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A deletion–contraction long exact sequence for chromatic symmetric homology

Azzurra Ciliberti

La Sapienza Università di Roma, Italy



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ABSTRACT

In Crew and Spirkel (2020), the authors generalize Stanley's chromatic symmetric function (Stanley, 1995) to vertex-weighted graphs. In this paper we find a categorification of their new invariant extending the definition of chromatic symmetric homology to vertex-weighted graphs. We prove the existence of a deletion–contraction long exact sequence for chromatic symmetric homology which gives a useful computational tool and allow us to answer two questions left open in Chandler et al. (2019). In particular, we prove that, for a graph G with n vertices, the maximal index with nonzero homology is not greater than $n - 1$. Moreover, we show that the homology is non-trivial for all the indices between the minimum and the maximum with this property.

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0. Introduction

The *chromatic symmetric function* X_G of a graph G , defined by Stanley in [7], is a remarkable combinatorial invariant which refines the chromatic polynomial. In [6], Sazdanovic and Yip categorified this invariant by defining a new homological theory, called the *chromatic symmetric homology* of G . This construction, inspired by Khovanov's categorification of the Jones polynomial [1], is obtained by assigning a graded representation of the symmetric group to every subgraph of G , and a differential to every cover relation in the Boolean poset of subgraphs of G . The chromatic symmetric homology $H_{*,*}(G)$ is then defined as the homology of this chain complex; its bigraded Frobenius series $\text{Frob}_G(q, t)$, when evaluated at $q = t = 1$, reduces to Stanley's chromatic symmetric

E-mail address: azzurra.ciliberti@uniroma1.it.

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function expressed in the Schur basis. This categorification has interesting properties which have been investigated in [2,3].

In [4], Logan Crew and Sophie Spirkl generalize Stanley's chromatic symmetric function [7] to vertex-weighted graphs (G, w) with the definition of the *weighted chromatic symmetric function* $X_{(G,w)}$. One of the primary motivations for extending the chromatic symmetric function to vertex-weighted graphs is the existence of a deletion–contraction relation in this setting, which, as known, holds for the chromatic polynomial, but does not hold for the chromatic symmetric function, as observed by Stanley in [7].

In this paper we generalize chromatic symmetric homology to vertex-weighted graphs. We obtain in this way a categorification of the weighted chromatic symmetric function that we call *weighted chromatic symmetric homology* and we denote by $H_{*,*}(G, w)$. The weighted chromatic symmetric homology specializes to the chromatic symmetric homology if $w = \mathbf{1}$ is the function assigning weight 1 to each vertex, i.e. if G is an unweighted graph.

Moreover, we prove the existence of a deletion–contraction long exact sequence for the weighted chromatic symmetric homology which lifts to homology the deletion–contraction relation that holds for the function defined by Crew and Spirkl.

In particular, we prove that

Theorem. *Let (G, w) be a vertex-weighted graph and let e be an edge of G . For each $j \geq 0$, there is a long exact sequence in homology*

$$\rightarrow H_{i,j}(G \setminus e, w) \rightarrow H_{i,j}(G, w) \rightarrow H_{i-1,j}(G/e, w/e) \rightarrow H_{i-1,j}(G \setminus e, w) \rightarrow \dots,$$

where $G \setminus e$ denotes the graph G with the edge e removed, G/e denotes the graph G with the edge e contracted to a point, and w/e denotes the weight function on G/e defined in Section 1.

The long exact sequence in homology gives a useful computational tool and allow us to answer two questions left open in [2].

Let $\text{span}_0(G)$ denote the homological span of the degree 0 chromatic symmetric homology of G . In [2], the authors formulate the following two conjectures.

Conjecture (C.5). *Given any graph G , chromatic symmetric homology groups $H_{i,0}(G; \mathbb{C})$ are non-trivial for all $0 \leq i \leq \text{span}_0(G) - 1$.*

Conjecture (C.6). *Let G be a graph with n vertices and m edges, and let b denote the number of blocks of G . Then $n - b \leq \text{span}_0(G) \leq n - 1$.*

Using the deletion–contraction long exact sequence for chromatic symmetric homology we show that Conjecture C.5 and a part of Conjecture C.6 are true, also for the case of vertex-weighted graphs.

In particular, denoting by $k_{\max}^j(G, w)$ the largest index k such that $H_{k,j}(G, w) \neq 0$ and by $k_{\min}^j(G, w)$ the smallest one ($k_{\min}^0(G, w)$ is always 0 in the case of simple graphs), we prove that

Theorem. *Given any graph (G, w) , chromatic symmetric homology groups $H_{i,j}(G, w; \mathbb{C})$ are non-trivial for all $k_{\min}^j(G, w) \leq i \leq k_{\max}^j(G, w)$, $j \geq 0$.*

Theorem. *Let (G, w) be a graph with n vertices and m edges. Then $k_{\max}^j(G, w) \leq n - 1$ for all $j \geq 0$. Moreover, if $m \geq 1$, $k_{\max}^0(G, w) \leq n - 2$, so $\text{span}_0(G) \leq n - 1$.*

The paper is organized as follows. In Section 1 we recall the definition and some basic properties of the weighted chromatic symmetric function. In Section 2 we build our categorification and prove the existence of a long exact sequence in homology that lifts the deletion–contraction relation for the weighted chromatic symmetric function. Finally, in Section 3, we present some applications of the mentioned sequence and we prove the last two theorems above.

1. Weighted chromatic symmetric function

Let G be a graph. Then $G \setminus e$ denotes the graph G with the edge e removed and G/e denotes the graph G with the edge e contracted to a point.

Definition 1. Define a *vertex-weighted graph* (G, w) to be a graph $G = (V(G), E(G))$ together with a vertex-weight function $w : V(G) \rightarrow \mathbb{N}$. The *weight* of a vertex $v \in V(G)$ is $w(v)$.

Remark 2. Let G be any graph. Then G can be viewed as the vertex-weighted graph $(G, \mathbf{1})$, where $\mathbf{1}$ is the function assigning weight 1 to each vertex.

Definition 3. Given a vertex-weighted graph (G, w) , we say that $F \subseteq V(G)$ is a *state* of G , and we define the *total weight* $w(F)$ of F to be $\sum_{v \in F} w(v)$. Moreover, we define the total weight $w(G)$ of G to be the total weight of $V(G)$.

The set $Q(G)$ of all the states of G has a structure of Boolean lattice, ordered by reverse inclusion. In the Hasse diagram of $Q(G)$, we direct an edge $e(F, F')$ from a subgraph F to a subgraph F' if and only if F' can be obtained by removing an edge from F .

In [4], Logan Crew and Sophie Spirkl generalize Stanley's chromatic symmetric function [7] to vertex-weighted graphs with the following definition:

Definition 4. Let (G, w) be a vertex-weighted graph. Then the *weighted chromatic symmetric function* is

$$X_{(G,w)}(x_1, x_2, \dots) = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}^{w(v)},$$

where the sum ranges over all proper colorings $\kappa : V(G) \rightarrow \mathbb{N}$ of G .

Remark 5. If G has a loop, then $X_{(G,w)} = 0$ for every $w : V(G) \rightarrow \mathbb{N}$. Moreover, if e_1, e_2 are edges of G with the same endpoints, then $X_{(G,w)} = X_{(G \setminus e_1, w)} = X_{(G \setminus e_2, w)}$ for every $w : V(G) \rightarrow \mathbb{N}$.

Remark 6. Note that $X_{(G, \mathbf{1})} = X_G$, where X_G is the usual chromatic symmetric function.

Recall that, if $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of a positive integer n , i.e. a non-increasing sequence of positive integers whose sum is n , the power sum symmetric function p_λ is defined as

$$p_\lambda(x_1, x_2, \dots) = p_{\lambda_1}(x_1, x_2, \dots) \cdots p_{\lambda_k}(x_1, x_2, \dots),$$

where $p_r(x_1, x_2, \dots) = x_1^r + x_2^r + \dots$, for $r \in \mathbb{N}$.

Let Λ_n be the \mathbb{Z} -module of the homogeneous symmetric functions of degree n . Then $\{p_\lambda \mid \lambda \text{ partition of } n\}$ is a basis for Λ_n . Another basis for Λ_n is given by the Schur symmetric functions $\{s_\lambda \mid \lambda \text{ partition of } n\}$. Moreover, let $\Lambda^{\mathbb{C}} = \bigoplus_{n \geq 0} \Lambda_n$ denote the space of symmetric functions in the indeterminates x_1, x_2, \dots .

Definition 7. Given a vertex-weighted graph (G, w) , and $F \subseteq E(G)$, we define $\lambda(G, w, F)$ to be the partition of $w(G)$ whose parts are the total weights of the connected components of (G', w) , where $G' = (V(G), F)$.

Lemma 8 ([4], Lemma 3). Let (G, w) be a vertex-weighted graph. Then

$$X_{(G,w)} = \sum_{F \subseteq E(G)} (-1)^{|F|} p_{\lambda(G,w,F)}.$$

One of the primary motivations for extending the chromatic symmetric function to vertex-weighted graphs is the existence of a deletion-contraction relation in this setting.

Definition 9. Let (G, w) be a vertex-weighted graph, and let $e = (v_1, v_2) \in E(G)$. We define $w/e : V(G/e) \rightarrow \mathbb{N}$ to be the modified weight function on G/e such that $w/e = w$ if e is a loop, and otherwise $(w/e)(v) = w(v)$ if $v \neq v_1, v_2$, and for the vertex v^* of G/e formed by the contraction, $(w/e)(v^*) = w(v_1) + w(v_2)$.

We have the following:

Theorem 10 ([4], Lemma 2). Let (G, w) be a vertex-weighted graph, and let $e \in E(G)$ be any edge. Then

$$X_{(G,w)} = X_{(G \setminus e, w)} - X_{(G/e, w/e)}.$$

Note that the deletion-contraction relation of [Theorem 10](#) does not give a similar relation for the ordinary chromatic symmetric function, since if we contract a non loop edge we do not get an ordinary chromatic symmetric function.

2. Weighted chromatic symmetric homology

Now we build a categorification of the invariant just introduced.

In this section we assume that the set of edges of G is ordered.

Let \mathfrak{S}_n denote the symmetric group on n elements. The irreducible representations of \mathfrak{S}_n over \mathbb{C} are indexed by the partitions of n , and are called *Specht modules*. Let \mathbf{S}^λ denote the Specht module indexed by λ .

The Grothendieck group R_n of representations of \mathfrak{S}_n is the free abelian group on the isomorphism classes $[\mathbf{S}^\lambda]$ of irreducible representations of \mathfrak{S}_n , modulo the subgroup generated by all $[V \oplus W] - [V] - [W]$. Let $R = \bigoplus_{n \geq 0} R_n$. If $[V] \in R_a$ and $[W] \in R_b$, define a multiplication in R by

$$[V] \circ [W] = [\text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}} V \otimes W].$$

Here the tensor product $V \otimes W$ is regarded as a representation of $\mathfrak{S}_n \times \mathfrak{S}_m$ in the obvious way: $(\sigma \times \tau) \cdot (v \otimes w) = \sigma \cdot v \otimes \tau \cdot w$; and $\mathfrak{S}_n \times \mathfrak{S}_m$ is regarded as a subgroup of \mathfrak{S}_{n+m} with \mathfrak{S}_n acting on the first n integers and \mathfrak{S}_m acting on the last m integers. The induced representation can be defined quickly by the formula

$$\text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} = \mathbb{C}[\mathfrak{S}_{n+m}] \otimes_{\mathbb{C}[\mathfrak{S}_n \times \mathfrak{S}_m]} (V \otimes W).$$

It is straightforward to verify that this product is well defined and makes R into a commutative, associative, graded ring with unit.

The morphism of graded rings given by sending the Specht modules to the Schur functions

$$ch : R \rightarrow \Lambda^{\mathbb{C}}, [\mathbf{S}^\lambda] \rightarrow s_\lambda$$

is an isomorphism.

Moreover, for $n \in \mathbb{N}$, we have

$$ch^{-1}(p_n) = \sum_{i=0}^{n-1} (-1)^i [\mathbf{S}^{(n-i, 1^i)}]. \quad (1)$$

For the proofs of these two last facts see [5], Section 7.3.

With the notation of [6], we define:

Definition 11. Let (G, w) be a vertex-weighted graph. Suppose $F \subseteq E(G)$ is a state with r connected components of total weights b_1^w, \dots, b_r^w respectively. To F , we assign the graded $\mathfrak{S}_{w(G)}$ -module

$$M_F^w = \text{Ind}_{\mathfrak{S}_{b_1^w} \times \dots \times \mathfrak{S}_{b_r^w}}^{\mathfrak{S}_{w(G)}} (\mathbf{L}_{b_1^w} \otimes \dots \otimes \mathbf{L}_{b_r^w}), \quad (2)$$

where \mathbf{L}_a denotes the q -graded \mathfrak{S}_a -module

$$\mathbf{L}_a = \bigoplus_{j=0}^{a-1} \mathbf{S}^{(a-j, 1^j)}, \quad (3)$$

and $\mathbf{S}^{(a-j, 1^j)}$ is the Specht module related to the partition $(a - j, 1^j)$ of the positive integer a . The grading is given by the index j .

Definition 12. For $i \geq 0$, the i th *weighted chain module* for (G, w) is

$$C_i(G, w) = \bigoplus_{|F|=i} M_F^w.$$

More precisely, since $M_F^w = \bigoplus_{j \geq 0} (M_F^w)_j$ is graded, then for $i, j \geq 0$, we define

$$C_{i,j}(G, w) = \bigoplus_{|F|=i} (M_F^w)_j.$$

Remark 13. Observe that $(M_F^w)_j = 0$ if $j \geq b_t^w$ for all $t = 1, \dots, r$.

Since the differential defined in [6] depends only on the b_i 's, we can define a differential in the same way, replacing the b_i 's with the b_i^w 's.

Let F be a state of G . Suppose $F' = F - e$ where $e \in E(G)$. We define the $\mathfrak{S}_{w(G)}$ -modules morphism $d_\epsilon^{(G,w)} : M_F^w \rightarrow M_{F'}^w$, i.e. the *per-edge maps*, in the following way.

There are two cases to consider:

Case 1 The edge e is incident to vertices in the same connected component of F' . Since M_F^w and $M_{F'}^w$ are equal, we define $d_\epsilon : M_F^w \rightarrow M_{F'}^w$ to be the identity map.

Case 2 The edge e is incident to vertices in different connected components of F' . First, consider the simplest case where F consists of one connected component and F' consists of two components A and B . Suppose $w(A) = a$ and $w(B) = b$, so that $a + b = w(G)$. Since, by Frobenius Reciprocity, $\text{Hom}_{\mathfrak{S}_{w(G)}}(M_F^w, M_{F'}^w) \cong \text{Hom}_{\mathfrak{S}_a \times \mathfrak{S}_b}(\Lambda^* T \oplus (\Lambda^* T)[1], \Lambda^* T)$, where $T = (\mathbf{S}^{(a-1, 1)} \otimes \mathbb{1}_{\mathfrak{S}_b}) \oplus (\mathbb{1}_{\mathfrak{S}_a} \otimes \mathbf{S}^{(b-1, 1)})$ (see [6], Lemma 2.6), we choose the element $d_\epsilon \in \text{Hom}_{\mathfrak{S}_{w(G)}}(M_F^w, M_{F'}^w)$ to be the map that corresponds to the $(\mathfrak{S}_a \times \mathfrak{S}_b)$ -module map that is the identity on $\Lambda^* T$ and zero on $(\Lambda^* T)[1]$. In the general case when F has more than one connected component, the definition of the per-edge map is achieved by recursion on the two-component case.

Suppose F is a state with r connected components B_1, \dots, B_r of total weights b_1^w, \dots, b_r^w . Further suppose that the removal of the edge $e \in E(G)$ decomposes B_r into two components A and B of total weights a and b respectively ($a + b = b_r^w$). Let $d_\zeta : \mathbf{L}_{\mathbf{b}, \mathbf{w}} \rightarrow \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{b_r^w}}(\mathbf{L}_a \otimes \mathbf{L}_b)$ be the per-edge map defined previously (note that $M_{B_r}^{b_r^w} = \mathbf{L}_{b_r^w}$, since B_r is connected), and let $\mathbf{N} = \mathbf{L}_{b_1^w} \otimes \dots \otimes \mathbf{L}_{b_{r-1}^w}$. The map $d_\epsilon : M_F^w \rightarrow M_{F'}^w$ is chosen to be

$$d_\epsilon = \text{Ind}_{\mathfrak{S}_{b_1^w} \times \dots \times \mathfrak{S}_{b_{r-1}^w} \times \mathfrak{S}_{b_r^w}}^{\mathfrak{S}_{w(G)}}(id_{\mathbf{N}} \otimes d_\zeta)$$

Definition 14. Let F and F' be states of G . Assume that $F' = F \setminus e$, $e \in E(F)$. The sign of $\epsilon = \epsilon(F, F')$, $\text{sgn}(\epsilon)$, is defined as $(-1)^k$, where k is the number of edges of F less than e .

Definition 15. For $i \geq 0$, define $d_i^{(G,w)} : C_i(G, w) \rightarrow C_{i-1}(G, w)$ letting

$$d_i^{(G,w)} = \sum_{\epsilon} \text{sgn}(\epsilon) d_\epsilon^{(G,w)},$$

where the sum is over all edges ϵ in the Hasse diagram of $Q(G)$ joining a state with i edges to a state with $i - 1$ edges. We also define $d_{i,j}^{(G,w)} : C_{i,j}(G, w) \rightarrow C_{i-1,j}(G, w)$ to be the map $d_i^{(G,w)}$ in the j th grading.

Proposition 16. The maps $d_i^{(G,w)}$ form a differential on the chain complex $C_*(G, w)$.

Proof. The proof is completely analogous to that of Proposition 2.10 of [6] replacing the b_i 's with the b_i^w 's. \square

Definition 17. For $i, j \geq 0$, the (i, j) -th *weighted chromatic symmetric homology* of (G, w) is

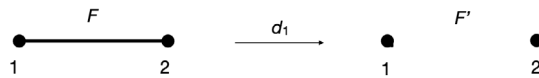
$$H_{i,j}(G, w) = \ker d_{i,j}^{(G,w)} / \operatorname{im} d_{i+1,j}^{(G,w)}.$$

Moreover, we define

$$H_i(G, w) = \bigoplus_{j \geq 0} H_{i,j}(G, w).$$

Remark 18. $H_{*,*}(G, \mathbf{1}) = H_{*,*}(G)$, where $H_{*,*}(G)$ is the usual chromatic symmetric homology.

Example 19. Let (K_2, w) be the segment with a vertex v_1 of weight 1 and the other v_2 of weight 2. The labels of the vertices indicate their weights.



We have

- $\diamond C_{1,0}(K_2, w) = (M_F^w)_0 = \mathbf{S}^{(3)};$
- $\diamond C_{0,0}(K_2, w) = (M_{F'}^w)_0 = \operatorname{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_1}^{\mathfrak{S}_3} \mathbf{S}^{(2)} \otimes \mathbf{S}^{(1)} = \mathbf{S}^{(3)} \oplus \mathbf{S}^{(2,1)};$
- $\diamond C_{1,1}(K_2, w) = (M_F^w)_1 = \mathbf{S}^{(2,1)};$
- $\diamond C_{0,1}(K_2, w) = (M_{F'}^w)_1 = \operatorname{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_1}^{\mathfrak{S}_3} \mathbf{S}^{(1,1)} \otimes \mathbf{S}^{(1)} = \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)};$
- $\diamond C_{1,2}(K_2, w) = (M_F^w)_2 = \mathbf{S}^{(1^3)};$
- $\diamond C_{0,2}(K_2, w) = 0.$

Therefore, $H_{1,0}(K_2, w) = H_{1,1}(K_2, w) = H_{0,2}(K_2, w) = 0$, $H_{0,0}(K_2, w) = \mathbf{S}^{(2,1)}$, $H_{0,1}(K_2, w) = H_{1,2}(K_2, w) = \mathbf{S}^{(1^3)}$.

In general,

- $\diamond C_{1,0}(K_2, w) = (M_F^w)_0 = \mathbf{S}^{(w(v_1)+w(v_2))};$
- $\diamond C_{0,0}(K_2, w) = (M_{F'}^w)_0 = \operatorname{Ind}_{\mathfrak{S}_{w(v_1)} \times \mathfrak{S}_{w(v_2)}}^{\mathfrak{S}_{w(v_1)+w(v_2)}} \mathbf{S}^{(w(v_2))} \otimes \mathbf{S}^{(w(v_1))} = \mathbf{S}^{(w(v_1)+w(v_2))} \oplus \bigoplus_{\lambda} (\mathbf{S}^{\lambda})^{m_{\lambda}}.$

We do not give the details about the \mathbf{S}^{λ} 's which appear in the last formula and their multiplicities. You can find an explanation of it in [5], Section 7.3. We say only that they are all different from $\mathbf{S}^{(w(v_1)+w(v_2))}$. Therefore, we have $H_{1,0}(K_2, w) = 0$ and $H_{0,0}(K_2, w) \neq 0$. Moreover, $H_{i,0}(K_2, w) = 0$ for any $i \geq 2$, since K_2 does not have any states with more than one edge.

Definition 20.

The *bigraded Frobenius series* of $H_{*,*}(G, w) = \bigoplus_{i,j \geq 0} H_{i,j}(G, w)$ is

$$\operatorname{Frob}_{(G,w)}(q, t) = \sum_{i,j \geq 0} (-1)^{i+j} t^i q^j \operatorname{ch}(H_{i,j}(G, w)).$$

Example 21. Let us consider the vertex-weighted graph of the previous example. We have

$$\operatorname{Frob}_{(K_2,w)}(q, t) = -(q + tq^2)s_{(1^3)} + s_{(2,1)}.$$

Lemma 22. For any vertex-weighted graph (G, w) ,

$$\sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(C_{i,j}(G, w)).$$

Proof. Let n be any positive integer. Any short exact sequence of \mathfrak{S}_n -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is split exact, so $B \cong A \oplus C$ and $\text{ch}(B) = \text{ch}(A) + \text{ch}(C)$.

Let $Z_{i,j}(G, w) = \ker d_{i,j}^{(G,w)}$ and $B_{i,j}(G, w) = \text{im } d_{i+1,j}^{(G,w)}$. For $i, j \geq 0$, we have short exact sequence $0 \rightarrow Z_{i,j}(G, w) \rightarrow C_{i,j}(G, w) \rightarrow B_{i-1,j}(G, w) \rightarrow 0$ and $0 \rightarrow B_{i,j}(G, w) \rightarrow Z_{i,j}(G, w) \rightarrow H_{i,j}(G, w) \rightarrow 0$, where $B_{-1,j}(G, w)$ is understood to be zero. Thus

$$\begin{aligned} \text{ch}(C_{i,j}(G, w)) &= \text{ch}(Z_{i,j}(G, w)) + \text{ch}(B_{i-1,j}(G, w)) \\ &= \text{ch}(H_{i,j}(G, w)) + \text{ch}(B_{i,j}(G, w)) + \text{ch}(B_{i-1,j}(G, w)). \end{aligned}$$

If we multiply this by $(-1)^{i+j}$ and we sum over all $i, j \geq 0$, we get:

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(C_{i,j}(G, w)) &= \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G, w)) + \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(B_{i,j}(G, w)) \\ &+ \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(B_{i-1,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G, w)) + \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(B_{i,j}(G, w)) \\ &- \sum_{t,j \geq 0} (-1)^{t+j} \text{ch}(B_{t,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G, w)). \quad \square \end{aligned}$$

Theorem 23. Weighted chromatic symmetric homology categorifies the weighted chromatic symmetric function. That is, for any vertex-weighted graph (G, w) ,

$$\text{Frob}_{(G,w)}(1, 1) = X_{(G,w)}.$$

Proof. Using Lemma 22, (4) and Lemma 8, we have

$$\begin{aligned} \text{Frob}_{(G,w)}(1, 1) &= \sum_{i,j \geq 0} (-1)^{i+j} \text{ch}(H_{i,j}(G, w)) = \sum_{i \geq 0} (-1)^i \left(\sum_{j \geq 0} (-1)^j \text{ch}(C_{i,j}(G, w)) \right) \\ &= \sum_{i \geq 0} (-1)^i \sum_{F \subseteq E(G): |F|=i} p_{\lambda(G,w,F)} = X_{(G,w)}. \quad \square \end{aligned}$$

Now we want to lift to homology the result of Theorem 10.

Proposition 24. Let (G, w) be a vertex-weighted graph and let e be an edge of G . For each $i, j \geq 0$, there is a short exact sequence of $\mathfrak{S}_{w(G)}$ -modules

$$0 \rightarrow C_{i,j}(G \setminus e, w) \rightarrow C_{i,j}(G, w) \rightarrow C_{i-1,j}(G/e, w/e) \rightarrow 0.$$

Proof. By definition

$$\begin{aligned} C_{i,j}(G \setminus e, w) &= \bigoplus_{|F|=i, F \subseteq E(G \setminus e)} (M_F^w)_j \\ &= \bigoplus_{|F|=i, F \subseteq E(G), e \notin F} (M_F^w)_j \\ &\subseteq \bigoplus_{|F|=i, F \subseteq E(G)} (M_F^w)_j = C_{i,j}(G, w). \end{aligned}$$

Therefore, there is a short exact sequence

$$0 \rightarrow C_{i,j}(G \setminus e, w) \xrightarrow{\iota_i} C_{i,j}(G, w) \xrightarrow{\pi_i} \frac{C_{i,j}(G, w)}{C_{i,j}(G \setminus e, w)} \rightarrow 0,$$

where ι_i is the inclusion and π_i is the projection to the quotient.

We have that

$$\frac{C_{i,j}(G, w)}{C_{i,j}(G \setminus e, w)} = \frac{\bigoplus_{|F|=i, F \subseteq E(G)} (M_F^w)_j}{\bigoplus_{|F|=i, F \subseteq E(G), e \notin F} (M_F^w)_j} \cong \bigoplus_{|F|=i, F \subseteq E(G), e \in F} (M_F^w)_j.$$

Since, if F is a state of (G, w) with i edges such that $e \in F$, then $M_F^w = M_{F/e}^{w/e}$, because the contraction does not change the total weight of the connected components of F , and F/e is a state of $(G/e, w/e)$ with $i - 1$ edges, we have that

$$\bigoplus_{|F|=i, F \subseteq E(G), e \in F} (M_F^w)_j = C_{i-1,j}(G/e, w/e),$$

and the theorem follows. \square

Remark 25. If G is an unweighted graph, for each $i, j \geq 0$, we have the following short exact sequence of $\mathfrak{S}_{|V(G)|}$ -modules

$$0 \rightarrow C_{i,j}(G \setminus e) \rightarrow C_{i,j}(G) \rightarrow C_{i-1,j}(G/e, \mathbf{1}/e) \rightarrow 0.$$

Proposition 26. Let (G, w) be a vertex-weighted graph and let e be an edge of G . For each $j \geq 0$, there is a short exact sequence of chain complexes

$$0 \rightarrow C_{*,j}(G \setminus e, w) \rightarrow C_{*,j}(G, w) \rightarrow C_{*-1,j}(G/e, w/e) \rightarrow 0.$$

Proof. With the notation of the proof of Proposition 24, we have to show that, for each $i \geq 0$, $d_i^{(G,w)} \circ \iota_i = \iota_{i-1} \circ d_i^{(G \setminus e, w)}$ and $d_{i-1}^{(G/e, w/e)} \circ \pi_i = \pi_{i-1} \circ d_i^{(G,w)}$. It is clear that the first equality holds. Let us look at the second.

If $i = 0, 1$, we have 0 on both sides. Consider $i \geq 2$. Since, if F is a state of (G, w) with i edges such that $e \in F$, then $M_F^w = M_{F/e}^{w/e}$, π_i is the map such that

$$\pi_{i, M_F^w} = \begin{cases} \text{id} & \text{if } e \in F, \\ 0 & \text{if } e \notin F. \end{cases}$$

Therefore,

$$\pi_{i-1} \circ d_i^{(G,w)} = \sum_{\epsilon} \text{sgn}(\epsilon) \pi_{i-1} \circ d_{\epsilon}^{(G,w)} = \sum_{\epsilon'} \text{sgn}(\epsilon') d_{\epsilon'}^{(G,w)}, \text{ where the last sum is over all the } \epsilon'$$

in the Hasse diagram of $Q(G, w)$ joining a state of (G, w) with i edges that contains e to a state of (G, w) with $i - 1$ edges that also contains e .

On the other hand, $d_{i-1}^{(G/e, w/e)} \circ \pi_i = \sum_{\epsilon''} \text{sgn}(\epsilon'') d_{\epsilon''}^{(G/e, w/e)}$, where the sum is over all the ϵ'' in the

Hasse diagram of $Q(G/e, w/e)$ joining a state of $(G/e, w/e)$ with $i - 1$ edges to a state of $(G/e, w/e)$ with $i - 2$ edges.

We know that, if F is a state of G with i edges such that $e \in F$, then $M_F^w = M_{F/e}^{w/e}$ and F/e is a state of $(G/e, w/e)$ with $i - 1$ edges. Therefore, if ϵ' is an edge in the Hasse diagram of $Q(G, w)$ connecting a state F of (G, w) with i edges that contains e with a state F' of (G, w) with $i - 1$ edges that also contains e ,

$$d_{\epsilon'}^{(G,w)} : M_F^w = M_{F/e}^{w/e} \rightarrow M_{F'}^w = M_{F'/e}^{w/e}$$

coincides with $d_{\epsilon''}^{(G/e, w/e)}$, where ϵ'' is an edge in the Hasse diagram of $Q(G/e, w/e)$ joining the state F/e of $(G/e, w/e)$ with $i - 1$ edges to the state F'/e of $(G/e, w/e)$ with $i - 2$ edges.

Since there is a bijection between the states of G with i edges that contains e and the states of $(G/e, w/e)$ with $i - 1$ edges, we have that the two sums coincide. Therefore,

$$d_{i-1}^{(G/e, w/e)} \circ \pi_i = \pi_{i-1} \circ d_i^{(G,w)}. \quad \square$$

Therefore, we have:

Theorem 27. Let (G, w) be a vertex-weighted graph and let e be an edge of G . For each $j \geq 0$, there is a long exact sequence in homology

$$\rightarrow H_{i,j}(G \setminus e, w) \rightarrow H_{i,j}(G, w) \rightarrow H_{i-1,j}(G/e, w/e) \xrightarrow{\gamma^*} H_{i-1,j}(G \setminus e, w) \rightarrow \dots \quad (4)$$

Proof. The short exact sequences of chain complexes in Proposition 26 induce for each $j \geq 0$ a long exact sequence in homology. \square

Remark 28. The specialization of the Frobenius series at $q = t = 1$ recovers the deletion-contraction relation of Theorem 10.

Remark 29. The description for γ^* follows from the standard diagram chasing argument in the zig-zag lemma and the result is as follows. It is the linear extension of the map that, given a state of $(G/e, w/e)$ with $i - 1$ edges, where $e = (v_e, w_e)$ is an edge of G that has been contracted to a point, expands $v_e = w_e$ by adding e with weight $w(v_e)$ at the vertex v_e and $w(w_e)$ at the vertex w_e and then deletes e . In this way we get a state of $(G \setminus e, w)$ with $i - 1$ edges.

Remark 30. If G is an unweighted graph, for each $j \geq 0$, we have the following long exact sequence in homology

$$\dots \rightarrow H_{i,j}(G \setminus e) \rightarrow H_{i,j}(G) \rightarrow H_{i-1,j}(G/e, \mathbf{1}/e) \xrightarrow{\gamma^*} H_{i-1,j}(G \setminus e) \rightarrow \dots$$

2.1. Properties of $H_{*,*}(G, w)$

The deletion-contraction long exact sequence allows us to give a different and faster proof of the following two properties of chromatic symmetric homology, contained in [6], and to extend them to the case of vertex-weighted graphs.

Proposition 31. If (G, w) contains a loop, then $H_{*,*}(G, w) = 0$.

Proof. Let (G, w) be a graph with a loop l . The exact sequence for (G, w) with respect to l is

$$\begin{aligned} \dots \rightarrow H_{i,j}(G/l, w/l) &\xrightarrow{\gamma^*} H_{i,j}(G \setminus l, w) \rightarrow H_{i,j}(G, w) \rightarrow \\ H_{i-1,j}(G/l, w/l) &\xrightarrow{\gamma^*} H_{i-1,j}(G \setminus l, w) \rightarrow \dots \end{aligned}$$

Using our description of the snake map γ^* in Remark 29, we get that the map $H_{i,j}(G/l, w/l) \xrightarrow{\gamma^*} H_{i,j}(G \setminus l, w)$ is the identity map. Therefore, $H_{i,j}(G, w) = 0$ for all i, j . \square

Proposition 32. Let (G, w) be a multigraph, i.e. a graph which is allowed to have multiple edges. Let e_1 and e_2 be two edges of (G, w) with the same endpoints. Then $H_{*,*}(G, w) = H_{*,*}(G - e_2, w)$.

Proof. In G/e_2 , e_1 becomes a loop so, by Proposition 31, $H_{i,j}(G/e_2, w/e_2) = 0$ for all i, j . It follows from the long exact sequence (4) that $H_{i,j}(G - e_2, w)$ and $H_{i,j}(G, w)$ are isomorphic modules. \square

Therefore, from now on we assume that G is simple, so without loops or multiple edges.

Given two vertex-weighted graphs (A, w_A) and (B, w_B) , let $(A + B, w_{A+B})$ denote their disjoint union, where

$$w_{A+B}(v) = \begin{cases} w_A(v), & \text{if } v \in V(A), \\ w_B(v), & \text{if } v \in V(B). \end{cases}$$

Proposition 33. For $i, j \geq 0$,

$$H_{i,j}(A + B, w_{A+B}) = \bigoplus_{\substack{p+r=i \\ q+s=j}} \text{Ind}_{\mathfrak{S}_{w_A(A)} \times \mathfrak{S}_{w_B(B)}}^{\mathfrak{S}_{w_A(A)+w_B(B)}} (H_{p,q}(A, w_A) \otimes H_{r,s}(B, w_B)).$$

Proof. The proof is completely analogous to the unweighted case. See [6], Proposition 3.3. \square

Remark 34. If (G, w) is a graph with homology $H_{i,j}(G, w) = \bigoplus_{\lambda} (\mathbf{S}^{\lambda})^{\oplus m_{\lambda}}$, then the homology of the disjoint union of G with a single vertex with weight w_v is

$$H_{i,j}(G + \bullet) = \bigoplus_{\mu} (\mathbf{S}^{\mu})^{\oplus m_{\lambda}},$$

where the sum is over all partitions μ which can be obtained by adding w_v boxes to the partitions λ indexing the irreducible factors of $H_{i,j}(G, w)$.

3. Applications

The deletion–contraction long exact sequence in homology has proved to be a useful computational tool. Moreover, we can use it to compute weighted chromatic symmetric homology starting from unweighted chromatic symmetric homology.

Example 35. Let (K_2, w) be the segment with a vertex of weight 1 and the other of weight 2. We can compute its homology using the deletion–contraction long exact sequence.

Let $G = P_3$ be the graph made of two segments with a vertex in common, and let $e \in E(G)$. We have that $(K_2, w) = G/e$ and $G \setminus e$ is the disjoint union of K_2 and an isolated vertex.

We have $H_{0,0}(G \setminus e) = H_{1,1}(G \setminus e) = \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)}$ and $H_{1,0}(G \setminus e) = 0$.

Moreover, we have $H_{0,0}(G) = H_{2,2}(G) = \mathbf{S}^{(1^3)}$, $H_{1,1}(G) = \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)}$ and $H_{0,1}(G) = H_{2,0}(G) = H_{2,1}(G) = 0$.

For $j = 0$, we have the following long exact sequence in homology:

$$0 \longrightarrow H_{1,0}(K_2, w) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_{0,0}(K_2, w) \longrightarrow \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)} \longrightarrow \mathbf{S}^{(1^3)} \longrightarrow 0,$$

from which we can conclude that $H_{1,0}(K_2, w) = 0$ and $H_{0,0}(K_2, w) = \mathbf{S}^{(2,1)}$.

For $j = 1$, we have the following long exact sequence in homology:

$$0 \longrightarrow H_{1,1}(K_2, w) \longrightarrow \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)} \longrightarrow \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)} \longrightarrow H_{0,1}(K_2, w) \longrightarrow 0,$$

from which we can conclude that $H_{1,1}(K_2, w) = 0$ and $H_{0,1}(K_2, w) = \mathbf{S}^{(1^3)}$.

For $j = 2$, we have the following long exact sequence in homology:

$$0 \longrightarrow \mathbf{S}^{(1^3)} \longrightarrow H_{1,2}(K_2, w) \longrightarrow 0 \cdots \longrightarrow 0,$$

from which we can conclude that $H_{1,2}(K_2, w) = \mathbf{S}^{(1^3)}$ and $H_{0,2}(K_2, w) = 0$.

Now, given a graph (G, w) , let $\text{span}_0(G, w)$ denote the homological span of the degree 0 weighted chromatic symmetric homology of (G, w) , i.e. of $H_{i,0}(G, w)$. We have $\text{span}_0(G, w) = k + 1$ where k is maximal among indices such that $H_{k,0}(G, w) \neq 0$, since we are assuming that G has no loops, so $H_{0,0}(G, w)$ is always nonzero.

In [2], the authors left open the following

Conjecture (C.6). Let G be a graph with n vertices and m edges, and let b denote the number of blocks of G . Then $n - b \leq \text{span}_0(G) \leq n - 1$.

We denote by $k_{\max}^j(G, w)$ the largest index k such that $H_{k,j}(G, w) \neq 0$ and by $k_{\min}^j(G, w)$ the smallest one. As observed earlier, $k_{\min}^0(G, w)$ is always 0.

Using the deletion-contraction long exact sequence for weighted chromatic symmetric homology (4) we can prove that

Theorem 36. *Let (G, w) be a graph with n vertices and m edges. Then $k_{\max}^j(G, w) \leq n - 1$ for all $j \geq 0$. Moreover, if $m \geq 1$, $k_{\max}^0(G, w) \leq n - 2$, so $\text{span}_0(G) \leq n - 1$.*

Proof. We prove that, if $i \geq 0$ is an index such that $H_{i,j}(G, w) \neq 0$, then we have $i \leq n - 1$.

We proceed by induction on the number $m \geq 0$ of edges of G . If $m = 0$, we have that the homology $H_{i,j}(G, w)$ is trivial for all $i > 0$, since we do not have any states with more than zero edges. Therefore, the first inequality holds.

Furthermore, if we require $m \geq 1$, at the base step we have to consider the case $m = 1$. It follows from Remark 34 that we can assume without loss of generality that G is connected, so, if $m = 1$, then G is a segment with two vertices and an edge between them. It follows from Example 19 that $k_{\max}^0(G, w) = 0$, so the second part of the theorem holds.

We now assume the statement true for any graph with $m - 1$ edges. Let $v(G)$ denote the number of vertices of G and $e(G)$ the number of edges of G . We have that $v(G \setminus e) = v(G)$ and $e(G \setminus e) = e(G) - 1 = m - 1$. Moreover, we have that $v(G/e) = v(G) - 1$ and $e(G/e) = e(G) - 1 = m - 1$.

Let $i > v(G) - 2$. Since $v(G \setminus e) = v(G)$, we have also that $i > v(G \setminus e) - 2$. By inductive hypothesis, we have $H_{i,j}(G \setminus e, w) = 0$. Moreover, since $i - 1 > v(G) - 3 = v(G/e) - 2$, by inductive hypothesis, we have $H_{i-1,j}(G/e, w/e) = 0$ and $H_{i,j}(G/e, w/e) = 0$.

From the deletion-contraction long exact sequence (4)

$$\cdots \longrightarrow H_{i,j}(G/e, w/e) \longrightarrow H_{i,j}(G \setminus e, w) \longrightarrow H_{i,j}(G, w) \longrightarrow H_{i-1,j}(G/e, w/e) \longrightarrow,$$

it follows that $H_{i,j}(G, w) = 0$. \square

In [2], the authors left open also the following

Conjecture (C.5). *Given any graph G , chromatic symmetric homology groups $H_{i,0}(G; \mathbb{C})$ are non-trivial for all $0 \leq i \leq \text{span}_0(G) - 1$, $j \geq 0$.*

Using the deletion-contraction long exact sequence, we can prove the following

Theorem 37. *Let (G, w) be a graph. Then $H_{i,j}(G, w; \mathbb{C})$ is non-trivial for all $k_{\min}^j(G, w) \leq i \leq k_{\max}^j(G, w)$, $j \geq 0$.*

Since $k_{\min}^0(G, w)$ is always 0, Theorem 37 shows in particular that Conjecture C.5 is true.

Proof. We proceed by induction on the number $m \geq 0$ of edges of G . If $m = 0$, we have that the homology $H_{i,j}(G, w)$ is trivial for all $i > 0$, since we do not have any states with more than zero edges. Therefore, the result holds.

Now assume the statement true for any graph with $m - 1$ edges.

If $k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)$, since $G \setminus e$ has $m - 1$ edges, by inductive hypothesis, we have that $H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \neq 0$. If $H_{k_{\max}^j(G, w)-1, j}^j(G/e, w/e) = 0$, then by inductive hypothesis, it is also $H_{k_{\max}^j(G, w), j}^j(G/e, w/e) = 0$. Therefore, by the deletion-contraction long exact sequence (4)

$$\begin{aligned} &\longrightarrow H_{k_{\max}^j(G, w), j}^j(G/e, w/e) \longrightarrow H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \longrightarrow H_{k_{\max}^j(G, w), j}^j(G, w) \\ &\longrightarrow H_{k_{\max}^j(G, w)-1, j}^j(G/e, w/e) \longrightarrow \cdots, \end{aligned}$$

we have $H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \cong H_{k_{\max}^j(G, w), j}^j(G, w)$.

Otherwise, $H_{k_{\max}^j(G, w)-1, j}^j(G/e, w/e) \neq 0$, so $k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1$.

If instead $k_{\max}^j(G \setminus e, w) < k_{\max}^j(G, w)$, we have $H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) = 0$ and $H_{k_{\max}^j(G, w), j}^j(G, w) \neq 0$. Therefore, by the deletion-contraction long exact sequence (4)

$$\cdots \longrightarrow H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \longrightarrow H_{k_{\max}^j(G, w), j}^j(G, w) \longrightarrow H_{k_{\max}^j(G, w)-1, j}^j(G/e, w/e) \longrightarrow \cdots,$$

we have that the map from $H_{k_{\max}(G,w),j}^j(G, w)$ to $H_{k_{\max}(G,w)-1,j}^j(G/e, w/e)$ is injective. Hence, $H_{k_{\max}(G,w),j}^j(G, w)$ is isomorphic to the image of this map, which is a non-trivial submodule of $H_{k_{\max}(G,w)-1,j}^j(G/e, w/e)$. It follows that

$$H_{k_{\max}(G,w)-1,j}^j(G/e, w/e) \neq 0 \text{ and } k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1.$$

Now assume $k_{\min}^j(G, w) \leq i \leq k_{\max}^j(G, w)$ and prove that $H_{i,j}(G, w)$ is non-trivial. As observed above, we have three cases to consider:

- (i) $k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)$ and $H_{k_{\max}(G,w),j}^j(G \setminus e, w) \cong H_{k_{\max}(G,w),j}^j(G, w)$;
- (ii) $k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)$ and $k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1$;
- (iii) $k_{\max}^j(G \setminus e, w) < k_{\max}^j(G, w)$ and $k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1$.

In case (i), $k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)$ and $H_{k_{\max}(G,w),j}^j(G \setminus e) \cong H_{k_{\max}(G,w),j}^j(G, w)$, so by inductive hypothesis we have that $H_{i,j}(G \setminus e, w)$ is non-trivial. It follows from (4), and for how the maps are defined, that also $H_{i,j}(G, w)$ is non-trivial.

In case (ii), if $k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)$ and $k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1$, then $i - 1 \leq k_{\max}^j(G, w) - 1 \leq k_{\max}^j(G/e, w/e)$. Therefore, by induction, $H_{i-1,j}(G/e, w/e)$ is non-trivial. Moreover, by induction, also $H_{i,j}(G \setminus e, w)$ is non-trivial. It follows from (4), and for how the maps are defined, that also $H_{i,j}(G, w)$ is non-trivial.

Finally, we consider the case (iii) with $k_{\max}^j(G \setminus e, w) < k_{\max}^j(G, w)$. We just have to see what happens if $k_{\max}^j(G \setminus e, w) < i \leq k_{\max}^j(G, w)$, since, if $i \leq k_{\max}^j(G \setminus e, w) < k_{\max}^j(G, w)$, as in the previous case, both $H_{i-1,j}(G/e, w/e)$ and $H_{i,j}(G \setminus e, w)$ are non-trivial, and so it is $H_{i,j}(G, w) \neq 0$. If $k_{\max}^j(G \setminus e, w) < i \leq k_{\max}^j(G, w)$, we have that $H_{i,j}(G \setminus e, w) = 0$. From the deletion-contraction long exact sequence (4)

$$\cdots \longrightarrow H_{i,j}(G \setminus e, w) \longrightarrow H_{i,j}(G, w) \longrightarrow H_{i-1,j}(G/e, w/e) \longrightarrow \cdots,$$

it follows that the map from $H_{i,j}(G, w)$ to $H_{i-1,j}(G/e, w/e)$ is injective. Moreover, since $i - 1 \leq k_{\max}^j(G, w) - 1 \leq k_{\max}^j(G/e, w/e)$, as proved above, by induction, $H_{i-1,j}(G/e, w/e)$ is non-trivial. Hence, for how the maps are defined, $H_{i,j}(G, w)$ is non-trivial. \square

3.1. Future directions

Chandler, Sazdanovic, Stella and Yip in [2] investigated the properties of chromatic symmetric homology with integer coefficients. They conjectured that a graph G is non-planar if and only if its chromatic symmetric homology in bidegree $(1, 0)$ contains \mathbb{Z}_2 -torsion. In [3], the authors showed that the chromatic symmetric homology of a finite non-planar graph contains \mathbb{Z}_2 -torsion in bidegree $(1, 0)$. We hope that these new tools will help to understand if this conjecture is true also in the other direction.

Moreover, we think that the deletion-contraction long exact sequence could simplify the computation of the homology, even in the unweighted case, and allow to study it better.

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