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To cite this article: Giacomo Bracci Testasecca and Andrea Pelissetto *J. Stat. Mech.* (2023) 043101

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Multicomponent gauge-Higgs models with discrete Abelian gauge groups

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Received 6 December 2022

Accepted for publication 18 March 2023

Published 12 April 2023

Online at stacks.iop.org/JSTAT/2023/043101

<https://doi.org/10.1088/1742-5468/acc72f>



Abstract. We consider a variant of the charge- Q compact Abelian-Higgs model, in which an N_f -dimensional complex vector is coupled with an Abelian \mathbb{Z}_q gauge field. For $N_f = 2$ and $Q = 1$ we observe several transition lines that belong to the $O(4)$, $O(3)$, and $O(2)$ vector universality classes, depending on the symmetry breaking pattern at the transition. The universality class is independent of q as long as $q \geq 3$. The universality class of the transition is uniquely determined by the behavior of the scalar fields; gauge fields do not play any role. We also investigate the system for $N_f = 15$ and $Q = 2$. In the presence of $U(1)$ gauge fields, the system undergoes transitions associated with charged fixed points of the Abelian-Higgs field theory. These continuous transitions turn into first-order ones when the $U(1)$ gauge fields are replaced by the discrete \mathbb{Z}_q fields: in the present compact model charged transitions appear to be very sensitive to the nature of the gauge fields.

Keywords: renormalization group, scalar electrodynamics, three-dimensional gauge field theories

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1. Introduction

Symmetries have always played a central role in the development of physical models. Global symmetries are commonly used to characterize the different phases of matter [1] and to understand the nature and properties of the transition lines between them. In the absence of local symmetries, the universal features of critical transitions can be understood using the traditional Landau-Ginzburg-Wilson (LGW) approach to critical phenomena [2–6]. It assumes that universal properties can be explained by a statistical field theory, in which a local order parameter plays the role of fundamental field. The universality classes are uniquely determined by the breaking pattern of the global symmetry, i.e. by the symmetry of the two phases coexisting at the transition, and by the transformation properties of the order parameter under the symmetry group.

Local gauge symmetries also play a fundamental role, both in high-energy physics [7], and in condensed-matter physics, where their applications span from superconductivity [8] to topological order and quantum phase transitions, see, e.g. [9–11]. The critical behavior of three-dimensional systems with global and local symmetries is, at present, not fully understood. First of all, in the presence of gauge degrees of freedom, one may have topologically ordered phases, that cannot be characterized in terms of local order parameters, whose existence is the basic tenet of the LGW approach [12–15]. Second, it is unclear under which conditions the gauge degrees of freedom play a role

at transitions where global symmetries are broken and matter fields show long-range correlations.

Critical transitions in three-dimensional lattice gauge theories can be classified in three different groups. First, there are topological transitions, where only the gauge degrees of freedom play a role and the global symmetry of the coexisting phases is the same. Matter fields are insensitive to the behavior of the gauge degrees of freedom and the critical behavior of the gauge modes is the same as in pure gauge models. Transitions of this type have been observed, e.g. in the three-dimensional Abelian Higgs (AH) model, both with noncompact fields [16] and with charge- Q scalar fields [17, 18].

Second, one should consider transitions in which only matter correlations are critical. At the transition gauge modes do not display long-range correlations, but are still crucial in determining the universality class of the critical behavior. Indeed, the gauge symmetry prevents non-gauge-invariant matter observables from developing long-range order: the gauge symmetry hinders some matter degrees of freedom—those that are not gauge invariant—from becoming critical. In this case, the gauge symmetry plays a crucial role in determining the order parameter of the transition and its transformation properties under the global symmetry group. However, in the absence of long-range gauge modes, one can still use the LGW approach, without including the gauge fields, to predict the critical behavior. The lattice CP^{N-1} model is an example of a $\text{U}(1)$ gauge model that shows this type of behavior [19, 20]. Transitions of this type also occur in several non-Abelian gauge models [21–24]. The finite-temperature transition of massless quantum chromodynamics has also been assumed to have these features [25].

Finally, there are transitions at which both matter and gauge correlations are critical. In this case an appropriate effective field-theory description of the critical behavior requires the inclusion of explicit gauge fields. Transitions of this type have been observed in very few cases; for instance, in the lattice AH model with noncompact gauge fields [16] (along the transition line that separates the Coulomb and the Higgs phase), and in the lattice AH model with compact gauge fields and scalar fields with charge $Q \geq 2$ [18] (along the transition line between the confined and the deconfined phase), when the number of components is sufficiently large (numerical simulations find the charged transition for $N_f \geq 10$). In the non-Abelian case, at present, there is some evidence of this type of behavior only in $\text{SU}(2)$ gauge theories coupled with scalar matter in the fundamental representation [22].

In this work we consider charge- Q N_f -component scalar fields interacting by means of a $\text{U}(N_f)$ -invariant Hamiltonian and couple them with \mathbb{Z}_q compact gauge fields. The model represents a generalization of the compact charge- Q AH model [18, 26], in which the gauge group $\text{U}(1)$ is replaced by its subgroup \mathbb{Z}_q . We determine the phase diagram of the model and the nature of the critical lines, with the purpose of verifying whether the previous considerations also apply to systems with discrete gauge groups.

We will study the model for two different values of the charge and number of components, which have been chosen on the basis of the results for $\text{U}(1)$ gauge models. First, we perform a numerical study of the model for $N_f = 2$ and $Q = 1$. In the presence of compact $\text{U}(1)$ gauge fields (it represents the standard compact lattice AH model), it shows a single critical line associated with the breaking of the global $\text{SU}(2)$ symmetry [27]. The transitions have an effective LGW description that predicts $\text{O}(3)$ vector behavior.

In the LGW approach to gauge systems, the nature of the gauge group is not relevant, and thus we expect the same critical behavior in \mathbb{Z}_q gauge models, provided the breaking pattern of the global symmetry is the same. Our numerical results for $q=3$ and $q=4$ are in full agreement with the LGW predictions: also for these values of q , transitions associated with the breaking of the $SU(2)$ global symmetry belong to the $O(3)$ vector universality class. In the discrete gauge model, the global symmetry group is larger. The model is also invariant under $U(1)/\mathbb{Z}_q$ global transformations. For $q=3$ and 4 , we find a second transition line where this symmetry breaks. Also along this transition line the gauge group does not play any role: the transitions belong to the $O(2)/XY$ vector universality class, as expected on the basis of the LGW analysis.

The second case of interest corresponds to systems with $N_f = 15$ and $Q = 2$. The corresponding $U(1)$ gauge-invariant model was studied in [18], finding a *charged* transition line, where both scalar and gauge degrees of freedom play a role. Gauge fields must be included in the effective description of the critical behavior and indeed the observed behavior is consistent with that predicted by the AH field theory [28–32]. A natural question is whether the *charged* transition survives if one replaces the continuous gauge group $U(1)$ with the discrete \mathbb{Z}_q group. In the case of a global $U(1)$ symmetry, it is well known that systems with microscopic \mathbb{Z}_q symmetry may have transitions in the $U(1)/O(2)$ universality class if $q \geq 4$ (see [33, 34] and references therein). At the transition one observes an enlargement of the global symmetry: The large-distance behavior is invariant under transformations that are not symmetries of the microscopic theory. Here, we investigate whether a similar phenomenon occurs for gauge symmetries, i.e. whether it is possible to have a *gauge symmetry enlargement*. Simulations with $q=6$ and $q=10$ indicate that no such symmetry enlargement occurs. We identify a transition line that separates two phases that have the same features as those coexisting along the charged transition line in the $U(1)$ gauge model, but in \mathbb{Z}_q gauge compact models these transitions turn out to be of first order for both values of q . Apparently, the microscopic model should be exactly $U(1)$ gauge invariant to allow the system to develop critical charged transitions.

The paper is organized as follows. In section 2 we define the model, while in section 3 we specify the quantities that are determined in the Monte Carlo simulations. Some limiting cases, useful to understand the general features of the phase diagram, are discussed in section 4, while the numerical results are presented in section 5 (results for $N_f = 2$) and in section 6 (results for $N_f = 15$). Finally, in section 7 we summarize the results and draw our conclusions.

2. The model

We consider a \mathbb{Z}_q gauge model coupled with an N_f -component charge- Q complex scalar field defined on a cubic lattice. The fundamental fields are complex N_f -dimensional vectors \mathbf{w}_x , satisfying $|\mathbf{w}_x| = 1$, associated with the sites of the lattice, and phases $\sigma_{x,\mu}$, $|\sigma_{x,\mu}| = 1$, associated with the lattice links. The gauge variables can only take q values. More precisely, we set

$$\sigma = \exp(2\pi i n/q), \quad (1)$$

where $n = 0, \dots, q-1$.

The Hamiltonian is

$$H = H_{\text{kin}} + H_{\text{g}}. \quad (2)$$

The first term is

$$H_{\text{kin}} = -J \operatorname{Re} \sum_{\mathbf{x}, \mu} (\bar{\mathbf{w}}_{\mathbf{x}} \cdot \mathbf{w}_{\mathbf{x}+\hat{\mu}} \sigma_{\mathbf{x}, \mu}^Q), \quad (3)$$

where the sum is over all lattice sites \mathbf{x} and directions μ ($\hat{\mu}$ are the corresponding unit vectors), and Q is the integer charge of the scalar field. Since $\sigma_{\mathbf{x}, \mu}^q = 1$, we can limit ourselves to charges Q satisfying $1 \leq Q \leq q/2$.

The second term is

$$H_{\text{g}} = -g \sum_{\mathbf{x}, \mu > \nu} \operatorname{Re} \Pi_{\mathbf{x}, \mu \nu}, \quad (4)$$

where the sum is over all lattice plaquettes, and the plaquette contribution is given by

$$\Pi_{\mathbf{x}, \mu \nu} = \sigma_{\mathbf{x}, \mu} \sigma_{\mathbf{x}+\hat{\mu}, \nu} \bar{\sigma}_{\mathbf{x}+\hat{\nu}, \mu} \bar{\sigma}_{\mathbf{x}, \nu}, \quad (5)$$

The partition function is

$$Z = \sum_{\{\mathbf{w}, \sigma\}} e^{-H/T}. \quad (6)$$

In the following we will use $\beta = J/T$ and $\kappa = g/T$ as independent variables.

The model is invariant under the local \mathbb{Z}_q transformations

$$\mathbf{w}_{\mathbf{x}} \rightarrow \lambda_{\mathbf{x}}^Q \mathbf{w}_{\mathbf{x}} \quad \sigma_{\mathbf{x}, \mu} \rightarrow \lambda_{\mathbf{x}} \sigma_{\mathbf{x}, \mu} \bar{\lambda}_{\mathbf{x}+\hat{\mu}}, \quad (7)$$

where $\lambda_{\mathbf{x}}$ are phases satisfying $\lambda_{\mathbf{x}}^q = 1$. It is also invariant under the global $U(N_f)$ transformations

$$\mathbf{w}_{\mathbf{x}} \rightarrow V \mathbf{w}_{\mathbf{x}}, \quad (8)$$

where $V \in U(N_f)$.

For $q/Q = 2$, the global symmetry is larger. Indeed, in this case $\sigma_{\mathbf{x}, \mu}^Q$ is real (it takes the values ± 1) and this allows us to rewrite the Hamiltonian as a vector Hamiltonian. We define a real field Φ^A with $2N_f$ components

$$\Phi_{\mathbf{x}}^a = \operatorname{Re} w_{\mathbf{x}}^a \quad \Phi_{\mathbf{x}}^{a+N_f} = \operatorname{Im} w_{\mathbf{x}}^a, \quad (9)$$

where $1 \leq a \leq N_f$. Since

$$\operatorname{Re} \bar{\mathbf{w}}_x \cdot \mathbf{w}_y = \Phi_x \cdot \Phi_y, \quad (10)$$

the scalar Hamiltonian can be rewritten as

$$H_{\text{kin}} = -J \sum_{x\mu} \sigma_{x,\mu}^Q \Phi_x \cdot \Phi_{x+\hat{\mu}}. \quad (11)$$

This Hamiltonian is invariant under the transformations $\Phi_x \rightarrow V \Phi_x$, where V is an $O(2N_f)$ matrix. For $Q=1$ and $q=2$, the scalar Hamiltonian corresponds to a particular \mathbb{RP}^{N-1} vector model ($N=2N_f$), in which the vector fields take values in the \mathbb{RP}^{N-1} manifold. For $N=3$ this model is relevant for liquid crystals [35, 36] and it has been studied, also in the presence of gauge interactions, in [37, 38].

3. The observables

We simulate the system using a combination of standard Metropolis updates of the scalar and gauge fields and of microcanonical updates of the scalar field. We compute the energy densities and the specific heats

$$\begin{aligned} E_k &= -\frac{1}{VJ} \langle H_{\text{kin}} \rangle, & C_k &= \frac{1}{VJ^2} (\langle H_{\text{kin}}^2 \rangle - \langle H_{\text{kin}} \rangle^2), \\ E_g &= -\frac{1}{Vg} \langle H_g \rangle, & C_g &= \frac{1}{Vg^2} (\langle H_g^2 \rangle - \langle H_g \rangle^2), \end{aligned} \quad (12)$$

where $V = L^3$.

We consider the two-point correlation function of the gauge-invariant tensor combination

$$T^{ab} = \bar{w}^a w^b - \frac{1}{N_f} \delta^{ab}, \quad (13)$$

defined as

$$G_T(\mathbf{x}, \mathbf{y}) = \sum_{ab} \langle T_x^{ab} T_y^{ba} \rangle. \quad (14)$$

The correlation function is invariant under local \mathbb{Z}_q transformations and global $U(N_f)$ transformations. We also consider the scalar-field combination

$$\Sigma_k^{a_1, \dots, a_k} = w^{a_1} \dots w^{a_k}. \quad (15)$$

If kQ is a multiple of q , this quantity is gauge invariant. Correspondingly, we consider the charge- k correlation function

$$G_k(\mathbf{x}, \mathbf{y}) = \operatorname{Re} \langle \bar{\Sigma}_{k,x} \cdot \Sigma_{k,y} \rangle = \operatorname{Re} \langle (\bar{\mathbf{w}}_x \cdot \mathbf{w}_y)^k \rangle.$$

The two local order parameters $T_{\mathbf{x}}^{ab}$ and $\Sigma_{\mathbf{x},k}^{a_1,\dots,a_k}$ belong to two different representations of the global symmetry group $U(N_f)$. In particular, $T_{\mathbf{x}}^{ab}$ is invariant under transformations belonging to the Abelian subgroup $U(1)$, while $\Sigma_{\mathbf{x},k}^{a_1,\dots,a_k}$ transforms nontrivially. For $q = 2Q$, the symmetry group is $O(2N_f)$. In this case, the gauge-invariant order parameters T^{ab} and Σ_2^{ab} are equivalent, being both related to

$$Q^{AB} = \Phi^A \Phi^B - \frac{1}{2N_f} \delta^{AB}, \quad (16)$$

where Φ is defined in equation (9). If

$$G_{\Phi}(\mathbf{x}, \mathbf{y}) = \sum_{AB} \langle Q_{\mathbf{x}}^{AB} Q_{\mathbf{y}}^{BA} \rangle, \quad (17)$$

one easily derives

$$\begin{aligned} G_T(\mathbf{x}, \mathbf{y}) &= \frac{2(N_f - 1)}{2N_f - 1} G_{\Phi}(\mathbf{x}, \mathbf{y}), \\ G_2(\mathbf{x}, \mathbf{y}) &= \frac{2N_f}{2N_f - 1} G_{\Phi}(\mathbf{x}, \mathbf{y}). \end{aligned} \quad (18)$$

The calculation of $G_k(\mathbf{x}, \mathbf{y})$ is particularly time-consuming when k and N_f are large. In some cases, we have used the following identity that relies on the $U(N_f)$ invariance of the theory:

$$G_k(\mathbf{x}, \mathbf{y}) = \frac{(k + N_f - 1)!}{k!(N_f - 1)!} \frac{\langle (\mathbf{a} \cdot \overline{\mathbf{w}})^k (\mathbf{b} \cdot \mathbf{w})^k \rangle}{(\mathbf{a} \cdot \mathbf{b})^k}, \quad (19)$$

where \mathbf{a} and \mathbf{b} are arbitrary N_f -dimensional vectors. Choosing vectors $\mathbf{a} = \mathbf{b} = (0, \dots, 0, 1, 0, \dots)$, we obtain

$$G_k(\mathbf{x}, \mathbf{y}) = \frac{(k + N_f - 1)!}{k!N_f!} \sum_{\alpha} \langle \overline{w}_{\alpha}^k w_{\alpha}^k \rangle. \quad (20)$$

To define the corresponding correlation lengths, we define the Fourier transform

$$\tilde{G}_{\#}(\mathbf{p}) = \frac{1}{V} \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} G_{\#}(\mathbf{x}, \mathbf{y}) \quad (21)$$

(V is the volume) of the two correlation functions. The corresponding susceptibilities and correlation lengths are defined as

$$\chi_{\#} = \tilde{G}_{\#}(\mathbf{0}), \quad (22)$$

$$\xi_{\#}^2 \equiv \frac{1}{4\sin^2(\pi/L)} \frac{\tilde{G}_{\#}(\mathbf{0}) - \tilde{G}_{\#}(\mathbf{p}_m)}{\tilde{G}_{\#}(\mathbf{p}_m)}, \quad (23)$$

where $\mathbf{p}_m = (2\pi/L, 0, 0)$. Note that, if one uses open boundary conditions, the choice of \mathbf{p}_m is somewhat arbitrary. Other choices, as long as they satisfy $|p_m| \sim 1/L$, would be equally valid.

In our finite-size scaling (FSS) analysis we use renormalization-group invariant quantities. We consider

$$R_{\xi, \#} = \xi_{\#}/L \quad (24)$$

and the Binder parameters

$$U_{\#} = \frac{\langle \mu_{2, \#}^2 \rangle}{\langle \mu_{2, \#} \rangle^2}, \quad \mu_{2, \#} = \sum_{\mathbf{x}\mathbf{y}} G_{\#}(\mathbf{x}, \mathbf{y}). \quad (25)$$

In the disordered phase, we have

$$U_T = \frac{N_f^2 + 1}{N_f^2 - 1}, \quad U_k = 1 + \frac{k!(N_f - 1)!}{(N_f + k - 1)!}, \quad (26)$$

while in the ordered phase, all these quantities converge to 1.

To determine the nature of the transition, one can consider the L dependence of the maximum $C_{\max}(L)$ of one of the specific heats. At a first-order transition, $C_{\max}(L)$ is proportional to the volume L^3 , while at a continuous transition it behaves as

$$C_{\max}(L) = aL^{\alpha/\nu} + C_{\text{reg}}. \quad (27)$$

The constant term C_{reg} , due to the analytic background, is the dominant contribution if $\alpha < 0$. The analysis of the L -dependence of $C_{\max}(L)$ may allow one to distinguish first-order and continuous transitions. However, experience with models that undergo weak first-order transitions indicates that in many cases the analysis of the specific heat is not conclusive [39, 40]. The maximum $C_{\max}(L)$ may start scaling as L^3 at values of L that are much larger than those at which simulations can be actually performed. A more useful quantity is a Binder parameter U . At first-order transitions, the maximum $U_{\max}(L)$ of U at fixed size L increases with the volume [39, 40]. On the other hand, U is bounded as $L \rightarrow \infty$ at a continuous phase transition. In this case, in the FSS limit, the Binder parameter as well as any renormalization-group invariant quantity R scales as

$$R(\beta, L) \approx f_R(X) + L^{-\omega} f_{c,R}(X) \quad X = (\beta - \beta_c)L^{1/\nu}, \quad (28)$$

where ω is the leading correction-to-scaling exponent. Thus, a first-order transition can be identified by verifying that $U_{\max}(L)$ increases with L , without the need of explicitly observing the linear behavior in the volume.

In the case of weak first-order transitions, the nature of the transition can also be understood from the combined analysis of U and R_{ξ} . At a continuous transition, in the FSS limit any renormalization-group invariant quantity R scales as

$$R(\beta, L) = F_R(R_{\xi}) + L^{-\omega} F_{c,R}(R_{\xi}) + \dots \quad (29)$$

where $F_R(x)$ is universal and $F_{c,R}(x)$ is universal apart from a multiplicative constant. The Binder parameter U does not obey this scaling relation at first-order transitions, because of the divergence of U for $L \rightarrow \infty$. Therefore, the absence of data collapse in plots of U versus R_ξ is an early indication of the first-order nature of the transition.

4. Some limiting cases

The phase diagram of the compact model with $U(1)$ gauge fields, i.e. in the limit $q \rightarrow \infty$, has been discussed at length in [17, 18, 26, 27], while systems with discrete gauge groups are discussed in [26, 37, 38, 41, 42]. To derive the phase diagram of the present model, it is useful to discuss some limiting cases.

In the limit $\kappa \rightarrow \infty$, the gauge degrees of freedom freeze and one can set $\sigma_{x,\mu} = 1$ on all links (when open boundary conditions are used, this is also true in a finite volume). Using the mapping (10), the scalar Hamiltonian becomes equivalent to that of the $O(2N_f)$ vector model, which undergoes a standard finite- β continuous transition for any $N_f \geq 1$.

For $\beta = 0$, there are no scalar fields and one obtains a pure gauge \mathbb{Z}_q model, that can be related by duality [43] to a \mathbb{Z}_q spin model, with a global \mathbb{Z}_q symmetry. The \mathbb{Z}_q gauge theory undergoes a topological transition at κ_c , which belongs to the same universality class as the corresponding transition in the \mathbb{Z}_q spin clock model. For $q = 2$ and 4 it belongs to the Ising universality class, for $q = 3$ it is of first order, while for $q \geq 5$ the critical behavior is the same as in the XY model, see [33, 44, 45]. Estimates of κ_c can be found in [46]. For $q = 2$, we can use duality and the results [47] for the standard Ising model to estimate $\kappa_c = 0.761413292(12)$. For $q \rightarrow \infty$, one has [18, 46]

$$\kappa_c \simeq \kappa_{gc} q^2, \quad (30)$$

where $\kappa_{gc} = 0.076051(2)$ is the critical coupling of the inverted XY model [48].

For $\kappa = 0$, there is no plaquette contribution. We can, first of all, simplify the scalar-field interaction term, defining

$$\tau_{x,\mu} = \sigma_{x,\mu}^M, \quad (31)$$

where M is the greatest common divisor of Q and q . The new field takes $p = q/M$ different values, i.e. it satisfies $\tau_{x,\mu}^p = 1$. If $r = Q/M$, the Hamiltonian becomes

$$H_{\text{kin}} = -J \operatorname{Re} \sum_{x\mu} \bar{\mathbf{w}}_x \cdot \mathbf{w}_{x+\hat{\mu}} \tau_{x,\mu}^r. \quad (32)$$

Note that p is always larger than 1, since $M \leq Q < q$. Thus, we obtain an effective model for charge- r scalars with \mathbb{Z}_p local gauge invariance.

For $p/r = q/Q = 2$ (in this case we have necessarily $p = 2$ and $r = 1$), the global invariance group of the model is the $O(2N_f)$ group and one obtains an effective \mathbb{RP}^{2N_f-1} model. The critical behavior of \mathbb{RP}^{N-1} models is well known: for any $N \geq 3$ they are expected to undergo a first-order transition [35, 36, 49], in agreement with standard LGW arguments.

The behavior of the model (32) for $p/r \neq 2$ is less clear. Since gauge fields are not dynamical for $\kappa = 0$ —they can be integrated out—the critical behavior is completely determined by an effective model in terms of a gauge-invariant order parameter. In this effective description only the global symmetry group is relevant. The gauge invariance group has the only role of selecting the appropriate gauge-invariant order parameter. In the present case the global symmetry group of the model is $U(N_f)/\mathbb{Z}_p$, which is not a simple group. Thus, it is possible to have transitions where only the $SU(N_f)$ symmetry is broken, transitions where only the Abelian subgroup $U(1)/\mathbb{Z}_p$ is broken, and transitions where both subgroups condense simultaneously. Thus, one or two transitions may occur as a function of β . In the following we will discuss the behavior of the model for $Q = 1, 2$ and two values of N_f , $N_f = 2$ and $N_f = 15$. In all cases numerical results show that two transitions occur: a first one at β_{c1} and a second one at $\beta_{c2} > \beta_{c1}$. At β_{c1} the order parameter T^{ab} condenses, while Σ_q is still disordered: the transition is clearly associated with the breaking of the $SU(N_f)$ symmetry. At β_{c2} , Σ_q correlations become critical, signalling the breaking of the Abelian symmetry. The small- β transition has the same symmetry-breaking pattern of the small- κ transitions observed in the model with $U(1)$ gauge symmetry [27]. Under the assumption that the nature of the transition only depends on the global symmetry breaking pattern, we can predict that the critical behavior is the same as that observed in [27]. For $N_f \geq 3$, only first-order transitions are possible, under the usual assumption that a cubic term in the LGW Hamiltonian signals discontinuous transitions (for a discussion of the validity of this assumption see [27] and references therein). For $N_f = 2$, continuous transitions are possible: they belong to the $O(3)$ vector universality class. The second, large- β transition is associated with the breaking of the Abelian $U(1)$ subgroup. Again, assuming that gauge fields are not dynamical, we predict that the transition belongs to the $O(2)/XY$ universality class, if it is continuous.

Let us finally consider the limit $\beta \rightarrow \infty$. In this case the scalar field orders so that

$$\mathbf{w}_x = \sigma_{x+\hat{\mu}}^Q \mathbf{w}_{x+\hat{\mu}}, \quad (33)$$

on each link. This relation implies

$$\mathbf{w}_x = \Pi_{x,\mu\nu}^Q \mathbf{w}_x \quad \Rightarrow \quad \Pi_{x,\mu\nu}^Q = 1 \quad (34)$$

on all plaquettes, which, in turn, implies that $\Pi_{x,\mu\nu}$ belongs to the \mathbb{Z}_M subgroup of \mathbb{Z}_q (M is the greatest common divisor of Q and q). Thus, modulo gauge transformations, for $\beta \rightarrow \infty$ we can take $\sigma_{x,\mu} = \exp(2\pi i n/M)$, with $n = 0, \dots, M$. If $M = 1$, the gauge fields are completely ordered on the line $\beta = \infty$. If, instead, $M \geq 2$, there is still a nontrivial gauge dynamics and the system behaves as a \mathbb{Z}_M gauge model, which undergoes a finite- κ topological phase transition.

5. Numerical results: critical behavior for $N_f = 2$ and $Q = 1$

In this section we discuss the phase diagram of the two-component model ($N_f = 2$) with charge-one ($Q = 1$) scalar fields. We consider three values of q , $q = 2, 3$, and 4. The

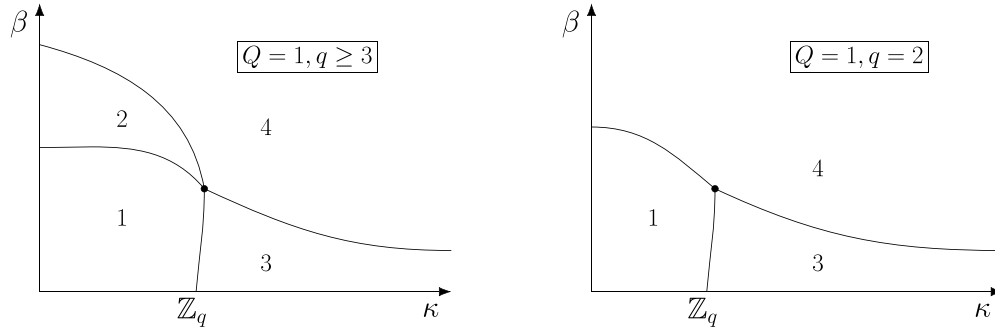


Figure 1. Expected phase diagram of the model for $Q = 1$. (Left) $q \geq 3$. (Right) $q = 2$. For $q \geq 3$ (left) there are four different phases. Scalar fields are disordered in phases 1 and 3, while they are ordered in phases 2 and 4. Transition lines 1–2 and 2–4 are associated with the breaking of the global symmetry $SU(N_f)$ and of the global symmetry $U(1)/\mathbb{Z}_q$, respectively; along the line 3–4 the simultaneous breaking of both global symmetries occurs. A topological \mathbb{Z}_q gauge transition occurs along the line 1–3. If $q = 2$ (right), phase 2 is missing and there is a single low- κ transition line 1–4 along which the global vector $O(2N_f)$ symmetry is broken.

results that we shall discuss below are consistent with the phase diagrams reported in figure 1. As expected—the global symmetry group is different in the two cases—the phase diagram for $q = 2$ is different from that of systems with $q \geq 3$.

To identify the different transition lines, we have performed runs at fixed κ , varying β . To determine the nature of the transitions that separate phases 1 and 2, and phases 2 and 4 (see figure 1) we have fixed $\kappa = 0.4$ for all q values. This value of κ has been chosen on the basis of the estimates of the critical κ_c for the pure gauge model ($\beta = 0$). Indeed, we have $\kappa_c = 0.761413292(12)$ (using duality and the results of [47]), 1.0844(2), 1.52276(4) [46] for $q = 2, 3$, and 4, respectively. The value of κ we use is significantly smaller than the values of κ_c reported above, guaranteeing that we are indeed studying the small- κ transition lines. To investigate the large- κ behavior we have instead fixed $\kappa = 1.5$ and 2 for $q = 2$ and 3, respectively.

In the runs at fixed $\kappa = 0.4$, we consider cubic lattices of linear size L with periodic boundary conditions. On the other hand, to determine the large- κ critical behavior, we consider open boundary conditions, to avoid slowly-decaying dynamic modes that are present in systems with periodic boundary conditions. Indeed, in the latter case, the Polyakov loops (the product of the gauge compact fields along nontrivial lattice paths that wrap around the lattice) have a very slow dynamics, if one uses algorithms with local updates. For open boundary conditions, Polyakov loops are not gauge invariant and thus their dynamics is not relevant for the estimation of gauge-invariant observables. A local algorithm is therefore efficient. Of course, open boundary conditions give rise to additional scaling corrections, due to the boundary, and thus larger systems are needed to obtain asymptotic results.

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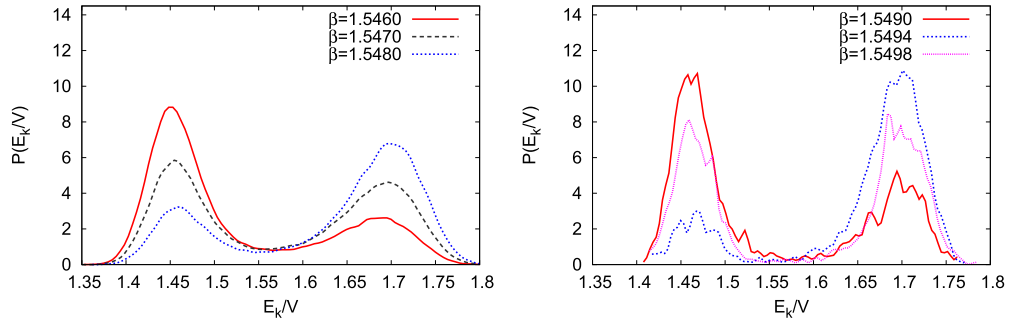


Figure 2. Histograms of the scalar energy density E_k/V for the model with $q=2$, $N_f=2$ along the line $\kappa=0.4$. (Left) Results for $L=16$. (Right) Results for $L=20$. The β values are reported in the legend.

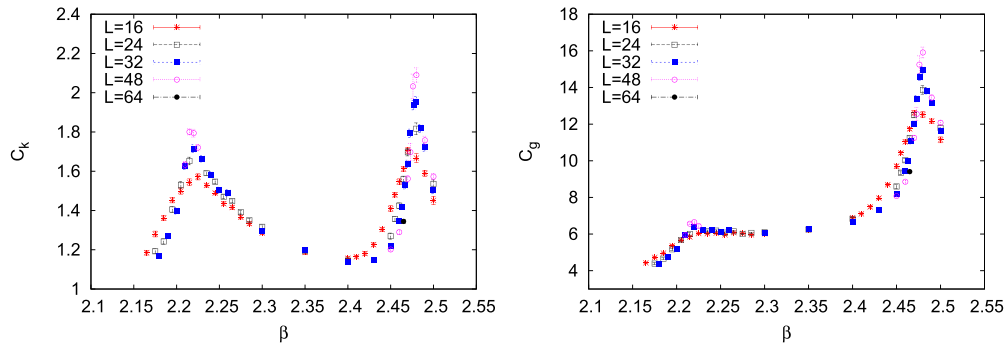


Figure 3. Plot of C_k (left) and C_g (right) versus β for the two-flavor model ($N_f=2$) with $q=3$ and $Q=1$, along the line $\kappa=0.4$.

5.1. Small- κ transitions

We have first considered the $q=2$ model, which has an enlarged $O(4)$ global symmetry. Along the line $\kappa=0.4$, simulations show the presence of a single first-order transition. The discontinuous nature is apparent from plots of the data as a function of the Monte Carlo time, which show the typical see-saw behavior, and from the histograms of the energy, which have a clear bimodal structure, see figure 2. This result is in agreement with the general theory. Indeed, for $q=2$ the model is equivalent to an RP^3 model, which is expected to undergo a first-order transition [35, 36, 49].

Let us now discuss the behavior for $q \geq 3$. As we have discussed in section 4, for $\kappa=0$, and therefore also for small values of κ , it is possible to have two different transitions as a function of β , associated with different breakings of the global symmetry. To verify this possibility, we have performed a scan in β . In figure 3, we report the specific heats as a function of β for $q=3$. There is a clear evidence of two transitions, one at $\beta \approx 2.2$, weakly coupled with the gauge degrees of freedom—the gauge specific heat is almost constant in the transition region—and a second one at $\beta \approx 2.45$. For $q=4$, we again observe two transitions, at $\beta \approx 2.3$ and 4.35 , respectively.

We first focus on the transition at $\beta \approx 2.2$ for $q=3$. We find that, while tensor correlations are critical (ξ_T increases rapidly at the transition), the gauge-invariant

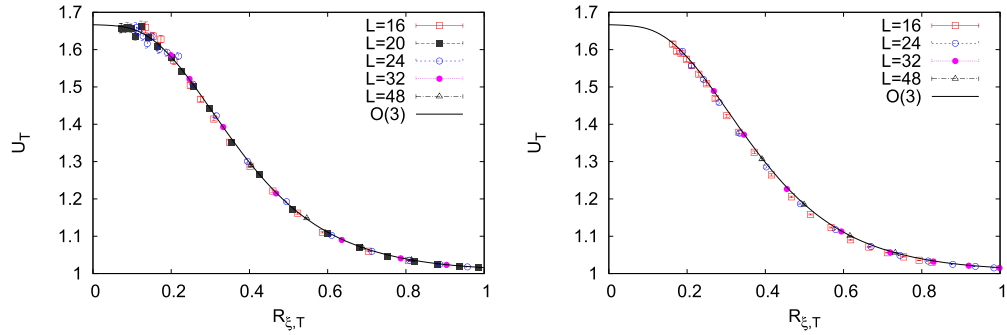


Figure 4. Plot of U_T versus $R_{\xi,T}$ for the two-flavor model ($N_f = 2$) along the line $\kappa = 0.4$ (small- β transition): (left) results for $q = 3$ (we only include data with $\beta < 2.3$); (right) results for $q = 4$. The continuous line is the universal curve for vector correlations in the O(3) vector model (see appendix of [50]).

charge-3 correlations are always short-ranged. This clearly indicates that the transition is associated with the breaking of the SU(2) global symmetry. In figure 4 we report U_T versus $R_{\xi,T}$ and compare the results with the universal curve appropriate for the O(3) universality class, reported in the appendix of [50]. Data fall quite precisely onto the O(3) curve, confirming the LGW prediction that the transition belongs to the O(3) universality class. Also for $q = 4$ the data close to the small- β transition fall on top of the O(3) curve, indicating that the transition belongs to the O(3) universality class for any $q > 3$, a result which is not surprising as the same O(3) transition occurs in the model with U(1) symmetry, i.e. for $q \rightarrow \infty$.

As an additional check we have analyzed $R_{\xi,T}$ and U_T , fitting the Monte Carlo data in the small- β transition region ($\beta < 2.3$ for $q = 3$) to equation (28). Scaling corrections have been neglected—as apparent from figure 4, they are small. The scaling function $F_R(x)$ has been parametrized with a polynomial. We obtain $\nu = 0.716(13)$ and $0.72(2)$ for $q = 3, 4$, respectively, in good agreement with the O(3) estimate $\nu = 0.71164(10)$ of [51]. To estimate the critical coupling β_c we have repeated the same fits fixing ν to the O(3) value. We obtain $\beta_c = 2.2155(3)$ and $2.3175(5)$ for $q = 3, 4$. Note that β_c depends weakly on q , indicating that the nonuniversal features of the transitions are only slightly dependent on q .

Finally, we have checked the behavior of the tensor susceptibility that is expected to scale as $L^{2-\eta}f(R_{\xi,T})$, where $f(x)$ is universal apart from a multiplicative rescaling. If the transition belongs to the O(3) universality class, data should scale provided we set $\eta = \eta_{O(3)}$, where $\eta_{O(3)} = 0.03624(8)$ [51] is the value it takes in the O(3) model. Results are shown in figure 5. We observe an excellent scaling. Moreover, on the scale of the figure, we do not see any dependence of the scaling curve on q . As observed for β_c , the q -dependence of nonuniversal amplitudes is small.

Let us now focus on the large- β transition. For both $q = 3$ and 4, charge- q correlations diverge. Since the SU(N_f) symmetry is broken, the transition is described by an effective scalar model in which the fundamental field is a phase and the interaction is \mathbb{Z}_q gauge invariant. The behavior of this model has been discussed in [42]. For small values of κ , the transition belongs to the XY universality class, if it is continuous. Moreover,

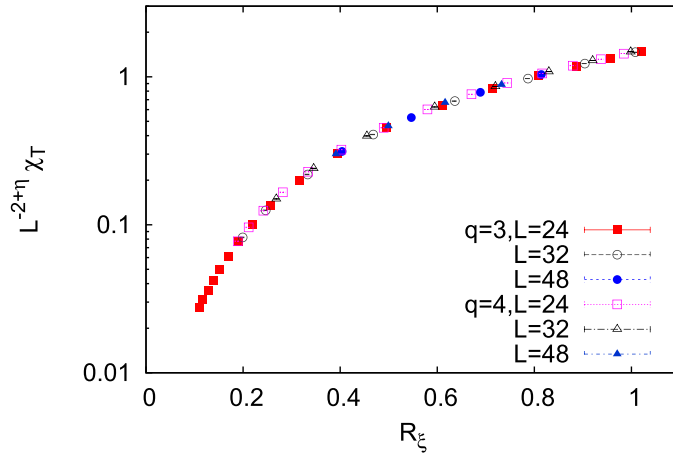


Figure 5. Plot of $L^{-2+\eta}\chi_T$ versus $R_{\xi,T}$ for the two-flavor model ($N_f = 2$) along the line $\kappa = 0.4$ (small- β transition): results for $q = 3$ (we only include data with $\beta < 2.3$) and $q = 4$. We use the $O(3)$ value [51] $\eta = 0.03624$.

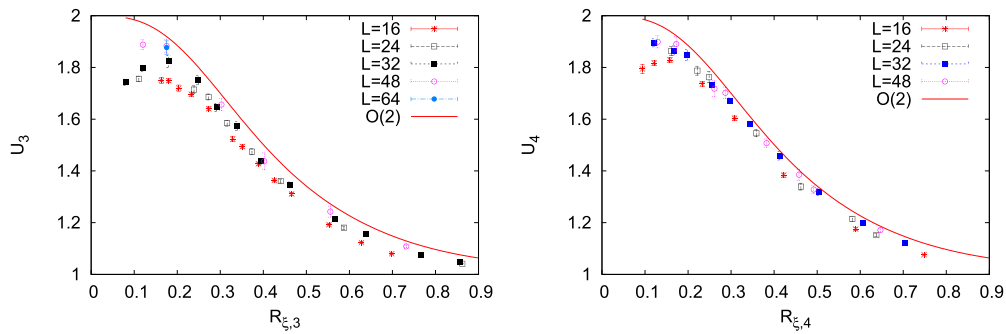


Figure 6. Scaling plots at the large- β critical transition, at fixed $\kappa = 0.4$, for $N_f = 2$ and $Q = 1$. (Left) Plot of U_3 versus $R_{\xi,3}$ for $q = 3$. (Right) Plot of U_4 versus $R_{\xi,4}$ for $q = 4$. We also report the universal scaling curve for vector correlations in the $O(2)$ vector model (see appendix of [50]).

the universal scaling function that expresses U_q in terms of $R_{\xi,q}$ is the same as the scaling function $U = f_U(R_\xi)$ for vector correlations in the XY model. To verify this prediction, we plot U_q versus $R_{\xi,q}$ for $q = 3, 4$ and compare the data with the universal XY vector curve reported in [50], see figure 6. Data are consistent with the XY curve (as L increases, data move closer to the XY curve), although significant corrections are apparently present, especially for $q = 3$, probably because of the presence of the nearby $O(3)$ transition. In order to have a stronger check of the correctness of the prediction, we have analyzed the scaling corrections. We consider the deviations

$$\Delta(R_{\xi,q}) = U_q - f_{XY}(R_{\xi,q}), \quad (35)$$

as a function of $R_{\xi,q}$; here $f_{XY}(R_\xi)$ is the universal XY curve for vector correlations. If $f_{XY}(R_{\xi,q})$ is the correct asymptotic behavior and ω is the leading correction-to-scaling

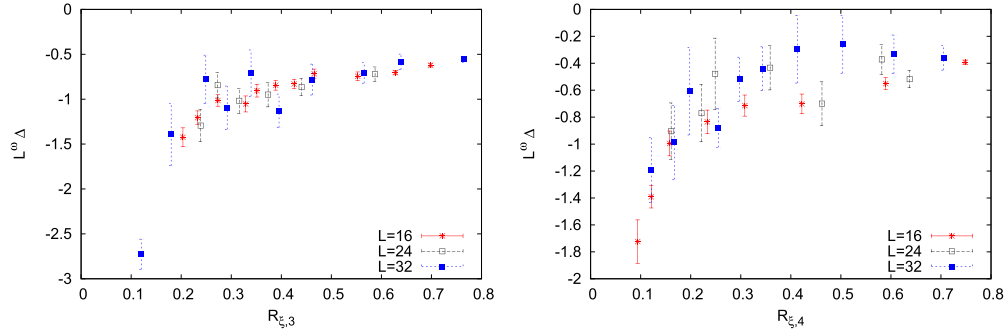


Figure 7. Scaling corrections at the large- β critical transition, at fixed $\kappa=0.4$, for $N_f=2$ and $Q=1$. Plot of $L^\omega \Delta(R_{\xi,q})$ versus $R_{\xi,q}$ for $q=3$ (left) and $q=4$ (right), where $\Delta(R_{\xi,q})$ is defined in equation (35). We set $\omega = \omega_{XY}$, where ω_{XY} is the XY value of the correction-to-scaling exponent: $\omega_{XY} = 0.789$ [45].

exponent, $L^\omega \Delta(R_{\xi,q})$ should scale for large L . We assume that ω is the same as the leading exponent ω_{XY} in the XY model⁴. In figure 7 we report the numerical results. Data scale reasonably, confirming that the transition belongs to the XY universality class.

Assuming an XY behavior, we have estimated the location of the critical transition. We have performed combined fits of U_q and $R_{\xi,q}$ to equation (28), including scaling corrections. Fixing ω and ν to the XY values, $\omega = 0.789$, $\nu = 0.6717$ [45], we obtain $\beta_c = 2.4769(4)$ and $\beta_c = 4.3370(15)$ for $q=3$ and 4, respectively.

5.2. Large- κ transition line

We have studied the large- κ behavior of the system for $q=2$ and $q=3$, performing runs at fixed $\kappa=1.5$ and $\kappa=2$, respectively. As we discussed above, we expect the same behavior as in the O(4) vector model, for all gauge-invariant observables. Note that, for $q=3$ this represents an enlargement of the global symmetry at the transition. In the presence of O(4) symmetry, tensor and charge- q correlations are related, see section 3, and thus they should be both critical along the large- κ transition line.

The numerical results are in agreement with these predictions. In figure 8 we compare the behavior of the tensor Binder parameter U_T as a function of $R_{\xi,T}$ in the gauge models and in the O(4) model. Data scale onto the same universal curve, as expected. We have also verified that charge- q correlations behave as in the O(4) model. For $q=2$ this is a consequence of the exact O(4) symmetry of the model. For $q=3$ this is confirmed by the results shown in figure 9.

⁴ *A priori* it is possible that $\omega < \omega_{XY}$, since in our model there are additional RG subleading operators due, e.g. to the gauge interactions, the interactions with the frozen $SU(N_f)$ modes, etc. They may be more relevant than the RG operator that controls the scaling corrections in the XY model. Our numerical results are in agreement with the assumption $\omega = \omega_{XY}$.

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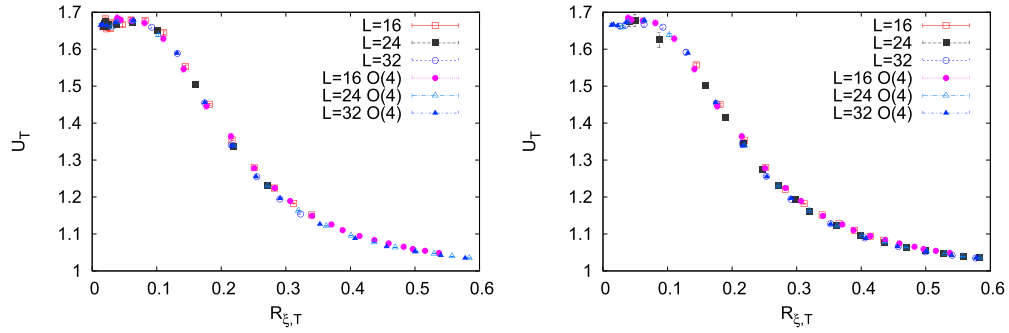


Figure 8. Plot of U_T versus $R_{\xi,T}$ for the two-flavor model ($N_f = 2$) along the large- κ transition line (open boundary conditions). (Left) Results for $q = 2$ (at fixed $\kappa = 1.5$). (Right) Results for $q = 3$ (at fixed $\kappa = 2.0$). The results for the gauge model are compared with the results for the $O(4)$ vector model.

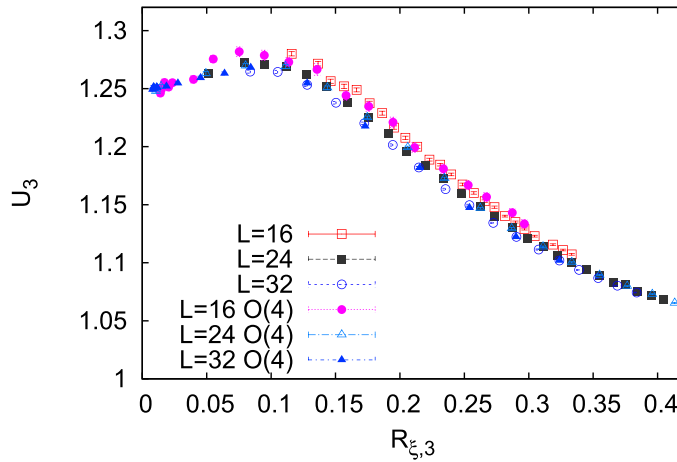


Figure 9. Plot of U_3 versus $R_{\xi,3}$ for the two-flavor model ($N_f = 2$) along the large- κ transition line, for $q = 3$ (at fixed $\kappa = 2$). The results for the gauge model are compared with the results for the $O(4)$ vector model.

5.3. Phase diagram

The numerical results presented above confirm that, for $N_f = 2$, the model has the phase diagram reported in figure 1. We expect the same phase diagram for any N_f as long as $Q = 1$. As far as the nature of the transition lines, for $q \geq 3$ we expect the small- κ , large- β transition, where the $U(1)/\mathbb{Z}_q$ symmetry is broken, to belong to the XY universality class in all cases. Analogously, the large- κ transitions should always belong to the $O(2N_f)$ universality class. For small κ and small β , transitions should be analogous to those observed in the CP^{N_f-1} model, and therefore they should be of first order for any $N_f \geq 3$.

If $Q \neq 1$ but M is 1 (M is the greatest common divisor of Q and q), the model should also have the phase diagram reported in figure 1. Instead, for $M \geq 2$ the phase diagram

is more complex (it will be discussed in the next section), because of the presence of a new topological transition line for large values of β .

6. Numerical results: critical behavior for $N_f = 15$ and $Q = 2$

In this section, we consider the model with charge $Q = 2$ and $N_f = 15$ for different values of q with the purpose of understanding whether it exhibits a continuous transition in the same universality class as the charged transition that occurs in $U(1)$ gauge invariant models [16–18]. We only consider even values of q . If q is odd, gauge fields are completely ordered for $\beta \rightarrow \infty$. In this case, the phase diagram should be analogous to that obtained for $Q = 1$, see figure 1, without the large- β topological transition line, whose presence is necessary to observe the charged transition in $U(1)$ gauge invariant models [16–18]. In most of the simulations we set $q = 10$. The numerical results are consistent with the phase diagram reported in figure 10. Comparing with figure 1, one observes the presence of a new line that starts at $\beta = \infty$ and that is associated with a \mathbb{Z}_Q (in our case Ising) topological transition. It intersects the two phases labeled 2 and 4 in figure 1. The additional topological transition is irrelevant for the breaking of the global symmetries: the global $SU(N_f)$ symmetry is broken in phases 2a, 2b, 4a, and 4b, while the $U(1)/\mathbb{Z}_q$ symmetry is broken in both phases 4a and 4b.

For $q = 4$ and $Q = 2$ the enlarged symmetry forbids the presence of phases 2a and 2b. The only difference with the case $Q = 1$ is the presence of the topological transition line that divides phase 4 in two different phases. The behavior of the phase diagram as q increases can be easily inferred. Since phases 4a, 4b and 3 are not present in the $U(1)$ gauge model, the multicritical point M_1 should move towards larger β values, while the multicritical point M_3 should move towards $\kappa = \infty$. For $q = \infty$ continuous transitions controlled by the charged fixed point are observed on the line starting at M_2 and ending at $\kappa = \infty$. Thus, if this transition survives for finite values of q , it should be observed on the line connecting M_2 with M_3 , which separates phase 1 from phase 2b.

6.1. Critical behavior for $q = 10$

For $q = 10$, we have first performed runs along lines with fixed κ . We have chosen three different values κ_i , and we have performed runs varying β at fixed $\kappa = \kappa_i$. The values κ_i have been chosen such as to probe the transitions $1 \rightarrow 2a \rightarrow 4a$ (runs at fixed κ_1), the transitions $1 \rightarrow 2b \rightarrow 4b$ (runs at fixed κ_2), and the transitions $3 \rightarrow 4b$ (runs at fixed κ_3), see the right panel of figure 10. The appropriate κ_i values have been determined by looking at the behavior for $\beta = 0$ and ∞ (in these two cases the system corresponds to a pure gauge model, as discussed in section 4). For $\beta = \infty$, the \mathbb{Z}_2 topological transition line starts at $\kappa \approx 0.76$, while, for $\beta = 0$, the \mathbb{Z}_{10} topological transition line starts [46] at $\kappa \approx 7.86$. Therefore, we have chosen $\kappa_1 = 0.4$, $\kappa_2 = 1.2$, and $\kappa_3 = 10$. For the runs at $\kappa = \kappa_1$ and κ_2 we have used periodic boundary conditions, while open boundary conditions have been used at κ_3 .

For $\kappa = 10$, a single transition is observed as β is increased, as expected. To confirm that the transition belongs to the $O(30)$ universality class we have determined the universal scaling curve of U_T versus $R_{\xi,T}$. The data are presented in figure 11, together

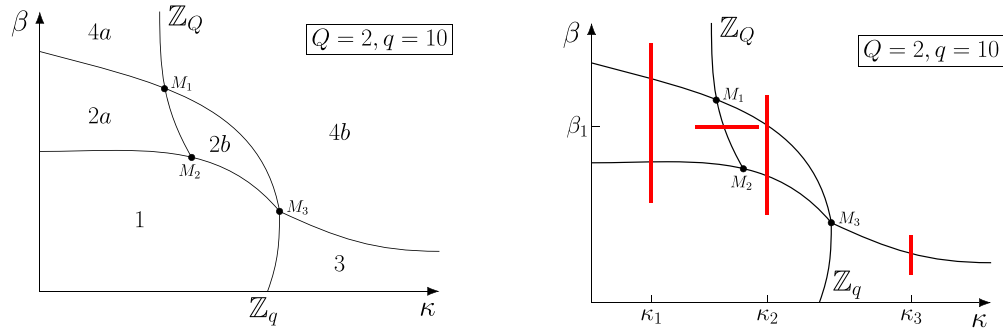


Figure 10. (Left) Sketch of the phase diagram of the model for $Q = 2$ and $q = 10$. We expect the same phase diagram for any even $q \geq 6$. For $q = 4$, the system has a larger $O(2N_f)$ global symmetry and phases $2a$ and $2b$ are missing. In the right panel we report the lines (red thick lines) along which runs have been performed. We have performed runs at fixed κ , at $\kappa_1 = 0.4$, $\kappa_2 = 1.2$, and $\kappa_3 = 10$, and a series of runs at fixed β , at $\beta = \beta_1 = 13$. The tensor correlation length diverges in phases $2a$, $2b$, $4a$, $4b$, signaling the breaking of the $SU(N_f)$ symmetry; the correlation length ξ_5 is finite in phases $2a$, $2b$, and diverges in phases $4a$, $4b$.

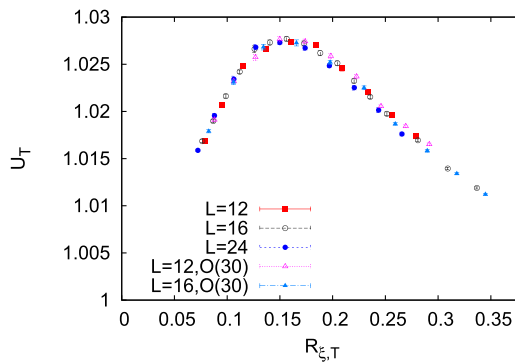


Figure 11. Plot of U_T versus $R_{\xi,5}$ for the model with $N_f = 15$, $Q = 2$, and $q = 10$. Results have been obtained varying β along the line $\kappa = 10$. The results are compared with analogous data for the $O(30)$ vector model.

with the analogous data for the $O(30)$ vector model. The results for the gauge model and for the vector model fall on top of each other, confirming that the transition is in the $O(30)$ universality class.

For both $\kappa = 0.4$ and $\kappa = 1.2$, we find two different transitions as β is increased. For $\kappa = 0.4$ we observe a very strong first-order transition for $\beta \approx 11$. The energy shows a strong hysteresis for $10.9 \lesssim \beta \lesssim 11$ and $11 \lesssim \beta \lesssim 11.3$ for $L = 6$ and 8 (for each β we perform runs of a few million iterations). Equilibrated results are only obtained on lattices of size $L = 4$. At the transition the $SU(N_f)$ symmetry is broken— ξ_T is large and increases rapidly with L on the large- β side of the transition—while the $U(1)/\mathbb{Z}_q$ symmetry is preserved—the correlation length ξ_5 is small and independent of L at the transition. At the second transition, observed at $\beta \approx 14.6$, ξ_5 diverges, signaling the breaking of the $U(1)/\mathbb{Z}_q$ global symmetry. We expect the transition to belong to the

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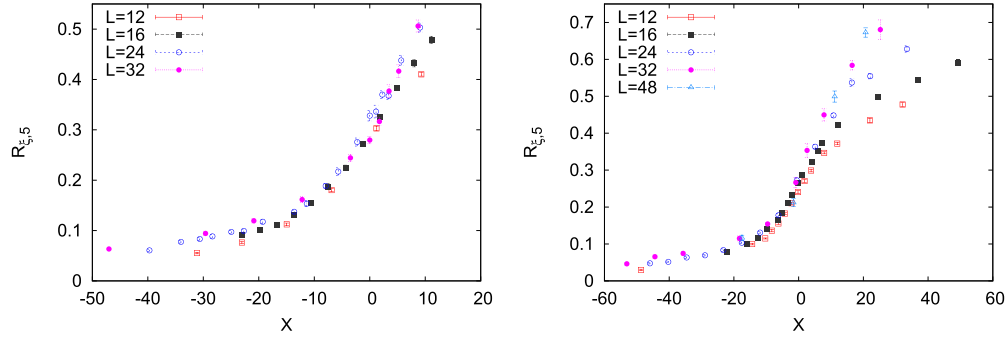


Figure 12. Plots of $R_{\xi,5}$ versus $X = (\beta - \beta_c)L^{1/\nu}$ for the model with $N_f = 15$, $Q = 2$, and $q = 10$. Results have been obtained varying β along the line $\kappa = 0.4$ (left) and $\kappa = 1.2$ (right). We set $\nu = 0.6717$ (the XY-model value [45]), $\beta_c = 14.57$ ($\kappa = 0.4$) and $\beta_c = 13.21$ ($\kappa = 1.2$).

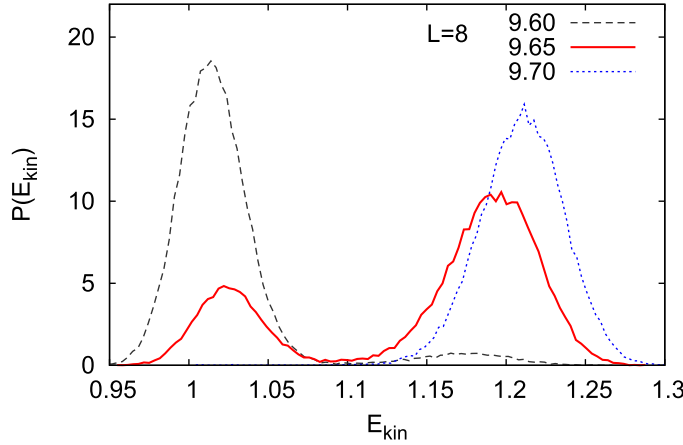


Figure 13. Plot of the probability distribution of E_{kin} for the model with $N_f = 15$, $Q = 2$, and $q = 10$. Results have been obtained for $\kappa = 1.2$ and a few values of β (they are reported in the legend in the figure).

XY universality class. In figure 12 (left panel) we plot $R_{\xi,5}$ versus $X = (\beta - \beta_c)L^{1/\nu}$ using the XY estimate [45] for the critical exponent ν , $\nu = 0.6717$. We observe a reasonable agreement, indicating that data are indeed consistent the XY behavior.

The same analysis has been repeated for $\kappa = 1.2$. As β increases, one first observes a transition at $\beta \approx 10$. The transition is of first order. The scalar-field energy E_{kin} shows a clear bimodal structure already for $L = 8$, see figure 13. The transition has the same features observed at $\kappa = 0.4$, indicating that it is associated with the breaking of the $\text{SU}(N_f)$ global symmetry. As β is further increased, a second transition occurs at $\beta \approx 13.2$, where the correlations ξ_5 diverges. The numerical data are again consistent with an XY behavior, see the right panel of figure 12.

Finally, to obtain an unambiguous check that the runs at $\kappa = 1.2$ are indeed probing the transitions $1 \rightarrow 2b \rightarrow 4b$, we have performed runs at fixed $\beta = 13$, varying κ . This value of β has been chosen on the basis of the results for $\kappa = 1.2$. Since the XY transition

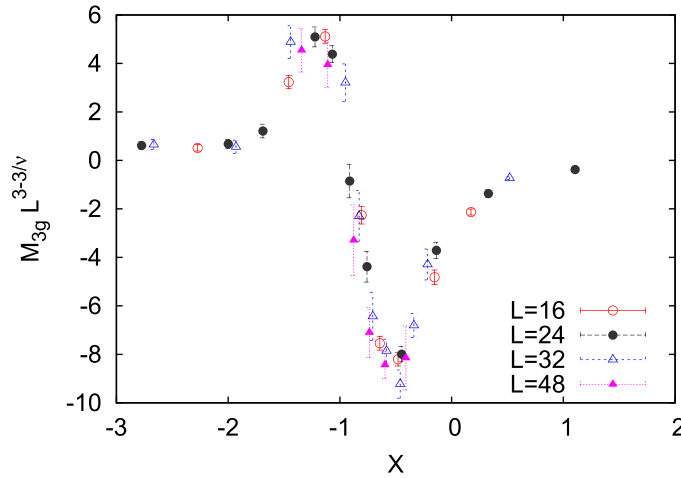


Figure 14. Scaling plot of $M_{3g} L^{3-3/\nu}$ versus $X = (\kappa - \kappa_c) L^{1/\nu}$ for the model with $N_f = 15$, $Q = 2$, and $q = 10$. Results have been obtained varying κ at fixed $\beta = 13$. We set $\nu = 0.629971$ (the Ising model value, see [52]) and $\kappa_c = 0.817878$.

is located at $\beta \approx 13.2$, if the runs at $\kappa = 1.2$ were probing the transitions $1 \rightarrow 2a \rightarrow 4a$, we would find the topological \mathbb{Z}_2 transition at $\kappa > 1.2$ when varying β (see figure 10). Instead, starting from $\kappa = 1.2$, we find a transition at a smaller value of κ , $\kappa \approx 0.81$, confirming the correct identification of the transition lines for $\kappa = 1.2$. The transition at $\kappa \approx 0.81$ has a clear topological nature: in the two coexisting phases, the $SU(N_f)$ symmetry is broken ($U_T \approx 1$ across the transition), while the $U(1)$ scalar modes are always disordered, $\xi_5 \lesssim 1$ in the transition region. Clearly, we are observing the transition line that separates phases $2a$ and $2b$. Since the transition is associated with the gauge modes, there are no local order parameters. Therefore, to characterize the critical behavior, we consider moments of the gauge energy. In particular, we consider the third cumulant [43]

$$M_{3g} = -\frac{1}{Vg^3} \langle (H_g - \langle H_g \rangle)^3 \rangle. \quad (36)$$

If the transition is continuous, this quantity should scale as

$$M_{3g} = L^{3/\nu-3} f(X) \quad X = (\kappa - \kappa_c) L^{1/\nu}. \quad (37)$$

If we take $\kappa_c = 0.8179$ and we use the Ising estimate [52] $\nu = 0.629971$, data scale nicely, see figure 14. The transition is therefore continuous in the Ising universality class.

The results of the runs at $\beta = 13$ confirm that the runs at $\kappa = 1.2 > \kappa_c(\beta = 13) = 0.8$ are indeed probing the transition lines $1 \rightarrow 2b \rightarrow 4b$. Since transitions along the $1-2b$ line are of first order, there is no evidence of the charged universality class. At variance with what happens with the global symmetry group—systems with \mathbb{Z}_q symmetry behave as $U(1)$ systems if $q \geq 4$ —in the case of the $U(1)$ gauge symmetry, no gauge symmetry enlargement occurs: the charged universality class can apparently be observed only for the $U(1)$ model, i.e. for $q = \infty$.

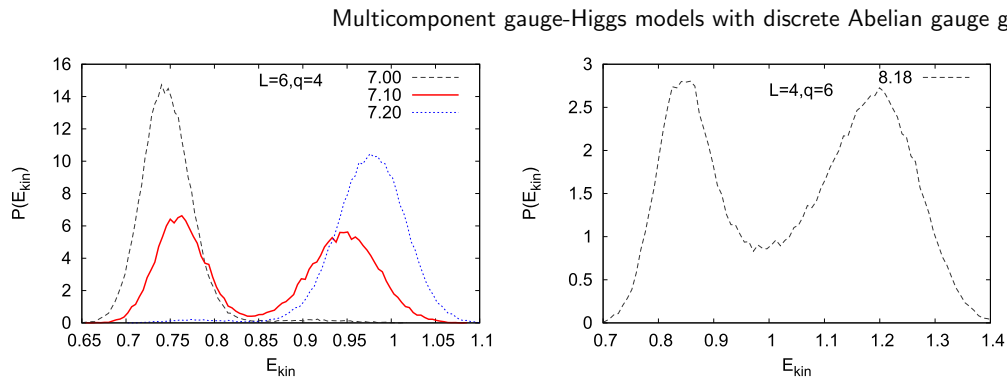


Figure 15. Plot of the probability distribution of E_{kin} for the model with $N_f = 15$ and $Q = 2$, for $\kappa = 1.2$ and different values of β (they are reported in the legend). Results for $q = 4$ and $q = 6$ are reported in the left and right panel, respectively. Periodic boundary conditions are used.

The runs at $\beta = 13$ also show that, along the line running from $\beta = \infty$ up to the multicritical point M_2 , the transitions are always topological in the Ising universality class. Scalar fields play no role, so that the same critical behavior is observed on the whole line starting at $\beta = \infty$ and ending at the multicritical point M_2 , see figure 10.

6.2. Phase behavior for $q = 4$ and $q = 6$

Beside the extensive simulations for $q = 10$, we also performed some simulations for $q = 4$ and $q = 6$. We chose $\kappa = 1.2$ in both cases and varied β , in order to identify the transitions 1–2b, 2b–4b for $q = 6$ and the single transition 1–4b for $q = 4$ (the enlarged $O(30)$ global symmetry forbids phases 2a and 2b), see figure 10. For $q = 4$ numerical simulations confirm the existence of a single transition. Using both open and periodic boundary conditions, we verify that the transition is of first order: the energy has a bimodal distribution already on small lattices, see the left panel of figure 15.

For $q = 6$ we would expect the same phase diagram as that shown in figure 10. Surprisingly, our data at fixed $\kappa = 1.2$ are consistent with a single first-order transition. The latent heat is large—the distribution of the energy is bimodal already for $L = 4$, see the right panel of figure 15—and we are not able to obtain equilibrated results for $L \geq 6$. On the large- β side of the transition, both ξ_T and ξ_3 increase rapidly with L (comparing data for $L = 8$ and 16, we estimate an effective behavior $\xi \sim L^{1.6}$), which would apparently indicate that both the $SU(N_f)$ and the $U(1)/\mathbb{Z}_6$ symmetry are broken. We do not see any reason why the model with $q = 6$ should have a phase diagram different from that of the model with $q = 10$, presented in figure 10. A possible explanation of the observed behavior that is consistent with the general picture obtained from the simulations with $q = 10$ is the following. It is possible that ξ_3 on the large- β side of the transition is finite (as expected in phase 2b), but large ($\xi_3 \gg 10$). In this case, our numerical simulations on lattices of size of order 10 would be unable to distinguish phase 2b from phase 4b.

7. Conclusions

In this paper we consider a variant of the usual three-dimensional compact AH model in which a complex N_f component scalar field of charge Q is coupled with a $U(1)$ Abelian gauge field. Here, gauge fields are phases that take only a set of q discrete values so that the model is \mathbb{Z}_q gauge invariant. Because of the smaller gauge group, the theory is globally invariant under a larger set of symmetry transformations. Beside $SU(N_f)$ transformations, there is also a nontrivial invariance under global $U(1)/\mathbb{Z}_q$ phase changes of the scalar fields. The larger invariance group allows for a richer phase diagram, with additional transition lines associated with the breaking of the $U(1)/\mathbb{Z}_q$ symmetry.

By means of numerical Monte Carlo simulations, we determine the phase diagram and investigate the nature of the different transition lines in two different cases. First, we consider the model with charge-1 two-component scalar fields; second, we study the model with scalar fields of charge $Q=2$ and $N_f=15$ components. These two sets of parameters have been chosen on the basis of the results for the $U(1)$ gauge AH model [18], which show that in these two cases gauge fields play a completely different role.

For $U(1)$ gauge fields, the transitions observed for $N_f=2$ and $Q=1$ are uniquely characterized by the critical fluctuations of the scalar fields. Gauge fields are only relevant in selecting the physical observables (i.e. gauge-invariant quantities), but do not determine the nature of the critical behavior, which can be predicted by using an effective LGW theory for a gauge-invariant order parameter, with no explicit gauge fields. Therefore, the universality class of the transition depends only on the transformation properties of the order parameter under the global $SU(2)$ symmetry and the symmetry breaking pattern at the transition. In the $SO(3)$ language [note that $SU(2)/\mathbb{Z}_2 = SO(3)$], the order parameter is an $SO(3)$ vector and the symmetry breaking pattern corresponds to $SO(3) \rightarrow SO(2)$. Therefore, one predicts the transition to belong to the $SO(3)$ vector universality class, a result that is fully supported by simulations (see, e.g. [27]). In the discrete gauge model, along the line where the $SU(2)$ symmetry is broken, we observe the same $SO(3)$ vector behavior for any $q \geq 3$. This result further confirms the correctness of the effective LGW effective theory and the irrelevance of the gauge fields along the $O(3)$ transition line.

As we already mentioned, the discrete model is also invariant under the $U(1)/\mathbb{Z}_q$ group. We find transition lines where this Abelian global symmetry is broken. They occur within the $SU(2)$ ordered phase. Again, gauge fields are irrelevant for the determination of the critical behavior: transitions belong to the XY universality class, as in systems without gauge fields. We also observe the presence of lines where both the $SU(2)$ group and the $U(1)/\mathbb{Z}_q$ group break simultaneously. On these lines the critical behavior belongs to the $O(4)$ universality class. We observe here an enlargement of the global symmetry group: the $O(4)$ group is the smallest simple group that has $U(2)$ as one of its subgroups. The symmetry enlargement is strictly related to the irrelevance of the \mathbb{Z}_q gauge interactions, as the model we consider is $O(4)$ invariant in the absence of gauge fields: the addition of \mathbb{Z}_q gauge fields represents an irrelevant perturbation (in the renormalization-group sense) of the $O(4)$ vector fixed point.

We have then considered the model with $Q = 2$ and $N_f = 15$. In the compact $U(1)$ gauge model, for these values of the parameters, there is a transition line where the $SU(N_f)$ symmetry is broken and transitions belong to the *charged* AH universality class [17, 18]. Gauge fields are crucial at the transition and the effective field theory model that describes the universal features of the transition is the continuum AH model with explicit gauge fields. We have studied whether these charged transitions also occur when the $U(1)$ group is replaced by one of its \mathbb{Z}_q subgroups, i.e. if there is a gauge symmetry enlargement, as it occurs for global $U(1)$ symmetries. Our numerical Monte Carlo results for $q = 6$ and $q = 10$ give a negative answer. In the compact models with discrete gauge groups that we consider, the transitions along the line where the $SU(N_f)$ symmetry is broken are always of first order. Apparently, in the compact AH model it is crucial that the microscopic model is $U(1)$ gauge invariant to observe the charged behavior, at variance with what happens along the *uncharged* transition lines, where the exact gauge symmetry plays little role.

Simulations were performed at the INFN Computation Center of the INFN, Sezione di Roma.

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