$F_H K = \bigcup_{x \in H} \triangle_x (K \cap (H^\perp + x)).$$

This symmetrization can be seen as a hybrid between Schwarz and Minkowski symmetrization, and the underlying operation was introduced by McMullen [McM99]. Like Minkowski symmetrization, it increases the volume, and in $\mathcal{K}^n$ it satisfies Properties 1, 2, 3, 4, while 2, 3, 4 fail to hold in $\mathcal{C}^n$. See Section 3.2 for some specific examples.

3.1.1 Symmetrization processes

Given a sequence of subspaces (H_m), a symmetrization $\diamondsuit$ on $\mathcal{E}$, and $K \in \mathcal{E}$, the corresponding *symmetrization process* is the sequence

$$\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K.$$

Theorem 3.2 tells us that, when $\diamondsuit = S$, for a fixed $K \in \mathscr{K}^n_n$ we can always find a sequence (H_m) of hyperplanes such that $S_{H_m} \cdots S_{H_1} K$ converges to a ball. This phenomenon is actually quite likely, probabilistically speaking. In 1986 Mani-Levitska [ML86] showed that for Steiner symmetrization, a randomly chosen symmetrization process for a convex body converges almost surely to a ball. Later, Van Schaftingen in [VS06]extended this result to compact sets, then Volčič pushed it to measurable sets [Vol13]. Coupier and Davydov [CD14] later proved, thanks to the inclusion between Steiner and Minkowski symmetrals, that analogous probabilistic properties hold for Minkowski symmetrization.

We do not focus on these probabilistic aspects, but they do certainly serve as a motivation for the study of these processes in the deterministic setting. As we mentioned, for Steiner symmetrization in $\mathbb{R}^2$, we can pick a sequence independently on the choice of K and have convergence to a ball. In general dimension, this property of Steiner symmetrization was proved by Klain in [Kla12], as a consequence of the following Theorem ([Kla12, Theorem 5.1]).

Theorem 3.4. *Given $K \in \mathscr{K}^n$ and a finite family $\mathcal{Q} = \{Q_1, ..., Q_l\})$ of hyperplanes, consider a sequence (H_m) of subspaces such that for every $m \in \mathbb{N}$, $H_m = Q_j$ for some $1 \leq j \leq l$. Then the sequence*

$$K_m := S_{H_m} ... S_{H_1} K$$

converges to a set $L \in \mathscr{K}^n$ in the Hausdorff metric. Moreover, L is symmetric with respect to Q_j for every Q_j appearing infinitely often in the sequence.

If we choose $l = n$ and unit vectors $N_1, \ldots, N_n$ normal to $Q_1, \ldots, Q_n$ such that the angles between them are irrational multiples of π, the limit of the symmetrization process is a ball. Less restrictive conditions are sufficient in the choice of the family $\mathcal{Q}$, as we will see later in Theorem 3.42.

This behavior of Steiner symmetrization extends as well to fiber, Minkowski, and Schwarz symmetrizations as well, as proved in [BGG22a, Theorem 5.6, 5.7, and 5.11].

Theorem 3.5. *Klain's Theorem holds if fiber, Minkowski, or Schwarz replace Steiner symmetrization. Moreover, the subspaces in $\mathcal{Q}$ can have different dimensions.*

Theorem 3.4 and its variations show two relevant facts: First, there exist, in general, sequences of subspaces that guarantee the convergence of the associated symmetrization process for every $K \in \mathscr{K}_n^n$. Secondly, the convergence to balls can be deduced consequence of a more delicate phenomenon, that is, the iteration of suitable hyperplanes in the sequence. These remarks justify the following two definitions.

Definition 3.6. If $\diamondsuit$ is a symmetrization on $\mathcal{E}$, a sequence (H_m) of subspaces of $\mathbb{R}^n$ is said to be *weakly $\diamondsuit$-universal* if for every $k \in \mathbb{N}$, the sequence of sets

$$K_{m,k} = \diamondsuit_{H_m} \cdots \diamondsuit_{H_k} K$$

converges for every $K \in \mathcal{E}$ with non-empty interior to a ball of radius $r(K,k)$. This quantity can change with respect to k, but if $r(K,k)$ is independent of k, then (H_m) is said to be *universal* for $\diamondsuit$.

Universal sequences were introduced in [CD14] and weakly-universal sequences in [BGG17]. The latter notion was needed since Steiner and Minkowski symmetrizations preserve V_n and V_1, respectively. Therefore for a fixed K, the volume of the limit ball is prescribed. For fiber symmetrization, for example, this is no longer the case.

The first relevant result concerning universal sequences was achieved in [CD14, Theorem 3.1] and reads as follows.

Theorem 3.7. *A sequence (H_m) of hyperplanes in $\mathbb{R}^n$ is universal for Steiner symmetrization in $\mathscr{K}_n^n$ if and only if it is universal for Minkowski symmetrization in $\mathscr{K}_n^n$.*

As we remarked earlier, Theorem 3.7 was instrumental in extending the results on convergence in probability from Steiner to Minkowski symmetrizations.

The next natural step was to extend these results in the family of compact sets of $\mathbb{R}^n$. In [BGG22a, Theorem 7.3 and 7.4], the following was achieved for Schwarz and Minkowski symmetrizations.

Theorem 3.8. *If a sequence (H_m) of subspaces is S- or M- universal in $\mathscr{K}_n^n$, then it is respectively S- or M-universal in $\mathcal{C}_n^n$.*

Since if a sequence (H_m) is universal in $\mathcal{C}_n^n$ it is trivially universal in $\mathscr{K}_n^n$, as observed in [Uli23] Theorem 3.8 together with Theorem 3.7 implies that the latter holds in $\mathcal{C}_n^n$.

3.2 Averages of Minkowski sums and combinations

It is reasonable to ask whether extensions like the ones in Theorem 3.8 can be achieved for Theorem 3.4. For Schwarz symmetrization, one has the following [BGG22a, Theorem 7.1]

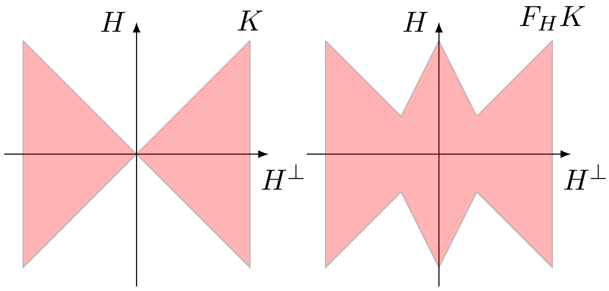

Figure 3.1

Theorem 3.9. *Klain's Theorem holds for Schwarz symmetrization in the family $\mathcal{C}^n$. Moreover, the limits of these processes are rotationally symmetric with respect to every $H \in \mathcal{Q}$.*

This section is devoted to obtaining an analog of Theorem 3.9 for Minkowski symmetrization, following [Uli21]. In doing so, we unravel the role of the convexification effect of the Minkowski addition, obtaining stronger results.

The first step is observing which properties fail to hold in this frame. We have the following simple example.

Example 3.10. *Consider in $\mathbb{R}^2$ the compact set $C = \{(-1,0),(1,0)\}$. This set is obviously symmetric with respect to the vertical axis, which we can identify with a subspace H. Then we have*

$$M_H C = \{(-1,0),(0,0),(1,0)\},$$

and thus the invariance for symmetric sets no longer holds. If we apply again the same symmetrization,

$$M_H(M_H C) = \{(-1,0),(-1/2,0),(0,0),(1/2,0),(1,0)\},$$

showing that the same happens to idempotence. In Figures 3.1 and 3.2, we see an example concerning the fiber symmetrization of a compact set in the plane.

Iterating this process for $C = \{(-1,0),(1,0)\}$, we see that in this case, there is no finite degree of idempotence, i.e., there does not exist an index $\ell \in \mathbb{N}$ such that

$$M_H^\ell C = M_H^{k+\ell} C$$

for every $k \in \mathbb{N}$, where in general

$$\underbrace{M_H \dots M_H}_{\ell\text{-times}} := M_H^\ell.$$

Moreover, the iterated symmetrals converge to the set given by conv(C).

The following lemma shows on which properties we can still count on.

Lemma 3.11. *Let $K \in \mathcal{C}^n$, H a subspace of $\mathbb{R}^n$. Then*

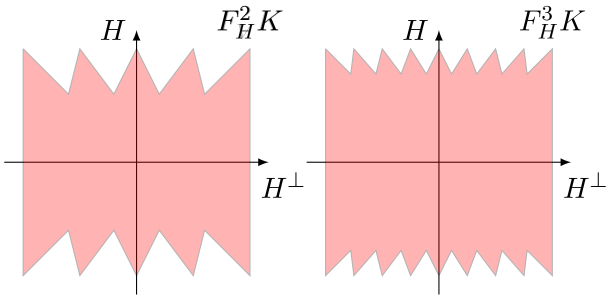

Figure 3.2

i) for every $v \in \mathbb{R}^n$

$$M_H(K+v) = M_H(K) + v|H,$$

ii) if K *is* H*-symmetric, then* $K \subseteq M_H K$*,*

iii) $K = M_H K$ *if and only if* K *is convex and* H*-symmetric.*

Proof. The first statement follows from the explicit calculations

$$M_H(K+v) = \frac{K+v+R_H(K+v)}{2} = \frac{K+R_H(K)}{2} + \frac{v|H^\perp + v|H - v|H^\perp + v|H}{2} = M_H(K) + v|H,$$

where we used the linearity of the reflections and the decomposition $v = v|H + v|H^\perp$.

For the second statement, by hypothesis, we have that $R_H K = K$, i.e., $R_H(x) \in K$ for every $x \in K$. Then, taking $x \in K$, $(x + R_H(R_H(x)))/2 = x \in M_H K$, concluding the proof.

Consider now K such that $K = M_H K$. Then obviously K must be H-symmetric, and $K = R_H K$. Then, for every $x, y \in K$, we have that $(x+y)/2 \in K$, thus for every point z in the segment $[x, y] := \{(1-t)x + ty\colon t \in [0,1]\}$, we can build a sequence by bisection such that it converges to z. K is compact. Henceforth it contains z. The other implication is trivial. □

Consider the iterated symmetral

$$K_m := M_H^m K = \underbrace{M_H ... M_H}_{\text{m-times}} K. \tag{3.2.1}$$

Then, (ii) in Lemma 3.11 implies that $K_m \subseteq K_{m+1}$ for every $m \in \mathbb{N}$. We now provide a first convergence result. In fact, it will be an immediate corollary of the results we will prove later in Section 3.2.1, but we present it for its self-contained proof. Moreover, it helps in understanding the underlying structure of Minkowski symmetrization.

Theorem 3.12. *Let* $K \in \mathcal{C}^n$*,* $H \in \mathcal{G}(n,i), 1 \le i \le n-1$*. Then the sequence* K_m *in* (3.2.1) *converges in the Hausdorff metric to the* H*-symmetric convex compact set*

$$L = conv(M_H K).$$

Proof. In this proof, $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ denotes the usual floor function, instead of the lower-boundary map of the previous chapter.

We observe preliminarily that for the properties of the convex envelope and Minkowski sum, we have $K_m \subseteq L$ for every $m \in \mathbb{N}$. Then we only need to prove that for every $x \in L$, we can find a sequence $x_m \in K_m$ such that $x_m \to x$. We can represent K as $\bar{K} + v, v \in K$, where $\bar{K}$ contains the origin. Since Minkowski symmetrization is invariant under H-orthogonal translations, we can take $v \in H$.

For every m we have $R_H K_m = K_m$, and thus we can write

$$K_{m+1} = M_H K_m = \frac{K_m + K_m}{2} = \frac{\overbrace{K_1 + ... + K_1}^{2^m-\text{times}}}{2^m}.$$

Considering the aforementioned representation of K, $R_H K = R_H \bar{K} + v$, and we have

$$K_m = \bar{K}_m + v, \qquad \text{where } \bar{K}_m := M_H^m \bar{K},$$

thus we can write every point $y \in K_m$ as $y = \bar{y} + v, \bar{y} \in \bar{K}_m$.

Given $x \in L$, by Carathéodory's Theorem [Sch14, 1.1.4] there exist $x_k \in K_1, t_k \in (0,1), k = 1, ..., n+1$ such that $\sum_{k=1}^{n+1} t_i = 1$ and

$$x = \sum_{k=1}^{n+1} t_k x_k = \sum_{k=1}^{n+1} t_k \bar{x}_k + v,$$

where $x_k = \bar{x}_k + v, \bar{x}_k \in \bar{K}_1$. For every t_k we consider its binary representation

$$t_k = \sum_{\ell=1}^{+\infty} \frac{a_{\ell,k}}{2^\ell}, \qquad a_{\ell,k} \in \{0,1\}$$

(we do not consider $\ell = 0$ because $t_i < 1$), and its m-th approximation given by the partial sum

$$t_{m,k} := \sum_{\ell=1}^{m} \frac{a_{\ell,k}}{2^\ell} = \frac{1}{2^m} \sum_{\ell=1}^{m} a_{\ell,k} 2^{m-\ell}.$$

We note for later use that $|t_k - t_{m,k}| \le 1/2^m$.

Calling $q_s := \lfloor 2^s/(n+1) \rfloor$ we now build the sequence

$$x_s := \sum_{k=1}^{n+1} t_{q_s,k} \bar{x}_k + v = \frac{1}{2^{q_s}} \sum_{k=1}^{n+1} \left(\sum_{\ell=1}^{q_s} a_{\ell,k} 2^{q_s-\ell} \right) \bar{x}_k + v,$$

where the $2^s - q_s(n+1)$ spare terms in $\bar{K}_1$ can be taken as the origin in the sum representing $\bar{K}_s$.

Then we have that every x_s belongs to K_s, and

$$|x - x_s| = |\bar{x} + v - (\bar{x}_s + v)| \le \sum_{k=1}^{n+1} |\bar{x}_k||t_k - t_{q_s,k}| \le$$
$$\frac{1}{2^{q_s}} \sum_{k=1}^{n+1} |\bar{x}_k| \le (n+1) \frac{\max_{y \in K_1} |y - v|}{2^{q_s}}.$$

Clearly $|x - x_s| \to 0$ as $s \to +\infty$, which concludes our proof. □

As an immediate consequence, we have the following result.

Corollary 3.13. *Let $K \in \mathcal{C}^n$, $H \in \mathcal{G}(n,i), 1 \leq i \leq n-1$. Then we have that the sequence*

$$K_m := F_H^m K = \underbrace{F_H ... F_H}_{m\text{-times}} K$$

converges in Hausdorff distance to the H-symmetric compact set

$$L = \bigcup_{x \in H} conv(F_H K \cap (x + H^\perp)).$$

Proof. Recalling the definition of fiber symmetrization

$$F_H K = \bigcup_{x \in H} \frac{1}{2}((K \cap (x + H^\perp)) + (R_H K \cap (x + H^\perp))) = \bigcup_{x \in H} M_{H^\perp, x}(K \cap (x + H^\perp)).$$

The result is a straightforward application of Theorem 3.12 to the sections of K. □

3.2.1 Compact generalization of Klain's Theorem for Minkowski symmetrization

From Theorem 3.12, we already saw how the convexification effect of Minkowski addition works when we iterate the same symmetrization. Now, we generalize the former result using the estimate in the Theorem of Shapley, Folkman, and Starr (Theorem 1.5).

Theorem 3.14. *Consider $K \in \mathcal{K}^n$ and a sequence of isometries $\{\mathbb{A}_m\}_{m \in \mathbb{N}}$. If the sequence*

$$K_m = \frac{1}{m} \sum_{j=1}^{m} \mathbb{A}_j K$$

converges, then the same happens for every compact set $C \in \mathcal{C}^n$ such that $conv(C) = K$. Moreover, the two sequences converge to the same limit.

Proof. First, note that orthogonal transformations and Minkowski addition commute with the convex envelope. Thus for $C_m = \sum_{j=1}^m \mathbb{A}_j C/m$, where $C \in \mathcal{C}^n$ and $\{\mathbb{A}_j\}$ is a sequence of isometries as in the hypothesis,

$$\text{conv}(C_m) = \text{conv}\left(\frac{1}{m}\sum_{j=1}^{m} \mathbb{A}_j C\right) = \frac{1}{m}\sum_{j=1}^{m} \mathbb{A}_j \text{conv}(C) = \frac{1}{m}\sum_{j=1}^{m} \mathbb{A}_j K = K_m.$$

We now apply Theorem 1.5 Theorem, obtaining

$$d_{\mathcal{H}}(C_m, K_m) = d_{\mathcal{H}}(C_m, \text{conv}C_m) \leq \frac{\sqrt{n}}{m} \max_{1 \leq j \leq m} D(\mathbb{A}_j C) = \frac{\sqrt{n}}{m} \max_{1 \leq j \leq m} D(C).$$

C is compact and thus bounded, hence $d_{\mathcal{H}}(C_m, K_m) \to 0$, completing the proof. In fact, compactness is not necessary, and boundedness would suffice, but this is beyond the interest of the present paper. □

Corollary 3.15. *Let K be a convex compact set and let (H_m) be a sequence of subspaces of $\mathbb{R}^n$ (not necessarily of the same dimension) such that the sequence of iterated symmetrals*

$$K_m := M_{H_m}...M_{H_1}K$$

converges to a convex compact set L in Hausdorff distance. Then the same happens for every compact set $\tilde{K}$ such that $conv(\tilde{K}) = K$, and the sequence $\tilde{K}_m$, defined as $\tilde{K}_m := M_{H_m}\dots M_{H_1}\tilde{K}$, converges to the same limit L.

Proof. We will show that the theorem holds proving that

$$d_{\mathcal{H}}(\tilde{K}_m, K_m) \to 0$$

for $m \to \infty$. First, we can write K_m as the mean of Minkowski sums of compositions of reflections of K. Indeed, we have

$$K_1 = \frac{K + R_{H_1}K}{2},$$

$$K_2 = \frac{K + R_{H_1}K + R_{H_2}(K + R_{H_1}K)}{4} = \frac{K + R_{H_1}K + R_{H_2}K + R_{H_2}R_{H_1}K}{4},$$

$$\dots$$

and so on. The same obviously holds for $\tilde{K}_m$. We call these compositions of reflections $\mathbb{A}_j, 1 \leq j \leq 2^m$, and defining $A_j := \mathbb{A}_j\tilde{K}$ we can write

$$\tilde{K}_m = \frac{1}{2^m}\sum_{j=1}^{2^m}\mathbb{A}_j\tilde{K} = \frac{1}{2^m}\sum_{j=1}^{2^m}A_j.$$

The proof follows applying Theorem 3.14. □

We obtain the generalization of Klain's result as a consequence of Corollary 3.15.

Corollary 3.16. *Let $K \in \mathcal{C}^n$, $\mathcal{F} = \{Q_1, ..., Q_s\} \subset \mathcal{G}(n, i), 1 \leq i \leq n-1$, (H_m) a sequence of elements of $\mathcal{F}$. Then the sequence*

$$K_m := M_{H_m}...M_{H_1}K$$

converges to a convex set L such that it is the limit of the same symmetrization process applied to $\bar{K} = conv(K)$. Moreover, L is symmetric with respect to all the subspaces of $\mathcal{F}$ appearing infinitely often in (H_m).

Proof. The proof follows at once from Theorem 3.5 and Corollary 3.15. □

We can use a similar method to generalize the following classical result from Hadwiger. See, for example, [Sch14, Theorem 3.3.5].

Theorem 3.17. *For each convex body $K \in \mathcal{K}^n_n$, there is a sequence of rotation means of K converging to a ball.*

Moreover, combining Theorems 3.17 and 3.14, we have the following.

Corollary 3.18. *For each compact set C such that $conv(C) \in \mathscr{K}^n_n$, there is a sequence of means of isometries C converging to a ball.*

3.2.2 The case of convex outer boundary

One of the main properties of Minkowski symmetrization is that, as a consequence of the Brunn-Minkowski inequality 1.1.1, it increases the volume. Indeed, for every measurable set $K \subset \mathbb{R}^n$ such that $\lambda_n(K) > 0$ and $M_H K$ is measurable, we have

$$\lambda_n(M_H K)^{1/n} = \lambda_n(1/2(K + R_H K))^{1/n} \geq \frac{1}{2}\lambda_n(K)^{1/n} + \frac{1}{2}\lambda_n(R_H K)^{1/n} = \lambda_n(K)^{1/n},$$

where equality holds if and only if K and $R_H K$ are homothetic convex sets from which sets of measure zero have been removed. We work only with compact sets. Therefore, the equality condition is possible if and only if the two sets are homothetic and convex. This happens if and only if $K = M_H K$, and thus we would like to state that the iteration of Minkowski symmetrization increases the volume until the sequence of symmetrals reaches $M_H \mathrm{conv}(K)$.

With Theorem 3.12 we proved that, regardless of the volume, the limit of $\tilde{K}_m$ is actually $M_H \mathrm{conv}(K)$, but now we raise one more question: can we obtain this limit in a finite number of iterations? Under which hypothesis is this possible?

We start by answering these questions for compact sets of $\mathbb{R}$. This case is more complicated than for similar objects in $\mathbb{R}^n, n \geq 2$, as we will prove later.

Lemma 3.19. *Let $K \in \mathbb{R}$ be a compact set such that conv$(K) = [a, b]$ with the following property:*

$$\exists \varepsilon > 0 \ s.t. \ [a, a + \varepsilon] \subset K \ or \ [b - \varepsilon, b] \subset K.$$

Then there exists an index $\ell \in \mathbb{N}$ depending on ε and $(b - a)$ such that

$$M_o^\ell K = M_o^{\ell+k} K$$

for every $k \in \mathbb{N}$.

Moreover, ℓ increases with $(b - a)$ and decreases if ε increases.

Proof. First consider the case $K \supseteq \{a\} \cup [b - \varepsilon, b]$. Then

$$M_o K \supseteq M_o(\{a\} \cup [b - \varepsilon, b]) \supset \left[\frac{a-b}{2}, \frac{a-b}{2} + \frac{\varepsilon}{2}\right] \cup \left[\frac{b-a}{2} - \frac{\varepsilon}{2}, \frac{b-a}{2}\right].$$

Easy calculations show that the same happens when $K \supseteq [a, a + \varepsilon] \cup \{b\}$. Then, naming

$$M := \frac{b-a}{2}, \quad m := \frac{b-a}{2} - \frac{\varepsilon}{2},$$

and we can work with a set containing a subset the form

$$[-M, -m] \cup [m, M] =: \tilde{K},$$

where $M - m = \varepsilon/2$.

If we apply the symmetrization, we obtain

$$M_o K \supseteq [-M, -m] \cup \left[\frac{m-M}{2}, \frac{M-m}{2}\right] \cup [m, M] = M_o \tilde{K}. \tag{3.2.2}$$

If $(M-m)/2 \geq m$, that is $m \leq M/3$, then $M_o K = \mathrm{conv}(K)$, and the result holds with $\ell = 1$.

In the general case, we can show by induction that the following inclusion holds

$$M_o^{k+1} K \supseteq M_o^{k+1} \tilde{K} \supseteq \bigcup_{j=0}^{2^{k+1}} \left[\frac{(2^{k+1}-j)m - jM}{2^{k+1}}, \frac{(2^{k+1}-j)M - jm}{2^{k+1}} \right],$$

where the first inclusion is trivial thanks to the monotonicity of Minkowski symmetrization. In particular we will show that

$$M_o^{k+1}\tilde{K} \supseteq M_o^k \tilde{K} \cup \bigcup_{j=1}^{2^k} \left[\frac{(2^{k+1}-2j+1)m - (2j-1)M}{2^{k+1}}, \frac{(2^{k+1}-2j+1)M - (2j-1)m}{2^{k+1}} \right],$$

which is the desired set. This inclusion is actually equality, but proving this fact is beyond our goal here.

If $k = 0$, by (3.2.2) that the inclusion holds. By inductive hypothesis, at the $(k+1)$-th step the means of adjacent intervals of $M_o^k \tilde{K}$ are given by

$$\frac{1}{2}\left[\frac{(2^k-(j+1))m - (j+1)M}{2^k}, \frac{(2^k-(j+1))M - (j+1)m}{2^k} \right] + \frac{1}{2}\left[\frac{(2^k-j)m - jM}{2^k}, \frac{(2^k-j)M - jm}{2^k} \right]$$
$$= \left[\frac{(2^{k+1}-2(j+1)+1)m - (2(j+1)-1)M}{2^{k+1}}, \frac{(2^{k+1}-2(j+1)+1)M - (2(j+1)-1)m}{2^{k+1}} \right]$$

for every $j = 0, ..., 2^k - 1$, giving us the elements of the union with odd indices.

Observe that $M_o^k \tilde{K}$ is invariant under reflection. Thus, thanks to Lemma 3.11 and the monotonicity of Minkowski symmetrization, we have $M_o^k \tilde{K} \subseteq M_o^{k+1} K$, concluding the induction.

Taking at the k-th step two adjacent intervals, we have that they are connected if

$$\frac{(2^k-(j+1))M - (j+1)m}{2^k} \geq \frac{(2^k-j)m - jM}{2^k}.$$

It follows that the condition for filling the whole segment $\mathrm{conv}(M_H^k K)$ is

$$\frac{m}{M} \leq \frac{2^k - 1}{2^k + 1}.$$

Observe that the dependence on the index j disappeared after calculations, confirming that this holds for every couple of adjacent intervals.

By hypothesis $M - m = \varepsilon/2$ and $(2^k-1)/(2^k+1) \to 1$. We have

$$\frac{m}{M} = 1 + \frac{m-M}{M} = 1 - \frac{\varepsilon}{2M},$$

then there exists $\ell \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2M} < \frac{2^\ell - 1}{2^\ell + 1},$$

thus $M_o^\ell K = \mathrm{conv}(K)$ for

$$\ell \geq \log_2\left(\frac{4M}{\varepsilon} - 1\right).$$

This set is convex and o-symmetric, thus is invariant under Minkowski symmetrization. The dependence from M and ε is clear from the last inequality. □

Notice that it is crucial that either a or b belong to an interval with positive measure contained in K. Indeed, if that was not the case, there would occur a situation analogous to the example presented in Example 3.10. Thus, there would be a part of the set which stabilizes itself only at the limit.

With wider generality, the previous Lemma holds for the means of Minkowski sums. Indeed, if $K \subset \mathbb{R}$, for every $x \in \mathbb{R}$ holds

$$\frac{1}{m}\sum_{j=1}^{m}(K - x) = \frac{1}{m}\sum_{j=1}^{m} K - x,$$

and taking x as the mean point of the extreme points of K we reduce ourself to the same context of the Lemma, which can be restated as follows.

Lemma 3.20. *Let $K \in \mathbb{R}$ be a compact set such that conv$(K) = [a, b]$ with the following property:*

$$\exists \varepsilon > 0 \ s.t.\ [a, a+\varepsilon] \cup [b-\varepsilon, b] \subset K.$$

Then there exist an index $\ell \in \mathbb{N}$ depending on ε and $(b-a)$ such that

$$\frac{1}{2^\ell}\sum_{j=1}^{2^\ell} K = \frac{1}{2^{\ell+k}}\sum_{j=1}^{2^{k+\ell}} K$$

for every $k \in \mathbb{N}$. Moreover, ℓ increases with $(b-a)$ and decreases if ε increases.

Proof. First, we remind the reader that, as we have seen in Theorem 3.12, when we iterate M_H, after the first symmetrization we are just computing the mean

$$\frac{1}{2^{m-1}}\sum_{j=1}^{2^{m-1}} M_H K = M_H^m K.$$

Moreover, we observe that the only difference with the previous Lemma is that we do not have the sum with the reflection, so we have to require in the hypothesis that both the end-points of K belong to segments included in K. Now we can work with a set

$$\tilde{K} := ([-M, -m] \cup [m, M]) + x$$

for a suitable $x \in \mathbb{R}$, and the rest of the proof follows as the previous one. □

A weaker property of these sets is to contain the boundary of their convex envelope. When $n \geq 2$, this is enough to prove the stronger and more general result in Corollary 3.23.

Lemma 3.21. *Let $K, L \in \mathcal{C}^n$ such that ∂K, ∂L are connected and $K \cap L \neq \emptyset$. If neither L is strictly contained in K nor K is strictly contained in L, then there exists $z \in \partial K \cap \partial L$.*

Proof. First, note that if K is a closed set and ∂K is connected, then K is connected. Moreover, $\mathbb{R}^n \setminus \mathrm{int}K$ is connected too.

Observe that if $K = L$, then $\partial K \cap \partial L = \partial K = \partial L \neq \emptyset$ and there would be nothing to prove. Thus, we can work in the hypothesis $K \neq L$.

We start proving that $\partial K \cap L \neq \emptyset$. Indeed, there exists $y \in L \setminus K$ and $x \in K \cap L$. Then since L is connected, there exists a continuous curve γ joining x, y. Now, γ must cross $\partial K \cap L$ going from one end (x, inside K) to the other (y, outside K) in a point u which belongs to the required intersection.

Now we prove that $\partial K \setminus L \neq \emptyset$. Indeed there exists $x \in K \setminus L$, and K and L are compact. Therefore, there exists $r > 0$ such that the ball $B(0, r)$ contains strictly K and L. Then, there exists a continuous curve γ' from x to the boundary of $B(0, r)$ that does not intersect ∂L because of the connectedness of $\mathbb{R}^n \setminus \mathrm{int}L$. Moreover, γ' must cross ∂K in a point v that does not belong to L. Hence, this point belongs to $\partial K \setminus L$.

Finally, since ∂K is connected, we can join u, v with a curve contained in ∂K from inside L to outside of it, crossing ∂L in at least one point $z \in \partial K \cap \partial L$. ☐

If A is a connected compact set, then we call the *external connected component* of $\mathbb{R}^n \setminus A$ the unbounded connected component of such a set. Then we note that as in [FLZ22], this result holds also for the boundary of the external connected component of $\mathbb{R}^n \setminus K$ and $\mathbb{R}^n \setminus L$. Moreover, we point out that the hypothesis of Lemma 3.21 immediately rules out the case $n = 1$. This is going to be an issue in Corollary 3.23 and Theorem 3.24.

We can now prove the following result.

Theorem 3.22. *Let K, L be compact sets with connected boundary such that, for every $x \in \mathbb{R}^n$, neither $K + x$ is strictly contained in $-L$ nor $-L$ is strictly contained in $K + x$. Then,*

$$K + L = \partial K + \partial L.$$

Proof. Let $x \in K + L$, then there exist $\kappa \in K$ and $\ell \in L$ such that $x = \kappa + \ell$. If we define $\tilde{K} := K + x - \kappa, \tilde{L} := -L + x + \ell$, we have that $x \in \tilde{K} \cap \tilde{L}$ hence $\tilde{K}$ and $\tilde{L}$ satisfy the hypothesis of Lemma 3.21. Thus $\partial\tilde{K} \cap \partial\tilde{L} \neq \emptyset$.

Let $z \in \partial\tilde{K} \cap \partial\tilde{L}$, then

$$z - x + \kappa \in \partial K, \ell - z + x \in \partial L.$$

Now

$$(z - x + \kappa) + (\ell - z + x) = \kappa + \ell,$$

proving our assertion. ☐

Corollary 3.23. *Let $K \in \mathcal{K}^n$ and H be a subspace of $\mathbb{R}^n$. Then*

$$M_H K = M_H \partial K, \tag{3.2.3}$$

In particular, if $C \in \mathcal{C}^n$ and $C \supseteq \partial conv(C)$, then $M_H C$ is convex, and

$$M_H conv(C) = M_H C.$$

The same holds for fiber symmetrization if H is not a hyperplane.

Proof. We first prove the result regarding Minkowski symmetrization. We apply Theorem 3.22 to $K/2$ and $R_H K/2$. Indeed, observe that the two sets are convex and thus with connected boundary. Moreover, since they have the same volume, no translate of one set is strictly contained in the other. Then, Theorem 3.22 yields $M_H K = M_H \partial K$.

Consider now a set $C \in \mathcal{C}^n$ with $\partial \text{conv}(C) \subseteq C$. From equation (3.2.3), $\partial \text{conv}(C) \subseteq \partial C$ and the monotonicity of Minkowski symmetrization, we infer

$$M_H C \supset \frac{\partial C + \partial R_H C}{2} \supseteq \frac{\partial \text{conv}(C) + \partial R_H \text{conv}(C)}{2} = M_H \text{conv}(C).$$

Since the reverse inclusion is trivial, this concludes the proof in the case of Minkowski symmetrization.

Regarding fiber symmetrization, note that if H was a hyperplane, then the sections are one-dimensional. Moreover, in Lemma 3.19, we proved that we need certain conditions on the boundary to obtain idempotence. In general, we know that fiber symmetrization preserves convexity. Therefore $F_H \text{conv} C$ is convex, and its boundary is given by the union of the boundaries of the sections by $H^\perp + x, x \in H$. If H is not a hyperplane, these sections are obtained by Minkowski symmetrization of convex sets of dimension greater or equal than two, completing the proof. □

We are now in a position to prove a version of Klain's Theorem for fiber symmetrization of compact sets, except for the case of sequences of hyperplanes.

Theorem 3.24. *Let $K \in \mathcal{C}^n$ such that $\partial conv(K) \subset K$, let $\mathcal{F} = \{Q_1, ..., Q_s\}$ be a family of subspaces of $\mathbb{R}^n$, $1 \leq dim(Q_i) \leq n-2$, and let (H_m) be a sequence such that $H_m \in \mathcal{F}$ for every $m \in \mathbb{N}$. Then the sequence*

$$K_m := F_{H_m} ... F_{H_1} K$$

converges to a convex set L, where L is the limit of the same symmetrization process applied to $conv(K)$. Thus L is symmetric with respect to all the subspaces of $\mathcal{F}$ appearing infinitely often in (H_m).

Proof. By Corollary 3.23 we have $F_{H_1} K = F_{H_1} \text{conv} K$. Therefore $F_{H_1} K \in \mathcal{K}^n_n$, and it suffices to apply to for the rest of the sequence Theorem 3.5, proving the claim. □

We conclude this section with another immediate application, a small addition to Klartag's following result (see [Kla04, Theorem 1.1]). The same generalization holds for similar results in [Kla02].

Theorem 3.25. *Let $n \geq 2, 0 < \epsilon < 1/2$, and let $K \subset \mathbb{R}^n$ be a compact set such that $K \supseteq \partial convK$. Then there exist $cn \log 1/\epsilon$ Minkowski symmetrizations with respect to hyperplanes, that transform K into a set $\tilde{K}$ that satisfies*

$$(1-\epsilon)w(K)B^n \subset \tilde{K} \subset (1+\epsilon)w(K)B^n,$$

where $c > 0$ is some numerical constant.

Proof. First, we consider the sequence given by the original statement of this theorem for the convex set $\mathrm{conv} M_H K$. As we have proved in Theorem 3.22, applying the first symmetrization, the resulting set will be $\mathrm{conv} M_H K$. The proof follows at once, considering the sequence constructed in [Kla04, Theorem 1.1]. □

3.3 Convergence in shape and stable sequences

It seems clear at this point, that symmetrization processes can present many different behaviors, and universal sequences are just a part of the zoo of phenomena we can encounter studying them. For example, in the last part of this section, we study instances where the convergence of symmetrization processes fails. We will present one in Example 3.41, which was given previously in [BKL+11] and [BF13]. This example shows that a dense sequence of directions, if accurately chosen, leads to a non-converging symmetrization process. We show, in particular, that the same construction works for the whole family of symmetrizations considered in Theorem 3.33.

For this family of non-converging sequences, when dealing with Steiner symmetrization of compact sets, convergence is still possible in a weaker sense, called convergence in shape.

Definition 3.26 (*Convergence in shape*)**.** Consider a family $\mathcal{E}$ of sets. Given $K \in \mathcal{E}$, a symmetrization $\diamondsuit$ on $\mathcal{E}$ and a sequence (H_m) of subspaces, the sequence of symmetrals

$$\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

is said to *converge in shape* if there exist a sequence $(\mathbb{A}_m)$ of rotations such that

$$\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

converges.

This section will be devoted to the study and the generalization of this kind of convergence (see Definition 3.26). The first result in this direction was achieved by Bianchi, Burchard, Gronchi, and Volčič in [BBGV12, Theorem 2.2] and reads as follows.

Theorem 3.27. *Let (u_m) be a sequence in $\mathbb{S}^{n-1}$ such that $u_m \cdot u_{m-1} = \cos \alpha_m$, where $\alpha_m \in (0, 2\pi)$ and $\sum_{m\in\mathbb{N}} \alpha_m^2 < \infty$. Let (H_m) be the corresponding sequence of hyperplanes given by $H_m = u_m^\perp$ for every $m \in \mathbb{N}$.*

Then there exists a sequence $(\mathbb{A}_m)$ of rotations such that for every non-empty compact set $K \subset \mathbb{R}^n$ the sets

$$K_m = \mathbb{A}_m S_{H_m} \cdots S_{H_1} K$$

converge in Hausdorff metric to a compact convex set L.

To glue together all these concepts, stable sequences were introduced in [Uli23] to study the instance of different limits arising other than the ball unconditionally from the seed of the sequence. Theorem 3.4 and the variations we showed are an example of this phenomenon. Thus, it make sense to introduce the following definition.

Definition 3.28 (*Stable* sequences)**.** If $\diamondsuit$ is a symmetrization on $\mathcal{E}$, a sequence (H_m) of subspaces is said to be $\diamondsuit$-*stable* (or *stable for the symmetrization* $\diamondsuit$) if for every $k \in \mathbb{N}$, the sequence defined for $m \geq k$ by

$$K_{m,k} = \diamondsuit_{H_m} \cdots \diamondsuit_{H_{k+1}} \diamondsuit_{H_k} K$$

converges for every $K \in \mathcal{E}$.

Definition 3.28, together with Definition 3.26, motivates the further definition.

Definition 3.29 (*Shape-stable* sequences)**.** If in Definition 3.26 for every $k \in \mathbb{N}, m \geq k$ the sequence

$$\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_k} K$$

converges, where $(\mathbb{A}_m)$ is independent of K and k, the sequence (H_m) of subspaces is shape-stable in $\mathcal{E}$ for $\diamondsuit$.

Examples of shape-stable sequences were presented in Theorem 3.27. Note that if $\mathbb{A}_m$ is the identity for every $m \in \mathbb{N}$, then a shape-stable sequence (H_m) is stable. To better understand these new concepts, we exhibit some examples.

Example 3.30 (Klain's Theorem)**.** *By Theorem 3.4 and its variations, for a finite family $\mathcal{D}$ of hyperplanes, a sequence (H_m) such that $H_m \in \mathcal{D}$ for every $m \in \mathbb{N}$ is stable for Minkowski, fiber, and Schwarz symmetrizations on $\mathcal{K}^n$.*

Example 3.31 (Stable sequences)**.** *Consider in $\mathbb{R}^2$ the square Q with vertices $(1,0)$, $(0,-1)$, $(-1,0)$, $(0,1)$ as in Figure 3.3. Consider the sequence (H_m) of lines where $H_m = span\{(1,0)\}$ when m is even, $H_m = span\{(0,1)\}$ when m is odd, for $m \geq 2$, while $H_1 = span\{(\sqrt{2}+1,1)\}$. Then (H_m) is stable in $\mathcal{K}^2$ for Minkowski symmetrization thanks to its version of Klain's Theorem. Now, observe that $M_{H_1}Q$ is the red octagon in the figure, and all the other symmetrizations leave this set unchanged so that the limit is exactly $M_{H_1}Q$. If we start from $m \geq 2$ instead, the limit is always Q.*

Example 3.32 (Shape-stable sequences)**.** *Consider in $\mathbb{R}^2$ an ellipse E centered at the origin and a sequence of lines as in Theorem 3.27. It is known that Steiner symmetrization preserves ellipses, and we can choose a direction v such that the symmetral with respect to the line H_1 parallel to v is a ball.*

If we consider a sequence (H_m) of lines starting from H_1 and then continuing as the sequence of Theorem 3.27, we infer that (H_m) is shape-stable in $\mathcal{K}^2$. Moreover, since $S_{H_1}E$ is a ball centered at the origin, the limit is of $S_{H_1}E$. If we skip the first symmetrizations, as was proved in [BBGV12, Example 2.1] (which we recall here in Example 3.41), we can choose the remaining directions such that the limit of the convergence in shape is not a ball.

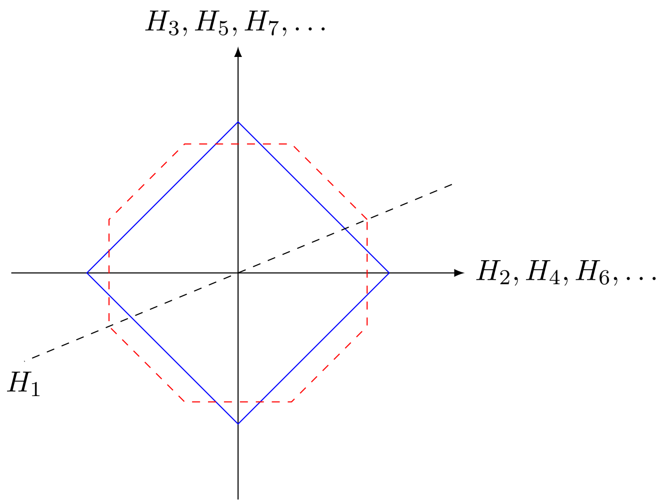

Figure 3.3: Different limits may arise from stable sequences.

At least in $\mathscr{K}_n^n$, it seems clear at this point that Steiner (this is equivalent to fiber symmetrization with respect to hyperplanes) and Minkowski symmetrizations play a peculiar role in determining convergence behaviors. Indeed, Fiber and Minkowski symmetrization can be considered the extremals of a certain family of symmetrizations, in a sense that Theorem 3.33 ([BGG17, Corollary 7.3]) will make clear.

Theorem 3.33 (Bianchi, Gardner, and Gronchi)**.** *Let $H \in \mathcal{G}(n,i), 1 \leq i \leq n-1$ and let $\mathcal{E} = \mathcal{K}^n$ or $\mathcal{K}^n_n$. If $\diamondsuit$ satisfies Properties 1, 3, and 4, then*

$$F_H K \subseteq \diamondsuit_H K \subseteq M_H K \tag{3.3.1}$$

for every $K \in \mathcal{E}$.

This explains why the following family of symmetrizations is of particular interest:

$$\mathcal{F} = \{ \text{ symmetrizations } \diamondsuit \mid \text{ Properties 1, 3, 4 hold } \},$$

where we refer to the properties presented in Section 3.1. The remainder of this chapter is dedicated to the study of this family of symmetrizations. As we show, these three properties are enough to characterize a full spectrum of convergence phenomena.

3.3.1 Shape-stable symmetrization processes

We start by noting some monotonicity properties for volume and mean width with respect to symmetrizations in $\mathcal{F}$.

Lemma 3.34. *Consider $H \in \mathcal{G}(n,i), 1 \leq i \leq n-1$, and a symmetrization $\diamondsuit_H : \mathcal{K}^n \to (\mathcal{K}^n)_H$ such that $\diamondsuit \in \mathcal{F}$. Then for every $K \in \mathcal{K}^n$ we have*

$$\lambda_n(\diamondsuit_H K) \geq \lambda_n(K), \quad V_1(K) \geq V_1(\diamondsuit_H K).$$

Proof. The first inequality is a consequence of the Brunn-Minkowski inequality (1.1.1). Indeed by definition if $K \in \mathcal{K}^n$, then every H-orthogonal section of $F_H K$ is a Minkowski symmetral of an H-orthogonal section of K, thus $\lambda_n(F_H K) \geq \lambda_n(K)$. Now, by (3.3.1) we have

$$\lambda_n(\diamondsuit_H K) \geq \lambda_n(F_H K) \geq \lambda_n(K).$$

For the second inequality, again in view of (3.3.1), we have $\diamondsuit_H K \subseteq M_H K$. Thus clearly $h_{\diamondsuit_H K}(u) \leq h_{M_H K}(u)$ for every $u \in \mathbb{S}^{n-1}$ and consequently

$$V_1(\diamondsuit_H K) \leq V_1(M_H K) = V_1(K),$$

completing the proof. □

We can now prove the following equivalence result. Note that, for this result, the dimension of the subspaces in the sequence is not relevant.

Theorem 3.35. *Let $\diamondsuit_0 \in \mathcal{F}$. If a sequence (H_m) of subspaces is shape-stable for $\diamondsuit_0$ in $\mathcal{K}^n$ with rotations $(\mathbb{A}_m)$, then for any $\diamondsuit \in \mathcal{F}$, (H_m) is shape-stable for $\diamondsuit$ in $\mathcal{K}^n$ with rotations $(\mathbb{A}_m)$.*

In particular, a sequence (H_m) of subspaces is shape-stable for $\diamondsuit \in \mathcal{F}$ if and only if the same property holds for fiber or Minkowski symmetrization. If each H_m is a hyperplane, the same conclusion holds for Steiner symmetrization.

Proof of Theorem 3.35. The outline of the proof is the following. We proceed by applying (3.3.1) multiple times. First, proving that if (H_m) is shape stable for $\diamondsuit_0$ in $\mathcal{K}^n$, then it is shape-stable for M in $\mathcal{K}^n$. After that, we show that if the same sequence is shape-stable for M in $\mathcal{K}^n$, then the same holds for any $\diamondsuit \in \mathcal{F}$. For a subspace H we denote by $\diamondsuit_{0,H}$ the symmetrization $\diamondsuit_0$ with respect to H.

Let (H_m) be a shape-stable sequence of subspaces for $\diamondsuit_0$ in $\mathcal{K}^n$ with rotations $(\mathbb{A}_m)$. We want to prove that for every $K \in \mathcal{K}^n$ the sequence of sets

$$K_m = \mathbb{A}_m M_{H_m} \cdots M_{H_1} K \tag{3.3.2}$$

converges. Suppose on the contrary that there exists $K \in \mathcal{K}^n$ such that for two subsequences (K_{m_j}) and (K_{m_l}) obtained by (3.3.2) one has $K_{m_j} \to L_1, K_{m_l} \to L_2$ where $L_1 \neq L_2$, $L_1, L_2 \in \mathcal{K}^n$.

Consider the sequence of bodies obtained by the same process starting from $K_r = K + B(0,r)$ instead of K, $r > 0$ fixed. This is done to cover both the full- and lower-dimensional cases at the same time.

Note that if H is a subspace and $\mathbb{A}$ is a rotation, for every $K \in \mathcal{K}^n$ we have

$$M_H(K + B(0,r)) = M_H K + B(0,r), \quad \mathbb{A}(K + B(0,r)) = \mathbb{A}K + B(0,r)$$

and thus

$$K_{r,m} = \mathbb{A}_m M_{H_m} \cdots M_{H_1}(K + B(0,r)) = \mathbb{A}_m M_{H_m} \cdots M_{H_1} K + B(0,r),$$

that is $K_{r,m} = K_m + B(0,r) = (K_m)_r$ and instead of L_1 and L_2 we have the limits $L_1 + B(0,r)$ and $L_2 + B(0,r)$. Note that $L_1 \neq L_2$ if and only if $L_1 + B(0,r) \neq L_2 + B(0,r)$ (see for example [Sch14, Theorem 1.7.5 (a)]).

Since $\lambda_n(K_r) > 0$, thanks to Lemma 3.34 the sequence of volumes $\lambda_n(K_m + B(0,r))$ is increasing and strictly positive. Moreover, it is bounded; indeed, from the compactness of K, there exists a ball $B(0,R)$ with $R > 0$ such that $K_r \subseteq B(0,R)$. From the monotonicity and symmetry invariance of Minkowski symmetrization $K_m + B(0,r) \subseteq B(0,R)$ for every $m \in \mathbb{N}$. Therefore $\lambda_n((K_m)_r)$ converges to a certain value $c_r > 0$.

Since $L_1 \neq L_2$, $\lambda_n((L_1)_r \Delta (L_2)_r) = \delta > 0$. Fix $0 < \varepsilon < \delta/2$. There exists an index ν such that $c_r - \lambda_n((K_m)_r) < \varepsilon$ for every $m \geq \nu$. If for $m > \nu$ we define

$$J_m = \mathbb{A}_m \diamondsuit_{0,H_m} \cdots \diamondsuit_{0,H_{\nu+1}} \mathbb{A}_\nu^{-1} (K_\nu)_r$$

$$= \mathbb{A}_m \diamondsuit_{0,H_m} \cdots \diamondsuit_{0,H_{\nu+1}} M_{H_\nu} \cdots M_{H_1} K_r,$$

then thanks to Theorem 3.33, $J_m \subseteq (K_m)_r$ and in particular we have $J_{m_j} \subseteq (K_{m_j})_r$ and $J_{m_l} \subseteq (K_{m_l})_r$. From the hypothesis the sequence (H_m) is shape-stable in $\mathcal{K}^n$ for $\diamondsuit_0$, so there exists $J \in \mathcal{K}^n$ such that $J_m \to J$. Clearly the same holds for (J_{m_j}) and (J_{m_l}). In particular $J \subseteq (L_1)_r$ and $J \subseteq (L_2)_r$ and for Lemma 3.34 $\lambda_n(J) \geq \lambda_n((K_\nu)_r)$. We infer

$$\lambda_n((L_1)_r \Delta (L_2)_r) = \lambda_n((L_1)_r \setminus (L_2)_r) + \lambda_n((L_2)_r \setminus (L_1)_r) \leq$$
$$\lambda_n((L_1)_r \setminus J) + \lambda_n((L_2)_r \setminus J) = 2c_r - 2\lambda_n(J) \leq 2c_r - 2\lambda_n((K_\nu)_r) < 2\varepsilon < \delta,$$

which is a contradiction, so $L_1 = L_2$. The same argument can be repeated for every truncated sequence

$$\mathbb{A}_m M_{H_m} \cdots M_{H_k} K$$

and consequently (H_m) is shape-stable for Minkowski symmetrization.

Now we prove that if a sequence is shape-stable in $\mathcal{K}^n$ for Minkowski symmetrization, then it is shape-stable for $\diamondsuit \in \mathcal{F}$ as well. Consider for $Z \in \mathcal{K}^n$ the sequence of sets

$$Z_m = \mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} Z.$$

If Z_m does not converge, we can find two different subsequences (Z_{m_j}) and (Z_{m_l}) converging respectively to W_1 and $W_2 \in \mathcal{K}^n$ with $W_1 \neq W_2$.

Thanks to Lemma 3.34 the sequence $V_1(Z_m)$ is non-negative and non-increasing, thus $V_1(Z_m) \to b$ for some $b \geq 0$. Then, since $W_1 \neq W_2$, we have $V_1(\mathrm{conv}(W_1 \cup W_2)) > b$. Note that the cases $W_1 \subset W_2$ and vice versa are automatically excluded since $b = V_1(W_1) = V_1(W_2)$ and the mean width is strictly monotone. Now, for every $\varepsilon > 0$ we can find $\nu \in \mathbb{N}$ such that $V_1(Z_m) - b < \varepsilon$ for every $m \geq \nu$. For every $m \geq \nu$, define the sequence of sets

$$V_m = \mathbb{A}_m M_{H_m} \cdots M_{H_{\nu+1}} \mathbb{A}_\nu^{-1} Z_\nu$$

$$= \mathbb{A}_m M_{H_m} \cdots M_{H_{\nu+1}} \diamondsuit_{H_\nu} \cdots \diamondsuit_{H_1} Z.$$

Then the sets V_m converge to some $V \in \mathcal{K}^n$ because (H_m) is shape-stable for Minkowski symmetrization. Minkowski symmetrization preserves the mean width, thus $V_1(V) = V_1(Z_\nu)$. Moreover, thanks to Theorem 3.33 we have $W_1, W_2 \subseteq V$ and since V is convex, $\mathrm{conv}(W_1 \cup W_2) \subseteq V$ and hence $V_1(V) \geq V_1(\mathrm{conv}(W_1 \cup W_2))$. Therefore

$$V_1(\mathrm{conv}(W_1 \cup W_2)) - b \leq V_1(V) - b = V_1(Z_\nu) - b < \varepsilon.$$

Since ε is arbitrary this inequality contradicts $V_1(\mathrm{conv}(W_1 \cup W_2)) > b$ and thus $W_1 = W_2$. Again the same process can be applied to the truncated sequences, concluding the proof. □

Now, the following result is just an easy corollary.

Theorem 3.36. *Let $\diamondsuit_0 \in \mathscr{F}$ be a symmetrization on $\mathcal{K}^n$. Then, if (H_m) is a $\diamondsuit_0$-stable sequence of subspaces of $\mathbb{R}^n$, it is $\diamondsuit$-stable for every symmetrization $\diamondsuit \in \mathscr{F}$.*

In particular, this holds for Steiner and Minkowski symmetrization when the H_m are hyperplanes.

Proof. Observe that if (H_m) is stable, then it is shape-stable with $\mathbb{A}_m$ equal to the identity for every m. The proof is then a straightforward application of Theorem 3.35. □

A second consequence is the following extension of Theorem 3.7. Note that the extension is twofold: The result is valid for the whole family $\mathcal{F}$ and the respective family of objects is $\mathcal{K}^n$ instead of $\mathcal{K}^n_n$.

Theorem 3.37. *Let $\diamondsuit_0 \in \mathcal{F}$. A sequence (H_m) of subspaces is weakly $\diamondsuit_0$-universal in $\mathcal{K}^n$ if and only if it is weakly $\diamondsuit$-universal for every $\diamondsuit \in \mathcal{F}$.*

Proof. The strategy is the same as Theorem 3.35, with the advantage of using Theorem 3.36.

If (H_m) is weakly $\diamondsuit_0$-universal in $\mathcal{K}^n$, then it is stable in $\mathcal{K}^n$. Using Theorem 3.36, this implies that (H_m) is stable for Minkowski symmetrization. Thus we only need to prove that for every $K \in \mathcal{K}^n$ and $k \in \mathbb{N}$, the limit L of the corresponding sequence of sets

$$K_m = M_{H_m} \cdots M_{H_k} K$$

is a ball, where $m \geq k$. Again the sequence of volumes $\lambda_n(K_m)$ is bounded and increasing, and therefore it converges to a certain $c \geq 0$. By the same argument employed in Theorem 3.35, we can suppose that $c > 0$, i.e. considering $K + B(0, r)$ for arbitrarily small $r > 0$ instead of K.

Since (H_m) is weakly $\diamondsuit_0$-universal, for every $\nu \geq k$ the sequence of sets

$$\diamondsuit_{0,H_m} \cdots \diamondsuit_{0,H_{\nu+1}} M_{H_\nu} \cdots M_{H_k} K$$

converges to a ball B_ν. Moreover $\lambda_n(B_\nu) \geq \lambda_n(K_\nu)$ by Lemma 3.34 and $B_\nu \subseteq L$ for every ν thanks to Theorem 3.33. Since $\lambda_n(K_m)$ increases to c, for every $\varepsilon > 0$ exists $\nu \in \mathbb{N}$ such that $\lambda_n(L \Delta B_\nu) < \varepsilon$. Therefore L is a ball.

Suppose now that (H_m) is weakly universal for Minkowski symmetrization in $\mathcal{K}^n$. This implies that (H_m) is stable for Minkowski symmetrization and hence also for $\diamondsuit$, by Theorem 3.36. Consider for $Z \in \mathcal{K}^n$ and $k \in \mathbb{N}$ the limit W of the sequence of sets

$$Z_m = \diamondsuit_{H_m} \cdots \diamondsuit_{H_k} Z$$

for $m \geq k$. Again $(V_1(Z_m))$ is a non-negative and non-increasing sequence; thus it converges to a value $b \geq 0$.

The sequence (H_m) is weakly universal for Minkowski symmetrization, thus for every $\nu \geq k$ we have a ball B_ν as the limit of the sequence of sets

$$M_{H_m} \cdots M_{H_{\nu+1}} \diamondsuit_{H_\nu} \cdots \diamondsuit_{H_k} Z.$$

Then $V_1(B_\nu) = V_1(Z_\nu)$ thanks to the properties of Minkowski symmetrization and for every ν Theorem 3.33 gives $W \subseteq B_\nu$. Since $V_1(B_\nu)$ decreases to $V_1(W)$, W must be a ball. □

Theorem 3.35 allows us to extend many known results for Steiner symmetrization to all $\diamondsuit \in \mathcal{F}$, in particular, Minkowski symmetrization with respect to hyperplanes. For example, we immediately have the following generalization of Theorem 3.27.

Corollary 3.38. *Let (H_m) be a sequence of hyperplanes with corresponding normals $u_m \in \mathbb{S}^{n-1}$. Consider $\diamondsuit \in \mathcal{F}$ and angles $\alpha_m \in (0, 2\pi)$ such that $u_m \cdot u_{m-1} = \cos\alpha_m$. If $\sum_{m\in\mathbb{N}} \alpha_m^2 < \infty$, then there exist rotations $\mathbb{A}_m$ such that for every non-empty compact convex set $K \subset \mathbb{R}^n$ the sets*

$$K_m = \mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

converge in Hausdorff metric to a set $L \in \mathcal{K}^n$.

For Minkowski symmetrization with respect to hyperplanes, we have a stronger result by Theorem 3.14 and 3.36.

Corollary 3.39. *If (H_m) is a shape-stable sequence of hyperplanes for Steiner symmetrization on $\mathcal{C}^n$, then it is shape-stable on $\mathcal{C}^n$ for Minkowski symmetrization. In particular, Theorem 3.27 holds for Minkowski symmetrization as well.*

Proof. First, observe that since $\mathcal{K}^n$ is closed in $\mathcal{C}^n$, the sequence (H_m) is shape-stable for Steiner symmetrization on $\mathcal{K}^n$ and by Theorem 3.35 it is shape-stable for Minkowski symmetrization on $\mathcal{K}^n$.

Now, to conclude the proof, we only have to express the shape-stable sequence as a sequence of means of isometries so that we can apply Theorem 1.5. To see this, note that for every $C \in \mathcal{C}^n$

$$C_1 = \mathbb{A}_1 M_{H_1} C = \mathbb{A}_1 \left(\frac{C + R_{H_1} C_1}{2} \right) = \frac{\mathbb{A}_1 C + \mathbb{A}_1 R_{H_1} C}{2}.$$

Iterating this process, we see that every C_m is a Minkowski mean of 2^m isometries of C.

Theorem 3.27 provides shape-stable sequences for Steiner symmetrization in $\mathcal{C}^n$, thus the same sequences are shape-stable in $\mathcal{C}^n$ for Minkowski symmetrization. □

We conclude with a further interesting consequence, a partial answer to the question: Does a converging sequence of hyperplanes induce a converging symmetrization process? We find a positive answer when one additional assumption is imposed.

Theorem 3.40. *Let* (H_m) *be a sequence of hyperplanes and consider the corresponding normals* $u_m \in \mathbb{S}^{n-1}$. *Let* $\diamondsuit \in \mathcal{F}$. *If the angles* $\alpha_m \in [0, \pi/2]$ *given by the relation* $|u_m \cdot u_{m-1}| = \cos\alpha_m$ *are such that*

$$\sum_{m\in\mathbb{N}} |\alpha_m| < \infty,$$

then (H_m) *converges and is* $\diamondsuit$*-stable on* $\mathcal{K}^n$.

Proof. We shall apply Corollary 3.38. The rotations $\mathbb{A}_m$ there are those constructed in Theorem 3.27; see the proof of [BBGV12, Theorem 2.2] which shows that $\mathbb{A}_m = \phi_m \cdots \phi_1$, where ϕ_m is a planar rotation by α_m degrees that fixes $u_m^\perp \cap e_1^\perp$ and is such that $\phi_m \mathbb{A}_{m-1} u_m = e_1$, for a fixed basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$.

First, we show that $(\mathbb{A}_m)$ is a Cauchy sequence in $GL(n)$ with the usual norm

$$\|\phi\| = \sup_{z\in\mathbb{S}^{n-1}} \|\phi z\|.$$

From the properties of the norm and the triangle inequality we obtain

$$\|\mathbb{A}_{m+k} - \mathbb{A}_m\| = \|\phi_{m+k}\cdots\phi_{m+1}\mathbb{A}_m - \mathbb{A}_m\| \leq \|\mathbb{A}_m\|\|\phi_{m+k}\cdots\phi_{m+1} - Id\| = \\ \|\phi_{m+k}\cdots\phi_{m+1} - Id\| \leq \|\phi_{m+k}\cdots\phi_{m+1} - \phi_{m+1}\| + \|\phi_{m+1} - Id\| \leq \\ \|\phi_{m+1}\|\|\phi_{m+k}\cdots\phi_{m+2} - Id\| + 2\sin(|\alpha_{m+1}|/2) \leq \cdots \leq 2\sum_{j=m+1}^{m+k} \sin(|\alpha_j|/2).$$

From the hypothesis, the series $\sum |\alpha_m|$ converges, proving the claim. Since $\mathbb{A}_m$ is a composition of rotations, $\mathbb{A}_m \in SO(n)$ for every m. As a subspace of $GL(n)$, $SO(n)$ is compact, and therefore it is complete. Thus the sequence $(\mathbb{A}_m)$ converges to some $\mathbb{A} \in SO(n)$.

By Corollary 3.38 the sequence of sets

$$\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

converges for every $K \in \mathcal{K}^n$ to a certain set L. We then have the estimate

$$d_{\mathcal{H}}(\diamondsuit_{H_m}\cdots\diamondsuit_{H_1}K, \mathbb{A}^{-1}L) \leq d_{\mathcal{H}}(\diamondsuit_{H_m}\cdots\diamondsuit_{H_1}K, \mathbb{A}_m^{-1}L) + d_{\mathcal{H}}(\mathbb{A}_m^{-1}L, \mathbb{A}^{-1}L).$$

By the isometry invariance of the Hausdorff distance

$$d_{\mathcal{H}}(\mathbb{A}_m\diamondsuit_{H_m}\cdots\diamondsuit_{H_1}K, L) = d_{\mathcal{H}}(\diamondsuit_{H_m}\cdots\diamondsuit_{H_1}K, \mathbb{A}_m^{-1}L)$$

and thus from the convergence of $\mathbb{A}_m\diamondsuit_{H_m}\cdots\diamondsuit_{H_1}K$ and $\mathbb{A}_m$ we infer that the sequence of sets $\diamondsuit_{H_m}\cdots\diamondsuit_{H_1}K$ converges to $\mathbb{A}^{-1}L$, concluding the proof. □

3.3.2 Counterexamples to convergence

As we have seen, Theorem 3.35 lets us extend Theorem 3.33 to all the symmetrizations $\diamondsuit \in \mathcal{F}$. The latter result arises from the study of a peculiar counterexample which we now briefly show. It can be found in different versions in [BBGV12], [BKL+11], and [BF13]. In the following examples the vectors $\{e_1, e_2\}$ form an orthonormal basis of $\mathbb{R}^2$.

Example 3.41. *Consider a sequence of angles* $\alpha_m \in (0, \pi/2)$ *such that*

$$\sum_{m\in\mathbb{N}} \alpha_m = \infty, \qquad \sum_{m\in\mathbb{N}} \alpha_m^2 < \infty. \tag{3.3.3}$$

We consider the sequence of directions in $\mathbb{R}^2$ *given by* $u_m = (\cos\beta_m, \sin\beta_m)$ *where*

$$\beta_m = \sum_{j=1}^{m} \alpha_j$$

with corresponding orthogonal lines $H_m = u_m^\perp$.

Let $0 < \gamma = \prod_{m\in\mathbb{N}} \cos\alpha_m$ *(which converges because of the second condition in* (3.3.3)*). We consider a compact set* $K \subset \mathbb{R}^2$ *with area* $0 < \lambda_2(K) < \pi(\gamma/2)^2$ *and containing a vertical unit segment* ℓ *centered at the origin. We claim that the sets*

$$K_m = S_{H_m} \cdots S_{H_1} K$$

do not converge. Indeed, consider the segments

$$\ell_m = P_{H_m} K_{m-1},$$

where the length of $\ell_m \subseteq H_m$ *converges to* $\gamma > 0$*. The sequence* (u_m) *of directions is dense in* $\mathbb{S}^1$*, thanks to* (3.3.3)*, and it does not converge. The same holds for the sequence* (H_m) *of lines. Thus for every* $\nu \in \mathbb{S}^1$*, we can find a subsequence* $(H_{m_k}^\perp)$ *such that the normal directions converge to* ν*. Then* ℓ_{m_k} *converges to a segment of length* $\gamma > 0$ *parallel to* ν*.*

Now, if K_m *converges, by the monotonicity of Steiner symmetrization the limit set must contain all these subsequences of diameters, and consequently a ball* B *of diameter* γ *centered at the origin. But we supposed* $\lambda_2(K) < \pi(\gamma/2)^2$*, thus* K_m *cannot converge.*

The peculiarity of the sequence involved in this example is that the corresponding directions are dense in $\mathbb{S}^1$, which could seem a reasonable sufficient condition for convergence to a ball. As was shown, this is not the case, even though in [BKL+11] it was proved for compact convex sets that a dense sequence of hyperplanes can be reordered to obtain a universal sequence. This was generalized in [Vol16] to generic compact sets.

In [BGG22b], Bianchi, Gardner, and Gronchi proved a characterization concerning the symmetry that a convex body needs to be a ball. The form we present includes the statements from [BGG22b, Theorem 3.2] for one-dimensional subspaces.

Theorem 3.42. *Let* $H_j \in \mathcal{G}(n,1), j = 1, \dots, n$*, be such that*
(i) at least two of them form an angle that is an irrational multiple of π*,*
(ii) $H_1 + \cdots + H_n = \mathbb{R}^n$*, and*
(iii) $H_1, \dots, H_n$ *cannot be partitioned into two mutually orthogonal non-empty subsets.*

If $E \subseteq \mathbb{S}^{n-1}$ *is non-empty, closed, and such that* $R_{H_j} E = E, j = 1, \dots, n$*, then* $E = \mathbb{S}^{n-1}$*.*

Hence, if $K \in \mathcal{K}_n^n$ *satisfies* $R_{H_j} K = K$ *for* $j = 1, \cdots, n$*, then* K *is a ball centered at the origin.*

We can use this theorem to find sequences of lines such that the corresponding symmetrization process, if it converges, tends to a ball. Indeed, consider a sequence (v_m) of directions with n accumulation points generating a family of lines $H_1, \dots, H_n$ as in the statement of Theorem 3.42. Consider a sequence (K_m) of convex bodies such that every K_m is symmetric with respect to $v_m^\perp$. Then, if the sequence converges, the limit must necessarily be a ball. We can use this fact to provide a new kind of counterexample.

Example 3.43. *Consider in $\mathbb{R}^2$ the two directions $w_1 = (1,0), w_2 = (\cos\alpha, \sin\alpha)$ such that $\alpha > 0$ is an irrational multiple of π. We consider $\gamma_m \in [0, \alpha]$, $m \in \mathbb{N}$, such that $\alpha_m = |\gamma_{m+1} - \gamma_m|$ is as in (3.3.3). Moreover we want α and 0 to be accumulation points of (γ_m).*

Consider the sequence (H_m) of lines given by $H_m = span\{(\cos\gamma_m, \sin\gamma_m)\}$. Then the corresponding sequence of directions has w_1 and w_2 as accumulation points.

Let K be a compact body centered at the origin with a diameter of unit length parallel to w_1 and consider the sequence of symmetrals

$$K_m = S_{H_m} \cdots S_{H_1} K.$$

As in Example 3.41 we can consider a sequence of segments

$$\ell_m = K_m \cap H_m$$

such that $\lambda_1(\ell_{m+1}) \geq \lambda_1(\ell_m)\cos\alpha_{m+1}$, thus $\lambda_1(\ell_m)$ converges to a value $\gamma > 0$ and in particular the two limits of the converging subsequences of (ℓ_m) respectively parallel to w_1 and w_2 have length greater than γ.

Using Theorem 3.42, if K_m converges, the limit must be a ball. If we choose $\lambda_2(K) < \pi(\gamma/2)^2$, the limit ball should contain a diameter of length γ, which is not possible. Therefore (K_m) cannot converge.

We conclude proving that Example 3.41 can be generalized for other symmetrizations, again thanks to Theorem 3.33.

Example 3.44. *Consider a set $K \in \mathcal{K}_2^2$ that contains a horizontal unit segment and has mean width $1/2\pi < V_1(K) < \gamma$, where γ is as in Example 3.41. In the hypothesis of Theorem 3.33, for $\diamondsuit \in \mathscr{F}$*

$$S_{U_m} \cdots S_{U_1} K \subseteq \diamondsuit_{U_m} \cdots \diamondsuit_{U_1} K \subseteq M_{U_m} \cdots M_{U_1} K,$$

where $U_j = span\{u_j\}$, and we used Steiner symmetrization because it is equivalent to fiber symmetrization relative to a hyperplane, which is our case working in $\mathbb{R}^2$.

In this way, we can exploit the first counterexample and the inclusion chain of Theorem 3.33 to guarantee that, if a limit exists for $\diamondsuit_{U_m} \cdots \diamondsuit_{U_1} K$ and $M_{U_m} \cdots M_{U_1} K$, proceeding as before it must contain a ball of diameter γ, and therefore this limit must have mean width greater than γ. In particular, this holds for the sequence of Minkowski symmetrals. But Minkowski symmetrization preserves mean width, which we supposed to be less than γ. This is a contradiction, and therefore there cannot be a limit.

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