NEW APPROACHES FOR STUDYING CONFORMAL EMBEDDINGS AND COLLAPSING LEVELS FOR W–ALGEBRAS

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Abstract. In this paper we prove a general result saying that under certain hypothesis an embedding of an affine vertex algebra into an affine W–algebra is conformal if and only if their central charges coincide. This result extends our previous result obtained in the case of minimal affine W-algebras [3]. We also find a sufficient condition showing that certain conformal levels are collapsing. This new condition enables us to find some levels k where $W_k(sl(N), x, f)$ collapses to its affine part when f is of hook or rectangular type. Our methods can be applied to non-admissible levels. In particular, we prove Creutzig's conjecture [18] on the conformal embedding in the hook type W-algebra $W_k(sl(n+m), x, f_{m,n})$ of its affine vertex subalgebra.

Quite surprisingly, the problem of showing that certain conformal levels are not collapsing turns out to be very difficult. In the cases when k is admissible and conformal, we prove that $W_k(sl(n+m), x, f_{m,n})$ is not collapsing. Then, by generalizing the results on semi-simplicity of conformal embeddings from [2], [5], we find many cases in which $W_k(sl(n+m), x, f_{m,n})$ is semi-simple as a module for its affine subalgebra at conformal level and we provide explicit decompositions.

1. Introduction

An embedding $i : U \to V$ of a vertex operator algebra (U, ω') into a vertex operator algebra (V, ω) is called conformal if $i(\omega') = \omega$. This definition is a natural generalization of a notion which was popular in physics literature in the mid 1980s, due to its relevance for string compactifications. In a series of papers [2, 4, 5, 7] we studied conformal embeddings associated to affine vertex algebras $V^{k'}(\mathfrak{g}'), V^k(\mathfrak{g})$ where $\mathfrak{g}$ is a basic Lie superalgebra (cf. §1.1) and $\mathfrak{g}'$ ranges over a suitable class of Lie subsuperalgebras. In [3] we initiated the study of conformal embeddings of affine vertex algebras in minimal affine W–algebras.

In this paper we study embeddings into more general affine W–algebras which are not of minimal type, as an application of an abstract criterion working for conformal vertex algebras (cf. Definition 2.1) with assumptions on strong generators of weight 2.

Affine W–algebras can be regarded as a far reaching generalization of the superconformal algebras arising in Conformal Field Theory. They are the vertex algebras $W^k(\mathfrak{g}, x, f)$, constructed in [31], [34] by quantum Hamiltonian reduction starting from a datum $(\mathfrak{g}, x, f)$ and $k \in \mathbb{C}$ (see §1.1). Two relevant features of the vertex algebras $W^k(\mathfrak{g}, x, f)$ are the following:

- $W^k(\mathfrak{g}, x, f)$ contains an affine vertex algebra $V^{\beta_k}(\mathfrak{g}^\natural)$, where $\mathfrak{g}^\natural$ is the centralizer in $\mathfrak{g}$ of an sl_2-triple containing f and β_k is the bilinear form on $\mathfrak{g}^\natural$ defined in [34, Theorem 2.1 (c)] (cf. §2.2 for notation).
- $W^k(\mathfrak{g}, x, f)$ admits a unique simple graded quotient $W_k(\mathfrak{g}, x, f)$, when $k \neq -h^\vee$.

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An important and extensively studied instance of this construction is the *minimal* case, i.e. $f = f_\theta$ is a root vector of a minimal even root θ. Motivated by our study of conformal embeddings, we found in [3] that a minimal simple affine W-algebra at certain levels can collapse to its affine vertex algebra. This leads to introduce the following definition. Denote by $\mathcal{V}(\mathfrak{g}^\natural)$ the image of $V^{\beta_k}(\mathfrak{g}^\natural)$ under the projection $W^k(\mathfrak{g}, x, f) \to W_k(\mathfrak{g}, x, f)$.

Definition 1.1. [3] We say that a level k is *collapsing* if $\mathcal{V}(\mathfrak{g}^\natural) = W_k(\mathfrak{g}, x, f)$.

A complete classification of collapsing levels in the minimal case was presented in [3]. Applications of collapsing levels to semi-simplicity of the category KL_k for affine vertex algebras were presented in [6], [8], [13]. Moreover, at collapsing levels, one has the structure of a vertex tensor category [21].

In the recent paper [17] T. Arakawa, J. van Ekeren, and A. Moreau started the study of collapsing levels in general affine W-algebras. They focus on the cases when k and the level $k^\natural$ of $\mathcal{V}(\mathfrak{g}^\natural)$ are both admissible. Admissibility implies the modularity of characters, which enables them to define the asymptotic datum of a representation. Then, if f belongs to the associated variety of $V^k(\mathfrak{g})$ (a certain affine Poisson variety), they provide a criterion for k to be collapsing in terms of asymptotic data. Although their method is not applicable if $k^\natural$ is not admissible, some of their results provide conjectures on the existence of collapsing levels beyond the admissible range. Moreover, in a recent physics paper on 4d SCFT, B. Li, D. Xie and W. Yan [36] study various quantities (central charges, flavor symmetry, Higgs branch, etc) which suggest the existence of new collapsing levels.

In the present paper we consider conformal embeddings and collapsing levels associated to affine W–algebras for general, not necessarily admissible levels. We are able to prove some conjectures of from [17] and [36] on collapsing levels. We focus on the following specific topics.

- A criterion for conformal embeddings and collapsing levels.
- Hook type W–algebras and rectangular W–algebras for $sl(N)$.
- Conformal vs collapsing levels.
- Decomposition of conformal embeddings: hook W–algebra case.

We now expand on each topic.

1.1. **A criterion for conformal embeddings and collapsing levels.** Let $\mathfrak{g}$ be a basic Lie superalgebra, i.e. a simple finite–dimensional Lie superalgebra with a reductive even part and a non-zero even invariant supersymmetric bilinear form $(.|.)$. Recall [34] that the universal affine W-algebra $W^k(\mathfrak{g}, x, f)$ of central charge $c(\mathfrak{g}, k)$ is associated to the datum $(\mathfrak{g}, x, f)$, where $\mathfrak{g}$ is a basic Lie superalgebra, x is an ad–diagonalizable element of $\mathfrak{g}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, f is an even nilpotent element of $\mathfrak{g}$ such that $[x, f] = -f$ and the eigenvalues of $\operatorname{ad} x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive. We will also assume that the datum $(\mathfrak{g}, x, f)$ is *Dynkin*, i.e. there is a $sl(2)$-triple $\{e, h, f\}$ and $x = \frac{1}{2}h$. Let $\mathfrak{g}^\natural$ be the centralizer of this $sl(2)$-triple. In [3] we proved in the minimal case that $\mathcal{V}(\mathfrak{g}^\natural)$ is conformal if and only if

$$c_{\mathfrak{g}^\natural} = c(\mathfrak{g}, k) = \frac{k\,\mathrm{sdim}\mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4, \tag{1.1}$$

where $c_{\mathfrak{g}^\natural}$ is the Sugawara central charge of the affine vertex subalgebra $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_\theta)$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. It is clear that (1.1) is a necessary condition for a conformal embedding. But proving that this condition is sufficient was highly non-trivial and it was a central part of [3]. Recall from [34] that a minimal W-algebra $W_k(\mathfrak{g}, x, f_\theta)$ is strongly generated by a Virasoro field L and fields $J^{\{u\}}, G^{\{v\}}$ of conformal weight $1, 3/2$, respectively. We proved that

(1) a sufficient condition for the embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_\theta)$ to be conformal is that the *Sugawara* conformal weight of the fields $G^{\{v\}}$ is $\frac{3}{2}$;

(2) the previous condition implies that the conformal level k is either $-\frac{2h^\vee}{3}$ or $-\frac{h^\vee-1}{2}$.

Statements (1) and (2) easily imply that (1.1) is a sufficient condition for conformal embedding.

In the present paper we provide a generalization of criterion (1.1) applying also to non-minimal affine W–algebras. Roughly speaking, we show that if one has enough knowledge about the weight two subspace of strong generators then one can check that the embedding is conformal by comparing central charges. More precisely, we prove the following general result for a conformal vertex algebra $W = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} W(n)$, of central charge c satisfying the technical hypothesis (3.1), (3.2), (3.3) below. Let L be the conformal vector of W and $L^{\mathfrak{a}}$ that of the vertex algebra $V(\mathfrak{a})$ attached to the Lie superalgebra $\mathfrak{a} = W(1)$ with λ-bracket (2.12). Note that $V(\mathfrak{a})$ is an affine vertex algebra in the sense of §2.2. In Theorem 3.1 we prove

Theorem 1.1. *Assume that W is strongly generated by $\mathfrak{a} = W(1)$ and by $\{L - L^{\mathfrak{a}}\} \cup S$ with S homogeneous and such that $(L - L^{\mathfrak{a}})(2)X = 0$ for $X \in S \cap W(2)$. Then $(L - L^{\mathfrak{a}})$ generates a proper ideal I in W if and only if $c = c_{\mathfrak{a}}$. In other words, W/I is non-zero and the image of $V(\mathfrak{a})$ in W/I is conformally embedded if and only if $c = c_{\mathfrak{a}}$.*

Note that the weight two subspace is an $\mathfrak{a}$-module, and a sufficient condition for our criterion to hold is that the trivial representation of $\mathfrak{a}$ appears with multiplicity one.

Next we prove a sufficient condition for a conformal level k to be collapsing. Assume that $c = c_{\mathfrak{a}}$ and let I be, as above, the proper ideal generated by $L - L^{\mathfrak{a}}$. Let $\pi_I : W \to W/I = \overline{W}$ be the quotient map. Set $V = span(S)$. Choose the strong generators in such a way that V is both $\mathfrak{a}$-stable and $L(0)$-stable. Decompose

$$V = \bigoplus_{i\in\mathcal{J}} V_i$$

into irreducible $\mathfrak{a}$-modules. Since $L(0)$ and $\mathfrak{a}$ commute, we can assume that V_i is homogeneous for $L(0)$ of conformal weight Δ_i. Let also C_i be the eigenvalue of $L^{\mathfrak{a}}$ on the highest weight vector of the $V(\mathfrak{a})$-module with top component V_i (cf. (3.8)). Then $\overline{W}$ is strongly generated by $\mathfrak{a} \oplus \sum_{i\in\mathcal{K}} \pi_I(V_i), \mathcal{K} \subset \mathcal{J}$. We assume $\mathcal{K}$ to be minimal in the sense that, if $\mathcal{T}$ is a proper subset of $\mathcal{K}$, then $\mathfrak{a} \oplus (\sum_{i\in\mathcal{T}} \pi_I(V_i))$ does not strongly generate $\overline{W}$. In Theorem 3.2 we prove

Theorem 1.2. *With the above assumptions, $C_j = \Delta_j$ for all $j \in \mathcal{K}$. In particular, if $C_i \neq \Delta_i$ for all $i \in \mathcal{J}$, then $\mathcal{K} = \emptyset$, hence $\overline{W}$ collapses to $V(\mathfrak{a})/(I \cap V(\mathfrak{a}))$.*

1.2. **Hook type W–algebras and rectangular W–algebras.** Hook type W–algebras recently appeared in [37] as coset vertex algebras, and also in the context of dualities and trialities of various vertex algebras [19, 20, 22]. In these papers the authors mainly studied the generic level case. Since conformal levels are not generic, we derive in Theorem 5.1 the structure of these W-algebras at arbitrary level, using only very general results by Kac and Wakimoto [34]. A similar approach is applied in Theorem 10.1 for the rectangular W–algebras.

We first apply Theorems 1.1 and 1.2 to the hook type W-algebra $W_k(\mathfrak{g}, x, f_{m,n})$ for $\mathfrak{g} = sl(m+n)$ (i.e., the partition representing the nilpotent element $f_{m,n}$ is the hook $(m, 1^n)$). The outcome is the following result (see Theorems 5.2, 6.2).

Theorem 1.3.

(1) *The embedding* $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{m,n})$ *is conformal if and only if*

$$k = k^{(i)}_{m,n}, \quad 1 \le i \le 4,$$

where $k^{(1)}_{m,n} = -\frac{m}{m+1}h^\vee$ *(*$n > 1$*),* $k^{(2)}_{m,n} = -\frac{(m-1)h^\vee - 1}{m}$ *(*$n \ge 1$*),* $k^{(3)}_{m,n} = -\frac{(m-2)h^\vee+1}{m-1}$ *(*$n \ge 1$*,* $m > 1$*),* $k^{(4)}_{m,n} = -\frac{(m-1)h^\vee}{m}$*.*

(2) *Levels* $k^{(3)}_{m,n}$ *(*$m \ne n-1$*) and* $k^{(4)}_{m,n}$ *are collapsing.*

The case $k = k^{(1)}_{m,n}$ solves a conjecture of Creutzig [18]. Next we consider the rectangular case and the W-algebra $W_k(\mathfrak{g}, x, f)$ for $\mathfrak{g} = sl(mq)$ (i.e., the partition representing the nilpotent element f is rectangular of shape (q^m)). In this case we obtain the following theorem (see Theorem 10.2).

Theorem 1.4.

(1) *The embedding* $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{m,n})$ *is conformal if and only if*

$$k = k^{[i]}_{m,q}, \quad 1 \le i \le 3,$$

where $k^{[1]}_{m,q} = -\frac{mq^2}{q+1}$*,* $k^{[2]}_{m,q} = -\frac{-mq^2+mq-1}{q}$*,* $k^{[3]}_{m,q} = -\frac{-mq^2+mq+1}{q}$*.*

(2) *Levels* $k^{[i]}_{m,n}$*,* $i = 1, 2, 3$ *are collapsing for all* $q \ge 2$ *and* $m \ge 3$ *and we have*

$$W_{k^{[1]}_{m,q}}(\mathfrak{g}, x, f_{m,n}) = V_{-\frac{mq}{q+1}}(sl(m)),$$
$$W_{k^{[2]}_{m,q}}(\mathfrak{g}, x, f_{m,n}) = V_{-1}(sl(m)),$$
$$W_{k^{[3]}_{m,q}}(\mathfrak{g}, x, f_{m,n}) = V_{1}(sl(m)).$$

We should mention that in the cases when $k^{[i]}_{m,q}$ is admissible, the results of the previous theorem are (or can be) obtained by using methods from [17]. Collapsing of the non-admissible W–algebra $W_{k^{[2]}_{m,q}}(sl(mq), x, f_{m,n})$ was conjectured in [17] (see Remark 8.4 below).

1.3. **Conformal vs collapsing levels.** Assume that k is a conformal level of an affine W-algebra $W^k(\mathfrak{g}, x, f)$. Then $\bar{L} = L - L^{\mathfrak{g}^\natural}$ is a singular vector in $W^k(\mathfrak{g}, x, f)$ and we can investigate the ideal $I = W^k(\mathfrak{g}, x, f).\bar{L}$ and the quotient vertex algebra $\overline{W}_k(\mathfrak{g}, x, f) = W^k(\mathfrak{g}, x, f)/I$. Since it can happen that $\overline{W}_k(\mathfrak{g}, x, f) \ne W_k(\mathfrak{g}, x, f)$ (cf. Remark 6.1), we introduce the following notion.

Definition 1.2. Let $\mathcal{V}(\mathfrak{g}^\natural)$ be the image of $V^{\beta_k}(\mathfrak{g}^\natural)$ in $\overline{W}_k(\mathfrak{g}, x, f)$. A conformal level k is called *strongly collapsing* if $\mathcal{V}(\mathfrak{g}^\natural) = \overline{W}_k(\mathfrak{g}, x, f)$.

The strongly collapsing levels are easy to detect using our Theorem 1.2. We do have examples of levels which are collapsing but not strongly collapsing (see Remark 6.1). These situations are difficult to detect in general, due to the lack of knowledge of explicit OPE formulas in non-minimal cases. Clearly, every strongly collapsing level is also collapsing. Levels described in Theorem 1.2 are strongly collapsing. So when we investigate whether a certain level is collapsing, we first check if such a level is strongly collapsing.

In the case of hook W–algebras we get:

Proposition 1.5. *Levels* $k^{(1)}_{m,n}$ *and* $k^{(2)}_{m,n}$ *are not strongly collapsing for* $W_k(sl(n+m), x, f_{m,n})$*.*

But it still may happen that some of these levels are collapsing. We conjecture:

Conjecture 1.6. $k = k^{(1)}_{m,n}$ *is always non-collapsing.*

Using different techniques, it was proved in [9] that the level $k^{(1)}_{p-1,2}$ is not collapsing. But we prove in Theorem 8.6 that $k^{(2)}_{3p,2}$ is collapsing. The main difference between these two cases is that $k^{(1)}_{p-1,2}$ is admissible, whereas $k^{(2)}_{3p,2}$ is not.

The peculiarity of the admissible case is that the maximal ideal in $V^k(sl(n+m))$ is generated by one singular vector. As a consequence, we prove in Theorem 9.6 that $\bar{L}$ generates the maximal ideal in $W^k(sl(n+m), x, f_{m,n})$. Thus if $k = k^{(i)}_{m,n}$, $i = 1, 2$ is admissible, then k is not collapsing. Therefore Conjecture 1.6 holds for admissible levels. We also present in Remark 8.3 some arguments explaining why the conjecture should hold in general.

1.4. **Decomposition of conformal embeddings: hook W–algebra case.** If the embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f)$ is conformal, it is natural to ask for the decomposition of $W_k(\mathfrak{g}, x, f)$ as a $\mathcal{V}(\mathfrak{g}^\natural)$-module. In general, describing this decomposition is a hard problem, open in most cases. Some decompositions are provided in our previous papers [2, 4, 5, 7]. In particular, the paper [5] shows decompositions of minimal affine W–algebras $W_k(\mathfrak{g}, x, f_\theta)$. It turns out that we can generalize some results of [5] to hook type W–algebras.

Theorem 1.7. *(cf. Theorem 8.2) Let $k = k^{(i)}_{m,n}$ for $i = 1, 2$ and assume also that k is non-collapsing. Assume that $\frac{m+1}{n-1} \notin \mathbb{Z}$ if $i = 1$ and $\frac{m}{n+1} \notin \mathbb{Z}$ if $i = 2$. Then*

$$W_k(\mathfrak{g}, x, f_{m,n}) = \bigoplus_{i \in \mathbb{Z}} W^{(i)}_k, \tag{1.2}$$

where each $W^{(i)}_k = \{v \in W_k \mid J(0)v = iv\}$ is an irreducible $V(gl(n))$–module (cf. (7.4) *for the definition of J). In particular, decomposition* (1.2) *holds if k is admissible and $n > 2$.*

1.5. **Future work.** The main purpose of the present paper is to introduce new general methods for studying conformal embeddings and collapsing levels for affine W–algebras. As an illustration, we apply our methods for the hook and rectangular type W-algebras of type A. It is worthwhile to note that, since the present paper appeared in preprint form, our results on collapsing levels have already been applied by physicists in the context of four dimensional $N = 2$ superconformal field theories [38].

The same methods can also be applied for hook type W–algebras of types B, C, D and for hook type W–superalgebras (see [22, Table 1] for the list of such vertex algebras). We hope to study these cases in our forthcoming papers.

The next important problem is to study decompositions of conformal embeddings. As we see in present paper, the first difficult step is to prove that a given conformal level is not collapsing. We obtained results in this direction using the fact that a certain conformal level is admissible, but we hope that our arguments can be extended to non-admissible cases. A natural approach would be to use λ–bracket for W–algebras. This approach works for algebras of low rank (cf. [12], [26]), where one can use computer calculations.

It was proved in [9] that $W_{k^{(1)}_{p-1,2}}(sl(p+1), f_{p-1,2}) = \mathcal{R}^{(p)}$, where $\mathcal{R}^{(p)}$ is a certain logarithmic vertex algebra realized in [1] as an extension of $L_{-2+1/p}(gl(2))$. We believe that recent explicit realizations, motivated by inverting of quantum Hamiltonian reduction (as in the case of $L_k(sl(3))$ from [10]), can be used to get explicit decompositions for conformal embeddings in some non-collapsing cases.

We believe that at each collapsing level, the category $KL_k(\mathfrak{g})$ (cf. [8, Definition 2.1]) is semi-simple. This is proved in [6, 7] in the case of collapsing levels for the minimal reduction. The same is true for some collapsing levels in non-minimal cases (cf. [13, 11]). In all these examples we also have that $KL_k(\mathfrak{g})$ is a rigid braided tensor category [21]. The proof of rigidity uses the conformal embedding of $\mathcal{V}(\mathfrak{g})$ into affine vertex algebras or minimal affine W–algebras. Our paper provides some tools for a natural generalization. Decompositions in Theorem 1.7 suggest that the $V(gl(n))$–modules in (1.2) are simple currents in a suitable tensor category. Unfortunately, our current methods are not sufficient to prove such statement. We hope to study this problem elsewhere.

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2. Setup

2.1. **Conformal vertex algebras.** Recall that a vector superspace is a $\mathbb{Z}/2\mathbb{Z}$–graded vector space $W = W_{\bar{0}} \oplus W_{\bar{1}}$. The elements in $W_{\bar{0}}$ (resp. $W_{\bar{1}}$) are called even (resp. odd). Set

$$p(v) = \begin{cases} 0 \in \mathbb{Z} & \text{if } v \in W_{\bar{0}}, \\ 1 \in \mathbb{Z} & \text{if } v \in W_{\bar{1}}. \end{cases}$$

We will regard $p(v)$ as an integer, not as a residue class. We will often use the notation

$$\sigma(u) = (-1)^{p(u)}u, \qquad p(u, v) = (-1)^{p(u)p(v)}. \tag{2.1}$$

Let W be a vertex algebra. We let

$$\begin{aligned} &Y : W \to (\operatorname{End} W)[[z, z^{-1}]], \\ &v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \qquad (v_n \in \operatorname{End} W), \end{aligned} \tag{2.2}$$

denote the state–field correspondence. We denote by $\mathbf{1}$ the vacuum vector in W and by T the translation operator.

Definition 2.1. A *conformal* vertex algebra is a vertex algebra W such that there exists a distinguished vector $L \in W_2$, called a Virasoro vector, satisfying the following conditions:

(2.3) $Y(L, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, $[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c\, I$,

(2.4) $L(-1) = T$,

(2.5) $L(0)$ is diagonalizable and its eigenspace decomposition has the form

$$W = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} W(n), \tag{2.6}$$

where

$$\dim W(n) < \infty \text{ for all } n \text{ and } W(0) = \mathbb{C}\mathbf{1}. \tag{2.7}$$

The number c is called the *central charge*.

2.2. **Affine vertex algebras.** Let $\mathfrak{a}$ be a Lie superalgebra equipped with an invariant supersymmetric bilinear form B. The universal affine vertex algebra $V^B(\mathfrak{a})$ is the universal enveloping vertex algebra of the non–linear Lie conformal superalgebra $R=(\mathbb{C}[T]\otimes\mathfrak{a})$ with λ-bracket given by

$$[a_\lambda b]=[a,b]+\lambda B(a,b),\ a,b\in\mathfrak{a}.$$

In the following, we shall say that a vertex algebra V is an affine vertex algebra if it is a quotient of some $V^B(\mathfrak{a})$. If $k\in\mathbb{C}$ and $\mathfrak{a}$ is simple, we will write simply $V^k(\mathfrak{a})$ for $V^{k(\cdot|\cdot)}(\mathfrak{a})$, where $(\cdot|\cdot)$ is a fixed normalized bilinear form. We will always assume that k is non–critical, i.e. $k\neq -h^\vee$. With this assumption, it is known that $V^k(\mathfrak{g})$ has a unique simple quotient, denoted by $V_k(\mathfrak{g})$ (see [28, 4.7 and Example 4.9b]). The vertex algebras $V^k(\mathfrak{g}),V_k(\mathfrak{g})$ are VOAs with Virasoro vector $L_\mathfrak{g}$ given by the Sugawara construction.

2.3. **Invariant Hermitian forms on conformal vertex algebras.** Let W be a conformal vertex algebra. If $a\in W(\Delta_a)$, set

$$(-1)^{L(0)}a=e^{\pi\sqrt{-1}\Delta_a}a,\quad \sigma^{1/2}(a)=e^{\frac{\pi}{2}\sqrt{-1}p(a)}a. \tag{2.8}$$

If ϕ is a conjugate linear involution of W, set

$$g=((-1)^{L(0)}\sigma^{1/2})^{-1}\phi. \tag{2.9}$$

Recall from [29] the following definition.

Definition 2.2. Let ϕ be a conjugate linear involution of a conformal vertex algebra W such that $\phi(L)=L$. Let g be as in (2.9). A Hermitian form $(\cdot\,,\cdot)$ on W is said to be *ϕ–invariant* if, for all $a\in W$,

$$(v,Y(a,z)u)=(Y(e^{zL(1)}z^{-2L(0)}ga,z^{-1})v,u),\quad u,v\in W. \tag{2.10}$$

Remark 2.1. The existence conditions for a ϕ–invariant Hermitian form on a vertex operator algebra W are discussed in [29, Theorem 4.3]. For a conformal vertex algebra these conditions reduce to

$$L(1)W(1)=\{0\}. \tag{2.11}$$

In such a case we normalize the form by requiring $(\mathbf{1},\mathbf{1})=1$.

Remark 2.2. The kernel of (any) ϕ–invariant Hermitian form on a conformal vertex algebra coincides with its maximal ideal.

Recall that $\mathfrak{a}=W(1)$ is a Lie superalgebra with bracket defined by

$$[a,b]=a_0b.$$

Let $\langle\cdot,\cdot\rangle$ be the bilinear form on $\mathfrak{a}$ defined by

$$\langle a,b\rangle=(g(a),b).$$

If (2.11) holds, $\mathfrak{a}$ is made of primary elements. It follows that

$$\langle a,b\rangle\mathbf{1}=(g(a),b)\mathbf{1}=(g(a)_{-1}\mathbf{1},b_{-1}\mathbf{1})\mathbf{1}=(\mathbf{1},a_1b)\mathbf{1}=a_1b,$$

so that we can write

$$[a_\lambda b]=[a,b]+\lambda\langle a,b\rangle\mathbf{1}. \tag{2.12}$$

Note that the form $\langle\cdot,\cdot\rangle$ is a supersymmetric invariant form. Indeed

$$\langle b,a\rangle=(g(b),a)=\overline{(a,g(b))}=\overline{g(a)_1g(b)}.$$

As shown in Lemma 3.1 of [29], one has

$$g(v_n u) = (-1)^{n+1} p(v,u) g(v)_n g(u), \tag{2.13}$$

so

$$\langle b, a\rangle = p(a,b)\overline{g(a_1 b)} = p(a,b) a_1 b = p(a,b)(g(a),b) = p(a,b)\langle a,b\rangle.$$

Since

$$\langle a, [b,c]\rangle = \langle a, b_0 c\rangle = (g(a), b_0 c) = (g(b)_0 g(a), c),$$

using (2.13) again, we find

$$\langle a, [b,c]\rangle = -p(b,a)(g(b_0 a), c) = -p(b,a)\langle [b,a], c\rangle = \langle [a,b], c\rangle.$$

Therefore, by (2.12), the vertex subalgebra generated by $\mathfrak{a}$ is an affine vertex algebra that, in this section, we denote by $V(\mathfrak{a})$.

We assume that $\mathfrak{a} = \bigoplus_{i=0}^r \mathfrak{a}_i$ with $\mathfrak{a}_0$ an even abelian Lie algebra (possibly $\{0\}$) and $\mathfrak{a}_i$ simple Lie algebras or basic Lie superalgebras.

Lemma 2.1. *Let $\mathfrak{a}'$ be the kernel of $\langle\cdot,\cdot\rangle$. Then $(\mathfrak{a}')_{-1}\mathbf{1}$ generates a proper ideal $I_{\mathfrak{a}'}$ of W.*

Proof. Since ϕ is an automorphism of W, if $a, b \in \mathfrak{a}$, then

$$\phi([a_\lambda b]) = [\phi(a)_\lambda \phi(b)]$$

so $\phi(\langle a,b\rangle\mathbf{1}) = \langle\phi(a),\phi(b)\rangle\mathbf{1}$ hence $\langle\phi(a),\phi(b)\rangle = \overline{\langle a,b\rangle}$. In particular $\phi(\mathfrak{a}') = \mathfrak{a}'$. It follows that

$$(\mathfrak{a}', \mathfrak{a}) = (g(\mathfrak{a}'), \mathfrak{a}) = \langle \mathfrak{a}', \mathfrak{a}\rangle = 0.$$

Since $\mathfrak{a} = W(1)$ and $\mathfrak{a}' \subset W(1)$, we have $(\mathfrak{a}', W) = 0$, hence $\mathfrak{a}'$ is in the kernel of the form $(\cdot,\cdot)$. We conclude using Remark 2.2. □

2.4. **Casimir elements and conformal vectors.** Let $\langle\cdot,\cdot\rangle_i$ be the nondegenerate invariant form on $\mathfrak{a}_i$ normalized as follows: if $\mathfrak{a}_i$ is an even simple Lie algebra then require $\langle\theta_i,\theta_i\rangle = 2$ (where θ_i is a long root of $\mathfrak{a}_i$). If $\mathfrak{a}_i$ is not even, then we let $\langle\cdot,\cdot\rangle_i$ be the form described explicitly in Table 6.1 of of [33]. Set $\mathfrak{a}_0' = \mathfrak{a}' \cap \mathfrak{a}_0$. On $\mathfrak{a}_0$ we choose a basis $\{a_i\}$ such that $\langle a_i, a_j\rangle = 0$ for $i \neq j$ and

$$\langle a_i, a_i\rangle = \begin{cases} 0 & 1 \le i \le \dim \mathfrak{a}_0', \\ 1 & \dim \mathfrak{a}_0' < i \le \dim \mathfrak{a}_0. \end{cases}$$

Set $\mathfrak{a}_0'' = span(a_i \mid i > \dim \mathfrak{a}' \cap \mathfrak{a}_0)$. Set $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_{|\mathfrak{a}_0\times\mathfrak{a}_0}$. Since $\langle\cdot,\cdot\rangle$ is invariant we have

$$\langle\cdot,\cdot\rangle_{|\mathfrak{a}_i\times\mathfrak{a}_i} = k_i\langle\cdot,\cdot\rangle_i \ (i \ge 1). \tag{2.14}$$

If $\langle\cdot,\cdot\rangle_0 = 0$ we let $k_0 = 0$ and $k_0 = 1$ otherwise, so that (2.14) holds also for $i = 0$.

It follows that $V(\mathfrak{a})$ is a quotient of $V^0(\mathfrak{a}_0') \otimes V^1(\mathfrak{a}_0'') \otimes (\bigotimes_{i>0} V^{k_i}(\mathfrak{a}_i))$ We assume that $k_i \in \mathbb{R}$ for all i.

For $i > 0$, let $C_{\mathfrak{a}_i}$ be the Casimir element of $\mathfrak{a}_i$ corresponding to $\langle\cdot,\cdot\rangle_i$ and let $2h_i^\vee$ be the eigenvalue of its action on $\mathfrak{a}_i$. For $i = 0$ we let

$$C_{\mathfrak{a}_0} = \begin{cases} 0 & \text{if } \mathfrak{a}_0 = \mathfrak{a}_0', \\ \sum_{i=\dim\mathfrak{a}_0'+1}^{\dim\mathfrak{a}_0} a_i^2 & \text{otherwise,} \end{cases}$$

so that $h_0^\vee = 0$.

We assume that, for $i>0$, k_i are non-critical, i.e. $k_i\neq -h_i^\vee$. This implies that $V^{k_i}(\mathfrak{a}_i)$ admits a Virasoro vector $L^{\mathfrak{a}_i}$ given by Sugawara construction. We set also

$$L^{\mathfrak{a}_0}=\frac{1}{2}\sum_{i=\dim\mathfrak{a}_0'+1}^{\dim\mathfrak{a}_0}:a_ia_i:.$$

This is a Virasoro vector for $V^1(\mathfrak{a}_0'')$. Define $L^{\mathfrak{a}}\in W$ to be the image of $\sum_{i\geq 0}L^{\mathfrak{a}_i}$ in $V(\mathfrak{a})$. In general this is not a Virasoro vector for $V(\mathfrak{a})$. Let $I_{\mathfrak{a}_0'}$ be the ideal generated by $\mathfrak{a}_0'$. By Lemma 2.1, the ideal $I_{\mathfrak{a}_0'}$ is proper. Let $\pi_{\mathfrak{a}_0'}:W\to W/I_{\mathfrak{a}_0'}$ be the quotient map. Then $\pi_{\mathfrak{a}_0'}(L^{\mathfrak{a}})$ is a Virasoro vector for $\pi_{\mathfrak{a}_0'}(V(\mathfrak{a}))$. Note that

$$L^{\mathfrak{a}}(2)L^{\mathfrak{a}}=\tfrac{1}{2}c_{\mathfrak{a}}\mathbf{1}.$$

By a slight abuse of terminology, we call $c_{\mathfrak{a}}$ the *central charge* of $L^{\mathfrak{a}}$.

3. A criterion for conformal embedding of affine vertex algebras into conformal vertex algebras

Let W be a conformal vertex algebra with conformal vector L. Make the following assumptions on W:

(3.1) there is a conjugate linear involution ϕ of W with $\phi(L)=L$;

(3.2) $W(\frac{1}{2})=\{0\}$;

(3.3) $\mathfrak{a}=W(1)$ consists of L–primary elements.

By Remark 2.1, there is a ϕ-invariant Hermitian form $(\cdot,\cdot)$ on W such that $(\mathbf{1},\mathbf{1})=1$.

We say that a subset $\mathfrak{V}$ of a vertex algebra W, homogeneous with respect to parity, strongly generates W if

$$W=span(:T^{j_1}(w_1)\cdots T^{j_r}(w_r):\mid r\in\mathbb{Z}_+,\ w_i\in\mathfrak{V}).$$

Our first result is the following general criterion for the existence of a conformal embedding

$$\widetilde{V}(\mathfrak{a})\hookrightarrow\overline{W},$$

where $\overline{W}=W/I$ is a nonzero quotient of W and $\widetilde{V}(\mathfrak{a})=V(\mathfrak{a})/(I\cap V(\mathfrak{a}))$. It is clear that such an embedding exists if and only if $L-L^{\mathfrak{a}}$ generates a proper ideal of W.

Theorem 3.1. *Assume that W is strongly generated by $\mathfrak{a}$ and by $\{L-L^{\mathfrak{a}}\}\cup S$ with S homogeneous with respect to the gradation (2.6) and such that $(L-L^{\mathfrak{a}})(2)X=0$ for $X\in S\cap W(2)$. Then $(L-L^{\mathfrak{a}})$ generates a proper ideal I in W if and only if $c=c_{\mathfrak{a}}$.*

Proof. As noted in the proof of Lemma 2.1, $\langle\phi(a),\phi(b)\rangle=\overline{\langle a,b\rangle}$. In particular $\phi(\mathfrak{a}')=\mathfrak{a}'$, thus the ideal $I_{\mathfrak{a}'}$ is ϕ–stable. Let $\pi_{\mathfrak{a}'}:W\to W/I_{\mathfrak{a}'}$ be the projection. It is clear that the ideal I generated by $L-L^{\mathfrak{a}}$ is proper in W if $\pi_{\mathfrak{a}'}(I)$ is proper in $W/I_{\mathfrak{a}'}$, we can therefore assume that $\mathfrak{a}'=\{0\}$, so that $k_i\neq 0$ for all i. Moreover, if $\phi(\mathfrak{a}_i)=\mathfrak{a}_j$, then, since $k_i\in\mathbb{R}$, we have $k_i=k_j$ and for $a,b\in\mathfrak{a}_i$,

$$\langle\phi(a),\phi(b)\rangle_j=\overline{\langle a,b\rangle}_i. \tag{3.4}$$

Let $\{x_i\}$ be a basis of $\mathfrak{a}$ and let $\{x^i\}$ its dual basis with respect to $\bigoplus_i\langle\cdot,\cdot\rangle_i$. Then, by (3.4),

$$\phi(\sum:x^ix_i:)=\sum:\phi(x^i)\phi(x_i):=\sum:x^ix_i:,$$

so $\phi(L^{\mathfrak{a}})=L^{\mathfrak{a}}$.

Set $U = span(X \in S \mid X \in W(2))$. Observe that

$$W(2) = \mathbb{C}(L - L^{\mathfrak{a}}) + U + V(\mathfrak{a}) \cap W(2). \tag{3.5}$$

We have

$$(L - L^{\mathfrak{a}}, U) = (\mathbf{1}, (L^{\mathfrak{a}} - L^{\mathfrak{a}})(2)U) = 0 \tag{3.6}$$

by our assumption that $(L - L^{\mathfrak{a}})(2)U = 0$.

If $c = c_{\mathfrak{a}}$ then

$$(L - L^{\mathfrak{a}}, L - L^{\mathfrak{a}}) = ((L - L^{\mathfrak{a}})(2)(L - L^{\mathfrak{a}}), \mathbf{1}) = \frac{1}{2}\overline{(c - c_{\mathfrak{a}})} = 0.$$

Since $L - L^{\mathfrak{a}} \in Com(V(\mathfrak{a}), W)$, we have

$$(L - L^{\mathfrak{a}}, V(\mathfrak{a}) \cap W(2)) = 0. \tag{3.7}$$

From (3.5)-(3.7) it follows that $(L - L^{\mathfrak{a}}, W(2)) = 0$. Since $(W(i), W(j)) = 0$ if $i \neq j$ we see that $L - L^{\mathfrak{a}}$ is in the kernel of the form $(\cdot, \cdot)$, which is a proper ideal.

Converesely, if $L - L^{\mathfrak{a}}$ generates a proper ideal I then $L - L^{\mathfrak{a}} = 0$ in W/I, hence $(L - L^{\mathfrak{a}})(2)(L - L^{\mathfrak{a}}) = \frac{1}{2}(c - c_{\mathfrak{a}})\mathbf{1} = 0$. This completes the proof. □

In the setting of Theorem 3.1, we can clearly assume that S is a space stable under the action of $\mathfrak{a}$. We make the further assumption that the action of $\mathfrak{a}$ on W is completely reducible, thus we can choose S so that $(\mathbb{C}L + \mathfrak{a}) \oplus S$ is a set of strong generators for W. Decomposing S as $\mathfrak{a}$-module, we can write $S = \oplus_{i \in \mathcal{J}} S_i$ with S_i irreducible and $\mathcal{J}$ some index set. Write

$$S_i = \bigotimes_j S_i^j$$

with S_i^j irreducible $\mathfrak{a}_j$–modules. Set

$$C_i = \sum_{j \geq 1} \frac{c_j^{(i)}}{2(k_j + h_j^\vee)} + (1 - \delta_{k_0,0})\frac{c_0^{(i)}}{2k_0}, \tag{3.8}$$

where $c_j^{(i)}$ is the eigenvalue of $C_{\mathfrak{a}_j}$ on S_i^j. Let M be a weight $V(\mathfrak{a})$–module such that $S_i \subset M^{top}$. Then

$$L^{\mathfrak{a}}(0)v = C_i v,\ v \in S_i \subset M^{top}. \tag{3.9}$$

Recall that we are assuming that S is homogeneous under the action of $L(0)$; since the actions of $\mathfrak{a}$ and $L(0)$ commute, we can clearly assume that also $S_i, i \in \mathcal{J}$ are homogeneous: we set Δ_i to be the eigenvalue of $L(0)$ on S_i. Note that we are not assuming that the action of $L_0^{\mathfrak{a}}$ on W is semisimple.

In the setting of Theorem 3.1, assume that $c = c_{\mathfrak{a}}$ and let I be the proper ideal generated by $L - L^{\mathfrak{a}}$. Let $\pi_I : W \to W/I = \overline{W}$ be the quotient map. Since $L = L^{\mathfrak{a}}$ on $\overline{W}$, the action of $L^{\mathfrak{a}}$ is semisimple on $\overline{W}$. Choose as above a set of strong generators $(\mathbb{C}(L - L^{\mathfrak{a}}) + \mathfrak{a}) \oplus \sum_{i \in \mathcal{J}} S_i$. Then $\overline{W}$ is strongly generated by

$$\mathfrak{a} \oplus \sum_{i \in \mathcal{K}} \pi_I(S_i)$$

and we can assume $\mathcal{K} \subset \mathcal{J}$ minimal.

Theorem 3.2. *With the above assumptions, $C_j = \Delta_j$ for all $j \in \mathcal{K}$. In particular, if $C_i \neq \Delta_i$ for all $i \in \mathcal{J}$, then $\overline{W} \cong \widetilde{V}(\mathfrak{a})$.*

Proof. Fix $\Delta_i, i \in \mathcal{K}$. Let $\overline{W}'$ be the vertex subalgebra generated by

$$\overline{W}(\Delta_i)^- = \sum_{n<\Delta_i} \overline{W}(n).$$

Note that, if $a \in \mathfrak{a}$ and $n \geq 0$, then $a_n\overline{W}(\Delta_i)^- \subset \overline{W}(\Delta_i)^-$. It follows that $\overline{W}'$ is strongly generated by $\overline{W}(\Delta_i)^-$. By the minimality of $\mathcal{K}$, this observation implies that $\pi_I(S_i) \cap \overline{W}' = \{0\}$. By construction, $a_n(\pi_I(S_i) + \overline{W}') = \overline{W}'$ in $\overline{W}/\overline{W}'$ if $n > 0$ and $a \in \mathfrak{a}$, hence, by (3.9),

$$L^{\mathfrak{a}}(0)(\pi_I(S_i) + \overline{W}') = C_i(\pi_I(S_i) + \overline{W}').$$

On the other hand, since $L = L^{\mathfrak{a}}$ in $\overline{W}$, we find

$$L^{\mathfrak{a}}(0)(\pi_I(S_i) + \overline{W}') = L(0)(\pi_I(S_i)) + \overline{W}' = \Delta_i(\pi_I(S_i) + \overline{W}').$$

Since $\pi_I(S_i) + \overline{W}' \neq 0$ the claim follows. □

4. Structure

We adopt the setting and notation of Section 1 of [34]. We let $W^k(\mathfrak{g}, x, f)$ be the universal W–algebra of level $k \in \mathbb{R}$ associated to the datum $(\mathfrak{g}, x, f)$, where $\mathfrak{g}$ is a basic Lie superalgebra, x is an ad–diagonalizable element of $\mathfrak{g}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, f is an even element of $\mathfrak{g}$ such that $[x, f] = -f$ and the eigenvalues of $\operatorname{ad} x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive. We also assume that the datum $(\mathfrak{g}, x, f)$ is *Dynkin*, i.e. there is a $sl(2)$-triple $\{e, h, f\}$ and $x = \frac{1}{2}h$. Let

$$\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{4.1}$$

be the grading of $\mathfrak{g}$ by $\operatorname{ad}(x)$–eigenspaces. We assume that $k \neq -h^\vee$ so that $W^k(\mathfrak{g}, x, f)$ has a Virasoro vector. Then $W^k(\mathfrak{g}, x, f)$ is a conformal vertex algebra in the sense of Definition 2.1. It is known that the vertex algebra structure of $W^k(\mathfrak{g}, x, f)$ depends only on the $G_{\bar{0}}$-orbit of f (where $G_{\bar{0}}$ is the adjoint group corresponding to $\mathfrak{g}_{\bar{0}}$), but the conformal structure does depend also on the grading (4.1).

We let:

$$\mathfrak{g}_+ = \bigoplus_{j>0} \mathfrak{g}_j\,, \quad \mathfrak{g}_- = \bigoplus_{j<0} \mathfrak{g}_j\,, \quad \mathfrak{g}_\leq = \mathfrak{g}_0 \oplus \mathfrak{g}_-\,. \tag{4.2}$$

The element f defines a skew-supersymmetric even bilinear form $\langle\,.\,,.\rangle_{ne}$ on $\mathfrak{g}_{1/2}$ by the formula:

$$\langle a, b\rangle_{ne} = (f|[a, b])\,. \tag{4.3}$$

Denote by $\mathfrak{g}^\natural$ the centralizer of f in $\mathfrak{g}_0$.

Denote by A_{ne} the vector superspace $\mathfrak{g}_{1/2}$ endowed with the bilinear form (4.3). Denote by A (resp. A^*) the vector superspace $\mathfrak{g}_+$ (resp. $\mathfrak{g}_+^*$) with the reversed parity, let $A_{ch} = A \oplus A^*$ and define an even skew-supersymmetric non-degenerate bilinear form $\langle.\,,.\rangle_{ch}$ on A_{ch} by

$$\begin{aligned} \langle A, A\rangle_{ch} &= 0 = \langle A^*, A^*\rangle_{ch}\,, \\ \langle a, b^*\rangle_{ch} &= -(-1)^{p(a)p(b^*)}\langle b^*, a\rangle_{ch} = b^*(a) \text{ for } a \in A, b^* \in A^*\,. \end{aligned} \tag{4.4}$$

Let $F(A_{ch})$ and $F(A_{ne})$ be the fermionic vertex algebras based on A_{ch}, A_{ne} respectively. Recall from [34] that $W^k(\mathfrak{g}, x, f)$ is a vertex subalgebra of $V^k(\mathfrak{g}) \otimes F(A_{ch}) \otimes F(A_{ne})$.

Choose a basis $\{u_\alpha\}_{\alpha\in\Sigma_j}$ of each $\mathfrak{g}_j$ in (4.1), and let $\Sigma = \coprod_{j\in\frac{1}{2}\mathbb{Z}}\Sigma_j$, $\Sigma_+ = \coprod_{j>0}\Sigma_j$. Let $m_\alpha = j$ if $\alpha \in \Sigma_j$. Denote by $\{\varphi_\alpha\}_{\alpha\in\Sigma_+}$ the corresponding basis of A and by $\{\varphi^\alpha\}_{\alpha\in\Sigma_+}$ the basis of A^* such that $\langle\varphi_\alpha, \varphi^\beta\rangle_{ch} = \delta_{\alpha\beta}$. Denote by $\{\Phi_\alpha\}_{\alpha\in\Sigma_+}$ the corresponding basis of A_{ne}, and by $\{\Phi^\alpha\}_{\alpha\in\Sigma_{1/2}}$ the dual basis with respect to $\langle .\, , .\rangle_{ne}$, i.e., $\langle\Phi_\alpha, \Phi^\beta\rangle_{ne} = \delta_{\alpha\beta}$. It will also be convenient to define Φ_u for any $u \in \sum_{\alpha\in\Sigma} c_\alpha u_\alpha \in \mathfrak{g}$ by letting $\Phi_u = \sum_{\alpha\in\Sigma_{1/2}} c_\alpha\Phi_\alpha$; similarly, for $u \in \mathfrak{g}_+$, we will use the notation φ_u.

Remark 4.1. Since we have chosen the grading to be Dynkin, we can apply [29, Lemma 7.3]. Thus a conjugate linear involution ϕ of $\mathfrak{g}$ fixing x and f and satisfying $(\phi(a)|\phi(b)) = \overline{(a|b)}$ descends to define a conjugate linear involution of $W^k(\mathfrak{g}, x, f)$ that we still denote by ϕ.

Set

$$\widehat{\mathfrak{g}^f} = \bigoplus_{j\leq 0}(\mathfrak{g}^f_j \otimes \mathbb{C}[t^{-1}]t^{-1+j}).$$

The space $\widehat{\mathfrak{g}^f}$ admits a grading by the action of $D = -t\frac{d}{dt}$. Moreover there is a natural action of $\mathfrak{g}^\natural$ on $\widehat{\mathfrak{g}^f}$. Let $S(\widehat{\mathfrak{g}^f})$ be the symmetric algebra of $\widehat{\mathfrak{g}^f}$, equipped with the natural action of $\mathfrak{g}^\natural$ and the action of D extended by derivations. We let D act on $W^k(\mathfrak{g}, x, f)$ by the action of $L(0)$. Recall that $W^k(\mathfrak{g}, x, f)(1)$ is isomorphic to $\mathfrak{g}^\natural$ as a Lie superalgebra, so we can define an action of $\mathfrak{g}^\natural$ on $W^k(\mathfrak{g}, x, f)$ by letting $a \in \mathfrak{g}^\natural \cong W^k(\mathfrak{g}, x, f)(1)$ act by a_0.

We say that a finite–dimensional sub–superspace $\mathfrak{V}$ of W is free in W if there is a basis $\mathcal{B} = \{w_1, \dots, w_s\}$ of $\mathfrak{V}$ such that the set

$$\mathcal{M}(\mathcal{B}) = \Big\{ :T^{j_1}(w_{i_1})\cdots T^{j_r}(w_{i_r}): \mid r \in \mathbb{Z}_+,\ i_1 \leq i_2 \cdots \leq i_r, i_j < i_{j+1} \text{ if } p(w_{i_j}) = 1 \Big\}$$

is linearly independent. Obviously, if $\mathcal{M}(\mathcal{B})$ is linearly independent for a basis $\mathcal{B}$ of $\mathfrak{V}$, then $\mathcal{M}(\mathcal{B}')$ is linearly independent for any basis $\mathcal{B}'$ of $\mathfrak{V}$. We say that $\mathfrak{V}$ strongly and freely generates W if it is free and strongly generates W.

Theorem 4.1. *Assume that $W^k(\mathfrak{g}, x, f)$ is completely reducible as a $\mathfrak{g}^\natural$–module. Then there is a $\mathbb{C}D \oplus \mathfrak{g}^\natural$–module isomorphism $\Psi : S(\widehat{\mathfrak{g}^f}) \to W^k(\mathfrak{g}, x, f)$ such that $\Psi(\mathfrak{g}^f)$ strongly and freely generates $W^k(\mathfrak{g}, x, f)$. Moreover one can choose Ψ so that $\Psi(f) = L$.*

Proof. The existence of a $\mathfrak{g}^\natural$–module isomorphism $\Psi : S(\widehat{\mathfrak{g}^f}) \to W^k(\mathfrak{g}, x, f)$ is highlighted in [19, Proposition 3.1] as a corollary of the proof of Theorem 4.1 in [34]. More precisely, Theorem 4.1 in [34] provides a vector space isomorphism between $S(\widehat{\mathfrak{g}^f})$ and $W^k(\mathfrak{g}, x, f)$, mapping a polynomial over $\widehat{\mathfrak{g}^f}$ into the normally ordered product of the corresponding generators of $W^k(\mathfrak{g}, x, f)$. In [19, Proposition 3.1] it is observed that it is possible to modify Ψ into a $\mathfrak{g}^\natural$-module map. We observe that the fact that one can choose Ψ so that $\Psi(\mathfrak{g}^f)$ strongly and freely generates $W^k(\mathfrak{g}, x, f)$ is also implicit in the construction given in Theorem 4.1 in [34]. Here we give a proof of this latter statement.

Recall from [34] that there is a $\mathbb{C}D$–module monomorphism $\Psi' : \mathfrak{g}^f \to W^k(\mathfrak{g}, x, f)$ such that $W^k(\mathfrak{g}, x, f)$ is strongly and freely generated by $\Psi'(\mathfrak{g}^f)$. We prove by induction on j that there is a $\mathbb{C}D$–module monomorphism $\Psi_j : \mathfrak{g}^f \to W^k(\mathfrak{g}, x, f)$ such that Ψ_j restricted to $\bigoplus_{i=0}^{j-1}\mathfrak{g}^f_{-i} \otimes t^{-i-1}$ is a $\mathbb{C}D \oplus \mathfrak{g}^\natural$–module homomorphism and such that $W^k(\mathfrak{g}, x, f)$ is strongly and freely generated by $\Psi_j(\mathfrak{g}^f)$.

For $j = 1$ it is enough to choose $\Psi_1 = \Psi'$. We now assume by induction the existence of Ψ_j and construct $\Psi_{j+\frac{1}{2}}$.

For each r choose a basis $\{a_i^r\}$ of $\mathfrak{g}_r^f$ and choose an order for

$$\widehat{\mathcal{B}} = \bigcup_{s,i,r} \{a_i^r \otimes t^{-s+r-1}\}.$$

so that we can write $\widehat{\mathcal{B}} = \{b_1, b_2, \dots, b_i, \dots\}$. Clearly $\widehat{\mathcal{B}}$ is an ordered basis for $\widehat{\mathfrak{g}^f}$. Extend Ψ_j to $\widehat{\mathfrak{g}^f}$ by setting $\Psi_j(a_i^r \otimes t^{-s+r-1}) = T^s(\Psi_j(a_i^r))$ and also to $S(\widehat{\mathfrak{g}^f})$ by mapping the ordered monomials $b_{i_1} \cdots b_{i_r}$ to $:\Psi_j(b_{i_1}) \cdots \Psi_j(b_{i_r}):$. By the induction hypothesis this extension of Ψ_j is an isomorphism of $\mathbb{C}D$–modules. Write

$$S(\widehat{\mathfrak{g}^f}) = \bigoplus_{n \in \mathbb{Z}_+} S(\widehat{\mathfrak{g}^f})(n)$$

for the grading defined by the action of D. Set

$$U = \Psi_j \left(S \left(\bigoplus_{i=0}^{j-1} (\mathfrak{g}^f_{-i} \otimes \mathbb{C}[t^{-1}]t^{-1-i}) \right) (j + \tfrac{1}{2}) \right).$$

Since Ψ_j is $\mathfrak{g}^\natural$–equivariant when restricted to $S\left(\bigoplus_{i=0}^{j-1}(\mathfrak{g}^f_{-i} \otimes \mathbb{C}[t^{-1}]t^{-1-i})\right)$, it follows that U is $\mathfrak{g}^\natural$–stable. Let V be a $\mathfrak{g}^\natural$–stable complement of U in $W^k(\mathfrak{g}, x, f)(j + \frac{1}{2})$.

Since, by [19, Proposition 3.1], $S(\widehat{\mathfrak{g}^f})(j + \frac{1}{2})$ and $W^k(\mathfrak{g}, x, f)(j + \frac{1}{2})$ are isomorphic as $\mathfrak{g}^\natural$–modules, we have

$$V \cong W^k(\mathfrak{g}, x, f)(j + \tfrac{1}{2})/U \cong S(\widehat{\mathfrak{g}^f})(j + \tfrac{1}{2}) \Big/ S \left(\bigoplus_{i=0}^{j-1} (\mathfrak{g}^f_{-i} \otimes \mathbb{C}[t^{-1}]t^{-1-i}) \right) (j + \tfrac{1}{2}) \cong \mathfrak{g}^f_{-j+\frac{1}{2}}.$$

Let $\phi : \mathfrak{g}^f_{-j+\frac{1}{2}} \to V$ be a $\mathfrak{g}^\natural$–module isomorphism. We define $(\Psi_{j+\frac{1}{2}})_{|\mathfrak{g}^f_{-j+\frac{1}{2}}} = \phi$ and $(\Psi_{j+\frac{1}{2}})_{|\mathfrak{g}^f_i} = \Psi_j$ for $i \neq -j + \frac{1}{2}$.

It remains to check that $\Psi_{j+\frac{1}{2}}(\mathfrak{g}^f)$ strongly and freely generates $W^k(\mathfrak{g}, x, f)$.

Since $\Psi_j(\mathfrak{g}^f)$ is free, the projection $p_V : \Psi_j(\mathfrak{g}^f_{-j+\frac{1}{2}}) \to V$ with respect to the decomposition $W^k(\mathfrak{g}, x, f)(j + \frac{1}{2}) = V \oplus U$ is injective, hence, since $\Psi_j(\mathfrak{g}^f_{-j+\frac{1}{2}})$ and V have the same dimension, also bijective. In particular the set $\{\widetilde{a}_i^{-j+\frac{1}{2}}\}$ such that

$$\{\widetilde{a}_i^{-j+\frac{1}{2}} \otimes t^{-j-\frac{1}{2}}\} = \phi^{-1}(\{p_V(\Psi_j(a_i^{j-\frac{1}{2}} \otimes t^{-j-\frac{1}{2}}))\})$$

is a basis of $\mathfrak{g}^f_{-j+\frac{1}{2}}$. Let $\widetilde{\mathcal{B}} = \{\widetilde{b}_1, \widetilde{b}_2, \dots,\}$ be the basis of $S(\widehat{\mathfrak{g}^f})$ constructed as $\widehat{\mathcal{B}}$ using the basis $\{\widetilde{a}_i^{-j+\frac{1}{2}}\} \cup (\bigcup_{r \neq -j+\frac{1}{2}} \{a_i^r\})$ of $\mathfrak{g}^f$.

We need to show that the monomials

$$:\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_1}) \cdots \Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_r}):$$

form a basis of $W^k(\mathfrak{g}, x, f)$. To this end, following [23], we grade $\widehat{\mathfrak{g}^f}$ by setting $\deg(a_i^r \otimes t^s) = r + 1$ and extend the grading to $S(\widehat{\mathfrak{g}^f})$. Set

$$S(\widehat{\mathfrak{g}^f})_p = \{a \in S(\widehat{\mathfrak{g}^f}) \mid \deg(a) \leq p\}.$$

Set also $W^k(\mathfrak{g},x,f)_p=\Psi_j(S(\widehat{\mathfrak{g}^f})_p)$, thus obtaining a filtration of $W^k(\mathfrak{g},x,f)$, which, according to [23] and Example 3.12 of [25], has the property that

$$:W^k(\mathfrak{g},x,f)_pW^k(\mathfrak{g},x,f)_q:\subset W^k(\mathfrak{g},x,f)_{p+q},$$

and

$$[(W^k(\mathfrak{g},x,f)_p)_\lambda(W^k(\mathfrak{g},x,f)_q)]\subset W^k(\mathfrak{g},x,f)_{p+q-\frac{1}{2}}.$$

It follows from (1.39) and (1.40) of [25], that the normal order $:\cdot:$ induces a (super)–commutative and (super)–associative product on $gr(W^k(\mathfrak{g},x,f))$. We define a grading of the resulting algebra $gr(W^k(\mathfrak{g},x,f))$ by giving degree 1 to $\Psi_j(a_i^{-j+\frac{1}{2}}\otimes t^s)$ and degree 0 to $\Psi_j(\oplus_{i\neq -j+\frac{1}{2}}(\mathfrak{g}^f_i\otimes\mathbb{C}[t^{-1}]t^{-1+i}))$. Write $gr(W^k(\mathfrak{g},x,f))^n$ for the space of degree n in $gr(W^k(\mathfrak{g},x,f))$. By construction we have that

$$\Psi_{j+\frac{1}{2}}(\widetilde{b}_i)=\Psi_j(b_i)+\sum c^i_{i_1,\dots,i_s}:\Psi_j(b_{i_1})\cdots\Psi_j(b_{i_s}):$$

with $\deg(b_{i_t})<j+\frac{1}{2}$, so, if $\deg(:\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_1})\cdots\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_r}):)=m$ and $:\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_1})\cdots\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_r}):+(W^k(\mathfrak{g},x,f))_{m-\frac{1}{2}}\in gr(W^k(\mathfrak{g},x,f))^n$ then

$$\begin{aligned}&:\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_1})\cdots\Psi_{j+\frac{1}{2}}(\widetilde{b}_{i_r}):+(W^k(\mathfrak{g},x,f))_{m-\frac{1}{2}}\\&\qquad=:\Psi_j(b_{i_1})\cdots\Psi_j(b_{i_r}):+(W^k(\mathfrak{g},x,f))_{m-\frac{1}{2}}\mod gr(W^k(\mathfrak{g},x,f))^{n-1}.\end{aligned}$$

This concludes the induction step.

It remains only to check that we can set up Ψ so that $\Psi(f)=L$. This is achieved by choosing in the construction of Ψ_2 a $\mathfrak{g}^\natural$–stable complement V' of $\mathbb{C}L\oplus\Psi_{\frac{3}{2}}(S(\mathfrak{g}^f_0\otimes\mathbb{C}[t^{-1}]t^{-1})(2))$, a $\mathfrak{g}^\natural$–stable complement V'' of $\mathbb{C}f$ in $\mathfrak{g}^f_1$, and defining ϕ by setting $\phi(f)=L$ and $\phi_{|V''}=\phi'$ with ϕ' any $\mathfrak{g}^\natural$–isomorphism between V'' and V'. □

5. Hook type W-algebras

We shall consider the special case $\mathfrak{g}=sl(m+n)$, and the nilpotent element $f=f_{m,n}$ determined by the partition $(m,1^n)$. In this case $\mathfrak{g}^\natural\cong gl(n)$. The labels of the weighted Dynkin diagram corresponding to a dominant semisimple element in the orbit of (x,f) are

$$m\text{ even: }\quad(\underbrace{1,\dots,1}_{\frac{m}{2}-1},\tfrac{1}{2},\underbrace{0,\dots,0}_{n-1},\tfrac{1}{2},\underbrace{1,\dots,1}_{\frac{m}{2}-1}),\tag{5.1}$$

$$m\text{ odd: }\quad(\underbrace{1,\dots,1}_{\frac{m-1}{2}},\underbrace{0,\dots,0}_{n},\underbrace{1,\dots,1}_{\frac{m-1}{2}}).\tag{5.2}$$

Remark 5.1. Note that when m is even the grading is not even; on the other hand, using Kac-Elashivili theory of good gradings [24], one easily checks that the semisimple element

$$x_{good}=diag\left(\tfrac{(m+2n)(m-1)}{2(m+n)},\tfrac{(m+2n)(m-1)}{2(m+n)}-1,\dots,-\tfrac{m(m-1)}{2(m+n)},\underbrace{-\tfrac{m(m-1)}{2(m+n)},\dots,-\tfrac{m(m-1)}{2(m+n)}}_{n}\right)$$

gives rise to an even good grading for $f_{m,n}$.

TABLE 1. m odd

	$\dim \mathfrak{g}_{-j}$	$\dim \mathfrak{g}^f_{-j}$
$\frac{m+1}{2} \le j \le m-1$	$m-j$	1
$j = \frac{m-1}{2}$	$2n + \frac{m+1}{2}$	$2n+1$
$1 \le j \le \frac{m-3}{2}$	$2n+m-j$	1
$j=0$	$n^2+2n+m-1$	n^2

TABLE 2. m even

	$\dim \mathfrak{g}_{-j}$	$\dim \mathfrak{g}^f_{-j}$
$1 \le j \le m-1$	$m-j$	1
$j = \frac{m-1}{2}$	$2n$	$2n$
$j = i+\frac{1}{2},\, 0 \le i < \frac{m}{2}-1$	$2n$	0
0	n^2+m-1	n^2

From (5.1), (5.2) it is easy to compute the dimension of $\mathfrak{g}_j$, hence also those of $\mathfrak{g}^f_j$, since

$$\dim \mathfrak{g}^f_j = \dim \mathfrak{g}_j - \dim \mathfrak{g}_{j-1} \text{ for } j \le 0. \tag{5.3}$$

These data are summed up in Tables 1,2.

Theorem 5.1. *Assume that k is not critical, i.e. $k \neq -n-m$. One can choose strong generators for the vertex algebra $W^k(\mathfrak{g}, x, f_{m,n})$ as follows:*

1. *$J^{\{a\}}$, $a \in \mathfrak{g}^\natural \cong gl(n)$; these generators are primary for L of conformal weight 1;*
2. *the Virasoro field L;*
3. *fields W_i, $i = 3, \dots, m$, of conformal weight i;*
4. *fields $G^{\pm}_i, i = 1, \dots, n$ of conformal weight $\frac{m+1}{2}$.*

The fields $G^{\pm}_i, i = 1, \dots, n$ are primary for both L and $V(\mathfrak{g}^\natural)$. As $sl(n)$–modules,

$$span_{\mathbb{C}}\{G^+_i, i = 1, \dots, n\} \cong \mathbb{C}^n, \quad span_{\mathbb{C}}\{G^-_i, i = 1, \dots, n\} \cong (\mathbb{C}^n)^*.$$

Finally, $gl(n)$ acts trivially on W_i.

Proof. We prove statements (1)–(4) by applying Theorem 4.1, hence we need to describe explicitly the structure of $\mathfrak{g}^f$ as a $\mathfrak{g}^\natural$–module. We choose

$$f = \left(\begin{array}{c|c} J_m & 0 \\ \hline 0 & 0 \end{array}\right), \quad x = \left(\begin{array}{ccccc|c} \frac{m-1}{2} & 0 & \cdots & 0 & & \\ 0 & \frac{m-3}{2} & \cdots & 0 & & \\ \vdots & \vdots & \ddots & \vdots & & 0 \\ 0 & 0 & \cdots & -\frac{m-1}{2} & & \\ \hline & & 0 & & & 0 \end{array}\right). \tag{5.4}$$

Here J_m the $m \times m$ is the Jordan block

$$J_m = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In our case

$$\mathfrak{g}^\natural=\left\{\left(\begin{array}{c|c}-\frac{tr(A)}{m}I_m & 0\\ \hline 0 & A\end{array}\right)\mid A\in gl(n)\right\}\cong gl(n), \tag{5.5}$$

and the 1-dimensional center is spanned by

$$\varpi=\left(\begin{array}{c|c}-\frac{n}{m}I_m & 0\\ \hline 0 & I_n\end{array}\right). \tag{5.6}$$

If m is odd

$$\mathfrak{g}^f_j=\mathbb{C}\left(\begin{array}{c|c}J_m^j & 0\\ \hline 0 & 0\end{array}\right),\ 1\le j\le m-1, j\neq\frac{m-1}{2},$$

and

$$\mathfrak{g}^f_{(m-1)/2}=\mathbb{C}\left(\begin{array}{c|c}J_m^{(m-1)/2} & 0\\ \hline 0 & 0\end{array}\right)\oplus\left\{\left(\begin{array}{c|c|c}0&0&0\\ \hline 0&0&v\\ \hline {}^tw&0&0\end{array}\right)\mid v,w\in\mathbb{C}^n\right\}.$$

If m is even

$$\mathfrak{g}^f_j=\mathbb{C}\left(\begin{array}{c|c}J_m^j & 0\\ \hline 0 & 0\end{array}\right),\ 1\le j\le m-1,$$

and

$$\mathfrak{g}^f_{(m-1)/2}=\left\{\left(\begin{array}{c|c|c}0&0&0\\ \hline 0&0&v\\ \hline {}^tw&0&0\end{array}\right)\mid v,w\in\mathbb{C}^n\right\}.$$

Note that

$$\left[\varpi,\left(\begin{array}{c|c|c}0&0&0\\ \hline 0&0&v\\ \hline {}^tw&0&0\end{array}\right)\right]=\left(\begin{array}{c|c|c}0&0&0\\ \hline 0&0&-\frac{n+m}{m}v\\ \hline \frac{n+m}{m}\,{}^tw&0&0\end{array}\right). \tag{5.7}$$

Now we check that $\mathfrak{g}^\natural$ acts semisimply on $W^k(\mathfrak{g},x,f_{m,n})$. Since $W^k(\mathfrak{g},x,f_{m,n})(s)$ is finite dimensional for every s, $[\mathfrak{g}^\natural,\mathfrak{g}^\natural]$ acts semisimply on $W^k(\mathfrak{g},x,f)$. To analyze the action of the center $\mathbb{C}\varpi$ on $V^k(\mathfrak{g})\otimes F(A_{ch})\otimes F(A_{ne})$ we use the explicit formula given in [34, (2.4), (2.7), Theorem 2.1 (a)]

$$J^{\{\varpi\}}=\varpi+\sum_{\beta\in\S_+}:\varphi_{[\varpi,u_\beta]}\varphi^\beta:+\tfrac{1}{2}\sum_{\alpha\in\Sigma_{1/2}}:\Phi^\alpha\Phi_{[u_\alpha,\varpi]}:.$$

If $v\in\mathfrak{g}_{1/2}$, then

$$\begin{aligned}\tfrac{1}{2}\sum_{\alpha\in\Sigma_{1/2}}[:\Phi^\alpha\Phi_{[u_\alpha,\varpi]}:_\lambda v]&=-\tfrac{1}{2}\sum_{\alpha\in\Sigma_{1/2}}[v_{-\lambda-T}:\Phi^\alpha\Phi_{[u_\alpha,\varpi]}:]\\&=-\tfrac{1}{2}\sum_{\alpha\in\Sigma_{1/2}}\langle v,u^\alpha\rangle_{ne}\Phi_{[u_\alpha,\varpi]}-\tfrac{1}{2}\sum_{\alpha\in\Sigma_{1/2}}\Phi^\alpha\langle v,[u_\alpha,\varpi]\rangle_{ne}=\Phi_{[\varpi,v]},\end{aligned}$$

hence the monomials $:T^{j_1}\Phi_{\alpha_1}\cdots T^{j_s}\Phi_{\alpha_s}:$ generate $F(A_{ne})$ and are eigenvectors for

$$\tfrac{1}{2}\sum_{\alpha\in\Sigma_{1/2}}:\Phi^\alpha\Phi_{[u_\alpha,\varpi]}:_0.$$

Similarly, if $v\in\mathfrak{g}_+$,

$$\sum_{\beta\in\S_+}[:\varphi_{[\varpi,u_\beta]}\varphi^\beta:_\lambda v]=\varphi_{[\varpi,v]},$$

hence $\sum_{\beta\in\S_+}:\varphi_{[\varpi,u_\beta]}\varphi^\beta :_0$ acts semisimply on $F(A_{ch})$. Since ϖ_0 acts semisimply on $V^k(\mathfrak{g})$, we conclude that $J_0^{\{\varpi\}}$ acts semisimply on $V^k(\mathfrak{g})\otimes F(A_{ch})\otimes F(A_{ne})$, hence also on $W^k(\mathfrak{g},x,f_{m,n})$. We can therefore apply Theorem 4.1: we set

$$W_i=\Psi(\left(\begin{array}{c|c} J_m^{i-1} & 0\\ \hline 0 & 0\end{array}\right)),\ 3\le i\le m,$$

and

$$G_i^+=\Psi(E_{m+i,1}),\ 1\le i\le n,\ G_i^-=\Psi(E_{m,m+i}),\ 1\le i\le n.$$

It remains to check that $G_i^\pm$ are primary: by the first part of the proof, $J_0^{\{\varpi\}}$ acts trivially on $W^k(\mathfrak{g},x,f)(n)$ for $n<\frac{m+1}{2}$. Since $J_0^{\{\varpi\}}G_i^\pm=\pm G_i^\pm$ for all i and $[J_0^{\{\varpi\}},L(n)]=0$, we conclude that, if $n>0$,

$$\pm L(n)G_i^\pm=J_0^{\{\varpi\}}L(n)G_i^\pm=0.$$

The same argument shows that, if $a\in\mathfrak{g}^\natural$, then $a_nG_i^\pm=0$ if $n>0$. □

Remark 5.2. In [19, Theorem 9.5] the authors prove the following stronger result: hook type W-algebras are unique, under certain non-degeneracy conditions.

Let $(\cdot|\cdot)$ denote the trace form on $sl(n+m)$. The Killing form of $\mathfrak{g}$ is $K_{\mathfrak{g}}(\cdot\,,\,\cdot)=2h^\vee(\cdot|\cdot)$ with $h^\vee=n+m$. Recall from [34] that for any W-algebra the central charge is given by the formula

$$c(\mathfrak{g},x,f,k)=\tfrac{k\dim\mathfrak{g}}{k+h^\vee}-12k(x|x)-\sum_{\alpha\in S_+}(12m_\alpha^2-12m_\alpha+2)-\tfrac12\dim\mathfrak{g}_{1/2}. \tag{5.8}$$

where $m_\alpha=j$ if $\alpha\in S_j$. In the hook case we write $c_{m,n}(k)$ for $c(\mathfrak{g},x,f_{m,n},k)$. To calculate explicitly $c_{m,n}(k)$ note that we have:

$$(x|x)=\frac12\binom{m+1}{3}.$$

Moreover, using the data in Table 1, one computes that the contribution of the last two terms in (5.8) is

$$\sum_{j=1}^{(m-3)/2}(12j^2-12j+2)(2n+m-j)+(3(m-1)^2-6(m-1)+2))(2n+\tfrac{m+1}{2})+\sum_{j=(m+1)/2}^{m-1}(12j^2-12j+2)(m-j)$$

if m is odd and

$$\sum_{j=1}^{m-1}(12j^2-12j+2)(m-j)+2n(3(m-1)^2-6(m-1)+2)+2n\sum_{j=0}^{m/2-2}(3(2j+1)^2-6(2j+1)+2)+n$$

if m is even. In any case, the following formula holds (see [38, Appendix C]).

(5.9)

$$c_{m,n}(k)=-\frac{k+k(1-m-n)(m+n)+(m+n)^2}{k+m+n}+m(k-m-n-m^2(1+k+m+n)+m(1+3(m+n))).$$

Let $\widetilde{W}_k(\mathfrak{g},x,f)$ be a quotient of $W^k(\mathfrak{g},x,f)$. We have the following homomorphisms:

$$V(\mathfrak{g}^\natural)\to W^k(\mathfrak{g},x,f)\to\widetilde{W}_k(\mathfrak{g},x,f). \tag{5.10}$$

Let us denote the image of the resulting homomorphism by $\mathcal{V}(\mathfrak{g}^\natural)$.

Theorem 5.2. *The embedding* $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{m,n})$ *is conformal if and only if*

$$k = k^{(i)}_{m,n}, \quad 1 \le i \le 4,$$

where

(1) $k^{(1)}_{m,n} = -n - m + \frac{n+m}{m+1} = -\frac{m}{m+1}h^\vee$ *and* $n > 1$,
(2) $k^{(2)}_{m,n} = -n - m + \frac{1+m+n}{m} = -\frac{(m-1)h^\vee - 1}{m}$ *and* $n \ge 1$,
(3) $k^{(3)}_{m,n} = \frac{-1+2m-m^2+2n-mn}{m-1} = -\frac{(m-2)h^\vee+1}{m-1} = -h^\vee + \frac{h^\vee-1}{m-1}$ *and* $n \ge 1$, $m > 1$,
(4) $k^{(4)}_{m,n} = -\frac{(m-1)h^\vee}{m} = -h^\vee + \frac{h^\vee}{m}$.

Remark 5.3. In the following cases we already proved that the above embeddings are conformal:

- $m = 1$, $k = k^{(1)}_{m,n} = -\frac{n+1}{2}$, since then $W_k(\mathfrak{g}, x, f_{m,n}) = V_k(sl(n+1))$ (cf. [2]).
- $m = 1$, $k = k^{(2)}_{m,n} = 1$, since then $W_k(\mathfrak{g}, x, f_{m,n}) = V_1(sl(n+1))$.
- $m = 2$, $k = k^{(1)}_{m,n} = -\frac{2h^\vee}{3}$ since then $W_k(\mathfrak{g}, f_{m,n}) = W_k(sl(n+2), f_\theta)$ and k is a conformal level (cf. [3], [5]).
- $m = 2$, $k = k^{(2)}_{m,n} = -\frac{h^\vee-1}{2}$ since then $W_k(\mathfrak{g}, f_{m,n}) = W_k(sl(n+2), f_\theta)$ and k is conformal level (cf. [3], [5]).
- $m = 2$, $k = k^{(3)}_{m,n} = -1$ since then $W_k(\mathfrak{g}, f_{m,n}) = W_k(sl(n+2), f_\theta)$ and k is a collapsing level (cf. [3], [5]).
- $m = 2$, $k = k^{(4)}_{m,n} = -\frac{h^\vee}{2}$ is collapsing level for $W_k(\mathfrak{g}, f) = W_k(sl(n+2), f_\theta)$.

Proof of Theorem 5.2. We apply Theorem 3.1. We choose the bilinear form on $\mathfrak{g}^\natural_0$ setting $\langle \varpi, \varpi \rangle_0 = (\frac{n}{m})^2 + 1$. We choose the conjugate linear involution ϕ to be the conjugation with respect to $sl(m+n, \mathbb{R})$, so that $\phi(x) = x, \phi(L) = L$. We choose

$$S = span\left(\{W_i \mid 3 \le i \le m-1\} \cup \{G^\pm_i \mid 1 \le i \le n\}\right). \tag{5.11}$$

If $(m+1)/2 \neq 2$, i.e. $m \neq 3$, then $S \cap W^k(\mathfrak{g}, x, f_{m,n})(2) = \emptyset$, hence the hypotheses of Theorem 3.1 are vacuously verified. If $m = 3$, we have to check that $(L - L^\alpha)(2)G^\pm_i = 0$: this readily follows from Theorem 5.1.

By Theorem 3.1, it is enough to equate the central charge (5.9) of $W_k(\mathfrak{g}, x, f_{m,n})$ and the central charge of $\mathcal{V}(\mathfrak{g}^\natural)$. By [34, Theorem 2.1 (c)], we have that $\mathcal{V}(\mathfrak{g}^\natural)$ is a quotient of $V^{k_0}(\mathbb{C}\varpi) \otimes V^{k_1}(sl(n))$, where

$$k_0 = k + \frac{(m-1)(m+n)}{m}, \quad k_1 = k + m - 1.$$

Moreover $h^\vee_1 = n, h^\vee_0 = 0$. We obtain the equations

$$c_{m,n}(k) = \frac{k_1(n^2-1)}{k_1+n} + (1 - \delta_{k_0,0}). \tag{5.12}$$

Solving for k, we get $k = k^{(i)}_{m,n}, 1 \le i \le 3$ if $k_0 \neq 0$. If instead $k_0 = 0$, then $k = k^{(4)}_{m,n}$ and one readily verifies that this value of k satisfies (5.12). □

6. Collapsing levels

Assume that k is a conformal level. Then $\bar{L} = L - L^{\mathfrak{g}^\natural}$ belongs to the maximal ideal of $W^k(\mathfrak{g}, x, f)$. We define

$$\overline{W}_k(\mathfrak{g}, x, f) = W^k(\mathfrak{g}, x, f)/W^k(\mathfrak{g}, x, f) \cdot (L - L^{\mathfrak{g}^\natural}).$$

Recall from the introduction that a level k is called *strongly collapsing* if $\mathcal{V}(\mathfrak{g}^\natural) = \overline{W}_k(\mathfrak{g}, x, f)$ (see (5.10)).

Remark 6.1. *Clearly, any strongly collapsing level is also a collapsing. In* [3, Example 4.4], *we showed that can exist collapsing levels which are not strongly collapsing. In particular, this holds for minimal affine W–algebras in the following cases:*

- $\mathfrak{g} = sl(3)$, $k = -1$;
- $\mathfrak{g} = sp(4)$, $k = -2$.

Theorem 6.2. *For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the following are strongly collapsing levels:*

(1) $k^{(3)}_{m,n}$, $n \neq m-1$,
(2) $k^{(4)}_{m,n}$.

Proof. We apply Theorem 3.2. By Theorem 5.1, we can choose S as in (5.11). Let $\mathbb{C}_t$ be the 1-dimensional representation of $\mathbb{C}\varpi$ with ϖ acting by $\varpi \cdot 1 = t$. Then, by (5.7), as a $(\mathbb{C}\varpi \oplus sl(n))$–module,

$$S = S_0 \oplus (\mathbb{C}_{\frac{n+m}{m}} \otimes \mathbb{C}^n) \oplus (\mathbb{C}_{-\frac{n+m}{m}} \otimes (\mathbb{C}^n)^*), \tag{6.1}$$

where S_0 is the isotypic component of the trivial representation of $\mathfrak{g}^\natural$. We now compute $C_1, \dots, C_m$ (cf. (3.8)), where $C_1, \dots, C_{m-2}$ correspond to the $m-2$ copies of the trivial representation occurring in S_0 and C_{m-1}, C_m to the rightmost factors in (6.1). We have

$$C_1 = \cdots = C_{m-2} = 0, \quad C_{m-1} = C_m = (1 - \delta_{k_0,0})\frac{m+n}{2mnk_0} + \frac{n^2-1}{2n(k_1+n)}.$$

A direct verification shows that if $k = k^{(3)}_{m,n}$ then

$$C_{m-1} = C_m = \frac{(m-1)\left(m+n^2+n-1\right)}{2n^2},$$

hence $C_{m-1} = C_m = \frac{m+1}{2}$ if and only if $n = m-1$.

If $k = k^{(4)}_{m,n}$ then $C_{m-1} = C_m = \frac{m\left(n^2-1\right)}{2n^2}$ hence, $C_{m-1} = C_m < \frac{m+1}{2}$. □

7. Non-collapsing conformal levels

Set

$$I = W^k(\mathfrak{g}, x, f).(L - L^{\mathfrak{g}^\natural}).$$

Theorem 7.1. *For the hook W-algebra $W(sl(n+m), x, f_{m,n})$ the conformal levels $k^{(1)}_{m,n}$ for $n > 1$ and $k^{(2)}_{m,n}$ are not strongly collapsing.*

Proof. One readily computes that, if $k = k^{(i)}_{m,n}$, $i = 1, 2$, then $C_{m-1} = C_m = \frac{m+1}{2}$. We need only to check whether

$$G^{\pm}_i \notin I.$$

Set $\bar{L} = W_2 = L - L^{\mathfrak{g}^\natural}$. Let $\tau \in \mathrm{span}_{\mathbb{C}}\{G^{\pm}_1, \dots, G^{\pm}_n\}$. Then for each $i \in \mathbb{Z}_{\geq 1}$:

$$\tau_i \bar{L} = -[\bar{L}(-2), \tau_i]\mathbf{1} = -\sum_{j=0}^{\infty} \binom{-1}{j} (\bar{L}(j-1)\tau)_{i-j-1}\mathbf{1} = 0. \tag{7.1}$$

The last equality follows since, as shown in Theorem 5.1, the fields $G^{\pm}_i, i = 1, \dots, n$ are primary for both L and $V(\mathfrak{g}^\natural)$, hence $\bar{L}(j)\tau = 0$ for $j \geq 0$. For $2 \leq p \leq m$, we set $W_p(s) =$

$(W_p)_{s+p-1}$. Then for $s \geq 1$ we have $W_p(s)\bar{L} = 0$. This is clear for $s \geq 3$; if $W_p(2)\bar{L} \neq 0$ then I would not be a proper ideal. Finally, if $s = 1$ then $W_p(2)\bar{L} \in gl(n)_{-1}\mathbf{1}$. If this vector is non-zero, then it belongs to $I \cap gl(n)_{-1}\mathbf{1}$. This is not possible, since

$$\mathcal{V}(gl(n)) = V^{-\frac{m+n}{1+m}}(\mathbb{C}\varpi) \otimes V^{-\frac{1+mn}{1+m}}(sl(n)) \qquad \text{if } k = k^{(1)}_{m,n},$$
$$\mathcal{V}(gl(n)) = V^{1}(\mathbb{C}\varpi) \otimes V^{\frac{1+n(1-m)}{m}}(sl(n)) \qquad \text{if } k = k^{(2)}_{m,n},$$

and the levels appearing in the right hand sides are always non-zero (recall that we are assuming $n > 1$).

Moreover $W_p(0)\bar{L} = a\bar{L} + u$ for certain $a \in \mathbb{C}$ and $u \in V(\mathfrak{g}^\natural) \cap I$. Assume that u is non-zero, then u is a subsingular vector in $V(\mathfrak{g}^\natural)$ of zero $\mathfrak{g}^\natural$-weight. Such subsingular vector cannot exist since $V(\mathfrak{g}^\natural)$ is not a vertex algebra at the critical level. Thus $u = 0$ and we conclude that $W_p(0)\bar{L} = a\bar{L}$.

Therefore $\bar{L}$ is a singular vector in $W^k(\mathfrak{g}, x, f)$. In particular, I is a quotient of a Verma module, hence it is linearly spanned by monomials

$$U_s = y_1(-q_1)\cdots y_t(-q_t)x_1(-n_1-1)\cdots x_r(-n_r-1)(g^1)_{-m_1}\cdots(g^s)_{-m_s}\bar{L},$$

where $y_i \in \{W_p, p = 2, \dots, m\}$, $x^i \in \mathfrak{g}^\natural$, $g^j \in \{G^{\pm}_1, \dots, G^{\pm}_n\}$, $q_i, n_i \in \mathbb{Z}_{\geq 0}$, and $m_i \in \mathbb{Z}$, $m_i \geq -\frac{m-1}{2}$, $m_1 \geq m_2 \geq \ldots \geq m_s$. Formula (7.1) implies $m_s \geq 0$, hence all m_i are non-negative integers. Denote the conformal weight of U_s by $\mathrm{wt}(U_s)$. Then we have

$$\mathrm{wt}(U_s) \geq 2 + s\frac{m+1}{2} + (m_1 + \cdots + m_s) - s \geq 2 + s\frac{m-1}{2}.$$

This implies that if $s \geq 1$, then $\mathrm{wt}(U_s) \geq \frac{m+3}{2}$. If $G^{\pm}_i \in I$, then it can be written as a a linear combination of elements U_{s_i} with $s_i \geq 1$ of conformal weight $\frac{m+1}{2}$. This is not possible. The claim follows. □

We set

$$A^{\pm} = \mathcal{V}(\mathfrak{g}^\natural).\mathrm{span}_{\mathbb{C}}\{G^{\pm}_i, i = 1, \dots, n\}, \tag{7.2}$$

and

$$\overline{W}_k(\mathfrak{g}, x, f) = \bigoplus_{\ell \in \mathbb{Z}} \overline{W}^{(\ell)}_k, \quad \overline{W}^{(\ell)}_k = \{v \in \overline{W}_k(\mathfrak{g}, x, f) \mid J(0)v = \ell v\}, \tag{7.3}$$

where

$$J = J^{\{-\frac{n+m}{m}\varpi\}} \tag{7.4}$$

(cf. (5.6)).

We believe (see Conjecture 1.6) that $k = k^{(1)}_{m,n}$ is never a collapsing level. We will prove this in the special case when k is admissible. The next result supports this conjecture, by showing that each component $\overline{W}^{(\ell)}_k$ is always non-zero. In Remark 8.3 we shall explain how this result is related with Conjecture 1.6.

Proposition 7.2. *Let $k = k^{(1)}_{m,n}$. For each $p \in \mathbb{Z}_{>0}$ we have*

$$:(G^{\pm}_i)^p: \neq 0 \quad \text{in} \quad \overline{W}_k(\mathfrak{g}, x, f_{m,n}).$$

In particular, each $\overline{W}^{(\ell)}_k$ is a non-zero $\overline{W}^{(0)}_k$–module.

Proof. By direct calculation we have

$$\begin{aligned} L^{\mathfrak{g}^\natural}(0) : (G_i^{\pm})^p : &= \left(-\frac{(m+1)p^2}{2n} + \frac{p(p+n)(m+1)}{2n} \right) : (G_i^{\pm})^p : \\ &= \frac{p(m+1)}{2} : (G_i^{\pm})^p : . \end{aligned}$$

This implies that for $s \geq 0$ we have

$$\bar{L}(s) : (G_i^{\pm})^p := 0.$$

Using the same arguments as in the proof of Theorem 7.1 we get

$$: (G_i^{\pm})^p :\notin I = W^k(\mathfrak{g}, x, f_{m,n})\bar{L}.$$

The claim follows. □

Remark 7.3. *The situation for $k = k_{m,n}^{(2)}$ is different. One can show that for $p \in \mathbb{Z}_{\geq 2}$ the following relation holds:*

$$: (G_i^{\pm})^p := 0 \quad \text{in } \overline{W}_{k_{m,n}^{(2)}}(\mathfrak{g}, x, f_{m,n}).$$

This result won't be used elsewhere in the paper.

8. Decompositions of conformal embeddings

Consider the embedding

$$\mathcal{V}(\mathfrak{g}^\natural) = V(\mathfrak{g}^\natural).\mathbf{1} \hookrightarrow W_k(\mathfrak{g}, x, f).$$

In the terminology of [2] and [5], the conformal embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f)$ is called finite (resp. infinite) if each $W_k(\mathfrak{g}, x, f)^{(i)}$ from (7.3) is finite (resp. infinite) sum of $\mathcal{V}(\mathfrak{g}^\natural)$–modules.

8.1. **Finite decompositions.** Let $M(k)$ be the Heisenberg vertex algebras $V^k(\mathbb{C}J)$ (cf. (7.4)). Recall that $V^{\beta_k}(\mathfrak{g}^\natural) \subset W^k(\mathfrak{g}, x, f)$ can be written as

$$V^{\beta_k}(\mathfrak{g}^\natural) = V^{k_1}(sl(n)) \otimes M(k_0).$$

Let $V_{sl(n)}(\mu)$ denote the finite-dimensional irreducible $sl(n)$-module of highest weight μ. Let $L_k^{sl(n)}(\lambda)$ be the irreducible $V^k(sl(n))$-module with top component $V_{sl(n)}(\lambda)$. Identify $\mathbb{C}$ with $(\mathbb{C}J)^*$ by $\lambda \mapsto \lambda(J)$, and let $M(k, r)$ be the $M(k)$-module of highest weight r. We use the tensor product decomposition

$$V_{sl(n)}(\omega_1) \otimes V_{sl(n)}(\omega_{n-1}) = V_{sl(n)}(\omega_1 + \omega_{n-1}) \oplus V_{sl(n)}(0). \tag{8.1}$$

Recall that $J(0)$ acts semi-simply on $W_k(\mathfrak{g}, x, f)$ and that

$$W_k(\mathfrak{g}, x, f_{m,n}) = \bigoplus_{i \in \mathbb{Z}} W_k(\mathfrak{g}, x, f_{m,n})^{(i)}, \quad W_k(\mathfrak{g}, x, f_{m,n})^{(i)} = \{v \in W_k(\mathfrak{g}, x, f_{m,n}),\ J(0)v = iv\}.$$

Lemma 8.1. *Assume that*

(1) *$\mathcal{V}(\mathfrak{g}^\natural)$ is conformally embedded in $W_k(\mathfrak{g}, x, f_{m,n})$ and the level k is not collapsing;*
(2) *$W_k(\mathfrak{g}, x, f)^{(0)}$ does not contain $\mathcal{V}(\mathfrak{g}^\natural)$–primitive vectors of weight $\mu = \omega_1 + \omega_{n-1}$.*

Then $W_k(\mathfrak{g}, x, f_{m,n})^{(0)}$ is a simple vertex algebra isomorphic to $V(\mathfrak{g}^\natural)$ and each $W_k(\mathfrak{g}, x, f_{m,n})^{(i)}$ is a non-trivial simple $V(\mathfrak{g}^\natural)$–module.

Proof. The proof is the same as the proof of [5, Theorem 6.2]. Since $\mathcal{V}(\mathfrak{g}^\natural)$ is conformally embedded into $W_k(\mathfrak{g}, x, f_{m,n})$ we have that $W_k(\mathfrak{g}, x, f_{m,n})$ is generated by $\mathcal{V}(\mathfrak{g}^\natural) + A^+ + A^-$ (see (7.2) for notation). The assumption of the lemma and the tensor product decomposition (8.1) give that

$$A^+ \cdot A^- \subset \mathcal{V}(\mathfrak{g}^\natural). \tag{8.2}$$

Since

$$W_k(\mathfrak{g}, x, f_{m,n})^{(0)} \subset A_1 \cdots A_r$$

with $A_i = A^\pm$ or $A = \mathcal{V}^{k^\natural}$ and

$$\sharp\{i \mid A_i = A^+\} = \sharp\{i \mid A_i = A^-\},$$

by (8.2) we conclude that $W_k(\mathfrak{g}, x, f_{m,n})^{(0)} = \mathcal{V}(\mathfrak{g}^\natural)$. Since $W_k(\mathfrak{g}, x, f_{m,n})^{(0)}$ is a simple vertex algebra, we have that $\mathcal{V}(\mathfrak{g}^\natural) = V(\mathfrak{g}^\natural)$ is simple, and each $W_k(\mathfrak{g}, x, f_{m,n})^{(i)}$ is a simple $V(\mathfrak{g}^\natural)$–module. □

Using the tensor product decomposition (8.1), we get that any $V(\mathfrak{g}^\natural)$–primitive vector of $sl(n)$–weight $\mu = \omega_1 + \omega_{n-1}$ has conformal weight

- $h^{(1)}_\mu = m + 1 + \frac{m+1}{n-1}$ if $k = k^{(1)}_{m,n}$,
- $h^{(2)}_\mu = m - \frac{m}{n+1}$ if $k = k^{(2)}_{m,n}$.

Theorem 8.2. *Let $k = k^{(i)}_{m,n}$ for $i = 1, 2$ and assume that k is non-collapsing. Assume also that $\frac{m+1}{n-1} \notin \mathbb{Z}$ if $i = 1$ and $\frac{m}{n+1} \notin \mathbb{Z}$ if $i = 2$. Then*

$$W_k = W_k(\mathfrak{g}, x, f_{m,n}) = \bigoplus_{i \in \mathbb{Z}} W^{(i)}_k, \tag{8.3}$$

and each $W^{(i)}_k = \{v \in W_k \mid J(0)v = iv\}$ is an irreducible $V(\mathfrak{g}^\natural)$–module. The summands in the r.h.s. of (8.3) have the form:

- $W^{(i)}_k = L^{sl(n)}_{k_1}(i\omega_1) \otimes M(k_0, i)$ *if* $i \geq 0$,
- $W^{(i)}_k = L^{sl(n)}_{k_1}(-i\omega_{n-1}) \otimes M(k_0, i)$ *if* $i < 0$.

In particular, $\mathcal{V}(\mathfrak{g}^\natural) \cong W_k(\mathfrak{g}, x, f_{m,n})^{(0)} = V(sl(n)) \otimes V^{k_0}(\mathbb{C}J)$ is a simple vertex algebra which is conformally embedded in $W_k(\mathfrak{g}, x, f_{m,n})$.

Proof. Assume that k is non-collapsing. Then by the simplicity of W_k we conclude that $A^+ \cdot A^- \neq \{0\}$ in W_k. This implies that $W^{(0)}_k$ has a primitive vector v_ν of $\mathfrak{g}^\natural$–weight $\nu \in \{0, \mu\}$. Since $h^{(i)}_\mu \notin \mathbb{Z}$, we conclude that $W^{(0)}_k$ can not contain primitive vector v_μ. Now the claim follows from Lemma 8.1.

□

Remark 8.3. *In Theorem 9.6 we shall prove that $k^{(i)}_{m,n}$, $i = 1, 2$, are non-collapsing whenever they are admissible. But we expect that the level $k = k^{(1)}_{m,n}$ is also non-collapsing for non-admissible level. Since we cannot prove this, let us contemplate what could happen if k is collapsing and $\frac{m+1}{n-1} \notin \mathbb{Z}$. Then:*

- $:(G^\pm_i)^p: \neq 0$ *in $\overline{W}_k(\mathfrak{g}, x, f_{m,n})$, for each $p \in \mathbb{Z}_{\geq 0}$ (cf. Proposition 7.2);*
- $A^+ \cdot A^- = \{0\}$.

This gives that $\Omega_{\pm p} =: (G^\pm_i)^p :$ is a non-zero singular vector in $\overline{W}_k(\mathfrak{g}, x, f_{m,n})$. In particular,

- *$\overline{W}_k(\mathfrak{g}, x, f_{m,n})$ has infinite-length as $W^k(\mathfrak{g}, x, f_{m,n})$–module.*

Since $W^k(\mathfrak{g}, x, f_{m,n}) = H_f(V^k(\mathfrak{g}))$, we conclude that:

- $V^k(\mathfrak{g})$ *is a module in* KL^k *of infinite length.*

So assuming that $k = k^{(1)}_{m,n}$ *is collapsing would imply the existence of highest weight modules in* KL^k *of infinite length. It is expected that this is not possible for non-critical levels. On the other hand, if* $k = k^{(2)}_{m,n}$ *is non-admissible, we will see in Proposition 8.6 that* k *can be collapsing.*

Remark 8.4. *Note that the levels* $k^\natural = -n + \frac{n-1}{m+1} = -n + \frac{n-1}{q}$, *for* $q = m+1$, *appeared in* [17, Remark 8.9]. *The authors conjectured that these levels are collapsing if* $n = pq$ *and that* $W_{k^\natural}(sl(n), f_1) = V_{-1}(sl(p))$ *for certain nilpotent element* f_1. *In Section 10 we prove that their conjecture is true except possibly for* $p = 2$.

8.2. **Infinite decompositions: the case $n = 2$.** We shall see that if the conformal level does not satisfy the condition of Theorem 8.2, then we can expect that the decomposition is infinite.

Let us consider here the case $m = p - 1$, $n = 2$. Then $k = k^{(1)}_{p-1,2}$ and $k_1 = -2 + \frac{1}{p}$. Clearly, the conditions of Theorem 8.2 are not satisfied. But according to Theorem 5.2 we have conformal embedding

$$V(\mathfrak{g}^\natural) = V_{-2+\frac{1}{p}}(sl(2)) \otimes M(k_0) \hookrightarrow W_k(\mathfrak{g}, x, f_{p-1,2}).$$

In the analysis of this conformal embedding, an important role is played by the vertex algebras $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$ introduced in [1]. Their properties are further studied in [9]. The vertex algebra $\mathcal{V}^{(p)}$ is realized as an extension of $V_{-2+\frac{1}{p}}(sl(2))$. It has $\mathbb{Z}$–gradation so that

$$\mathcal{V}^{(p)} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{V}^{(p)}_\ell,$$
$$\mathcal{V}^{(p)}_\ell = \bigoplus_{s=0}^{\infty} L^{sl(2)}_{-2+1/p}(((|\ell| + 2s)\omega_1).$$

$\mathcal{V}^{(p)}$ is generated by the generators e, f, h of $V_{-2+1/p}(sl(2))$ and four primary fields $\tau^\pm, \overline{\tau}^\pm$ of conformal weight $\frac{3p}{4}$. In the case $p = 2$, $\mathcal{V}^{(p)}$ is isomorphic to the small $N = 4$ superconformal vertex algebra with central charge $c = -9$.

For each $q \in \frac{1}{2}\mathbb{Z}$, let F_q denote the (generalized) lattice vertex algebra $V_{\mathbb{Z}\varphi} = M_\varphi(1) \otimes \mathbb{C}[\mathbb{Z}\varphi]$ such that $\langle \varphi, \varphi \rangle = q$. Here $M_\varphi(1)$ is the Heisenberg vertex algebra of level q generated by the field $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi(n) z^{-n-1}$ and $\mathbb{C}[\mathbb{Z}\varphi]$ is the group algebra of the lattice $\mathbb{Z}\varphi$. Let $M_\varphi(1, r)$ denotes the irreducible $M_\varphi(1)$–module on which $\varphi(0)$ acts as $r \cdot \mathrm{Id}$. Then

$$F_q = \bigoplus_{\ell \in \mathbb{Z}} F^{(\ell)}_q, \quad F^{(\ell)}_q \cong M_\varphi(1, q\ell).$$

Finally, recall that the vertex algebra $\mathcal{R}^{(p)}$ is realized as the following subalgebra of $\mathcal{V}^{(p)} \otimes F_{-\frac{p}{2}}$:

$$\mathcal{R}^{(p)} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{V}^{(p)}_\ell \otimes F^{(\ell)}_{-\frac{p}{2}} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{V}^{(p)}_\ell \otimes M_\varphi(1, -\ell \tfrac{p}{2}).$$

The vertex algebra $\mathcal{R}^{(p)}$ admits the decomposition

$$\mathcal{R}^{(p)} = \bigoplus \mathcal{R}^{(p)}_{(\ell)},$$

such that each $\mathcal{R}^{(p)}_{(\ell)} = \mathcal{V}^{(p)}_\ell \otimes M_\varphi(1, -\ell\frac{p}{2})$ is a direct sum of infinitely many non-isomorphic $V_{-2+\frac{1}{p}}(sl(2)) \otimes M_\varphi(1)$–modules. The next theorem is proved in [9, Theorem 10]:

Theorem 8.5. [9] *Let* $k = k^{(1)}_{p-1,2} = -\frac{p^2-1}{p}$*. Then* $W_k(sl(p+1), f_{p-1,2}) = \mathcal{R}^{(p)}$*. In particular, level* k *is not collapsing.*

As a consequence, the decomposition of conformal embedding is infinite in this case. Some important special cases were considered in our previous papers:

- the case $p = 2$ was treated in [1], when it was proved that $W_k(\mathfrak{g}, x, f) = V_{-3/2}(sl(3)) = \mathcal{R}^{(2)}$. This result is used in [2] to show that the decomposition of the conformal embedding $gl(2) \hookrightarrow sl(3)$ is infinite for $k = -3/2$;
- the case $p = 3$ was treated in [5], where it was proved that $W_k(\mathfrak{g}, x, f) = W_k(\mathfrak{g}, x, f_\theta) = \mathcal{R}^{(3)}$.

8.3. **More collapsing levels.** Let $k = k^{(2)}_{m,n}$ for $m = 3p$ and $n = 2$. We see that again the conditions of Theorem 8.2 are not satisfied. Then we have:

- $k_1 = -2 + \frac{1}{p}$.
- The embedding
$$V(\mathfrak{g}^\natural) = V_{-2+\frac{1}{p}}(sl(2)) \otimes M(k_0) \hookrightarrow \overline{W}_k(\mathfrak{g}, x, f_{3p,2})$$
is conformal.

Proposition 8.6. *Level* $k = k^{(2)}_{3p,2} = -3p - 1 + \frac{1}{p}$ *is collapsing and*
$$W_k(sl(3p+2), x, f_{3p,2}) = V_{-2+\frac{1}{p}}(sl(2)) \otimes M(k_0).$$

Proof. Set $W_k = W_k(sl(3p+2), x, f_{3p,2})$. Assume that k is not collapsing. Then W_k is strongly generated by generators of $V_{-2+\frac{1}{p}}(sl(2)) \otimes M(k_0)$ and four primary fields $G^{\pm}_i$, $i = 1, 2$, of conformal weight $(3p+1)/2$.

Assume next that $W^{(0)}_k$ contains a primitive vector u of $\mathfrak{g}^\natural$–weight $\mu = \omega_1 + \omega_{n-1}$. Then the conformal weight of u is $h^{(2)}_\mu = 2p$. Since u is strongly generated by affine generators and $G^{\pm}_i$, we have that
$$u = \sum_{i,j=1}^{2} a_{i,j} (G^+_i)_{-n_0} (G^-_j) + u',$$
where $n_0 > 0$, $a_{i,j} \in \mathbb{C}$ (not all zero) and $u' \in V_{-2+\frac{1}{p}}(sl(2)) \otimes M(k_0)$. This implies that the conformal weight $h^{(2)}_\mu \geq 3p + 1$. This is a contradiction.

Therefore $W^{(0)}_k$ cannot contain a primitive vector of $\mathfrak{g}^\natural$–weight μ. Lemma 8.1 implies that $W^{(0)}_k = V_{k_1}(sl(2)) \otimes M(k_0)$, where $k_1 = -2 + 1/p$, and
$$W^{(i)}_k = L^{sl(2)}_{k_1}(|i|\omega_1) \otimes M(k_0, i).$$
Therefore
$$W_k \cong \bigoplus_{i \in \mathbb{Z}} L^{sl(2)}_{k_1}(|i|\omega_1) \otimes M(k_0, i). \tag{8.4}$$

Using arguments from [21, Section 5] we can conclude that, since $KL^{sl(2)}_{k_1}$ is a braided tensor category, then the $V_{k_1}(sl(2))$–modules and the $M(k_0)$–modules appearing in the decomposition (8.4) must have same fusion rules. This implies that all modules $L^{sl(2)}_{k_1}(i\omega_1)$ are simple currents in $KL^{sl(2)}_{k_1}$. This is a contradiction since the fusion ring of $KL^{sl(2)}_{k_1}$ is equivalent to Grothendieck ring of finite dimensional $sl(2)$–modules (cf. [9]). Therefore the modules $L_{k_1}^{sl(2)}(i\omega_1)$ are not simple currents for $i > 0$. We have therefore proved that k is a collapsing level. □

9. The cases when the conformal level is admissible

Recall the following definition.

Definition 9.1. A level k for $\mathfrak{g} = sl(n+m)$ is said to be admissible if $k+h^\vee = \frac{p'}{p}$, $p, p' \in \mathbb{Z}_{\geq 1}$, $(p, p') = 1$ and $p' \geq h^\vee = n+m$.

In this section we study the cases when $k = k_{m,n}^{(1)}$ or $k = k_{m,n}^{(2)}$ are admissible rational numbers.

We can choose the root system so that the $sl(2)$–triple corresponding to highest root θ is

$$e_\theta = \left(\begin{array}{c|c} (J_m^{tr})^{m-1} & 0 \\ \hline 0 & 0 \end{array}\right), \quad h_\theta = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & -1 & \\ \hline & & 0 & & 0 \end{array}\right), \quad f_\theta = \left(\begin{array}{c|c} J_m^{m-1} & 0 \\ \hline 0 & 0 \end{array}\right). \tag{9.1}$$

The following results from [32] and [16] hold for any simple Lie algebra $\mathfrak{g}$.

Proposition 9.1. [32] *Assume that k is an admissible level. Then*

$$V_k(\mathfrak{g}) = V^k(\mathfrak{g})/J^k(\mathfrak{g}),$$

where the maximal ideal $J^k(\mathfrak{g}) = V^k(\mathfrak{g}).\Omega_k$ is generated by the singular vector Ω_k, which is the unique (up to a scalar factor) singular vector in $V^k(\mathfrak{g})$ of $\mathfrak{g}$–highest weight $\mu^{(k)} = (p'+1-h^\vee)\theta$, and conformal weight $d_{(k)} = p(p'+1-h^\vee)$.

Denote by H_f the quantum Hamiltonian reduction functor. In [35] Kac and Wakimoto state the following conjecture:

$$H_f(V_k(\mathfrak{g})) \text{ is either zero or isomorphic to } W_k(\mathfrak{g}, x, f). \tag{9.2}$$

Recall from [15] that if $k = -h^\vee + p'/p$ is admissible, then the associated variety of the simple affine vertex algebra $V_k(\mathfrak{g})$ is the closure of a nilpotent orbit $\mathbb{O}_k$, depending just on p. Moreover, if we set $N_p = \{x \in \mathfrak{g} \mid \mathrm{ad}(x)^{2p} = 0\}$, then

$$\overline{\mathbb{O}}_k = N_p.$$

The following result has been proved in [16, Theorem 7.8].

Theorem 9.2. *(Arakawa-Van Ekeren) Assume that k is an admissible level, $f \in \overline{\mathbb{O}}_k$ and f admits an even good grading. Then* (9.2) *holds.*

The following result is a consequence of results from [14] (see also [16, Remark 7.10]).

Proposition 9.3. *Assume that* (9.2) *holds for k admissible. Then $H_f(J^k(\mathfrak{g}))$ is a submodule of $W^k(\mathfrak{g}, x, f) = H_f(V^k(\mathfrak{g}))$ which is generated by a singular vector Ω_k^W of conformal weight*

$$d_k^W = (p'+1-h^\vee)(p - (h_\theta|x)).$$

Moreover, $H_f(J^k(\mathfrak{g}))$ is either a maximal ideal in $W^k(\mathfrak{g}, x, f)$ or $H_f(J^k(\mathfrak{g})) = W^k(\mathfrak{g}, x, f)$. If $f \in \overline{\mathbb{O}}_k$ and f admits an even good grading, then the above statement holds.

Proof. It is proved in [14] that H_f is an exact functor acting on KL_k. This implies that that

$$H_f(V_k(\mathfrak{g})) = H_f(V^k(\mathfrak{g}))/H_f(J^k(\mathfrak{g})) = W^k(\mathfrak{g}, x, f)/H_f(J^k(\mathfrak{g})).$$

By Conjecture 9.2 $H_f(J^k(\mathfrak{g}))$ is either a maximal ideal or $H_f(J^k(\mathfrak{g})) = W^k(\mathfrak{g}, x, f)$. In any case, we have that $H_f(J^k(\mathfrak{g}))$ is a non-zero submodule in $W^k(\mathfrak{g}, x, f)$. Then the lowest conformal weight of $H_f(J^k(\mathfrak{g}))$ is given by the formula given in [31, Remark 2.3], which can be expressed as $d_k^W = (p' + 1 - h^\vee)(p - (h_\theta|x))$. The final claim follows from Theorem 9.2. □

By (5.4), we have $(h_\theta|x) = m - 1$. Since $h^\vee = n + m$, we get

Lemma 9.4. *We have:*

1. $k = k^{(1)}_{m,n}$ *is admissible if and only if* $(n - 1, m + 1) = 1$.
2. $k = k^{(2)}_{m,n}$ *is admissible if and only if* $(n + 1, m) = 1$.

In both cases, if k is admissible we have $d_k^W = 2$.

Lemma 9.5. *We have:*

$$f = f_{m,n} \in N_m \subset N_{m+1}.$$

Proof. Recall that the height of a nilpotent element f is $\max\{n \mid ad(f)^n \neq 0\}$. If $\{e, h, f\}$ is a $sl(2)$–triple containing f, and one chooses the set of positive roots so that h is dominant, then the height of f is $\theta(h)$, with θ the highest root. The dominant element corresponding to $f_{m,n}$ is $h = 2x$ and the Dynkin labels of x are given in (5.1) and (5.2). It follows that the height of $f_{m,n}$ is $2(m - 1)$. □

Th next theorem completely describes the structure of $W_k(\mathfrak{g}, x, f)$ in the admissible case:

Theorem 9.6. *Assume that* $k = k^{(1)}_{m,n}$ *or* $k = k^{(2)}_{m,n}$ *and that k is admissible. We have:*

1. $W_k(\mathfrak{g}, x, f_{m,n}) = \overline{W}_k(\mathfrak{g}, x, f_{m,n})$.
2. *The level k is not collapsing.*
3. $W_k(\mathfrak{g}, x, f_{m,n})$ *admits the decomposition given in Theorem 8.2 provided that* $k \neq k^{(1)}_{p-1,2}$.
4. *If* $k = k^{(1)}_{p-1,2}$, *the decomposition is given in Theorem 8.5.*

Proof. Let $k = k^{(1)}_{m,n}$ be admissible. Then, by Lemma 9.4, $\overline{\mathbb{O}}_k = N_{m+1}$, hence, by Lemma 9.5, $f_{m,n} \in \overline{\mathbb{O}}_k$. In Remark 5.1 we exhibited a good grading for $f_{m.n}$. Therefore, by Proposition 9.3 we have that $H_f(V_k(\mathfrak{g}))$ is zero or $W_k(\mathfrak{g}, x, f_{m,n})$. By Lemma 9.4 the former possibility is excluded, since $d_k^W = 2$. In particular, it follows that the maximal ideal in $W^k(\mathfrak{g}, x, f_{m,n})$ is generated by a singular vector Ω_k^W of conformal weight 2. Since the highest weight of Ω_k with respect to $\mathfrak{g}$ is $(p + 1 - h^\vee)\theta$, using (9.1) one shows that the singular vector Ω_k^W must have $\mathfrak{g}^\natural$–weight 0. Since k_1 is not critical, the only possible candidate for a singular vector of conformal weight 2 of $\mathfrak{g}^\natural$–weight 0 is $\bar{L}$. Therefore $\Omega_k^W = \bar{L}$. This implies that the maximal ideal is $W^k(\mathfrak{g}, x, f_{m,n})$ is $W^k(\mathfrak{g}, x, f_{m,n}).\bar{L}$, and thus $W_k(\mathfrak{g}, x, f_{m,n}) = \overline{W}_k(\mathfrak{g}, x, f_{m,n})$, proving (1). Therefore, by Theorem 7.1, k is non-collapsing. Since $n > 2$ and $(n - 1, m + 1) = 1$, we conclude that $\frac{m+1}{n-1} \notin \mathbb{Z}$. Therefore the assumptions of Theorem 8.2 are satisfied and we get assertion (3). The case $k = k^{(2)}_{m,n}$ is dealt with along the same lines.

□

	$\dim \mathfrak{g}_{-j}$	$\dim \mathfrak{g}^f_{-j}$
$1 \le j \le q-1$	$m^2(q-j)$	m^2
$j=0$	m^2q-1	m^2-1

Table 3.

10. The rectangular case and the Arakawa-Van Ekeren-Moreau question

Let $f = f_{[m,q]} = diag(\underbrace{J_q, \dots, J_q}_{m \text{ times}})$ be the nilpotent element of $sl(qm)$ with Jordan block decomposition corresponding to the partition (q^m). Let $x = diag(\underbrace{\frac{q-1}{2}, \frac{q-3}{2}, \dots, \frac{1-1}{2}}_{m \text{ times}})$ be the corresponding Dynkin element, whose weighted Dynkin diagram is

$$(\underbrace{\underbrace{0,\dots,0}_{m-1},1,\underbrace{0,\dots,0}_{m-1},1,\dots\dots,\underbrace{0,\dots,0}_{m-1}}_{q-1},1,\underbrace{0,\dots,0}_{m-1}).$$

Using formula (5.3) to obtain Table 3, we get

$$\mathfrak{g}^f = \begin{pmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \dots & \vdots \\ A_{m1} & \dots & A_{mm} \end{pmatrix} \tag{10.1}$$

where

$$A_{ij} = \alpha_0^{ij} Id + \alpha_1^{ij} J_q + \dots + \alpha_{q-1}^{ij} J_q^{q-1}.$$

Indeed, the matrices in the r.h.s. of (10.1) are contained in $\mathfrak{g}^f$; comparing dimensions we have the equality. Moreover

$$\mathfrak{g}^\natural = \begin{pmatrix} \alpha_0^{11} Id & \dots & \alpha_0^{1m} Id \\ \vdots & \dots & \vdots \\ \alpha_0^{m1} Id & \dots & -\sum_{j=1}^{m-1} \alpha_0^{jj} Id \end{pmatrix} \cong sl(m), \tag{10.2}$$

$$\mathfrak{g}^f_{-r} = \begin{pmatrix} \alpha_r^{11} J_q^r & \dots & \alpha_r^{1m} J_q^r \\ \vdots & \dots & \vdots \\ \alpha_r^{m1} J_q^r & \dots & \alpha_r^{mm} J_q^r \end{pmatrix} \cong gl(m), \quad 1 \le r \le q-1. \tag{10.3}$$

Applying Theorem 4.1 we obtain

Theorem 10.1. *One can choose strong generators for the vertex algebra $W^k(\mathfrak{g}, x, f_{[m,q]})$ as follows:*

1. *$J^{\{a\}}$, $a \in \mathfrak{g}^\natural \cong sl(m)$; these generators are primary for L of conformal weight 1;*
2. *the Virasoro field L;*
3. *fields W_i, $3 \le i \le q$, of conformal weight i;*
4. *fields $G_i^{j,s}$, $2 \le i \le q$, $1 \le s, j \le m^2-1$, of conformal weight i.*

The action of $\mathfrak{g}^\natural$ on W_i (resp. $G_i^{j,s}$) is trivial (resp. adjoint).

According to [34], the central charge for L in $W^k(sl(mq), f_{[m,q]}, x)$ is

$$C(k) = \frac{k\left(m\left(q-q^3\right)(k+mq)+m^2q^2-1\right)}{k+mq} - m^2 q\left(q^3-2q^2+1\right).$$

The hypothesis of Theorem 3.1 are satisfied since we can choose $S \cap W^k(\mathfrak{g}, f_{[m,q]}, x)(2)$ to be, with the notation of Theorem 4.1, $\Psi(sl(m))$. Thus the embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, x, f_{[m,q]})$ is conformal if and only if k is a solution of the equation $C(k) = c_{\mathfrak{g}^\natural}$. One readily computes that

$$c_{\mathfrak{g}^\natural} = \frac{\left(m^2-1\right) q(k+mq-m)}{q(k+mq-m)+m},$$

hence the conformal levels are

$$k^{[1]}_{m,q} = -\frac{mq^2}{q+1},\ k^{[2]}_{m,q} = \frac{-mq^2+mq-1}{q},\ k^{[3]}_{m,q} = \frac{-mq^2+mq+1}{q}. \tag{10.4}$$

Theorem 10.2. *Levels $k^{[i]}_{m,q}$, $i = 1, 2, 3$ are collapsing for all $q \geq 2$ and $m \geq 2$, except possibly level $k^{[2]}_{2,q}$. More precisely:*

$$W_{k^{[1]}_{m,q}}(sl(mq), x, f_{[m,q]}) = V_{-\frac{mq}{q+1}}(sl(m)), \tag{10.5}$$

$$W_{k^{[2]}_{m,q}}(sl(mq), x, f_{[m,q]}) = V_{-1}(sl(m)),\ m \geq 3, \tag{10.6}$$

$$W_{k^{[3]}_{m,q}}(sl(mq), x, f_{[m,q]}) = V_1(sl(m)). \tag{10.7}$$

Proof. We use Theorem 3.2. Recall that $k_1 = qk + mq^2 - mq$. Also remark that, by (10.3), $\mathfrak{g}^f$ decomposes as a sum of trivial and adjoint representations. Since $\mathfrak{g}^\natural$ is simple, we have (cf. (3.8))

$$C_i = \begin{cases} 0 & \text{if } S_i \text{ is trivial,} \\ \frac{2m}{2(k_1+m)} & \text{if } S_i \text{ is adjoint.} \end{cases}$$

Assume $C_i \neq 0$. If $k = k^{[1]}_{m,q}$, then $C_i = q + 1$, which cannot equate Δ_i (see Theorem 3.2), since the maximal conformal weight of a generator is q. If $k = k^{[2]}_{m,q}$, then $C_i = \frac{m}{m-1}$, which might be integral iff $m = 2$. If $k = k^{[3]}_{m,q}$, then $C_i = \frac{m}{m+1}$, which is never integral. □

Remark 10.3. *Note that*

- $k^{[1]}_{m,q}$ *is admissible if and only if* $(m, q + 1) = 1$.
- $k^{[2]}_{m,q}$ *is never admissible. The result (10.6) gives a positive answer to a question by T. Arakawa, J. van Ekeren and A. Moreau* [17, Remark 8.9] *in the case* $m \geq 3$.
- $k^{[3]}_{m,q}$ *is always admissible. The result (10.7) is proved by different methods in* [17].

Theorem 10.2 cannot be applied for $m = 2$. In the case $m = q = 2$, $f_{[m,q]}$ coincides with the short nilpotent element f_{sh}. It is proved in [12], using explicit OPE formulas, that $W_{-\frac{5}{2}}(sl(4), f_{sh})$ is isomorphic to an orbifold of the rank two Weyl vertex algebra. So $k^{[2]}_{2,2} = -5/2$ is not collapsing.

Conjecture 10.4. *Level $k = k^{[2]}_{2,q} = \frac{-2q^2+2q-1}{q}$ is not collapsing for $q \geq 2$ and*

$$W_k(sl(2q), f_{[2,q]}, x) = W_{-\frac{5}{2}}(sl(4), f_{sh}).$$

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