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Energy-Balance PBC of nonlinear dynamics under sampling and delays ⋆

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Abstract: The paper provides a new class of passivity-based controllers (PBCs) for stabilizing sampled-data input-delayed dynamics at a desired equilibrium via energy-balancing (EB) and reduction. Given a nonlinear dynamics under piecewise constant and retarded input, we first exhibit a new dynamics (the reduced dynamics) that is free of delays and equivalent to the original one. Accordingly, we design the digital controller assigning a suitable energetic behaviour to the reduced delay-free model with a stable target equilibrium. Then, it is proved that such a controller solves the EB-PBC problem on the original retarded system. The results are illustrated over a simple mechanical system.

Keywords: Sampled-Data Control, Sampled-Data and Delays, Delay Compensation.

1. INTRODUCTION

Passivity-Based Control (PBC) yields a natural and powerful framework for controlling dynamic systems by exploiting their physical properties. Basically, the approach stands in assigning a suitable energetic behaviour with a minimum at the target equilibrium one whats to stabilize at (Ortega et al., 2021, 2001; van der Schaft, 2020; Willems, 2007).

Despite the practical interest in several applications (e.g., teleoperation, robotics or network systems Chopra et al. (2022); Paredes et al. (2021); Zhou and Lin (2014)), only few works are devoted to analyzing the (possibly simultaneous) effects of both sampling and time-delays on the energetic properties of the plant and, consequently, the design (Chopra, 2008; Fridman and Shaked, 2002; Li et al., 2002; Mahmoud and Ismail, 2004; Niculescu and Lozano, 2001; Thomas et al., 2021). Among these, (Mattioni et al., 2018a,b) deal with stabilization at the origin through the notion of reduction (Mazenc and Malisoff, 2014; Mazenc et al., 2014; Mazenc and Normand-Cyrot, 2013). The underlying idea consists of constructing a new dynamics (the reduced dynamics) that is free of delays and equivalent, in terms of stabilizability, to the original delayed one. Such a dynamics preserves the drift of the retarded one but exhibits a transformed control vector field that is explicitly parameterized by the delay. Consequently, when the original dynamics free of delays is passive, the corresponding reduced dynamics is passive with respect to a new output mapping but with the same energy function. In addition, it was shown in Mattioni et al. (2020b) that those properties allow to deduce passivity of the original retarded system with respect to new output and energy functionals. Accordingly, passivity-based arguments for stabilization can be fruitfully applied.

Exploiting the approach proposed in Mattioni et al. (2018b), the contribution of this paper concerns sampled-data stabilization of nonlinear dynamics under input delays at a desired equilibrium via Energy-Balance (EB) PBC. In doing so, we assume that an EB-PBC stabilizer exists for the nominal continuous-time dynamics free of delays and sampling. With no further hypothesis, we design the digital control assigning to the reduced model the same target energetic behavior as in the nominal case, with a minimum at the target equilibrium. As a consequence, such a feedback asymptotically stabilizes the retarded original system as well with the expected energetic behavior despite the effect of both sampling and delays. The proposed sampled-data solution takes the form of a series expansion in powers of both the sampling period and delay around the continuous-time delay-free solution. As a further contribution, when the sampling period tends to zero, the proposed design provides a new and original PBC feedback for continuous-time retarded dynamics so partially extending the results in Mattioni et al. (2020a) on PBC for the LTI case.

The paper is organized as follows. The problem is settled in Section 2 and the main results in Section 3 for continuous-time and sampled-data dynamics under input delays. A simple example is developed in Section 4 to illustrate the result with comparisons with respect to prediction-based control laws. Section 5 concludes the paper.

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Notations. $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0. For any vector $z \in \mathbb{R}^n$, $\|z\|$ and $z^\top$ define respectively the norm and transpose of z. Given a full rank matrix $B \in \mathbb{R}^{n\times m}$ with $n > m$, $B^\dagger = (B^\top B)^{-1}B^\top$ denotes the pseudo inverse. I and I_d denote respectively the identity matrix (or function, depending on the context) and identity operator of suitable dimensions. Given a twice continuously differentiable function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\nabla S(\cdot)$ represents its gradient (column) vector. For $v, w \in \mathbb{R}^n$, the discrete gradient of $S(\cdot)$ is defined as

$$\bar{\nabla} V|_v^w = \int_0^1 \nabla V(v + s(w-v))\mathrm{d}s$$

satisfying $V(w) - V(v) = (w-v)^\top \bar{\nabla} V|_v^w$ with $\bar{\nabla} V|_v^v = \nabla V(v)$. Given a vector field f on $\mathbb{R}^n$, $\mathrm{L}_f = \sum_{i=1}^n \frac{\partial}{\partial x_i}$, denotes the Lie derivative operator, and recursively, $L_f^i = \mathrm{L}_f \circ \mathrm{L}_f^{i-1}$ with $\mathrm{L}_f^0 = I_d$. Given two vector fields f and g on $\mathbb{R}^n$, $ad_f g = (\mathrm{L}_f \mathrm{L}_g - \mathrm{L}_g \mathrm{L}_f)I_d$ denotes their Lie-bracket and, recursively, $ad_f^i g = ad_f \circ ad_f^{i-1} g$ and $ad_f^0 g = g$. For $\delta > 0$, $e^{\delta \mathrm{L}_f} = I_d + \sum_{i>0} \frac{\delta^i}{i!}\mathrm{L}_f^i$ denotes the Lie exponential operator. Given any smooth function $H(\cdot) : \mathbb{R}^n \to \mathbb{R}$, one gets by the Exchange Theorem $H(e^{\delta \mathrm{L}_f} x) = e^{\delta \mathrm{L}_f} H(x) = H(x) + \sum_{i>0} \frac{\delta^i}{i!} \mathrm{L}_f^i H(x)$. A function $R(x, \delta) : \mathcal{B} \times \mathbb{R} \to \mathbb{R}^n$ is said in $\mathcal{O}(\delta^p)$, with $p \geq 1$, if it can be written as $R(x,\delta) = \delta^{p-1}\tilde{R}(x,\delta)$ for all $x \in \mathcal{B}$ and there exist a function $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ s.t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x,\delta)| \leq \theta(\delta)$.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 The reduction and the reduced dynamics

We consider time-delay systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t-\tau) \tag{1}$$

with $u \in \mathbb{R}^m$ and a costant $\tau = N\delta$ for some $N \in \mathbb{N}$ and $\delta > 0$. We assume inputs piecewise constant over intervals of length δ, the sampling period that is $u(t) = u(k\delta)$ for $t \in [k\delta, (k+1)\delta[$. In this setting, the dynamics (1) is described at all sampling instants $t = k\delta$ with $k \in \mathbb{N}$ by the so-called sampled-data equivalent model

$$x^+(u_{-N}) = F^\delta(x, u_{-N}) \tag{2}$$

with

$$F^\delta(x,u) = e^{\delta(\mathrm{L}_f + \mathrm{L}_{gu})}x = x + \sum_{i>0} \frac{\delta^i}{i!}\big(\mathrm{L}_f + \mathrm{L}_{gu}\big)x$$

and, for simplicity of notations, $x = x(k\delta)$, $u_{-N} = u((k-N)\delta)$, $u = u_{-0} = u(k\delta)$. Accordingly, $x^+(u_{-N}) = x((k+1)\delta))$ defines the one step-ahead state evolution starting from $t = k\delta$. Also, for the sake of brevity, we rewrite

$$F^\delta(x,u) = F_0^\delta(x) + g^\delta(x,u)u$$

with $F_0^\delta(x) := F^\delta(x,0)$, $g^\delta(x,u)u := F^\delta(x,u) - F_0^\delta(x)$. When $\tau = 0$, we refer to (1) and (2) as the corresponding delay free continuous-time and sampled-data dynamics that are respectively given by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{3}$$

$$x^+(u) = F^\delta(x,u). \tag{4}$$

In the following, we address the problem of stabilizing (1) at a desired equilibrium $x_\star \in \mathbb{R}^n$ under piecewise constant control; namely, we seek for a control $u = \gamma_\tau^\delta(x, u_{-N}, \dots, u_{-1})$ making $x_\star$ asymptotically stable for the equivalent sampled-data model (2).

To this end, we introduce the reduction variable associated to the sampled-data delayed dynamics as in (Mattioni et al., 2017)

$$\eta = F_0^{-\tau}(\cdot) \circ F^{\frac{\tau}{N}}(\cdot, u_{-1}) \circ \dots \circ F^{\frac{\tau}{N}}(x, u_{-N})$$

initialized at $\eta(0) = x(0)$, evolving as the (delay-free) reduced dynamics

$$\eta^+(u) = F_\tau^\delta(\eta, u) \tag{5}$$

with

$$\begin{aligned} F_\tau^\delta(\eta,u) =& F_0^{-\tau}(\cdot) \circ F^\delta(\cdot, u) \circ F_0^\tau(\eta) \\ =& F_0^\delta(\eta) + g_\tau^\delta(\eta,u)u. \end{aligned}$$

As $\tau \to 0$, one gets $\eta \to x$ and $F_\tau^\delta(\cdot,u) \to F^\delta(\cdot,u)$.

Remark 2.1. In the Linear Time Invariant (LTI) case

$$\dot{x}(t) = Ax(t) + Bu(t-\tau)$$

one gets the sampled-data equivalent model

$$x^+(u_{-N}) = A^\delta x + B^\delta u_{-N}$$

with $A^\delta = e^{A\delta}$ and $B^\delta = \int_0^\delta e^{As}B\mathrm{d}s$. Accordingly, the reduction variable and reduced dynamics are given by

$$\eta = x + \sum_{\ell=k-N}^{k-1} A^{(k-1-N-\ell)\delta}B^\delta u_{\ell-k}$$

$$\eta^+(u) = A^\delta \eta + A^{-\tau}B^\delta u.$$

Remark 2.2. The expressions above highlight that, contrarily to prediction, the original and the reduced dynamics share the free evolution; namely, as $u \equiv 0$, $\eta \equiv x$.

Remark 2.3. Initialization issues that are typical of prediction (see, e.g., Deng et al. (2022); Karafyllis and Krstic (2017)) are by construction overcome by reduction since one can easily set $\eta(0) = x(0)$. This is not the case for prediction. In that case, defining $z(t) = x(t+\tau)$, it is generally required to explicitly compute $z(0) = x(\tau) = e^{\tau \mathrm{L}_f}x|_{x(0)}$. In general, such a quantity cannot be computed in closed form and only approximations are available so making prediction not robust in general.

In this setting, it was proved in (Mattioni et al., 2017, Proposition 1) that all feedback laws $u = u_\tau^\delta(\eta)$ making $\eta_\star = x_\star$ asymptotically stable for the sampled-data reduced dynamics (5) guarantee asymptotic stabilization of $x_\star$ for the time-delay sampled-data one (2). This general result opens toward new perspectives involving the way the reduction-based feedback should be designed based on the reduced model (5) while exploiting, as much as possible, the delay-free properties of the plant; that is, the ones of the delay-free models (3)-(4).

2.2 Energy-Balancing and Problem Statement

We set the problem of stabilizing $x_\star \in \mathbb{R}^n$ for the dynamics (1) via sampled-data passivation over its reduced model (5). To this end, the following standing assumption is made on the continuous-time delay free model.

Assumption 1. (Delay-free stabilizability). The delay-free continuous-time system (3) is passive and zero-state detectable (ZSD) with storage function $H(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

and output $y = g^\top(x)\nabla H(x)$ with $x_\star \in \mathbb{R}^n$ being asymptotically stabilizable via Energy-Balancing Passivity Based Control (EB-PBC); namely, there exist $H_a(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $u(\cdot) : \mathbb{R}^n \to \mathbb{R}$ solution to

$$-\nabla^\top H(x)g(x)u(x) = \nabla^\top H_a(x)\Big(f(x) + g(x)u(x)\Big) \quad (6)$$

with $\nabla H_a(x_\star) = -\nabla H(x_\star)$ and $H(x_\star) = -H_a(x_\star)$.

In the following, we address the problem of stabilizing $x_\star \in \mathbb{R}^n$ for the time-delay sampled-data model (2) (equivalently, for (1)) via discrete-time EB-PBC over the reduced dynamics (2) by assigning the same storage function as in the continuous-time delay-free case (i.e., Assumption (1)). As proposed in Mattioni et al. (2021), we look for a sampled-data reduction-based feedback of the form

$$u = u^\delta_\tau(\eta) + v$$

making the closed-loop reduced system

$$\eta^+(u^\delta_\tau(\eta) + v) = F^\delta_\tau(\eta, u^\delta_\tau(\eta) + v) \quad (7a)$$

$$Y^\delta_\tau(\eta, v) = \frac{1}{\delta}\big(g^\delta_\tau(\eta, v)\big)^\top \bar{\nabla} H_d\big|^{\eta^+(u^\delta_\tau(\eta)+v)}_{\eta^+(u^\delta_\tau(\eta))} \quad (7b)$$

passive with dissipation inequality

$$H_d(\eta^+(u^\delta_\tau(\eta) + v)) - H_d(\eta) \leq \delta v^\top Y^\delta_\tau(\eta, v). \quad (8)$$

At this point, stabilization is achieved via the additional damping control $v = v^\delta_\tau(\eta)$ solution to the damping equality

$$v + \kappa Y^\delta_\tau(\eta, v) = 0, \quad \kappa > 0. \quad (9)$$

Remark 2.4. We set the target energy-function to be assigned equal to the one set in the continuous-time delay-free design. This choice is motivated by the fact that, in general, such a function catches the physical properties of the system at the desired equilibrium (e.g., the potential energy for mechanical systems) that one wants to preserve despite the effects of sampling and delays.

3. MAIN RESULT

As shown in (Mattioni et al., 2021, Proposition 3.1), for the purely (delay-free) discrete-time case, stabilization via EB-PBC over the reduced (delay-free) model (5) is ensured provided that the equality

$$\bar{\nabla}^\top H_d\big|^{\eta^+(u)}_{\eta^+} g^\delta_\tau(\eta, u) = H_a(\eta^+) - H_a(\eta). \quad (10)$$

admits a solution $u = u^\delta_\tau(\eta)$ when setting $\eta^+ = \eta^+(0) = F^\delta_0(\eta)$. Equation (10) is parameterized by both δ and τ, that is, the sampling period and the delay length. Moreover, the following limit equalities hold true:

(1) when both $\tau \to 0$ and $\delta \to 0$, the delay-free continuous-time equality (6) is recovered;
(2) when $\tau \to 0$, (10) recovers the delay-free sampled-data equality

$$\frac{1}{\delta}\bar{\nabla}^\top H_d\big|^{x^+(u)}_{x^+} g^\delta(x, u) = -\frac{1}{\delta}\big(H_a(x^+) - H_a(x)\big) \quad (11)$$

that admits a unique solution under Assumption 1 (Mattioni et al., 2021);
(3) when $\delta \to 0$, (10) gets the form

$$\nabla^\top H_d(\eta)g_\tau(\eta)u = -\nabla^\top H_a(\eta)f(\eta) \quad (12)$$

with [1]

$$g_\tau(\eta) = e^{\tau ad_f}g(\eta) = g(\eta) + \sum_{i>0}\frac{\tau^i}{i!}ad^i_f g(\eta) \quad (13)$$

and the continuous-time reduced model

$$\dot{\eta}(t) = f(\eta(t)) + g_\tau(\eta(t))u(t). \quad (14)$$

3.1 Reduction-based EB-PBC in continuous time

We first prove the existence of a continuous-time reduction-based control for stabilizing reduced dynamics (14) (and thus (1)) at the desired equilibrium; namely, we prove that a solution to (12) exists.

Theorem 3.1. (Continuoustime reduction-based EB-PBC). Let the retarded dynamics (1) verify Assumption 1 with reduced model (14) and $x_\star \in \mathbb{R}^n$ be an equilibrium to stabilize. Then, the control

$$u_\tau(\eta) = -\Big(\nabla^\top H_d(\eta)g_\tau(\eta)\Big)^\dagger \nabla^\top H_a(\eta)f(\eta) \quad (15)$$

is a solution to the continuous-time reduced EBE (12). In addition, the following holds:

(i) the reduced model (14) with output

$$y_\tau = g^\top_\tau(\eta)\nabla H_d(\eta)$$

and under the feedback $u = u_\tau(\eta) + v$ is passive with storage function $H_d(\eta) = H(\eta) + H_a(\eta)$;
(ii) the damping control

$$v = -\kappa g^\top_\tau(\eta)\nabla H_d(\eta) \quad (16)$$

makes $\eta_\star = x_\star$ asymptotically stable for the reduced dynamics and thus for the retarded dynamics (1).

Proof. To show that (12) admits a solution, one has to show that $\nabla^\top H_d(\eta)g_\tau(\eta)$ is full rank, at least in a neighborhood of $\eta_\star = x_\star$. To this end, exploiting (13) one rewrites

$$g^\top_\tau(\eta)\nabla H_d(\eta) = g^\top(\eta)H_d(\eta) + \sum_{i>0}\frac{\tau^i}{i!}\big(ad^i_f g(\eta)\big)^\top \nabla H_d(\eta).$$

From Assumption 1, $g^\top(\cdot)\nabla H_d(\cdot)$ is full rank as passivity requires relative degree one (Byrnes et al., 1991). Accordingly, by the series expansion above around the delay-free mapping, one concludes that (12) admits a solution (at least in a neighborhood of $\eta_\star$). (**i**) holds true computing

$$\dot{H}_d(\eta) = \nabla^\top H_d(\eta)\Big(f(\eta) + g_\tau(\eta)\big(u_\tau(\eta) + v\big)\Big) \leq v^\top y_\tau.$$

(**ii**) follows when substituting (16) into the inequality above so getting $\dot{H}_d(\eta) \leq 0$ with asymptotic stability of $\eta_\star = x_\star$ guaranteed by ZSD of the delay-free system (see (Mattioni et al., 2018b)). □

Remark 3.1. As natural in this case, the solution (15) to (12) is not unique. Different solutions might arise depending on the particular case of study. For instance, when applied to a fully actuated port-Hamiltonian (pH) system, (12) is solved by

$$u(\eta) = g^\dagger_\tau(\eta)\big(J(\eta) - R(\eta)\big)\nabla H_a(\eta).$$

with $J(\eta) + J^\top(\eta) = 0$ and $R(\eta) = R^\top(\eta) \succeq 0$.

[1] With notational abuse, we denote $e^{\tau ad_f}g := \big(e^{\tau ad_f}g_1 \; \dots \; e^{\tau ad_f}g_m\big)$ when fixing $g = \big(g_1 \; \dots \; g_m\big)$.

Remark 3.2. The control (15)-(16) is explicitly depending on $g_\tau(\eta)$ that is defined through its series expansion in powers of $\tau > 0$, the delay length. Even if the computation of such vector field might be tough, approximations can be easily computed by truncating the corresponding series expansion at an arbitrary finite order $p \geq 0$; namely, one gets

$$g_{[p],\tau}(\eta) = g(\eta) + \sum_{i=1}^{p} \frac{\tau}{i!} ad_f^i g(\eta)$$

and the corresponding feedback law

$$u_{[p],\tau}(\eta) = -\left(\nabla^\top H_d(\eta) g_\tau^{[p]}(\eta)\right)^\dagger \nabla^\top H_a(\eta) f(\eta) \quad (17a)$$

$$v_{[p],\tau} = -\kappa g_{[p],\tau}^\top(\eta) \nabla H_d(\eta). \quad (17b)$$

Remark 3.3. As $\tau \to 0$, the continuous-time reduction-based feedback (15)-(16) recovers the delay-free continuous-time counterpart set in Assumption 1 with damping component $u = -\kappa g^\top(x) \nabla H_d(x)$.

3.2 Reduction-based EB-PBC under sampling

Theorem 3.1 highlights that Assumption 1 also guarantees the existence of a continuous-time reduction-based EB-PBC, solution to (12), stabilizing the retarded dynamics (1) at the desired equilibrium. Starting from this, we show that Assumption 1 is also sufficient to further guarantee the existence of a sampled-data reduction-based EB-PBC solution to the sampled-data reduced equality (10). To this end, we rewrite (10) as

$$H_d(F_\tau^\delta(\eta, u)) - H_d(\eta) + H_a(F_0^\delta(\eta)) - H_a(\eta) = 0 \quad (18)$$

to highlight its series expansion in powers of $\delta > 0$. The main theorem can now be stated.

Theorem 3.2. (Sampled reduction-based EB-PBC). Let the retarded dynamics (1) verify Assumption 1 with $x_\star \in \mathbb{R}^n$ the equilibrium to stabilize and (5) the sampled-data reduced model. Then, there exists $\delta^\star > 0$ such that, for all $\delta \in [0, \delta^\star[$, the sampled-data reduced EBE (10) admits a unique solution $u_\tau^\delta : \mathbb{R}^n \to \mathbb{R}^m$ in the form of a series expansion in powers of δ around the continuous-time reduced EB-PBC (15); i.e., one gets

$$u_\tau^\delta(\eta) = u_\tau(\eta) + \sum_{i>0} \frac{\delta^i}{(i+1)!} u_\tau^i(\eta) \quad (19)$$

In addition, the following holds:

(i) the reduced system (7) is passive with storage function $H_d(\eta) = H(\eta) + H_a(\eta)$ and dissipation inequality (8);
(ii) the damping feedback

$$v_\tau^\delta(\eta) = v_\tau(\eta) + \sum_{i>0} \frac{\delta^i}{(i+1)!} v_\tau^i(\eta) \quad (20)$$

defined as the unique solution $v = v_\tau^\delta(\eta)$ to the damping equality (9) exists and makes $\eta_\star = x_\star$ asymptotically stable for the reduced dynamics (5) and thus for the retarded system (2).

Proof. Existence of a unique solution to the sampled reduced equality (10) (or, equivalently, (18)) follows rewriting (18) as a formal series equality $\delta \mathcal{Q}_\tau^\delta(\eta, u) = 0$ with

$$\mathcal{Q}_\tau^\delta(\eta, u) = \mathcal{Q}_\tau^0(\eta, u) + \sum_{i>0} \frac{\delta^i}{(i+1)!} \mathcal{Q}_\tau^i(\eta, u)$$

and $\mathcal{Q}_\tau^0(\eta, u) = \nabla^\top H_d(\eta) g_\tau(\eta) u + \nabla^\top H_a(\eta) f(\eta)$. Accordingly, as $\delta \to 0$, the equality above recovers the continuous-time reduced EBE (12) that is solved by (15). Thus, by the implicit function theorem, the result follows because

$$\lim_{\delta \to 0} \mathcal{Q}_\tau^\delta(\eta, u) = \nabla^\top H_d(\eta) g_\tau(\eta)$$

is full rank around $\eta_\star = x_\star$. (**i**) follows by computing the one-step increment of the storage function along the closed-loop system (7) that yields

$$\begin{aligned}\Delta H_d(\eta) :=& H_d(F_\tau^\delta(\eta, u_\tau^\delta(\eta) + v)) - H_d(\eta)\\ \leq& H_d(F_\tau^\delta(\eta, u_\tau^\delta(\eta) + v)) - H_d(F_\tau^\delta(\eta, u_\tau^\delta(\eta)))\\ =& \bar{\nabla}^\top H_d|_{\eta^+(u_\tau^\delta(\eta))}^{\eta^+(u_\tau^\delta(\eta)+v)} g_\tau^\delta(\eta, v) v\end{aligned}$$

and thus passivity. As far as (**ii**) is concerned, existence of a solution to (9) follows again by the Implicit Function Theorem by rewriting it as a formal series equality

$$\mathcal{S}_\tau^\delta(\eta, v) = \mathcal{S}_\tau^0(\eta, v) + \sum_{i>0} \frac{\delta^i}{(i+1)!} \mathcal{S}_\tau^i(\eta, v)$$

with $\mathcal{S}_\tau^0(\eta, v) = v + \kappa \nabla^\top H_d(\eta) g_\tau(\eta)$ and because

$$\lim_{\delta \to 0} \mathcal{S}_\tau^\delta(\eta, v) = I$$

is invertible. Substituting the corresponding feedback into the dissipation inequality (8), one gets $\Delta H_d(\eta) \leq 0$ and thus the result by zero-state detectability of the continuous-time counterpart (Theorem 3.1) and (Mattioni et al., 2018b, Theorem 5.2)). □

The reduction EB-PBC feedback gets the form

$$u = u_\tau^\delta(\eta) + v_\tau^\delta(\eta) \quad (21)$$

with energy-shaping and damping components defined as the solutions to the reduced equality (10) and (9) respectively, starting from the continuous-time components in (15) and (16).

3.3 Some constructive aspects

As usual under sampling, (21) gets the form of an asymptotic series expansion in powers of δ with an infinite number of terms. Thus, exact solutions are hard to compute in practice. However, the proof of Theorem 3.2 establishes an iterative and constructive procedure allowing to compute all terms of the corresponding series expansions (11) and (9) solving, at each step, a linear equality in the corresponding unknowns. To this end, one substitutes (19) and (20) into the corresponding series expansions and equates the terms with the same powers of δ so getting, for the first terms

$$\nabla^\top H_d(\eta) g_\tau(\eta) u_\tau(\eta) + \nabla^\top H_a(\eta) f(\eta) = 0$$
$$v_\tau(\eta) + \kappa g_\tau^\top(\eta) \nabla H_d(\eta) = 0$$
$$\nabla^\top H_d(\eta) g_\tau(\eta) \big(u_\tau^1(\eta) - \dot{u}_\tau(\eta)\big) = 0$$
$$v_\tau^1(\eta) - \dot{v}_\tau^1(\eta) + \kappa \mathrm{L}_g \mathrm{L}_{f_d} H_d(\eta) = 0$$

with $\dot{u}_\tau(\eta) = \nabla u_\tau(\eta) f_d(\eta)$, $f_d(\eta) = f(\eta) + g_\tau(\eta) u_\tau(\eta)$ and $\dot{v}_\tau(\eta) = \nabla v_\tau(\eta)\big(f_d(\eta) + g_\tau(\eta) v_\tau(\eta)\big)$.

Remark 3.4. As $\delta \to 0$ one naturally recovers the continuous-time solution. On the other side, as $\tau \to 0$, the sampled-data controller recovers the sampled-data delay-free EB-PBC proposed in Mattioni et al. (2021).

By the discussion above, even when the continuous-time components in Theorem 3.1 are exactly computable, only digital controllers defined as truncations on (21) at all desired finite orders $q \geq 0$ can be implemented in practice. To this end, we define the q^{th}*-order approximate reduction-based EB-PBC* as

$$u_{\tau}^{[q],\delta}(\eta) = u_{\tau}(\eta) + \sum_{i=0}^{q} \frac{\delta^i}{(i+1)!} u_{\tau}^{i}(\eta) \tag{22a}$$

$$v_{\tau}^{[q],\delta}(\eta) = v_{\tau}(\eta) + \sum_{i=0}^{q} \frac{\delta^i}{(i+1)!} v_{\tau}^{i}(\eta) \tag{22b}$$

When neither the continuous-time nor the sampled-data components are computable exactly, starting from Remark 3.2, we define the *(p,q)-order approximate reduction-based EB-PBC* as

$$u_{[p],\tau}^{[q],\delta}(\eta) = u_{[p],\tau}(\eta) + \sum_{i=0}^{q} \frac{\delta^i}{(i+1)!} u_{[p],\tau}^{i}(\eta) \tag{23a}$$

$$v_{[p],\tau}^{[q],\delta}(\eta) = v_{[p],\tau}(\eta) + \sum_{i=0}^{q} \frac{\delta^i}{(i+1)!} v_{[p],\tau}^{i}(\eta) \tag{23b}$$

deduced from (22) when substituting the continuous-time part as the approximation in (17).

4. THE PENDULUM AS AN EXAMPLE

Consider a simple pendulum described by

$$\dot{x}(t) = J\nabla H(x(t)) + Bu(t-\tau),\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{24}$$

with $x = (x_1\ x_2)^\top \in \mathbb{R}^2$, $x_1 = q$ and $x_2 = p$ the position and momentum, $H(x) = \frac{1}{2}x_2^2 + 1 - \cos x_1$ the energy (storage) function and $u \in \mathbb{R}$ the input force. The control objective is to design a digital feedback so to stabilize the pendulum in the upward position that is at $x_\star = (\pi\ 0)^\top$.

Delay-free design. When $\tau = 0$ the dynamics (24) satisfies Assumption 1 with

$$H_a(x_1) = 2\cos x_1, \quad H_d(x) = \frac{1}{2}x_2^2 + 1 + \cos x_1 \tag{25}$$

so that $u = -2\sin x_1 - \kappa x_2$ makes $x_\star$ asymptotically stable.

The sampled-data equivalent model (2) associated to (24) is in $\mathcal{O}(\delta^3)$ given by

$$x^+(u_{-N}) = x + \delta \begin{pmatrix} x_2 \\ u_{-N} - \sin x_1 \end{pmatrix} + \frac{\delta^2}{2} \begin{pmatrix} u_{-N} - \sin x_1 \\ -x_2 \cos x_1 \end{pmatrix}.$$

Continuous-time design. When $\delta \to 0$, the continuous-time reduced model is of the form (14) with in $\mathcal{O}(\tau^2)$

$$g_\tau(\eta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(\tau^2)$$

so getting the energy-shaping and damping controllers

$$u_{[1],\tau}(\eta) = \frac{2x_2 \sin x_1}{x_2 + \tau \sin x_1},\ v_{[1],\tau}(\eta) = -\kappa\big(x_2 + \tau \sin x_1\big).$$

Sampled-data reduction-based design. For the sake of simplicity, let us fix $\tau = \delta$ and thus $N = 1$. In this case, we compute the reduction variable in $\mathcal{O}(\tau^3)$

$$\eta = x + \tau \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_{-1} - \frac{\tau^2}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_{-1} \tag{26}$$

and the corresponding reduced dynamics in $\mathcal{O}(\delta^3)$

$$\eta^+(u) = \eta + \delta \begin{pmatrix} \eta_2 \\ -\sin \eta_1 + u \end{pmatrix} - \frac{\delta^2}{2} \begin{pmatrix} \sin \eta_1 + u \\ \eta_2 \cos \eta_1 \end{pmatrix}.$$

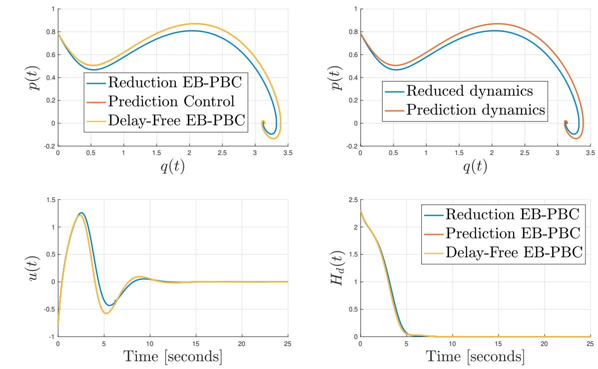

Fig. 1. The retarded pendulum with $\delta = \tau = 5 \times 10^{-4}$.

From Theorem 3.2, the stabilizing feedback is of the form (19)-(20) with the first terms provided by

$$\begin{aligned} u_{[1],\tau}^{1}(\eta) =& 4(\eta_2 + \tau \sin(\eta_1))^{-3}\Big(2\eta_2^4 \cos(\eta_1) + \tau\big(2\eta_2 \sin^3 \eta_1 \\ & - \eta_2^3 \sin 2\eta_1\big)\Big) \\ v_{[1],\tau}^{1}(\eta) =& \kappa\big(\eta_2 + \tau \sin \eta_1\big)^{-1}\big(\eta_2(\kappa\eta_2 - \sin \eta_1) \\ & + \tau(\sin^2 \eta_1 - \eta_2^2 \cos \eta_1 + 2\kappa\eta_2 \sin \eta_1)\big) \end{aligned}$$

with the subscript $_{[1]}$ indicating that they have been computed based on the first order approximation of the continuous-time component (see Remark 3.2).

Simulations. Simulations were performed fixing $\tau = \delta$ (i.e., when $N = 1$) to test the approximate proposed reduction-based controller (with $p = q = 1$) with approximate reduction computed as (26) in $\mathcal{O}(\delta^2)$. The corresponding sampled-data controller is compared with (continuous-time) prediction-based implementation of the nominal delay free EB-PBC controller; that is

$$\begin{aligned} u =& 2\sin z_1 - \kappa z_2 \\ \dot{z}_1 =& z_2, \quad \dot{z}_2 = -\sin z_1 + u \end{aligned}$$

with, denoting $z = (z_1\ z_2)^\top$, the approximate initial condition in $\mathcal{O}(\tau^2)$ (see Remark 2.3)

$$z(0) = x(0) + \tau \begin{pmatrix} x_2(0) \\ -\sin x_1(0) \end{pmatrix} - \frac{\tau^2}{2} \begin{pmatrix} \sin x_1(0) \\ x_2(0)\cos x_2(0) \end{pmatrix}.$$

The results are reported in Figs. 1-2 with $x(0) = (0\ \frac{\pi}{4})$ and increasing delays and sampling periods. Prediction is sensitive to approximations of the initial condition (that is not exactly computable in this case) so that stabilization of the equilibrium is not achieved even when the delay is small (Figure 2). Also, note that in this case, the prediction dynamics converges to the desired equilibrium whereas the the original trajectories diverge. On the other side, reduction well performs with respect to both sampling and delays even approximate model and feedback laws.

5. CONCLUSIONS AND PERSPECTIVES

In this paper, a new class of EB-PBCs has been designed for coping with both sampling and time delays in the inputs. The design is performed over the so-called reduced dynamics that is delay-free and equivalent, in terms of stabilizability, to the original retarded one. For, we have assumed the case of entire delay homogeneously acting

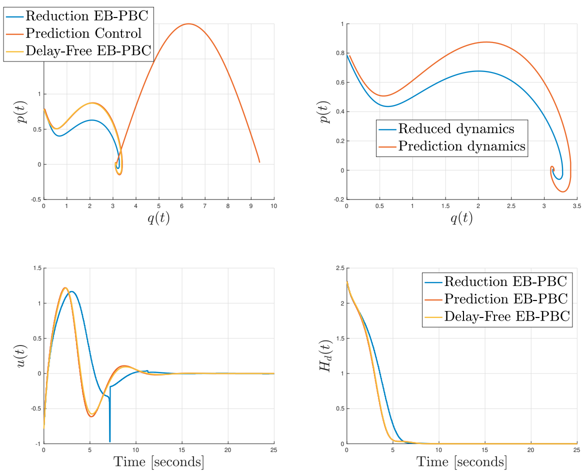

Fig. 2. The retarded pendulum with $\delta = \tau = 5 \times 10^{-2}$.

over all inputs. More general cases (e.g., $\tau = N\delta + \sigma$ with $\sigma \in [0, \delta[$ and multi-channel delays) are straightforward using the arguments in Mattioni et al. (2017, 2018c). Perspectives concern the generalization of this method to larger classes of delays (e.g., distributed delays) with a formal study on the properties under approximate controllers when considering both inter and non-uniform sampling (Di Ferdinando et al., 2022; Liu et al., 2022).

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