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L_∞ morphisms and semiregularity

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Introduction

The starting point of this thesis is the Buchweitz–Flenner semiregularity map, introduced in 1999 by Buchweitz and Flenner [15, 16] and generalising the semiregularity maps investigated by Severi, Kodaira and Spencer, and Bloch. For a coherent sheaf $\mathcal{F}$ on a complex manifold X the Buchweitz–Flenner semiregularity map

$$\sigma\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \prod_{k\geq 0} H^{k+2}(X,\Omega^k_X)$$

is defined in terms of the Atiyah class $\operatorname{At}(\mathcal{F}) \in \operatorname{Ext}^1_X(\mathcal{F},\mathcal{F}\otimes\Omega^1_X)$ of $\mathcal{F}$, introduced in 1957 by Atiyah [3] as the obstruction to the existence of a holomorphic connection on $\mathcal{F}$, namely a $\mathbb{C}$-linear map of sheaves

$$\nabla\colon \mathcal{F} \to \mathcal{F}\otimes\Omega^1_X$$

such that the Leibniz rule holds:

$$\nabla(f\cdot e) = f\cdot\nabla(e) + e\otimes df, \quad \forall f\in\mathcal{O}_X, e\in\mathcal{F},$$

where $d\colon \mathcal{O}_X\to\Omega^1_X$ denotes the universal derivation. More explicitly, via the Yoneda product

$$\operatorname{Ext}^i_X(\mathcal{F},\mathcal{F}\otimes\Omega^p_X)\times\operatorname{Ext}^j_X(\mathcal{F},\mathcal{F}\otimes\Omega^q_X)\to\operatorname{Ext}^{i+j}_X(\mathcal{F},\mathcal{F}\otimes\Omega^{p+q}_X), \qquad (a,b)\mapsto a\smile b,$$

it is possible to construct the powers, and hence the exponential, of the opposite of the Atiyah class

$$\exp(-\operatorname{At}(\mathcal{F}))\in\prod_{q\geq 0}\operatorname{Ext}^q_X(\mathcal{F},\mathcal{F}\otimes\Omega^q_X)\,.$$

When X is smooth, every coherent sheaf has locally finite projective dimension and the trace maps are well-defined

$$\operatorname{Tr}\colon \operatorname{Ext}^i_X(\mathcal{F},\mathcal{F}\otimes\Omega^j_X)\to H^i(X,\Omega^j_X), \qquad i,j\geq 0\,.$$

As proved by Atiyah for vector bundles and by Illusie in the general case [3, 42], when X is a projective manifold, then with respect to the Hodge decomposition in cohomology, the trace of the exponential of the opposite of the Atiyah class is the Chern character

$$\operatorname{ch}(\mathcal{F}) = \operatorname{Tr}(\exp(-\operatorname{At}(\mathcal{F})))\,.$$

The Buchweitz–Flenner semiregularity map is defined by the formula:

$$\sigma\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F})\to\prod_{k\geq 0}H^{k+2}(X,\Omega^k_X), \quad \sigma(x) = \operatorname{Tr}(\exp(-\operatorname{At}(\mathcal{F}))\smile x). \tag{*}$$

In order to explain the importance of this map in deformation theory, it is useful to give a brief history of semiregularity maps.

The concept of semiregularity is due to Severi, who called a curve C in a surface S semiregular if the restriction map $H^0(S,\omega_S)\to H^0(C,\omega_S|_C)$, where ω_S denotes the canonical sheaf of S, is surjective and proved, in modern terminology, that the Hilbert scheme of S is smooth at every semiregular curve [71].

For a smooth hypersurface Z in a compact complex manifold X, Kodaira and Spencer [46] introduced a semiregularity map

$$\sigma_{KS}\colon H^1(Z, \mathcal{N}_{Z|X}) \to H^2(X, \mathcal{O}_X),$$

which is induced by the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(Z) \longrightarrow \mathcal{N}_{Z|X} \longrightarrow 0.$$

They proved that, if σ_{KS} is injective, then the Hilbert scheme of X is smooth at Z. The cohomology group $H^1(Z, \mathcal{N}_{Z|X})$ is an obstruction space for the functor of embedded deformations of $Z \subset X$, see e.g. [35, Example 11.0.1]. This means that for every small extension of finitely generated local Artin C-algebras $0 \to \mathsf{C} \to A \to B \to 0$ and every embedded deformation of Z over $\operatorname{Spec}(B)$, there is a canonically defined obstruction $u \in H^1(Z, \mathcal{N}_{Z|X})$, which vanishes if and only if the deformation lifts to $\operatorname{Spec}(A)$. From Kodaira and Spencer's proof it is possible to infer the more general statement that the obstructions to embedded deformations of Z in X are contained in the kernel of the semiregularity map σ_{KS}, see e.g. [59, Thm. 8.1.5].

In 1972, Bloch defined a semiregularity map

$$\sigma_B\colon H^1(Z, \mathcal{N}_{Z|X}) \to H^{p+1}(X, \Omega_X^{p-1})$$

for every locally complete intersection Z of codimension p in a smooth projective variety X and proved, by using variations of Hodge structures, that every *simple* obstruction to embedded deformations of Z in X is annihilated by σ_B [12].

An obstruction is called simple if it comes from a simple small extension, i.e., from a small extension $0 \to \mathsf{C} \to A \to B \to 0$ such that the differential map $d\colon \mathsf{C} \to \Omega_{A/\mathsf{C}} \otimes_A B$ in the second exact sequence of Kähler differentials is injective. In general, simple obstructions do not generate the whole obstruction space, but in characteristic zero their vanishing is sufficient to ensure smoothness. Hence, if the Bloch semiregularity map is injective, then Z has unobstructed embedded deformations in X.

The Buchweitz–Flenner semiregularity map is important both for the variational Hodge conjecture and for the deformation theory of coherent sheaves. In this work we are interested in its application to deformation theory; Buchweitz and Flenner's result regarding the variational Hodge conjecture is stated in Chapter 3.

The deformation theory of coherent sheaves has been studied extensively, see e.g [8, 24, 35, 39, 62]. Every problem in infinitesimal deformation theory can be formally described by a functor of Artin rings, namely a covariant functor F from the category of local Artin rings to the category of sets such that $F(\mathsf{K}) = \{*\}$. We denote by $\mathbf{Art}_\mathsf{K}$ the category of local Artin K-algebras with residue field K, which can be thought of as infinitesimal thickenings of a point.

Let $\mathcal{F}$ be coherent sheaf on a smooth separated scheme X of finite type over a field K of characteristic zero. An infinitesimal deformation of $\mathcal{F}$ over $A \in \mathbf{Art}_\mathsf{K}$ is given by a coherent sheaf of $\mathcal{O}_X \otimes A$-modules $\mathcal{F}_A$ on $X \times \operatorname{Spec} A$, flat over A, with a morphism of sheaves of $\mathcal{O}_X \otimes A$-modules $\pi\colon \mathcal{F}_A \to \mathcal{F}$ inducing an isomorphism $\mathcal{F}_A \otimes_A \mathsf{K} \cong \mathcal{F}$. Two deformations $\mathcal{F}_A, \mathcal{F}'_A$ are isomorphic if there exists an isomorphism of sheaves of $\mathcal{O}_X \otimes A$-modules $f\colon \mathcal{F}_A \to \mathcal{F}'_A$ that commutes with the morphisms to $\mathcal{F}$. Thus, studying the infinitesimal deformations of $\mathcal{F}$ corresponds to studying the following deformation functor of Artin rings:

Definition 0.0.1. The functor of infinitesimal deformations of the coherent sheaf $\mathcal{F}$ is

$$\operatorname{Def}_\mathcal{F}\colon \mathbf{Art}_\mathsf{K} \to \mathbf{Set},$$

$$\operatorname{Def}_\mathcal{F}(A) = \left\{ (\mathcal{F}_A, \pi) \;\middle|\; \begin{array}{c} \mathcal{F}_A \text{ is a coherent sheaf of } \mathcal{O}_X \otimes A\text{-modules, flat over } A \\ \pi\colon \mathcal{F}_A \to \mathcal{F} \text{ induces an isomorphism } \mathcal{F}_A \otimes_A \mathsf{K} \cong \mathcal{F} \end{array} \right\} / \sim.$$

It is well known that the tangent space to the functor $\mathrm{Def}_{\mathcal{F}}$ is given by $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{F})$, and $\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})$ is a complete obstruction space.

The Buchweitz–Flenner semiregularity map is connected to the deformation theory of coherent sheaves, in the same way as the semiregularity maps of Severi, Kodaira–Spencer and Bloch are connected with embedded deformations of a subvariety. Explicitly, we have this key result:

Theorem 0.0.2 (Buchweitz–Flenner). *The semiregularity map of a coherent sheaf $\mathcal{F}$ on a smooth projective variety X annihilates all* simple *obstructions to deformations of $\mathcal{F}$. In particular, if the Buchweitz-Flenner map σ defined by Equation (*) is injective, then $\mathcal{F}$ has unobstructed deformations.*

This theorem relies on the fact that in characteristic zero the vanishing of simple obstructions is enough to ensure the smoothness of the deformation functor, as does Bloch's result.

Buchweitz and Flenner left open the problem of whether their semiregularity map annihilates all obstructions to deformations of a coherent sheaf, but they conjectured that this should be true and suggested a strategy to prove it. This strategy, outlined in [16], is to realise each component of the semiregularity map

$$\sigma_k\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^{k+2}(X,\Omega^k_X), \quad \sigma_k(x) = \frac{(-1)^k}{k!}\,\mathrm{Tr}(\mathrm{At}(\mathcal{F})^k x), \qquad k \geq 0,$$

as the obstruction map of a morphism of deformation theories with unobstructed target, which would automatically imply the annihilation of all obstructions.

This can be done easily for 0th component of the semiregularity map, which is just the trace map, recovering a result by Mukai [64] and Artamkin [2]:

Theorem 0.0.3 (Artamkin). *Let $\mathcal{F}$ be a coherent sheaf on a complex projective manifold X. Then the 0th semiregularity map (=trace) $\sigma_0 = \mathrm{Tr}\colon \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \to H^2(X,\mathcal{O}_X)$ annihilates all obstructions to deformations of $\mathcal{F}$.*

This thesis is based on on a series of articles written together with Ruggero Bandiera and Marco Manetti [4, 5] with Marco Manetti [50] and alone [49], where the main goal was to employ the strategy suggested by Buchweitz and Flenner for all the higher components of the semiregularity map, i.e., to prove that each $\sigma_k\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^{k+2}(X,\Omega^k_X)$ is the obstruction map of a morphism of deformation theories with unobstructed target, and hence it annihilates all the obstructions to deformations of the coherent sheaf $\mathcal{F}$.

Buchweitz and Flenner suggested that the unobstructed target should be given by an intermediate Jacobian or by Deligne cohomology.

Intermediate Jacobians have been used in [26] and [40] as the target of the Abel–Jacobi map and of the Bloch semiregularity map. More precisely, in [26] Fiorenza and Manetti proved that the Abel–Jacobi map is the tangent map of a morphism of deformation theories, where the target is an intermediate Jacobian, and in [40] Iacono and Manetti proved that the Bloch semiregularity map for a locally complete intersection subvariety with extendable normal bundle is the obstruction map of a morphism of deformation theories with target an intermediate Jacobian, and hence that it annihilates all the obstructions to embedded deformations.

In the setting of derived algebraic geometry, Pridham [68] proved that the Buchweitz–Flenner semiregularity map can be realised as the tangent of a generalised Abel–Jacobi map on the derived moduli stack of perfect complexes on X, with target given by an analogue of Deligne cohomology, which entails that, for every coherent sheaf $\mathcal{F}$ on a complex projective manifold, the semiregularity map σ annihilates all obstructions.

In characteristic zero, the homotopy category of DG-Lie algebras is one of the possible frameworks to study deformation theory. This approach, due to Deligne, Drinfeld and others, which follows the principle *"In characteristic 0, a deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG-Lie algebras giving the same deformation theory"*, has been investigated thoroughly by Goldman and Millson, Hinich, Kontsevich, Manetti and many others [33, 37, 47, 55, 59], and has been formalised independently by Lurie and Pridham [52, 67]. In this setting, a deformation functor corresponds to a quasi-isomorphism class of DG-Lie algebras, and a morphism of deformation functors corresponds to a morphism in the homotopy category of DG-Lie algebras, or equivalently to an L_∞ morphism between DG-Lie algebras.

The relation between DG-Lie algebras and functors of Artin rings is obtained via Maurer–Cartan equation and gauge equivalence. Precisely, given a DG-Lie algebra L and $A \in \mathbf{Art}_{\mathsf{K}}$ with maximal ideal $\mathfrak{m}_A$, the deformation functor associated to L is defined as

$$\mathrm{Def}_L\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}, \quad \mathrm{Def}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\} / \sim_{\text{gauge}} .$$

The strategy of this approach is to find a DG-Lie algebra L controlling a geometric deformation problem, namely such that Def_L is isomorphic to the deformation functor of the geometric problem considered. The DG-Lie algebra L then contains information about the deformation problem: for instance, the first cohomology group $H^1(L)$ is equal to the Zariski tangent space of the local moduli space, while the second cohomology group $H^2(L)$ is a complete obstruction space.

For instance, the deformation theory of a coherent sheaf $\mathcal{F}$ is controlled by $\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ considered as an element in the homotopy category of differential graded Lie algebras, see e.g. [24, 39, 62]. When X is a complex manifold, if $\mathcal{F}$ admits a finite locally free resolution (e.g. if X is projective)

$$0 \to \mathcal{E}^{-n} \to \cdots \to \mathcal{E}^0 \to \mathcal{F} \to 0$$

then a representative of $\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ is given by the Dolbeault complex $A^{0,}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$.

Morphisms of DG-Lie algebras and more generally L_∞ morphisms between DG-Lie algebras induce morphisms between the associated deformation functors. They also induce morphisms in cohomology, giving in degrees 1 and 2 the tangent and obstruction map respectively.

In this framework, to show that each component of the Buchweitz–Flenner semiregularity map is the obstruction map of a morphism of deformation theories, we need to show that there exists a sequence of L_∞ morphisms between DG-Lie algebras whose linear components induce in cohomology the components of the semiregularity map. If the DG-Lie algebra which is the target of this L_∞ morphism is abelian, i.e., it has trivial bracket, then its associated deformation functor is unobstructed and we automatically obtain that each component of the semiregularity map annihilates all obstructions to deformations of the coherent sheaf.

In view of the discussion about the unobstructed target and intermediate Jacobians, the goal of our works [4, 49, 50] was to construct a sequence of L_∞ morphisms whose linear components induce in cohomology the components of the *modified Buchweitz–Flenner semiregularity map*

$$\tau_k\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \xrightarrow{\sigma_k} H^{k+2}(X,\Omega^k_X) = H^2(X,\Omega^k_X[k]) \xrightarrow{i_k} \mathsf{H}^2(X,\Omega^{\leq k}_X[2k]), \quad k \geq 0,$$

where $\Omega^{\leq k}_X = (\oplus_{i=0}^k \Omega^i_X[-i], \partial)$ denotes the truncated holomorphic de Rham complex and i_k is induced by the inclusion of complexes $\Omega^k_X[k] \subset \Omega^{\leq k}_X[2k]$.

The main result of [4] was the construction of canonical L_∞ liftings

$$\sigma^k\colon A^{0,}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \qquad A^{\leq k,}_X[2k]$$

of the modified Buchweitz–Flenner semiregularity maps for a coherent sheaf $\mathcal{F}$ equipped with a finite locally free resolution $\mathcal{E}$ on a complex manifold X. Then the modified Buchweitz–Flenner semiregularity maps are obstruction maps of a morphism of deformation theories with unobstructed target, and we obtain:

Theorem 0.0.4 (=Corollaries 4.5.7 and 4.5.8)**.** *Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X admitting a locally free resolution. Then for every $k \geq 0$ the semiregularity map*

$$\tau_k\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \mathsf{H}^{2+2k}(X,\Omega_X^{\leq k}), \qquad x \mapsto \frac{1}{k!}\operatorname{Tr}(\operatorname{At}(\mathcal{F})^k \cdot x),$$

annihilates obstructions to deformations of $\mathcal{F}$.

If the Hodge to de Rham spectral sequence of X degenerates at E_1, then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map

$$\sigma_k\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^{k+2}(X,\Omega_X^k), \qquad x \mapsto \frac{1}{k!}\operatorname{Tr}(\operatorname{At}(\mathcal{F})^k \cdot x).$$

This was achieved by considering curved DG-pairs, abstract algebraic structures which encode the geometric situation of a complex of locally free sheaves equipped with a connection of type $(1,0)$. A curved DG-pair is the data of a curved DG-algebra and of a Lie ideal which is closed for the derivation and contains the curvature. It is possible to associate to a curved DG-pair an Atiyah class and abstract semiregularity maps, and to introduce Chern–Simons classes, which we used to construct L_∞ liftings of the abstract semiregularity maps.

This construction of L_∞ liftings of semiregularity maps can be also employed effectively in other contexts. Consider the situation of a Lie algebroid $\mathcal{A}$ over a smooth separated scheme X of finite type over K, i.e., the data of a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{A}$ equipped with a K-linear bracket $[-,-]\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and a morphism of sheaves of $\mathcal{O}_X$-modules $a\colon \mathcal{A}\to\Theta_X$, which commutes with brackets, and such that the Leibniz rule holds:

$$[l,fm] = a(l)(f)m + f[l,m], \quad \forall l,m\in\mathcal{A},\ f\in\mathcal{O}_X.$$

Define a Lie pair $(\mathcal{L},\mathcal{A})$ of Lie algebroids to be an inclusion of Lie algebroids $\mathcal{A}\subset\mathcal{L}$ such that the quotient is locally free. An $\mathcal{A}$-module is a locally free sheaf $\mathcal{E}$ on X equipped with a flat $\mathcal{A}$-connection ∇, namely a morphism of $\mathcal{O}_X$-modules

$$\nabla\colon \mathcal{A}\to \mathcal{H}om_{\mathsf{K}}(\mathcal{E},\mathcal{E}), \quad l\mapsto\nabla_l$$

$$\nabla_l(fe) = a(l)(f)e + f\nabla_l(e), \quad \forall f\in\mathcal{O}_X,\ l\in\mathcal{A},\ e\in\mathcal{E}$$

with the property that $\nabla_l\nabla_m - \nabla_m\nabla_l = \nabla_{[l,m]}$ for all $l,m\in\mathcal{A}$.

Given an $\mathcal{A}$-module $(\mathcal{E},\nabla)$ and a Lie pair $(\mathcal{L},\mathcal{A})$, it is possible to define the Atiyah class

$$\operatorname{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \in \mathsf{H}^1(\mathcal{A};(\mathcal{L}/\mathcal{A})\ \ \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$$

as the primary obstruction to the extension of ∇ to a flat $\mathcal{L}$-connection. Here, for any $\mathcal{A}$-module $(\mathcal{F},\nabla\ )$, we denote by $\mathsf{H}\ (\mathcal{A};\mathcal{F})$ the cohomology of the complex $(\Omega\ (\mathcal{A})\otimes\mathcal{F},\nabla\ )$, where $\Omega\ (\mathcal{A})$ is the de Rham DG-algebra of $\mathcal{A}$. More precisely, the Atiyah class $\operatorname{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is the obstruction to the extension of ∇ to an $\mathcal{L}$-connection compatible with the $\mathcal{A}$-module structure, i.e., to a connection $\nabla : \mathcal{L}\to\mathcal{H}om_{\mathsf{K}}(\mathcal{E},\mathcal{E})$ such that $[\nabla_l,\nabla_a] = \nabla_{[l,a]}$ for every $l\in\mathcal{L}$ and $a\in\mathcal{A}$.

In [5] we constructed L_∞ liftings of semiregularity maps associated to a locally free $\mathcal{A}$-module $\mathcal{E}$ and a Lie pair $(\mathcal{L},\mathcal{A})$, which are defined as

$$\tau_k\colon \mathsf{H}^2(\mathcal{A};\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \to \mathsf{H}^{2+k}\left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \right), \quad \tau_k(x) = \frac{1}{k!}\operatorname{Tr}(\operatorname{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x).$$

We also proved that the DG-Lie algebra of derived sections of the sheaf of DG-Lie algebras $\Omega\ (\mathcal{A})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ controls the deformations of the $\mathcal{A}$-module $\mathcal{E}$, and then, by the principles explained above, the semiregularity maps τ_k annihilate all obstructions to the deformations of the $\mathcal{A}$-module $\mathcal{E}$, provided that a certain spectral sequence associated to the Lie pair degenerates at E_1.

Chapter 1 The first chapter contains some algebraic preliminaries: the basic definitions and theory of DG-Lie algebras, graded coalgebras, L_∞-algebras and L_∞ morphisms. It also contains a brief recall of the Thom–Whitney totalisation of a semicosimplicial complex of vector spaces.

Chapter 2 The second chapter begins with a brief introduction to deformation functors and obstruction theory, with an emphasis on simple obstructions. The deformation functors associated to DG-Lie algebras and semicosimplicial Lie algebras are described. The last section concerns the deformation theory of coherent sheaves, and three DG-Lie algebras controlling this deformation problem are given.

Chapter 3 This chapter consists of a history of the semiregularity maps of Severi, Kodaira–Spencer, Bloch and Buchweitz–Flenner and an account of their importance in deformation theory. The annihilation of all obstructions for the Buchweitz–Flenner semiregularity map is discussed thoroughly.

Chapter 4 The core of this part is based on [4] and it contains the construction of a sequence of canonical L_∞ morphisms associated to a curved DG-pair via Chern–Simons classes. A particular case of this construction, where the curved DG-pair is obtained via a connection of type $(1,0)$ on a complex of locally free sheaves on a complex manifold, allows to construct canonical L_∞ liftings of all the components of the Buchweitz–Flenner semiregularity map, and therefore to prove that the semiregularity map annihilates all obstructions to deformations of a coherent sheaf on a complex manifold. Also contained in this chapter is a part based on [50], where we used connections of type $(1,0)$ on complexes of locally free sheaves to construct a lifting of the first component of the semiregularity map via an explicit computation.

Chapter 5 This chapter is based on [49], where the results of [50] were extended to the algebraic case. The situation considered is that of a transitive DG-Lie algebroid $(\mathcal{A},\rho)$ over a smooth separated scheme X of finite type over a field K of characteristic 0. Using simplicial methods, it is possible to define a notion of connection on the kernel of the anchor map ρ, and to construct an L_∞ morphism between DG-Lie algebras $f\colon \mathsf{R}\Gamma(X,\mathrm{Ker}\rho) \quad \mathsf{R}\Gamma(X,\Omega_X^{\leq 1}[2])$ associated to a connection and to a cyclic form on the DG-Lie algebroid, from which one obtains a lifting of the first component of the semiregularity map.

Chapter 6 The last chapter is based on [5] and it concerns more general semiregularity maps, defined for a Lie pair $(\mathcal{L},\mathcal{A})$ and a locally free module over a Lie algebroid $\mathcal{A}$ on a smooth separated scheme of finite type over a field K of characteristic zero.. We determine a DG-Lie algebra controlling the deformations of the $\mathcal{A}$-module, and prove that these semiregularity maps annihilate the obstructions if a certain spectral sequence associated to the Lie pair degenerates at E_1. By considering the trivial Lie pair $(\Theta_X,0)$ one can recover the results of the Chapter 4 for a locally free sheaf in the algebraic setting.

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Notation

By K we denote a fixed field of characteristic 0; unless otherwise specified every (graded) vector space is intended over K.

The term differential graded (DG) means graded over the integers and with differential of degree +1. The degree of a homogeneous element x in a graded vector space will be denoted $\bar{x}$. We adopt the Grothendieck–Verdier formalism for degree shifting: given a DG-vector space $(V = \oplus_n V^n, d_V)$ and an integer p, we define the DG-vector space $(V[p], d_{V[p]})$ by setting $V[p]^n = V^{n+p}$, $d_{V[p]} = (-1)^p d_V$.

All rings are commutative and unitary, unless specified. For any local ring R, we denote by $\mathfrak{m}_R$ its maximal ideal. Let $\mathbf{Art}_{\mathsf{K}}$ denote the category with objects Artin local K-algebras with residue field K, and morphisms given by local morphisms of K-algebras.

If L is a DG-Lie algebra, $H\;(L)$ always denotes the cohomology of the underlying complex of vector spaces, which inherits a graded bracket from the one on L.

For every pair of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we denote by $\mathcal{H}om_{\mathsf{K}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the sheaves of K-linear morphisms and $\mathcal{O}_X$-linear morphisms respectively. The $\mathcal{O}_X$-module structure on $\mathcal{G}$ induces an $\mathcal{O}_X$-module structure both on $\mathcal{H}om_{\mathsf{K}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. We also write $\mathcal{E}nd_{\mathsf{K}}(\mathcal{F})$ and $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ for $\mathcal{H}om_{\mathsf{K}}(\mathcal{F}, \mathcal{F})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ respectively.

For two complexes of $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{F}$ we denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ the graded sheaf of $\mathcal{O}_X$-linear morphisms

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = \bigoplus_i \mathcal{H}om^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}), \quad \mathcal{H}om^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = \prod_j \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^j, \mathcal{F}^{i+j}).$$

For a complex manifold, $\mathcal{A}^{p,q}_X$ denotes the sheaf of differential forms of type p, q, and $\mathcal{A}^{p,q}_X(\mathcal{E})$ denotes the sheaf of differential forms of type p, q with coefficients in a locally free sheaf $\mathcal{E}$. The global sections of these sheaves will be denoted by $A^{p,q}_X$ and $A^{p,q}_X(\mathcal{E})$ respectively.

Chapter 1

DG-Lie algebras and L_∞ morphisms

In characteristic zero, the homotopy category of DG-Lie algebras is one of the possible frameworks to study deformation theory; this chapter contains a basic introduction to DG-Lie algebras and L_∞ morphisms, with the objective of deformation theory in view. Deformation functors associated to DG-Lie algebras will be treated in the next chapter.

In the first section, the basic definitions and examples of DG-Lie algebras are given. The third section concerns L_∞ algebras and morphisms; one definition of L_∞ morphism used here relies on the properties of coalgebras and their coderivations, described in Section 1.2. In the last section, we describe semicosimplicial objects and review the definition and of some of the main properties of the Thom–Whitney totalisation functor.

The main reference for this chapter is [59]. For more details on the Thom–Whitney totalisation we refer also to [23, 24, 27, 40].

1.1 DG-Lie algebras

A graded vector space is a vector space with a $\mathbb{Z}$-graded direct sum decomposition $V = \bigoplus_{n \in \mathbb{Z}} V^n$. If $V = \bigoplus_{n \in \mathbb{Z}} V^n$ is a graded vector space, we denote by $\bar{a}$ the degree of a non-zero homogeneous element a: in other words $\bar{a} = n$ whenever $a \neq 0$ and $a \in V^n$. It is implicitly assumed that if a formula contains the degree symbols $\bar{a}, \bar{b}, \dots$ then all the elements $a, b, \dots$ involved are homogeneous and different from 0.

Definition 1.1.1. A differential graded vector space (a DG-vector space for short), or complex of vector spaces, is the data of a graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V^n$ together with a linear map $d\colon V \to V$, called differential, such that $d(V^n) \subset V^{n+1}$ for every n and $d^2 = dd = 0$.

$$\cdots \longrightarrow V^{i-1} \xrightarrow{d} V^i \xrightarrow{d} V^{i+1} \longrightarrow \cdots$$

A morphism $f\colon (V, d_V) \to (W, d_W)$ of DG-vector spaces is a linear map $f\colon V \to W$ such that $f(V^n) \subset W^n$ for every n and $d_W f = f d_V$.

$$\begin{tikzcd}
\cdots \arrow[r] & V^{i-1} \arrow[r, "d_V"] \arrow[d, "f"] & V^i \arrow[r, "d_V"] \arrow[d, "f"] & V^{i+1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & W^{i-1} \arrow[r, "d_W"'] & W^i \arrow[r, "d_W"'] & W^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

A complex of vector spaces (V, d) is called bounded above if there exists $n \in \mathbb{Z}$ such that $V^k = 0$ for all $k > n$, bounded below if there exists $m \in \mathbb{Z}$ such that $V^k = 0$ for all $k < m$, and bounded if it is bounded both above and below.

The tensor product of differential graded vector spaces is defined as follows:

$$V \otimes W = \bigoplus_{n \in \mathbb{Z}} (V \otimes W)^n, \quad (V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j,$$

with differential $d_{V\ W}(v \otimes w) = d_V v \otimes w + (-1)^{\overline{v}} v \otimes d_W w$.

The Hom complex of DG-vector spaces V, W is the DG-vector space

$$\mathrm{Hom}_{\mathsf{K}}(V, W) = \bigoplus_{n\ \mathsf{Z}} \mathrm{Hom}^n_{\mathsf{K}}(V, W), \quad \mathrm{Hom}^n_{\mathsf{K}}(V, W) = \prod_{i\ \mathsf{Z}} \mathrm{Hom}_{\mathsf{K}}(V^i, W^{i+n})$$

with the differential

$$df(v) = d_W(f(v)) - (-1)^{\overline{f}} f(d_V v).$$

For every DG-vector space (V, d), $Z\ (V) = \mathrm{Ker}\, d$ denotes the graded subspace of cocycles, $B\ (V) = d(V)$ denotes the graded subspace of coboundaries and $H\ (V) = Z\ (V)/B\ (V)$ is the cohomology of V. The cohomology of V can be considered as a DG-vector space with trivial differential. A morphism $f\colon V \to W$ of DG-vector spaces naturally induces a map in cohomology $f\colon H\ (V) \to H\ (W)$. Morphisms of DG-vector spaces $V \to W$ correspond exactly to the elements of $Z^0(\mathrm{Hom}_{\mathsf{K}}(V, W))$.

Definition 1.1.2. 1. A morphism of DG-vector spaces $f\colon V \to W$ is a quasi-isomorphism if it induces an isomorphism in cohomology, i.e. if $f\colon H\ (V) \to H\ (W)$ is an isomorphism.

2. A DG-vector space V is called acyclic if $H\ (V) = 0$.

3. Two morphisms of DG-vector spaces $f, g\colon V \to W$ are homotopic if there exists $h \in \mathrm{Hom}^{-1}(V, W)$ such that $f - g = d_W h + h d_V$.

4. A DG-vector space V is contractible if the identity is a coboundary in $\mathrm{Hom}\ (V, V)$, i.e there exists $h \in \mathrm{Hom}^{-1}(V, V)$ such that $dh + hd = \mathrm{Id}_V$. It is equivalent to saying that the identity is homotopic to the zero morphism.

Given a DG-vector space (V, d_V) and an integer p we define the DG-vector space $(V[p], d_{V[p]})$ by setting

$$V[p]^n = V^{n+p}, \qquad d_{V[p]} = (-1)^p d_V.$$

For instance, $\mathsf{K}[-n]$ is the complex that has K in degree n and 0 in degrees different from n. The tautological map $s\colon V \to V[-1]$ of degree 1, defined in each degree n as the identity map $V^n \to V[-1]^{n+1} = V^n$ is called a suspension; more generally, for any integer p there exists a tautological morphism $s\colon V \to V[p]$ of degree $-p$, defined in each degree n as the identity map $V \to V[p]^{n-p} = V^n$; the definition of $d_{V[p]}$ implies that s^{-p} is a cocycle in $\mathrm{Hom}\ (V, V[p])$.

Definition 1.1.3. A double complex or bicomplex of vector spaces C is a family $\{C^{p,q}\}$, $p, q \in \mathsf{Z}$, of vector spaces, together with maps

$$d^v\colon C^{p,q} \to C^{p,q+1}, \quad d^h\colon C^{p,q} \to C^{p+1,q}, \quad \forall\ p, q \in \mathsf{Z},$$

called vertical and horizontal differential respectively, such that

$$(d^v)^2 = (d^h)^2 = d^v d^h + d^h d^v = 0.$$

$$\begin{tikzcd}
 & \cdots & \cdots & \\
\cdots \arrow[r] & C^{p,q+1} \arrow[u] \arrow[r, "d^h"] & C^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{p,q} \arrow[u, "d^v"] \arrow[r, "d^h"'] & C^{p+1,q} \arrow[u, "d^v"'] \arrow[r] & \cdots \\
 & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}$$

Definition 1.1.4. Define the product total complex $\mathrm{Tot}^{\prod}(C)$ associated to a double complex C as

$$\mathrm{Tot}^{\prod}(C)^n = \prod_{p+q=n} C^{p,q},$$

with differential the sum of the horizontal and vertical differentials. Define the sum total complex $\mathrm{Tot}\ \ (C)$ as

$$\mathrm{Tot}\ \ (C)^n = \bigoplus_{p+q=n} C^{p,q}, \qquad d = d^v + d^h.$$

Remark 1.1.5. A double complex (C, d^h, d^v) is called a first quadrant double complex if it is concentrated in the first quadrant of the plane, i.e. if $C^{p,q} = 0$ if $p < 0$ or $q < 0$.

Notice that for a first quadrant double complex C one has that $\mathrm{Tot}^{\prod}(C) = \mathrm{Tot}\ \ (C)$, because there are finitely many terms in any diagonal.

Definition 1.1.6. A **differential graded Lie algebra**, or **DG-Lie algebra**, over a field K of characteristic zero is the data of a DG-vector space (L, d) with a bilinear bracket $[-,-]\colon L \times L \to L$ satisfying the following conditions:

1. $[-,-]$ is homogeneous graded skew-symmetric of degree 0. This means that:
   - $[L^i, L^j] \subset L^{i+j}$,
   - $[a, b] + (-1)^{\overline{a}\overline{b}}[b, a] = 0$ for every a, b homogeneous,
2. (Leibniz identity) $d[a, b] = [da, b] + (-1)^{\overline{a}}[a, db]$ for every a, b homogeneous;
3. (Jacobi identity) every triple of homogeneous elements a, b, c satisfies the equality $[a, [b, c]] = [[a, b], c] + (-1)^{\overline{a}\overline{b}}[b, [a, c]]$.

A DG-Lie algebra with trivial differential $d = 0$ is simply referred to as a graded Lie algebra, while a DG-Lie algebra is called **abelian** if its bracket is trivial. A morphism of differential graded Lie algebras is a morphism of DG-vector spaces commuting with brackets.

Example 1.1.7. Every Lie algebra can be considered as a DG-Lie algebra concentrated in degree 0, with trivial differential. If $L = \bigoplus L^i$ is a DG-Lie algebra, L^0 is a Lie algebra in the usual sense.

Definition 1.1.8. A graded algebra over a field K of characteristic zero is the data of a graded vector space A with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, called a product, that satisfies the following conditions:

1. $\overline{ab} = \overline{a} + \overline{b}$;
2. $(ab)c = a(bc)$.

A **differential graded algebra**, or **DG-algebra**, is the data of a graded algebra A as above and of a degree 1 map $d\colon A \to A$, called differential, such that $d^2 = 0$ and the Leibniz rule holds:

$$d(ab) = d(a)b + (-1)^{\overline{a}} a d(b), \quad \forall a, b \in A.$$

A commutative DG-algebra is a DG-algebra such that the graded commutativity holds:

$$ab = (-1)^{\overline{a}\overline{b}} ba, \quad \forall a, b \in A.$$

Remark 1.1.9. Notice that every (differential) graded associative algebra is also a (differential) graded Lie algebra, with bracket given by the graded commutator $[a, b] = ab - (-1)^{\overline{a}\overline{b}} ba$.

Example 1.1.10. Given a DG-Lie algebra L and a commutative differential graded algebra A, the DG-vector space $L \otimes A$ has a natural structure of DG-Lie algebra, with bracket given by

$$[x \otimes a, y \otimes b] = (-1)^{\overline{a}\ \overline{y}}[x, y] \otimes ab.$$

Example 1.1.11. Let V be a DG-vector space, then the Hom complex $\mathrm{Hom}_{\mathsf{K}}(V, V)$ has a natural structure of differential graded Lie algebra, with the bracket equal to the graded commutator

$$[f, g] = fg - (-1)^{\bar{f}\,\bar{g}} gf,$$

and the differential equal to the adjoint operator $[d, -]$, where d is the differential of V.

Example 1.1.12. Let A be a DG-algebra over the field K. The DG-Lie algebra of derivations of A is the DG-Lie subalgebra of $\mathrm{Hom}_{\mathsf{K}}(A, A)$ defined by:

$$\mathrm{Der}_{\mathsf{K}}(A, A) = \bigoplus_n \mathrm{Der}^n_{\mathsf{K}}(A, A) \subset \mathrm{Hom}_{\mathsf{K}}(A, A),$$
$$\mathrm{Der}^n_{\mathsf{K}}(A, A) = \{\phi \in \mathrm{Hom}^n_{\mathsf{K}}(A, A) \mid \phi(ab) = \phi(a)b + (-1)^{n\bar{a}} a\phi(b)\}.$$

Notice that the differential of A is a derivation of degree $+1$.

Let L be a differential graded Lie algebra, then the derivations of L

$$\mathrm{Der}_{\mathsf{K}}(L, L) = \bigoplus_n \mathrm{Der}^n_{\mathsf{K}}(L, L),$$

$$\mathrm{Der}^n_{\mathsf{K}}(L, L) = \{\phi \in \mathrm{Hom}^n_{\mathsf{K}}(L, L) \mid \phi[a, b] = [\phi(a), b] + (-1)^{n\bar{a}}[a, \phi(b)]\}$$

form a DG-Lie algebra, with bracket equal to the graded commutator $[\phi, \psi] = \phi\psi - (-1)^{\bar{\phi}\,\bar{\psi}}\psi\phi$ and differential equal to $[d, -]$.

The cohomology of a DG-Lie algebra is defined as the cohomology of the underlying DG-vector space. For any DG-Lie algebra $(L, d, [-, -])$ and every $x, y \in L$ we have that if $dx = dy = 0$, then $d[x, y] = 0$, and if $dy = 0$, then $[dx, y] = d[x, y]$. Therefore the bracket of L factors to a bracket in $H\ (L)$, inducing a graded Lie algebra structure on the cohomology of L. If $f\colon L \to M$ is a morphism of differential graded Lie algebras, then $f\colon H\ (L) \to H\ (M)$ is a morphism of graded Lie algebras.

Definition 1.1.13. A quasi-isomorphism of DG-Lie algebras is a morphism of DG-Lie algebras which is a quasi-isomorphism of the underlying DG-vector spaces.

Definition 1.1.14. The descending central series $L^{[n]}$, $n \geq 1$, of a DG-Lie algebra L is defined as $L^{[1]} = L$ and

$$L^{[n]} = \mathrm{Span}\{[a_1, [\cdots [a_{n-1}, a_n] \cdots]] \mid a_1, \cdots, a_n \in L\}, \quad n \geq 2.$$

Equivalently, it is defined by the recursive formulas $L^{[1]} = L$ and $L^{[n]} = [L, L^{[n-1]}]$.

A DG-Lie algebra L is **nilpotent** if $L^{[n]} = 0$ for some $n > 0$.

In particular, for every differential graded Lie algebra L and every proper ideal I of an Artin local K-algebra, the DG-Lie algebra $L \otimes I$ is nilpotent.

1.2 Graded coalgebras

Definition 1.2.1. A **graded coalgebra** is the data (C, Δ) of a graded vector space C together with a morphism of graded vector spaces $\Delta\colon C \to C \otimes C$ such that $(\mathrm{Id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id}_C)\Delta$. The map Δ is called coproduct and the above property is called coassociativity.

The twist map $\mathrm{tw}\colon V \otimes W \to W \otimes V$ is defined as

$$\mathrm{tw}(v \otimes w) = (-1)^{\bar{v}\,\bar{w}} w \otimes v, \quad v \in V, w \in W. \tag{1.2.1}$$

Definition 1.2.2. A graded coalgebra (C, Δ) is called cocommutative if $\mathrm{tw} \circ \Delta = \Delta$.

Given a graded coalgebra (C, Δ), using coassociativity, one can define the iterated coproducts $\Delta^n : C \to C^{\ \ n+1}$ as

$$\Delta^0 = \mathrm{Id}_C, \quad \Delta^n = (\mathrm{Id}_C \otimes \Delta^{n-1}) \circ \Delta.$$

Definition 1.2.3. A graded coalgebra (C, Δ) is called conilpotent if $\Delta^n = 0$ for $n >> 0$. It is called locally conilpotent if $C = \bigcup_n \operatorname{Ker} \Delta^n$.

Definition 1.2.4. A morphism of graded coalgebras $F : (C, \Delta) \to (B, \Gamma)$ is a morphism of graded vector spaces $F : C \to B$ such that $\Gamma F = F^{\ \ 2}\Delta$.

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "F"'] & C \otimes C \arrow[d, "F\quad F"] \\
B \arrow[r, "\Gamma"'] & B \otimes B
\end{tikzcd}$$

Definition 1.2.5. Let (C, Δ) be a graded coalgebra. A morphism of graded vector spaces $p : C \to V$ is called a **cogenerator** of C if for every $x \in C$, $c \neq 0$, there exists $n > 0$ such that $p^{\ \ n}\Delta^{n-1}(x) \neq 0$ in $V^{\ \ n}$. Equivalently, $p : C \to V$ is a cogenerator of C if the linear map

$$(p, p^{\ \ 2}\Delta, p^{\ \ 3}\Delta^2, \ldots) : C \longrightarrow \prod_{n>0} V^{\ \ n}$$

is injective. For a cogenerator $p : C \to V$ and a linear map $f : B \to C$, the composition $pf : B \to V$ is called the **corestriction** of f to p.

Proposition 1.2.6. *Let $p : B \to V$ be a cogenerator of a graded coalgebra (B, Γ). Then every morphism of graded coalgebras $\phi : (C, \Delta) \to (B, \Gamma)$ is uniquely determined by its corestriction $p\phi : C \to V$.*

Proof. Given a morphism of graded coalgebras $F : (C, \Delta) \to (B, \Gamma)$ for every $n \geq 0$ one has that

$$\Gamma^n F = F^{\ \ n+1}\Delta^n : C \to B^{\ \ n+1}.$$

Given two morphisms of graded coalgebras $F, G : C \to B$ such that $pF = pG$ we have

$$p^{\ \ n+1}\Gamma^n F = p^{\ \ n+1}F^{\ \ n+1}\Delta^n = (pF)^{\ \ n+1}\Delta^n = (pG)^{\ \ n+1}\Delta^n = p^{\ \ n+1}G^{\ \ n+1}\Delta^n = p^{\ \ n+1}\Gamma^n G$$

and the claim follows. □

Definition 1.2.7. Given a morphism of graded coalgebras $F : (C, \Delta) \to (B, \Gamma)$ the set of F-coderivations of degree n is

$$\mathrm{Coder}^n(C, B; F) := \{Q \in \mathrm{Hom}^n_{\mathbb{K}}(C, B) \mid \Gamma Q = (F \otimes Q + Q \otimes F)\Delta\}$$

and we set

$$\mathrm{Coder}\ \ (C, B; F) = \bigoplus_{n\ \ \mathbb{Z}} \mathrm{Coder}^n(C, B; F).$$

We denote $\mathrm{Coder}\ \ (C, C; \mathrm{Id}_C)$ by $\mathrm{Coder}\ \ (C)$.

Given a graded vector space V, the twist map of (1.2.1)

$$\mathrm{tw} : V \otimes W \to W \otimes V, \qquad \mathrm{tw}(v \otimes w) = (-1)^{\bar{v}\,\bar{w}} w \otimes v,$$

extends naturally, for every $n \geq 0$, to a right action of the symmetric group Σ_n on the nth tensor power of V:

$$\mathrm{tw} : V^{\ \ n} \times \Sigma_n \to V^{\ \ n}.$$

More explicitly, for $v_1, \ldots, v_n$ homogeneous elements and $\sigma \in \Sigma_n$ we have:

$$\mathrm{tw}(v_1 \otimes \cdots \otimes v_n, \sigma) = \pm(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}),$$

where the above sign $\pm$ is equal to the signature of the restriction of σ to the subset of indices i such that v_i has odd degree.

Definition 1.2.8. Given a permutation $\sigma \in \Sigma_n$, a graded vector space V and non-trivial homogeneous elements $v_1, \dots, v_n \in V \setminus \{0\}$, the **Koszul sign** $\varepsilon(\sigma, V; v_1, \dots, v_n) = \pm 1$ is defined by the relation

$$\mathrm{tw}(v_1 \otimes \cdots \otimes v_n, \sigma) = \varepsilon(\sigma, V; v_1, \dots, v_n)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})$$

The **antisymmetric Koszul sign** $\chi(\sigma, V; v_1, \dots, v_n) = \pm 1$ is the product of the Koszul sign and the signature of the permutation:

$$\chi(\sigma, V; v_1, \dots, v_n) = (-1)^{\sigma} \varepsilon(\sigma, V; v_1, \dots, v_n).$$

By convention, if $v_i = 0$ for some index i we set

$$\varepsilon(\sigma, V; v_1, \dots, v_n) = \chi(\sigma, V; v_1, \dots, v_n) = 0\,.$$

Definition 1.2.9 (Shuffles)**.** Given two non-negative integers $p, q \geq 0$ with $p + q > 0$, a (p, q)**-shuffle** is a permutation σ of the set $\{1, \dots, p+q\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q)\ .$$

The subset of (p, q)-shuffles is denoted by $S(p, q) \subset \Sigma_{p+q}$.

The (p, q)-shuffles are $\binom{p+q}{q}$ in number.

Example 1.2.10. For every $n > 0$ one has that $S(0, n) = S(n, 0) = \{\mathrm{Id}\}$, while $S(1, 1) = \Sigma_2$ and the three $(2, 1)$-shuffles are:

$$(1, 2, 3), \qquad (1, 3, 2), \qquad (2, 3, 1)\ .$$

It is possible to associate to a graded vector space V a cocommutative coalgebra, the symmetric coalgebra of V. This is the graded coalgebra $(S(V), \mathfrak{l})$, where $S(V) = \bigoplus_{n>0} V^{\odot n}$, and $\mathfrak{l} \colon S(V) \to S(V) \otimes S(V)$ is given by

$$\mathfrak{l}(v_1 \odot \cdots \odot v_n) = \sum_{a=1}^{n-1} \sum_{\sigma \in S(a, n-a)} \varepsilon(\sigma)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(a)}) \otimes (v_{\sigma(a+1)} \odot \cdots \odot v_{\sigma(n)}), \qquad (1.2.2)$$

where $S(a, n-a)$ denotes the set of $(a, n-a)$-shuffles and ε the symmetric Koszul sign.

The projection map $p_V \colon S(V) \to V$ is a cogenerator.

Lemma 1.2.11. *Every morphism of locally conilpotent cocommutative coalgebras* $F \colon (C, \Delta) \to (S(V), \mathfrak{l})$ *is uniquely determined by its corestriction* $f = p_V F \colon C \to V$.

Every coderivation of a reduced symmetric coalgebra $Q \in \mathrm{Coder}\ (S(V))$ *is uniquely determined by its corestriction* $p_V Q = \sum_{n>0} q_n \colon S(V) \to V$.

For a proof we refer to [59, 11.5].

Definition 1.2.12. A **differential graded coalgebra** is the data of a graded coalgebra C together with a coderivation $d \in \mathrm{Coder}^1(C, C)$, called a differential, such that $d^2 = 0$. A morphism of differential graded coalgebras is a morphism of graded coalgebras commuting with differentials.

Definition 1.2.13. The bar construction of a DG-Lie algebra $(L, d, [-,-])$ is the cocommutative DG-coalgebra $(S(L[1]), \mathfrak{l}, Q)$, where $S(L[1])$ is the symmetric coalgebra of the graded vector space $L[1]$, with the coproduct defined in (1.2.2) and the coderivation $Q \colon S(L[1]) \to S(L[1])$ is defined via Lemma 1.2.11 by

$$q_i \colon L[1]^{\odot i} \to L[1], \quad q_1(x) = -dx, \quad q_2(x, y) = (-1)^{\overline{x}}[x, y], \quad q_i = 0 \ \ \forall i \geq 3.$$

1.3 L_∞ and $L_\infty[1]$-algebras

Definition 1.3.1. An L_∞ **structure** on a graded vector space L is a sequence of skew-symmetric maps

$$l_n\colon L^{\ n} \to L, \qquad \deg(l_n) = 2-n, \quad n>0,$$

such that for every $n>0$ and every sequence of homogeneous vectors $x_1,\dots,x_n$ we have:

$$\sum_{k=1}^{n}(-1)^{n-k}\sum_{\sigma\ \ S(k,n-k)}\chi(\sigma)\, l_{n-k+1}(l_k(x_{\sigma(1)},\dots,x_{\sigma(k)}),x_{\sigma(k+1)},\dots,x_{\sigma(n)})=0, \tag{1.3.1}$$

where $S(k,n-k)$ are the $(k,n-k)$ shuffles of Definition 1.2.9 and χ is the antisymmetric Koszul sign. An L_∞**-algebra** $(L,l_1,l_2,\dots)$ is a graded vector space L equipped with an L_∞ structure $l_1,l_2,\dots$.

The first equations of (1.3.1) are, for $n=1,2,3$:

1. $l_1(l_1(x_1))=0$;
2. $\sum_{\sigma\ \ S(2,0)}\chi(\sigma)l_1(l_2(x_{\sigma(1)},x_{\sigma(2)}))-\sum_{\sigma\ \ S(1,1)}\chi(\sigma)l_2(l_1(x_{\sigma(1)}),x_{\sigma(2)})=0$;
3. $\sum_{\sigma\ \ S(1,2)}\chi(\sigma)l_3(l_1(x_{\sigma(1)}),x_{\sigma(2)},x_{\sigma(3)})-\sum_{\sigma\ \ S(2,1)}\chi(\sigma)l_2(l_2(x_{\sigma(1)},x_{\sigma(2)}),x_{\sigma(3)})+$
   $+\sum_{\sigma\ \ S(3,0)}\chi(\sigma)l_1(l_3(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}))=0.$

Remark 1.3.2. Notice that from the first equation for an L_∞-algebra $(L,l_1,l_2,\dots)$ we have $\deg(l_1)=1$ and $l_1^2=0$. Therefore (L,l_1) is a DG-vector space, and it is possible to consider its cohomology $H^{\ }(L)$.

Example 1.3.3. Differential graded Lie algebras are L_∞-algebras with l_1 equal to the differential, l_2 equal to the Lie bracket and $l_n=0$ for every $n>2$. In fact, the axioms of a differential graded Lie algebra $(L,d,[-,-])$ over a field of characteristic 0:

1. $d(d(x_1))=0$;
2. $d[x_1,x_2]-([dx_1,x_2]-(-1)^{\bar{x}_1\bar{x}_2}[dx_2,x_1])=0$;
3. $[[x_1,x_2],x_3]-(-1)^{\bar{x}_2\bar{x}_3}[[x_1,x_3],x_2]+(-1)^{\bar{x}_1(\bar{x}_2+\bar{x}_3)}[[x_2,x_3],x_1]=0$;

may be rewritten as:

1. $d(d(x_{\sigma(1)}))=0$;
2. $\sum_{\sigma\ \ S(2,0)}\chi(\sigma)d[x_{\sigma(1)},x_{\sigma(2)}]-\sum_{\sigma\ \ S(1,1)}\chi(\sigma)[dx_{\sigma(1)},x_{\sigma(2)}]=0$;
3. $\sum_{\sigma\ \ S(2,1)}\chi(\sigma)[[x_{\sigma(1)},x_{\sigma(2)}],x_{\sigma(3)}]=0.$

Definition 1.3.4. The **décalage isomorphisms** of a graded vector space V are the linear isomorphisms

$$\text{déc}\colon \operatorname{Hom}^n_{\mathbb{K}}(V^{\ k},V)\longrightarrow \operatorname{Hom}^{n-k+1}_{\mathbb{K}}((sV)^{\ k},sV), \qquad n,k\in\mathbb{Z},\ k\geq 0,$$

defined by imposing the commutativity of the diagrams

$$\begin{tikzcd} V^{\ k} \arrow[r, "f"] \arrow[d, "s^{\ k}"] & V \arrow[d, "s"] \\ (sV)^{\ k} \arrow[r, "\text{déc}(f)"] & sV \end{tikzcd}$$

or, equivalently, by the formula

$$\text{déc}(f)(sv_1,\dots,sv_k)=(-1)^{\sum_i (k-i)\bar{v_i}}sf(v_1,\dots,v_k)\,.$$

The importance of décalage isomorphisms is expressed by the following proposition.

Proposition 1.3.5. *The décalage isomorphisms exchange symmetric map into skew-symmetric maps and conversely. For every $n,k\in\mathsf{Z}$, $k\geq 0$, we have two linear isomorphisms:*

$$\begin{aligned}\text{déc}\colon \operatorname{Hom}^n_{\mathsf{K}}(V^{\ k},V)&\longrightarrow \operatorname{Hom}^{n-k+1}_{\mathsf{K}}((sV)^{\ k},sV),\\ \text{déc}\colon \operatorname{Hom}^n_{\mathsf{K}}(V^{\ k},V)&\longrightarrow \operatorname{Hom}^{n-k+1}_{\mathsf{K}}((sV)^{\ k},sV).\end{aligned}\tag{1.3.2}$$

Proof. See [59, 10.6]. □

The equations (1.3.1) take a simpler form after a décalage isomorphism, so it is convenient to encode these new equations into a new algebraic structure called $L_\infty[1]$ structure, which is equivalent to the old one.

Definition 1.3.6. An $L_\infty[1]$ **structure** on a graded vector space V is a sequence of symmetric linear maps

$$q_n\colon V^{\ n}\to V,\qquad \deg(q_n)=1,\quad n>0,$$

such that for every $n>0$ and every sequence of homogeneous vectors $v_1,\dots,v_n$:

$$\sum_{k=1}^{n}\sum_{\sigma\quad S(k,n-k)}\varepsilon(\sigma)\,q_{n-k+1}(q_k(v_{\sigma(1)},\dots,v_{\sigma(k)}),v_{\sigma(k+1)},\dots,v_{\sigma(n)})=0\,.\tag{1.3.3}$$

An $L_\infty[1]$**-algebra** $(V,q_1,q_2,\dots)$ is a graded vector space equipped with an $L_\infty[1]$ structure.

For notational convenience we shall sometimes denote an $L_\infty[1]$-algebra $(V,q_1,q_2,\dots)$ by the pair (V,q), where

$$q\in\operatorname{Hom}^1_{\mathsf{K}}(\oplus_{n\geq1}V^{\ n},V),\qquad q=\sum_{n=1}^{\infty}q_n\,.$$

Lemma 1.3.7. *For every graded vector space V, the opposite of the décalage isomorphisms give a canonical bijection from the set of $L_\infty[1]$ structures on V and the set of L_∞ structures on $sV=V[-1]$:*

$$-\text{déc}\colon\{L_\infty[1]\ \textit{structures on}\ V\}\longrightarrow\{L_\infty\ \textit{structures on}\ sV\}.$$

More explicitly, in the notation above, the bijection is given by:

$$l_k=-\text{déc}(q_k),\qquad l_k(sv_1,\dots,sv_k)=-(-1)^{\sum_i(k-i)\bar{v_i}}sq_k(v_1,\dots,v_k).$$

It is possible to give an equivalent, more concise definition of $L_\infty[1]$-algebras in terms of symmetric coalgebras, see e.g. [56, Chapter IX]:

Definition 1.3.8. An $L_\infty[1]$-algebra is a pair (V,Q), where V is a graded vector space, and $Q\in\operatorname{Coder}(S(V))$ such that $Q^2=\frac{1}{2}[Q,Q]=0$.

Equivalently, $(S(V),Q)$ is a DG-coalgebra, as in Definition 1.2.12

An advantage of this second definition is the possibility of defining L_∞-morphisms easily:

Definition 1.3.9. An L_∞-morphism of $L_\infty[1]$-algebras $f\colon(V,q)\to(W,r)$ is the corestriction of a morphism of symmetric coalgebras $F\colon(S(V),Q)\to(S(W),R)$ such that $FQ=RF$.

Denoting by f_n, $n > 0$, the components of f, i.e.,

$$f = \sum_{n>0} f_n, \qquad f_n \in \mathrm{Hom}^0(V^{\; n}, W),$$

we shall call f_1 the *linear part* of f, f_2 the *quadratic part* and so on.

An L_∞ morphism of $L_\infty[1]$-algebras can be written more explicitly as follows:

Proposition 1.3.10. *Given two $L_\infty[1]$-algebras $(V, q_1, q_2, \dots)$ and $(W, r_1, r_2, \dots)$, a sequence of linear maps $f_n \in \mathrm{Hom}^0(V^{\; n}, W)$ gives an L_∞-morphism $f = \sum f_n$ if and only if for every $v_1, \dots, v_n \in V$ we have*

$$\begin{aligned} &\sum_{i=1}^{n} r_i F_n^i(v_1 \odot \cdots \odot v_n) \\ &\qquad = \sum_{i=1}^{n} \sum_{\sigma \quad S(i,n-i)} \varepsilon(\sigma) f_{n-i+1}(q_i(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \odot \cdots \odot v_{\sigma(n)}), \end{aligned} \tag{1.3.4}$$

where the maps $F_n^i \colon V^{\; n} \to W^{\; i}$ are defined recursively by the formulas $F_n^1 = f_n$ and

$$F_n^i(v_1 \odot \cdots \odot v_n) = \frac{1}{i} \sum_{a=1}^{n-i+1} \sum_{\sigma \quad S(a,n-a)} \varepsilon(\sigma) f_a(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(a)}) \odot F_{n-a}^{i-1}(v_{\sigma(a+1)} \odot \cdots \odot v_{\sigma(n)}).$$

Definition 1.3.11. An **L_∞-morphism** of L_∞-algebras is an L_∞-morphism of the corresponding $L_\infty[1]$-algebras.

Given two L_∞-algebras $(H, h_1, h_2, \dots)$ and $(L, l_1, l_2, \dots)$, via the décalage isomorphisms of Definition 1.3.4:

$$\text{déc}\colon \mathrm{Hom}^0_{\mathbb{K}}(H[1]^{\; n}, L[1]) \longrightarrow \mathrm{Hom}^{1-n}_{\mathbb{K}}(H^{\; n}, L),$$

every L_∞-morphism $g\colon (H, h_1, h_2, \dots) \quad (L, l_1, l_2, \dots)$ is given by a sequence of maps

$$g_n \in \mathrm{Hom}^{1-n}_{\mathbb{K}}(H^{\; n}, L), \qquad n \geq 1,$$

such that the maps $f_n = \text{déc}^{-1}(g_n)$ satisfy the condition of Proposition 1.3.10.

The general expression of the equations satisfied by the maps g_n is complicated and outside the scope of this thesis. However, in the following chapters, it will be useful to consider the special case of a L_∞ morphism between DG-Lie algebras, as in the following definition.

Definition 1.3.12. Let $(V, \delta, [-,-])$ and $(L, d, \{-,-\})$ be DG-Lie algebras over the same field. An L_∞ morphism $g\colon V \quad L$ is a sequence of linear maps $g_n\colon V^{\; n} \to L$, $n \geq 1$, with g_n of degree $1-n$ such that g_1 is a morphism of complexes, while for every $n \geq 2$ and every $v_1, \dots, v_n \in V$ homogeneous we have

$$\begin{aligned} &\frac{1}{2} \sum_{p=1}^{n-1} \sum_{\sigma \quad S(p,n-p)} \chi(\sigma)(-1)^{(1-n+p)(\overline{v_{\sigma(1)}} + \cdots + \overline{v_{\sigma(p)}} - p)} \left\{ g_p(v_{\sigma(1)}, \dots, v_{\sigma(p)}), g_{n-p}(v_{\sigma(p+1)}, \dots, v_{\sigma(n)}) \right\} \\ &\qquad + d g_n(v_1, \dots, v_n) = (-1)^{n-1} \sum_{\sigma \quad S(1,n-1)} \chi(\sigma) g_n(\delta(v_{\sigma(1)}), v_{\sigma(2)}, \dots, v_{\sigma(n)}) \\ &\qquad\qquad + (-1)^{n-2} \sum_{\sigma \quad S(2,n-2)} \chi(\sigma) g_{n-1}([v_{\sigma(1)}, v_{\sigma(2)}], v_{\sigma(3)}, \dots, v_{\sigma(n)}). \end{aligned}$$

When we deal with L_∞ morphisms of DG-Lie algebras where the target L is abelian, $\{-,-\} = 0$ and the above definition reduces to:

Definition 1.3.13. Let $(V, \delta, [-,-])$ be a DG-Lie algebra and (L, d) an abelian DG-Lie algebra. An L_∞ morphism $g\colon V \quad L$ is a sequence of maps $g_n\colon V^{\; n} \to L$, $n \geq 1$, with g_n of degree $1-n$ such that the following conditions C_n, $n = 1, 2, 3, \dots$, are satisfied:

C_1 $g_1\delta = dg_1$;

C_n, $n \geq 2$ for every $v_1, \dots, v_n \in V$ homogeneous we have

$$\begin{aligned} dg_n(v_1, \dots, v_n) = (-1)^{n-1} & \sum_{\sigma \ S(1,n-1)} \chi(\sigma) g_n(\delta v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}) \\ & + (-1)^{n-2} \sum_{\sigma \ S(2,n-2)} \chi(\sigma) g_{n-1}([v_{\sigma(1)}, v_{\sigma(2)}], v_{\sigma(3)}, \dots, v_{\sigma(n)}). \end{aligned}$$

Notice that if $g_n = 0$ for every $n \geq N$ then C_n is trivially satisfied for every $n > N$.

Remark 1.3.14. Notice that condition C_1 entails that the linear component g_1 induces a map in cohomology $g_1 \colon H\ (V) \to H\ (M)$. It is clear that the cohomology $H\ (M)$ of an abelian DG-Lie algebra M is an abelian graded Lie algebra. Condition C_2 can be written as

$$g_1([v_1, v_2]) = dg_2(v_1, v_2) + g_2(\delta v_1, v_2) + (-1)^{\overline{v_1}} g_2(v_1, \delta v_2),$$

which implies that the map induced by g_1 in cohomology is a morphism of graded Lie algebras.

Definition 1.3.15. An L_∞-morphism $f \colon (V, q_1, \cdots) \quad (W, r_1, \cdots)$ is called a weak equivalence if its linear component $f_1 \colon (V, q_1) \to (W, r_1)$ is a quasi-isomorphism of DG-vector spaces.

Theorem 1.3.16. *Two DG-Lie algebras are weakly equivalent as L_∞-algebras if and only if they are quasi-isomorphic as DG-Lie algebras.*

For the proof, we refer to [51, 11.4] or to [59, 12.6].

1.4 Semicosimplicial objects and the Thom–Whitney totalisation

For every integer $n \geq 0$, consider the finite set $[n] = \{0, 1, \cdots, n\}$, equipped with the usual order relation. Let $\overrightarrow{\Delta}$ be the category whose objects are $[0] = \{0\}, [1] = \{0, 1\}, [2] = \{0, 1, 2\}$, etc. and whose morphisms are the strictly monotone maps. For instance, $\mathrm{Mor}_{\overrightarrow{\Delta}}([n-1], [n])$ contains exactly $n + 1$ morphisms called face maps, namely:

$$\delta_k \colon [n-1] \to [n], \quad \delta_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } p \geq k \end{cases} \qquad k = 0, \cdots, n.$$

The face maps satisfy the semicosimplicial identities:

$$\delta_l \delta_k = \delta_{k+1} \delta_l \quad \text{for every } l \leq k.$$

Every strictly monotone map $f \colon [n] \to [n+k], k > 0$, admits a unique factorisation as

$$f = \delta_{i_1} \cdots \delta_{i_k}, \quad n + k \geq i_2 > i_2 > \cdots > i_k \geq 0.$$

Definition 1.4.1. Let $\mathcal{C}$ be a category. A **semicosimplicial object** in $\mathcal{C}$ is a covariant functor $A \colon \overrightarrow{\Delta} \to \mathcal{C}$. Equivalently, a semicosimplicial object A is a diagram

$$A: \qquad A_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} A_1 \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\to\!\!\!\to\!\!\!\to}}} A_2 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \ \cdots$$

where each A_i is an object of $\mathcal{C}$, and, for each $n > 0$, there are $n+1$ face operators $\delta_k \colon A_{n-1} \to A_n$, $k = 0, \cdots, n$, which are morphisms in the category $\mathcal{C}$ and satisfy the semicosimplicial identities.

A morphism of semicosimplicial objects is a natural transformation of functors; equivalently, a morphism $f \colon A \to B$ of semicosimplicial objects is a sequence of morphisms $f_n \colon A_n \to B_n$ such that $\delta_k f_{n-1} = f_n \delta_k$ for every k, n.

Definition 1.4.2. A **semicosimplicial Lie algebra** over a field K is a semicosimplicial object in the category of Lie algebras over K; it is represented by a diagram of Lie algebras

$$\mathfrak{g}: \qquad \mathfrak{g}_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathfrak{g}_1 \underset{\delta_2}{\overset{\delta_0}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\delta_1}}}}} \mathfrak{g}_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

in which every face operator δ_i is a morphism of Lie algebras.

Analogously, a **semicosimplicial DG-Lie algebra** over K is a semicosimplicial object in the category of DG-Lie algebras.

The Thom–Whitney totalisation is a functor from the category of semicosimplicial DG-vector spaces to the category of DG-vector spaces. For every $n \geq 0$ consider

$$A_n = \frac{\mathsf{K}[t_0, \dots, t_n, dt_0, \dots, dt_n]}{(1 - \sum_i t_i, \sum_i dt_i)}$$

the commutative differential graded algebra of polynomial differential forms on the affine standard n-simplex, and the maps

$$\delta_k \colon A_n \to A_{n-1}, \quad 0 \leq k \leq n \qquad \delta_k(t_i) = \begin{cases} t_i & i < k \\ 0 & i = k \\ t_{i-1} & i > k. \end{cases}$$

Definition 1.4.3. The Thom–Whitney totalisation of a semicosimplicial DG-vector space V

$$V: \qquad V_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} V_1 \underset{\delta_2}{\overset{\delta_0}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\delta_1}}}}} V_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

is the DG-vector space

$$\operatorname{Tot}(V) = \{(x_n) \in \prod_{n \geq 0} A_n \otimes_{\mathsf{K}} V_n \mid (\delta_k \otimes \operatorname{Id})x_n = (\operatorname{Id} \otimes \delta_k)x_{n-1} \text{ for every } 0 \leq k \leq n\},$$

with differential induced by the one on $\prod_{n \geq 0} A_n \otimes V_n$. To simplify notation, we will sometimes denote this differential by $d_{\mathsf{Tot}} = d_A + d_V$, where d_A denotes the differential of polynomial differential forms, and d_V the differential on V.

If $f \colon V \to W$ is a morphism of semicosimplicial DG-vector spaces, then $\operatorname{Tot}(f) \colon \operatorname{Tot}(V) \to \operatorname{Tot}(W)$ is defined as the restriction of the map

$$\prod \operatorname{Id} \otimes f \colon \prod_{n \geq 0} A_n \otimes_{\mathsf{K}} V_n \to \prod_{n \geq 0} A_n \otimes_{\mathsf{K}} W_n.$$

The Tot functor is exact: given semicosimplicial DG-vector spaces V, W, Z and morphisms $f \colon V \to W$, $g \colon W \to Z$ such that for every $n \geq 0$ the sequence

$$0 \longrightarrow V_n \xrightarrow{f} W_n \xrightarrow{g} Z_n \longrightarrow 0$$

is exact, one obtains an exact sequence

$$0 \longrightarrow \operatorname{Tot}(V) \xrightarrow{f} \operatorname{Tot}(W) \xrightarrow{g} \operatorname{Tot}(Z) \longrightarrow 0,$$

see e.g. [23, 59].

Given two semicosimplicial DG-vector spaces V and W, then $\operatorname{Tot}(V \times W)$ is naturally isomorphic to $\operatorname{Tot}(V) \times \operatorname{Tot}(W)$. An important consequence is the preservation of multiplicative structures; in particular, we will use the fact that the functor Tot sends semicosimplicial DG-Lie algebras to DG-Lie algebras.

To fix notation we recall here the definition of the Čech complex and its sheafified version.

Definition 1.4.4. Let X be smooth separated scheme of finite type over the field K, let $\mathcal{U}=\{U_i\}$, $i\in I$, be an affine open cover of X, and denote by $U_{i_1\cdots i_n}=U_{i_1}\cap\cdots\cap U_{i_n}$.

1. The Čech complex of a quasi-coherent sheaf $\mathcal{F}$ is

$$C^p(\mathcal{U},\mathcal{F})=\prod_{i_0<\cdots<i_p}\mathcal{F}(U_{i_0\cdots i_p}),\quad p\in\mathsf{N},\ i_0,\dots,i_p\in I$$

$$\check{d}\colon C^p(\mathcal{U},\mathcal{F})\to C^{p+1}(\mathcal{U},\mathcal{F}),\quad \check{d}(\alpha)_{i_0\cdots i_{p+1}}=\sum_{k=0}^{p+1}(-1)^k\alpha_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}. \tag{1.4.1}$$

2. Let $(\mathcal{E},d_{\mathcal{E}})$ be a finite complex of quasi-coherent sheaves, the Čech hypercomplex is

$$(C(\mathcal{U},\mathcal{E}))^p=\bigoplus_{q\geq 0}C^q(\mathcal{U},\mathcal{E}^{p-q})$$

$$C^p(\mathcal{U},\mathcal{E}^k)=\prod_{i_0<\cdots<i_p}\mathcal{E}^k(U_{i_0\cdots i_p}),\quad p\in\mathsf{N},\ i_0,\dots,i_p\in I,\ k\in\mathsf{Z}$$

with differential $\check{d}+d_{\mathcal{E}}$.

3. The sheafified Čech complex of $\mathcal{F}$ is the complex of $\mathcal{O}_X$-modules given by

$$(\mathscr{C}(\mathcal{U},\mathcal{F}),\check{d}),\quad \mathscr{C}^p(\mathcal{U},\mathcal{F})=\prod_{i_0<\cdots<i_p}(j_{i_0\cdots i_p})\,\mathcal{F}|_{U_{i_0\cdots i_p}},\quad p\in\mathsf{N},\ i_0,\dots,i_p\in I,$$

where $\check{d}$ is defined as in (1.4.1) and $j_{i_0\cdots i_p}\colon U_{i_0\cdots i_p}\to X$ denotes the inclusion map.

4. The sheafified Čech hypercomplex of $(\mathcal{E},d_{\mathcal{E}})$ is defined as

$$(\mathscr{C}(\mathcal{U},\mathcal{E}),\check{d}+d_{\mathcal{E}}),\quad (\mathscr{C}(\mathcal{U},\mathcal{E}))^p=\bigoplus_{q\geq 0}\mathscr{C}^q(\mathcal{U},\mathcal{E}^{p-q}).$$

In the above situation, the Čech cohomology, defined as the cohomology of the Čech (hyper)complex, is isomorphic to the (hyper)cohomology of the (complex of) sheaves, and the sheafified Čech complex is a resolution of the sheaf, see e.g. [34, III.4].

Example 1.4.5. Let $(\mathcal{E},\delta)$ be a bounded below complex of quasi-coherent sheaves on a smooth separated scheme X of finite type over the field K, and let $\mathcal{U}=\{U_i\}$ be an open affine cover of X. Consider the semicosimplicial DG-vector space of Čech cochains:

$$\mathcal{E}(\mathcal{U}):\quad \prod_i\mathcal{E}(U_i)\ \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\ \prod_{i<j}\mathcal{E}(U_{ij})\ \overset{\delta_0,\,\delta_1,\,\delta_2}{\rightrightarrows\!\!\!\to}\ \prod_{i<j<k}\mathcal{E}(U_{ijk})\ \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}\ \cdots,$$

$$\delta_s\colon\prod_{i_0<\cdots<i_n}\mathcal{E}(U_{i_0\cdots i_n})\to\prod_{i_0<\cdots<i_{n+1}}\mathcal{E}(U_{i_0\cdots i_{n+1}}),\quad (\delta_s\alpha)_{i_0\cdots i_{n+1}}=\alpha_{i_0\cdots\widehat{i_s}\cdots i_{n+1}}|_{U_{i_0\cdots i_{n+1}}}.$$

According to the **Whitney integration theorem**, there exists a natural quasi-isomorphism

$$I\colon\ \mathrm{Tot}(\mathcal{U},\mathcal{E})\to C(\mathcal{U},\mathcal{E})$$

where $C(\mathcal{U},\mathcal{E})$ is the hypercomplex of Čech cochains of Definition 1.4.4 (see [76] for the C^∞ version, [21, 31, 59, 66] for the algebraic version used here). Therefore the cohomology of $\mathrm{Tot}(\mathcal{U},\mathcal{E})$ is isomorphic to the hypercohomology of the complex of sheaves $\mathcal{E}$ and then the quasi-isomorphism class of $\mathrm{Tot}(\mathcal{U},\mathcal{E})$ does not depend on the affine open cover, since $H^i(\mathrm{Tot}(\mathcal{U},\mathcal{E}))=\mathsf{H}^i(X,\mathcal{E})$ and the map I commutes with refinements of affine covers.

For our later application it is important to point out that there exists a natural inclusion of DG-vector spaces $\Gamma(X,\mathcal{E}\ ) \to \operatorname{Tot}(\mathcal{U},\mathcal{E}\ )$ such that the restriction of I to $\Gamma(X,\mathcal{E}\ )$ is the natural inclusion map

$$i\colon \Gamma(X,\mathcal{E}\ ) \to \prod_i \mathcal{E}\ (U_i), \qquad i(s) = \{s_{|U_i}\}.$$

In fact, $\delta_0 i = \delta_1 i$, therefore

$$\delta_{j_k}\delta_{j_{k-1}}\cdots\delta_{i_1} i = \delta_0^k i, \quad \text{for every } 0 \le j_s \le s,$$

and this implies that

$$\iota\colon \Gamma(X,\mathcal{E}\ ) \to \operatorname{Tot}(\mathcal{U},\mathcal{E}\ ), \quad \iota(a) = (1\otimes i(a), 1\otimes \delta_0 i(a), 1\otimes \delta_0^2 i(a),\ldots) \tag{1.4.2}$$

is a properly defined injective morphism of DG-vector spaces.

For later use we also point out that for every quasi-coherent sheaf $\mathcal{F}$ and every affine open cover $\mathcal{U}$, the inclusion $\Gamma(X,\mathcal{F}) \subset \operatorname{Tot}(\mathcal{U},\mathcal{F})$ induces an isomorphism $\Gamma(X,\mathcal{F}) \cong H^0(\operatorname{Tot}(\mathcal{U},\mathcal{F}))$.

Chapter 2

Infinitesimal deformations and obstructions

This chapter begins with a brief review of functors of Artin rings, deformation functors and obstruction theories, with a particular emphasis on simple obstructions. To a differential graded Lie algebra L one can associate two functors of Artin rings MC_L and Def_L, called Maurer–Cartan and deformation functor respectively. These are defined in Section 2.4, together with their tangent spaces and natural obstruction theory. The references for Sections 2.1–2.4 are [22, 59].

Section 2.5 concerns the deformation theory of coherent sheaves. In particular, using locally free and injective resolutions of a coherent sheaf $\mathscr{F}$, three DG-Lie models for $\mathsf{R}\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{F})$ are outlined.

2.1 Deformation functors

For a fixed field K we denote by $\mathbf{Art}_{\mathsf{K}}$ the category of Artin local K-algebras with residue field K, with morphisms local morphisms inducing isomorphisms on the residue field. For every local ring R, $\mathfrak{m}_R$ will denote its maximal ideal. In this chapter, unadorned tensor products are taken over the field K. We denote by **Set** the category of sets (to avoid foundational problems we always work in a fixed universe), and by $0 \in \mathbf{Set}$ the singleton, i.e., the terminal object in the category of sets.

Definition 2.1.1. A **functor of Artin rings** is a covariant functor $F\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ such that $F(\mathsf{K}) = 0$.

Functors of Artin rings are objects of a category where the morphisms are natural transformations.

Example 2.1.2. A vector space V over the field K induces a functor of Artin rings

$$V\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}, \qquad V(A) = V \otimes_{\mathsf{K}} \mathfrak{m}_A.$$

The category $\mathbf{Art}_{\mathsf{K}}$ has fibre products: the fibre product of two maps $f\colon S \to R$ and $g\colon S \to R$ in $\mathbf{Art}_{\mathsf{K}}$ is the set-theoretic fibre product $S \times_R S$, which is an Artin local K-algebra with residue field K and maximal ideal given by $\mathfrak{m}_S \times_R \mathfrak{m}_S$.

Definition 2.1.3. Let $F\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ be a functor of Artin rings. For every fibre product

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r, "\beta"] & A \end{tikzcd}$$

in $\mathbf{Art}_{\mathsf{K}}$ consider the induced map

$$\eta\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C)\,.$$

The functor F is called a **deformation functor** if:

1. η is surjective, whenever β is surjective;
2. η is bijective, whenever $A = \mathsf{K}$.

Definition 2.1.4. A natural transformation $\phi\colon F \to G$ of functors of Artin rings is called **smooth** if for every surjective morphism $B \to A$ in the category $\mathbf{Art}_{\mathsf{K}}$, the natural map

$$F(B) \to G(B) \times_{G(A)} F(A)$$

is surjective. A functor of Artin rings F is called **smooth** or **unobstructed** if the natural transformation $F \to 0$ is smooth; equivalently F is smooth if $F(B) \to F(A)$ is surjective for every surjective morphism $B \to A$ in $\mathbf{Art}_{\mathsf{K}}$.

Definition 2.1.5. Let $F\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ be a deformation functor. The set

$$T^1F = F\left(\frac{\mathsf{K}[\varepsilon]}{(\varepsilon^2)}\right)$$

is called the **tangent space** of F.

The tangent space of a deformation functor has a natural structure of vector space, and for every natural transformation of deformation functors $F \to G$, the induced map $T^1F \to T^1G$ is linear; for a proof see e.g. [59, 3.4.2].

2.2 Obstruction theory

In this context, by obstructions we intend obstructions for a deformation functor F to be smooth, i.e., obstructions to the existence of a lifting of $a \in F(A)$ to $F(B)$, for any surjection $B \to A$ in $\mathbf{Art}_{\mathsf{K}}$.

Definition 2.2.1. A morphism $\alpha\colon B \to A$ in $\mathbf{Art}_{\mathsf{K}}$ is a small surjection if α is surjective and its kernel is annihilated by the maximal ideal $\mathfrak{m}_B$.

If $\alpha\colon B \to A$ is surjective with kernel $I = \ker(\alpha)$ there exists an integer $s > 0$ such that $\mathfrak{m}_B^s I = 0$, and then α factors as the composition of the finite sequence of small surjections

$$B \to \frac{B}{\mathfrak{m}_B^{s-1} I} \to \cdots \to \frac{B}{\mathfrak{m}_B I} \to A\,.$$

Definition 2.2.2. A **small extension** e in $\mathbf{Art}_{\mathsf{K}}$ is an exact sequence of abelian groups

$$e\colon \quad 0 \to M \xrightarrow{\varphi} B \xrightarrow{\alpha} A \to 0,$$

such that α is a morphism in the category $\mathbf{Art}_{\mathsf{K}}$, φ is a morphism of B-modules and the ideal $\varphi(M)$ is annihilated by the maximal ideal $\mathfrak{m}_B$. In particular M is a finite dimensional vector space over $B/\mathfrak{m}_B = \mathsf{K}$.

A small extension as above is called **principal** if $M = \mathsf{K}$.

A small extension is a small surjection together with a framing of its kernel. Every surjective morphism in $\mathbf{Art}_{\mathsf{K}}$ is a finite composition of small surjections arising from principal small extensions.

A morphism of small extensions is a commutative diagram

$$\begin{tikzcd} e_1\colon & 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\alpha_M"] & B_1 \arrow[r] \arrow[d, "\alpha_B"] & A_1 \arrow[r] \arrow[d, "\alpha_A"] & 0 \\ e_2\colon & 0 \arrow[r] & M_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0 \end{tikzcd} \tag{2.2.1}$$

with α_A and α_B morphisms in $\mathbf{Art}_{\mathsf{K}}$. The push-out of a small extension

$$e\colon\ 0 \to M \xrightarrow{\varphi} B \xrightarrow{\alpha} A \to 0,$$

by a morphism of finite dimensional K-vector spaces $g\colon M \to N$ is the small extension

$$g\ e\colon\ 0 \to N \xrightarrow{(0,\mathsf{Id})} \frac{B \oplus N}{\{(\varphi(m), -g(m)) \mid m \in M\}} \xrightarrow{g\ \alpha} A \to 0\ , \qquad g\ \alpha(b,n) = \alpha(b)\ .$$

Definition 2.2.3. Let F be a functor of Artin rings. An **obstruction theory** for F with values in a vector space V is the data, for every small extension in $\mathbf{Art}_{\mathsf{K}}$

$$e\colon\quad 0 \to M \to B \to A \to 0\ ,$$

of an **obstruction map**, a map $v_e\colon F(A) \to V \otimes M$ with the base change property with respect to morphisms of small extensions: this means that for every morphism of small extensions

$$\begin{tikzcd} e_1\colon & 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\alpha_M"] & B_1 \arrow[r] \arrow[d, "\alpha_B"] & A_1 \arrow[r] \arrow[d, "\alpha_A"] & 0 \\ e_2\colon & 0 \arrow[r] & M_2 \arrow[r] & B_2 \arrow[r] & A_2 \arrow[r] & 0 \end{tikzcd}$$

we have $v_{e_2}(\alpha_A(a)) = (\mathrm{Id}_V \otimes \alpha_M)(v_{e_1}(a))$ for every $a \in F(A_1)$.

The name obstruction theory is motivated by the following:

Lemma 2.2.4. *Let (V, v_e) be an obstruction theory for a functor of Artin rings F, and let*

$$e\colon\quad 0 \to M \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0$$

be a small extension. If an element $a \in F(A)$ lifts to $F(B)$, i.e., if $a = \beta(b)$ for some $b \in F(B)$, then $v_e(a) = 0$.

Proof. Immediate consequence of the base change property applied to the morphism of small extensions

$$\begin{tikzcd} & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r, "\mathsf{Id}"] \arrow[d, "\mathsf{Id}"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ e\colon & 0 \arrow[r] & M \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & A \arrow[r] & 0\,. \end{tikzcd}$$

In particular, we have $v_e = 0$ whenever there exists a morphism $s\colon A \to B$ in $\mathbf{Art}_{\mathsf{K}}$ such that $\beta s = \mathrm{Id}_A$. $\square$

Every obstruction theory is uniquely determined by its behaviour on principal small extensions. In fact, let

$$e\colon\ 0 \to M \to B \to A \to 0$$

be a small extension and $a \in F(A)$; the obstruction $v_e(a) \in V \otimes M$ is uniquely determined by the values $(\mathrm{Id}_V \otimes f)v_e(a) \in V$, where f varies along a basis of $\mathrm{Hom}_{\mathsf{K}}(M, \mathsf{K})$. By the base change property, $(\mathrm{Id}_V \otimes f)v_e(a) = v_{f\ e}(a)$, where $f\ e$ is the push-out extension

$$f\ e\colon\quad 0 \to \mathsf{K} \to \frac{B \oplus \mathsf{K}}{\{(m, -f(m)) \mid m \in M\}} \to A \to 0.$$

Definition 2.2.5. An obstruction theory (V, v_e) for F is called **complete** if the converse of Lemma 2.2.4 holds; i.e., the lifting exists if and only if the obstruction vanishes.

Thus, a functor of Artin rings F is smooth if and only if the trivial obstruction theory $(0,0)$ is complete; complete obstruction theories play an essential role when we want to check the smoothness of a natural transformation.

An obstruction theory (O_F, ob_e) for F is called **universal** if, for every obstruction theory (V, v_e), there exists a unique morphism of obstruction theories $(O_F, ob_e) \to (V, v_e)$.

It is clear that if the universal obstruction theory (O_F, ob_e) exists then it is unique up to isomorphism and it is uniquely determined by the functor F; the vector space O_F is called the **obstruction space** of F.

Theorem 2.2.6. *Let F be a deformation functor, then:*

1. *there exists a universal obstruction theory (O_F, ob_e) for F that is complete;*
2. *every element of the obstruction space O_F is of the form $ob_e(a)$, for some principal extension*

$$e\colon \quad 0 \to \mathsf{K} \to B \to A \to 0$$

and some $a \in F(A)$.

For the proof, we refer to [59, 3.6.7].

2.3 Simple obstructions

There exist some special classes of small extensions for which it is easier to compute the corresponding obstruction maps, and which can be enough to give some information on the smoothness of a deformation functor. Of particular relevance for the following chapters will be the simple extensions and obstructions.

Let F be a deformation functor and let $\mathcal{C}$ be a class of small extensions in $\mathbf{Art}_{\mathsf{K}}$, and denote by O_F the universal obstruction space of Theorem 2.2.6. The obstructions arising from $\mathcal{C}$ are defined as the obstructions $f(ob_e(x)) \in O_F = O_F \otimes \mathsf{K}$, with

$$e\colon \quad 0 \to M \to A \to B \to 0$$

a small extension in $\mathcal{C}$, $x \in F(B)$ and $f\colon M \to \mathsf{K}$ a linear map.

Definition 2.3.1. A **curvilinear** extension is a small extension in $\mathbf{Art}_{\mathsf{K}}$ that is isomorphic to

$$0 \to \mathsf{K} \xrightarrow{t^n} \frac{\mathsf{K}[t]}{(t^{n+1})} \to \frac{\mathsf{K}[t]}{(t^n)} \to 0$$

for some $n \geq 2$. The curvilinear obstructions of a deformation functor are the obstructions arising from the curvilinear extensions.

Theorem 2.3.2 ([22])**.** *Let $F\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ be a deformation functor. Then F is smooth if and only if for every integer $n \geq 2$ the natural map*

$$F\left(\frac{\mathsf{K}[t]}{(t^{n+1})}\right) \to F\left(\frac{\mathsf{K}[t]}{(t^n)}\right)$$

is surjective. In other words, a deformation functor is smooth if and only if every curvilinear obstruction vanishes.

Proof. For the proof, we refer to [22]. □

For a deformation functor F, let O_F^c denote the vector subspace of O_F generated by curvilinear obstructions. Notice that the above theorem says that $O_F = 0$ if and only if $O_F^c = 0$, but does not imply in general that $O_F = O_F^c$.

Associated to every small extension $0 \to M \to A \to B \to 0$ is the second fundamental sequence of Kähler differentials (see [61, Theorem 58]):

$$M \xrightarrow{d} \Omega_{A/\mathsf{K}} \otimes_A B \longrightarrow \Omega_{B/\mathsf{K}} \longrightarrow 0,$$

where $d(x) = d_{A/\mathsf{K}}(x) \otimes 1$ for $x \in M$.

Definition 2.3.3. A small extension is **simple** if the natural map $d\colon M \to \Omega_{A/\mathsf{K}} \otimes_A B$ is injective. Simple obstructions are the ones arising from simple extensions.

Over a field of characteristic 0 every curvilinear extension is simple.

For every $A \in \mathbf{Art}_{\mathsf{K}}$ and every A-module M we denote by $A \oplus M$ the trivial extension, with multiplication rule $(a, m)(b, n) = (ab, an + bm)$. Notice that $A \oplus M \in \mathbf{Art}_{\mathsf{K}}$ if and only if M is finitely generated as an A-module.

Definition 2.3.4. A small extension in $\mathbf{Art}_{\mathsf{K}}$ is called **semitrivial** if it is isomorphic to

$$0 \to K \to A \oplus M \xrightarrow{\alpha} A \oplus N \to 0$$

for some $A \in \mathbf{Art}_{\mathsf{K}}$ and some short exact sequence $0 \to K \to M \xrightarrow{\beta} N \to 0$ of finitely generated A-modules with $\mathfrak{m}_A K = 0$; the morphism α is the trivial extension of β, i.e., $\alpha(a, m) = (a, \beta(m))$. A **semitrivial obstruction** is an obstruction arising from a semitrivial extension.

Example 2.3.5. For every $n \geq 1$ the small extension

$$0 \to \mathsf{K} \xrightarrow{x^n y} \frac{\mathsf{K}[x, y]}{(x^{n+1}, y^2)} \to \frac{\mathsf{K}[x, y]}{(x^{n+1}, x^n y, y^2)} \to 0$$

is semitrivial and isomorphic to

$$0 \to \mathsf{K} \xrightarrow{x^n y} A \oplus Ay \to A \oplus \frac{A}{(x^n)} y \to 0, \qquad \text{where} \quad A = \frac{\mathsf{K}[x]}{(x^{n+1})} .$$

Lemma 2.3.6. *Every semitrivial small extension is simple.*

Proof. In the setup of Definition 2.3.4, the projection $\pi\colon A \oplus M \to M$ is a derivation and then there exists a morphism of $A \oplus M$-modules $\phi\colon \Omega_{A\ \ M/\mathsf{K}} \to M$ such that $\phi(a\, db) = a\, \pi(b)$ for every $a, b \in A \oplus M$. Since $KM = 0$ we have $\phi(K\Omega_{A\ \ M/\mathsf{K}}) = 0$ and then ϕ factors through

$$\Omega_{A\ \ M/\mathsf{K}} \otimes_{A\ \ M} A \oplus N = \Omega_{A\ \ M/\mathsf{K}} \otimes_{A\ \ M} \frac{A \oplus M}{K} = \frac{\Omega_{A\ \ M/\mathsf{K}}}{K\Omega_{A\ \ M/\mathsf{K}}} .$$

The proof follows by observing that the composition $\phi d\colon K \to M$ is the inclusion of K into M. ☐

Denoting by O_F^{semi} and O_F^{simple} the obstructions arising from from the semitrivial and simple extensions respectively, from Lemma 2.3.6 it follows that $O_F^{semi} \subseteq O_F^{simple}$.

Proposition 2.3.7. *Let F be a deformation functor, then every obstruction of F arising from a simple small extension is semitrivial.*

Proof. Given a simple small extension $0 \to J \to A \xrightarrow{f} B \to 0$ we have a morphism of small extensions

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "g"] & B \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & J \arrow[r] & B \oplus (\Omega_{A/\mathsf{K}} \otimes_A B) \arrow[r] & B \oplus \Omega_{B/\mathsf{K}} \arrow[r] & 0
\end{tikzcd}$$

where $g(a) = (f(a), da)$ and $h(b) = (b, db)$. The bottom row is a semitrivial extension and the conclusion follows by the base change of obstructions maps. ☐

Putting together Lemma 2.3.6 and Proposition 2.3.7, we obtain that $O_F^{semi} = O_F^{simple} \subseteq O_F$.

Theorem 2.3.8 (T^1-lifting trick)**.** *Let K be a field of characteristic 0 and $F\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ a deformation functor. Then F is smooth if and only if for every $n \geq 2$ the map*

$$F\left(\frac{\mathsf{K}[x,y]}{(x^n, y^2)}\right) \to F\left(\frac{\mathsf{K}[x,y]}{(x^n, x^{n-1}y, y^2)}\right)$$

is surjective.

Proof. Apply Theorem 2.3.2 and base change to the morphisms of small extensions:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{K} \arrow[r, "t^n"] \arrow[d, "n"] & \frac{\mathsf{K}[t]}{(t^{n+1})} \arrow[r] \arrow[d, "t\to x+y"] & \frac{\mathsf{K}[t]}{(t^n)} \arrow[r] \arrow[d, "t\to x+y"] & 0 \\
0 \arrow[r] & \mathsf{K} \arrow[r, "x^{n-1}y"] & \frac{\mathsf{K}[x,y]}{(x^n, y^2)} \arrow[r] & \frac{\mathsf{K}[x,y]}{(x^n, x^{n-1}y, y^2)} \arrow[r] & 0\,,
\end{tikzcd}$$

where the bottom small extension is semitrival, as seen in Example 2.3.5. □

Corollary 2.3.9 (Abstract T^1-lifting theorem)**.** *Over a field K of characteristic 0, a deformation functor is smooth if and only if every semitrivial obstruction vanishes.*

Proof. For every $n \geq 1$ the small extension

$$0 \to \mathsf{K} \xrightarrow{x^n y} \frac{\mathsf{K}[x,y]}{(x^{n+1}, y^2)} \to \frac{\mathsf{K}[x,y]}{(x^{n+1}, x^n y, y^2)} \to 0 \tag{2.3.1}$$

is semitrivial and isomorphic to

$$0 \to \mathsf{K} \xrightarrow{x^n y} A \oplus Ay \to A \oplus \frac{A}{(x^n)}y \to 0, \qquad \text{where} \quad A = \frac{\mathsf{K}[x]}{(x^{n+1})}\,.$$

Therefore, if all semitrivial obstructions vanish, so do obstructions coming from small extensions of the form (2.3.1), and the Corollary follows from Theorem 2.3.8. □

Remark 2.3.10. From Corollary 2.3.9 and from the fact that $O_F^{semi} = O_F^{simple}$ it follows that in characteristic zero the vanishing of simple obstructions is enough to ensure the smoothness of the deformation functor.

2.4 Maurer–Cartan and deformation functors associated to a DG-Lie algebra

In this section we describe the Maurer–Cartan equation of a differential graded Lie algebra L, which is used to define the Maurer–Cartan elements of L and to construct the Maurer–Cartan functor $\mathrm{MC}_L\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$. The deformation functor Def_L is obtained as the quotient of the Maurer–Cartan functor MC_L modulo the gauge action. The tangent spaces and natural obstruction theory of these functors are also described.

Definition 2.4.1. Given a differential graded Lie algebra $(L, d, [-,-])$, the Maurer–Cartan equation is given by

$$dx + \frac{1}{2}[x,x] = 0.$$

An element $x \in L^1$ which satisfies the Maurer–Cartan equation is called a **Maurer–Cartan element**. The set of Maurer–Cartan elements is denoted by

$$\mathrm{MC}(L) = \left\{ x \in L^1 \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}.$$

Definition 2.4.2. Let L be a DG-Lie algebra. The **Maurer–Cartan functor** associated to L is the functor $\mathrm{MC}_L\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$,

$$\mathrm{MC}_L(A) = \mathrm{MC}(L^1 \otimes \mathfrak{m}_A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}.$$

Notice that the DG-Lie algebra structure on $L \otimes \mathfrak{m}_A$ is induced by the one on L, as in Example 1.1.10.

Definition 2.4.3. Let L be a DG-Lie algebra and $A \in \mathbf{Art}_{\mathsf{K}}$. Two elements x, y in $L^1 \otimes \mathfrak{m}_A$ are **gauge equivalent** if there exists $a \in L^0 \otimes \mathfrak{m}_A$ such that

$$y = e^a * x := x + \sum_{n \geq 0} \frac{[a,-]^n}{(n+1)!}([a,x] - da).$$

The operator $*$ is called the **gauge action** of the group $\exp(L^0 \otimes \mathfrak{m}_A)$: in fact $e^a * (e^b * x) = e^{a \bullet b} * x$, where $\bullet$ is the Baker–Campbell–Hausdorff product.

Notice that the gauge action is a perturbation of the adjoint action: if the differential d is trivial, it reduces to the adjoint action.

The gauge action preserves Maurer–Cartan elements: for a proof we refer to [59, 6.3.4].

Definition 2.4.4. The **deformation functor** associated to a DG-Lie algebra L is

$$\mathrm{Def}_L\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$$

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}(L^1 \otimes \mathfrak{m}_A)}{\exp(L^0 \otimes \mathfrak{m}_A)} = \frac{\left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}}{\text{gauge action}}.$$

It is clear that for any DG-Lie algebra L, the functors MC_L and Def_L are functors of Artin rings, as in Definition 2.1.1. Moreover, the functors MC_L and Def_L are deformation functors, as in Definition 2.1.3; for a proof of this fact we refer to [59, Chapter 6].

Example 2.4.5. Consider a DG-Lie algebra L and A in $\mathbf{Art}_{\mathsf{K}}$ such that the DG-Lie algebra $L \otimes \mathfrak{m}_A$ is abelian. Then one has that

$$\mathrm{Def}_L(A) = H^1(L) \otimes \mathfrak{m}_A;$$

in fact, when the DG-Lie algebra is abelian the Maurer–Cartan equation reduces to $dx = 0$ and then $\mathrm{MC}_L(A) = Z^1(L) \otimes \mathfrak{m}_A$. If $a \in L^0 \otimes \mathfrak{m}_A$ and $x \in L^1 \otimes \mathfrak{m}_A$ we have

$$e^a * x = x + \sum_{n \geq 0} \frac{\mathrm{ad}(a)^n}{(n+1)!}([a,x] - da) = x - da$$

and then $\mathrm{Def}_L(A) = \dfrac{Z^1(L) \otimes \mathfrak{m}_A}{d(L^0 \otimes \mathfrak{m}_A)} = H^1(L) \otimes \mathfrak{m}_A$.

From this follows the fact that the **tangent space** to the deformation functor Def_L is given by

$$T^1 \mathrm{Def}_L := \mathrm{Def}_L(\mathsf{K}[\varepsilon]) = H^1(L) \otimes \mathsf{K}\varepsilon, \qquad \varepsilon^2 = 0, \tag{2.4.1}$$

and the useful fact:

Lemma 2.4.6. *If L is an abelian DG-Lie algebra, the associated deformation functor Def_L is smooth.*

Proof. As seen in Example 2.4.5, $\mathrm{Def}_L(A) \cong H^1(L) \otimes \mathfrak{m}_A$, and for every surjection $B \to A$ in $\mathbf{Art}_{\mathsf{K}}$, the map $H^1(L) \otimes \mathfrak{m}_B \to H^1(L) \otimes \mathfrak{m}_A$ is surjective. □

Proposition 2.4.7. *Let L be a di erential graded Lie algebra. The deformation functor MC_L carries a natural complete obstruction theory with values in the vector space $H^2(L)$.*

Moreover, this complete obstruction theory is also a complete obstruction theory for the functor Def_L.

Proof. Consider a small extension

$$e\colon \quad 0 \to M \to B \to A \to 0\,,$$

and take $x \in \mathrm{MC}_L(A)$, i.e. $x \in L^1 \otimes \mathfrak{m}_A$ such that $dx + \frac{1}{2}[x,x] = 0$. We can lift x to $\widetilde{x} \in L^1 \otimes \mathfrak{m}_B$, and define

$$h = d\widetilde{x} + \frac{1}{2}[\widetilde{x},\widetilde{x}] \in L^2 \otimes M.$$

Then

$$dh = d^2(\widetilde{x}) + [d\widetilde{x},\widetilde{x}] = [h,\widetilde{x}] - \frac{1}{2}[[\widetilde{x},\widetilde{x}],\widetilde{x}],$$

and by the fact that $[L^2 \otimes M, L^1 \otimes \mathfrak{m}_B] = 0$ and by the Jacobi identity we obtain $dh = 0$. We define $v_e(x) = [h] \in H^2(L \otimes M) = H^2(L) \otimes M$; this does not depend on the choice of the lifting, because if we take y another lifting of x, we have that $y = \widetilde{x} + z$, $z \in L^1 \otimes M$, and

$$dy + \frac{1}{2}[y,y] = h + dz.$$

Then $(H^2(L), v_e)$ is a complete obstruction theory for the functor MC_L.

Given a surjective morphism $\alpha\colon A \to B$ in the category $\mathbf{Art}_{\mathsf{K}}$, an element $x \in \mathrm{MC}_L(B)$ can be lifted to $\mathrm{MC}_L(A)$ if and only if its equivalence class $[x] \in \mathrm{Def}_L(B)$ can be lifted to $\mathrm{Def}_L(A)$. In fact, if $x \in \mathrm{MC}_L(B)$ lifts to some $y \in \mathrm{MC}_L(A)$, it is plain that $[y] \in \mathrm{Def}_L(A)$ lifts $[x] \in \mathrm{Def}_L(B)$. Vice versa, if $[x]$ lifts to $\mathrm{Def}_L(A)$ then there exists $y \in \mathrm{MC}_L(A)$ and $b \in L^0 \otimes \mathfrak{m}_B$ such that $\alpha(y) = e^b * x$. Then one can lift b to an element $a \in L^0 \otimes \mathfrak{m}_A$ and then $x_0 = e^{-a} * y$ is a lifting of x.

□

Every morphism $f\colon L \to M$ of differential graded Lie algebras induces a natural transformation of the associated Maurer–Cartan functors, which is compatible with the gauge action, and therefore induces a natural transformation of the associated deformation functors $f\colon \mathrm{Def}_L \to \mathrm{Def}_M$.

Remark 2.4.8. Let $f\colon L \to M$ be a morphism of DG-Lie algebras, and let $f\colon H^i(L) \to H^i(M)$ be the maps induced in cohomology by f, and consider in particular $f\colon H^2(L) \to H^2(M)$. This map commutes with obstructions, and hence if Def_M is smooth, f annihilates all obstructions to Def_L.

In particular, if M is an abelian DG-Lie algebra, by Lemma 2.4.6 the functor Def_M is smooth and the map f annihilates all obstructions to Def_L.

Proposition 2.4.9. *Let $L \to N$ be a quasi-isomorphism of di erential graded Lie algebras. Then the induced morphism $\mathrm{Def}_L \to \mathrm{Def}_N$ is an isomorphism.*

This follows from the *Standard smoothness criterion* of [22].

Finally, to work in the homotopy category of DG-Lie algebras, we need the following result:

Theorem 2.4.10. *Every L_∞ morphism between DG-Lie algebras $f\colon L \quad M$ induces a natural transformation of functors*

$$\mathrm{MC}_f\colon \mathrm{MC}_L \to \mathrm{MC}_M, \quad x \mapsto \sum_n \frac{1}{n!} f_n(x^{\quad n}),$$

that factors to a natural transformation

$$\mathrm{Def}_f\colon \mathrm{Def}_L \to \mathrm{Def}_M\,.$$

If f is a weak equivalence of L_∞-algebras as in Definition 1.3.15, then $\mathrm{Def}_f\colon \mathrm{Def}_L \to \mathrm{Def}_M$ is an isomorphism of functors.

Proof. See [59, Section 13.1]. □

2.4.1 Deformation functors associated to semicosimplicial Lie algebras

Let $\mathfrak{h}$ be semicosimplicial Lie algebra over K, as in Definition 1.4.2

$$\mathfrak{h}: \qquad \mathfrak{h}_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathfrak{h}_1 \underset{\delta_2}{\overset{\delta_0}{\overset{\delta_1}{\rightrightarrows}}} \mathfrak{h}_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots .$$

One can associate to $\mathfrak{h}$ two functors of Artin rings $Z^1_{\mathfrak{h}}, H^1_{\mathfrak{h}}\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$, which here are described in brief; for more details see [27, 59]. The functor of non-abelian 1-cocycles $Z^1_{\mathfrak{h}}$ is defined as

$$Z^1_{\mathfrak{h}}(A) = \{e^x \in \exp(\mathfrak{h}_1 \otimes \mathfrak{m}_A) \mid e^{\delta_1(x)} = e^{\delta_2(x)}e^{\delta_0(x)}\}.$$

For every $A \in \mathbf{Art}_{\mathsf{K}}$ there is a left action of $\exp(\mathfrak{h}_0 \otimes \mathfrak{m}_A)$ on $Z^1_{\mathfrak{h}}(A)$

$$\exp(\mathfrak{h}_0 \otimes \mathfrak{m}_A) \times Z^1_{\mathfrak{h}}(A) \to Z^1_{\mathfrak{h}}(A), \quad (e^a, e^x) \mapsto e^{\delta_1(a)}e^x e^{-\delta_0(a)}.$$

The functor $H^1_{\mathfrak{h}}\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ is then defined as

$$H^1_{\mathfrak{h}}(A) = \frac{Z^1_{\mathfrak{h}}(A)}{\exp(\mathfrak{h}_0 \otimes \mathfrak{m}_A)}.$$

A corollary of Hinich's theorem on descent of Deligne groupoids [36] relates the Thom–Whitney totalisation of Section 1.4 and the functor $H^1_{\mathfrak{g}}$.

Proposition 2.4.11 (Hinich)**.** *For every semicosimplicial Lie algebra $\mathfrak{g}$ there exists a natural isomorphism of functors $H^1_{\mathfrak{g}} \cong \operatorname{Def}_{\mathsf{Tot}(\mathfrak{g})}$.*

For the proof we refer to [36] or [59].

2.5 Deformations of coherent sheaves

Let $\mathcal{F}$ be a coherent sheaf on a smooth separated scheme X of finite type over a field K of characteristic zero. This sections concerns the deformation theory of coherent sheaves: in particular, three DG-Lie algebras controlling this deformation problem are given, under different hypotheses on $\mathcal{F}$ and on X.

Definition 2.5.1. A deformation of $\mathcal{F}$ over $A \in \mathbf{Art}_{\mathsf{K}}$ is a pair $(\mathcal{F}_A, \alpha)$ where $\mathcal{F}_A$ is a coherent sheaf of $\mathcal{O}_X \otimes A$-modules, flat over A, and $\alpha\colon \mathcal{F}_A \to \mathcal{F}$ is morphism of $\mathcal{O}_X \otimes A$-modules inducing an isomorphism $\mathcal{F}_A \otimes_A \mathsf{K} \cong \mathcal{F}$.

Two deformations $(\mathcal{F}_A, \alpha)$ and $(\mathcal{F}_A, \alpha\ )$ are isomorphic if there exists an isomorphism of $\mathcal{O}_X \otimes A$-modules $f\colon \mathcal{F}_A \to \mathcal{F}_A$ commuting with the maps to $\mathcal{F}$, i.e. such that $\alpha\, f = \alpha$.

This defines a functor of Artin rings:

$$\operatorname{Def}_{\mathcal{F}}\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}, \quad \operatorname{Def}_{\mathcal{F}}(A) = \{\text{isomorphism classes of deformations of } \mathcal{F} \text{ over } A\}.$$

For a proof of the fact that $\operatorname{Def}_{\mathcal{F}}$ is a deformation functor as in Definition 2.1.3, we refer e.g. to [35, 3.19].

2.5.1 DG-Lie representatives of $\mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$

By an injective resolution of a coherent sheaf $\mathcal{F}$ we mean an injective quasi-isomorphism $\mathcal{F} \to \mathcal{I}\ $, with $(\mathcal{I}\ , d_{\mathcal{I}})$ a complex of injective $\mathcal{O}_X$-modules concentrated in positive degree.

Definition 2.5.2. Define $\mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ as the quasi-isomorphism class of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}\ , \mathcal{I}\ )$, for $\mathcal{F} \to \mathcal{I}\ $ any injective resolution of the coherent sheaf $\mathcal{F}$.

The definition is well-posed, because the quasi-isomorphism class of the DG-Lie algebra $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$ is independent from the choice of the injective resolution of $\mathcal{F}$. In fact, given two injective resolutions $\mathcal{F} \to \mathcal{I}$ and $\mathcal{F} \to \mathcal{J}$, it is well known that there exists a morphism of complexes $f\colon \mathcal{I} \to \mathcal{J}$, unique up to homotopy, lifting the identity on $\mathcal{F}$. Then it is possible to conclude that the DG-Lie algebras $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J})$ are quasi-isomorphic using the following lemma:

Lemma 2.5.3. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a quasi-isomorphism of bounded below complexes of injective $\mathcal{O}_X$-modules, then the associative DG-algebras $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J})$ are quasi-isomorphic.*

Proof. Recall that the mapping cone $\mathcal{C} = \mathrm{cone}(f)$ of f is defined by $\mathcal{C}^n = \mathcal{I}^{n+1} \oplus \mathcal{J}^n$, $d_{\mathcal{C}}(x, y) = (-dx, f(x)+dy)$. The map f is a quasi-isomorphism between bounded below complexes of injective objects, therefore its mapping cone is a bounded below, acyclic complex of injective objects and by applying Lemma A.3.1 to its identity map we can see it is contractible.

Consider first the case where moreover f is degreewise split injective and its cokernel $\mathcal{K} = \mathrm{Coker}(f)$ is contractible. In this case, we can identify $\mathcal{I}$ with the image $f(\mathcal{I}) \subset \mathcal{J}$, and consider the associative DG-algebra

$$L = \{\alpha \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J}) \mid \alpha(\mathcal{I}) \subseteq \mathcal{I}\},$$

which fits inside the short exact sequence

$$0 \to L \xrightarrow{i} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{K}) \to 0,$$

where i a morphism of DG-algebras. Since f is degreewise split injective, there exists a surjective morphism of DG-algebras

$$p\colon L \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}), \quad g \mapsto p(g) = g|_{\mathcal{I}},$$

which fits inside the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{J}) \to L \xrightarrow{p} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \to 0.$$

Since the complex $\mathcal{K}$ is contractible, so are $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{J})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{K})$, therefore the maps p, i are quasi-isomorphisms of DG-algebras.

In the general case, consider the mapping cylinder $\mathcal{H} := \mathrm{cyl}(f)$, defined by $\mathcal{H}^n = \mathcal{I}^{n+1} \oplus \mathcal{I}^n \oplus \mathcal{J}^n$, $d_{\mathcal{H}}(x, y, z) = (-dx, dy - x, f(x) + dz)$. There are two natural inclusions $\mathcal{I} \to \mathcal{H}$, $\mathcal{J} \to \mathcal{H}$, which are quasi-isomorphisms of bounded below complexes of injective objects and are also degreewise split injective. Therefore, we can apply the first part to obtain a zigzag of quasi-isomorphisms of DG-algebras:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \xleftarrow{p} L \xrightarrow{i} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{H}) \xleftarrow{i} L \xrightarrow{p} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J}).$$

□

If there exists a finite locally free resolution $\mathcal{E} \to \mathcal{F}$ of the coherent sheaf $\mathcal{F}$, it can be used to construct a particularly useful DG-Lie algebra model for $\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$: the resolution $\mathcal{E} \to \mathcal{F}$ gives a sheaf of DG-Lie algebras $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ on X, and taking the Thom–Whitney totalisation of Definition 1.4.3 with respect to an affine open cover $\mathcal{U}$ one obtains a DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$.

In the next proposition, we prove that it is in fact a DG-Lie model for the derived endomorphisms of $\mathcal{F}$.

Proposition 2.5.4. *Let $\mathcal{E} \to \mathcal{F}$ be a finite locally free resolution of the coherent sheaf $\mathcal{F}$, and let $\mathcal{F} \to \mathcal{I}$ be a resolution of injective $\mathcal{O}_X$-modules. There is a quasi-isomorphism of DG-Lie algebras between $\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$.*

Proof. Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X. It is well known (or it can be deduced by applying Lemma A.3.1) that the complexes $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )$ are quasi-isomorphic. Therefore, considering the semicosimplicial DG-vector spaces

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )(\mathcal{U}): \quad \prod_i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )(U_i) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{i<j} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )(U_{ij}) \underset{\delta_2}{\overset{\delta_0}{\substack{\rightarrow\\ \rightarrow \\ \rightarrow}}} \cdots$$

and

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(\mathcal{U}): \quad \prod_i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_i) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{i<j} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_{ij}) \underset{\delta_2}{\overset{\delta_0}{\substack{\rightarrow\\ \rightarrow \\ \rightarrow}}} \cdots,$$

for every $n \geq 0$ there is a quasi-isomorphism of DG-vector spaces

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )(\mathcal{U})_n \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(\mathcal{U})_n.$$

By a corollary of Whitney's integration theorem, see [59, 7.4.6], this gives a quasi-isomorphism of DG-vector spaces

$$\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )) \longrightarrow \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )).$$

Consider now the double complex

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) \to \prod_i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_i) \xrightarrow{\delta_0 - \delta_1} \prod_{i<j} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_{ij}) \to \cdots, \tag{2.5.1}$$

which is acyclic in the horizontal direction, i.e., for every n the complex

$$0 \to \mathrm{Hom}^n_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) \to \prod_i \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_i) \xrightarrow{\delta_0 - \delta_1} \prod_{i<j} \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_{ij}) \to \cdots$$

is acyclic. This follows by the fact that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G},\mathcal{Q})$ is flasque for any injective $\mathcal{O}_X$-module $\mathcal{Q}$, see [32, Ch. II, 7.3.2] and by Leray's theorem on acyclic covers. This double complex is contained in the first quadrant, because $\mathcal{H}om^{-k}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}^{-k}) = 0$ for all $k > 0$.

Consider the total complex associated to the double complex of (2.5.1), as in Definition 1.1.4 (as seen in Remark 1.1.5, for a first quadrant double complex the sum and product total complexes coincide). Since the double complex of (2.5.1) is contained in the first quadrant, the acyclicity in the horizontal direction implies that the associated total complex is acyclic. This implies that there is a quasi-isomorphism of complexes

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) \to C\ (\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )),$$

where $C\ (\mathcal{U}, -)$ denotes the Čech hypercomplex of Definition 1.4.4. Since the complex of sheaves $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )$ is bounded below, we can apply the Whitney integration theorem of Example 1.4.5 to obtain a quasi-isomorphism of complexes

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) \longrightarrow \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )),$$

induced by the restriction maps $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )(U_i)$.

Likewise, the restriction maps $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ ) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )(U_i)$ induce a map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ ) \to \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )),$$

which is a morphism of DG-Lie algebras. We then obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ ) & \longrightarrow & \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ ) & \longrightarrow & \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}\ )), \end{array}$$

where the first vertical map is a quasi-isomorphism by Lemma A.3.1, and hence the map $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ ) \to \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ ))$ is a quasi-isomorphism of DG-Lie algebras.

Therefore, to obtain the main claim, it is now enough to prove that there is a quasi-isomorphism of DG-Lie algebras between $\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ ))$ and $\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$.

Let $U = \operatorname{Spec} R$ be an affine open set, with R a commutative unitary K-algebra. To prove the proposition, it is enough to prove that for an R-module M equipped with a finite projective resolution $p\colon P\ \to M$ and an injective resolution $i\colon M \to I\ $, the DG-Lie algebras $\mathrm{Hom}_R(P\ ,P\ )$ and $\mathrm{Hom}_R(I\ ,I\ )$ are quasi-isomorphic.

Consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker} p \longrightarrow P\ \xrightarrow{\ p\ } M \longrightarrow 0, \tag{2.5.2}$$

where the complex $\operatorname{Ker} p$ is acyclic because p is a surjective quasi-isomorphism. Applying the functor $\mathrm{Hom}_R(-, I\ )$ to the exact sequence (2.5.2), we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, I\ ) \xrightarrow{\ p\ } \mathrm{Hom}_R(P\ ,I\ ) \longrightarrow \mathrm{Hom}_R(\operatorname{Ker} p, I\ ) \longrightarrow 0,$$

which is exact on the right because I is a bounded below complex of injective modules. By Lemma A.3.1 any morphism of complexes from an acyclic complex to a bounded below complex formed by injective elements is homotopic to the zero morphism, and therefore the complex $\mathrm{Hom}_R(\operatorname{Ker} p, I\ )$ is acyclic. This implies that the map p is a quasi-isomorphism of complexes.

Similarly, we can apply the functor $\mathrm{Hom}_R(P\ ,-)$ to the exact sequence (2.5.2) to obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(P\ ,\operatorname{Ker} p) \longrightarrow \mathrm{Hom}_R(P\ ,P\ ) \xrightarrow{\ p\ } \mathrm{Hom}_R(P\ ,M) \longrightarrow 0,$$

which is exact on the right because P is a bounded complex of projective R-modules. By Lemma A.3.1 the complex $\mathrm{Hom}_R(P\ ,\operatorname{Ker} p)$ is acyclic and the map p is a quasi-isomorphism.

Consider now the short exact sequence of bounded below complexes

$$0 \longrightarrow M \xrightarrow{\ i\ } I\ \longrightarrow \operatorname{Coker} i \longrightarrow 0, \tag{2.5.3}$$

where $\operatorname{Coker} i$ is acyclic because i is an injective quasi-isomorphism. As above, one can apply the functors $\mathrm{Hom}_R(-, I\ )$ and $\mathrm{Hom}_R(P\ ,-)$ to the sequence (2.5.3), and obtain, using Lemma A.3.1 again, that $i\ \colon \mathrm{Hom}_R(I\ ,I\ ) \to \mathrm{Hom}_R(M, I\ )$ is a surjective quasi-isomorphism and that $i\ \colon \mathrm{Hom}_R(P\ ,M) \to \mathrm{Hom}_R(P\ ,I\ )$ is an injective quasi-isomorphism.

Denote by f the composition $f = ip\colon P\ \to M \to I\ $. It follows from the above that the maps

$$f\ \colon \mathrm{Hom}_R(P\ ,P\ ) \to \mathrm{Hom}_R(P\ ,I\ ), \quad f\ \colon \mathrm{Hom}_R(I\ ,I\ ) \to \mathrm{Hom}_R(P\ ,I\ ) \tag{2.5.4}$$

are quasi-isomorphisms of complexes, and therefore $\operatorname{cone}(f\ )$ and $\operatorname{cone}(f\ )$ are acyclic complexes.

Let C denote the cone of the map $f\colon P\ \to I\ $, and let N denote the DG-Lie algebra of endomorphisms of the cone, $N := \mathrm{Hom}_R(\operatorname{cone}(f), \operatorname{cone}(f)) = \mathrm{Hom}_R(C\ ,C\ )$. As a graded vector space, $C\ = P\ [1] \oplus I\ $, and the differential is given by

$$d_C = \begin{pmatrix} d_{P[1]} & 0 \\ f[1] & d_I \end{pmatrix},$$

so the differential of N is given by $[d_C, -]$. Elements of N can be represented by matrices of the form

$$\begin{pmatrix} a & b \\ c & e \end{pmatrix} \qquad \begin{array}{ll} a \in \mathrm{Hom}_R(P\ [1], P\ [1]), & b \in \mathrm{Hom}_R(I\ ,P\ [1]), \\ c \in \mathrm{Hom}_R(P\ [1], I\ ), & e \in \mathrm{Hom}_R(I\ ,I\ ). \end{array}$$

Following the proof of [70, Lemma 4.4], it is easy to see that the subspace $L \subset N$ of matrices of the form

$$\begin{pmatrix} a & 0 \\ c & e \end{pmatrix} \qquad a \in \operatorname{Hom}_R(P\ [1], P\ [1]),\ c \in \operatorname{Hom}_R(P\ [1], I\ ),\ e \in \operatorname{Hom}_R(I\ , I\ )$$

is a sub-DG-Lie-algebra of N. One can check that the projections

$$\pi_1\colon L \to \operatorname{Hom}_R(P\ , P\ ), \qquad \pi_1 \begin{pmatrix} a & 0 \\ c & e \end{pmatrix} = (-1)^{\overline{a}} a,$$

$$\pi_2\colon L \to \operatorname{Hom}_R(I\ , I\ ), \qquad \pi_2 \begin{pmatrix} a & 0 \\ c & e \end{pmatrix} = e$$

are surjective morphisms of DG-Lie algebras.

We claim that $\operatorname{Ker} \pi_1$ and $\operatorname{Ker} \pi_2$ are acyclic, which implies that π_1 and π_2 are quasi-isomorphisms, so that the proof of the proposition is complete. In particular, we will prove that up to a shift $\operatorname{Ker} \pi_1$ is isomorphic to $\operatorname{cone}(f\ )$ and $\operatorname{Ker} \pi_2$ to $\operatorname{cone}(f\ )$, where f and f are the quasi-isomorphisms of (2.5.4). We only prove the statement for $\operatorname{Ker} \pi_1$, as the proof for $\operatorname{Ker} \pi_2$ is analogous.

The kernel of π_1 is given by matrices of the form

$$\begin{pmatrix} 0 & 0 \\ c & e \end{pmatrix} \qquad c \in \operatorname{Hom}_R(P\ [1], I\ ),\ e \in \operatorname{Hom}_R(I\ , I\ ),$$

and the differential is $[d_C, -]$, inherited from $\operatorname{Hom}_R(C\ , C\ )$. The cone of $f\ \colon \operatorname{Hom}_R(I\ , I\ ) \to \operatorname{Hom}_R(P\ , I\ )$ is the complex $K\ := \operatorname{cone}(f\ ) = \operatorname{Hom}_R(I\ , I\ )[1] \oplus \operatorname{Hom}_R(P\ , I\ )$ with differential

$$d_K = \begin{pmatrix} d_{\mathsf{Hom}_R(I\ ,I\ )[1]} & 0 \\ f\ [1] & d_{\mathsf{Hom}_R(P\ ,I\ )} \end{pmatrix}.$$

We claim that the map

$$\psi\colon (\operatorname{Ker} \pi_1, [d_C, -]) \to (K\ [-1], -d_K), \qquad \begin{pmatrix} 0 & 0 \\ c & e \end{pmatrix} \to (e, -c)$$

is an isomorphism. It is clear it is an isomorphism of graded vector spaces, since as as a graded vector space

$$\operatorname{Ker} \pi_1 \cong \operatorname{Hom}_R(P\ [1], I\ ) \oplus \operatorname{Hom}_R(I\ , I\ ) \cong \operatorname{Hom}_R(I\ , I\ ) \oplus \operatorname{Hom}_R(P\ , I\ )[-1].$$

We need only to prove that the map ψ commutes with the differentials. Consider an element

$$\begin{pmatrix} 0 & 0 \\ c & e \end{pmatrix}$$

of degree j in $\operatorname{Ker} \pi_1$, so that $c \in \operatorname{Hom}^j_R(P\ [1], I\ ) \cong \operatorname{Hom}^{j-1}_R(P\ , I\ )$ and $e \in \operatorname{Hom}^j_R(I\ , I\ )$. The differential of this element is given by

$$\begin{pmatrix} 0 & 0 \\ d_I c - (-1)^j c d_{P[1]} - (-1)^j e f[1] & d_I e - (-1)^j e d_I \end{pmatrix},$$

and then

$$\psi\left(\left[d_C, \begin{pmatrix} 0 & 0 \\ c & e \end{pmatrix}\right]\right) = (d_I e - (-1)^j e d_I, -d_I c + (-1)^j c d_{P[1]} + (-1)^j e f[1]).$$

On the other hand, the differential of $(e,-c)$ is given by

$$-\begin{pmatrix} d_{\mathrm{End}\ (I\ )[1]} & 0 \\ f\ [1] & d_{\mathrm{Hom}\ (P\ ,I\ )} \end{pmatrix}\begin{pmatrix} e \\ -c \end{pmatrix} = -\begin{pmatrix} -(d_I e-(-1)^j e d_I) \\ f\ [1](e)-d_I c+(-1)^{j-1} c d_P \end{pmatrix}$$
$$= \begin{pmatrix} d_I e-(-1)^j e d_I \\ -f\ [1](e)+d_I c+(-1)^j c d_P, \end{pmatrix}$$

and we have the claim by the fact that $d_{P[1]}=-d_P$ and $f\ [1](e)=(-1)^j e f[1]$. $\square$

In the case where $\mathcal{F}$ is a coherent sheaf on a complex manifold X admitting a finite locally free resolution $\mathcal{E}\ \to\mathcal{F}$, there is another representative for $\mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ given by the Dolbeault model

$$(A_X^{0,}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )),[-,-],\overline{\partial}+[d_{\mathcal{E}},-]).$$

This DG-Lie algebra is obtained as the global sections of the sheaf of DG-Lie algebras

$$\mathcal{A}_X^{0,}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))=\mathcal{A}_X^{0,}\ \otimes_{\mathcal{O}_X}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ).$$

Here, $\mathcal{A}_X^{0,q}$ denotes the sheaf of forms of type $(0,q)$ on X, the degree of an element of $A_X^{0,q}(\mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ is $q+n$, and $\overline{\partial}$ denotes the Dolbeault differential.

The bracket in $\mathcal{A}_X^{0,}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ and $A_X^{0,}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ is given by:

$$[\omega\cdot f,\eta\cdot g]=(\omega\cdot f)(\eta\cdot g)-(-1)^{(\overline{\omega}+\overline{f})(\overline{\eta}+\overline{g})}(\eta\cdot g)(\omega\cdot f)=(-1)^{\overline{f}\overline{\eta}}(\omega\wedge\eta)\cdot[f,g]\,,$$
$$\forall\ \omega,\eta\in\mathcal{A}_X^{0,}\ ,\quad f,g\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ).$$

Proposition 2.5.5. *The DG-Lie algebra*

$$(A_X^{0,}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )),[-,-],\overline{\partial}+[d_{\mathcal{E}},-])$$

is quasi isomorphic to $\operatorname{Tot}(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$.

Proof. Let $\mathcal{U}$ be a Stein open cover, and consider the semicosimplicial DG-Lie algebras

$$\begin{array}{llllllll}
\mathfrak{h}: & \prod_i \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_i) & \longrightarrow & \prod_{i,j}\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ij}) & \substack{\longrightarrow\\ \longrightarrow} & \prod_{i,j,k}\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ijk}) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{g}: & \prod_i A^{0,}_{U_i}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )) & \rightrightarrows & \prod_{i,j} A^{0,}_{U_{ij}}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \prod_{i,j,k} A^{0,}_{U_{ijk}}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \cdots,
\end{array}$$

where for every $n\geq 0$, the inclusion $\mathfrak{h}_n\to\mathfrak{g}_n$ is a quasi-isomorphism. By a corollary of Whitney's integration theorem, see [59, 7.4.6], there is then a quasi-isomorphism of DG-Lie algebras

$$\operatorname{Tot}(\mathcal{U},\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))\to\operatorname{Tot}(\mathcal{U},\mathcal{A}_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))).$$

By Leray's theorem on acyclic covers, the Čech hypercomplex of $\mathcal{A}_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))$ is quasi-isomorphic to $A_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))$ via the restriction map

$$A_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))\longrightarrow\prod_i A^{0,}_{U_i}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))\xrightarrow{\delta_0-\delta_1}\prod_{i,j}A^{0,}_{U_{ij}}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))\ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \cdots.$$

By Whitney's integration theorem the restriction maps induce a quasi-isomorphism of DG-Lie algebras $A_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))\to\operatorname{Tot}(\mathcal{U},A_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )))$, and therefore we have a zigzag of quasi-isomorphisms of DG-Lie algebras

$$A_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ ))\to\operatorname{Tot}(\mathcal{U},\mathcal{A}_X^{0,}\ (\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )))\leftarrow\operatorname{Tot}(\mathcal{U},\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )).$$

$\square$

Therefore, via Theorem 2.5.4, $A_X^{0,}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ is a DG-Lie model for $\mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$.

2.5.2 The DG-Lie models for $\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ control the deformations of $\mathcal{F}$

Let $\mathcal{F}$ be a coherent sheaf on a smooth separated scheme X of finite type over a field K of characteristic zero, and let $\mathcal{F} \to \mathcal{I}\ $ be an injective resolution of $\mathcal{O}_X$-modules. In the following, we prove that the DG-Lie algebra $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\ ,\mathcal{I}\ )$ controls the deformations of $\mathcal{F}$.

Definition 2.5.6. Let $(\mathcal{I}\ , d_{\mathcal{I}})$ be a complex of injective sheaves of $\mathcal{O}_X$-modules, a deformation of $(\mathcal{I}\ , d_{\mathcal{I}})$ over $A \in \mathbf{Art}_{\mathsf{K}}$ is the datum of a complex of sheaves $(\mathcal{I}\ \otimes A, d_A)$ such that d_A reduces to $d_{\mathcal{I}}$ modulo $\mathfrak{m}_A$. An isomorphism of deformations $(\mathcal{I}\ \otimes A, d_A)$ and $(\mathcal{I}\ \otimes A, d'_A)$ is an isomorphism of complexes of sheaves $f\colon (\mathcal{I}\ \otimes A, d_A) \to (\mathcal{I}\ \otimes A, d'_A)$ which reduces to the identity modulo $\mathfrak{m}_A$.

Proposition 2.5.7. *Let $(\mathcal{I}\ , d_{\mathcal{I}})$ be an injective resolution of the coherent sheaf $\mathcal{F}$. For every deformation $(\mathcal{I}\ \otimes A, d_A)$ of $(\mathcal{I}\ , d_{\mathcal{I}})$ over A, $\mathcal{H}^0_{d_A}(\mathcal{I}\ \otimes A)$ is a deformation of $\mathcal{F}$ over A.*

Vice versa, for every deformation $(\mathcal{F}_A, \alpha)$ of $\mathcal{F}$ over A, there exists a deformation $(\mathcal{I}\ \otimes A, d_A)$ of $(\mathcal{I}\ , d_{\mathcal{I}})$ such that $\mathcal{H}^0_{d_A}(\mathcal{I}\ \otimes A) \cong \mathcal{F}_A$.

Proof. Notice that $\mathcal{I}^k \otimes A$ is flat over A for every k, because the functor $\mathcal{I}^k \otimes_{\mathsf{K}} A \otimes_A - \cong \mathcal{I}^k \otimes_{\mathsf{K}} -$ is exact. By upper semicontinuity of the cohomology and by the fact that is reduces to $(\mathcal{I}\ , d_{\mathcal{I}})$ modulo $\mathfrak{m}_A$, the complex $(\mathcal{I}\ \otimes A, d_A)$ is exact except in level zero, so that the following sequence is exact:

$$0 \longrightarrow \mathrm{Ker}(d^0_A) \longrightarrow \mathcal{I}^0 \otimes A \xrightarrow{d^0_A} \mathcal{I}^1 \otimes A \xrightarrow{d^1_A} \mathcal{I}^2 \otimes A \longrightarrow \cdots.$$

By Corollary A.2.8, $\mathrm{Ker}(d^0_A)$ is flat over A. The map $\mathrm{Ker}(d^0_A) \to \mathcal{F}$ is induced by the isomorphism $\mathrm{Ker}(d^0_A) \otimes_A \mathsf{K} \to \mathcal{F}$.

Vice versa, let $(\mathcal{F}_A, \alpha)$ be a deformation of $\mathcal{F}$ over $A \in \mathbf{Art}_{\mathsf{K}}$. We proceed by induction on the length $l(A)$ of the Artin ring, see Definition A.2.1. For $l(A) = 1$, $A = \mathsf{K}$ and there is nothing to prove. Let

$$0 \to \mathsf{K} \to A \to B \to 0$$

be a small extension. The sheaf $\mathcal{F}_B := \mathcal{F}_A \otimes_A B$ is a deformation of $\mathcal{F}$ over B, and by inductive hypothesis there exists a differential d_B on $\mathcal{I}\ \otimes B$ such that $\mathcal{H}^0_{d_B}(\mathcal{I}\ \otimes B) \cong \mathcal{F}_B$. Then there exists a diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, hook, "\varepsilon"'] & \mathcal{F}_A \arrow[r] \arrow[d, dashed, "\varepsilon_A"'] & \mathcal{F}_B \arrow[r] \arrow[d, hook, "\varepsilon_B"] & 0 \\
0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^0 \otimes A \arrow[r] & \mathcal{I}^0 \otimes B \arrow[r] & 0
\end{tikzcd}$$

where the first row is obtained by applying $\mathcal{F}_A \otimes_A -$ to the small extension, which is an exact functor because $\mathcal{F}_A$ is flat over A.

The claim is that there exists a map $\varepsilon_A\colon \mathcal{F}_A \to \mathcal{I}^0 \otimes A$ making the above diagram commutative. Notice that if ε_A exists, it is injective by the Five Lemma. There is an exact sequence

$$\mathrm{Hom}_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0 \otimes A) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0 \otimes B) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0).$$

We would like to say that $\mathrm{Ext}^1_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0) = 0$ by Corollary A.2.10, however the argument used relies on the existence of projective resolutions. We must then proceed locally; let $U = \operatorname{Spec} R \subset X$ be any affine open set, then by Corollary A.2.10 $\mathrm{Ext}^1_{R\ A}(\mathcal{F}_A|_{U\times \operatorname{Spec} A}, \mathcal{I}^0|_{U\times\operatorname{Spec} A}) = 0$ and by [34, III.6.2], we have that $\mathcal{E}xt^1_{X\times\operatorname{Spec} A}(\mathcal{F}_A, \mathcal{I}^0) = 0$. There is an isomorphism $\mathcal{H}om_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0) \cong i\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^0)$, where i denotes the inclusion $X \to X \times \operatorname{Spec} A$. By [32, Ch. II, 7.3.2], the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^0)$ is flasque because $\mathcal{I}^0$ is injective, and by [34, II.1.16], $i\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^0)$ is also flasque. Then $H^1(X \times \operatorname{Spec} A, \mathcal{H}om_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0)) = 0$, and by the local to global ext spectral sequence, we finally obtain $\mathrm{Ext}^1_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0) = 0$.

This means that there exists $\varepsilon_A\colon \mathcal{F}_A \to \mathcal{I}^0 \otimes A$, which reduces to ε modulo $\mathfrak{m}_A$.

We can iterate this procedure, by considering $\operatorname{Coker} \varepsilon_A$ in the place of $\mathcal{F}_A$; notice that it is flat over A by Corollary A.2.6.

□

Proposition 2.5.8. *Let $(\mathcal{I}\ , d_{\mathcal{I}})$ be an injective resolution of $\mathcal{F}$, then the isomorphism classes of deformations of $\mathcal{F}$ over $A \in \mathbf{Art}_{\mathsf{K}}$ are in bijective correspondence with the isomorphism classes of deformations of $(\mathcal{I}\ , d_{\mathcal{I}})$ over A.*

Proof. It is clear that isomorphic deformations of $(\mathcal{I}\ , d_{\mathcal{I}})$ give isomorphic deformations of $\mathcal{F}$.

Vice versa, let $(\mathcal{I}\ \otimes A, d_A)$ and $(\mathcal{I}\ \otimes A, d_{A})$ be deformations of $(\mathcal{I}\ , d_{\mathcal{I}})$ over A such that there exists an isomorphism

$$\mathcal{F}_A := \mathcal{H}^0_{d_A}(\mathcal{I}\ \otimes A) \xrightarrow{f} \mathcal{H}^0_{d_{A}}(\mathcal{I}\ \otimes A) =: \mathcal{F}_{A}.$$

We claim that this isomorphism f can be lifted to an isomorphism of complexes $(\mathcal{I}\ \otimes A, d_A) \to (\mathcal{I}\ \otimes A, d_{A})$, which reduces to the identity modulo $\mathfrak{m}_A$.

We proceed by induction on $l(A)$; let $0 \to \mathsf{K} \to A \to B \to 0$ be a small extension, and let $\mathcal{F}_B := \mathcal{F}_A \otimes_A B$ and $\mathcal{F}_{B} := \mathcal{F}_{A} \otimes_A B$. By inductive hypothesis, there exists an isomorphism F_B

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}^0 \arrow[r] \arrow[d, equal] & \mathcal{I}^0 \otimes A \arrow[r] \arrow[d, dashed, "F_A"] & \mathcal{I}^0 \otimes B \arrow[r] \arrow[d, "F_B"] & 0 \\
0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^0 \otimes A \arrow[r] & \mathcal{I}^0 \otimes B \arrow[r] & 0
\end{tikzcd}$$

which reduces to the identity modulo $\mathfrak{m}_B$ and such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_B \arrow[r, "\varepsilon_B"] \arrow[d, "f\ {}_A\mathrm{Id}_B"'] & \mathcal{I}^0 \otimes B \arrow[d, "F_B"] \\
0 \arrow[r] & \mathcal{F}_{B} \arrow[r, "\varepsilon_{B}"'] & \mathcal{I}^0 \otimes B.
\end{tikzcd} \tag{2.5.5}$$

Since $\mathcal{I}^0 \otimes A$ is flat over A, with the same argument used in the proof of Proposition 2.5.7, relying on Corollary A.2.10, we can lift F_B to $F_A \colon \mathcal{I}^0 \otimes A \to \mathcal{I}^0 \otimes A$, which is an isomorphism by the Five Lemma and which reduces to the identity modulo $\mathfrak{m}_A$.

However, it is not in general true that F_A makes the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_A \arrow[r, "\varepsilon_A"] \arrow[d, "f"'] & \mathcal{I}^0 \otimes A \arrow[d, "F_A"] \\
0 \arrow[r] & \mathcal{F}_{A} \arrow[r, "\varepsilon_{A}"'] & \mathcal{I}^0 \otimes A.
\end{tikzcd}$$

We have instead that $(F_A\varepsilon_A - \varepsilon_{A} f) \otimes_A \mathrm{Id}_B = 0$, because of (2.5.5). Then $F_A\varepsilon_A - \varepsilon_{A} f$ factors through $\mathcal{I}^0$, and via the isomorphism $\mathrm{Hom}_{\mathcal{O}_X\ A}(\mathcal{F}_A, \mathcal{I}^0) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^0)$ we obtain a map $x \colon \mathcal{F} \to \mathcal{I}^0$ such that $F_A\varepsilon_A - \varepsilon_{A} f = ix\alpha$:

$$\begin{tikzcd}
& \mathcal{I}^0 \arrow[dr, dashed, "y"] & \\
\mathcal{F} \arrow[ur, hook, "\varepsilon"] \arrow[rr, "x"] & & \mathcal{I}^0 \arrow[d, hook, "i"] \\
\mathcal{F}_A \arrow[u, "\alpha"] \arrow[urr] \arrow[rr, "F_A\varepsilon_A - \varepsilon_{A} f"'] & & \mathcal{I}^0 \otimes A,
\end{tikzcd}$$

Since $\mathcal{I}^0$ is injective, there exists a map y such that $y\varepsilon = x$, so that

$$iy\varepsilon\alpha = F_A\varepsilon_A - \varepsilon_{A} f.$$

Let $\pi\colon \mathcal{I}^0 \otimes A \to \mathcal{I}^0$, and define $y^A := y\pi$. Let $\widetilde{F_A} := F_A - iy^A$, then

$$\widetilde{F_A}\varepsilon_A = F_A\varepsilon_A - iy^A\varepsilon_A = F_A\varepsilon_A - iy\pi\varepsilon_A = F_A\varepsilon_A - iy\varepsilon\alpha = F_A\varepsilon_A - F_A\varepsilon_A + \varepsilon_A f = \varepsilon_A f,$$

so that $\widetilde{F_A}$ is the desired lifting.

To iterate this procedure, we consider the map $F_B\colon \mathcal{I}^1 \otimes B \to \mathcal{I}^1 \otimes B$ obtained by inductive hypothesis. □

Theorem 2.5.9. *Let $\mathcal{F}$ be a coherent sheaf and let $\mathcal{F} \to \mathcal{I}$ be an injective resolution, then the DG-Lie algebra $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$ controls the deformations of $\mathcal{F}$.*

Proof. By Propositions 2.5.7 and 2.5.8, there is an isomorphism of functors $\operatorname{Def}_{\mathcal{F}} \cong \operatorname{Def}_{(\mathcal{I}, d_{\mathcal{I}})}$. It is then easy to see that deformations of $(\mathcal{I}, d_{\mathcal{I}})$ are controlled by the DG-Lie algebra $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$. In fact, given $A \in \mathbf{Art}_{\mathsf{K}}$, the datum $(\mathcal{I} \otimes A, d_A)$ is a deformation of $(\mathcal{I}, d_{\mathcal{I}})$ if and only if d_A reduces to $d_{\mathcal{I}}$ modulo $\mathfrak{m}_A$ and $d_A^2 = 0$. This is equivalent to saying that $d_A = d_{\mathcal{I}} \otimes \operatorname{Id} + \xi$, with $\xi \in \operatorname{Hom}^1_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \otimes \mathfrak{m}_A$ and

$$0 = d_A^2 = (d_{\mathcal{I}} \otimes \operatorname{Id} + \xi)^2 = d\xi + \frac{1}{2}[\xi, \xi] = 0,$$

where $d = [d_{\mathcal{I}}, -]$ denotes the differential of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \otimes \mathfrak{m}_A$. Then a deformation corresponds to a Maurer–Cartan element of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \otimes \mathfrak{m}_A$.

Two deformations $(\mathcal{I} \otimes A, d_A = d_{\mathcal{I}} \otimes \operatorname{Id} + \xi)$ and $(\mathcal{I} \otimes A, d_A = d_{\mathcal{I}} \otimes \operatorname{Id} + \xi)$ are isomorphic if and only if there exists an isomorphism of complexes $\phi\colon (\mathcal{I} \otimes A, d_A) \to (\mathcal{I} \otimes A, d_A)$ which reduces to the identity modulo $\mathfrak{m}_A$. Then ϕ is of the form $\phi = \operatorname{Id} + \eta$, with $\eta \in \operatorname{Hom}^0_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \otimes \mathfrak{m}_A$, and since we are in characteristic zero, $\phi = e^a$, with $a \in \operatorname{Hom}^0_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I}) \otimes \mathfrak{m}_A$. The commutativity of ϕ with differentials translates to the equation

$$d_A e^a = e^a d_A \quad \Longleftrightarrow \quad d_{\mathcal{I}} + \xi = e^a(d_{\mathcal{I}} + \xi)e^{-a} \quad \Longleftrightarrow \quad \xi = e^a * \xi,$$

so that the deformations are isomorphic if and only if the corresponding Maurer–Cartan elements are gauge equivalent.

□

Corollary 2.5.10. *The tangent space to the functor $\operatorname{Def}_{\mathcal{F}}$ is given by $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$ and obstructions are contained in $\operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F})$.*

Proof. This follows from the description of the tangent space in (2.4.1), from Proposition 2.4.7, and from the well-known fact that the cohomology of the complex $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{I})$ computes $\operatorname{Ext}_X(\mathcal{F}, \mathcal{F})$. □

In the next part, we discuss the fact that the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ controls the deformations of a coherent sheaf $\mathcal{F}$ equipped with a finite locally free resolution $\mathcal{E} \to \mathcal{F}$. Notice that this follows automatically from Theorem 2.5.9 and Propositions 2.4.9 and 2.5.4, but we want to give here an idea of how the isomorphism

$$\operatorname{Def}_{\mathsf{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))} \to \operatorname{Def}_{\mathcal{F}}$$

works, based on the description contained in [24].

The first step for the Thom–Whitney model $\operatorname{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is to show that locally on an affine open set $U = \operatorname{Spec}(R)$, the deformations of $\mathcal{F}|_U$ are controlled by the DG-Lie algebra $\operatorname{Hom}_R(P, P)$ of graded endomorphisms of a projective resolution $P \to M$, where $\mathcal{F}|_U = \widetilde{M}$.

Let R be a commutative unitary algebra over a field K of characteristic zero, and let M be an R-module.

Definition 2.5.11. An infinitesimal deformation of M over $A \in \mathbf{Art}_{\mathsf{K}}$ is a pair (M_A, α), where M_A is a $R \otimes A$-module, flat over A, and $\alpha\colon M_A \to M$ is a morphism of $R \otimes A$-modules inducing an isomorphism $M_A \otimes_A \mathsf{K} \to M$.

Two deformations (M_A, α) and $(M_A, \alpha\ )$ are isomorphic if there exists an isomorphism of $R \otimes A$-modules $f\colon M_A \to M_A$ such that the following diagram commutes:

$$\begin{tikzcd} M_A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & M_A \arrow[dl, "\alpha"] \\ & M. & \end{tikzcd}$$

The map α is a morphism of $R \otimes A$-modules, where M is considered as an $R \otimes A$-module by restriction of scalars via the ring map $\theta\colon R \otimes A \to R \otimes \mathsf{K} = R$ induced by $\pi\colon A \to A/\mathfrak{m}_A = \mathsf{K}$. The isomorphism $M_A \otimes_A \mathsf{K} \to M$ is induced by α via the isomorphism

$$\mathrm{Hom}_{R\ \ A}(M_A, M) \cong \mathrm{Hom}_R(M_A \otimes_{R\ \ A} R, M) \cong \mathrm{Hom}_R(M_A \otimes_A \mathsf{K}, M), \tag{2.5.6}$$

induced by the ring map θ above.

The definition is functorial in A, and one can define the functor $\mathrm{Def}_M\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$, which associates to every $A \in \mathbf{Art}_{\mathsf{K}}$ the isomorphism classes of deformations of M over A.

Let now $(P\ , \partial)$ be a complex of projective R-modules.

Definition 2.5.12. An infinitesimal deformation of $(P\ , \partial)$ over $A \in \mathbf{Art}_{\mathsf{K}}$ is a complex of $R \otimes A$-modules $(P\ \otimes A, \partial_A)$ which reduces to $(P\ , \partial)$ modulo $\mathfrak{m}_A$. Two infinitesimal deformations $(P\ \otimes A, \partial_A)$ and $(P\ \otimes A, \partial_A)$ are isomorphic if there exists an isomorphism of complexes of $R \otimes A$-modules which reduces to the identity modulo $\mathfrak{m}_A$.

Theorem 2.5.13. *Let* $(P\ , \partial) \xrightarrow{\varphi} M$ *be a projective resolution of the* R*-module* M*. Then the isomorphism classes of deformations of* M *as an* R*-module are in bijective correspondence with the isomorphism classes of deformations of the complex* $(P\ , \partial)$*.*

Proof. We begin by proving that for every deformation $(P\ \otimes A, \partial_A)$ of $(P\ , \partial)$ over $A \in \mathbf{Art}_{\mathsf{K}}$, the module $H^0_{\partial_A}(P\ \otimes A)$ is a deformation of M over $A \in \mathbf{Art}_{\mathsf{K}}$.

By upper semicontinuity of the cohomology and by the fact that the complex $(P\ \otimes A, \partial_A)$ reduces to the resolution $P\ \to M$ modulo $\mathfrak{m}_A$, the complex is exact except at the zero level, hence there is an exact sequence:

$$\cdots \longrightarrow P^{-2} \otimes A \xrightarrow{\partial_A} P^{-1} \otimes A \xrightarrow{\partial_A} P^0 \otimes A \longrightarrow \mathrm{Coker}(\partial_A) \longrightarrow 0.$$

Each $P^i \otimes A$ is flat over A, because we have that $P^i \otimes A \otimes_A - \cong P^i \otimes -$, which is an exact functor. Then we can apply Corollary A.2.8 to obtain that $\mathrm{Coker}(\partial_A)$ is flat over A. The map $\mathrm{Coker}(\partial_A) \to M$ is induced by the isomorphism $\mathrm{Coker}(\partial_A) \otimes_A \mathsf{K} \cong M$, as in (2.5.6).

Vice versa, let (M_A, α) be a deformation of M over A. Notice that the map $\alpha\colon M_A \to M$ is surjective: in fact, by (2.5.6), α is the composition of the isomorphism $M_A \otimes_A \mathsf{K} \to M$ with the canonical surjective map $M_A \to M_A \otimes_A \mathsf{K}$ induced by $A \to A/\mathfrak{m}_A = \mathsf{K}$. Then, by the projectivity of $P^0 \otimes A$, see Lemma A.1.2, there exists a map φ_A of $R \otimes A$-modules

$$\begin{tikzcd} & M_A \arrow[d, "\alpha"] \\ P^0 \otimes A \arrow[ur, dashed, "\varphi_A"] \arrow[r, "\varphi\pi"] & M, \end{tikzcd}$$

where π denotes the map $\pi\colon P^0 \otimes A \to P^0$. This map reduces to $\varphi\colon P^0 \to M$ when tensoring by $- \otimes_A \mathsf{K}$, so it is surjective, by Lemma A.2.5.

The sequence

$$0 \longrightarrow \mathrm{Ker}(\varphi_A) \longrightarrow P^0 \otimes A \xrightarrow{\varphi_A} M_A \longrightarrow 0$$

is exact: by Corollary A.2.6 $\operatorname{Ker}(\varphi_A)$ is flat over A and the map $\operatorname{Ker}(\varphi_A)\otimes_A \mathsf{K} \to P^0 \otimes A \otimes_A \mathsf{K} = P^0$ is injective. Hence there is an isomorphism $\operatorname{Ker}(\varphi_A) \otimes_A \mathsf{K} \cong \operatorname{Ker}(\varphi)$ which defines a surjective morphism $\operatorname{Ker}(\varphi_A) \to \operatorname{Ker}(\varphi)$. We can iterate the above construction with this map and the surjective morphism $P^{-1} \otimes A \to \operatorname{Ker}(\varphi)$. In this way, we obtain a differential ∂_A on $P \otimes A$ such that $H^0_{\partial_A}(P \otimes A) \cong M$.

Next, we need to show that if (M_A, α) and (M_A, α) are isomorphic deformations of M, and ∂_A, ∂_A are differentials on $P \otimes A$ such that $H^0_{\partial_A}(P \otimes A) \cong M_A$ and $H^0_{\partial_A}(P \otimes A) \cong M_A$, then $(P \otimes A, \partial_A)$ and $(P \otimes A, \partial_A)$ are isomorphic deformations of P .

Let $f\colon M_A \to M_A$ be an isomorphism such that $\alpha f = \alpha$; we need to construct a lifting of f which is an isomorphism of complexes of $R \otimes A$-modules and which reduces to the identity modulo $\mathfrak{m}_A$. Consider the pullback

$$\begin{tikzcd}
P^0 \times_M M_A \arrow[r] \arrow[d] & M_A \arrow[d, "\alpha"] \\
P^0 \arrow[r, "\varphi"'] & M.
\end{tikzcd}$$

By the universal property of the pullback, there exist maps u, v making the following diagrams commute:

$$\begin{tikzcd}
P^0 \otimes A \arrow[dr, dashed, "v"] \arrow[drr, bend left, "\varphi_A"] \arrow[ddr, bend right, "\pi"'] & & \\
& P^0 \times_M M_A \arrow[r] \arrow[d] & M_A \arrow[d, "\alpha"] \\
& P^0 \arrow[r, "\varphi"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
P^0 \otimes A \arrow[dr, dashed, "u"] \arrow[drr, bend left, "f\varphi_A"] \arrow[ddr, bend right, "\pi"'] & & \\
& P^0 \times_M M_A \arrow[r] \arrow[d] & M_A \arrow[d, "\alpha"] \\
& P^0 \arrow[r, "\varphi"'] & M.
\end{tikzcd}$$

We claim that the map v is surjective. Take (x, y) in $P^0 \times_M M_A$, i.e. $x \in P^0$ and $y \in M_A$ such that $\varphi(x) = \alpha(y)$. Both the maps π and φ_A are surjective, so there exist $z, t \in P^0 \otimes A$ such that $\pi(z) = x$ and $\varphi_A(t) = y$. Then $\alpha \varphi_A(z) = \varphi\pi(z) = \varphi(x)$ and $\alpha \varphi_A(t) = \alpha(y)$, so that $\varphi_A(t - z)$ belongs to the kernel of α , which is $M_A \otimes_A \mathfrak{m}_A$. The morphism

$$\varphi_A\colon P^0 \otimes_A \mathfrak{m}_A \to M_A \otimes_A \mathfrak{m}_A$$

is surjective, so there exists w in $P^0 \otimes_A \mathfrak{m}_A$ such that $\varphi_A(w) = \varphi_A(t-z)$, and then $v(z-w) = (x, y)$.

Since v is surjective, by the projectivity of $P^0 \otimes A$ there exists a map F such that the diagram commutes:

$$\begin{tikzcd}
& P^0 \otimes A \arrow[d, two heads, "v"] \\
P^0 \otimes A \arrow[ur, dashed, "F"] \arrow[r, "u"] & P^0 \otimes_M M_A.
\end{tikzcd}$$

It is immediate to check that the map F is such that $f\varphi_A = \varphi_A F$, and that it reduces to the identity when tensored by $- \otimes_A \mathsf{K}$. It is also an isomorphism, by Lemma A.2.5. The construction can be iterated by considering the isomorphism of $R \otimes A$-modules $\operatorname{Ker}(\varphi_A) \to \operatorname{Ker}(\varphi_A)$ given by F.

□

Proposition 2.5.14. *Deformations of a complex of projective R-modules (P, ∂) are controlled by the DG-Lie algebra $L = \operatorname{Hom}_R(P, P)$, i.e., there is an isomorphism $\operatorname{Def}_{(P, \partial)} \cong \operatorname{Def}_L$.*

Proof. A deformation of the complex (P, ∂) over $A \in \mathbf{Art}_{\mathsf{K}}$ is the datum of a degree one map $\partial_A\colon P \otimes A \to P \otimes A$ which squares to zero and which reduces to ∂ modulo $\mathfrak{m}_A$. The last condition means that ∂_A is of the form $\partial_A = \partial \otimes \mathrm{Id}_A + \xi$, with $\xi\colon P\ \otimes A \to P\ \otimes \mathfrak{m}_A$. Equivalently, ξ belongs to $\mathrm{Hom}^1_R(P\ ,P\ ) \otimes \mathfrak{m}_A$. Since ∂_A has to square to zero,

$$0 = \partial_A^2 = (\partial \otimes \mathrm{Id}_A + \xi)^2 = (\partial \otimes \mathrm{Id}_A)^2 + \xi^2 + [\partial \otimes \mathrm{Id}_A, \xi] = \xi^2 + [\partial \otimes \mathrm{Id}_A, \xi] = d(\xi) + \frac{1}{2}[\xi, \xi],$$

so that ξ is a Maurer–Cartan element of $\mathrm{Hom}_R(P\ ,P\ ) \otimes \mathfrak{m}_A$.

Two deformations $(P\ \otimes A, \partial_A)$ and $(P\ \otimes A, \partial_A)$ of the complex $(P\ , \partial)$ are isomorphic if there exists an isomorphism of complexes $\phi\colon (P\ \otimes A, \partial_A) \to (P\ \otimes A, \partial_A)$ which reduces to the identity modulo $\mathfrak{m}_A$. This means that ϕ is of the form $\phi = \mathrm{Id} + \eta$, with $\eta \in \mathrm{Hom}^0_R(P\ ,P\ ) \otimes \mathfrak{m}_A$. Since the DG-Lie algebra $L \otimes \mathfrak{m}_A$ is nilpotent, and the characteristic is zero, ϕ is of the form $\phi = e^a$, with $a \in \mathrm{Hom}^0_R(P\ ,P\ ) \otimes \mathfrak{m}_A$. The commutativity of ϕ with differentials is given by the equation

$$\partial_A e^a = e^a \partial_A \quad \iff \quad \partial_A = e^a \partial_A e^{-a} \quad \iff \quad \xi\ = e^a(\partial \otimes \mathrm{Id} + \xi)e^{-a} - \partial \otimes \mathrm{Id},$$

which is exactly $\xi\ = e^a * \xi$, where $*$ denotes the gauge action of Definition 2.4.3. □

Corollary 2.5.15. *Deformations of an R-module M are controlled by the DG-Lie algebra $L = \mathrm{Hom}_R(P\ ,P\ )$, where $P\ \to M$ is projective resolution.*

Remark 2.5.16. The quasi-isomorphism class of the DG-Lie algebra $\mathrm{Hom}_R(P\ ,P\ )$ does not depend on the choice of a projective resolution P of M; for a proof see [70, Lemma 4.4].

We pass now on to the global case: let $\mathcal{F}$ be a coherent sheaf on a smooth separated scheme X of finite type over a field K of characteristic zero, admitting a finite locally free resolution $\mathcal{E}\ \to \mathcal{F}$. Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X and let R_i be K-algebras such that $\mathrm{Spec}(R_i) = U_i$ for every i. Since the sheaves $\mathcal{F}, \mathcal{E}^j|_{U_i}$ are coherent, on every U_i they are of the form $\mathcal{F}|_{U_i} = \widetilde{M_i}$, $\mathcal{E}^j|_{U_i} = \widetilde{E^j_i}$ for E^j_i projective R_i-modules.

For every open set $U_{i_0\cdots i_k}$ one has that $\mathcal{E}\ |_{U_{i_0\cdots i_k}} \to \mathcal{F}|_{U_{i_0\cdots i_k}}$ is a projective resolution. Following [24], we can use the results above to deform the sheaf $\mathcal{F}$ locally and then glue together the local deformations.

By Theorem 2.5.13 and Proposition 2.5.14, a deformation of $\mathcal{F}|_{U_i}$ over $A \in \mathbf{Art}_{\mathsf{K}}$ is equivalent to a deformation of the complex $\mathcal{E}\ |_{U_i}$, hence it corresponds to an element $l_i \in \mathrm{MC}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_i) \otimes \mathfrak{m}_A)$ modulo the gauge action.

Denote by d the differential $[d_{\mathcal{E}}, -]$, and let $l = \{l_i\} \in \prod_i \mathcal{E}nd^1(\mathcal{E}\ )(U_i) \otimes \mathfrak{m}_A$ be such that $dl_i + \frac{1}{2}[l_i, l_i] = 0$ for all $i \in I$, so that l_i defines a deformation of the complex $(\mathcal{E}\ |_{U_i}, d_{\mathcal{E}})$ for every i. We need these deformations to glue, hence we need isomorphisms between the deformed complexes on the double intersections U_{ij}. These isomorphisms must reduce to the identity modulo $\mathfrak{m}_A$, and so are of the form

$$e^{m_{ij}}\colon (\mathcal{E}\ |_{U_{ij}} \otimes A, d_{\mathcal{E}} + l_j) \to (\mathcal{E}\ |_{U_{ij}} \otimes A, d_{\mathcal{E}} + l_i), \quad m_{ij} \in \prod_{i<j} \mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ij}) \otimes \mathfrak{m}_A.$$

The condition of compatibility with the differentials translates to $(d_{\mathcal{E}} + l_i|_{U_{ij}})e^{m_{ij}} = e^{m_{ij}}(d_{\mathcal{E}} + l_j|_{U_{ij}})$, which can be rewritten as $l_i|_{U_{ij}} = e^{m_{ij}} * l_j|_{U_{ij}}$ for all $i < j$.

The delicate part is that it is not necessary to glue the deformed complexes $(\mathcal{E}\ |_{U_i} \otimes A, d_{\mathcal{E}} + l_i)$, but their cohomology sheaves. Therefore the isomorphisms $\{e^{m_{ij}}\}$ do not have to satisfy the cocycle condition, but rather the cocycle condition up to homotopy. Taking the logarithm and denoting by $\bullet$ the Baker–Campbell–Hausdorff product, this condition can be expressed as

$$m_{jk}|_{U_{ijk}} \bullet -m_{ik}|_{U_{ijk}} \bullet m_{ij}|_{U_{ijk}} = [d_{\mathcal{E}} + l_j|_{U_{ijk}}, n_{ijk}] = dn_{ijk} + [l_j|_{U_{ijk}}, n_{ijk}]$$

for some $n = \{n_{ijk}\} \in \prod_{i<j<k} \mathcal{E}nd^{-1}_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ijk}) \otimes \mathfrak{m}_A$.

The above data (l, m) ensures that the the sheaves $\mathcal{F}_{A,U_i} := \mathcal{H}^0(\mathcal{E}\ |_{U_i} \otimes A, d_{\mathcal{E}} + l_i)$ glue to give a deformation $\mathcal{F}_A$ of $\mathcal{F}$ over A, and every deformation of $\mathcal{F}$ can be obtained in this way; we refer again to [24] for details of this correspondence.

The next step is understanding how isomorphisms of deformations of $\mathcal{F}$ translate to this data. Let $\mathcal{F}_A$ and $\mathcal{F}_A$ be isomorphic deformations of $\mathcal{F}$, with isomorphism $f\colon \mathcal{F}_A \to \mathcal{F}_A$ and corresponding to the data (l, m) and $(l\ , m\ )$ respectively. The map f restricts to isomorphisms $\mathcal{F}_A(U_i) \to \mathcal{F}_A(U_i)$ for every i, which can be lifted to isomorphisms to the resolutions $f_i\colon (\mathcal{E}\ |_{U_i} \otimes A, d_{\mathcal{E}} + l_i) \to (\mathcal{E}\ |_{U_i} \otimes A, d_{\mathcal{E}} + l_i)$. Since every f_i reduces to the identity modulo $\mathfrak{m}_A$, these isomorphisms must be of the form

$$e^{a_i}\colon (\mathcal{E}\ |_{U_i} \otimes A, d_{\mathcal{E}} + l_i) \to (\mathcal{E}\ |_{U_i} \otimes A, d_{\mathcal{E}} + l_i), \quad a = \{a_i\} \in \prod_i \mathcal{E}nd^0(\mathcal{E}\ )(U_i) \otimes \mathfrak{m}_A.$$

Since they have to be compatible with differentials, the condition $e^{a_i} * l_i = l_i$ has to be satisfied for all i.

Lastly, the isomorphisms $\{e^{a_i}\}$ have to commute with the isomorphisms $\{e^{m_{ij}}\}, \{e^{m_{ij}}\}$ in cohomology. This means that for every $i < j$ the composition $e^{-m_{ij}} e^{-a_i} e^{m_{ij}} e^{a_j}$ is homotopic to the identity, therefore, taking the logarithm:

$$-m_{ij} \bullet -a_i|_{U_{ij}} \bullet m_{ij} \bullet a_j|_{U_{ij}} = [d_{\mathcal{E}} + l_j|_{U_{ij}}, b_{ij}] = db_{ij} + [l_j|_{U_{ij}}, b_{ij}]$$

for some $b = \{b_{ij}\} \in \prod_{i<j} \mathcal{E}nd^{-1}_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ij}) \otimes \mathfrak{m}_A$.

Denoting by $\mathfrak{l}$ the semicosimplicial DG-Lie algebra associated to $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )$:

$$\mathfrak{l}: \quad \prod_i \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_i) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{i,j} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ij}) \underset{\delta_2}{\overset{\delta_0}{\overset{\delta_1}{\rightrightarrows\!\!\!\to}}} \prod_{i,j,k} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )(U_{ijk}) \quad \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \quad \cdots .$$

the deformation data can be summarised as follows

$$\left\{ (l, m) \in (\mathfrak{l}_0^1 \oplus \mathfrak{l}_1^0) \otimes \mathfrak{m}_A \;\middle|\; \begin{array}{c} dl + \frac{1}{2}[l, l] = 0, \\ \delta_1 l = e^m * \delta_0 l, \\ \delta_0 m \bullet -\delta_1 m \bullet \delta_2 m = dn + [\delta_2\delta_0 l, n] \quad \exists n \in \mathfrak{l}_2^{-1} \otimes \mathfrak{m}_A. \end{array} \right\}$$

This data defines a functor $Z^1_{sc}(\exp \mathfrak{l})\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$. One can then define an equivalence relation on $Z^1_{sc}(\exp \mathfrak{l})(A)$: two elements (l, m) and $(l\ , m\ )$ are equivalent if and only if there exists elements $a \in \mathfrak{l}_0^0 \otimes \mathfrak{m}_A$, $b \in \mathfrak{l}_1^{-1} \otimes \mathfrak{m}_A$ such that

$$e^a * l = l\ , \quad -m \bullet -\delta_1 a \bullet m\ \bullet \delta_0 a = db + [\delta_0 l, b].$$

We refer again to [24] for a proof of the fact that this is an equivalence relation.

This defines a functor

$$H^1_{sc}(\exp \mathfrak{l})\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}, \quad H^1_{sc}(\exp \mathfrak{l})(A) := \frac{Z^1_{sc}(\exp \mathfrak{l})(A)}{\sim}.$$

Notice if the semicosimplicial DG-Lie algebra $\mathfrak{l}$ is concentrated in degree zero, i.e., it is a semicosimplicial Lie algebra, these functors reduce to the functors of Subsection 2.4.1.

When $\mathcal{U}$ is an affine open cover, the above considerations can be rephrased as an isomorphism of functors

$$\mathrm{Def}_{\mathcal{F}} \cong H^1_{sc}(\exp \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}\ )).$$

Theorem 2.5.17 ([24, Theorem 7.6]). *Let $\mathfrak{l}$ be a semicosimplicial DG-Lie algebra such that $H^j(\mathfrak{l}_i) = 0$ for all $i \geq 0$ and $j < 0$. Then there is a natural isomorphism of functors*

$$\mathrm{Def}_{\mathsf{Tot}(\mathfrak{l})} \cong H^1_{sc}(\exp \mathfrak{l}).$$

In particular the tangent space to $H^1_{sc}(\exp \mathfrak{l})$ is $H^1(\mathrm{Tot}(\mathfrak{l}))$ and obstructions are contained in $H^2(\mathrm{Tot}(\mathfrak{l}))$.

Since negative Ext groups between coherent sheaves are always trivial, as a corollary of the above theorem one obtains that the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}^\cdot))$ controls the deformations of the sheaf $\mathcal{F}$.

Let X be now a complex manifold, $\mathcal{F}$ a coherent sheaf and $\mathcal{E}^\cdot \to \mathcal{F}$ a finite locally free resolution.

Proposition 2.5.18. *The Dolbeault model* $(A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)), [-,-], \overline{\partial} + [d_{\mathcal{E}}, -])$ *controls the deformations of the coherent sheaf* $\mathcal{F}$*.*

Proof. By Propositions 2.5.4 and 2.5.5, there is a quasi-isomorphism of DG-Lie algebras between $A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot))$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^\cdot, \mathcal{I}^\cdot)$. By Theorem 2.5.9 there is an isomorphism of functors of Artin rings $\mathrm{Def}_{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^\cdot, \mathcal{I}^\cdot)} \to \mathrm{Def}_{\mathcal{F}}$, and finally by Proposition 2.4.9 two quasi-isomorphic DG-Lie algebras give isomorphic deformation functors. □

Following [24] again, the isomorphism between the deformation functor associated to the Dolbeault DG-Lie algebra $A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot))$ and the functor of deformations of $\mathcal{F}$ can be described concretely. Let A be in $\mathbf{Art}_{\mathsf{K}}$, and consider a Maurer-Cartan element $\xi \in \mathrm{MC}(A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)) \otimes \mathfrak{m}_A)$. The element ξ belongs to $(A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)))^1 \otimes \mathfrak{m}_A$, so, via the isomorphism

$$\begin{aligned}\mathcal{A}_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)) \otimes \mathfrak{m}_A &\cong \mathcal{H}om_{\mathcal{A}_X^{0,\cdot}}(\mathcal{A}_X^{0,\cdot}(\mathcal{E}), \mathcal{A}_X^{0,\cdot}(\mathcal{E})) \otimes \mathfrak{m}_A \\ &\cong \mathcal{H}om_{\mathcal{A}_X^{0,\cdot} \otimes A}(\mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes A, \mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes \mathfrak{m}_A),\end{aligned}$$

the element ξ belongs to $\mathrm{Hom}^1_{\mathcal{A}_X^{0,\cdot} \otimes A}(\mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes A, \mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes \mathfrak{m}_A)$, so it induces a degree one map $\xi \colon \mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes A \to \mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes A$ with image contained in $\mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes \mathfrak{m}_A$. Therefore $\overline{\partial} + d_{\mathcal{E}} + \xi$ is a degree one map from $\mathcal{A}_X^{0,\cdot}(\mathcal{E}) \otimes A$ to itself which reduces to $\overline{\partial} + d_{\mathcal{E}}$ modulo $\mathfrak{m}_A$, and it squares to zero precisely because ξ satisfies the Maurer-Cartan equation. Then the isomorphism of functors is given as follows:

$$\begin{gathered}\mathrm{Def}_{A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot))}(A) \to \mathrm{Def}_{\mathcal{F}}(A) \\ \xi \in \mathrm{MC}(A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)) \otimes \mathfrak{m}_A) \mapsto \mathcal{H}^0(\mathcal{A}_X^{0,\cdot}(\mathcal{E}^\cdot) \otimes A, \overline{\partial} + d_{\mathcal{E}} + \xi).\end{gathered}$$

Chapter 3

Semiregularity maps

This chapter concerns the semiregularity maps studied by Severi, Kodaira and Spencer, Bloch, and Buchweitz and Flenner, and their relevance in the deformation theory of subvarieties and coherent sheaves. In the first two sections, a brief history of the semiregularity maps of Severi, Kodaira–Spencer and Bloch is outlined. The Atiyah class of a coherent sheaf is described in Section 3.3, and some representatives of it for locally free sheaves are given. Finally, the Buchweitz–Flenner semiregularity map is described in Section 3.4, and the annihilation of obstructions is discussed in Section 3.5.

3.1 The semiregularity maps of Severi and Kodaira–Spencer

The notion of semiregularity was introduced in 1944 by Severi [71], who defined a curve C on a surface S to be semiregular if the canonical linear system of the surface cuts out a complete linear system on C. This can be rephrased in the following way: denoting by ω_S the canonical sheaf of the surface S, a curve C on S is semiregular if the restriction map

$$r\colon H^0(S,\omega_S) \to H^0(C,\omega_S|_C)$$

is surjective. By Serre duality, this is equivalent to asking for the map

$$\sigma_S\colon H^1(C,\mathcal{O}_C(C)) \cong H^1(C,\mathcal{N}_{C|S}) \to H^2(S,\mathcal{O}_S),$$

the **Severi semiregularity map**, to be injective.

In 1959 Kodaira and Spencer [46] generalised Severi's definition to compact submanifolds of codimension 1 of a complex manifold. They defined a submanifold Z of X of codimension 1 to be semiregular if the map

$$v\colon H^1(X,\mathcal{O}_X(Z)) \to H^1(Z,\mathcal{O}_Z(Z)) \cong H^1(Z,\mathcal{N}_{Z|X})$$

induced by the restriction $\mathcal{O}_X(Z) \to \mathcal{O}_Z(Z)$ has zero image. In the above, $\mathcal{N}_{Z|X}$ denotes the normal bundle of Z in X.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(Z) \longrightarrow \mathcal{O}_Z(Z) \cong \mathcal{N}_{Z|X} \longrightarrow 0,$$

obtained by tensoring the ideal sheaf sequence of $Z \subset X$ with the invertible sheaf $\mathcal{O}_X(Z)$. From the resulting long exact sequence in cohomology

$$\cdots \longrightarrow H^1(X,\mathcal{O}_X(Z)) \xrightarrow{v} H^1(Z,\mathcal{N}_{Z|X}) \xrightarrow{\sigma_{KS}} H^2(X,\mathcal{O}_X) \longrightarrow \cdots$$

one can see that the map v has image zero if and only if the **Kodaira–Spencer semiregularity map**

$$\sigma_{KS}\colon H^1(Z,\mathcal{N}_{Z|X}) \to H^2(X,\mathcal{O}_X)$$

is injective.

The property of semiregularity was considered by Severi, Kodaira and Spencer in the context of completeness of characteristic systems, for which we follow the notation of [43].

Let D be an effective locally principal divisor on a complete variety X over an algebraically closed field K, and let $\mathcal{D}$ be a flat family of deformations of D, i.e. the data of a variety W, a point $w \in W$ and an effective divisor on $X \times_{\mathsf{K}} W$, flat over W, whose fibre over w is equal to D.

Definition 3.1.1. The characteristic map of $\mathcal{D}$ is the linear map

$$\rho\colon \Theta_w(W) \to H^0(D, \mathcal{N}_{D|X})$$

and its characteristic system is the linear system on D cut out by the image of ρ.

The characteristic system is said to be complete if the map ρ is surjective.

For a rigorous definition of the characteristic map we refer to [46] or to [65, Lecture 22]; intuitively it is obtained in the following way: to every $v \in \Theta_w(W)$ it is associated a canonical map $f\colon \operatorname{Spec} \mathsf{K}[\varepsilon]/(\varepsilon^2) \to W$, which induces a family of divisors $\mathcal{D}_f \subset X \times_{\mathsf{K}} \operatorname{Spec} \mathsf{K}[\varepsilon]/(\varepsilon^2)$ extending D, i.e., an element of $H^0(D, \mathcal{N}_{D|X})$.

The connection between completeness of the characteristic system and smoothness of the functor of embedded deformations of $D \subset X$ is discussed for instance in [43, p. 305]. In particular, when the parameter space W is smooth at w and the characteristic map ρ is surjective, the functor of embedded deformations is smooth.

The main result of Severi's work [71] was the proof of the theorem of completeness of characteristic systems of complete continuous systems for semiregular curves on algebraic surfaces, which therefore implies:

Theorem 3.1.2 (Severi)**.** *Every semiregular curve $C \subset S$ has unobstructed embedded deformations.*

Kodaira and Spencer proved the theorem of completeness of characteristic systems of complete continuous systems for semiregular submanifolds of codimension 1 of higher dimensional manifolds, which states the existence of a flat family of deformations of a codimension 1 submanifold Z whose characteristic system is complete and whose parameter space is smooth at the point representing a semiregular Z. As before, from this follows the result:

Theorem 3.1.3 (Kodaira–Spencer)**.** *Every semiregular hypersurface has unobstructed embedded deformations.*

From Kodaira and Spencer's proof of the above theorem, the following more general result, not stated explicitly in [46], can be deduced; see in particular [46, proof of Theorem 1, p. 488]:

Theorem 3.1.4 (Kodaira–Spencer, implicit)**.** *The Kodaira–Spencer semiregularity map*

$$\sigma_{KS}\colon H^1(Z, \mathcal{N}_{Z|X}) \to H^2(X, \mathcal{O}_X)$$

annihilates every obstruction to embedded deformations.

3.2 The Bloch semiregularity map

In 1972 Bloch [12] constructed a semiregularity map for locally complete intersection subschemes of a smooth complex projective variety, which reduces to Kodaira and Spencer's semiregularity map when the codimension is 1.

Let X be a smooth projective variety over C, let $Z \subset X$ be a locally complete intersection of codimension q, and denote by $\mathcal{N}_{Z|X}$ the normal bundle of Z in X. The **Bloch semiregularity map**

$$\sigma_B\colon H^1(Z, \mathcal{N}_{Z|X}) \to H^{q+1}(X, \Omega_X^{q-1}),$$

is constructed as follows, as described in [12, 16]. Let $m := \dim Z$, so that $\dim X = m + q$, and denote by ω_X and ω_Z be the canonical sheaves of X and Z respectively. Denote by $\mathcal{I}$ the ideal sheaf of Z in X, and let $\bar{d}\colon \mathcal{I}/\mathcal{I}^2 \cong \mathcal{N}_{Z|X} \to \Omega^1_X \otimes \mathcal{O}_Z$ be the map induced by the second exact sequence of Kähler differentials, see [61, Theorem 58]. Then there is a natural pairing

$$\Omega^{m+1}_X \times \Lambda^{q-1}\mathcal{N}_{Z|X} \xrightarrow{\mathrm{Id} \times \Lambda^{q-1}\bar{d}} \Omega^{m+1}_X \times \Omega^{q-1}_X \otimes \mathcal{O}_Z \longrightarrow \omega_X \otimes \mathcal{O}_Z,$$

or equivalently there exists a map

$$\Omega^{m+1}_X \to \Lambda^{q-1}\mathcal{N}_{Z|X} \otimes \omega_X.$$

By the adjunction formula $\omega_Z \cong \det \mathcal{N}_{Z|X} \otimes \omega_X$ and by the fact that $\Lambda^{q-1}\mathcal{N}_{Z|X} \cong \mathcal{N}_{Z|X} \otimes \det \mathcal{N}_{Z|X}$, this gives a map

$$\Omega^{m+1}_X \to \mathcal{N}_{Z|X} \otimes \omega_Z.$$

In cohomology we obtain

$$H^{m-1}(X, \Omega^{m+1}_X) \to H^{m-1}(Z, \mathcal{N}_{Z|X} \otimes \omega_Z),$$

and by Serre duality this gives a map

$$\sigma_B\colon H^1(Z, \mathcal{N}_{Z|X}) \to H^{q+1}(X, \Omega^{q-1}_X),$$

which is exactly the Bloch semiregularity map.

Like Severi and Kodaira–Spencer, Bloch defines $Z \subset X$ to be semiregular if the semiregularity map for Z is injective. His main result is:

Theorem 3.2.1 (Bloch)**.** *Every semiregular locally complete intersection subvariety has unobstructed embedded deformations.*

Theorem 3.2.2 (Bloch, implicit)**.** *The semiregularity map σ_B annihilates every* simple *obstruction to embedded deformations of $Z \subset X$.*

As in the case of Theorem 3.1.4, by implicit we mean that the above result is not stated explicitly in [12], but can be inferred from the proof of Theorem 3.2.1.

As discussed in Section 2.3, simple obstructions do not in general generate the whole obstruction space, but in characteristic zero their vanishing is enough to ensure smoothness, see e.g. [58, 59]. Bloch's proof is based on Hodge theory and the Gauss–Manin connection. The restriction to simple obstructions is then natural in this context, as these tools can be used only for smooth proper families of manifolds over certain bases, for instance smooth bases.

It is worth mentioning that Bloch's semiregularity map has an application to the variational Hodge conjecture. In fact, Bloch proved the variational Hodge conjecture for cycle classes of the form $[Z]$, for a semiregular locally complete intersection $Z \subset X$.

Theorem 3.2.3 (Bloch)**.** *Let $f\colon X \to S$ be a smooth projective morphism with S smooth, connected of finite type over $\mathbb{C}$. Let $0 \in S$ and let $z \in \Gamma(S, \mathrm{R}^{2p}f\ (\Omega_{X/S}))$ be a horizontal section of the de Rham cohomology. Suppose $z_0 = z|_{X_0} \in H^{2p}_{DR}(X_0, \mathbb{C})$ is algebraic, representing a local complete intersection $Z_0 \subseteq X_0$ which is semiregular in X_0. Then $z_s = z|_{X_s}$ is algebraic for all $s \in S$.*

3.3 The Atiyah class

The Atiyah class of a coherent sheaf of $\mathcal{O}_X$-modules was introduced by Atiyah in 1957 [3] and is defined as follows. Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X over $\mathbb{C}$, and consider the Atiyah sequence

$$0 \longrightarrow \mathcal{F} \otimes \Omega^1_X \longrightarrow J^1_X(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow 0, \tag{3.3.1}$$

where the sheaf $J^1_X(\mathcal{F})$ of 1-jets or principal parts of $\mathcal{F}$ is defined as a sheaf of C-modules as $J^1_X(\mathcal{F}) = \mathcal{F} \oplus \mathcal{F} \otimes \Omega^1_X$, with $\mathcal{O}_X$-action given by

$$f \cdot (s, \sigma) = (fs, f\sigma + s \otimes df) \quad f \in \mathcal{O}_X,\ s \in \mathcal{F},\ \sigma \in \mathcal{F} \otimes \Omega^1_X.$$

The map $\mathcal{F} \otimes \Omega^1_X \to J^1_X(\mathcal{F})$ is given by the inclusion $i_2(\sigma) = (0, \sigma)$, while the map $J^1_X(\mathcal{F}) \to \mathcal{F}$ is given by $p_1(s, \sigma) = s$.

Definition 3.3.1. The Atiyah class $\mathrm{At}(\mathcal{F}) \in \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F} \otimes \Omega^1_X)$ of the coherent sheaf $\mathcal{F}$ is the extension class of the short exact sequence (3.3.1).

Denote by d the universal derivation $d\colon \mathcal{O}_X \to \Omega^1_X$.

Definition 3.3.2. A global algebraic connection on a coherent sheaf $\mathcal{F}$ is a C-linear map of sheaves

$$\nabla\colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$$

satisfying

$$\nabla(fe) = e \otimes df + f\nabla(e), \quad \forall f \in \mathcal{O}_X, e \in \mathcal{F}.$$

One can easily see that the Atiyah sequence splits if and only if there exists a global algebraic connection on a $\mathcal{F}$. In fact, the sequence splits if and only if there exists a morphism of sheaves $t\colon \mathcal{F} \to J^1(\mathcal{F})$ such that $p_1 t = \mathrm{Id}_{\mathcal{F}}$. Then t is of the form $\mathrm{Id}_{\mathcal{F}} + u$, with $\mathrm{Id}_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}$ and $u\colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$,

$$(\mathrm{Id}_{\mathcal{F}} + u)(fs) = (fs, u(fs)) = f \cdot (s, u(s)) = (fs, fu(s) + s \otimes df), \quad \forall f \in \mathcal{O}_X,\ s \in \mathcal{F},$$

and u is exactly an algebraic connection on $\mathcal{F}$. Hence the Atiyah class of $\mathcal{F}$ is the obstruction to the existence of a global algebraic connection on $\mathcal{F}$.

The Atiyah class can be constructed more generally for any object E in the bounded derived category of coherent sheaves $D(X)$. Denoting by $\mathcal{I}$ the ideal sheaf of the diagonal $\Delta\colon X \to X \times X$, the ideal sheaf short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{X \times X} \longrightarrow \Delta_* \mathcal{O}_X \longrightarrow 0$$

induces the short exact sequence

$$e\colon \quad 0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_{X \times X}/\mathcal{I}^2 \longrightarrow \Delta_* \mathcal{O}_X \longrightarrow 0, \tag{3.3.2}$$

where there is an isomorphism $\mathcal{I}/\mathcal{I}^2 \cong \Delta_* \Omega^1_X$. Denoting by $p, q\colon X \times X \to X$ the two projections and by $\Phi_{\mathcal{K}}(E) = \mathsf{R}p_* (\mathcal{K} \otimes^{\mathsf{L}} q^* E)$ the Fourier–Mukai transform with kernel $\mathcal{K} \in D(X \times X)$, the Atiyah class of E is given by

$$\mathrm{At}(E) = [\Phi_e(E)] = [\mathsf{R}p_* (e \otimes^{\mathsf{L}} q^* E)] \in \mathrm{Ext}^1_X(E, E \otimes \Omega^1_X). \tag{3.3.3}$$

When $E = \mathcal{F}$ is a coherent sheaf, the exact sequence (3.3.2) gives rise to an exact sequence

$$\Phi_e(\mathcal{F})\colon \quad 0 \longrightarrow \Phi_{\Delta_* \Omega^1_X}(\mathcal{F}) \longrightarrow \Phi_{\mathcal{O}_{X \times X}/\mathcal{I}^2}(\mathcal{F}) \longrightarrow \Phi_{\Delta_* \mathcal{O}_X}(\mathcal{F}) \longrightarrow 0,$$

and using the derived projection formula for the map $\Delta\colon X \to X \times X$ and the fact that Δ_* is an exact functor,

$$\Phi_{\Delta_* \Omega^1_X}(\mathcal{F}) = \mathsf{R}p_* (\Delta_* \Omega^1_X \otimes^{\mathsf{L}} q^* \mathcal{F}) = \mathsf{R}p_* \mathsf{R}\Delta_* (\Omega^1_X \otimes^{\mathsf{L}} \mathsf{L}\Delta^* q^* \mathcal{F}) = \Omega^1_X \otimes \mathcal{F},$$

and analogously $\Phi_{\Delta_* \mathcal{O}_X}(\mathcal{F}) = \mathcal{F}$, so that

$$\Phi_e(\mathcal{F})\colon \quad 0 \longrightarrow \mathcal{F} \otimes \Omega^1_X \longrightarrow \Phi_{\mathcal{O}_{X \times X}/\mathcal{I}^2}(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Equivalently, following [60], since the terms of the sequence (3.3.2) are supported on the diagonal one may consider it as a sequence of sheaves of $\mathcal{O}_X - \mathcal{O}_X$-bimodules on X; the two $\mathcal{O}_X$-module structures coincide on $\mathcal{I}/\mathcal{I}^2$ and $\Delta\ \mathcal{O}_X$ but not on $\mathcal{O}_{X\times X}/\mathcal{I}^2$. The Atiyah class of $\mathcal{F}$ can be thought as obtained by tensoring $\mathcal{F}$ with the exact sequence (3.3.2) using the left $\mathcal{O}_X$-module structure and considering it via the right $\mathcal{O}_X$-module structure.

For a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on a complex manifold X the Atiyah class $\mathrm{At}(\mathcal{E}) \in H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \otimes \Omega^1_X)$ is the obstruction to the existence of a holomorphic connection and it is represented by cohomology class of the $(1,1)$ component of the curvature of any connection of type $(1,0)$ on $\mathcal{E}$.

Denote by $\mathcal{A}^{p,q}_X$ the sheaf of differential forms of type (p,q), by $\mathcal{A}^{p,q}_X(\mathcal{E})$ the sheaf of differential forms of type (p,q) with coefficients in $\mathcal{E}$, and by $A^{p,q}_X$ and $A^{p,q}_X(\mathcal{E})$ their respective global sections. Denote by $d\colon \mathcal{A}_X^{\cdot} \to \mathcal{A}_X^{\cdot+1}$ the de Rham differential and by $\overline{\partial}$ both the Dolbeault differential $\overline{\partial}\colon \mathcal{A}_X^{\cdot,\cdot} \to \mathcal{A}_X^{\cdot,\cdot+1}$ and the operator $\overline{\partial}\colon \mathcal{A}_X^{\cdot,\cdot}(\mathcal{E}) \to \mathcal{A}_X^{\cdot,\cdot+1}(\mathcal{E})$.

Definition 3.3.3. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules on a complex manifold X.

1. A holomorphic connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map of sheaves $\nabla\colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ such that
$$\nabla(fs) = f\nabla(s) + df \otimes s, \quad \forall s \in \mathcal{E},\ f \in \mathcal{O}_X.$$
2. A connection of type $(1,0)$ on $\mathcal{E}$, also called a connection compatible with the holomorphic structure, is a $\mathbb{C}$-linear sheaf homomorphism
$$D\colon \mathcal{A}^0_X(\mathcal{E}) \to \mathcal{A}^1_X(\mathcal{E}) = \mathcal{A}^{1,0}_X(\mathcal{E}) \oplus \mathcal{A}^{0,1}_X(\mathcal{E})$$
such that the Leibniz rule
$$D(fs) = d(f)s + fD(s), \quad f \in \mathcal{A}^0_X,\ s \in \mathcal{A}^0_X(\mathcal{E})$$
holds and such that
$$D = D^{1,0} + \overline{\partial}, \quad D^{1,0}\colon \mathcal{A}^{0,0}_X(\mathcal{E}) \to \mathcal{A}^{1,0}_X(\mathcal{E}), \quad \overline{\partial}\colon \mathcal{A}^{0,0}_X(\mathcal{E}) \to \mathcal{A}^{0,1}_X(\mathcal{E}).$$

Notice that a holomorphic connection on $\mathcal{E}$ is the same as an algebraic connection on $\mathcal{E}$ (Definition 3.3.2), the name just changes depending on the complex or algebraic situation. Therefore holomorphic connections exist if and only if the Atiyah sequence of (3.3.1) splits. On the other hand, by a partitions of unity argument it is easy to see that connections of type $(1,0)$ always exist.

Lemma 3.3.4. *A representative of the Atiyah class for a locally free sheaf of $\mathcal{O}_X$-modules on a complex manifold is given by the cohomology class of the component of type $(1,1)$ of the curvature of a connection of type $(1,0)$.*

Proof. Let D be a connection of type $(1,0)$ on $\mathcal{E}$, then $D - \overline{\partial} = D^{1,0}$ restricts to a holomorphic connection $\mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ if and only if it sends holomorphic sections to holomorphic sections, i.e., if and only if $[\overline{\partial}, D - \overline{\partial}] = [\overline{\partial}, D] = 0$. The component of type $(1,1)$ of the curvature
$$R = D^2 = ((D-\overline{\partial}) + \overline{\partial})^2 = (D-\overline{\partial})^2 + [\overline{\partial}, D-\overline{\partial}] \in A^{2,0}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \oplus A^{1,1}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$$
is $[\overline{\partial}, D-\overline{\partial}] = [\overline{\partial}, D]$. This is trivial in the cohomology of the complex $(A^{1,\cdot}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})), \overline{\partial})$ if and only if there exists $\varphi \in A^{1,0}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ such that $[\overline{\partial}, D - \varphi] = 0$, so that $D^{1,0} - \varphi$ restricts to a holomorphic connection on $\mathcal{E}$.

Vice versa, if there exists a holomorphic connection $\nabla\colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$, it can be extended to a connection of type $(1,0)$
$$\nabla + \overline{\partial}\colon \mathcal{A}^0_X(\mathcal{E}) \to \mathcal{A}^1_X(\mathcal{E}),$$
which is such that $[\overline{\partial}, \nabla + \overline{\partial}] = [\overline{\partial}, \overline{\partial}] = 0$, because ∇ sends holomorphic sections to holomorphic sections. $\square$

The Chern connection is an example of connection of type $(1,0)$ and its curvature is only of type $(1,1)$, so the cohomology class of the curvature of the Chern connection represents the Atiyah class, see e.g. [38, 4.3.10].

The above construction can be generalised to a finite complex of locally free sheaves by defining connections of type $(1,0)$ for finite complexes of locally free sheaves; this a key ingredient of Chapter 4.

It is possible to give a representative of $\operatorname{At}(\mathcal{E})$ in Čech cohomology, for a locally free sheaf $\mathcal{E}$ on a smooth separated scheme X of finite type over the field K.

Lemma 3.3.5. *A representative of the Atiyah class of a locally free sheaf is given by the Čech cocycle* $\{\alpha_{i_0i_1} = \nabla_{i_0}|_{U_{i_0i_1}} - \nabla_{i_1}|_{U_{i_0i_1}}\}$*, where* ∇_i *are local algebraic connections.*

Proof. Let $\mathcal{U} = \{U_i\}$ be a an affine open cover of X such that the restriction of the Atiyah sequence

$$0 \longrightarrow \mathcal{E} \otimes \Omega^1_X|_{U_i} \longrightarrow J^1_X(\mathcal{E})|_{U_i} \longrightarrow \mathcal{E}|_{U_i} \longrightarrow 0,$$

splits for every i, and let $\nabla_i \colon \mathcal{E}|_{U_i} \to \mathcal{E} \otimes \Omega^1_X|_{U_i}$ be local algebraic connections. In the Čech complex

$$(C\ \ (\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \check{d})$$

one can define the cocycle

$$\alpha \in C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \quad \alpha_{i_0i_1} = \nabla_{i_0}|_{U_{i_0i_1}} - \nabla_{i_1}|_{U_{i_0i_1}}.$$

Its cohomology class is trivial if and only if there exists a global algebraic connection on $\mathcal{E}$: in fact, $\alpha = \check{d}\beta$ if and only if $\{\nabla_i - \beta_i\}$ glue to a global algebraic connection. □

It is possible to generalise this construction to a finite complex of locally free sheaves; this is done in [39, Section 10] and will be used in Lemma 4.5.4 to prove the equivalence with our definition of Atiyah class.

When X is smooth, every coherent sheaf $\mathcal{F}$ has locally finite projective dimension and there exist trace maps

$$\operatorname{Tr}^p \colon \operatorname{Ext}^p_X(\mathcal{F}, \mathcal{F} \otimes \Omega^q_X) \to H^p(X, \Omega^q_X), \quad \forall\, p, q \geq 0.$$

Combining the Yoneda product with the exterior product on Ω_X one obtains products

$$\grave{}\ \colon \operatorname{Ext}^a_X(\mathcal{F}, \mathcal{F} \otimes \Omega^b_X) \times \operatorname{Ext}^c_X(\mathcal{F}, \mathcal{F} \otimes \Omega^d_X) \to \operatorname{Ext}^{a+c}_X(\mathcal{F}, \mathcal{F} \otimes \Omega^{b+d}_X),$$

which allow to construct powers of the Atiyah class

$$\operatorname{At}^p(\mathcal{F}) \in \operatorname{Ext}^p_X(\mathcal{F}, \mathcal{F} \otimes \Omega^p_X)$$

and then the exponential of the Atiyah class, which is called the Atiyah–Chern character

$$e^{\mathsf{At}(\mathcal{F})} := \sum_{p\geq 0} \frac{\operatorname{At}^p(\mathcal{F})}{p!} \in \prod_{p \geq 0} \operatorname{Ext}^p_X(\mathcal{F}, \mathcal{F} \otimes \Omega^p_X).$$

Theorem 3.3.6 (Atiyah [3], Illusie [42])**.** *For the Chern character of a coherent sheaf* $\mathcal{F}$

$$\operatorname{ch}(\mathcal{F}) = \operatorname{Tr}(e^{-\mathsf{At}(\mathcal{F})}) \in \prod_{p \geq 0} H^p(X, \Omega^p_X).$$

3.4 The Buchweitz–Flenner semiregularity map

The third generalisation of the semiregularity map is due to Buchweitz and Flenner in 1999 [15, 16], and it is defined not for a subvariety but more generally for a coherent sheaf.

Definition 3.4.1. Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X, the **Buchweitz–Flenner semiregularity map** of $\mathcal{F}$ is defined as

$$\sigma\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \prod_{p\geq 0} H^{p+2}(X,\Omega^p_X), \quad \sigma(c) = \operatorname{Tr}(e^{-\operatorname{At}(\mathcal{F})} \smile c).$$

It can be convenient to write the map as the sum of its components:

$$\sigma = \sum_{p\geq 0}\sigma_p\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \prod_{p\geq 0} H^{p+2}(X,\Omega^p_X), \quad \sigma_p\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^{p+2}(X,\Omega^p_X),$$

$$\sigma_p(c) = \operatorname{Tr}\left(\frac{(-1)^p \operatorname{At}^p(\mathcal{F}) \smile c}{p!}\right),$$

It is not immediate to see how the Buchweitz–Flenner semiregularity map generalises the Bloch semiregularity map σ_B; if $Z \subset X$ has codimension q, Buchweitz and Flenner proved that σ_B is the composition of the $(q-1)$st component of the semiregularity map for the sheaf $\mathcal{O}_Z$,

$$\sigma_{q-1}\colon \operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z) \to H^{q+1}(X,\Omega^{q-1}_X),$$

and the natural map $H^1(Z,\mathcal{N}_{Z|X}) \to \operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z)$.

The map $H^1(Z,\mathcal{N}_{Z|X}) \to \operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z)$ can be obtained by considering the extension defined by the first fundamental neighbourhood of Z in X

$$\mathcal{Z}^{(2)}: \qquad 0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_X/\mathcal{I}^2 \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where $\mathcal{I}$ denotes the ideal sheaf of $Z \subset X$. This induces a map $\operatorname{Ext}^1_X(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Z) \to \operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z)$ which can be composed with the forgetful map $\operatorname{Ext}^1_Z(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Z) \to \operatorname{Ext}^1_X(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_Z)$.

The map $H^1(Z,\mathcal{N}_{Z|X}) \to \operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z)$ also has an interpretation in deformation theory, as the obstruction map associated to the forgetful natural transformation from the functor of embedded deformations of Z inside X to the functor of deformations of $\mathcal{O}_Z$ as an $\mathcal{O}_X$-module. As recalled before, see e.g. [35], the tangent space of the functor of embedded deformations of Z inside X is isomorphic to $H^0(Z,\mathcal{N}_{Z|X})$, and there exists a complete obstruction theory with values in $H^1(Z,\mathcal{N}_{Z|X})$. For the functor of deformations of a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, the tangent space is given by $\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F})$ and there exists a complete obstruction theory with values in $\operatorname{Ext}^2_X(\mathcal{F},\mathcal{F})$, see Corollary 2.5.10. The map $H^1(Z,\mathcal{N}_{Z|X}) \to \operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z)$ is the obstruction map of the morphism of deformation theories $\operatorname{Hilb}^Z_X \to \operatorname{Def}_{\mathcal{O}_Z}$.

Buchweitz and Flenner's main result in deformation theory is the following:

Theorem 3.4.2 ([16])**.** *The semiregularity map of a coherent sheaf $\mathcal{F}$ on a smooth projective variety X*

$$\sigma\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \prod_{p\geq 0} H^{p+2}(X,\Omega^p_X), \quad \sigma(c) = \operatorname{Tr}(e^{-\operatorname{At}(\mathcal{F})} \smile c)$$

annihilates all simple *obstructions to deformations of $\mathcal{F}$. In particular if σ is injective, then $\mathcal{F}$ has unobstructed deformations.*

As remarked above, in characteristic zero the vanishing of the simple obstructions is enough to ensure smoothness, see Remark 2.3.10.

Like Bloch's, Buchweitz–Flenner's semiregularity map has an application to the variational Hodge conjecture. Buchweitz and Flenner call a coherent sheaf $\mathcal{F}$ k-semiregular if the component σ_k of the semiregularity map is injective. They proved the variational Hodge conjecture for cycles that are representable as the $(k+1)$st component of the Chern character of a k-semiregular sheaf $\mathcal{F}$.

3.5 Annihilation of all obstructions

Buchweitz and Flenner left unanswered the question of whether their semiregularity map annihilates all obstructions to deformations of a coherent sheaf. The strategy they suggested in [16] to prove this is to realise each component of the semiregularity map as the obstruction map of a morphism of deformation theories, the target one unobstructed, which would automatically imply the annihilation of all obstructions:

> *"Ideally, the semiregularity map, say, for a module $\mathcal{F}$ should correspond to a morphism between two deformation theories so that it maps the obstruction space $\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})$ into the obstruction space of some other deformation theory. It seems quite clear that this second deformation theory should be given in terms of the intermediate Jacobians, or, more naturally, by Deligne cohomology."*

This strategy can be employed for the 0th component of the semiregularity map, which is simply the trace map, recovering a result by Artamkin [2] and Mukai [64]:

Theorem 3.5.1 (Artamkin)**.** *Let $\mathcal{F}$ be a coherent sheaf on a complex projective manifold X. Then the 0th semiregularity map (=trace) $\sigma_0 = \mathrm{Tr}^2\colon \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \to H^2(X,\mathcal{O}_X)$ annihilates all obstructions to deformations of $\mathcal{F}$.*

This map σ_0 has a natural interpretation from the point of view of deformation theory [41]. Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X over an algebraically closed field of characteristic 0, so that in particular $\mathcal{F}$ admits a finite locally free resolution. Consider the trace map $\mathrm{Tr}\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \to \mathcal{O}_X$, and the induced maps in hypercohomology

$$\mathrm{Tr}^i\colon \mathrm{Ext}^i_X(\mathcal{F},\mathcal{F}) \to H^i(X,\mathcal{O}_X), \quad i \geq 0,$$

see e.g. [2, 44]. A deformation of the sheaf $\mathcal{F}$ naturally induces a deformation of the determinant line bundle $\det\mathcal{F}$, hence there exists a morphism of deformation functors $\mathrm{Def}_{\mathcal{F}} \to \mathrm{Def}_{\det\mathcal{F}}$. The tangent and obstructions spaces to $\mathrm{Def}_{\mathcal{F}}$ are $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{F})$ and $\mathrm{Ext}^2_X(\mathcal{F},\mathcal{F})$ respectively, while the tangent and obstruction spaces to $\mathrm{Def}_{\det\mathcal{F}}$ are $H^1(X,\mathcal{O}_X)$ and $H^2(X,\mathcal{O}_X)$ respectively. The maps Tr^1 and Tr^2 have an interpretation in deformation theory: they are induced on tangent and obstructions spaces by the morphism $\mathrm{Def}_{\mathcal{F}} \to \mathrm{Def}_{\det\mathcal{F}}$. In characteristic zero, the functor $\mathrm{Def}_{\det\mathcal{F}}$ is smooth (see e.g. [65, Lecture 25]) and hence by the argument outlined in Remark 2.4.8 the trace map $\mathrm{Tr}\colon \mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \to H^2(X,\mathcal{O}_X)$ annihilates all obstructions, and one recovers Theorem 3.5.1.

When trying to prove that each higher component σ_k of the semiregularity map is the obstruction map of a morphism of deformation theories with unobstructed target, it is necessary to identify such an unobstructed target. In the quoted text above, Buchweitz and Flenner suggest that this should be an intermediate Jacobian or given by Deligne cohomology. This last strategy is used by Pridham in [68] in the setting of derived algebraic geometry, where using as target an analogue of Deligne cohomology defined in terms of cyclic homology, he proves the following:

Theorem 3.5.2 (Pridham)**.** *For every coherent sheaf $\mathcal{F}$ on a complex projective manifold, the semiregularity map σ annihilates all obstructions.*

On the other hand, Fiorenza and Manetti proved that the Abel–Jacobi map is the tangent map of a morphism of deformation theories, where the target is an intermediate Jacobian [26], and Iacono and Manetti proved that the Bloch semiregularity map for a locally complete intersection subvariety with extendable normal bundle is the obstruction map of a morphism of deformation theories with target an intermediate Jacobian, and hence that it annihilates all obstructions to embedded deformations [40], see also [57].

Denoting by $\Omega_X^{\leq p-1}$ the truncated de Rham complex, the tangent space and obstruction space to the p-th intermediate Jacobian $J^p(X)$ are given by the vector spaces

$$\mathsf{H}^{2p-1}(X, \mathcal{O}_X \xrightarrow{d_{dR}} \cdots \xrightarrow{d_{dR}} \Omega_X^{p-1}) \cong \mathsf{H}^1(X, \Omega_X^{\leq p-1}[2(p-1)])$$

and

$$\mathsf{H}^{2p}(X, \mathcal{O}_X \xrightarrow{d_{dR}} \cdots \xrightarrow{d_{dR}} \Omega_X^{p-1}) \cong \mathsf{H}^2(X, \Omega_X^{\leq p-1}[2(p-1)])$$

respectively. Since the k-th component of the Buchweitz–Flenner semiregularity has target $H^{k+2}(X, \Omega_X^k) \cong H^2(X, \Omega_X^k[k])$, it makes sense to compose it with the map induced in hypercohomology by the inclusion of complexes $\iota_k \colon \Omega_X^k[k] \to \Omega_X^{\leq k}[2k]$:

$$\iota_k \colon H^2(X, \Omega_X^k[k]) \to \mathsf{H}^2(X, \Omega_X^{\leq k}[2k]).$$

Notice that when the Hodge–de Rham spectral sequence of X degenerates as E_1 (for instance when X is a complex projective manifold) this map is injective, and hence σ_k and $\iota_k\sigma_k$ have the same kernel. It is convenient to call $\iota_k\sigma_k$ the k-th component of the ***modified Buchweitz–Flenner semiregularity map***:

$$\iota_k\sigma_k \colon \operatorname{Ext}_X^2(\mathcal{F}, \mathcal{F}) \xrightarrow{\sigma_k} H^{k+2}(X, \Omega_X^k) = H^2(X, \Omega_X^k[k]) \xrightarrow{\iota_k} \mathsf{H}^2(X, \Omega_X^{\leq k}[2k]).$$

Choosing the approach to deformation theory via DG-Lie algebras and L_∞-morphisms, as described in Section 2.4, to show that each component of the modified Buchweitz–Flenner semiregularity map is the obstruction map of a morphism of deformation theories, we need to show that there exists an L_∞ morphism between DG-Lie algebras that induces each component of the modified semiregularity map in cohomology. If the DG-Lie algebra that is the target of this L_∞ morphism is abelian, i.e. it has trivial bracket, then its associated deformation functor is unobstructed and we obtain automatically that each component of the semiregularity map annihilates all obstructions to deformations of a coherent sheaf. This is the strategy that will be employed in the next chapter.

Chapter 4

L_∞ liftings of semiregularity maps

This chapter, based on the paper [4], describes the construction of L_∞ liftings of each component of the Buchweitz–Flenner semiregularity map for coherent sheaves on complex manifolds. This is done by introducing the notion of Chern–Simons classes for curved DG-pairs, and by proving that a particular case of this general construction, where the curved DG-pair is provided by a connection of type $(1,0)$ on a finite complex of locally free sheaves, provides canonical L_∞ liftings of the components of the Buchweitz–Flenner semiregularity map. In view of the discussion in Section 3.5, this implies the fact that the Buchweitz–Flenner semiregularity map annihilates all obstructions to deformations of a coherent sheaf.

In Section 4.1 curved DG-pairs and their associated Atiyah classes and semiregularity maps are described. Chern–Simons classes of curved DG-algebras are introduced in Section 4.2, while the proof of the main result is contained in Section 4.3. Section 4.4 contains explicit formulas for the components of the L_∞ morphisms in the case the curved DG-pair is split. The geometric application is contained in Section 4.5, where a notion of connection of type $(1,0)$ on a finite complex of locally free sheaves is introduced. This gives rise to a curved DG-pair, whose associated Atiyah class is the usual Atiyah class of Section 3.3. This particular case of the construction allows to construct L_∞ liftings of the components of the Buchweitz–Flenner semiregularity map, and finally Section 4.5 contains our main result regarding deformation theory.

Finally, in Section 4.6, which is based on [50], we give an alternative way of lifting the first component of the semiregularity map, based on the properties of cyclic forms.

4.1 Atiyah classes and semiregularity maps for curved DG-pairs

Let A be a graded associative algebra, as in Definition 1.1.8. For every vector subspace $E \subset A$ we shall denote by $E^{(k)}$, $k \geq 1$, the linear span of all the products $e_1 \cdots e_k$, with $e_i \in E$ for every i, and by EA the linear span of all the products ea, with $e \in E$ and $a \in A$.

Definition 4.1.1. A **curved DG-algebra** is the datum $(A, d, \cdot, R)$ of a graded associative algebra $(A, \cdot)$ together with a degree one derivation $d\colon A \to A^{+1}$ and a degree two element $R \in A^2$, called **curvature**, such that

$$d(R) = 0, \qquad d^2(x) = [R, x] = R \cdot x - x \cdot R \quad \forall\, x \in A.$$

For notational simplicity we shall write (A, d, R) in place of $(A, d, \cdot, R)$ when the product $\cdot$ is clear from the context. We denote by $[A, A] \subset A$ the linear span of all the graded commutators $[a, b] = ab - (-1)^{\bar{a}\bar{b}} ba$. Following [30] we call $A/[A, A]$ the **cyclic space** of A and we denote by

$$\mathrm{tr}\colon A \to A/[A, A]$$

the quotient map. Notice that $[A, A]$ is a homogeneous Lie ideal and then the cyclic space inherits a natural structure of DG-Lie algebra with trivial bracket.

Definition 4.1.2. Let $A = (A, d, R)$ be a curved DG-algebra. A **curved Lie ideal** in A is a homogeneous Lie ideal $I \subset A$ such that $d(I) \subset I$ and $R \in I$.

A **curved DG-pair** is the data (A, I) of a curved DG-algebra A equipped with a curved Lie ideal I.

In particular, for every curved DG-pair (A, I), the quotient A/I is a (non-curved) DG-Lie algebra and, for every $k \geq 1$, the subset $I^{(k)}A$ is an associative bilateral ideal of A.

Example 4.1.3. It is useful to briefly anticipate from Section 4.5 the following paradigmatic geometric example of curved DG-pair. Let $\mathcal{E}$ be a holomorphic vector bundle on a complex manifold X equipped with a connection of type $(1, 0)$ as in Definition 3.3.3, and denote by $R \in A_X^{1,1}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \oplus A_X^{2,0}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ the curvature. Denoting by d the induced connection on the associated bundle $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$, we have that $(A_X^{,}\,(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})), d, R)$ is a curved DG-algebra and $I = A_X^{>0,}\,(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ is a curved Lie ideal. In this case the DG-Lie algebra $A/I = A_X^{0,}\,(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ is the Dolbeault resolution of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ and controls the deformations of the vector bundle $\mathcal{E}$, see Proposition 2.5.18 or [28]. Notice that I is also an associative ideal and $I^{(k)}A = I^{(k)} = A_X^{\geq k,}\,(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ for every $k > 0$.

The classical theory of Atiyah classes and the above example suggest the introduction of the following objects associated to a curved DG-pair.

Definition 4.1.4. Let $A = (A, d, R)$ be a curved DG-algebra and $I \subset A$ a curved Lie ideal. The **Atiyah cocycle** of the pair (A, I) is the class of R in the DG-vector space $\dfrac{I + I^{(2)}A}{I^{(2)}A}$. The **Atiyah class** of the pair (A, I) is the cohomology class of the Atiyah cocycle:

$$\operatorname{At}(A, I) = [R] \in H^2\left(\frac{I + I^{(2)}A}{I^{(2)}A}\right).$$

Definition 4.1.5. Let $A = (A, d, R)$ be a curved DG-algebra and $I \subset A$ a curved Lie ideal. For every integer $k \geq 0$, we introduce the morphism of complexes of vector spaces

$$\sigma_1^k\colon \frac{A}{I} \to \frac{A}{[A, A] + I^{(k+1)}A}[2k], \qquad \sigma_1^k(x) = \frac{1}{k!}\operatorname{tr}(R^k x)\,.$$

Notice that σ_1^k depends only on the Atiyah cocycle of the pair (A, I), while the induced map in cohomology

$$\sigma_1^k\colon H\ \left(\frac{A}{I}\right) \to H^{2k+}\ \left(\frac{A}{[A, A] + I^{(k+1)}A}\right), \qquad \sigma_1^k(x) = \frac{1}{k!}\operatorname{tr}(\operatorname{At}(A, I)^k x)\,,$$

depends only on the Atiyah class.

The **semiregularity map** of the curved DG-pair (A, I) is defined as the degree 2 component

$$\sigma_1^k\colon H^2\left(\frac{A}{I}\right) \to H^{2k+2}\left(\frac{A}{[A, A] + I^{(k+1)}A}\right), \qquad \sigma_1^k(x) = \frac{1}{k!}\operatorname{tr}(\operatorname{At}(A, I)^k x)\,,$$

of the above map.

Remark 4.1.6. The name semiregularity map is clearly motivated by the analogous definition by Buchweitz and Flenner [16], see Definition 3.4.1. More precisely, every morphism σ_1^k factors as the composition of two morphisms of differential graded vector spaces

$$\frac{A}{I} \xrightarrow{\tau_1^k} \frac{I^{(k)}A + [A, A]}{[A, A] + I^{(k+1)}A} \hookrightarrow \frac{A}{[A, A] + I^{(k+1)}A}, \qquad \tau_1^k(x) = \frac{1}{k!}\operatorname{tr}(R^k x)\,,$$

and the direct generalisation of Buchweitz–Flenner's semiregularity maps should be the maps induced by τ_1^k in the group $H^2(A/I)$, up to signs. However, several geometric considerations about Abel–Jacobi maps, see Section 3.5 and the introduction of [40], strongly suggest that, from the point of view of deformation theory, the right objects to consider are the maps σ_1^k.

Remark 4.1.7. For every $x \in A^1$ one can consider the twisted derivation $d_x = d + [x, -]$ and an easy computation shows that $A_x := (A, d_x, R_x)$ remains a curved DG-algebra with curvature $R_x = R + d(x) + \frac{1}{2}[x, x]$. In particular, if $x \in I$ then I is a curved Lie ideal also for A_x, the derivations d, d_x induce the same differential in A/I and $A/[A, A]$, the difference $R_x - R$ is exact in $A/[A, A]$ and therefore the semiregularity maps of the pairs (A, I) and (A_x, I) induce the same map in cohomology.

Since $[A, A]+I^{(k+1)}A$ is a Lie ideal, the space $\dfrac{A}{[A, A] + I^{(k+1)}A}[2k]$ inherits from A a structure of DG-Lie algebra with trivial bracket and it is obvious that σ_1^0 is a morphism of DG-Lie algebras. It is easy to see that in general σ_1^k is not a morphism of DG-Lie algebras for $k > 0$. It is therefore natural to ask whether σ_1^k is the linear component of an L_∞ morphism.

In Section 4.3 we prove the following result:

Theorem 4.1.8 (=Corollary 4.3.10)**.** *Let I be a curved Lie ideal of a curved DG-algebra (A, d, R) and denote by $\pi\colon A \to A/I$ the projection. Then to every $k \geq 0$ and every morphism of graded vector spaces $s\colon A/I \to A$ such that $\pi s = \mathrm{Id}_{A/I}$ it is canonically associated an L_∞ morphism*

$$\sigma^k\colon \frac{A}{I} \qquad \frac{A}{[A, A] + I^{(k+1)}A}[2k]$$

with linear component the map σ_1^k.

The proof of the above theorem is constructive and an explicit description of the higher components of σ^k is possible but rather cumbersome for general sections s. In Section 4.4 we study the higher components of the L_∞ morphism of Theorem 4.1.8 under the additional assumption that s is a morphism of graded Lie algebras. This hypothesis is satisfied in most of the applications and has the effect of a dramatic simplification of the algebraic and combinatorial aspects.

Definition 4.1.9 ([69])**.** A *trace map* on a curved DG-algebra (A, d, R) is the data of a complex of vector spaces (C, δ) and a morphism of graded vector spaces $\mathrm{Tr}\colon A \to C$ such that $\mathrm{Tr} \circ d = \delta \circ \mathrm{Tr}$ and $\mathrm{Tr}([A, A]) = 0$.

Thus every trace map $\mathrm{Tr}\colon A \to C$ factors to a morphism of abelian DG-Lie algebras $A/[A, A] \to C$ and we have the following immediate consequence of the above theorem.

Corollary 4.1.10. *Let I be a curved Lie ideal of a curved DG-algebra (A, d, R) and let $\mathrm{Tr}\colon A \to C$ be a trace map. Then for every $k \geq 0$ there exists an L_∞ morphism*

$$\eta^k\colon \frac{A}{I} \qquad \frac{C}{\mathrm{Tr}(I^{(k+1)}A)}[2k]$$

with linear component

$$\eta_1^k\colon \frac{A}{I} \to \frac{C}{\mathrm{Tr}(I^{(k+1)}A)}[2k], \qquad \eta_1^k(x) = \frac{1}{k!}\,\mathrm{Tr}(R^k x)\,.$$

We then have a clear application of the above results to deformation theory. In the situation of Corollary 4.1.10, denote for simplicity by C_k the quotient complex $C_k := C/\,\mathrm{Tr}(I^{(k+1)}A)$, and suppose that a given deformation problem is controlled by the DG-Lie algebra A/I. Then the L_∞ morphism η^k induces a morphism of deformation functors

$$\eta^k\colon\ \mathrm{Def}_{A/I} \to \mathrm{Def}_{C_k[2k]}$$

that at the level of tangent and obstruction spaces gives the maps

$$H^i(A/I) \to H^{2k+i}(C_k), \qquad x \mapsto \frac{1}{k!}\,\mathrm{Tr}(\mathrm{At}(A, I)^k x), \qquad i = 1, 2.$$

Since $C_k[2k]$ is abelian, the deformation functor $\mathrm{Def}_{C_k[2k]}$ is unobstructed and therefore the above map $H^2(A/I) \to H^{2k+2}(C_k)$ annihilates the obstructions, see Remark 2.4.8.

4.2 Curved DG-algebras and Chern–Simons classes

The general theory of Chern–Simons classes for differential graded associative algebras [69] extends naturally to the curved case.

Let (A, d, R) be a curved DG-algebra. Then for every $x \in A^1$ we have the twisted curved DG-algebra $A_x := (A, d_x, R_x)$, where

$$d_x(a) = d(a) + [x, a], \quad R_x = R + d(x) + x^2 = R + d(x) + \frac{1}{2}[x, x].$$

Let t be a central indeterminate of degree 0, and consider the family of polynomials

$$P(t)_x^k = \sum_{i=1}^{k} R_{tx}^{i-1} x R_{tx}^{k-i} = \sum_{i=1}^{k} (R + td(x) + t^2x^2)^{i-1} x (R + td(x) + t^2x^2)^{k-i} \in A[t]\,,$$

with $k \geq 0$ an integer and $x \in A^1$.

Lemma 4.2.1. *In the above notation, for every $k \geq 0$ and every $x \in A^1$ we have*

$$R_x^k - R^k = d\left(\int_0^1 P(t)_x^k dt\right) + \left[x, \int_0^1 tP(t)_x^k dt\right].$$

Proof. In the graded algebra $A[t]$ consider the derivations $\partial_t = \dfrac{d}{dt}$ and $d_{tx} = d + [tx, -]$. Since $R_x^k - R^k = \int_0^1 \partial_t(R_{tx}^k)dt$ it is sufficient to prove that

$$\partial_t(R_{tx}^k) = d(P(t)_x^k) + [x, tP(t)_x^k] = d_{tx}(P(t)_x^k)\,. \tag{4.2.1}$$

Since $d^2(x) = [R, x]$, $d(x^2) = \frac{1}{2}d[x, x] = [d(x), x]$, $[x^2, x] = 0$, we have

$$d_{tx}(R_{tx}) = d(R + td(x) + t^2x^2) + [tx, R + td(x) + t^2x^2] = td^2(x) + t^2d(x^2) - t[R + td(x), x] = 0$$

and

$$\partial_t(R_{tx}) = \partial_t(R + td(x) + t^2x^2) = d(x) + 2tx^2 = d_{tx}(x)\,.$$

By the Leibniz formula, for every $k \geq 0$ we have

$$d_{tx}(P(t)_x^k) = \sum_{i=1}^{k} d_{tx}\left(R_{tx}^{i-1} x R_{tx}^{k-i}\right) = \sum_{i=1}^{k} R_{tx}^{i-1} d_{tx}(x) R_{tx}^{k-i} = \sum_{i=1}^{k} R_{tx}^{i-1} \partial_t(R_{tx}) R_{tx}^{k-i} = \partial_t(R_{tx}^k).$$

□

Denote by $\operatorname{tr}\colon A \to A/[A, A]$ the projection; this is the universal trace of A in the sense that every trace map $A \to C$ is induced from tr by a unique morphism of DG-vector spaces $A/[A, A] \to C$. For notational simplicity we denote by $a \overset{\mathrm{tr}}{=} b$ the fact that $\operatorname{tr}(a) = \operatorname{tr}(b)$.

Following the theory of Chern classes, we can define the (universal) Chern character

$$\operatorname{ch}(A) = \sum_{k \geq 0} \operatorname{ch}(A)_k, \qquad \operatorname{ch}(A)_k \in H^{2k}\left(\frac{A}{[A, A]}\right),$$

where $\operatorname{ch}(A)_k$ is the cohomology class of $\dfrac{1}{k!}\operatorname{tr}(R^k)$.

Similarly, following Chern–Simons' theory [18, 30, 69], it also makes sense to define the (universal) **Chern–Simons class**

$$\mathrm{cs} = \sum_{k > 0} \mathrm{cs}_{2k-1}, \qquad \mathrm{cs}_{2k-1}\colon A^1 \to (A/[A, A])^{2k-1},$$

$$\mathrm{cs}_{2k-1}(x) = \frac{1}{(k-1)!} \operatorname{tr} \int_0^1 R_{tx}^{k-1} x\, dt \in \left(\frac{A}{[A, A]}\right)^{2k-1}, \qquad k \geq 1,\ x \in A^1,$$

where as before $R_{tx} := R + td(x) + t^2x^2$, $t \in \mathsf{K}$, $x \in A^1$, denotes the curvature of the twisted curved DG-algebra $A_{tx} := (A, d_{tx}, R_{tx})$.

Lemma 4.2.2. *The Chern character is invariant under twisting: more precisely for every $x \in A^1$ and every $k \geq 1$ we have*

$$d(\mathrm{cs}_{2k-1}(x)) = \frac{1}{k!}\,\mathrm{tr}\,(R_x^k - R^k).$$

Proof. Immediate consequence of Lemma 4.2.1 since

$$R_{tx}^{k-1}x \overset{\mathrm{tr}}{=} \frac{1}{k}\sum_{i=1}^{k} R_{tx}^{i-1}xR_{tx}^{k-i}$$

and therefore

$$\mathrm{cs}_{2k-1}(x) = \frac{1}{k!}\,\mathrm{tr}\int_0^1 P(t)_x^k dt\,.$$

□

For the explicit computations in the following Section 4.4, it will be useful to introduce the elements

$$W(x)^{k+1} = \frac{1}{k!}\int_0^1 R_{tx}^k x\,dt = \frac{1}{k!}\int_0^1 (R + td(x) + t^2x^2)^k x\,dt \quad \in A^{2k+1}, \quad x \in A^1,\ k \geq 0, \tag{4.2.2}$$

as a representative set of liftings to A of Chern–Simons classes.

4.3 Convolution algebras and L_∞ liftings of σ_1^k

Our next step is to prove that curved DG-algebras are preserved by taking convolution with the bar construction of a DG-Lie algebra.

For a graded vector space V we shall denote by $V[1]$ the same vector space with the degrees shifted by -1. More precisely, if $v \in V$ is homogenous of degree $\overline{v}$, then the degree of v in $V[1]$ is $\overline{v} - 1$. Unless otherwise specified, for any $v \in V[1]$ we shall denote by $\overline{v}$ the degree of v as an element of V.

In the following we adopt the following sign convention for the décalage isomorphisms: given a pair of graded vector spaces V, W, for every $i > 0$, $k \in \mathsf{Z}$ we consider the isomorphisms:

$$\begin{aligned} &\text{déc}\colon \operatorname{Hom}^k_{\mathsf{K}}(V^{\ i}, W) \to \operatorname{Hom}^{k+i-1}_{\mathsf{K}}(V[1]^{\ i}, W[1]),\\ &\text{déc}(f)(v_1, \dots, v_i) = (-1)^{k+i-1+\sum_{s=1}^{i}(i-s)(\overline{v_s}-1)} f(v_1, \dots, v_i). \end{aligned} \tag{4.3.1}$$

In particular, for $i = 1$ and $k = 0$ the décalage isomorphism is the identity.

Let $(L, \overline{\partial}, [-,-])$ be a DG-Lie algebra. Then there exists a counital DG-coalgebra structure on the symmetric coalgebra $S(L[1])$, where the differential $Q\colon S(L[1]) \to S(L[1])$ is given in Taylor coefficients $q_i\colon L[1]^{\ i} \to L[1]$ by

$$q_1(x) = -\overline{\partial}(x), \qquad q_2(x,y) = (-1)^{\overline{x}}[x,y], \qquad q_i = 0 \ \text{ for } i \neq 1, 2,$$

where $\overline{x}$ denotes the degree of x in L, see also Definition 1.2.13. In other words q_1 and q_2 are the images of $\overline{\partial}$ and $[-,-]$ under the décalage isomorphisms (4.3.1).

More precisely, see e.g. [48, 59], Q decomposes as $Q = Q_0 + Q_1$, where $Q_0, Q_1 : S(L[1]) \to S(L[1])$ are the coderivations defined by $Q_0(1) = Q_1(1) = 0$ and

$$Q_0(x_1 \odot \cdots \odot x_n) = \sum_{i=1}^{n}(-1)^{i+\overline{x_1}+\cdots+\overline{x_{i-1}}}x_1 \odot \cdots \odot \overline{\partial}(x_i) \odot \cdots \odot x_n,$$

$$Q_1(x_1 \odot \cdots \odot x_n) = \sum_{\tau\ \ S(2,n-2)} \varepsilon(\tau)(-1)^{\overline{x_{\tau(1)}}}[x_{\tau(1)}, x_{\tau(2)}] \odot x_{\tau(3)} \odot \cdots \odot x_{\tau(n)},$$

for every $x_1, \dots, x_n \in L[1]$, $n \geq 1$, where we denote by $S(i, n-i)$ the set of $(i, n-i)$-shuffles (Definition 1.2.9), i.e., permutations $\tau \in S_n$ such that $\tau(1) < \cdots < \tau(i)$ and $\tau(i+1) < \cdots < \tau(n)$, and by $\varepsilon(\tau)$ the symmetric Koszul sign defined by the identity $x_{\tau(1)} \odot \cdots \odot x_{\tau(n)} = \varepsilon(\tau)\, x_1 \odot \cdots \odot x_n$ in the symmetric power $L[1]^{\odot n}$, see Definition 1.2.8.

In particular, for every $x, y \in L[1]$ we have

$$Q(x) = -\overline{\partial}(x), \qquad Q(x \odot y) = -\overline{\partial}(x) \odot y - (-1)^{\overline{x}-1} x \odot \overline{\partial}(y) + (-1)^{\overline{x}}[x, y]\,.$$

Notice also that $Q_1(L[1]) = 0$, $Q_0(L[1]^{\odot i}) \subset L[1]^{\odot i}$ and $Q_1(L[1]^{\odot i}) \subset L[1]^{\odot i-1}$ for every i.

Given a curved DG algebra (A, d, R), we shall denote by

$$\mathbf{C}(L, A)_i := \operatorname{Hom}_{\mathsf{K}}\left(L[1]^{\odot i}, A\right), \qquad \mathbf{C}(L, A) = \bigoplus_{i \geq 0} \mathbf{C}(L, A)_i \subset \operatorname{Hom}_{\mathsf{K}}\left(S(L[1]), A\right).$$

The unshuffle coproduct $\Delta \colon S(L[1]) \to S(L[1])^{\otimes 2}$ and the algebra product $m \colon A^{\otimes 2} \to A$ induce an associative product $f \star g := m(f \otimes g)\Delta$ on the space $\mathbf{C}(L, A)$, called the ***convolution product***. More explicitly, if $f \in \mathbf{C}(L, A)_i$ and $g \in \mathbf{C}(L, A)_j$, then $f \star g \in \mathbf{C}(L, A)_{i+j}$ is defined by

$$\begin{aligned}
&(f \star g)(x_1, \dots, x_{i+j}) \\
&\quad = \sum_{\tau \in S(i,j)} \varepsilon(\tau)(-1)^{\overline{g}(\overline{x_{\tau(1)}} + \cdots + \overline{x_{\tau(i)}} - i)} f\big(x_{\tau(1)}, \dots, x_{\tau(i)}\big) g\big(x_{\tau(i+1)}, \dots, x_{\tau(i+j)}\big) \\
&\quad = \sum_{\tau \in S_{i+j}} \frac{\varepsilon(\tau)}{i!j!}(-1)^{\overline{g}(\overline{x_{\tau(1)}} + \cdots + \overline{x_{\tau(i)}} - i)}\, f\big(x_{\tau(1)}, \dots, x_{\tau(i)}\big) g\big(x_{\tau(i+1)}, \dots, x_{\tau(i+j)}\big).
\end{aligned} \tag{4.3.2}$$

In particular, for $a, b \in A = \operatorname{Hom}_{\mathsf{K}}\left(L[1]^{\odot 0}, A\right)$ and $f \in \operatorname{Hom}_{\mathsf{K}}\left(L[1]^{\odot 1}, A\right)$ we have

$$a \star b = ab, \qquad a \star f = af, \qquad [a, f]_\star(x) = [a, f(x)],$$

where $[-,-]_\star$ is the graded commutator of $\star$.

On the algebra $\mathbf{C}(L, A)$ we can define the degree one derivations

$$\delta_0, \delta_1, \delta \in \operatorname{Hom}^1_{\mathsf{K}}(\mathbf{C}(L, A), \mathbf{C}(L, A))$$

induced by the derivation d on A and by the coderivations Q_0, Q_1, Q on $S(L[1])$ respectively. Namely, given $f \in \mathbf{C}(L, A)$, we put

$$\delta_0(f) := df - (-1)^{\overline{f}} f Q_0, \qquad \delta_1(f) := (-1)^{\overline{f}+1} f Q_1, \qquad \delta(f) := \delta_0(f) + \delta_1(f) = df - (-1)^{\overline{f}} f Q.$$

Notice that $\delta_0(\mathbf{C}(L, A)_i) \subset \mathbf{C}(L, A)_i$ and $\delta_1(\mathbf{C}(L, A)_i) \subset \mathbf{C}(L, A)_{i+1}$ for every i.

Defining a weight gradation in $\mathbf{C}(L, A)$ by setting the elements in $\mathbf{C}(L, A)_i$ of weight i we have that $\delta = \delta_0 + \delta_1$ is precisely the weight decomposition of the derivation δ.

More explicitly, given $f \in \mathbf{C}(L, A)_i$, then $\delta_0(f) \in \mathbf{C}(L, A)_i$ and $\delta_1(f) \in \mathbf{C}(L, A)_{i+1}$ are defined by:

$$\begin{aligned}
\delta_0(f)(x_1, \dots, x_i) &= df(x_1, \dots, x_i) \\
&\qquad + (-1)^{\overline{f}} f(\overline{\partial} x_1, \dots, x_i) + \cdots + (-1)^{\overline{f} + \overline{x_1} + \cdots + \overline{x_{i-1}} + i - 1} f(x_1, \dots, \overline{\partial} x_i), \\
\delta_1(f)(x_1, \dots, x_{i+1}) &= (-1)^{\overline{f}+1} \sum_{\tau \in S(2, i-1)} \varepsilon(\tau)(-1)^{\overline{x_{\tau(1)}}} f([x_{\tau(1)}, x_{\tau(2)}], x_{\tau(3)}, \dots, x_{\tau(i+1)}).
\end{aligned}$$

Finally, we continue to denote by $R \in \mathbf{C}(L, A)_0$ the degree two element corresponding to the curvature $R \in A$ under the isomorphism

$$\mathbf{C}(L, A)_0 := \operatorname{Hom}_{\mathsf{K}}\left(L[1]^{\odot 0}, A\right) = \operatorname{Hom}_{\mathsf{K}}(\mathsf{K}, A) = A.$$

(in other words, $R(1) = R$ and $R(x_1, \dots, x_i) = 0$ whenever $i > 0$).

Proposition 4.3.1. *In the above situation, the data* $(\mathbf{C}(L,A),\delta,\star,R)$ *is a curved DG-algebra.*

Proof. Using the fact that d is an algebra derivation and Q_0, Q_1, Q are coalgebra coderivations, it is easy to check that δ_0,δ_1,δ are algebra derivations with respect to the convolution product $\star$. For instance, given $f\in\mathbf{C}(L,A)_i$ and $g\in\mathbf{C}(L,A)_j$ we have

$$\begin{aligned}\delta(f\star g) &= dm(f\otimes g)\Delta-(-1)^{\overline{f}+\overline{g}}m(f\otimes g)\Delta Q\\ &= m(d\otimes\mathrm{Id}+\mathrm{Id}\otimes d)(f\otimes g)\Delta-(-1)^{\overline{f}+\overline{g}}m(f\otimes g)(Q\otimes\mathrm{Id}+\mathrm{Id}\otimes Q)\Delta\\ &= m(df\otimes g+(-1)^{\overline{f}}f\otimes dg)\Delta-(-1)^{\overline{f}+\overline{g}}m((-1)^{\overline{g}}fQ\otimes g+f\otimes gQ)\Delta\\ &= m(\delta(f)\otimes g+(-1)^{\overline{f}}f\otimes\delta(g))\Delta=\delta(f)\star g+(-1)^{\overline{f}}f\star\delta(g).\end{aligned}$$

Moreover, using the fact that $d^2=[R,-]$ and $Q_0^2=Q_1^2=Q^2=0$, one readily checks that

$$\delta(R)=\delta_0(R)=0,\qquad \delta_1^2=\delta_0\delta_1+\delta_1\delta_0=0,\qquad \delta^2=(\delta_0)^2=[R,-]_\star.$$

□

Definition 4.3.2. In the above notation, we shall call $(\mathbf{C}(L,A),\delta,\star,R)$ the **convolution (curved DG) algebra** associated with the curved DG-algebra A and the DG-Lie algebra L.

Definition 4.3.3. A morphism of curved DG-algebras is a morphism of graded algebras that commutes with the derivations and and curvatures:

$$f\colon (A_1,d_1,R_1)\to(A_2,d_2,R_2),\qquad fd_1=d_2f,\quad f(R_1)=R_2.$$

Remark 4.3.4. If $f\colon A_1\to A_2$ is a morphism of curved DG-algebras then the induced map $\mathbf{C}(L,A_1)\to\mathbf{C}(L,A_2)$ is a morphism of curved DG-algebras. Similarly, if $M\to L$ is a morphism of DG-Lie algebras (or, more in general, an L_∞ morphism), then the induced map $\mathbf{C}(L,A)\to\mathbf{C}(M,A)$ is a morphism of curved DG-algebras.

Remark 4.3.5. Given a degree one element $x\in L^1=L[1]^0$, there is an associated morphism of graded associative algebras

$$\begin{aligned}\mathrm{ev}_x:\mathbf{C}(L,A) &\to A,\\ f\in\mathbf{C}(L,A)_i &\mapsto \mathrm{ev}_x(f):=\frac{1}{i!}f(x,\dots,x).\end{aligned}$$

In fact, if $f\in\mathbf{C}(L,A)_i$ and $g\in\mathbf{C}(L,A)_j$, then

$$\begin{aligned}\mathrm{ev}_x(f\star g) &= \frac{1}{(i+j)!}(f\star g)(x,\dots,x)=\frac{1}{(i+j)!}\binom{i+j}{i}f(x,\dots,x)g(x,\dots,x)\\ &= \frac{1}{i!}f(x,\dots,x)\frac{1}{j!}g(x,\dots,x)=\mathrm{ev}_x(f)\,\mathrm{ev}_x(g)\,.\end{aligned}$$

It is also clear that ev_x sends the curvature $R\in\mathbf{C}(L,A)_0$ to the curvature $R\in A^2$.

In general ev_x is not a morphism of curved DG algebras, but it is so when $x\in\mathrm{MC}(L)$. In fact if $\overline{\partial}x+[x,x]/2=0$, then for every $f\in\mathbf{C}(L,A)_i$ we have:

$$\begin{aligned}\mathrm{ev}_x(\delta(f)) &= \frac{1}{i!}\delta_0(f)(x,\dots,x)+\frac{1}{(i+1)!}\delta_1(f)(x,\dots,x)\\ &= \frac{1}{i!}df(x,\dots,x)+\frac{(-1)^{\overline{f}}}{(i-1)!}f(\overline{\partial}x,x,\dots,x)+\frac{(-1)^{\overline{f}}}{2(i-1)!}f([x,x],x,\dots,x)\\ &= d\,\mathrm{ev}_x(f)\,.\end{aligned}$$

For every graded subspace $E\subset A$, we denote by $\mathbf{C}(L,E)=\bigoplus_i\mathrm{Hom}_{\mathsf{K}}(L[1]^{\odot i},E)\subset\mathbf{C}(L,A)$.

Lemma 4.3.6. *If I is a curved Lie ideal of A, then $\mathbf{C}(L,I)$ is a curved Lie ideal of $\mathbf{C}(L,A)$. Moreover $\mathbf{C}(L,I)^{(k)}\mathbf{C}(L,A)\subset\mathbf{C}(L,I^{(k)}A)$ for every k, and $[\mathbf{C}(L,A),\mathbf{C}(L,A)]\subset\mathbf{C}(L,[A,A])$.*

Proof. Immediate from the definitions and from the fact that, since the unshuffle coproduct Δ is graded cocommutative, given $f,g\in\mathbf{C}(L,A)$, every element in the image of $[f,g]$ is a linear combination of elements of type

$$\begin{aligned} &m(f\otimes g-(-1)^{\overline{f}\overline{g}}g\otimes f)(x\otimes y+(-1)^{(\overline{x}-1)(\overline{y}-1)}y\otimes x)\\ &\quad=(-1)^{\overline{g}(\overline{x}-1)}[f(x),g(y)]+(-1)^{(\overline{x}+\overline{g}-1)(\overline{y}-1)}[f(y),g(x)]. \end{aligned}$$

□

Let (A,d,R) be a curved DG-algebra with a curved Lie ideal $I\subset A$ and denote by $\pi\colon A\to A/I$ the projection.

Given a DG-Lie algebra $L=(L,\overline{\partial},[-,-])$ together with a morphism of graded vector spaces $s\colon L\to A$, the latter can be seen as an element of degree $+1$ in $\mathbf{C}(L,A)_1=\operatorname{Hom}_{\mathbb{K}}(L[1],A)$, and then it gives a sequence of Chern–Simons forms

$$W(s)^{k+1}=\frac{1}{k!}\int_0^1(R+t\delta(s)+t^2s\star s)^k\star s\,dt\quad\in\mathbf{C}(L,A)^{2k+1}=\operatorname{Hom}^{2k+1}_{\mathbb{K}}(S(L[1]),A),\quad k\ge 0.$$

Notice that

$$W(s)^{k+1}=\sum_{i=1}^{2k+1}W(s)^{k+1}_i,\quad\text{with } W(s)^{k+1}_i\in\operatorname{Hom}^{2k+1}_{\mathbb{K}}(L[1]^{\odot i},A)\text{ and }W(s)^{k+1}_1=\frac{R^ks}{k!}\,. \tag{4.3.3}$$

Lemma 4.3.7. *In the above situation, let $s\colon L\to A$ be a morphism of graded vector spaces such that the composition $\pi s\colon L\to A/I$ is a morphism of DG-Lie algebras. Then $\delta(s)+s\star s\in\mathbf{C}(L,I)$ and*

$$\delta W(s)^{k+1}\in[\mathbf{C}(L,A),\mathbf{C}(L,A)]+\mathbf{C}(L,I)^{(k+1)}\quad\forall k\ge 0.$$

Moreover, $\delta(s)+s\star s\in\mathbf{C}(L,I)_1$ if and only if s is a morphism of graded Lie algebras.

Proof. By Lemma 4.2.2

$$\delta(W(s)^{k+1})\overset{\mathrm{tr}}{=}\frac{1}{(k+1)!}((R+\delta(s)+s\star s)^{k+1}-R^{k+1}).$$

Since $R\in\mathbf{C}(L,I)$, it is sufficient to show that $\delta(s)+s\star s$ belongs to $\mathbf{C}(L,I)$.

Since $\delta_0(s)\in\mathbf{C}(L,A)_1$ and $\delta_1(s),s\star s\in\mathbf{C}(L,A)_2$, the condition $\delta(s)+s\star s\in\mathbf{C}(L,I)$ is equivalent to:

$$\pi\delta_0s(b_1)=0,\quad\pi\delta_1s(b_1,b_2)+\pi(s\star s)(b_1,b_2)=0,\quad\forall b_1,b_2\in L\,.$$

By definition, $\delta_0s(b_1)=ds(b_1)-s(\overline{\partial}b_1)$, so that

$$\pi\delta_0s(b_1)=\pi ds(b_1)-\pi s(\overline{\partial}b_1)=d\pi s(b_1)-\pi s\overline{\partial}b_1=0\,.$$

On the other hand,

$$\begin{aligned}(s\star s)(b_1,b_2)&=m(s\otimes s)(b_1\otimes b_2+(-1)^{(\overline{b_1}-1)(\overline{b_2}-1)}b_2\otimes b_1)\\ &=(-1)^{\overline{b_1}-1}s(b_1)s(b_2)+(-1)^{(\overline{b_1}-1)(\overline{b_2}-1)+\overline{b_2}-1}s(b_2)s(b_1)\\ &=(-1)^{\overline{b_1}-1}[s(b_1),s(b_2)].\end{aligned}$$

Since πs and π are morphisms of graded Lie algebras we have

$$\pi\delta_1s(b_1,b_2)=(-1)^{\overline{b_1}}\pi s([b_1,b_2])=(-1)^{\overline{b_1}}[\pi s(b_1),\pi s(b_2)]=(-1)^{\overline{b_1}}\pi[s(b_1),s(b_2)]\,.$$

The same computation shows that $\delta_1s+s\star s=0$ if and only if s is a morphism of graded Lie algebras. □

Remark 4.3.8. In fact, the above computations also prove the converse of the first part of Lemma 4.3.7: namely, that for a morphism of graded vector spaces $s\colon L \to A$ we have $\delta(s)+s\star s \in \mathbf{C}(L,I)$ if and only if the composition πs is a morphism of DG-Lie algebras.

By Lemma 4.3.6 we have a natural morphism of differential graded vector spaces

$$\theta_k\colon \frac{\mathbf{C}(L,A)}{[\mathbf{C}(L,A),\mathbf{C}(L,A)]+\mathbf{C}(L,I)^{(k+1)}\mathbf{C}(L,A)} \to \mathrm{Hom}_{\mathsf{K}}\left(S(L[1]), \frac{A}{[A,A]+I^{(k+1)}A}\right)$$

and then, for every $s \in \mathbf{C}(L,A)^1$ and every $k \geq 0$, it is defined the element

$$\theta_k(W(s)^{k+1}) \in \mathrm{Hom}^{2k+1}_{\mathsf{K}}\left(S(L[1]), \frac{A}{[A,A]+I^{(k+1)}A}\right)$$

that we can view as

$$\theta_k(W(s)^{k+1}) \in \mathrm{Hom}^{0}_{\mathsf{K}}\left(S(L[1]), \frac{A}{[A,A]+I^{(k+1)}A}[2k+1]\right).$$

Theorem 4.3.9. *In the above situation, suppose that $\pi s\colon L \to A/I$ is a morphism of DG-Lie algebras. Then $\theta_k(W(s)^{k+1})$ is the corestriction of an L_∞ morphism*

$$L \qquad \frac{A}{[A,A]+I^{(k+1)}A}[2k]$$

with linear Taylor coefficient $\sigma_1^k \pi s$, where σ_1^k is the morphism from Definition 4.1.5, and all Taylor coefficients $L[1]^{\ i} \to \dfrac{A}{[A,A]+I^{(k+1)}A}[2k+1]$ of degree $i \geq 2k+2$ vanishing.

Proof. Recall that an L_∞ morphism $f : L \quad M$ between two DG-Lie algebras is the same as a morphism of DG coalgebras $F : S(L[1]) \to S(M[1])$ between their bar constructions, see Section 1.3. By cofreeness of $S(M[1])$, the correspondence sending F to its corestriction $f = pF$, where we denote by $p\colon S(M[1]) \to M[1]$ the natural projection, establishes a bijection between the set of morphisms of graded coalgebras $F : S(L[1]) \to S(M[1])$ and the set of morphism of graded vector spaces $f : S(L[1]) \to M[1]$: in general, compatibility with the bar differentials translates into a countable sequence of algebraic equations in f, see e.g. Section 1.3, [6, 59]. However, in the particular situation we are concerned with, that is, when the bracket on M is trivial, the situation simplifies considerably, and we have that $f \in \mathrm{Hom}^0_{\mathsf{K}}(S(L[1]), M[1])$ is the corestriction of an L_∞ morphism $F : S(L[1]) \to S(M[1])$ if and only if $r_1 f = fQ$, where we denote by r_1 the shifted differential $r_1(m) = -d_M(m)$ on $M[1]$ and by Q the bar differential on $S(L[1])$. In other words, when M has trivial bracket the L_∞ morphisms $L \quad M$ are in bijective correspondence with the set of 0-cocycles in the complex $\mathrm{Hom}_{\mathsf{K}}(S(L[1]), M[1])$.

On the other hand, by Lemma 4.3.7 the image of $W(s)^{k+1}$ onto $\frac{\mathbf{C}(L,A)}{[\mathbf{C}(L,A),\mathbf{C}(L,A)]+\mathbf{C}(L,I)^{(k+1)}\mathbf{C}(L,A)}$ is a degree $(2k+1)$ cocycle. Therefore, in order to conclude it is sufficient to define the desired L_∞ morphism as the image of $\theta_k(W(s)^{k+1})$ under the natural isomorphism of differential graded vector spaces

$$\mathrm{Hom}_{\mathsf{K}}\left(S(L[1]), \frac{A}{[A,A]+I^{(k+1)}A}\right)[2k+1] = \mathrm{Hom}_{\mathsf{K}}\left(S(L[1]), \frac{A}{[A,A]+I^{(k+1)}A}[2k+1]\right).$$

Finally, the last two statements about the Taylor coefficients follow immediately from the definitions and (4.3.3). □

Corollary 4.3.10. *Let I be a curved Lie ideal of a curved DG-algebra A and denote by $B := A/I$. For every morphism of graded vector spaces $s\colon B \to A$ such that $\pi s = \mathrm{Id}_B$, the image of $W(s)^{k+1} \in \mathrm{Hom}^{2k+1}_{\mathsf{K}}(S(B[1]), A)$ onto*

$$\mathrm{Hom}^{2k+1}_{\mathsf{K}}\left(S(B[1]), \frac{A}{[A,A]+I^{(k+1)}A}\right) = \mathrm{Hom}^{0}_{\mathsf{K}}\left(S(B[1]), \frac{A}{[A,A]+I^{(k+1)}A}[2k+1]\right),$$

is an L_∞ morphism

$$\sigma^k = (\sigma^k_1, \sigma^k_2, \ldots, \sigma^k_{2k+1}, 0, 0, \ldots)\colon B \qquad \frac{A}{[A,A]+I^{(k+1)}A}[2k]$$

with linear Taylor coe cient σ^k_1 and all Taylor coe cients of degree $\geq 2k+2$ vanishing.

Remark 4.3.11. It follows by Remark 4.3.5 that the induced push-forward on Maurer–Cartan elements

$$\mathrm{MC}(\sigma^k)\colon\ \mathrm{MC}(B) \to \mathrm{MC}\left(\frac{A}{[A,A]+I^{(k+1)}A}[2k]\right) = Z^{2k+1}\left(\frac{A}{[A,A]+I^{(k+1)}A}\right),$$

$$\mathrm{MC}(\sigma^k)(x) := \sum_{i=1}^{2k+1} \frac{1}{i!}\sigma^k_i(x, \ldots, x) = \mathrm{tr}\left(\mathrm{ev}_x\left(W^{k+1}(s)\right)\right) = \mathrm{tr}\left(W^{k+1}(s(x))\right),$$

sends the Maurer–Cartan element $x \in B^1$ to the residue modulo $\mathrm{tr}\left(I^{(k+1)}A\right)$ of the Chern–Simons class $\mathrm{cs}_{2k+1}\left(s(x)\right) \in A/[A,A]$.

For general s an explicit combinatorial description of the higher Taylor coefficients σ^k_i, although possible, is quite complicated. In the next section we study these components under the additional assumption that $s\colon B \to A$ is a morphism of graded Lie algebras, or equivalently, by Lemma 4.3.7, that $\delta(s) + s \star s \in \mathbf{C}(B, I)_1$. This is often satisfied in concrete examples: for instance, in the geometric example that we shall consider in the following Section 4.5.

4.4 Explicit formulas in the split case.

Let $\mathsf{K}\langle Z_0, Z_1, Z_2\rangle$ be the associative algebra of noncommutative polynomials in Z_0, Z_1, Z_2. Following [30], we denote by $\Sigma[Z_0^p, Z_1^q, Z_2^r] \in \mathsf{K}\langle Z_0, Z_1, Z_2\rangle$, $p, q, r \in \mathsf{N}$, the symmetric functions: by definition $\Sigma[Z_0^p, Z_1^q, Z_2^r]$ is the sum of all the words in Z_0, Z_1, Z_2 having p factors Z_0, q factors Z_1 and r factors Z_2. For instance, $\Sigma[Z_0^0, Z_1^0, Z_2^0] = 1$ and $\Sigma[Z_0^1, Z_1^2, Z_2^0] = Z_0Z_1^2 + Z_1Z_0Z_1 + Z_1^2Z_0$.

For every $k \geq 0$ we define homogeneous polynomials $V^k(Z_0, Z_1, Z_2) \in \mathsf{K}\langle Z_0, Z_1, Z_2\rangle$ by the formula

$$V^k(Z_0, Z_1, Z_2) = \frac{1}{k!}\int_0^1 (Z_0 + tZ_1 + (t^2 - t)Z_2)^k\, dt\,.$$

Notice that for every curved DG-algebra (A, d, R) and every $x \in A^1$ we have

$$W(x)^{k+1} = V^k(R, d(x) + x^2, x^2)x\,. \tag{4.4.1}$$

It is also useful to assign to each variable Z_i the weight i, and denote by $V^k = \sum_{i=0}^{2k} V^k_i$ the associated isobaric decomposition. Notice that every monomial in Z_0, Z_1, Z_2 of weight i with r occurrences of the variable Z_2 (hence $i - 2r \geq 0$ occurrences of the variable Z_1) appears in V^k_i with coefficient

$$\frac{1}{k!}\int_0^1 t^{i-r}(t-1)^r dt = \frac{(-1)^r r!\,(i-r)!}{k!\,(i+1)!} = \frac{(-1)^r}{k!\,(i+1)}\binom{i}{r}^{-1}.$$

Therefore, for every $0 \leq i \leq 2k$

$$V^k_i = \sum_{\substack{p+q+r=k\\ q+2r=i}} \frac{(-1)^r r!\,(i-r)!}{k!\,(i+1)!}\,\Sigma[Z_0^p, Z_1^q, Z_2^r]\,.$$

For instance, one checks that for $0 \le i \le 2k \le 6$ the above formula for V_i^k gives:

$$V_0^0 = 1, \qquad V_0^1 = Z_0, \qquad V_1^1 = \frac{1}{2}Z_1, \qquad V_2^1 = -\frac{1}{6}Z_2,$$
$$V_0^2 = \frac{1}{2}Z_0^2, \quad V_1^2 = \frac{1}{4}(Z_0Z_1 + Z_1Z_0), \qquad V_2^2 = \frac{1}{6}Z_1^2 - \frac{1}{12}(Z_0Z_2 + Z_2Z_0),$$
$$V_3^2 = -\frac{1}{24}(Z_1Z_2 + Z_2Z_1), \qquad V_4^2 = \frac{1}{60}\,Z_2^2,$$
$$V_0^3 = \frac{1}{6}Z_0^3, \quad V_1^3 = \frac{1}{12}(Z_0^2Z_1 + Z_0Z_1Z_0 + Z_1Z_0^2),$$
$$V_2^3 = \frac{1}{18}(Z_0Z_1^2 + Z_1Z_0Z_1 + Z_1^2Z_0) - \frac{1}{36}(Z_0^2Z_2 + Z_0Z_2Z_0 + Z_2Z_0^2),$$
$$V_3^3 = \frac{1}{24}Z_1^3 - \frac{1}{72}(Z_0Z_1Z_2 + Z_0Z_2Z_1 + Z_1Z_0Z_2 + Z_1Z_2Z_0 + Z_2Z_0Z_1 + Z_2Z_1Z_0),$$
$$V_4^3 = -\frac{1}{120}(Z_1^2Z_2 + Z_1Z_2Z_1 + Z_2Z_1^2) + \frac{1}{180}(Z_0Z_2^2 + Z_2Z_0Z_2 + Z_2^2Z_0),$$
$$V_5^3 = \frac{1}{360}(Z_1Z_2^2 + Z_2Z_1Z_2 + Z_2^2Z_1), \qquad V_6^3 = -\frac{1}{840}Z_2^3.$$

Definition 4.4.1. A **split curved DG-algebra** is the datum of a curved DG-algebra $A = (A, d, R)$ equipped with a direct sum decomposition

$$A = B \oplus I,$$

where $B \subset A$ is a graded Lie subalgebra and $I \subset A$ is a curved Lie ideal.

We shall denote by $\imath\colon B \to A$ the inclusion and by $P\colon A \to B$ the projection with kernel I (in particular, $\imath, P$ are morphisms of graded Lie algebras) and by $P\quad := \mathrm{id}_A - P\colon A \to I$. We shall also denote by

$$\overline{\partial} := Pd\colon B \to B, \qquad \nabla := P\quad d\colon B \to I\,.$$

Notice in particular that since $\mathrm{Ker}(P) = I$ is d-closed, then the identity $Pd = PdP$ holds, and in particular

$$\overline{\partial}^2 = (Pd)^2 = Pd^2 = P[R, -] = 0,$$

since I is a Lie ideal and $R \in I$. Thus $(B, \overline{\partial}, [-,-])$ is a DG-Lie algebra and the natural map $(B, \overline{\partial}) \to (A/I, d)$ is an isomorphism of DG-Lie algebras. Moreover,

$$d\nabla + \nabla\overline{\partial} = [R, -] : B \to I, \tag{4.4.2}$$

since for every $b \in B$ we have

$$[R, b] = d^2(b) = d\nabla(b) + d\overline{\partial}(b) = d\nabla(b) + \nabla\overline{\partial}(b) + \overline{\partial}\overline{\partial}(b)\,.$$

Lemma 4.4.2. *Let $A = B \oplus I$ be a split curved DG-algebra and consider the inclusion $\imath\colon B \hookrightarrow A$ as an element of $\mathbf{C}(B, A)_1$. Then for every $x, y \in B$ we have*

$$\imath \star \imath \in \mathrm{Hom}^2_{\mathbb{K}}\left(B[1]^{\;2}, A\right) = \mathbf{C}(B, A)^2_2, \qquad \imath \star \imath(x, y) = (-1)^{\overline{x}-1}[x, y],$$
$$\delta(\imath) + \imath \star \imath = \nabla \in \mathrm{Hom}^2_{\mathbb{K}}\left(B[1], I\right) \subset \mathbf{C}(B, I)^2_1, \qquad (\delta(\imath) + \imath \star \imath)(x) = \nabla(x),$$
$$R \in \mathbf{C}(B, I)^2_0.$$

In particular $W(\imath)^{k+1} = \sum_{i=1}^{2k+1} W(\imath)_i^{k+1}$ with

$$W(\imath)_i^{k+1} \in \mathbf{C}(B, A)_i, \qquad W(\imath)_i^{k+1} = V_{i-1}^k(R, \nabla, \imath \star \imath) \star \imath\,.$$

Proof. We have already proved that $\imath\star\imath(x,y)=(-1)^{\overline{x}-1}[x,y]$ in the proof of Lemma 4.3.7. It remains to show that $(\delta(\imath)+\imath\star\imath)(x)=\nabla(x)$. Again by Lemma 4.3.7 we have $\delta(\imath)+\imath\star\imath=\delta_0(\imath)\in\mathbf{C}(B,I)_1$ and then for every $x\in B$

$$(\delta(\imath)+\imath\star\imath)(x)=\delta_0(\imath)(x)=(d\imath-\imath\overline{\partial})(x)=\nabla(x)\,.$$

The last claim follows from Equation (4.4.1). □

Corollary 4.4.3. *Let $A=B\oplus I$ be a split curved DG-algebra with inclusion morphism $\imath\colon B\hookrightarrow A$. For every i,k with $1\le i\le 2k+1$ denote by $\sigma_i^k\in\operatorname{Hom}^0_{\mathbb{K}}\left(B[1]^{\ \ i},\frac{A}{[A,A]+I^{(k+1)}A}[2k+1]\right)$ the image of $V_{i-1}^k(R,\nabla,\imath\star\imath)\star\imath$ under the trace map*

$$\mathbf{C}(B,A)_i^{2k+1}\xrightarrow{\mathrm{tr}}\operatorname{Hom}^0_{\mathbb{K}}\left(B[1]^{\ \ i},\frac{A}{[A,A]+I^{(k+1)}A}[2k+1]\right).$$

Then

$$\sigma^k=(\sigma_1^k,\sigma_2^k,\ldots,\sigma_{2k+1}^k,0,0,\ldots)\colon B\qquad\frac{A}{[A,A]+I^{(k+1)}A}[2k]$$

is an L_∞ morphism with linear component σ_1^k.

Example 4.4.4. More explicitly, the Taylor coefficients

$$\sigma_i^k:B[1]^{\ \ i}\to\frac{A}{[A,A]+I^{k+1}A}[2k+1]$$

are given on the diagonal, i.e., when all the arguments equal a certain $x\in B^1$, by the formula

$$\frac{1}{i!}\sigma_i^k(x,\ldots,x)=\mathrm{tr}\left(V_{i-1}^k(R,\nabla(x),x^2)x\right),$$

and in general is given by the above formula via graded polarization.

For instance, using the previous explicit formulas for the non-commutative polynomials V_{i-1}^k (together with the cyclic invariance of the trace), we see that for $k\le 3$ the L_∞ morphism from Corollary 4.4.3 is given explicitly as follows. For $k=0$, we have the DG-Lie algebra morphism

$$\sigma^0\colon(B,\overline{\partial},[-,-])\to\left(\frac{A}{[A,A]+IA},d,0\right),\qquad\sigma^0(x)=\mathrm{tr}(x)\,.$$

For $k=1$ we have the L_∞ morphism

$$\sigma^1=(\sigma_1^1,\sigma_2^1,\sigma_2^1,0,0,\ldots)\colon(B,\overline{\partial},[-,-])\qquad\left(\frac{A}{[A,A]+I^{(2)}A}[2],d,0\right)$$

given by:

$$\begin{aligned}\sigma_1^1(x)&=\mathrm{tr}\left(Rx\right),\\ \sigma_2^1(x_1,x_2)&=\sum_{\tau\quad S_2}\frac{\varepsilon(\tau)}{2}\;\mathrm{tr}\left(\nabla(x_{\tau(1)})x_{\tau(2)}\right),\\ \sigma_3^1(x_1,x_2,x_3)&=\sum_{\tau\quad S_3}-\frac{\varepsilon(\tau)}{6}\;\mathrm{tr}\left(x_{\tau(1)}x_{\tau(2)}x_{\tau(3)}\right),\end{aligned}\tag{4.4.3}$$

where we denote by $\varepsilon(\tau)$ the symmetric Koszul sign, defined by the identity $x_{\tau(1)}\odot\cdots\odot x_{\tau(i)}=\varepsilon(\tau)\,x_1\odot\cdots\odot x_i$ in the symmetric power $B[1]^{\ \ i}$. Hence we recover, in a more general framework, the formulas of [50] with the curvature in place of the Atiyah cocycle, see Remark 4.4.5 below.

For $k=2$ we have the L_∞ morphism

$$\sigma^2=(\sigma_1^2,\ldots,\sigma_5^2,0,0,\ldots)\colon(B,\overline{\partial}[-,-])\qquad\left(\frac{A}{[A,A]+I^{(3)}A}[4],d,0\right)$$

given by:

$$
\begin{aligned}
\sigma_1^2(x) &= \frac{1}{2}\,\mathrm{tr}\left(R^2 x\right),\\
\sigma_2^2(x_1,x_2) &= \sum_{\tau\ \ S_2} \frac{\varepsilon(\tau)}{4}\,\mathrm{tr}\left(R\nabla(x_{\tau(1)})x_{\tau(2)} + \nabla(x_{\tau(1)})Rx_{\tau(2)}\right),\\
\sigma_3^2(x_1,x_2,x_3) &= \sum_{\tau\ \ S_3} \frac{\varepsilon(\tau)}{6}\,\mathrm{tr}\left((-1)^{\overline{x_{\tau(1)}}+1}\nabla(x_{\tau(1)})\nabla(x_{\tau(2)})x_{\tau(3)} - Rx_{\tau(1)}x_{\tau(2)}x_{\tau(3)}\right),\\
\sigma_4^2(x_1,x_2,x_3,x_4) &= \sum_{\tau\ \ S_4} -\frac{\varepsilon(\tau)}{12}\,\mathrm{tr}\left(\nabla(x_{\tau(1)})x_{\tau(2)}x_{\tau(3)}x_{\tau(4)}\right),\\
\sigma_5^2(x_1,x_2,x_3,x_4,x_5) &= \sum_{\tau\ \ S_5} \frac{\varepsilon(\tau)}{60}\,\mathrm{tr}\left(x_{\tau(1)}x_{\tau(2)}x_{\tau(3)}x_{\tau(4)}x_{\tau(5)}\right).
\end{aligned}
$$

Finally, for $k = 3$ we have the L_∞ morphism

$$
\sigma^3 = (\sigma_1^3,\ldots,\sigma_7^3,0,0,\ldots)\colon (B,\overline{\partial},[-,-]) \qquad \left(\frac{A}{[A,A]+I^{(4)}A}[6], d, 0\right)
$$

given by

$$
\sigma_i^3(x_1,\ldots,x_i) = \sum_{\tau\ \ S_i} \varepsilon(\tau)\,\mathrm{tr}(P_i(\tau)),
$$

where:

$$
P_1(\tau) = \frac{1}{6}R^3 x_1,
$$

$$
P_2(\tau) = \frac{1}{12}\left(R^2\nabla(x_{\tau(1)}) + R\nabla(x_{\tau(1)})R + \nabla(x_{\tau(1)})R^2\right)x_{\tau(2)}
$$

$$
\begin{aligned}
P_3(\tau) = &-\frac{1}{18}\,R^2 x_{\tau(1)}x_{\tau(2)}x_{\tau(3)} - \frac{1}{36}\,Rx_{\tau(1)}x_{\tau(2)}Rx_{\tau(3)}\\
&+\frac{1}{18}\,(-1)^{\overline{x_{\tau(1)}}+1}\Big(R\nabla(x_{\tau(1)})\nabla(x_{\tau(2)}) + \nabla(x_{\tau(1)})R\nabla(x_{\tau(2)}) + \nabla(x_{\tau(1)})\nabla(x_{\tau(2)})R\Big)x_{\tau(3)},
\end{aligned}
$$

$$
\begin{aligned}
P_4(\tau) = &\frac{1}{24}\left((-1)^{\overline{x_{\tau(2)}}+1}\nabla(x_{\tau(1)})\nabla(x_{\tau(2)})\nabla(x_{\tau(3)})x_{\tau(4)}\right)\\
&-\frac{1}{36}\left(\nabla(x_{\tau(1)})Rx_{\tau(2)}x_{\tau(3)}x_{\tau(4)} + R\nabla(x_{\tau(1)})x_{\tau(2)}x_{\tau(3)}x_{\tau(4)}\right)\\
&-\frac{1}{72}\left((-1)^{\overline{x_{\tau(1)}}+\overline{x_{\tau(2)}}}Rx_{\tau(1)}x_{\tau(2)}\nabla(x_{\tau(3)})x_{\tau(4)} + \nabla(x_{\tau(1)})x_{\tau(2)}x_{\tau(3)}Rx_{\tau(4)}\right),
\end{aligned}
$$

$$
\begin{aligned}
P_5(\tau) = &\frac{1}{60}\left(Rx_{\tau(1)}x_{\tau(2)}x_{\tau(3)}x_{\tau(4)}x_{\tau(5)} - (-1)^{\overline{x_{\tau(1)}}+1}\nabla(x_{\tau(1)})\nabla(x_{\tau(2)})x_{\tau(3)}x_{\tau(4)}x_{\tau(5)}\right)\\
&-\frac{1}{120}\left((-1)^{\overline{x_{\tau(1)}}+\overline{x_{\tau(2)}}+\overline{x_{\tau(3)}}+1}\nabla(x_{\tau(1)})x_{\tau(2)}x_{\tau(3)}\nabla(x_{\tau(4)})x_{\tau(5)}\right),
\end{aligned}
$$

$$
P_6(\tau) = \frac{1}{120}\left(\nabla(x_{\tau(1)})x_{\tau(2)}x_{\tau(3)}x_{\tau(4)}x_{\tau(5)}x_{\tau(6)}\right),
$$

$$
P_7(\tau) = -\frac{1}{840}\left(x_{\tau(1)}x_{\tau(2)}x_{\tau(3)}x_{\tau(4)}x_{\tau(5)}x_{\tau(6)}x_{\tau(7)}\right).
$$

Remark 4.4.5. In the above setup, suppose that I is a bilateral associative ideal. Then the morphism σ_i^k depends only on the class of R in $I/I^{(2)}$ if and only if either $i \le 2$ or $i \ge 2k$. This partially explains the unsuccessful attempts of the last two authors to extend the formulas of [50], described in the following Section 4.6, involving the Atiyah form instead of the curvature, to the case $k > 1$.

4.5 Connections of type $(1,0)$ and curved DG-pairs

Let X be a complex manifold and let

$$\mathcal{E}^\cdot: \qquad 0 \to \mathcal{E}^p \xrightarrow{d_{\mathcal{E}}} \mathcal{E}^{p+1} \xrightarrow{d_{\mathcal{E}}} \cdots \xrightarrow{d_{\mathcal{E}}} \mathcal{E}^q \to 0, \qquad p \le q \in \mathbb{Z}, \quad d_{\mathcal{E}}^2 = 0,$$

be a fixed finite complex of locally free sheaves of $\mathcal{O}_X$-modules. We denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)$ the graded sheaf of $\mathcal{O}_X$-linear endomorphisms of $\mathcal{E}^\cdot$:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot) = \bigoplus_i \mathcal{H}om^i_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot), \qquad \mathcal{H}om^i_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot) = \prod_j \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^j, \mathcal{E}^{i+j}).$$

Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)$ is a sheaf of locally free DG-Lie algebras over $\mathcal{O}_X$, with the bracket equal to the graded commutator

$$[f, g] = fg - (-1)^{\overline{f}\,\overline{g}} gf$$

and the differential given by

$$f \mapsto [d_{\mathcal{E}}, f] = d_{\mathcal{E}} f - (-1)^{\overline{f}} f d_{\mathcal{E}}.$$

For every a, b, r denote by $\mathcal{A}_X^{a,b}(\mathcal{E}^r) \simeq \mathcal{A}_X^{a,b} \otimes_{\mathcal{O}_X} \mathcal{E}^r$ the sheaf of differential forms of type (a,b) with coefficients in $\mathcal{E}^r$, and by $\overline{\partial}\colon \mathcal{A}_X^{a,b}(\mathcal{E}^r) \to \mathcal{A}_X^{a,b+1}(\mathcal{E}^r)$ the Dolbeault differential.

We consider

$$\mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot) = \bigoplus_{a,b,r} \mathcal{A}_X^{a,b}(\mathcal{E}^r)$$

as a graded sheaf of $\mathcal{A}_X^{\cdot}$ modules, where the elements of $\mathcal{A}_X^{a,b}(\mathcal{E}^r)$ have degree $a+b+r$. It is useful to use the dot symbol $\cdot$ to denote the natural left multiplication map

$$\mathcal{A}_X^{\cdot} \times \mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot) \xrightarrow{\cdot} \mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot).$$

The differential $d_{\mathcal{E}}$ extends naturally to a differential

$$d_{\mathcal{E}}\colon \mathcal{A}_X^{a,b}(\mathcal{E}^r) \to \mathcal{A}_X^{a,b}(\mathcal{E}^{r+1}), \qquad d_{\mathcal{E}}(\phi \cdot e) = (-1)^{\overline{\phi}} \phi \cdot d_{\mathcal{E}}(e), \qquad \phi \in \mathcal{A}_X^{a,b},\ e \in \mathcal{E}^r.$$

We have that $\overline{\partial}^2 = d_{\mathcal{E}}^2 = 0$ and $[\overline{\partial}, d_{\mathcal{E}}] = \overline{\partial} d_{\mathcal{E}} + d_{\mathcal{E}} \overline{\partial} = 0$, so that $\overline{\partial} + d_{\mathcal{E}}$ is a differential in $\mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot)$. Therefore the space of $\mathbb{C}$-linear morphisms of sheaves

$$\operatorname{Hom}_{\mathbb{C}}(\mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot), \mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot))$$

carries a natural structure of differential graded associative algebra: the product is given by composition and the differential is the graded commutator with $\overline{\partial} + d_{\mathcal{E}}$.

Denoting by $A_X^{a,b}(-)$ the global sections of $\mathcal{A}_X^{a,b}(-)$ we have two differential graded subalgebras

$$A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)) \subset A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot)) \subset \operatorname{Hom}_{\mathbb{C}}(\mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot), \mathcal{A}_X^{\cdot}(\mathcal{E}^\cdot)),$$

where for

$$\omega, \eta \in \mathcal{A}_X^{\cdot}, \quad f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\cdot, \mathcal{E}^\cdot), \quad e \in \mathcal{E}^\cdot$$

one has that

$$(\omega \cdot f)(\eta \cdot e) = (-1)^{\overline{f}\,\overline{\eta}}(\omega \wedge \eta) \cdot f(e)\,,$$

so that the elements of $A_X^{a,b}(\mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ have degree $a+b+n$.

For every $a \in A_X^{\cdot}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ we have

$$\overline{\partial} a = [\overline{\partial}, a] \tag{4.5.1}$$

where the bracket on the right is intended in the DG-Lie algebra $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}_X^{\cdot}\ (\mathcal{E}\ ), \mathcal{A}_X^{\cdot}\ (\mathcal{E}\ ))$. In fact, for $\omega, \eta \in \mathcal{A}_X^{\cdot}$, $f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )$ and $e \in \mathcal{E}$ we have:

$$\begin{aligned}
[\overline{\partial}, \omega \cdot f](\eta \cdot e) &= \overline{\partial}((-1)^{\overline{f}\,\overline{\eta}}\omega \wedge \eta \cdot f(e)) - (-1)^{\overline{\omega}+\overline{f}}(\omega \cdot f)(\overline{\partial}(\eta) \cdot e) \\
&= (-1)^{\overline{f}\,\overline{\eta}}\overline{\partial}(\omega) \wedge \eta \cdot f(e) + (-1)^{\overline{f}\,\overline{\eta}+\overline{\omega}}\omega \wedge \overline{\partial}(\eta) \cdot f(e) - (-1)^{\overline{\omega}+\overline{f}\,\overline{\eta}}\omega \wedge \overline{\partial}(\eta) \cdot f(e) \\
&= (\overline{\partial}(\omega) \cdot f)(\eta \cdot e).
\end{aligned}$$

The composition product in $A_X^{0,\cdot}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ and $A_X^{\cdot}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ works in the following way:

$$\omega, \eta \in \mathcal{A}_X^{\cdot}\ , \qquad f, g \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )$$

$$(\omega \cdot f)(\eta \cdot g) = (-1)^{\overline{f}\,\overline{\eta}}(\omega \wedge \eta) \cdot fg\,,$$

and the commutator is

$$[\omega \cdot f, \eta \cdot g] = (\omega \cdot f)(\eta \cdot g) - (-1)^{(\overline{\omega}+\overline{f})(\overline{\eta}+\overline{g})}(\eta \cdot g)(\omega \cdot f) = (-1)^{\overline{f}\,\overline{\eta}}(\omega \wedge \eta) \cdot [f, g]\,.$$

The above commutator and the differential $[d_{\mathcal{E}} + \overline{\partial}, -] = [d_{\mathcal{E}}, -] + \overline{\partial}$ give $A_X^{0,\cdot}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ and $A_X^{\cdot}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ a structure of DG-Lie algebra.

We now define an obvious generalisation of the notion of connection of type $(1,0)$ to complexes of locally free sheaves.

Definition 4.5.1. Let $(\mathcal{E}\ , d_{\mathcal{E}})$ be a finite complex of locally free sheaves on X.

1. A **connection** on $\mathcal{E}$ is a $\mathbb{C}$-linear morphism of graded sheaves of degree $+1$

$$\nabla \colon \mathcal{E}\ \to \mathcal{A}_X^{\cdot}\ (\mathcal{E}\ )$$

   such that $\nabla(fe) = d_{dR}(f) \cdot e + f \cdot \nabla(e)$ for every $f \in \mathcal{O}_X$, $e \in \mathcal{E}$. Here d_{dR} denotes the de Rham differential.

2. A connection ∇ as above is called **of type** $(1,0)$ if $\nabla(e) - \overline{\partial}(e) \in \oplus_k \mathcal{A}_X^{1,k}(\mathcal{E}^{i-k})$ for every i and every $e \in \mathcal{E}^i$.

Thus, a connection ∇ is of type $(1,0)$ if and only if $\nabla = \overline{\partial} + \sum_k \nabla^{1,k}$, with $\nabla^{1,k} \colon \mathcal{E}^i \to \mathcal{A}_X^{1,k}(\mathcal{E}^{i-k})$ for every i.

Notice that $\nabla^{1,k}$ is $\mathcal{O}_X$-linear for every $k > 0$, and that, denoting by $\partial = d_{dR} - \overline{\partial} \colon \mathcal{A}_X^{\cdot} \to \mathcal{A}_X^{\cdot+1,\cdot}$,

$$\nabla^{1,0}(fe) = \partial(f) \cdot e + f \cdot \nabla^{1,0}(e), \quad \forall f \in \mathcal{O}_X,\ e \in \mathcal{E}\ .$$

As in the nongraded case (see e.g. [45]), every connection ∇ extends uniquely to a $\mathbb{C}$-linear morphism of graded sheaves of total degree $+1$

$$\nabla \colon \mathcal{A}_X^{\cdot}\ (\mathcal{E}\ ) \to \mathcal{A}_X^{\cdot}\ (\mathcal{E}\ )$$

such that $\nabla(\phi \cdot \omega) = d_{dR}(\phi) \cdot \omega + (-1)^{\overline{\phi}}\phi \cdot \nabla(\omega)$, for every $\phi \in \mathcal{A}_X^{\cdot}$, $\omega \in \mathcal{A}_X^{\cdot}\ (\mathcal{E}\ )$.

It is clear that giving a connection ∇ of type $(1,0)$ with $\nabla^{1,k} = 0$ for every $k > 0$ is the same as giving a classical connection of type $(1,0)$ on every $\mathcal{E}^i$ as in Definition 3.3.3. In particular, connections of type $(1,0)$ always exist.

Denote by $A = A_X^{\cdot}\ (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ the graded associative algebra of global differential forms with values in the graded sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )$. Every element of A may be naturally interpreted as an endomorphism of the sheaf $\mathcal{A}_X^{\cdot}\ (\mathcal{E}\ )$.

Lemma 4.5.2. *For any connection ∇ of type $(1,0)$ the adjoint operator*

$$d = [\nabla + d_{\mathcal{E}}, -]\colon A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \to A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$$

is a well-defined derivation.

Moreover $R = (\nabla + d_{\mathcal{E}})^2 = \dfrac{1}{2}[\nabla + d_{\mathcal{E}}, \nabla + d_{\mathcal{E}}]$ belongs to $A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, and the triple $(A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})), d, R)$ is a curved DG-algebra.

Proof. It is clear that $[d_{\mathcal{E}}, -]$ is a well-defined derivation of $A = A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. Since ∇ is a connection of type $(1,0)$, it can be written as

$$\nabla = \overline{\partial} + \nabla^{1,0} + \sum_{k\geq 1} \nabla^{1,k},$$

with $\nabla^{1,k}$ that is $\mathcal{O}_X$-linear for every $k \neq 0$. Therefore for every $k \neq 0$, $\nabla^{1,k}$ belongs to $A_X^{1,k}(\mathcal{H}om_{\mathcal{O}_X}^{-k}(\mathcal{E},\mathcal{E}))$, which implies that $[\nabla^{1,k}, -]$ is an inner derivation of A for every $k \neq 0$. It then suffices to show that $[\overline{\partial} + \nabla^{1,0}, -]$ is a derivation of A. For brevity, denote $D := \overline{\partial} + \nabla^{1,0}$.

Since every $\mathcal{E}^i$ is locally free, we can describe $A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ as the set of morphisms of graded sheaves $h\colon \mathcal{A}_X^{\cdot}(\mathcal{E}) \to \mathcal{A}_X^{\cdot}(\mathcal{E})$ that are $\mathcal{A}_X^{\cdot}$-linear, i.e., $h(f\cdot s) = (-1)^{\overline{h}\,\overline{f}} f \cdot h(s)$, $f \in \mathcal{A}_X^{\cdot}$, $s \in \mathcal{A}_X^{\cdot}(\mathcal{E})$.

Thus, for every $f \in \mathcal{A}_X^{\cdot}$ and $s \in \mathcal{A}_X^{\cdot}(\mathcal{E})$ we have

$$\begin{aligned}
[D,h](f\cdot s) &= (-1)^{\overline{h}\,\overline{f}} D(f\cdot h(s)) - (-1)^{\overline{h}} h(d_{dR}f\cdot s + (-1)^{\overline{f}} f \cdot D(s)) \\
&= (-1)^{\overline{h}\,\overline{f}}(d_{dR}f \cdot h(s) + (-1)^{\overline{f}} f \cdot D(h(s))) - (-1)^{\overline{h}+\overline{h}(\overline{f}+1)} d_{dR} f\cdot h(s) + \\
&\quad - (-1)^{\overline{h}+\overline{f}+\overline{h}\,\overline{f}} f \cdot h(D(s)) \\
&= (-1)^{(\overline{h}+1)\overline{f}} f\cdot [D,h](s),
\end{aligned}$$

which proves that $[D,h] \in A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. For $h \in A_X^{p,q}(\mathcal{H}om_{\mathcal{O}_X}^{j}(\mathcal{E},\mathcal{E}))$ according to (4.5.1) we have

$$[\nabla^{1,0}, h] = [D,h] - \overline{\partial}(a) \in A_X^{p+1,q}(\mathcal{H}om_{\mathcal{O}_X}^{j}(\mathcal{E},\mathcal{E}))\,. \tag{4.5.2}$$

For the second part, notice that

$$\begin{aligned}
R &= \frac{1}{2}[\nabla + d_{\mathcal{E}}, \nabla + d_{\mathcal{E}}] = \frac{1}{2}\left[\overline{\partial} + d_{\mathcal{E}} + \sum_{k\geq 0}\nabla^{1,k}, \overline{\partial} + d_{\mathcal{E}} + \sum_{j\geq 0}\nabla^{1,j}\right] = \\
&= \left[\overline{\partial} + d_{\mathcal{E}}, \sum_{k\geq 0}\nabla^{1,k}\right] + \frac{1}{2}\left[\sum_{k\geq 0}\nabla^{1,k}, \sum_{j\geq 0}\nabla^{1,j}\right] =: R_1 + R_2.
\end{aligned}$$

We begin by showing that $R_1 = [\overline{\partial} + d_{\mathcal{E}}, \sum_{k\geq 0}\nabla^{1,k}]$ belongs to $A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. In fact, R_1 can be be written as

$$R_1 = \left[\overline{\partial} + d_{\mathcal{E}}, \sum_{k\geq 0}\nabla^{1,k}\right] = [\overline{\partial} + d_{\mathcal{E}}, \nabla^{1,0}] + \left[\overline{\partial} + d_{\mathcal{E}}, \sum_{k\geq 1}\nabla^{1,k}\right],$$

and since $\nabla^{1,k}$ belongs to $A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ for every $k \neq 0$, it is clear that the second part belongs to $A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$.

It then remains to show that $u := [\overline{\partial} + d_{\mathcal{E}}, \nabla^{1,0}]$ also belongs to $A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. The element $u\colon \mathcal{A}_X^{\cdot}(\mathcal{E}) \to \mathcal{A}_X^{\cdot}(\mathcal{E})$ is a morphism of graded sheaves of even degree and we need to show that it is $\mathcal{A}_X^{\cdot}$-linear. By (4.5.2) we have $[d_{\mathcal{E}}, \nabla^{1,0}] \in A_X^{1,0}(\mathcal{H}om_{\mathcal{O}_X}^{1}(\mathcal{E},\mathcal{E}))$ and we only need to prove that $[\overline{\partial}, \nabla^{1,0}]$ is $\mathcal{A}_X^{\cdot}$-linear.

For $f \in \mathcal{A}_X^{\cdot}$, $s \in \mathcal{A}_X^{\cdot}(\mathcal{E})$ we have:

$$\begin{aligned}
[\overline{\partial}, \nabla^{1,0}](fs) &= \overline{\partial}(\partial f \cdot s + (-1)^{\overline{f}} f \cdot \nabla^{1,0}(s)) + \nabla^{1,0}(\overline{\partial} f \cdot s + (-1)^{\overline{f}} f \cdot \overline{\partial} s) \\
&= \overline{\partial}(\partial f) \cdot s + (-1)^{\overline{f}+1} \partial f \cdot \overline{\partial} s + (-1)^{\overline{f}} \overline{\partial} f \cdot \nabla^{1,0}(s) + f \cdot \overline{\partial}(\nabla^{1,0}(s)) \\
&\quad + \partial(\overline{\partial} f) \cdot s + (-1)^{\overline{f}+1} \overline{\partial} f \cdot \nabla^{1,0}(s) + (-1)^{\overline{f}} \partial f \cdot \overline{\partial} s + f \cdot \nabla^{1,0}(\overline{\partial}(s)) \\
&= f[\overline{\partial}, \nabla^{1,0}](s).
\end{aligned}$$

Therefore $[\overline{\partial}, \nabla^{1,0}]$ belongs to $A_X^{1,1}(\mathcal{H}om^0_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$.

Now, we show that R_2 belongs to $A_X^{2,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$: we can write it as

$$R_2 = \frac{1}{2}\left[\sum_{k\geq 0} \nabla^{1,k}, \sum_{j\geq 0} \nabla^{1,j}\right] = \frac{1}{2}[\nabla^{1,0}, \nabla^{1,0}] + \left[\nabla^{1,0}, \sum_{j\geq 1} \nabla^{1,j}\right] + \frac{1}{2} \sum_{j,k\geq 1} [\nabla^{1,k}, \nabla^{1,j}].$$

The last term is clearly in $A_X^{2,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ because $\nabla^{1,k}$ belongs to $A_X^{1,k}(\mathcal{H}om^{-k}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ for $k \neq 0$. The middle term is also in $A_X^{2,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, by the first part of the claim. For the first term and $f \in \mathcal{A}_X^{\cdot}$, $s \in \mathcal{A}_X^{\cdot}(\mathcal{E})$ we have that

$$\begin{aligned}
&\frac{1}{2}[\nabla^{1,0}, \nabla^{1,0}](fs) = (\nabla^{1,0})^2(fs) = \nabla^{1,0}(\partial f \cdot s + (-1)^{\overline{f}} f \cdot \nabla^{1,0}(s)) = \\
&\partial^2 f \cdot s + (-1)^{\overline{f}+1} \partial f \cdot \nabla^{1,0}(s) + (-1)^{\overline{f}} \partial f \cdot \nabla^{1,0}(s) + f(\nabla^{1,0})^2(s) = f(\nabla^{1,0})^2(s),
\end{aligned}$$

and we have the claim. □

To every connection $\nabla = \overline{\partial} + \sum_k \nabla^{1,k} : \mathcal{A}_X^{\cdot}(\mathcal{E}) \to \mathcal{A}_X^{\cdot}(\mathcal{E})$ of type $(1,0)$ it is associated a curved DG-pair (A, I) as in Definition 4.1.2, according to the following construction. Denote by $A = A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ the graded associative algebra of global differential forms with values in the graded sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$. By Lemma 4.5.2 the adjoint operator

$$d = [\nabla + d_{\mathcal{E}}, -] : A \to A$$

is a well defined derivation and $R = (\nabla + d_{\mathcal{E}})^2 = \dfrac{1}{2}[\nabla + d_{\mathcal{E}}, \nabla + d_{\mathcal{E}}]$ belongs to A, so that the triple (A, d, R) is a curved DG-algebra.

Writing $\nabla = \overline{\partial} + \sum_k \nabla^{1,k}$, since $(\overline{\partial} + d_{\mathcal{E}})^2 = 0$ we have

$$R = R_1 + R_2, \qquad R_i \in A_X^{i,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).$$

In particular, R belongs to the ideal $I = A_X^{>0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. Moreover, $R_1 = [\overline{\partial} + d_{\mathcal{E}}, \sum_k \nabla^{1,k}]$ and then the connection ∇ is such that $[d_{\mathcal{E}} + \overline{\partial}, \nabla] = 0$ if and only if $R_1 = 0$.

Lemma 4.5.3. *The Atiyah class*

$$\mathrm{At}(A, I) \in H^2(A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))) = \mathrm{Ext}^1_X(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}).$$

of the curved DG-pair $(A, I) = ((A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})), d, R), A_X^{>0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$ *is trivial if and only if* $\mathcal{E}$ *admits a connection* $\widetilde{\nabla}$ *of type* $(1,0)$ *such that* $[d_{\mathcal{E}} + \overline{\partial}, \widetilde{\nabla}] = 0$.

Proof. Since $I^{(k)} = I^{(k)}A = A_X^{\geq k,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ the Atiyah class $\mathrm{At}(A, I)$ of the curved DG-pair is precisely the cohomology class of R_1 in the complex $(A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})), [\overline{\partial} + d_{\mathcal{E}}, -])$; notice that this complex is the Dolbeault resolution of the complex $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X \otimes \mathcal{E})$ and therefore

$$\mathrm{At}(A, I) \in H^2(A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))) = \mathrm{Ext}^1_X(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}).$$

Since two connections of type $(1,0)$ differ by a degree 1 element of $A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, the Atiyah class is independent from the choice of the connection. Conversely, for every degree 1 element $\phi \in A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ the map $\widetilde{\nabla} = \nabla + \phi$ is again a connection of type $(1,0)$ with 1-component of the curvature $\widehat{R}_1 = R_1 + [d_{\mathcal{E}} + \overline{\partial}, \phi]$: it follows that $\mathrm{At}(A, I) = 0$ if and only if $\mathcal{E}$ admits a a connection $\widetilde{\nabla}$ of type $(1,0)$ such that $[d_{\mathcal{E}} + \overline{\partial}, \widetilde{\nabla}] = 0$. □

Lemma 4.5.4. *The class* $\mathrm{At}(A,I)$ *is the same as the usual Atiyah class of the complex* $\mathscr{E}$ *as an object in the derived category of* X*.*

Proof. We show that our definition is equivalent to the one of [39, Section 10.1], where a representative of the Atiyah class is given via Čech cohomology.

First, we recall the definition of the complex of 1-jets of $\mathscr{E}$, and of two variants. For $\mathscr{E}$ a finite complex of locally free sheaves, a representative of the Atiyah class of (3.3.3) is given by the class of the short exact sequence

$$0 \longrightarrow \mathscr{E} \otimes \Omega^1_X \longrightarrow J^1(\mathscr{E}) \xrightarrow{p_1} \mathscr{E} \longrightarrow 0 \tag{4.5.3}$$

where $J^1(\mathscr{E})$ is the complex of 1-jets of $\mathscr{E}$, defined as $(\mathscr{E} \oplus \mathscr{E} \otimes \Omega^1_X, (d_{\mathscr{E}}, d_{\mathscr{E}} \otimes \mathrm{Id}))$ as a complex of sheaves of $\mathbf{C}$-modules, with $\mathscr{O}_X$-action given by:

$$f \cdot (e,\sigma) = (fe, f\sigma + e \otimes d_{dR} f) \quad f \in \mathscr{O}_X,\ e \in \mathscr{E},\ \sigma \in \mathscr{E} \otimes \Omega^1_X.$$

Denoting by $\mathscr{C}(\mathcal{U},\mathscr{E})$ the sheafified version of the Čech complex, recalled in Definition 1.4.4, and by i the inclusions $i\colon \mathscr{E} \to \mathscr{C}(\mathcal{U},\mathscr{E})$ and $i\colon \mathscr{E} \otimes \Omega^1_X \to \mathscr{C}(\mathcal{U},\mathscr{E} \otimes \Omega^1_X)$, we can define the complex $\widetilde{J}^1(\mathcal{U},\mathscr{E}) = (\mathscr{E} \oplus \mathscr{C}(\mathcal{U},\mathscr{E} \otimes \Omega^1_X), (d_{\mathscr{E}}, \check{d} + d_{\mathscr{E}}))$ a a complex of sheaves of $\mathbf{C}$-modules, with $\mathscr{O}_X$-action given by

$$f \cdot (e,\sigma) = (fe, f\sigma + i(e \otimes d_{dR} f)) \quad f \in \mathscr{O}_X,\ e \in \mathscr{E},\ \sigma \in \mathscr{C}(\mathcal{U},\mathscr{E} \otimes \Omega^1_X).$$

The quasi-isomorphism $i\colon \mathscr{E} \to \mathscr{C}(\mathcal{U},\mathscr{E})$ induces a quasi-isomorphism $J^1(\mathscr{E}) \to \widetilde{J}^1(\mathcal{U},\mathscr{E})$. The complex $\widetilde{J}^1(\mathcal{U},\mathscr{E})$ fits into the short exact sequence

$$0 \longrightarrow \mathscr{C}(\mathcal{U},\mathscr{E} \otimes \Omega^1_X) \longrightarrow \widetilde{J}^1(\mathcal{U},\mathscr{E}) \xrightarrow{p_1} \mathscr{E} \longrightarrow 0, \tag{4.5.4}$$

which also represents the Atiyah class of $\mathscr{E}$ as an object in the derived category, because of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \otimes \Omega^1_X \arrow[r] \arrow[d, "i"] & J^1(\mathscr{E}) \arrow[r] \arrow[d, "i"] & \mathscr{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{C}(\mathcal{U},\mathscr{E} \otimes \Omega^1_X) \arrow[r] & \widetilde{J}^1(\mathcal{U},\mathscr{E}) \arrow[r] & \mathscr{E} \arrow[r] & 0.
\end{tikzcd}$$

Similarly, denoting by j the inclusions $j\colon \mathscr{E} \to \mathscr{A}^{0,}_X(\mathscr{E})$ and $j\colon \mathscr{E} \otimes \Omega^1_X \to \mathscr{A}^{1,}_X(\mathscr{E})$, we can define the complex $\mathscr{A}^{0,}_X(J^1(\mathscr{E})) := (\mathscr{E} \oplus \mathscr{A}^{0,}_X(\mathscr{E} \otimes \Omega^1_X), (d_{\mathscr{E}}, \overline{\partial} + d_{\mathscr{E}}))$ as a complex of sheaves of $\mathbf{C}$-modules, with $\mathscr{O}_X$-action given by:

$$f \cdot (e,\sigma) = (fe, f\sigma + j(e \otimes d_{dR} f)) \quad f \in \mathscr{O}_X,\ e \in \mathscr{E},\ \sigma \in \mathscr{A}^{0,}_X(\mathscr{E} \otimes \Omega^1_X).$$

This complex is quasi-isomorphic to $J^1(\mathscr{E})$ via the map j and it sits inside the short exact sequence

$$0 \longrightarrow \mathscr{A}^{1,}_X(\mathscr{E}) \longrightarrow \mathscr{A}^{0,}_X(J^1(\mathscr{E})) \xrightarrow{p_1} \mathscr{E} \longrightarrow 0, \tag{4.5.5}$$

which also represents the Atiyah class, in view of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \otimes \Omega^1_X \arrow[r] \arrow[d, "j"] & J^1(\mathscr{E}) \arrow[r] \arrow[d, "j"] & \mathscr{E} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{A}^{1,}_X(\mathscr{E}) \arrow[r] & \mathscr{A}^{0,}_X(J^1(\mathscr{E})) \arrow[r] & \mathscr{E} \arrow[r] & 0.
\end{tikzcd}$$

The representative in Čech cohomology of the Atiyah class given in [39] is constructed as follows, as a generalisation of the construction in Lemma 3.3.5. Choose an affine open cover $\mathcal{U} = \{U_i\}$ of X such that the restriction of the short exact sequence (4.5.3)

$$0 \longrightarrow \mathcal{E}^k \otimes \Omega^1_X \longrightarrow J^1(\mathcal{E}^k) \xrightarrow{p_1} \mathcal{E}^k \longrightarrow 0$$

splits on U_i for every i, k, and denote by $D^k_i \colon \mathcal{E}^k|_{U_i} \to \mathcal{E}^k \otimes \Omega^1_X|_{U_i}$ a set of local holomorphic connections. Defining

$$\alpha \in C^1(\mathcal{U}, \mathcal{H}om^0_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \quad \alpha^{q}_{i_0 i_1} = D^q_{i_0}|_{U_{i_0 i_1}} - D^q_{i_1}|_{U_{i_0 i_1}},$$

$$\alpha \in C^0(\mathcal{U}, \mathcal{H}om^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \quad \alpha_i{}^{q} = d_{\mathcal{E}} D^q_i - D^{q+1}_i d_{\mathcal{E}},$$

one can see that $\alpha = \alpha + \alpha$ is a cocycle in the Čech hypercomplex

$$(C (\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \check{d} + [d_{\mathcal{E}}, -]),$$

and it is a representative of the Atiyah class $\mathrm{At}(\mathcal{E})$ according to [39].

The cocycle α is trivial in cohomology if and only if there exists $\beta \in \big(C (\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X))\big)^0$ such that $\check{d}(D-\beta)+[d_{\mathcal{E}}, D-\beta] = 0$ in the complex $(C (\mathcal{U}, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \check{d}+[d_{\mathcal{E}}, -])$. Notice that $\beta = \sum_{k \geq 0} \beta^k$, with $\beta^k \in C^k(\mathcal{U}, \mathcal{H}om^{-k}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X))$, with β^k in general not trivial for $k \geq 1$. There is an isomorphism of complexes of vector spaces

$$(C (\mathcal{U}, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)), \check{d} + [d_{\mathcal{E}}, -]) \cong (\mathrm{Hom}_{\mathbb{C}}(\mathcal{E}, \mathcal{C} (\mathcal{U}, \mathcal{E} \otimes \Omega^1_X)), \check{d} + [d_{\mathcal{E}}, -])$$

and then $D := D - \beta$ is exactly a $\mathbb{C}$-linear morphism of complexes of sheaves of degree zero $D : \mathcal{E} \to \mathcal{C} (\mathcal{U}, \mathcal{E})$ such that the Leibniz identity

$$D (fe) = fD (e) + i(e \otimes d_{dR} f), \quad \forall f \in \mathcal{O}_X, \; e \in \mathcal{E}$$

holds. In fact, the cohomology class of α is trivial if and only if there exists such a map D. From D we can construct a morphism of complexes of sheaves

$$\mathrm{Id}_{\mathcal{E}} \oplus D : \mathcal{E} \to \widetilde{J}^1(\mathcal{U}, \mathcal{E}),$$

and the existence of the map D is equivalent to the existence of a morphism of complexes of sheaves $\psi \colon \mathcal{E} \to \widetilde{J}^1(\mathcal{U}, \mathcal{E})$ such that $p_1 \psi = \mathrm{Id}_{\mathcal{E}}$, i.e., a splitting of the short exact sequence (4.5.4).

The argument for our definition of Atiyah class is completely analogous. Let ∇ be a connection of type (1,0) on $\mathcal{E}$ and consider its Atiyah class, i.e., the cohomology class of $R_1 = [\overline{\partial} + d_{\mathcal{E}}, \nabla]$ in the complex $A^{1,}_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$. As seen in Lemma 4.5.3, this class is trivial if and only if there exists a connection ∇ of type (1,0) on $\mathcal{E}$ such that $[\overline{\partial} + d_{\mathcal{E}}, \nabla] = 0$. In view of the isomorphism of complexes

$$(A^{1,}_X(\mathcal{H}om_{\mathbb{C}}(\mathcal{E}, \mathcal{E})), \overline{\partial} + [d_{\mathcal{E}}, -]) \cong (\mathrm{Hom}_{\mathbb{C}}(\mathcal{E}, \mathcal{A}^{1,}_X(\mathcal{E})), \overline{\partial} + [d_{\mathcal{E}}, -])$$

the existence of such ∇ is equivalent to the existence of a $\mathbb{C}$-linear map of complexes of sheaves $\nabla : \mathcal{E} \to \mathcal{A}^{1,}_X(\mathcal{E})$ such that the Leibniz rule

$$\nabla (fe) = f\nabla (e) + j(e \otimes d_{dR} f), \quad \forall f \in \mathcal{O}_X, \; e \in \mathcal{E}$$

is satisfied. This is equivalent to the existence of a map of complexes of sheaves $\phi \colon \mathcal{E} \to \mathcal{A}^{0,}_X(J^1(\mathcal{E}))$ such that $p_1 \phi = \mathrm{Id}_{\mathcal{E}}$, i.e., a splitting of the short exact sequence (4.5.5).

□

Remark 4.5.5. For every $x \in A$ we have that

$$[R_1, x] = \left[\overline{\partial} + d_{\mathcal{E}}, \left[\sum_k \nabla^{1,k}, x\right]\right] + \left[\sum_k \nabla^{1,k}, [\overline{\partial} + d_{\mathcal{E}}, x]\right].$$

If $x \in A$ is such that $[\overline{\partial} + d_{\mathcal{E}}, x] = 0$, then $[R_1, x] = [\overline{\partial} + d_{\mathcal{E}}, [\sum_k \nabla^{1,k}, x]]$ and this immediately implies that the Atiyah class is a central element in the cohomology of the differential graded algebra $\mathrm{Gr}_I(A) = \oplus_k \dfrac{I^{(k)}}{I^{(k+1)}}$, cf. [16, Prop. 3.12].

Lemma 4.5.6. *The $A_X^{\cdot,\cdot}$-linear extension of the usual trace map*

$$\mathrm{Tr}\colon A_X^{\cdot,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{\cdot}, \mathcal{E}^{\cdot})) \to A_X^{\cdot,\cdot}, \qquad \mathrm{Tr}(\phi \cdot f) = \phi\,\mathrm{Tr}(f),$$

is a trace map in the sense of Definition 4.1.9. In fact, for every $h \in A_X^{\cdot,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{\cdot}, \mathcal{E}^{\cdot}))$ we have

$$\mathrm{Tr}(dh) = \mathrm{Tr}([\nabla + d_{\mathcal{E}}, h]) = \mathrm{Tr}([\nabla, h]) = d_{dR}\,\mathrm{Tr}(h).$$

Proof. Let $\nabla = \overline{\partial} + \sum_{k\geq 0} \nabla^{1,k}$ be a connection of type $(1,0)$ on $\mathcal{E}^{\cdot}$, and denote $D = \overline{\partial} + \nabla^{1,0}$. Since $\nabla^{1,k}$ is $\mathcal{O}_X$-linear for all $k \neq 0$, $\mathrm{Tr}([\nabla^{1,k}, h]) = [\mathrm{Tr}(\nabla^{1,k}), \mathrm{Tr}(h)] = 0$ for all $k \neq 0$ and $\mathrm{Tr}([\nabla, h]) = \mathrm{Tr}([D, h])$.

By linearity it is sufficient to consider the case $h = \eta \cdot g$, with $\eta \in \mathcal{A}_X^{\cdot,\cdot}$ and $g \in \mathcal{H}om^n_{\mathcal{O}_X}(\mathcal{E}^{\cdot}, \mathcal{E}^{\cdot})$. It is clear that it is enough to consider g of degree 0, and by linearity we may assume g concentrated in one degree, i.e., $g = g_l\colon \mathcal{E}^l \to \mathcal{E}^l$. Let $e_1, \dots, e_m$ be a local basis of holomorphic sections for $\mathcal{E}^l$, and let

$$g(e_i) = \sum_j a_{ij} e_j, \quad D(e_i) = \sum_j \omega_{ij} e_j, \quad \mathrm{Tr}(g) = (-1)^l \sum_i a_{ii}.$$

Then

$$d_{dR}\,\mathrm{Tr}(\eta \cdot g) = d_{dR}(\eta\,\mathrm{Tr}(g)) = d_{dR}\eta\,\mathrm{Tr}(g) + (-1)^{\overline{\eta}} \eta d_{dR}\,\mathrm{Tr}(g),$$

$$\begin{aligned}
[D, \eta \cdot g](e_i) &= D(\sum_j \eta a_{ij} e_j) - (-1)^{\overline{\eta}} (\eta \cdot g)(\sum_j \omega_{ij} e_j) \\
&= \sum_j d_{dR}(\eta) a_{ij} e_j + \sum_j (-1)^{\overline{\eta}} \eta \wedge d_{dR}(a_{ij}) e_j + \sum_j (-1)^{\overline{\eta}} \eta a_{ij} \wedge D(e_j) - (-1)^{\overline{\eta}} \sum_j \eta \wedge \omega_{ij} g(e_j) \\
&= \sum_k d_{dR}(\eta) a_{ik} e_k + \sum_k (-1)^{\overline{\eta}} \eta \wedge d_{dR}(a_{ik}) e_k + \sum_{j,k} (-1)^{\overline{\eta}} \eta a_{ij} \wedge \omega_{jk} e_k - (-1)^{\overline{\eta}} \sum_{j,k} \eta \wedge \omega_{ij} a_{jk} e_k\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Tr}([D, \eta \cdot g]) &= (-1)^l \sum_i \left(d_{dR}\eta a_{ii} + (-1)^{\overline{\eta}} \eta d_{dR}(a_{ii}) + \sum_j (-1)^{\overline{\eta}} \Big(\eta \wedge \omega_{ji} a_{ij} - \eta \wedge \omega_{ij} a_{ji} \Big) \right) \\
&= (-1)^l \sum_i \left(d_{dR}\eta a_{ii} + (-1)^{\overline{\eta}} \eta d_{dR}(a_{ii}) \right) = d_{dR}\eta\,\mathrm{Tr}(g) + (-1)^{\overline{\eta}} \eta d_{dR}\,\mathrm{Tr}(g).
\end{aligned}$$

□

Assume now that the complex $\mathcal{E}^{\cdot}$ is a resolution of a coherent sheaf $\mathcal{F}$. Then $\mathrm{At}(A, I)$ is equal to the Atiyah class $\mathrm{At}(\mathcal{F})$ of $\mathcal{F}$ and the DG-Lie algebra

$$\frac{A}{I} = A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{\cdot}, \mathcal{E}^{\cdot}))$$

is precisely Dolbeault's model of the DG-Lie algebra controlling deformations of $\mathcal{F}$, see Proposition 2.5.18 and [24, Section 8].

Consider the trace map $\mathrm{Tr}\colon A \to A_X^{\cdot}$: for every $k \geq 0$ we have $\mathrm{Tr}(I^{(k+1)}A) = A_X^{>k,}$, and then the map σ_1^k from Definition 4.1.5 becomes

$$\sigma_1^k\colon A_X^{0,}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E} \ ,\mathcal{E} \ )) \to A_X^{\leq k,}[2k], \qquad \sigma_1^k(x) = \frac{1}{k!}\mathrm{Tr}(R_1^k x)\,.$$

Therefore, at the cohomology level σ_1^k induces the composition of

$$\mathrm{Ext}_X(\mathcal{F},\mathcal{F}) \to H^{\ +k}(X,\Omega_X^k), \qquad x \mapsto \frac{1}{k!}\mathrm{Tr}\left(\mathrm{At}(\mathcal{F})^k \cdot x\right),$$

with the natural map $j\colon H^{\ +k}(X,\Omega_X^k) \to \mathsf{H}^{2k+}(A_X^{\leq k,})$.

By Corollary 4.1.10 the map σ_1^k is the linear component of an L_∞ morphism, and since the deformation functor associated to the abelian DG-Lie algebra $A_X^{\leq k,}[2k]$ has trivial obstructions (see Lemma 2.4.6) we immediately obtain the following result.

Corollary 4.5.7. *Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X admitting a locally free resolution. Then for every $k \geq 0$ the semiregularity map*

$$\mathrm{Ext}_X^2(\mathcal{F},\mathcal{F}) \to \mathsf{H}^{2k+2}(A_X^{\leq k,}), \qquad x \mapsto \frac{1}{k!}\mathrm{Tr}(\mathrm{At}(\mathcal{F})^k \cdot x),$$

annihilates obstructions to deformations of $\mathcal{F}$.

Proof. Since X is assumed smooth, by Hilbert's syzygy theorem, if $\mathcal{F}$ admits a locally free resolution, then it also admits a finite locally free resolution, see e.g. [44, V.3.11]. □

Corollary 4.5.8. *Let $\mathcal{F}$ be a coherent sheaf on a complex projective manifold X. Then for every $k \geq 0$ the semiregularity map*

$$\mathrm{Ext}_X^2(\mathcal{F},\mathcal{F}) \to H^{k+2}(X,\Omega_X^k), \qquad x \mapsto \frac{1}{k!}\mathrm{Tr}(\mathrm{At}(\mathcal{F})^k \cdot x),$$

annihilates obstructions to deformations of $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is projective every coherent sheaf admits a locally free resolution. Moreover, the Hodge to de Rham spectral sequence degenerates at E_1 and therefore the natural map $H^{k+2}(X,\Omega_X^k) \to \mathsf{H}^{2k+2}(A_X^{\leq k,})$ is injective. □

4.6 Cyclic forms and an alternative way of lifting σ_1^1

This section contains the results from [50], where we constructed a lifting of the first component of Buchweitz–Flenner semiregularity map with direct computations.

As in the previous section, let

$$\mathcal{E} \ : \qquad 0 \to \mathcal{E}^p \xrightarrow{d_{\mathcal{E}}} \mathcal{E}^{p+1} \xrightarrow{d_{\mathcal{E}}} \cdots \xrightarrow{d_{\mathcal{E}}} \mathcal{E}^q \to 0$$

be a fixed finite complex of locally free sheaves on a complex manifold X and let ∇ be a connection of type $(1,0)$ on $\mathcal{E}$, as in Definition 4.5.1.

Lemma 4.6.1. *In the above setup, for every $a \in A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E} \ ,\mathcal{E} \ ))$ we have:*

$$[[\nabla,\overline{\partial}],a] = [\nabla,\overline{\partial}a] + \overline{\partial}[\nabla,a]\,.$$

Proof. By (4.5.1), we have that $[[\nabla,\overline{\partial}],a] = [\nabla,[\overline{\partial},a]] + [\overline{\partial},[\nabla,a]] = [\nabla,\overline{\partial}a] + \overline{\partial}[\nabla,a]$. □

Definition 4.6.2. By a cyclic (bilinear) form on the sheaf of DG-Lie algebras $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )$ we mean a graded symmetric $\mathcal{O}_X$-bilinear product of degree 0

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ) \times \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ) \xrightarrow{-,-} \mathcal{O}_X,$$

such that

$$\langle f, [g,h]\rangle = \langle [f,g], h\rangle \qquad \forall\, f,g,h\,.$$

Equivalently, for every f,g,h we have

$$\langle [f,g], h\rangle + (-1)^{\overline{f}\,\overline{g}}\langle g, [f,h]\rangle = 0,$$

i.e., $\langle -,-\rangle$ is invariant under the adjoint action. In particular

$$\langle [d_{\mathcal{E}}, g], h\rangle + (-1)^{\overline{g}}\langle g, [d_{\mathcal{E}}, h]\rangle = 0\,. \tag{4.6.1}$$

Notice that (4.6.1) is equivalent to the fact that the bilinear form $\langle -,-\rangle$ is closed in the dual of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )^{\ 2}$.

Every cyclic form on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )$ has a natural extension to

$$A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))^{\ 2} \xrightarrow{-,-} A_X^{\cdot}\,, \qquad \langle \phi f, \psi g\rangle = (-1)^{\overline{f}\,\overline{\psi}}\phi\wedge\psi\langle f,g\rangle,$$

and it is immediate to check that, for $f,g \in A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$

$$\overline{\partial}\langle f,g\rangle = \langle \overline{\partial} f, g\rangle + (-1)^{\overline{f}}\langle f, \overline{\partial} g\rangle, \tag{4.6.2}$$

and then $\langle -,-\rangle$ is $\overline{\partial} + [d_{\mathcal{E}}, -]$ closed. Cyclic forms have received a lot of attention in several recent papers; for instance cyclic forms that are nondegenerate in cohomology play a central role in the proof of the formality conjecture for polystable sheaves on projective surfaces with torsion canonical bundles, given in [7].

Definition 4.6.3. We shall say that a connection ∇ of type $(1,0)$ on $\mathcal{E}$ is compatible with the cyclic form $\langle -,-\rangle$ if

$$\langle [\nabla, f], g\rangle + (-1)^{\overline{f}}\langle f, [\nabla, g]\rangle = d_{dR}\langle f,g\rangle\,,$$

or equivalently if

$$\langle [\nabla - \overline{\partial}, f], g\rangle + (-1)^{\overline{f}}\langle f, [\nabla - \overline{\partial}, g]\rangle = \partial\langle f,g\rangle\,,$$

for every $f,g \in A_X^{\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$.

Example 4.6.4. According to Lemma 4.5.6, for every $a,b \in \mathbb{C}$ the form

$$\langle f,g\rangle = a\,\mathrm{Tr}(fg) + b\,\mathrm{Tr}(f)\,\mathrm{Tr}(g)$$

is a cyclic form of degree 0 compatible with every connection of type (1,0).

In the following we consider the shifted quotient $\dfrac{A_X^{\cdot}}{A_X^{\geq 2,\cdot}}[2]$ of the de Rham complex by the 2nd subcomplex of the Hodge filtration as a DG-Lie algebra with trivial bracket. We denote $u = [D - \overline{\partial}, \overline{\partial} + d_{\mathcal{E}}] = [D, \overline{\partial} + d_{\mathcal{E}}] \in A_X^{1,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ the Atiyah cocycle.

Theorem 4.6.5. *Let $\mathcal{E}$ be a finite complex of locally free sheaves on a complex manifold X and let $\langle -,-\rangle$ be a cyclic form of degree 0 on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )$ which is compatible with a connection D of type $(1,0)$. Then there is an L_∞ morphism between DG-Lie algebras over the field $\mathbb{C}$*

$$g\colon A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \qquad \frac{A_X^{\cdot}}{A_X^{\geq 2,\cdot}}[2]$$

with components

$$g_1(f) = \langle u, f\rangle = \langle f, u\rangle \in A_X^{1,\cdot}[2],$$

$$g_2(f,g) = \frac{1}{2}\Big(\langle[\nabla - \overline{\partial}, f], g\rangle - (-1)^{\overline{f}\,\overline{g}}\langle[\nabla - \overline{\partial}, g], f\rangle\Big) \in A_X^{1,\cdot}[2],$$

$$g_3(f,g,h) = -\frac{1}{2}\langle f, [g,h]\rangle \in A_X^{0,\cdot}[2],$$

and $g_n = 0$ *for every* $n > 3$.

Notice that the definition of g only involves the DG-Lie structure of $A_X^{0,\cdot}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ and not the associative composition product.

Proof. Since the theorem gives explicit formulas for the components g_n, the proof reduces to a straightforward computation. Since $g_n = 0$ for every $n \geq 4$ we need to check the conditions C_n of Definition 1.3.13 for $n = 1,2,3,4$. For C_1 we have to prove that

$$dg_1(a) = g_1([d_{\mathcal{E}}, a] + \overline{\partial} a).$$

This follows from the fact that $[d_{\mathcal{E}}, u] + \overline{\partial} u = 0$ and that on the subcomplex $A_X^{1,\cdot} \subseteq \dfrac{A_X^{\cdot}}{A_X^{\geq 2,\cdot}}$ we have $d = \overline{\partial}$:

$$\begin{aligned} g_1([d_{\mathcal{E}}, a] + \overline{\partial} a) &= \langle u, [d_{\mathcal{E}}, a] + \overline{\partial} a\rangle = -\langle [d_{\mathcal{E}}, u], a\rangle + \overline{\partial}\langle u, a\rangle - \langle \overline{\partial} u, a\rangle \\ &= -\langle [d_{\mathcal{E}}, u] + \overline{\partial} u, a\rangle + \overline{\partial}\langle u, a\rangle = \overline{\partial}\langle u, a\rangle = dg_1(a). \end{aligned}$$

The condition C_2 is

$$g_2([d_{\mathcal{E}}, a_1] + \overline{\partial} a_1, a_2) + (-1)^{\overline{a_1}} g_2(a_1, [d_{\mathcal{E}}, a_2] + \overline{\partial} a_2) = g_1([a_1, a_2]) - dg_2(a_1, a_2).$$

On the left hand side we have

$$\begin{aligned} & g_2([d_{\mathcal{E}}, a_1] + \overline{\partial} a_1, a_2) + (-1)^{\overline{a_1}} g_2(a_1, [d_{\mathcal{E}}, a_2] + \overline{\partial} a_2) \\ &= \frac{1}{2}\Big(\langle[\nabla - \overline{\partial}, [d_{\mathcal{E}}, a_1]], a_2\rangle + \langle[\nabla - \overline{\partial}, \overline{\partial} a_1], a_2\rangle - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_2}}\langle[\nabla - \overline{\partial}, a_2], [d_{\mathcal{E}}, a_1]\rangle \\ &\quad - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_2}}\langle[\nabla - \overline{\partial}, a_2], \overline{\partial} a_1\rangle\Big) + \frac{1}{2}(-1)^{\overline{a_1}}\Big(\langle[\nabla - \overline{\partial}, a_1], [d_{\mathcal{E}}, a_2]\rangle + \langle[\nabla - \overline{\partial}, a_1], \overline{\partial} a_2\rangle \\ &\quad - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_1}}\langle[\nabla - \overline{\partial}, [d_{\mathcal{E}}, a_2]], a_1\rangle - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_1}}\langle[\nabla - \overline{\partial}, \overline{\partial} a_2], a_1\rangle\Big) \\ &= \frac{1}{2}\Big(\langle[[\nabla - \overline{\partial}, d_{\mathcal{E}}], a_1]), a_2\rangle - \langle[d_{\mathcal{E}}, [\nabla - \overline{\partial}, a_1]]), a_2\rangle + \langle[\nabla - \overline{\partial}, \overline{\partial} a_1], a_2\rangle + \\ &\quad - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_2}}\langle[\nabla - \overline{\partial}, a_2], [d_{\mathcal{E}}, a_1]\rangle - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_2}}\langle[\nabla - \overline{\partial}, a_2], \overline{\partial} a_1\rangle + \\ &\quad + (-1)^{\overline{a_1}}\langle[\nabla - \overline{\partial}, a_1], [d_{\mathcal{E}}, a_2]\rangle + (-1)^{\overline{a_1}}\langle[\nabla - \overline{\partial}, a_1], \overline{\partial} a_2\rangle - (-1)^{\overline{a_1}\,\overline{a_2}}\langle[[\nabla - \overline{\partial}, d_{\mathcal{E}}], a_2]), a_1\rangle \\ &\quad + (-1)^{\overline{a_1}\,\overline{a_2}}\langle[d_{\mathcal{E}}, [\nabla - \overline{\partial}, a_2]]), a_1\rangle - (-1)^{\overline{a_1}\,\overline{a_2}}\langle[\nabla - \overline{\partial}, \overline{\partial} a_2], a_1\rangle\Big) \\ &= \langle[\nabla - \overline{\partial}, d_{\mathcal{E}}], [a_1, a_2]\rangle + \frac{1}{2}\Big(\langle[\nabla - \overline{\partial}, \overline{\partial} a_1], a_2\rangle - (-1)^{\overline{a_1}\,\overline{a_2} + \overline{a_2}}\langle[\nabla - \overline{\partial}, a_2], \overline{\partial} a_1\rangle \\ &\quad + (-1)^{\overline{a_1}}\langle[\nabla - \overline{\partial}, a_1], \overline{\partial} a_2\rangle - (-1)^{\overline{a_1}\,\overline{a_2}}\langle[\nabla - \overline{\partial}, \overline{\partial} a_2], a_1\rangle\Big). \end{aligned}$$

Using Lemma 4.6.1 and Remark 4.5.5, the right hand side is:

$$g_1([a_1, a_2]) - dg_2(a_1, a_2) = \langle[\nabla - \overline{\partial}, d_{\mathcal{E}}] + [\nabla - \overline{\partial}, \overline{\partial}], [a_1, a_2]\rangle - \frac{1}{2}\Big(\langle\overline{\partial}[\nabla - \overline{\partial}, a_1], a_2\rangle$$

$$
\begin{aligned}
&- (-1)^{\overline{a_1}}\langle[\nabla-\overline{\partial},a_1],\overline{\partial}a_2\rangle - (-1)^{\overline{a_1}\ \overline{a_2}}\langle\overline{\partial}[\nabla-\overline{\partial},a_2],a_1\rangle + (-1)^{\overline{a_1}\ \overline{a_2}+\overline{a_2}}\langle[\nabla-\overline{\partial},a_2],\overline{\partial}a_1\rangle\Big)\\
&= \langle[\nabla-\overline{\partial},d_{\mathcal{E}}],[a_1,a_2]\rangle - \frac{1}{2}\Big(\langle\overline{\partial}[\nabla-\overline{\partial},a_1],a_2\rangle - (-1)^{\overline{a_1}}\langle[\nabla-\overline{\partial},a_1],\overline{\partial}a_2\rangle\\
&- (-1)^{\overline{a_1}\ \overline{a_2}}\langle\overline{\partial}[\nabla-\overline{\partial},a_2],a_1\rangle + (-1)^{\overline{a_1}\ \overline{a_2}+\overline{a_2}}\langle[\nabla-\overline{\partial},a_2],\overline{\partial}a_1\rangle\Big) + \frac{1}{2}\Big(\langle[[\nabla-\overline{\partial},\overline{\partial}],a_1],a_2\rangle+\\
&- (-1)^{\overline{a_1}\ \overline{a_2}}\langle[[\nabla-\overline{\partial},\overline{\partial}],a_2],a_1\rangle\Big)\\
&= \langle[\nabla-\overline{\partial},d_{\mathcal{E}}],[a_1,a_2]\rangle + \frac{1}{2}\Big(-\langle\overline{\partial}[\nabla-\overline{\partial},a_1],a_2\rangle + (-1)^{\overline{a_1}}\langle[\nabla-\overline{\partial},a_1],\overline{\partial}a_2\rangle\\
&+ (-1)^{\overline{a_1}\ \overline{a_2}}\langle\overline{\partial}[\nabla-\overline{\partial},a_2],a_1\rangle - (-1)^{\overline{a_1}\ \overline{a_2}+\overline{a_2}}\langle[\nabla-\overline{\partial},a_2],\overline{\partial}a_1\rangle + \langle[\nabla-\overline{\partial},\overline{\partial}a_1],a_2\rangle\\
&- (-1)^{\overline{a_1}\ \overline{a_2}}\langle[\nabla-\overline{\partial},\overline{\partial}a_2],a_1\rangle + \langle\overline{\partial}[\nabla-\overline{\partial},a_1],a_2\rangle - (-1)^{\overline{a_1}\ \overline{a_2}}\langle\overline{\partial}[\nabla-\overline{\partial},a_2],a_1\rangle\Big)\\
&= \langle[\nabla-\overline{\partial},d_{\mathcal{E}}],[a_1,a_2]\rangle + \frac{1}{2}\Big((-1)^{\overline{a_1}}\langle[\nabla-\overline{\partial},a_1],\overline{\partial}a_2\rangle - (-1)^{\overline{a_1}\ \overline{a_2}+\overline{a_2}}\langle[\nabla-\overline{\partial},a_2],\overline{\partial}a_1\rangle\\
&+ \langle[\nabla-\overline{\partial},\overline{\partial}a_1],a_2\rangle - (-1)^{\overline{a_1}\ \overline{a_2}}\langle[\nabla-\overline{\partial},\overline{\partial}a_2],a_1\rangle\Big),
\end{aligned}
$$

and this proves C_2. For C_3 we need to check that

$$
\begin{aligned}
dg_3(a_1,a_2,a_3) &= g_3([d_{\mathcal{E}},a_1]+\overline{\partial}a_1,a_2,a_3) - (-1)^{\overline{a_1}\ \overline{a_2}}g_3([d_{\mathcal{E}},a_2]+\overline{\partial}a_2,a_1,a_3)\\
&+ (-1)^{\overline{a_3}(\overline{a_1}+\overline{a_2})}g_3([d_{\mathcal{E}},a_3]+\overline{\partial}a_3,a_1,a_2) - g_2([a_1,a_2],a_3) + (-1)^{\overline{a_2}\ \overline{a_3}}g_2([a_1,a_3],a_2)\\
&- (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}g_2([a_2,a_3],a_1).
\end{aligned}
$$

Using the compatibility of the connection and the cyclic form, the terms involving g_2 can be expanded as:

$$
\begin{aligned}
&- g_2([a_1,a_2],a_3) + (-1)^{\overline{a_2}\ \overline{a_3}}g_2([a_1,a_3],a_2) - (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}g_2([a_2,a_3],a_1)\\
&= -\frac{1}{2}(\langle[\nabla-\overline{\partial},[a_1,a_2]],a_3\rangle - (-1)^{\overline{a_3}(\overline{a_1}+\overline{a_2})}\langle[\nabla-\overline{\partial},a_3],[a_1,a_2]\rangle)\\
&+ \frac{1}{2}(-1)^{\overline{a_2}\ \overline{a_3}}(\langle[\nabla-\overline{\partial},[a_1,a_3]],a_2\rangle - (-1)^{\overline{a_2}(\overline{a_1}+\overline{a_3})}\langle[\nabla-\overline{\partial},a_2],[a_1,a_3]\rangle)\\
&- \frac{1}{2}(-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}(\langle[\nabla-\overline{\partial},[a_2,a_3]],a_1\rangle - (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}\langle[\nabla-\overline{\partial},a_1],[a_2,a_3]\rangle)\\
&= -\frac{1}{2}\Big(\langle[[\nabla-\overline{\partial},a_1],a_2],a_3\rangle + (-1)^{\overline{a_1}}\langle[a_1,[\nabla-\overline{\partial},a_2]],a_3\rangle - (-1)^{\overline{a_3}(\overline{a_1}+\overline{a_2})}\langle[\nabla-\overline{\partial},a_3],[a_1,a_2]\rangle\\
&- (-1)^{\overline{a_2}\ \overline{a_3}}\langle[[\nabla-\overline{\partial},a_1],a_3],a_2\rangle - (-1)^{\overline{a_2}\ \overline{a_3}+\overline{a_1}}\langle[a_1,[\nabla-\overline{\partial},a_3]],a_2\rangle\\
&+ (-1)^{\overline{a_1}\ \overline{a_2}}\langle[\nabla-\overline{\partial},a_2],[a_1,a_3]\rangle + (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}\langle[[\nabla-\overline{\partial},a_2],a_3],a_1\rangle\\
&+ (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})+\overline{a_2}}\langle[a_2,[\nabla-\overline{\partial},a_3]],a_1\rangle - \langle[\nabla-\overline{\partial},a_1],[a_2,a_3]\rangle\Big) = -\frac{1}{2}\partial\langle a_1,[a_2,a_3]\rangle.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&g_3([d_{\mathcal{E}},a_1]+\overline{\partial}a_1,a_2,a_3) - (-1)^{\overline{a_1}\ \overline{a_2}}g_3([d_{\mathcal{E}},a_2]+\overline{\partial}a_2,a_1,a_3)+\\
&\quad + (-1)^{\overline{a_3}(\overline{a_1}+\overline{a_2})}g_3([d_{\mathcal{E}},a_3]+\overline{\partial}a_3,a_1,a_2)\\
&\quad = -\frac{1}{2}(\langle[d_{\mathcal{E}},a_1],[a_2,a_3]\rangle + (-1)^{\overline{a_1}}\langle a_1,[[d_{\mathcal{E}},a_2],a_3]\rangle + (-1)^{\overline{a_1}+\overline{a_2}}\langle a_1,[a_2,[d_{\mathcal{E}},a_3]]\rangle+\\
&\quad + \langle\overline{\partial}a_1,[a_2,a_3]\rangle + (-1)^{\overline{a_1}}\langle a_1,[\overline{\partial}a_2,a_3]\rangle + (-1)^{\overline{a_1}+\overline{a_2}}\langle a_1,[a_2,\overline{\partial}a_3]\rangle) = -\frac{1}{2}\overline{\partial}\langle a_1,[a_2,a_3]\rangle
\end{aligned}
$$

so that we obtain

$$
dg_3(a_1,[a_2,a_3]) = -\frac{1}{2}d\langle a_1,[a_2,a_3]\rangle = -\frac{1}{2}\overline{\partial}\langle a_1,[a_2,a_3]\rangle - \frac{1}{2}\partial\langle a_1,[a_2,a_3]\rangle.
$$

Lastly, the condition C_4 is

$$\begin{aligned} & g_3([a_1,a_2],a_3,a_4) - (-1)^{\overline{a_2}\,\overline{a_3}} g_3([a_1,a_3],a_2,a_4) + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})} g_3([a_1,a_4],a_2,a_3) \\ &\quad + (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})} g_3([a_2,a_3],a_1,a_4) - (-1)^{\overline{a_3}\,\overline{a_4}+\overline{a_1}\,\overline{a_2}+\overline{a_1}\,\overline{a_4}} g_3([a_2,a_4],a_1,a_3) \\ &\quad + (-1)^{(\overline{a_1}+\overline{a_2})(\overline{a_3}+\overline{a_4})} g_3([a_3,a_4],a_1,a_2) = 0. \end{aligned}$$

We have that

$$\begin{aligned} & \frac{1}{2}\langle [a_1,a_2],[a_3,a_4]\rangle - (-1)^{\overline{a_2}\,\overline{a_3}}\frac{1}{2}\langle [a_1,a_3],[a_2,a_4]\rangle + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\frac{1}{2}\langle [a_1,a_4],[a_2,a_3]\rangle \\ &\quad + (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}\frac{1}{2}\langle [a_2,a_3],[a_1,a_4]\rangle - (-1)^{\overline{a_3}\,\overline{a_4}+\overline{a_1}\,\overline{a_2}+\overline{a_1}\,\overline{a_4}}\frac{1}{2}\langle [a_2,a_4],[a_1,a_3]\rangle \\ &\quad + (-1)^{(\overline{a_1}+\overline{a_2})(\overline{a_3}+\overline{a_4})}\frac{1}{2}\langle [a_3,a_4],[a_1,a_2]\rangle \\ &= \langle [a_1,a_2],[a_3,a_4]\rangle - (-1)^{\overline{a_2}\,\overline{a_3}}\langle [a_1,a_3],[a_2,a_4]\rangle + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle [a_1,a_4],[a_2,a_3]\rangle \\ &= \langle a_1,[a_2,[a_3,a_4]]\rangle - (-1)^{\overline{a_2}\,\overline{a_3}}\langle a_1,[a_3,[a_2,a_4]]\rangle + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle a_1,[a_4,[a_2,a_3]]\rangle \\ &= \langle a_1,[a_2,[a_3,a_4]] - (-1)^{\overline{a_2}\,\overline{a_3}}[a_3,[a_2,a_4]] - [[a_2,a_3],a_4]\rangle = 0\,. \end{aligned}$$

□

Remark 4.6.6. If in Theorem 4.6.5 we use the cyclic form $\langle f,g\rangle = \mathrm{Tr}(fg)$ we obtain the following L_∞ morphism

$$g\colon A_X^{0,}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \qquad \frac{A_X^{,}}{A_X^{\geq 2,}}[2]$$

with components

$$\begin{aligned} g_1(f) &= \mathrm{Tr}(uf) \in A_X^{1,}[2] \\ g_2(f,g) &= \frac{1}{2}\,\mathrm{Tr}\left([\nabla-\overline{\partial},f]g - (-1)^{\overline{f}\overline{g}}[\nabla-\overline{\partial},g]f\right) \in A_X^{1,}[2] \\ g_3(f,g,h) &= -\frac{1}{2}\,\mathrm{Tr}(f[g,h]) \in A_X^{0,}[2], \end{aligned}$$

and $g_n = 0$ for every $n > 3$, which can be seen to be the same as the L_∞ morphism in (4.4.3).

Remark 4.6.7. The homotopy class of the L_∞ morphism g depends on the choice of the connection. This also holds for holomorphic connections, that is for connections where the Atiyah cocycle vanishes $u = 0$ and therefore $g_1 = 0$. This implies that g_2 factors to a bilinear graded skew-symmetric map in cohomology

$$g_2\colon \mathrm{Ext}^i_X(\mathcal{E}\ ,\mathcal{E}\ ) \times \mathrm{Ext}^j_X(\mathcal{E}\ ,\mathcal{E}\ ) \to \mathsf{H}^{i+j+1}\left(\frac{A_X^{,}}{A_X^{\geq 2,}}\right)$$

that depends only on the homotopy class of g.

In order to see that the above maps depend on the connection it is sufficient to consider the example of a trivial bundle of rank 2 over an elliptic curve X. In this case, since Ω^1_X is trivial, every holomorphic connection is of type $D = d + \theta$, where θ is a 2×2 matrix with values in $H^0(X,\Omega^1_X)$ and then $\nabla = \overline{\partial} + [\theta,-]$. Similarly $\mathrm{Ext}^0_X(\mathcal{E}\ ,\mathcal{E}\ )$ is identified with the Lie algebra $M_{2,2}(\mathbf{C})$ of 2×2 matrices with constant coefficients and therefore

$$g_2(a,b) = -\frac{1}{2}([\theta,a]b - [\theta,b]a) \in H^0(\Omega^1_X) = \mathsf{H}^1\left(\frac{A_X^{,}}{A_X^{\geq 2,}}\right).$$

If dz is a generator of $H^0(X,\Omega_X^1)$ and $\theta = Cdz$, with $C \in M_{2,2}(\mathbf{C})$, the conclusion follows by observing that the rank of the bilinear map

$$M_{2,2}(\mathbf{C}) \times M_{2,2}(\mathbf{C}) \to \mathbf{C}, \quad (A,B) \mapsto \frac{1}{2}\operatorname{Tr}([C,A]B - [C,B]A) = \operatorname{Tr}(C[A,B]),$$

is equal to 0 when C is a multiple of the identity and is 2 otherwise.

Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X equipped with a finite locally free resolution $\mathcal{E}$. Then, via Theorem 4.6.5 and Remark 4.6.6, every connection of type $(1,0)$ on the resolution $\mathcal{E}$ gives a lifting of

$$\tau_1\colon \operatorname{Ext}_X(\mathcal{F},\mathcal{F}) \to \mathsf{H}\ (X,\Omega_X^{\le 1}[2])$$

to an L_∞ morphism

$$g\colon A_X^{0,}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \quad \frac{A_X^{,}}{A_X^{>1,}}[2]\,.$$

Corollary 4.6.8. *Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X admitting a locally free resolution. Then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map*

$$\tau_1\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \mathsf{H}^2(X,\Omega_X^{\le 1}[2]).$$

If the Hodge to de Rham spectral sequence of X degenerates at E_1, then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map

$$\sigma_1\colon \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^3(X,\Omega_X^1), \qquad \sigma_1(a) = -\operatorname{Tr}(\operatorname{At}(\mathcal{F})\circ a).$$

Proof. By the syzygy theorem it is not restrictive to assume that $\mathcal{F}$ admits a finite locally free resolution $\mathcal{E}$. The map τ_1 lifts to an L_∞ morphism

$$g\colon A_X^{0,}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \quad \frac{A_X^{,}}{A_X^{\ge 2,}}[2]$$

and we have that the linear component g_1 commutes with obstruction maps of the associated deformation functors. By construction the DG-Lie algebra $\frac{A_X^{,}}{A_X^{\ge 2,}}[2]$ has trivial bracket and hence every obstruction of the associated deformation functor is trivial, as seen in Lemma 2.4.6.

If the Hodge to de Rham spectral sequence of X degenerates at E_1 then the inclusion of complexes $A_X^{1,}[2] \subset \frac{A_X^{,}}{A_X^{\ge 2,}}[2]$ is injective in cohomology

$$H^3(X,\Omega_X^1) \hookrightarrow \mathsf{H}^2(X,\Omega_X^{\le 1}[2])$$

and the maps σ,τ have the same kernel. $\square$

Chapter 5

Cyclic forms on DG-Lie algebroids and semiregularity

This chapter is based on the paper [49]; the initial objective was to carry out in the algebraic case a construction analogous the one done for the complex case in [50], Section 4.6.

More in general, given a transitive DG-Lie algebroid $(\mathcal{A}, \rho)$ over a smooth separated scheme X of finite type over a field K of characteristic 0 it is possible to define a simplicial notion of connection $\nabla\colon \mathsf{R}\Gamma(X, \mathrm{Ker}\rho) \to \mathsf{R}\Gamma(X, \Omega^1_X[-1] \otimes \mathrm{Ker}\rho)$ and to construct an L_∞ morphism between DG-Lie algebras $f\colon \mathsf{R}\Gamma(X, \mathrm{Ker}\rho) \quad \mathsf{R}\Gamma(X, \Omega^{\leq 1}_X[2])$ associated to a connection and to a cyclic form on the DG-Lie algebroid.

In a particular case of this construction, we obtain a lifting of the first component of the modified Buchweitz–Flenner semiregularity map in the algebraic context, which implies that this map annihilates all obstructions to deformations of coherent sheaves on X admitting a finite locally free resolution. In Section 5.3, another application is given, to the deformation theory of (Zariski) principal bundles on X.

5.1 DG-Lie algebroids, connections and extension cocycles

The goal of this section is to define K-linear operators

$$\nabla\colon\ \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \to \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$$

called connections on the kernel $\mathcal{L}$ of the anchor map of a transitive DG-Lie algebroid. In order to construct a connection, we introduce the notion of simplicial lifting of the identity. The section ends with the definition of the extension cocycle associated to a simplicial lifting of the identity, which generalises the notion of Atiyah cocycle. Different notions of Atiyah classes for DG-Lie algebroids have been considered elsewhere in the literature, see e.g. [9, 13, 63].

Let X be a smooth separated scheme of finite type over a field K of characteristic zero, and let $\Theta_X, \Omega^1_X = \Omega^1_{X/\mathsf{K}}$ denote its tangent and cotangent sheaves respectively. Often it will be useful to consider the cotangent sheaf as a trivial complex of sheaves concentrated in degree one, so as to have an inclusion $\Omega^1_X[-1] \to \Omega_X$, where $\Omega_X = \oplus_p \Omega^p_X[-p]$ denotes the algebraic de Rham complex.

Definition 5.1.1. A **DG-Lie algebroid** over X is a complex of sheaves of $\mathcal{O}_X$-modules $\mathcal{A}$ equipped with a K-bilinear bracket $[-,-]\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which defines a DG-Lie algebra structure on the spaces of sections, and with a morphism of complexes of $\mathcal{O}_X$-modules $\rho\colon \mathcal{A} \to \Theta_X$, called the anchor map, such that the induced map on the spaces of sections is a homomorphism of DG-Lie algebras. Moreover for any sections a_1, a_2 of $\mathcal{A}$ and f of $\mathcal{O}_X$, the following Leibniz identity holds:

$$[a_1, f a_2] = f[a_1, a_2] + \rho(a_1)(f)a_2.$$

Example 5.1.2. The sheaf Θ_X is a trivial example of a DG-Lie algebroid concentrated in degree zero, with anchor map given by the identity. A DG-Lie algebroid over $\operatorname{Spec}\mathsf{K}$ is exactly a DG-Lie algebra over the field K. Every sheaf of DG-Lie algebras over $\mathcal{O}_X$ can be considered as a DG-Lie algebroid over X with trivial anchor map.

Definition 5.1.3. Let $(\mathcal{A},\rho)$ and $(\mathcal{B},\sigma)$ be DG-Lie algebroids over X. A morphism of DG-Lie algebroids $\varphi\colon \mathcal{A}\to\mathcal{B}$ is a morphism of complexes of sheaves which preserves brackets and commutes with the anchor maps:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & \mathcal{B} \arrow[dl, "\sigma"] \\ & \Theta_X . & \end{tikzcd}$$

Let $(\mathcal{A},\rho)$ be a DG-Lie algebroid over X and assume that $\mathcal{L}=\operatorname{Ker}\rho$ is a finite complex of locally free sheaves. Notice that on $\mathcal{L}$ there is a naturally induced graded Lie bracket: for sections x,y of $\mathcal{L}$

$$[x,y] := [i(x),i(y)],$$

where $i\colon \mathcal{L}\to\mathcal{A}$ denotes the inclusion. This bracket is $\mathcal{O}_X$-linear, in fact for any sections x,y of $\mathcal{L}$ and f of $\mathcal{O}_X$ one has

$$\begin{aligned}[x,fy] &:= [i(x),i(fy)] = [i(x),fi(y)] = f[i(x),i(y)] + \rho(i(x))(f)y\\ &= f[i(x),i(y)] = f[x,y],\end{aligned}$$

so that $\mathcal{L}$ is a sheaf of DG-Lie algebras over $\mathcal{O}_X$.

Definition 5.1.4. [54, Chapter 3] A DG-Lie algebroid $(\mathcal{A},\rho)$ over X is **transitive** if the anchor map $\rho\colon\mathcal{A}\to\Theta_X$ is surjective.

Let now $(\mathcal{A},\rho)$ be a transitive DG-Lie algebroid over X, consider the short exact sequence of complexes of sheaves

$$0 \longrightarrow \mathcal{L} \xrightarrow{\ i\ } \mathcal{A} \xrightarrow{\ \rho\ } \Theta_X \longrightarrow 0$$

and tensor it with the shifted cotangent sheaf $\Omega^1_X[-1]$ to obtain the short exact sequence

$$0 \longrightarrow \Omega^1_X[-1]\otimes\mathcal{L} \xrightarrow{\mathsf{Id}\ \ i} \Omega^1_X[-1]\otimes\mathcal{A} \xrightarrow{\mathsf{Id}\ \ \rho} \Omega^1_X[-1]\otimes\Theta_X \longrightarrow 0. \tag{5.1.1}$$

Because of the isomorphism

$$\Omega^1_X[-1]\otimes\Theta_X \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X,\Omega^1_X[-1]) \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X,\Omega^1_X)[-1],$$

one can consider $\mathrm{Id}_{\Omega^1}\in\Gamma(X,\Omega^1_X[-1]\otimes\Theta_X)$ as an element of degree one.

Definition 5.1.5. A **lifting of the identity** is a global section D in $\Gamma(X,\Omega^1_X[-1]\otimes\mathcal{A})$ such that $(\mathrm{Id}\otimes\rho)(D)=\mathrm{Id}_{\Omega^1}\in\Gamma(X,\Omega^1_X[-1]\otimes\Theta_X)$.

Since the map $\mathrm{Id}\otimes\rho$ is not in general surjective on global sections, a lifting of the identity does not always exist. However a germ of a lifting of the identity, i.e., a preimage of Id_{Ω^1} in $\Omega^1_X[-1]\otimes\mathcal{A}$, always exists.

Example 5.1.6. For particular DG-Lie algebroids, the notion of lifting of the identity can be related to the more familiar notion of algebraic connection. Let $(\mathcal{E}\ ,\delta_{\mathcal{E}})$ be a finite complex of locally free sheaves. Following [41, Section 5], define the complex of derivations of pairs

$$\mathcal{D}\ \ (X,\mathcal{E}\ ) = \left\{(h,u)\in\Theta_X\times\mathcal{H}om_{\mathsf{K}}(\mathcal{E}\ ,\mathcal{E}\ )\ \middle|\ u(fe)=fu(e)+h(f)e,\quad \forall f\in\mathcal{O}_X,\ e\in\mathcal{E}\ \right\}.$$

The complex $\mathcal{D}\ \ (X, \mathcal{E}\ \ )$ is a finite complex of coherent sheaves and the natural map

$$\alpha \colon \mathcal{D}\ \ (X, \mathcal{E}\ \ ) \to \Theta_X, \quad (h, u) \mapsto h,$$

which is called the anchor map, is surjective, see [41]. The graded Lie bracket is defined as

$$[(h, u), (h\ , u\ )] = ([h, h\ ], [u, u\ ]),$$

where the (graded) Lie brackets on Θ_X and $\mathcal{H}om_{\mathsf{K}}(\mathcal{E}\ \ , \mathcal{E}\ \ )$ are the (graded) commutators of the composition products. For $f \in \mathcal{O}_X$ we then have that

$$\begin{aligned}
[(h, u), f(h\ , u\ )] &= [(h, u), (fh\ , fu\ )] = ([h, fh\ ], [u, fu\ ]) \\
&= (h(f)h\ + fhh\ - fh\ h, fuu\ + h(f)u\ - (-1)^{\overline{u}\overline{u}}fu\ u) \\
&= (h(f)h\ + f[h, h\ ], h(f)u\ + f[u, u\ ]) \\
&= f([h, h\ ], [u, u\ ]) + h(f)(h\ , u\ ) \\
&= f[(h, u), (h\ , u\ )] + \alpha((h, u))(f)(h\ , u\ ),
\end{aligned}$$

hence $(\mathcal{D}\ \ (X, \mathcal{E}\ \ ), \alpha)$ is a transitive DG-Lie algebroid over X. Define an algebraic connection on the complex of locally free sheaves $\mathcal{E}\ \ $ as the data for every i of an algebraic connection on $\mathcal{E}^i$, i.e., a K-linear map $D \colon \mathcal{E}^i \to \Omega^1_X \otimes \mathcal{E}^i$ such that for $e \in \mathcal{E}^i, f \in \mathcal{O}_X$

$$D(fe) = d_{dR}f \otimes e + fD(e),$$

where d_{dR} denotes the universal derivation $d_{dR} \colon \mathcal{O}_X \to \Omega^1_X$. A global algebraic connection on $\mathcal{E}$ need not exist. The kernel of the anchor map α is the sheaf of DG-Lie algebras $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ \ , \mathcal{E}\ \ )$, the graded sheaf of $\mathcal{O}_X$-linear endomorphisms of $\mathcal{E}\ \ $, with bracket equal to the graded commutator

$$[f, g] = fg - (-1)^{\overline{f}\overline{g}}gf,$$

and differential given by

$$g \mapsto [\delta_{\mathcal{E}}, g] = \delta_{\mathcal{E}}g - (-1)^{\overline{g}}g\delta_{\mathcal{E}}\,.$$

The short exact sequence in (5.1.1) in this case is isomorphic to

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ \ , \Omega^1_X[-1] \otimes \mathcal{E}\ \ ) \xrightarrow{g \to (0,g)} \mathcal{J}_{\Omega^1} \xrightarrow{(\beta,g) \to \beta} \mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X, \Omega^1_X[-1]) \to 0\,,$$

where the complex $\mathcal{J}_{\Omega^1}$ is defined as the subcomplex of

$$\mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X, \Omega^1_X[-1]) \times \mathcal{H}om_{\mathsf{K}}(\mathcal{E}\ \ , \Omega^1_X[-1] \otimes \mathcal{E}\ \ )$$

of elements (β, v), with $\beta \in \mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X, \Omega^1_X[-1])$ and $v \in \mathcal{H}om_{\mathsf{K}}(\mathcal{E}\ \ , \Omega^1_X[-1] \otimes \mathcal{E}\ \ )$ such that

$$v(fx) = fv(x) + \beta(f) \otimes x,$$

for all $x \in \mathcal{E}\ \ $ and $f \in \mathcal{O}_X$. In this case a lifting of the identity is exactly a global algebraic connection on the complex of sheaves $\mathcal{E}\ \ $: via the isomorphism $\Omega^1_X[-1] \otimes \mathcal{D}\ \ (X, \mathcal{E}\ \ ) \cong \mathcal{J}_{\Omega^1}$ a lifting of the identity D corresponds to K-linear maps $D\ : \mathcal{E}^i \to \Omega^1_X[-1] \otimes \mathcal{E}^i$ for all i such that $D\ (fe) = fD\ (e) + d_{dR}(f) \otimes e$ for all $f \in \mathcal{O}_X$ and $e \in \mathcal{E}^i$.

We now define connections on $\mathcal{L} = \operatorname{Ker} \rho$, the kernel of the anchor map of a transitive DG-Lie algebroid $(\mathcal{A}, \rho)$ over X. Assume that $\mathcal{L}$ is a finite complex of locally free sheaves and fix an affine open cover $\mathcal{U} = \{U_i\}$ of X. The short exact sequence

$$0 \longrightarrow \Omega^1_X[-1] \otimes \mathcal{L} \xrightarrow{\mathsf{Id}\ \ i} \Omega^1_X[-1] \otimes \mathcal{A} \xrightarrow{\mathsf{Id}\ \ \rho} \Omega^1_X[-1] \otimes \Theta_X \longrightarrow 0$$

gives a short exact sequence of the corresponding semicosimplicial complexes of Čech cochains. Consider the Thom–Whitney totalisation functor Tot described in Section 1.4, which is exact and hence gives an exact sequence

$$0 \longrightarrow \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}) \xrightarrow{\mathsf{Id} \otimes i} \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \xrightarrow{\mathsf{Id} \otimes \rho} \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X) \longrightarrow 0.$$

Denote by d the differential on $\mathcal{A}$ and $\mathcal{L}$, which can be extended to $\Omega^1_X[-1] \otimes \mathcal{A}$ and to $\Omega^1_X[-1] \otimes \mathcal{L}$ by setting

$$d(\eta \otimes x) = (-1)^{\overline{\eta}} \eta \otimes dx = -\eta \otimes dx.$$

Denote by d_{Tot} the differentials on all the above Tot complexes: for $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ and $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ the differential d_{Tot} is equal to $d_A + d$, while for $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$ one has that d_{Tot} is just d_A, where d_A is the differential of polynomial differential forms on the affine simplex, see Definition 1.4.3.

Because of the natural inclusion of global sections in the totalisation, see Example 1.4.5, Id_{Ω^1} belongs to $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$, where it has degree one.

Definition 5.1.7. A **simplicial lifting of the identity** is an element D of $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ such that $(\mathrm{Id} \otimes \rho)(D) = \mathrm{Id}_{\Omega^1}$ in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$.

It is clear that a simplicial lifting of the identity always exists and that D has degree one in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$.

Remark 5.1.8. Notice that via the isomorphism

$$\Omega^1_X[-1] \otimes \Theta_X = \Omega^1_X[-1] \otimes \mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X, \Omega^1_X[-1]),$$

we have that $(\mathrm{Id} \otimes \rho)(D) = d_{dR} \in \operatorname{Tot}(\mathcal{U}, \mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X, \Omega^1_X[-1]))$.

In order to define a connection on $\mathcal{L}$, it is necessary to define a Lie bracket

$$[-,-] \colon \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \times \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \to \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}),$$

induced by the bracket of the following lemma.

Lemma 5.1.9. *There exists a well defined* K*-bilinear bracket*

$$[-,-] \colon (\Omega^1_X[-1] \otimes \mathcal{A}) \times \mathcal{L} \to \Omega^1_X[-1] \otimes \mathcal{L}.$$

Proof. Denote by $i \colon \mathcal{L} \to \mathcal{A}$ the inclusion, take $\eta \otimes a$ with $\eta \in \Omega^1_X[-1]$ and $a \in \mathcal{A}$, and define for $x \in \mathcal{L}$

$$[\eta \otimes a, x] := \eta \otimes [a, i(x)].$$

Notice that the Leibniz identity in Definition 5.1.1 implies that

$$[fa_1, a_2] = f[a_1, a_2] - (-1)^{\overline{a_1}\ \overline{a_2}} \rho(a_2)(f) a_1.$$

Hence the bracket $[\eta \otimes a, x]$ is well defined: for any $f \in \mathcal{O}_X$

$$\begin{aligned}
[\eta \otimes fa, x] &= \eta \otimes [fa, x] = \eta \otimes \left(f[a,x] - (-1)^{\overline{a}\ \overline{x}} \rho(x)(f)a\right) = \eta \otimes f[a,x] \\
&= f\eta \otimes [a,x] = [f\eta \otimes a, x].
\end{aligned}$$

It is clear that $[\eta \otimes a, x]$ belongs to $\Omega^1_X[-1] \otimes \mathcal{L}$:

$$(\mathrm{Id} \otimes \rho)([\eta \otimes a, x]) = (\mathrm{Id} \otimes \rho)(\eta \otimes [a, x]) = \eta \otimes [\rho(a), \rho(x)] = 0.$$

$\square$

Since the functor Tot preserves products, the map

$$[-,-] \colon (\Omega^1_X[-1] \otimes \mathcal{A}) \times \mathcal{L} \to \Omega^1_X[-1] \otimes \mathcal{L}$$

induces a K-bilinear map

$$[-,-] \colon \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \times \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \to \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}), \tag{5.1.2}$$

which is defined component-wise as the restriction of

$$A_n \otimes \prod (\Omega^1_X[-1] \otimes \mathcal{A})(U_{i_0\cdots i_n}) \times A_n \otimes \prod \mathcal{L}(U_{i_0\cdots i_n}) \xrightarrow{[-,-]} A_n \otimes \prod (\Omega^1_X[-1] \otimes \mathcal{L})(U_{i_0\cdots i_n})$$

$$[\eta_n \otimes (t_{i_0\cdots i_n}), \phi_n \otimes (u_{i_0\cdots i_n})] = \eta_n\phi_n \otimes ([(-1)^{\overline{\phi_n}\ \overline{t_{i_0\cdots i_n}}} t_{i_0\cdots i_n}, u_{i_0\cdots i_n}]),$$

for η_n, ϕ_n in A_n, $t_{i_0\cdots i_n}$ in $(\Omega^1_X[-1] \otimes \mathcal{A})(U_{i_0\cdots i_n})$ and $u_{i_0\cdots i_n}$ in $\mathcal{L}(U_{i_0\cdots i_n})$.

Definition 5.1.10. A **connection** on $\mathcal{L}$ is the adjoint operator of a simplicial lifting of the identity $D \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$

$$\nabla = [D, -]\colon \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \to \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}),$$

where $[-,-]\colon \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \times \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \to \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ is the bracket in (5.1.2). It is a K-linear operator.

We will now examine the relationship between connections and particular representatives of extension classes. The short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{A} \xrightarrow{\rho} \Theta_X \longrightarrow 0 \tag{5.1.3}$$

gives an extension class $[u_\rho] \in \operatorname{Ext}^1_X(\Theta_X, \mathcal{L})$. It is possible to give a representative of $[u_\rho]$ in the totalisation $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ with respect to an affine open cover $\mathcal{U}$ of X.

Definition 5.1.11. An **extension cocycle** u of the transitive DG-Lie algebroid $\mathcal{A}$ is the differential of a simplicial lifting of the identity D in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$, $u = d_{\mathsf{Tot}} D$.

Notice that u belongs to $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$:

$$(\operatorname{Id} \otimes \rho)u = (\operatorname{Id} \otimes \rho) d_{\mathsf{Tot}}(D) = d_{\mathsf{Tot}}(\operatorname{Id} \otimes \rho)D = d_{\mathsf{Tot}} \operatorname{Id}_{\Omega^1} = 0,$$

where the last equality is a consequence of the fact that $\operatorname{Id}_{\Omega^1}$ is a global section and $\Omega^1_X[-1] \otimes \Theta_X$ has trivial differential (see Example 1.4.5). Note that u has degree two in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ and that $d_{\mathsf{Tot}} u = d_{\mathsf{Tot}} d_{\mathsf{Tot}} D = 0$.

Using the isomorphisms

$$\Omega^1_X[-1] \otimes \mathcal{L} \cong \mathcal{H}om_{\mathcal{O}_X}(\Theta_X[1], \mathcal{L}) \cong \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{L})[-1],$$

the cohomology class of u belongs to

$$\begin{aligned} H^2(\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})) &\cong H^2(\operatorname{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{L})[-1])) \cong \mathsf{H}^2(X, \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{L})[-1]) \\ &\cong \mathsf{H}^1(X, \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{L})) \cong \operatorname{Ext}^1_X(\Theta_X, \mathcal{L}). \end{aligned}$$

This cohomology class does not depend on the chosen simplicial lifting of the identity: if D and D are two simplicial liftings of the identity in $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$, we have that $(\operatorname{Id} \otimes \rho)(D - D) = \operatorname{Id}_{\Omega^1} - \operatorname{Id}_{\Omega^1} = 0$, so $D - D$ belongs to $\operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ and $d_{\mathsf{Tot}} D$ and $d_{\mathsf{Tot}} D$ differ by the coboundary $d_{\mathsf{Tot}}(D - D)$. It is easy to see that the cohomology class of u is trivial if and only if the short exact sequence in (5.1.3) splits.

Lemma 5.1.12. *Let* $\nabla\colon \operatorname{Tot}(\mathcal{U}, \mathcal{L}) \to \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ *be a connection on* $\mathcal{L}$*, associated to the simplicial lifting of the identity* $D \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$*. Let* $u = d_{\mathsf{Tot}} D$ *be the corresponding extension cocycle, then for every* x *in* $\operatorname{Tot}(\mathcal{U}, \mathcal{L})$ *we have that*

$$\nabla(d_{\mathsf{Tot}} x) = [u, x] - d_{\mathsf{Tot}} \nabla(x).$$

Proof. Recall that d denotes the differential of $\mathcal{A}$ and $\mathcal{L}$, which can be extended to $\Omega^1_X[-1] \otimes \mathcal{A}$ and to $\Omega^1_X[-1] \otimes \mathcal{L}$ by setting $d(\eta \otimes x) = (-1)^{\overline{\eta}} \eta \otimes dx = -\eta \otimes dx$. It is easy to see that for the K-bilinear map $[-,-]\colon (\Omega^1_X[-1] \otimes \mathcal{A}) \times \mathcal{L} \to \Omega^1_X[-1] \otimes \mathcal{L}$ of Lemma 5.1.9,

$$d[\eta \otimes a, x] = [d(\eta \otimes a), x] + (-1)^{\overline{\eta} + \overline{a}} [\eta \otimes a, dx].$$

A straightforward calculation then shows that for $z \in \operatorname{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ and $w \in \operatorname{Tot}(\mathcal{U}, \mathcal{L})$,

$$d_{\mathsf{Tot}}[z, w] = [d_{\mathsf{Tot}} z, w] + (-1)^{\overline{z}} [z, d_{\mathsf{Tot}} w],$$

and the conclusion follows from the fact $u = d_{\mathsf{Tot}} D$. $\square$

5.2 Cyclic forms and L_∞ morphisms

This section describes cyclic forms on DG-Lie algebroids and illustrates how DG-Lie algebroid representations give rise to cyclic forms. We then discuss induced cyclic forms on the Thom–Whitney totalisation and the property of d_{Tot}-closure. The central result is the construction of a L_∞ morphism associated to a connection and to a d_{Tot}-closed cyclic form for a transitive DG-Lie algebroid. This allows us to state the results of Section 4.6, [50] for a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme of finite type over a field K of characteristic zero.

Let $\mathcal{A}$ be a DG-Lie algebroid over a smooth separated scheme X of finite type over a field K of characteristic zero, with anchor map $\rho\colon \mathcal{A} \to \Theta_X$. Assume that the kernel of the anchor map $\mathcal{L}$ is a finite complex of locally free sheaves. Notice that for any $a \in \mathcal{A}$ and $x \in \mathcal{L}$, the bracket $[a, x]$ belongs to $\mathcal{L}$:

$$\rho([a, x]) = [\rho(a), \rho(x)] = 0.$$

Definition 5.2.1. A cyclic bilinear form on a DG-Lie algebroid $(\mathcal{A}, \rho)$ is a graded symmetric $\mathcal{O}_X$-bilinear product of degree zero on $\mathcal{L} = \operatorname{Ker} \rho$

$$\langle -, -\rangle\colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$$

such that for all sections x, y of $\mathcal{L}$ and a of $\mathcal{A}$

$$\langle [a, x], y\rangle + (-1)^{\bar{a}\,\bar{x}}\langle x, [a, y]\rangle = \rho(a)(\langle x, y\rangle). \tag{5.2.1}$$

Notice that the definition implies that for all $x, y, z \in \mathcal{L}$

$$\langle x, [y, z]\rangle = \langle [x, y], z\rangle. \tag{5.2.2}$$

These two properties will be discussed after giving some examples.

In the following two examples, the cyclicity of the forms will follow from Lemma 5.2.5.

Example 5.2.2. An example of cyclic form on $(\mathcal{A}, \rho)$ is induced by the Killing form. Consider the adjoint representation as a morphism of sheaves of DG-Lie algebras

$$\operatorname{ad}\colon \mathcal{L} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}), \quad a \mapsto [a, -]$$

and consider the trace map $\operatorname{Tr}\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to \mathcal{O}_X$, which is morphism of sheaves of DG-Lie algebras (when considering $\mathcal{O}_X$ as a trivial sheaf of DG-Lie algebras). Then one can define the form

$$\langle -, -\rangle\colon \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X, \quad (x, y) \mapsto \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y).$$

Example 5.2.3. For the DG-Lie algebroid $\mathcal{D}(X, \mathcal{E})$ of Example 5.1.6,

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{D}(X, \mathcal{E}) \longrightarrow \Theta_X \longrightarrow 0,$$

a natural bilinear form on $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ is induced by the trace map $\operatorname{Tr}\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$ as follows:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \times \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X, \quad \langle f, g\rangle := -\operatorname{Tr}(fg).$$

Example 5.2.3 explains the definition of cyclic form: (5.2.2) reflects the cyclicity property of the trace map $\operatorname{Tr}\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$, $\operatorname{Tr}(ab) = (-1)^{\bar{a}\bar{b}}\operatorname{Tr}(ba)$, while (5.2.1) is related to the properties of the extension of the trace map to $\mathcal{D}(X, \mathcal{E})$, for which we refer to [41].

The Leibniz identity of Definition 5.1.1

$$[a, fx] = f[a, x] + \rho(a)(f)x \quad \forall a \in \mathcal{A},\ x \in \mathcal{L},\ f \in \mathcal{O}_X$$

can be restated by noticing that for all a in $\mathcal{A}$ the operator $(\rho(a), [a, -])$ belongs to $\mathcal{D}(X, \mathcal{L})$ of Example 5.1.6. Hence there is a morphism of DG-Lie algebroids

$$\operatorname{ad}\colon \mathcal{A} \to \mathcal{D}(X, \mathcal{L}).$$

The morphism $\operatorname{ad}\colon \mathcal{A} \to \mathcal{D}(X, \mathcal{L})$ restricts to the morphism $\operatorname{ad}\colon \mathcal{L} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ of Example 5.2.2, so that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "i"] \arrow[d, "\mathrm{ad}"] & \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\mathrm{ad}"] & \Theta_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L}) \arrow[r] & \mathcal{D}(X,\mathcal{L}) \arrow[r, "\alpha"] & \Theta_X \arrow[r] & 0. \end{tikzcd}$$

This motivates the following definition:

Definition 5.2.4. A representation of a DG-Lie algebroid $(\mathcal{A},\rho)$ over X is a morphism of DG-Lie algebroids $\theta\colon \mathcal{A}\to\mathcal{D}(X,\mathcal{E})$, where $\mathcal{E}$ is a finite complex of locally free sheaves over X:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\theta"] \arrow[dr, "\rho"'] & & \mathcal{D}(X,\mathcal{E}) \arrow[dl, "\alpha"] \\ & \Theta_X. & \end{tikzcd}$$

Every representation $\theta\colon\mathcal{A}\to\mathcal{D}(X,\mathcal{E})$ induces a form $\langle -,-\rangle_\theta\colon \mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$: for any $x\in\mathcal{L}$ we have that $\alpha\circ\theta(x)=\rho(x)=0$, so that

$$\theta|_{\mathcal{L}}\colon \mathcal{L}\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}),$$

and using the trace map $\mathrm{Tr}\colon\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\to\mathcal{O}_X$ we can define for x,y sections of $\mathcal{L}$

$$\langle x,y\rangle_\theta := \mathrm{Tr}(\theta(x)\theta(y)).$$

Forms obtained in this way are cyclic:

Lemma 5.2.5. *For any DG-Lie algebroid representation $\theta\colon\mathcal{A}\to\mathcal{D}(X,\mathcal{E})$ the induced form $\langle -,-\rangle_\theta\colon\mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ is cyclic.*

Proof. For $a\in\mathcal{A}$ and $x,y\in\mathcal{L}$

$$\begin{aligned}\langle [a,x],y\rangle_\theta + (-1)^{\overline{a}\,\overline{x}}\langle x,[a,y]\rangle_\theta &= \mathrm{Tr}(\theta([a,x])\theta(y)+(-1)^{\overline{a}\,\overline{x}}\theta(x)\theta([a,y]))\\ &= \mathrm{Tr}([\theta(a),\theta(x)]\theta(y)+(-1)^{\overline{a}\,\overline{x}}\theta(x)[\theta(a),\theta(y)])\\ &= \mathrm{Tr}(\theta(a)\theta(x)\theta(y)-(-1)^{\overline{a}(\overline{x}+\overline{y})}\theta(x)\theta(y)\theta(a))\\ &= \mathrm{Tr}([\theta(a),\theta(x)\theta(y)]).\end{aligned}$$

Notice that if $\overline{a}\neq 0$ then a belongs to $\mathcal{L}$, so that $\theta(a)$ belongs to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ and it is clear that $\mathrm{Tr}([\theta(a),\theta(x)\theta(y)])=0$, by the properties of the trace map.

The only remaining non-trivial case is when $\overline{a}=\overline{x}+\overline{y}=0$. Let $\{e_i^k\}$ with $i=1,\cdots,n_k$ be a local basis of $\mathcal{L}^k$ and let

$$\theta(a)(e_i^k)=\sum_j A_{ij}^k e_j^k,\quad \theta(x)\theta(y)(e_i^k)=\sum_j B_{ij}^k e_j^k,\quad A_{ij}^k, B_{ij}^k\in\mathcal{O}_X,$$

then

$$\begin{aligned}{[\theta(a),\theta(x)\theta(y)]}(e_i^k) &= \theta(a)\theta(x)\theta(y)(e_i^k)-\theta(x)\theta(y)\theta(a)(e_i^k)\\ &= \theta(a)(\sum_j B_{ij}^k e_j^k)-\theta(x)\theta(y)(\sum_j A_{ij}^k e_j^k)\\ &= \sum_j B_{ij}^k\theta(a)(e_j^k)+\sum_j(\alpha\circ\theta)(a)(B_{ij}^k)e_j^k-\sum_{j,s}A_{ij}^kB_{js}^ke_s^k\\ &= \sum_{j,s}B_{ij}^kA_{js}^ke_s^k+\sum_j\rho(a)(B_{ij}^k)e_j^k-\sum_{j,s}A_{ij}^kB_{js}^ke_s^k.\end{aligned}$$

The trace of $[\theta(a),\theta(x)\theta(y)]$ is hence equal to

$$\sum_k(-1)^k(\sum_{j,i}B_{ij}^kA_{ji}^k+\sum_i\rho(a)(B_{ii}^k)-\sum_{j,i}A_{ij}^kB_{ji}^k)=\sum_{k,i}(-1)^k\rho(a)(B_{ii}^k)=$$

$$\rho(a)(\sum_{k,i}(-1)^k B^k_{ii}) = \rho(a)\operatorname{Tr}(\theta(x)\theta(y)) = \rho(a)(\langle x,y\rangle_\theta).$$

□

For every $i \geq 0$, let $\Omega^i_X[-i]$ denote the sheaf Ω^i_X seen as a trivial complex concentrated in degree i. Any cyclic form $\langle -,-\rangle\colon \mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ can be extended to a collection of $\mathcal{O}_X$-bilinear forms

$$\langle -,-\rangle\colon (\Omega^i_X[-i]\otimes\mathcal{L})\times(\Omega^j_X[-j]\otimes\mathcal{L})\to\Omega^{i+j}_X[-i-j],\quad i,j\geq 0,$$

according to the Koszul sign rule, by setting for $x,y\in\mathcal{L}$, $\omega\in\Omega^i_X[-i]$, an $\eta\in\Omega^j_X[-j]$

$$\langle\omega\otimes x,\eta\otimes y\rangle = (-1)^{\overline{x}j}\omega\wedge\eta\langle x,y\rangle.$$

It is immediate to see that this form is cyclic in the sense that

$$\langle[b,x],y\rangle + (-1)^{\overline{b}\overline{x}}\langle x,[b,y]\rangle = (\operatorname{Id}\otimes\rho)(b)(\langle x,y\rangle)\quad \forall b\in\Omega^1_X[-1]\otimes\mathcal{A},\quad \forall x,y\in\mathcal{L},$$

where the bracket is the one of Lemma 5.1.9, and the anchor map has been extended to $\Omega^1_X[-1]\otimes\mathcal{A}$ by setting $(\operatorname{Id}\otimes\rho)(\omega\otimes a) := \omega\otimes\rho(a)$.

Definition 5.2.6. A cyclic form $\langle -,-\rangle\colon\mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ is d-**closed** if for all $z,w\in\mathcal{L}$

$$\langle dz,w\rangle + (-1)^{\overline{z}}\langle z,dw\rangle = 0.$$

Lemma 5.2.7. *For any DG-Lie algebroid representation $\theta\colon\mathcal{A}\to\mathcal{D}(X,\mathcal{E})$ the induced cyclic form $\langle -,-\rangle_\theta\colon\mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ is d-closed.*

Proof. Since $\theta|_{\mathcal{L}}\colon\mathcal{L}\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$ is a morphism of DG-Lie algebroids it commutes with differentials: for $x\in\mathcal{L}$,

$$\theta(dx) = d_{\mathcal{H}om(\mathcal{E},\mathcal{E})}(\theta(x)) = [d_{\mathcal{E}},\theta(x)].$$

For x,y sections of $\mathcal{L}$

$$\begin{aligned}\langle dx,y\rangle_\theta + (-1)^{\overline{x}}\langle x,dy\rangle_\theta &= \operatorname{Tr}(\theta(dx)\theta(y) + (-1)^{\overline{x}}\theta(x)\theta(dy))\\ &= \operatorname{Tr}([d_{\mathcal{E}},\theta(x)]\theta(y) + (-1)^{\overline{x}}\theta(x)[d_{\mathcal{E}},\theta(y)])\\ &= \operatorname{Tr}(d_{\mathcal{E}}\theta(x)\theta(y) - (-1)^{\overline{x}+\overline{y}}\theta(x)\theta(y)d_{\mathcal{E}}) = \operatorname{Tr}([d_{\mathcal{E}},\theta(x)\theta(y)]) = 0.\end{aligned}$$

□

It follows from the properties of the Thom–Whitney totalisation functor Tot that every collection of cyclic forms $\langle -,-\rangle\colon(\Omega^i_X[-i]\otimes\mathcal{L})\times(\Omega^j_X[-j]\otimes\mathcal{L})\to\Omega^{i+j}_X[-i-j]$, with $i,j\geq 0$, induces a collection of $\mathbb{K}$-bilinear forms

$$\langle -,-\rangle\colon\ \operatorname{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes\mathcal{L})\times\operatorname{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes\mathcal{L})\to\operatorname{Tot}(\mathcal{U},\Omega^{i+j}_X[-i-j]).$$

Recalling Definition 1.4.3, the required forms are induced component-wise by the restriction of

$$A_n\otimes\prod_{i_0\cdots i_n}(\Omega^i_X[-i]\otimes\mathcal{L})(U_{i_0\cdots i_n})\times A_n\otimes\prod_{i_0\cdots i_n}(\Omega^j_X[-j]\otimes\mathcal{L})(U_{i_0\cdots i_n})\to A_n\otimes\prod_{i_0\cdots i_n}\Omega^{i+j}_X[-i-j](U_{i_0\cdots i_n}),$$

$$\langle\eta_n\otimes(x_{i_0\cdots i_n}),\omega_n\otimes(y_{i_0\cdots i_n})\rangle := \eta_n\omega_n(\langle(-1)^{\overline{\omega_n}\ \overline{(x_{i_0\cdots i_n})}}x_{i_0\cdots i_n},y_{i_0\cdots i_n}\rangle),$$

with η_n,ω_n in A_n, $x_{i_0\cdots i_n}$ in $(\Omega^i_X[-i]\otimes\mathcal{L})(U_{i_0\cdots i_n})$ and $y_{i_0\cdots i_n}$ in $(\Omega^j_X[-j]\otimes\mathcal{L})(U_{i_0\cdots i_n})$.

Let $(\Omega_X = \oplus_p\Omega^p_X[-p], d_{dR})$ denote the algebraic de Rham complex. In the following, when working with $\operatorname{Tot}(\mathcal{U},\Omega_X)$ the differential is denoted by d_{Tot} if $\Omega_X = \oplus_p\Omega^p_X[-p]$ is considered as complex with trivial differential, and by $d_{\mathsf{Tot}}+d_{dR}$ if it is considered as a complex with the de Rham differential.

Lemma 5.2.8. *The form induced on the totalisation by a cyclic form $\langle -,-\rangle\colon \mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ is cyclic: for all $b\in\operatorname{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})$ and $z,w\in\operatorname{Tot}(\mathcal{U},\mathcal{L})$ one has that*

$$\langle [b,z],w\rangle + (-1)^{\overline{b}\overline{z}}\langle z,[b,w]\rangle = (\operatorname{Id}\otimes\rho)(b)(\langle z,w\rangle).$$

Moreover if the form $\langle -,-\rangle\colon \mathcal{L}\times\mathcal{L}\to\mathcal{O}_X$ is d-closed (Definition 5.2.6), then for the induced form, for $z\in\operatorname{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes\mathcal{L})$ and $w\in\operatorname{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes\mathcal{L})$ we have that

$$\langle d_{\mathsf{Tot}}z,w\rangle + (-1)^{\overline{z}}\langle z,d_{\mathsf{Tot}}w\rangle = d_{\mathsf{Tot}}\langle z,w\rangle.$$

*The above condition will be called d_{Tot}-**closure**.*

Proof. For the first item, since everything is defined component-wise, it suffices to prove that for every $a\in A_n\otimes\prod(\Omega^1_X[-1]\otimes\mathcal{A})(U_{i_0\cdots i_n})$ and every $x,y\in A_n\otimes\prod\mathcal{L}(U_{i_0\cdots i_n})$

$$\langle [a,x],y\rangle + (-1)^{\overline{a}\ \overline{x}}\langle x,[a,y]\rangle = (\operatorname{Id}\otimes\rho)(a)(\langle x,y\rangle)$$

for every $n\geq 0$. By linearity let $a=\omega_n\otimes z_n$, with $\omega_n\in A_n$ and $z_n\in\prod(\Omega^1_X[-1]\otimes\mathcal{A})(U_{i_0\cdots i_n})$; let $x=\eta_n\otimes x_n$ and $y=\phi_n\otimes y_n$, with η_n,ϕ_n in A_n and x_n,y_n in $\prod\mathcal{L}(U_{i_0\cdots i_n})$. Then

$$\begin{aligned}
&\langle [a,x],y\rangle + (-1)^{\overline{a}\ \overline{x}}\langle x,[a,y]\rangle =\\
&\langle [\omega_n\otimes z_n,\eta_n\otimes x_n],\phi_n\otimes y_n\rangle + (-1)^{\overline{a}\ \overline{x}}\langle \eta_n\otimes x_n,[\omega_n\otimes z_n,\phi_n\otimes y_n]\rangle =\\
&(-1)^{\overline{\eta_n}\ \overline{z_n}}\langle \omega_n\eta_n\otimes [z_n,x_n],\phi_n\otimes y_n\rangle +\\
&+(-1)^{\overline{a}\ \overline{x}}\langle \eta_n\otimes x_n,(-1)^{\overline{\phi_n}\overline{z_n}}\omega_n\phi_n\otimes [z_n,y_n]\rangle =\\
&(-1)^{\overline{\phi_n}(\overline{z_n}+\overline{x_n})+\overline{\eta_n}\ \overline{z_n}}(\omega_n\eta_n\phi_n\langle [z_n,x_n],y_n\rangle +\\
&+(-1)^{\overline{\omega_n}\ \overline{\eta_n}+\overline{z_n}\ \overline{x_n}}\eta_n\omega_n\phi_n\langle x_n,[z_n,y_n]\rangle) =\\
&(-1)^{\overline{\phi_n}(\overline{z_n}+\overline{x_n})+\overline{\eta_n}\ \overline{z_n}}\omega_n\eta_n\phi_n(\langle [z_n,x_n],y_n\rangle + (-1)^{\overline{x_n}\ \overline{z_n}}\langle x_n,[z_n,y_n]\rangle) =\\
&(-1)^{\overline{\phi_n}(\overline{z_n}+\overline{x_n})+\overline{\eta_n}\ \overline{z_n}}\omega_n\eta_n\phi_n(\operatorname{Id}\otimes\rho)(z_n)(\langle x_n,y_n\rangle) = (\operatorname{Id}\otimes\rho)(a)(\langle x,y\rangle).
\end{aligned}$$

For the second item, recall that d_{Tot} is the differential on $\operatorname{Tot}(\mathcal{U},\Omega_X)$ when considering Ω_X as a complex with trivial differential. Again, since everything is defined component-wise, it is sufficient to prove that

$$\langle d_{\mathsf{Tot}}(\eta_n\otimes x_n),\omega_n\otimes y_n\rangle + (-1)^{\overline{x_n}+\overline{\eta_n}}\langle \eta_n\otimes x_n,d_{\mathsf{Tot}}(\omega_n\otimes y_n)\rangle = d_{\mathsf{Tot}}\langle \eta_n\otimes x_n,\omega_n\otimes y_n\rangle,$$

for $\eta_n,\omega_n\in A_n$ and $x_n\in\prod(\Omega^i_X[-i]\otimes\mathcal{L})(U_{i_0\cdots i_n})$ and $y_n\in\prod(\Omega^j_X[-j]\otimes\mathcal{L})(U_{i_0\cdots i_n})$. Then

$$\begin{aligned}
&\langle d_{\mathsf{Tot}}(\eta_n\otimes x_n),\omega_n\otimes y_n\rangle + (-1)^{\overline{x_n}+\overline{\eta_n}}\langle \eta_n\otimes x_n,d_{\mathsf{Tot}}(\omega_n\otimes y_n)\rangle =\\
&\langle d_{A_n}\eta_n\otimes x_n + (-1)^{\overline{\eta_n}}\eta_n\otimes dx_n,\omega_n\otimes y_n\rangle +\\
&+(-1)^{\overline{x_n}+\overline{\eta_n}}\langle \eta_n\otimes x_n,d_{A_n}\omega_n\otimes y_n + (-1)^{\overline{\omega_n}}\omega_n\otimes dy_n\rangle\\
&=(-1)^{\overline{\omega_n}\ \overline{x_n}}d_{A_n}(\eta_n)\omega_n\langle x_n,y_n\rangle + (-1)^{\overline{\eta_n}+\overline{\omega_n}(\overline{x_n}+1)}\eta_n\omega_n\langle dx_n,y_n\rangle +\\
&+(-1)^{\overline{x_n}+\overline{\eta_n}+(\overline{\omega_n}+1)\overline{x_n}}\eta_n d_{A_n}(\omega_n)\langle x_n,y_n\rangle +\\
&+(-1)^{\overline{x_n}+\overline{\eta_n}+\overline{\omega_n}+\overline{\omega_n}\ \overline{x_n}}\eta_n\omega_n\langle x_n,dy_n\rangle =\\
&(-1)^{\overline{\omega_n}\ \overline{x_n}}d_{A_n}(\eta_n\omega_n)\langle x_n,y_n\rangle + (-1)^{\overline{\eta_n}+\overline{\omega_n}(\overline{x_n}+1)}\eta_n\omega_n(\langle dx_n,y_n\rangle +\\
&(-1)^{\overline{x_n}}\langle x_n,dy_n\rangle) = (-1)^{\overline{\omega_n}\ \overline{x_n}}d_{A_n}(\eta_n\omega_n)\langle x_n,y_n\rangle =\\
&d_{A_n}\langle \eta_n\otimes x_n,\omega_n\otimes y_n\rangle = d_{\mathsf{Tot}}\langle \eta_n\otimes x_n,\omega_n\otimes y_n\rangle,
\end{aligned}$$

where d_{A_n} denotes the differential on A_n, the differential graded algebra of polynomial differential forms on the affine n-simplex.

$\square$

Corollary 5.2.9. *Let D in $\operatorname{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})$ be a simplicial lifting of the identity and let*

$$\nabla=[D,-,]\colon\ \operatorname{Tot}(\mathcal{U},\mathcal{L})\to\operatorname{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L})$$

be its associated connection, as in Definition 5.1.10. Then for any cyclic form

$$\langle -,-\rangle\colon\ \operatorname{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes\mathcal{L})\times\operatorname{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes\mathcal{L})\to\operatorname{Tot}(\mathcal{U},\Omega^{i+j}_X[-i-j]),$$

with $i,j\geq 0$, we have

$$\langle\nabla(x),y\rangle+(-1)^{\overline{x}}\langle x,\nabla(y)\rangle=d_{dR}\langle x,y\rangle$$

for $x,y\in\operatorname{Tot}(\mathcal{U},\mathcal{L})$.

Proof. It follows from the cyclicity of the form and by Remark 5.1.8. □

The next part is dedicated to defining an L_∞ morphism associated to a connection and to a d_{Tot}-closed cyclic form on a transitive DG-Lie algebroid.

Recall that since the functor Tot sends semicosimplicial DG-Lie algebras to DG-Lie algebras, the complex $\operatorname{Tot}(\mathcal{U},\mathcal{L})$ is a DG-Lie algebra. The complex of $\mathcal{O}_X$-modules $\Omega^{\leq 1}_X=\mathcal{O}_X\xrightarrow{d_{dR}}\Omega^1_X$ can be considered as a sheaf of abelian DG-Lie algebras, and hence it gives rise to a semicosimplicial abelian DG-Lie algebra; therefore the complex $\operatorname{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2])$ is an abelian DG-Lie algebra.

Theorem 5.2.10. *Let $(\mathcal{A},\rho)$ be a transitive DG-Lie algebroid over a smooth separated scheme X of finite type over a field K of characteristic zero. Let $\mathcal{L}=\operatorname{Ker}\rho$ be a finite complex of locally free sheaves and let $\langle -,-\rangle\colon\ \operatorname{Tot}(\mathcal{U},\Omega^i_X[-i]\otimes\mathcal{L})\times\operatorname{Tot}(\mathcal{U},\Omega^j_X[-j]\otimes\mathcal{L})\to\operatorname{Tot}(\mathcal{U},\Omega^{i+j}_X[-i-j])$, $i,j\geq 0$, be a cyclic form which is d_{Tot}-closed. For every simplicial lifting of the identity $D\in\operatorname{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{A})$ there exists a L_∞ morphism between DG-Lie algebras on the field K*

$$f\colon\ \operatorname{Tot}(\mathcal{U},\mathcal{L})\qquad\operatorname{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2])$$

with components

$$\begin{aligned}
f_1(x)&=\langle u,x\rangle,\\
f_2(x,y)&=\frac{1}{2}(\langle\nabla(x),y\rangle-(-1)^{\overline{x}\ \overline{y}}\langle\nabla(y),x\rangle),\\
f_3(x,y,z)&=-\frac{1}{2}\langle x,[y,z]\rangle,\\
f_n&=0\ \forall n\geq 4,
\end{aligned}$$

where $\nabla=[D,-]\colon\ \operatorname{Tot}(\mathcal{U},\mathcal{L})\to\operatorname{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L})$ denotes the connection associated to the simplicial lifting of the identity D, and $u=d_{\mathsf{Tot}}D$ its extension cocycle.

Proof. The strategy of the proof it to check that the conditions C_i of Definition 1.3.13 hold for $n=1,2,3,4$. In fact, since $f_n=0$ for $n\geq 4$, the conditions are automatically satisfied for $n\geq 5$.

Denote by d_{Tot} the differential on $\operatorname{Tot}(\mathcal{U},\mathcal{L})$, and by $d_{\mathsf{Tot}}+d_{dR}$ the differential on $\operatorname{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2])$. Condition C_1 requires that

$$f_1(d_{\mathsf{Tot}}x)=(d_{\mathsf{Tot}}+d_{dR})f_1(x);$$

notice however that since u belongs to $\operatorname{Tot}(\mathcal{U},\Omega^1_X[-1]\otimes\mathcal{L})$, $(d_{\mathsf{Tot}}+d_{dR})f_1=d_{\mathsf{Tot}}f_1$ in $\operatorname{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2])$. Then by the d_{Tot}-closure of the cyclic form and by the fact that u is closed:

$$\begin{aligned}
f_1(d_{\mathsf{Tot}}x)=\langle u,d_{\mathsf{Tot}}x\rangle&=(-1)^{\overline{u}}d_{\mathsf{Tot}}\langle u,x\rangle-(-1)^{\overline{u}}\langle d_{\mathsf{Tot}}u,x\rangle=d_{\mathsf{Tot}}\langle u,x\rangle\\
&=d_{\mathsf{Tot}}f_1(x).
\end{aligned}$$

For $n=2$ the condition is

$$(C_2):\quad f_2(d_{\mathsf{Tot}}x,y)+(-1)^{\overline{x}}f_2(x,d_{\mathsf{Tot}}y)=f_1([x,y])-(d_{\mathsf{Tot}}+d_{dR})f_2(x,y).$$

By definition of f_2 we have again that $(d_{\mathsf{Tot}} + d_{dR})f_2 = d_{\mathsf{Tot}} f_2$, and then, using Lemma 5.1.12,

$$\begin{aligned}
&f_2(d_{\mathsf{Tot}}x, y) + (-1)^{\overline{x}} f_2(x, d_{\mathsf{Tot}}y) = \\
&\frac{1}{2}\big(\langle \nabla(d_{\mathsf{Tot}}x), y\rangle - (-1)^{(\overline{x}+1)\overline{y}}\langle \nabla(y), d_{\mathsf{Tot}}x\rangle + (-1)^{\overline{x}}\langle \nabla(x), d_{\mathsf{Tot}}y\rangle + \\
&\qquad - (-1)^{\overline{x}\ \overline{y}}\langle \nabla(d_{\mathsf{Tot}}y), x\rangle\big) = \\
&\frac{1}{2}\big(- \langle d_{\mathsf{Tot}}\nabla(x), y\rangle + \langle [u, x], y\rangle - (-1)^{(\overline{x}+1)\overline{y}}\langle \nabla(y), d_{\mathsf{Tot}}x\rangle \\
&\qquad + (-1)^{\overline{x}}\langle \nabla(x), d_{\mathsf{Tot}}y\rangle + (-1)^{\overline{x}\ \overline{y}}\langle d_{\mathsf{Tot}}\nabla(y), x\rangle - (-1)^{\overline{x}\ \overline{y}}\langle [u, y], x\rangle\big) = \\
&\frac{1}{2}\big(\langle u, [x, y]\rangle - (-1)^{\overline{x}\ \overline{y}}\langle u, [y, x]\rangle - d_{\mathsf{Tot}}\langle \nabla(x), y\rangle + (-1)^{\overline{x}\ \overline{y}} d_{\mathsf{Tot}}\langle \nabla(y), x\rangle\big) = \\
&\langle u, [x, y]\rangle - \frac{1}{2} d_{\mathsf{Tot}}\big(\langle \nabla(x), y\rangle - (-1)^{\overline{x}\ \overline{y}}\langle \nabla(y), x\rangle\big) = f_1([x, y]) - d_{\mathsf{Tot}} f_2(x, y).
\end{aligned}$$

Condition C_3 is the following:

$$\begin{aligned}
&(d_{\mathsf{Tot}} + d_{dR}) f_3(x, y, z) = \\
&f_3(d_{\mathsf{Tot}}x, y, z) - (-1)^{\overline{x}\ \overline{y}} f_3(d_{\mathsf{Tot}}y, x, z) + (-1)^{\overline{z}(\overline{x}+\overline{y})} f_3(d_{\mathsf{Tot}}z, x, y) + \\
&- f_2([x, y], z) + (-1)^{\overline{y}\ \overline{z}} f_2([x, z], y) - (-1)^{\overline{x}(\overline{y}+\overline{z})} f_2([y, z], x),
\end{aligned}$$

and we begin by noting that by the d_{Tot}-closure

$$\begin{aligned}
&f_3(d_{\mathsf{Tot}}x, y, z) - (-1)^{\overline{x}\ \overline{y}} f_3(d_{\mathsf{Tot}}y, x, z) + (-1)^{\overline{z}(\overline{x}+\overline{y})} f_3(d_{\mathsf{Tot}}z, x, y) = \\
&-\frac{1}{2}\big(\langle d_{\mathsf{Tot}}x, [y, z]\rangle - (-1)^{\overline{x}\ \overline{y}}\langle d_{\mathsf{Tot}}y, [x, z]\rangle + (-1)^{\overline{z}(\overline{x}+\overline{y})}\langle d_{\mathsf{Tot}}z, [x, y]\rangle\big) = \\
&-\frac{1}{2}\big(\langle d_{\mathsf{Tot}}x, [y, z]\rangle - (-1)^{\overline{x}\ \overline{y}}\langle [d_{\mathsf{Tot}}y, x], z\rangle + (-1)^{\overline{x}+\overline{y}}\langle [x, y], d_{\mathsf{Tot}}z\rangle\big) = \\
&-\frac{1}{2}\big(\langle d_{\mathsf{Tot}}x, [y, z]\rangle + (-1)^{\overline{x}}\langle [x, d_{\mathsf{Tot}}y], z\rangle + (-1)^{\overline{x}+\overline{y}}\langle x, [y, d_{\mathsf{Tot}}z]\rangle\big) = \\
&-\frac{1}{2}\big(\langle d_{\mathsf{Tot}}x, [y, z]\rangle + (-1)^{\overline{x}}\langle x, d_{\mathsf{Tot}}[y, z]\rangle\big) = -\frac{1}{2} d_{\mathsf{Tot}}\langle x, [y, z]\rangle = \\
&d_{\mathsf{Tot}} f_3(x, y, z).
\end{aligned}$$

On the other hand, by Corollary 5.2.9

$$\begin{aligned}
&- f_2([x, y], z) + (-1)^{\overline{y}\ \overline{z}} f_2([x, z], y) - (-1)^{\overline{x}(\overline{y}+\overline{z})} f_2([y, z], x) = \\
&-\frac{1}{2}\big(\langle \nabla([x, y]), z\rangle - (-1)^{\overline{z}(\overline{x}+\overline{y})}\langle \nabla(z), [x, y]\rangle - (-1)^{\overline{y}\ \overline{z}}\langle \nabla([x, z]), y\rangle + \\
&+ \quad (-1)^{\overline{x}\ \overline{y}}\langle \nabla(y), [x, z]\rangle + (-1)^{\overline{x}(\overline{y}+\overline{z})}\langle \nabla([y, z]), x\rangle - \langle \nabla(x), [y, z]\rangle\big) = \\
&-\frac{1}{2}\big(\langle [\nabla(x), y], z\rangle + (-1)^{\overline{x}}\langle [x, \nabla(y)], z\rangle - (-1)^{\overline{z}(\overline{x}+\overline{y})}\langle \nabla(z), [x, y]\rangle + \\
&\quad - (-1)^{\overline{y}\ \overline{z}}\langle [\nabla(x), z], y\rangle - (-1)^{\overline{y}\ \overline{z}+\overline{x}}\langle [x, \nabla(z)], y\rangle + (-1)^{\overline{x}\ \overline{y}}\langle \nabla(y), [x, z]\rangle + \\
&\quad + (-1)^{\overline{x}(\overline{y}+\overline{z})}\langle [\nabla(y), z], x\rangle + (-1)^{\overline{x}(\overline{y}+\overline{z})+\overline{y}}\langle [y, \nabla(z)], x\rangle - \langle \nabla(x), [y, z]\rangle\big) = \\
&-\frac{1}{2}\big(\langle \nabla(x), [y, z]\rangle + (-1)^{\overline{x}}\langle x, \nabla([y, z])\rangle\big) = -\frac{1}{2} d_{dR}\langle x, [y, z]\rangle = d_{dR} f_3(x, y, z).
\end{aligned}$$

Lastly, condition C_4 is

$$\begin{aligned}
&f_3([a_1, a_2], a_3, a_4) + (-1)^{(\overline{a_1}+\overline{a_2})(\overline{a_3}+\overline{a_4})} f_3([a_3, a_4], a_1, a_2) + \\
&\quad + (-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})} f_3([a_2, a_3], a_1, a_4) - (-1)^{\overline{a_3}\ \overline{a_4}+\overline{a_1}\ \overline{a_2}+\overline{a_1}\ \overline{a_4}} f_3([a_2, a_4], a_1, a_3) \\
&\quad - (-1)^{\overline{a_2}\ \overline{a_3}} f_3([a_1, a_3], a_2, a_4) + (-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})} f_3([a_1, a_4], a_2, a_3) = 0
\end{aligned}$$

By the graded Jacobi identity we have that

$$-\frac{1}{2}\big(\langle [a_1, a_2], [a_3, a_4]\rangle - (-1)^{\overline{a_2}\ \overline{a_3}}\langle [a_1, a_3], [a_2, a_4]\rangle +$$

$$
\begin{aligned}
&+(-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle[a_1,a_4],[a_2,a_3]\rangle+(-1)^{(\overline{a_1}+\overline{a_2})(\overline{a_3}+\overline{a_4})}\langle[a_3,a_4],[a_1,a_2]\rangle+\\
&-(-1)^{\overline{a_3}\ \overline{a_4}+\overline{a_1}\ \overline{a_2}+\overline{a_1}\ \overline{a_4}}\langle[a_2,a_4],[a_1,a_3]\rangle+(-1)^{\overline{a_1}(\overline{a_2}+\overline{a_3})}\langle[a_2,a_3],[a_1,a_4]\rangle)\\
&=-(\langle[a_1,a_2],[a_3,a_4]\rangle-(-1)^{\overline{a_2}\ \overline{a_3}}\langle[a_1,a_3],[a_2,a_4]\rangle+\\
&+(-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle[a_1,a_4],[a_2,a_3]\rangle)=-(\langle a_1,[a_2,[a_3,a_4]]\rangle+\\
&-(-1)^{\overline{a_2}\ \overline{a_3}}\langle a_1,[a_3,[a_2,a_4]]\rangle+(-1)^{\overline{a_4}(\overline{a_2}+\overline{a_3})}\langle a_1,[a_4,[a_2,a_3]]\rangle)=\\
&-\langle a_1,[a_2,[a_3,a_4]]-(-1)^{\overline{a_2}\ \overline{a_3}}[a_3,[a_2,a_4]]-[[a_2,a_3],a_4]\rangle=0\,.
\end{aligned}
$$

□

We can now state the results of Section 4.6 for a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme X of finite type on a field K of characteristic zero.

Remark 5.2.11. It is not very restrictive to require that a coherent sheaf on X has a finite locally free resolution: in fact, by [34, III, Exercises 6.8, 6.9] every coherent sheaf on a smooth, Noetherian, integral, separated scheme admits a finite locally free resolution.

Let $(\mathcal{E},d_{\mathcal{E}})$ be a finite complex of locally free sheaves. Consider the DG-Lie algebroid of derivations of pairs $\mathcal{D}(X,\mathcal{E})$ of Example 5.1.6, [41], and the short exact sequence

$$0\longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\longrightarrow \mathcal{D}(X,\mathcal{E})\xrightarrow{\ \alpha\ }\Theta_X\longrightarrow 0;$$

it was noted in Example 5.1.6 that by tensoring with $\Omega^1_X[-1]$ one obtains

$$0\to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\Omega^1_X[-1]\otimes\mathcal{E})\xrightarrow{g\to(0,g)}\mathcal{G}_{\Omega^1}\xrightarrow{(\beta,g)\to\beta}\mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X,\Omega^1_X[-1])\to 0\,,$$

Fixing an affine open cover $\mathcal{U}$ of X and applying the Tot functor, we get the short exact sequence

$$0\longrightarrow \mathrm{Tot}(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\Omega^1_X[-1]\otimes\mathcal{E}))\longrightarrow \mathrm{Tot}(\mathcal{U},\mathcal{G}_{\Omega^1})\longrightarrow \mathrm{Tot}(\mathcal{U},\mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X,\Omega^1_X[-1]))\longrightarrow 0.$$

In Example 5.1.6 we have remarked that a lifting of the identity in $\mathcal{G}_{\Omega^1}$ is equivalent to a global algebraic connection on every component $\mathcal{E}^i$; hence a lifting to $\mathrm{Tot}(\mathcal{U},\mathcal{G}_\Omega)$ of the universal derivation $d_{dR}\colon \mathcal{O}_X\to\Omega^1_X[-1]$ in $\mathrm{Tot}(\mathcal{U},\mathcal{D}er_{\mathsf{K}}(\mathcal{O}_X,\Omega^1_X[-1]))$ can be termed a **simplicial connection** on the complex of locally free sheaves $\mathcal{E}$. As seen in Example 5.2.3, a natural cyclic form to consider is the one induced by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\times\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\to\mathcal{O}_X,\quad (a,b)\mapsto -\mathrm{Tr}(ab),$$

where $\mathrm{Tr}\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\to\mathcal{O}_X$ is the usual trace map. Then the L_∞ morphism of Theorem 5.2.10 yields :

Corollary 5.2.12. *Let $\mathcal{E}$ be a finite complex of locally free sheaves on a smooth separated scheme X of finite type over a field K of characteristic zero. For every simplicial connection $D\in\mathrm{Tot}(\mathcal{U},\mathcal{G}_{\Omega^1})$ there exists an L_∞ morphism between DG-Lie algebras on the field K*

$$g\colon\ \mathrm{Tot}(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\qquad \mathrm{Tot}(\mathcal{U},\Omega^{\leq 1}_X[2])$$

with components

$$
\begin{aligned}
g_1(x)&=-\mathrm{Tr}(ux),\\
g_2(x,y)&=-\frac{1}{2}\mathrm{Tr}(\nabla(x)y-(-1)^{\overline{x}\ \overline{y}}\nabla(y)x),\\
g_3(x,y,z)&=\frac{1}{2}\mathrm{Tr}(x,[y,z]),\\
g_n&=0\ \forall n\geq 4.
\end{aligned}
$$

Corollary 5.2.13. *Let $\mathcal{F}$ be a coherent sheaf admitting a finite locally free resolution $\mathcal{E}$ on a smooth separated scheme X of finite type over a field K of characteristic zero. Then every simplicial connection on the resolution $\mathcal{E}$ gives a lifting of the map*

$$\tau_1\colon \mathrm{Ext}_X(\mathcal{F},\mathcal{F}) \to \mathsf{H}\ (X, \Omega_X^{\leq 1}[2])$$

to an L_∞ morphism

$$g\colon \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \qquad \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]).$$

Corollary 5.2.14. *Let $\mathcal{F}$ be a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme X of finite type over a field K of characteristic zero. Then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map*

$$\tau_1\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \mathsf{H}^2(X, \Omega_X^{\leq 1}[2]).$$

If the Hodge to de Rham spectral sequence of X degenerates at E_1, then every obstruction to the deformations of $\mathcal{F}$ belongs to the kernel of the map

$$\sigma_1\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to H^3(X, \Omega_X^1), \qquad \sigma_1(a) = -\,\mathrm{Tr}(\mathrm{At}(\mathcal{F})\circ a).$$

Proof. If $\mathcal{E}$ is a finite locally free resolution of $\mathcal{F}$, the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$ controls the deformations of $\mathcal{F}$, see Section 2.5, [24]. According to Corollary 5.2.13, the map

$$\tau_1\colon \mathrm{Ext}^2_X(\mathcal{F},\mathcal{F}) \to \mathsf{H}^2(X, \Omega_X^{\leq 1}[2])$$

lifts to an L_∞ morphism

$$g\colon \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \qquad \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]),$$

whose linear component g_1 commutes with obstruction maps of the associated deformation functors. By construction the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$ is abelian and therefore every obstruction of the associated deformation functor is trivial.

If the Hodge to de Rham spectral sequence of X degenerates at E_1 then the inclusion of complexes $\mathrm{Tot}(\mathcal{U}, \Omega_X^1[1]) \to \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$ is injective in cohomology, so that $H^3(X, \Omega_X^1) \hookrightarrow \mathsf{H}^2(X, \Omega_X^{\leq 1}[2])$ and the maps σ and τ have the same kernel. □

Remark 5.2.15. In the setting of Theorem 5.2.10, if the cyclic form is induced by a DG-Lie algebroid representation $\theta\colon \mathcal{A} \to \mathcal{D}\ (X, \mathcal{E}\ )$, the L_∞ morphism can be obtained up to a sign from the the L_∞ morphism of Corollary 5.2.12 as follows. Let $D \in \mathrm{Tot}(\mathcal{U}, \Omega_X^1[-1]\otimes\mathcal{A})$ denote a simplicial lifting of the identity, and denote by $\mathrm{Id}\otimes\theta\colon \mathrm{Tot}(\mathcal{U}, \Omega_X^1[-1]\otimes\mathcal{A}) \to \mathrm{Tot}(\mathcal{U}, \Omega_X^1[-1]\otimes \mathcal{D}\ (X, \mathcal{E}\ )) \cong \mathrm{Tot}(\mathcal{U}, \mathcal{G}_\Omega)$ the induced map on the totalisation. Denoting as usual by α the anchor map of the transitive DG-Lie algebroid $\mathcal{D}\ (X, \mathcal{E}\ )$, it is clear that $(\mathrm{Id}\otimes\theta)(D)$ is a simplicial lifting of the identity in $\mathrm{Tot}(\mathcal{U}, \Omega_X^1[-1]\otimes\mathcal{D}\ (X, \mathcal{E}\ ))$:

$$\begin{aligned}(\mathrm{Id}\otimes\alpha)(\mathrm{Id}\otimes\theta)(D) &= \mathrm{Id}\otimes(\alpha\circ\theta)(D) = (\mathrm{Id}\otimes\rho)(D)\\ &= \mathrm{Id}_{\Omega^1} \in \mathrm{Tot}(\mathcal{U}, \Omega_X^1[-1]\otimes\Theta_X).\end{aligned}$$

Let $u = d_{\mathsf{Tot}}D \in \mathrm{Tot}(\mathcal{U}, \Omega_X^1[-1]\otimes\mathcal{L})$ denote the extension cocycle associated to D, then

$$(\mathrm{Id}\otimes\theta)(u) = (\mathrm{Id}\otimes\theta)(d_{\mathsf{Tot}}D) = d_{\mathsf{Tot}}(\mathrm{Id}\otimes\theta)(D).$$

Therefore the L_∞ morphism $f\colon \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \qquad \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$ associated to D and to $\langle -,-\rangle_\theta$ is the composition of the DG-Lie algebra morphism

$$\theta\colon \mathrm{Tot}(\mathcal{U}, \mathcal{L}) \to \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ ))$$

and of the L_∞ morphism

$$-g\colon \mathrm{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}\ ,\mathcal{E}\ )) \qquad \mathrm{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$$

associated to the simplicial lifting of the identity $(\mathrm{Id}\otimes\theta)(D)$ and to the cyclic form $(a,b)\mapsto \mathrm{Tr}(ab)$.

5.3 The L_∞ morphism for the Atiyah Lie algebroid of a principal bundle

Since Lie algebroids arise naturally in connection with principal bundles, we give an application of the L_∞ morphism constructed in Theorem 5.2.10 to the deformation theory of principal bundles.

Let X be a smooth separated scheme of finite type over an algebraically closed field K of characteristic zero, let G be an affine algebraic group with Lie algebra $\mathfrak{g}$, and let $P \to X$ be a principal G-bundle on X. By G-principal bundle we mean a G-fibration which is locally trivial for the Zariski topology, see e.g. [73]. We begin by finding a DG-Lie algebra that controls the deformations of P, using an argument similar to those in [11, 59, 74] . Let $\mathbf{Art}_{\mathsf{K}}$ be the category of Artin local K-algebras with residue field K. For any A in $\mathbf{Art}_{\mathsf{K}}$ denote by $\mathfrak{m}_A$ its maximal ideal and by 0 the closed point in $\operatorname{Spec} A$.

Definition 5.3.1. [11, 20] An infinitesimal deformation of P over $A \in \mathbf{Art}_{\mathsf{K}}$ is the data of a principal G-bundle $P_A \to X \times \operatorname{Spec} A$ and an isomorphism $\theta\colon i\ (P_A) \cong P$.

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p"'] & P_A \arrow[d, "p_A"] \\ X \arrow[r, "i"] & X \times \operatorname{Spec} A. \end{tikzcd}$$

Two deformations (P_A, θ) and $(P_A, \theta\ )$ are isomorphic if there exists an isomorphism of principal G-bundles $\lambda\colon P_A \to P_A$ such that $\theta = \theta\ \circ i\ (\lambda)$.

This defines a functor $\operatorname{Def}_P\colon \mathbf{Art}_{\mathsf{K}} \to \mathbf{Set}$ such that $\operatorname{Def}_P(A)$ is the set of isomorphism classes of deformations of P over $A \in \mathbf{Art}_{\mathsf{K}}$. For every $A \in \mathbf{Art}_{\mathsf{K}}$, the set $\operatorname{Def}_P(A)$ contains the trivial deformation $P \times \operatorname{Spec} A \to X \times \operatorname{Spec} A$.

Fix an open cover $\mathcal{U} = \{U_i\}$ of X such that P is trivial on every U_i, and let $\{g_{ij}\colon U_{ij} \to G\}$ denote the transition functions for P. Let $\mathfrak{g}$ be the Lie algebra of G.

- Let $\operatorname{ad} P = P \times^G \mathfrak{g}$ denote the adjoint bundle of P, with transition functions $\{\operatorname{Ad}_{g_{ij}}\}$, and let $\mathcal{ad}(P)$ denote the sheaf of sections of the vector bundle $\operatorname{ad} P$.
- The group G acts on itself by conjugation; denote by $\operatorname{Ad} P = P \times^G G$ the associated bundle corresponding to this action. Recall that $\Gamma(X, \operatorname{Ad}(P)) \cong \operatorname{Gauge}(P)$, where $\operatorname{Gauge}(P)$ is the group of bundle automorphisms of P.

There is a one to one correspondence between first order deformations of P, i.e., deformations over $\mathsf{K}[t]/(t^2) \in \mathbf{Art}_{\mathsf{K}}$, and $H^1(X, \mathcal{ad}(P))$, see e.g. [20, 74]. This implies that on every affine open set the deformations of P are trivial.

Lemma 5.3.2. *Let* $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$ *be the associated bundle induced by the action* $\operatorname{Ad} \otimes \operatorname{Id}\colon G \times \mathfrak{g} \otimes \mathfrak{m}_A \to \mathfrak{g} \otimes \mathfrak{m}_A$. *Then there is an isomorphism*

$$\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)) \cong \Gamma(\mathcal{ad}(P)) \otimes \mathfrak{m}_A.$$

Proof. A section of $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$ is the data of

$$\{\omega_i\colon U_i \to \mathfrak{g} \otimes \mathfrak{m}_A \mid \omega_i(p) = (\operatorname{Ad}_{g_{ij}(p)} \otimes \operatorname{Id})\omega_j(p) \quad \forall p \in U_{ij}\}.$$

Let $t_1, \cdots, t_n$ be a basis of the finite dimensional vector space $\mathfrak{m}_A$, then for every $p \in U_i$ one can write $\omega_i(p) = \sum_k h_{i,k}(p) \otimes t_k$. Since the action of G on $\mathfrak{g} \otimes \mathfrak{m}_A$ is defined as

$$g \cdot (x \otimes t) = \operatorname{Ad}_g(x) \otimes t,$$

the maps $h_{i,k}$ are such that $h_{i,k}(p) = \operatorname{Ad}_{g_{ij}(p)} h_{j,k}(p)$ for every $p \in U_{ij}$.

An element of $\Gamma(\mathcal{ad}(P))\otimes\mathfrak{m}_A$ is a finite sum $\sum_k \eta_k\otimes t_k$, with η_k sections of $\operatorname{ad}P$, so that each η_k is the data of

$$\{\eta_{k,i}\colon U_i\to\mathfrak{g}\mid \eta_{k,i}(p)=\operatorname{Ad}_{g_{ij}(p)}\eta_{k,j}(p)\quad\forall p\in U_{ij}\}.$$

Then, setting $(\eta_{k,i}\otimes t_k)(p)=\eta_{k,i}(p)\otimes t_k$ for every $p\in U_i$, the data $\{\eta_{k,i}\otimes t_k\colon U_i\to\mathfrak{g}\otimes\mathfrak{m}_A\}$ is exactly a section of $P\times^G(\mathfrak{g}\otimes\mathfrak{m}_A)$.

□

Lemma 5.3.3. *For every $A\in\mathbf{Art}_\mathbb{K}$ there is an isomorphism of groups*

$$\exp(\Gamma(\mathcal{ad}(P))\otimes\mathfrak{m}_A)\cong\left\{\begin{array}{c}\textit{automorphisms of the trivial}\\ \textit{deformation } P\times\operatorname{Spec}A\end{array}\right\}.$$

Proof. Denote by $G^0(A)$ the group of morphisms $f\colon\operatorname{Spec}A\to G$ such that $f(0)=\operatorname{Id}_G$, and recall that there is an isomorphism of groups $\exp(\mathfrak{g}\otimes\mathfrak{m}_A)\cong G^0(A)$ (see e.g. [72, Section 10]). The group structure on $G^0(A)$ is induced by the group structure on G, while $\exp(\mathfrak{g}\otimes\mathfrak{m}_A)$ is a group with the Baker–Campbell–Hausdorff product. By Lemma 5.3.2

$$\Gamma(\mathcal{ad}(P))\otimes\mathfrak{m}_A\cong\Gamma(P\times^G(\mathfrak{g}\otimes\mathfrak{m}_A)),$$

so that we can work with $\exp(\Gamma(P\times^G(\mathfrak{g}\otimes\mathfrak{m}_A)))$. Consider the associated bundle $P\times^G G^0(A)$, induced by the adjoint action of G on $G^0(A)$; the isomorphism $\exp(\mathfrak{g}\otimes\mathfrak{m}_A)\cong G^0(A)$ induces an isomorphism $\exp(\Gamma(P\times^G(\mathfrak{g}\otimes\mathfrak{m}_A)))\cong\Gamma(P\times^G G^0(A))$. In fact, a section of $P\times^G(\mathfrak{g}\otimes\mathfrak{m}_A)$ is the data of

$$\{\eta_i\colon U_i\to\mathfrak{g}\otimes\mathfrak{m}_A\mid \eta_i(p)=(\operatorname{Ad}_{g_{ij}(p)}\otimes\operatorname{Id})\eta_j(p)\quad\forall p\in U_{ij}\},$$

and composing with the exponential $\exp\colon\mathfrak{g}\otimes\mathfrak{m}_A\to G^0(A)$ we obtain

$$\{\exp\circ\eta_i\colon U_i\to G^0(A)\mid \exp\circ\eta_i(p)=g_{ij}(p)\exp\circ\eta_j(p)g_{ij}(p)^{-1}\quad\forall p\in U_{ij}\}.$$

Notice that this data is equivalent to

$$\left\{\lambda_i\colon U_i\times\operatorname{Spec}A\to G\;\middle|\;\begin{array}{c}\lambda_i(p,0)=\operatorname{Id}_G\quad\forall p\in U_i,\\ \lambda_i(p)=g_{ij}(p)\lambda_j(p)g_{ij}(p)^{-1}\quad\forall p\in U_{ij}\end{array}\right\},$$

which is a section of the associated bundle $\operatorname{Ad}(P\times\operatorname{Spec}A)=(P\times\operatorname{Spec}A)\times^G G$, where G acts on itself by conjugation.

For any G-principal bundle Q the global sections of the associated bundle $\operatorname{Ad}(Q)=Q\times^G G$ correspond to bundle automorphisms of Q. Therefore the $\{\lambda_i\}$ give an element $F\in\operatorname{Gauge}(P\times\operatorname{Spec}A)$, and the condition $\lambda_i(p,0)=\operatorname{Id}_G$ for all $p\in U_i$ is equivalent to the fact that the automorphism F induces the identity when restricted to P, so that F is an automorphism of the trivial deformation. □

In the following, $Z^1_{\mathcal{ad}(P)(\mathcal{U})}$ and $H^1_{\mathcal{ad}(P)(\mathcal{U})}$ will denote the functors associated to the semicosimplicial Lie algebra $\mathcal{ad}(P)(\mathcal{U})$, defined in Subsection 2.4.1.

Proposition 5.3.4. *Let $\mathcal{U}=\{U_i\}$ be an a ne open cover of X and let $\mathcal{ad}(P)(\mathcal{U})$ be the semicosimplicial Lie algebra of Čech cochains:*

$$\textstyle\prod_i\mathcal{ad}(P)(U_i)\rightrightarrows\prod_{i,j}\mathcal{ad}(P)(U_{ij})\;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\;\prod_{i,j,k}\mathcal{ad}(P)(U_{ijk})\;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\;\cdots.$$

There is a natural isomorphism of functors $H^1_{\mathcal{ad}(P)(\mathcal{U})}\to\operatorname{Def}_P$.

Proof. Recall that all deformations of P on an affine open set are trivial, as mentioned above. Fix $A \in \mathbf{Art}_{\mathsf{K}}$; by Lemma 5.3.3 an element f of $Z^1_{\mathcal{ad}(P)(\mathcal{U})}(A)$ is the data for every U_{ij} of isomorphisms $f_{ij}\colon P|_{U_{ij}} \times \operatorname{Spec} A \to P|_{U_{ij}} \times \operatorname{Spec} A$, which restrict to the identity $P|_{U_{ij}} \to P|_{U_{ij}}$ and such that $f_{ik} = f_{ij} f_{jk}$ for all i, j, k.

The last condition means that the $\{f_{ij}\}$ glue to obtain a principal G-bundle $P_A \to X \times \operatorname{Spec} A$ and isomorphisms $f_i\colon P_A|_{U_i \times \mathsf{Spec}\, A} \to P|_{U_i} \times \operatorname{Spec} A$ such that $f_{ij} = f_i f_j^{-1}$. Such isomorphisms coincide when restricted to $\overline{f_i}\colon i\ (P_A|_{U_i \times \mathsf{Spec}\, A}) \to P|_{U_i}$ and hence glue to an isomorphism of principal bundles $i\ (P_A) \to P$. This means that an element of $Z^1_{\mathcal{ad}(P)(\mathcal{U})}(A)$ gives a locally trivial deformation of P over $A \in \mathbf{Art}_{\mathsf{K}}$.

An element of $\exp(\prod_i \mathcal{ad}(P)(U_i) \otimes \mathfrak{m}_A)$ is again by Lemma 5.3.3 the data, for every U_i, of automorphisms $\lambda_i\colon P|_{U_i} \times \operatorname{Spec} A \to P|_{U_i} \times \operatorname{Spec} A$ which restrict to the identity $P|_{U_i} \to P|_{U_i}$. Two elements $f = \{f_{ij}\}, h = \{h_{ij}\}$ of $Z^1_{\mathcal{ad}(P)(\mathcal{U})}(A)$ are equivalent under the action of $\lambda \in \exp(\prod_i \mathcal{ad}(P)(U_i) \otimes \mathfrak{m}_A)$ if and only if $h_{ij} = \lambda_i f_{ij} \lambda_j^{-1}$ for all i, j.

$$
\begin{tikzcd}
P_A|_{U_i \times \mathsf{Spec}\, A} \arrow[r, "f_i"] \arrow[d, "\lambda"'] & P|_{U_i} \times \operatorname{Spec} A \arrow[d, "\lambda_i"] \\
P_A|_{U_i \times \mathsf{Spec}\, A} \arrow[r, "h_i"'] & P|_{U_i} \times \operatorname{Spec} A
\end{tikzcd}
$$

This can be expressed as $h_i^{-1} \lambda_i f_i = h_j^{-1} \lambda_j f_j$, which means that the $\{\lambda_i\}$ glue to a bundle isomorphism $\lambda\colon P_A \to P_A$, where P_A is the deformation corresponding to $\{f_{ij}\}$, and P_A to $\{h_{ij}\}$. Since each λ_i restricts to the identity on $P|_{U_i}$, λ is an isomorphism of deformations. □

Corollary 5.3.5. *If $\mathcal{U} = \{U_i\}$ is an affine open cover of X, there is an isomorphism*

$$\operatorname{Def}_P \cong \operatorname{Def}_{\mathsf{Tot}(\mathcal{U}, \mathcal{ad}(P))},$$

i.e., the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, \mathcal{ad}(P))$ controls the deformations of P.

Proof. Consequence of Propositions 5.3.4 and 2.4.11 . □

We now specialise the L_∞ morphism of Section 5.2 to the Atiyah Lie algebroid of the principal G-bundle P.

A Lie algebroid is DG-Lie algebroid (Definition 5.1.1) concentrated in degree zero. Consider the **Atiyah Lie algebroid** of the principal bundle P introduced in [3], which is a Lie algebroid structure on the sheaf $\mathcal{Q}$ of sections of the vector bundle $Q = \Theta_P / G$, the quotient of the tangent bundle of the total space Θ_P by the canonical induced G-action. There is a canonical short exact sequence of locally free sheaves over X

$$0 \longrightarrow \mathcal{ad}(P) \longrightarrow \mathcal{Q} \xrightarrow{\ \rho\ } \Theta_X \longrightarrow 0, \tag{5.3.1}$$

where $\mathcal{ad}(P)$ denotes the sheaf of sections of the adjoint bundle $\operatorname{ad} P = P \times^G \mathfrak{g}$ and $\rho\colon \mathcal{Q} \to \Theta_X$ is the anchor map. The vector bundle Q is the bundle of invariant tangent vector fields on P, and the Lie bracket on $\mathcal{Q}$ is induced by the Lie bracket of vector fields.

Definition 5.3.6. [3] A connection on the principal bundle $P \to X$ is a splitting of the exact sequence in (5.3.1). The **Atiyah class** of P is the extension class $\operatorname{At}_X(P) \in \operatorname{Ext}^1_X(\Theta_X, \mathcal{ad}(P)) \cong H^1(X, \Omega^1_X \otimes \mathcal{ad}(P))$ of the short exact sequence (5.3.1).

Therefore the Atiyah class $\operatorname{At}_X(P)$ is trivial if and only if there exists a connection on P.

Let Ω^1_X denote the cotangent sheaf, and $\Omega^1_X[-1]$ the cotangent sheaf considered as a trivial complex of sheaves concentrated in degree one. As in Section 5.1, one can tensor the short exact sequence (5.3.1) with $\Omega^1_X[-1]$ to obtain a short exact sequence of complexes of sheaves

$$0 \longrightarrow \Omega^1_X[-1] \otimes \mathcal{ad}(P) \longrightarrow \Omega^1_X[-1] \otimes \mathcal{Q} \xrightarrow{\mathsf{Id}\ \ \rho} \Omega^1_X[-1] \otimes \Theta_X \longrightarrow 0.$$

Fix an affine open cover $\mathcal{U} = \{U_i\}$ of X; as in Section 5.1 the short exact sequence above induces a short exact sequence of DG-vector spaces

$$0 \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{ad}(P)) \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{Q}) \xrightarrow{\mathrm{Id}\otimes\rho} \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X) \longrightarrow 0,$$

and we denote by d_{Tot} the differentials of the above complexes.

It is easily seen that a lifting of the identity $\mathrm{Id}_{\Omega^1} \in \Gamma(X, \Omega^1_X[-1] \otimes \Theta_X)$ to $D \in \Gamma(X, \Omega^1_X[-1] \otimes \mathcal{Q})$ is equivalent to a splitting of the exact sequence in (5.3.1). Hence in the case of a principal bundle P, a lifting of the identity can be identified with a connection on P. Therefore we call a preimage of Id_{Ω^1} in $\Omega^1_X[-1] \otimes \mathcal{Q}$ a germ of a connection on P, and we use the following terminology:

Definition 5.3.7. A **simplicial connection** on the principal bundle P is a lifting D in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{Q})$ of the identity Id_{Ω^1} in $\mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$.

Definition 5.3.8. The **Atiyah cocycle** of P is

$$u = d_{\mathsf{Tot}} D \in \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{ad}(P)).$$

It is natural to use the name Atiyah cocycle instead of extension cocycle of Definition 5.1.11, because its cohomology class is equal to the Atiyah class of Definition 5.3.6.

As in Definition 5.1.10, given a simplicial connection $D \in \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{Q})$ it is possible to define an adjoint operator

$$\nabla = [D, -] \colon \mathrm{Tot}(\mathcal{U}, \mathcal{ad}(P)) \to \mathrm{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{ad}(P)).$$

A cyclic form on the Atiyah Lie algebroid $\mathcal{Q}$ is a symmetric bilinear form $\langle -, - \rangle \colon \mathcal{ad}(P) \times \mathcal{ad}(P) \to \mathcal{O}_X$ such that for all $x, y \in \mathcal{ad}(P)$ and $q \in \mathcal{Q}$,

$$\langle [q, x], y \rangle + \langle x, [q, y] \rangle = \rho(q)(\langle x, y \rangle),$$

where $\rho \colon \mathcal{Q} \to \Theta_X$ is the anchor map of the Atiyah Lie algebroid $\mathcal{Q}$.

Example 5.3.9. The cyclic form induced by the adjoint representation of a DG-Lie algebroid of Example 5.2.2 in this case can be constructed in an equivalent way, starting from the Killing form of the Lie algebra $\mathfrak{g}$ of the group G:

$$K \colon \mathfrak{g} \otimes_{\mathsf{K}} \mathfrak{g} \to \mathsf{K}, \quad K(g, h) = \mathrm{Tr}(\mathrm{ad}\, g\, \mathrm{ad}\, h).$$

Take x, y in $\mathcal{ad}(P)(U)$ and let $U = \bigcup_i U_i$ with U_i open sets trivialising the principal bundle P, then

$$x = \{x_i \colon U_i \to \mathfrak{g} \mid x_i(p) = \mathrm{Ad}_{g_{ij}(p)}\, x_j(p) \quad \forall p \in U_{ij}\},$$

and analogously for y. Define $\langle x, y \rangle$ as $\{\langle x_i, y_i \rangle \colon U_i \to \mathsf{K}\}$, where for $p \in U_i$,

$$\langle x_i, y_i \rangle(p) = K(x_i(p), y_i(p)).$$

This is well defined because the Killing form is invariant under automorphisms of the Lie algebra $\mathfrak{g}$, so that for $p \in U_{ij}$

$$K(x_i(p), y_i(p)) = K(\mathrm{Ad}_{g_{ij}(p)}\, x_j(p), \mathrm{Ad}_{g_{ij}(p)}\, y_j(p)) = K(x_j(p), y_j(p)).$$

Recall that Tot preserves multiplicative structures, hence $\mathrm{Tot}(\mathcal{U}, \mathcal{ad}(P))$ is a DG-Lie algebra. In the sequel, $\mathrm{Tot}(\mathcal{U}, \Omega^{\leq 1}_X[2]) = \mathrm{Tot}(\mathcal{U}, \mathcal{O}_X[2] \xrightarrow{d_{dR}} \Omega^1_X[1])$ is considered as a DG-Lie algebra with trivial bracket; its differential is denoted $d_{\mathsf{Tot}} + d_{dR}$. Theorem 5.2.10 then yields the following.

Corollary 5.3.10. *For every simplicial connection D on a principal bundle P on a smooth separated scheme X of finite type over an algebraically closed field K of characteristic zero, endowed with a d_{Tot}-closed cyclic form $\langle -, - \rangle \colon \mathrm{Tot}(\mathcal{U}, \Omega^i_X[-i] \otimes \mathcal{ad}(P)) \times \mathrm{Tot}(\mathcal{U}, \Omega^j_X[-j] \otimes$*

$a\mathscr{d}(P)) \to \operatorname{Tot}(\mathcal{U}, \Omega_X^{i+j}[-i-j])$, $i, j \geq 0$, *there exists an* L_∞ *morphism of DG-Lie algebras on the field* K

$$f\colon \operatorname{Tot}(\mathcal{U}, a\mathscr{d}(P)) \qquad \operatorname{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]),$$

with components

$$\begin{aligned} f_1(x) &= \langle u, x\rangle, \\ f_2(x,y) &= \frac{1}{2}(\langle \nabla(x), y\rangle - (-1)^{\overline{x}\ \overline{y}}\langle \nabla(y), x\rangle), \\ f_3(x,y,z) &= -\frac{1}{2}\langle x, [y,z]\rangle, \\ f_n &= 0 \ \forall n \geq 4. \end{aligned}$$

As seen in Remark 1.3.14, the linear component f_1 of the L_∞ morphism induces a map of graded Lie algebras

$$f_1\colon H\ \ (\operatorname{Tot}(\mathcal{U}, a\mathscr{d}(P))) \to H\ \ (\operatorname{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])),$$

which, since the open cover $\mathcal{U}$ is affine, becomes

$$f_1\colon H\ \ (X, a\mathscr{d}(P)) \to \mathsf{H}\ \ (X, \Omega_X^{\leq 1}[2]).$$

Corollary 5.3.11. *Let P be a principal bundle on a smooth separated scheme X of finite type over an algebraically closed field K of characteristic zero and let*

$$\operatorname{Tot}(\mathcal{U}, \Omega_X^i[-i] \otimes a\mathscr{d}(P)) \times \operatorname{Tot}(\mathcal{U}, \Omega_X^j[-j] \otimes a\mathscr{d}(P)) \xrightarrow{\langle -,- \rangle} \operatorname{Tot}(\mathcal{U}, \Omega_X^{i+j}[-i-j]),$$

for $i, j \geq 0$, be a d_{Tot}-closed cyclic form. Then every obstruction to the deformations of P belongs to the kernel of the map

$$f_1\colon H^2(X, a\mathscr{d}(P)) \to \mathsf{H}^2(X, \Omega_X^{\leq 1}[2]), \quad f_1(x) = \langle \operatorname{At}(P), x\rangle,$$

where $\operatorname{At}(P)$ denotes the Atiyah class of the principal bundle P.

Proof. The proof is analogous to the one of Corollary 5.2.14: the linear component of the L_∞ morphism of DG-Lie algebras of Corollary 5.3.10 induces a morphism in cohomology which commutes with obstruction maps of the associated deformation functors, and the deformation functor associated to an abelian DG-Lie algebra has trivial obstructions. By Corollary 5.3.5, if $\mathcal{U} = \{U_i\}$ is an affine open cover of X, the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, a\mathscr{d}(P))$ controls the deformations of P and an obstruction space is $H^2(\operatorname{Tot}(\mathcal{U}, a\mathscr{d}(P))) \cong H^2(X, a\mathscr{d}(P))$. Since the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])$ is abelian, we obtain that f_1 annihilates all obstructions. □

Chapter 6

Semiregularity maps and deformations of modules over Lie algebroids

This chapter is based on the paper [5], where we generalise the results of [4] to produce L_∞ liftings of all the components of a semiregularity map associated to a locally free $\mathcal{A}$-module and a Lie pair $(\mathcal{L}, \mathcal{A})$ on a smooth separated scheme X of finite type over a field K of characteristic 0. For every flat inclusion of Lie algebroids $\mathcal{A} \subset \mathcal{L}$ we introduce semiregularity maps and prove that they annihilate obstructions, provided that the Leray spectral sequence of the pair $(\mathcal{L}, \mathcal{A})$ degenerates at E_1. We also determine a DG-Lie algebra controlling deformations of a locally free module over a Lie algebroid $\mathcal{A}$. The main results are described in detail in Section 6.1, after giving the necessary definitions.

By considering the Lie pair $(\Theta_X, 0)$, one obtains the classical Buchweitz–Flenner semiregularity map and hence the results of Chapter 4 for a locally free sheaf in the algebraic setting.

6.1 Outline of the main results

The main goal is to extend the results of [4], Chapter 4 to locally free modules over a Lie algebroid $\mathcal{A}$ on a smooth separated scheme X of finite type over a field K of characteristic 0, which is the data of $(\mathcal{A}, [-,-], a)$, where $\mathcal{A}$ is a locally free coherent sheaf of $\mathcal{O}_X$-modules, $[-,-]$ is a K-linear Lie bracket on $\mathcal{A}$, $a\colon \mathcal{A} \to \Theta_X$ is a morphism of sheaves of $\mathcal{O}_X$-modules commuting with the brackets and the Leibniz rule holds:

$$[l, fm] = a(l)(f)m + f[l,m], \qquad \forall\, l, m \in \mathcal{A}, f \in \mathcal{O}_X.$$

By definition, a locally free $\mathcal{A}$-module is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, and

$$\nabla\colon \mathcal{A} \to \mathcal{H}om_{\mathsf{K}}(\mathcal{E}, \mathcal{E}), \quad l \mapsto \nabla_l,$$

is a flat $\mathcal{A}$-connection, i.e., an $\mathcal{O}_X$-linear map such that:

$$\nabla_l(fe) = a(l)(f)e + f\nabla_l(e), \quad \forall\, l \in \mathcal{A},\ f \in \mathcal{O}_X,\ e \in \mathcal{E}$$

and such that its curvature $\nabla^2(l,m) = [\nabla_l, \nabla_m] - \nabla_{[l,m]}$ vanishes identically.

When $\mathcal{A} = \Theta_X$ with anchor map the identity, then the notion of $\mathcal{A}$-connection reduces to the usual definition of algebraic connection. As seen in Section 3.3, the Atiyah class of a locally free sheaf can be defined as the obstruction to the existence of a global algebraic connection. In other words, the Atiyah class of $\mathcal{E}$ can be defined as the obstruction to the lifting of the (unique) 0-connection on $\mathcal{E}$ to a Θ-connection; in view of the generalisation we can also write $\mathrm{At}(\mathcal{E}) = \mathrm{At}_{\Theta/0}(\mathcal{E})$.

By a straightforward generalisation, we can replace Θ with $\mathcal{A}$ and define $\mathrm{At}_{\mathcal{A}/0}(\mathcal{E})$ as the obstruction to the existence of an $\mathcal{A}$-connection on $\mathcal{E}$; however, this generalisation does not lead to anything new from the point of view of semiregularity maps and deformation theory.

Instead, we are interested here in the definition of a class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ in the following situation:

1. $\mathcal{A} \subset \mathcal{L}$ is an inclusion of Lie algebroids such that the quotient sheaf $\mathcal{L}/\mathcal{A}$ is locally free;

2. $(\mathcal{E}, \nabla)$ is a locally free $\mathcal{A}$-module.

In the above situation the quotient sheaf $\mathcal{L}/\mathcal{A}$ carries a natural structure of $\mathcal{A}$-module given by the *Bott connection* $\nabla^B \colon \mathcal{A} \to \mathcal{E}nd_{\mathsf{K}}(\mathcal{L}/\mathcal{A}, \mathcal{L}/\mathcal{A})$, $\nabla^B_a(x) = [a, x] \pmod{\mathcal{A}}$. Thus, for every $r \geq 0$, the sheaf $\bigwedge^r(\mathcal{L}/\mathcal{A})\ \ \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ carries a natural structure of $\mathcal{A}$-module.

Denoting by $\mathsf{H}\ \ (\mathcal{A}; \mathcal{F})$ the Lie algebroid cohomology of $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module $(\mathcal{F}, \nabla)$, i.e., the hypercohomology of the complex $(\Omega\ \ (\mathcal{A}) \otimes \mathcal{F}, \nabla)$, where $\Omega\ \ (\mathcal{A})$ is the de Rham algebra of $\mathcal{A}$, we prove in particular that:

1. $\mathsf{H}^1(\mathcal{A}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is the space of first order deformations of $\mathcal{E}$ as an $\mathcal{A}$-module;

2. $\mathsf{H}^2(\mathcal{A}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is a complete obstruction space for deformations of $\mathcal{E}$ as an $\mathcal{A}$-module;

3. the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \in \mathsf{H}^1(\mathcal{A}; (\mathcal{L}/\mathcal{A})\ \ \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is properly defined.

The first two items above are proved by showing that the DG-Lie algebra of derived sections of the sheaf of DG-Lie algebras $\Omega\ \ (\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ controls deformations of $\mathcal{E}$ as an $\mathcal{A}$-module. The Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is the primary obstruction to the extension of ∇ to a flat $\mathcal{L}$-connection. More precisely, $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is the obstruction to the extension of ∇ to an $\mathcal{L}$-connection $\nabla : \mathcal{L} \to \mathcal{H}om_{\mathsf{K}}(\mathcal{E}, \mathcal{E})$ such that $[\nabla_l, \nabla_a] = \nabla_{[l,a]}$ for every $l \in \mathcal{L}$ and $a \in \mathcal{A}$, cf. [17].

By analogy with the classical case, we define the semiregularity maps

$$\tau_k \colon \mathsf{H}^2(\mathcal{A}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \to \mathsf{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})\ \ \right), \qquad \tau_k(x) = \frac{1}{k!}\mathrm{Tr}(\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),$$

and we use the main result of [4] in order to prove that every τ_k annihilates obstructions, provided that the Leray spectral sequence (Definition 6.5.2) of the pair $(\mathcal{L}, \mathcal{A})$ degenerates at E_1.

6.2 Semiregularity maps for curved DG-algebras

We introduce here a slight simplification of the notion of curved DG-pair of Chapter 4, in which the ideal I is a bilateral associative ideal of the curved algebra, instead of a Lie ideal. This property is satisfied in all the geometric applications considered and it has the advantage of slightly simplifying notation and calculations.

Definition 6.2.1. Let $A = (A, d, R)$ be a curved DG-algebra, as in Definition 4.1.1. An (associative) **curved ideal** in A is homogeneous bilateral ideal $I \subset A$ such that $d(I) \subset I$ and $R \in I$.

By an (associative) **curved DG-pair** we mean the data (A, I) of a curved DG-algebra A equipped with a curved ideal I.

In particular, for every curved DG-pair (A, I), the quotient A/I is a (non-curved) associative DG-algebra, and therefore also a DG-Lie algebra. Writing $I^{(k)}$, $k \geq 0$, for the kth power of I, we have that $I^{(k)}$ is an associative bilateral ideal of A for every k. The differential graded algebra $\mathrm{Gr}_I A = \bigoplus_{k\geq 0} \dfrac{I^{(k)}}{I^{(k+1)}}$ is non-curved, since $d(I) \subset I$ and $d^2(I) \subset I^{(2)}$, the derivation d factors through differentials

$$d \colon \frac{I^{(k)}}{I^{(k+1)}} \to \frac{I^{(k)}}{I^{(k+1)}}, \qquad d^2 = 0.$$

The Atiyah cocycle and class of a curved associative DG-pair are defined analogously to Definition 4.1.4:

Definition 6.2.2. Let $A = (A, d, R)$ be a curved DG-algebra and $I \subset A$ a curved ideal. The **Atiyah cocycle** of the pair (A, I) is the class of R in the DG-vector space $\dfrac{I}{I^{(2)}}$. The **Atiyah class** of the pair (A, I) is the cohomology class of the Atiyah cocycle:

$$\mathrm{At}(A, I) = [R] \in H^2\left(\frac{I}{I^{(2)}}\right).$$

For every $x \in I$ of degree 1, we can consider the twisted derivation $d_x := d + [x, -]$ with curvature $R_x = R + dx + \dfrac{1}{2}[x, x]$. Then I remains a curved ideal of the twisted curved DG-algebra (A, d_x, R_x).

Lemma 6.2.3. *The Atiyah class of the pair (A, d_x, R_x, I) does not depend on the choice of $x \in I$. The Atiyah class $\mathrm{At}(A, I)$ is trivial if and only if there exists $x \in I$ of degree 1 such that R_x belongs to $I^{(2)}$.*

Proof. Firstly, notice that the differential on the algebra $\mathrm{Gr}_I A$ does not depend on the choice of $x \in I$: since x belongs to I the adjoint operator $[x, -]$ sends $I^{(k)}$ to $I^{(k+1)}$, and so $d = d_x := d + [x, -]$ in $\dfrac{I^{(k)}}{I^{(k+1)}}$. In $\dfrac{I}{I^{(2)}}$, one has that $[x, x] = 0$, so that

$$R_x - R = R + dx + \frac{1}{2}[x, x] - R = dx,$$

and the cohomology classes of R and R_x in $H\left(\dfrac{I}{I^{(2)}}\right)$ coincide.

Let now $x \in I$ be such that $R_x = R + dx + \dfrac{1}{2}[x, x]$ belongs to $I^{(2)}$. Then $R + dx$ also belongs to $I^{(2)}$ and $R = -dx$ in $\dfrac{I}{I^{(2)}}$, so that the Atiyah class is trivial. Conversely, let $R = dx$ in $\dfrac{I}{I^{(2)}}$, then $R - dx$ belongs to $I^{(2)}$, and so does $R_{-x} = R - dx + \dfrac{1}{2}[x, x]$. □

Assume now there are given a curved DG-algebra (A, d, R), a curved ideal I and a trace map $\mathrm{Tr}\colon A \to C$ as in Definition 4.1.9. Consider the decreasing filtration $C_k = \mathrm{Tr}(I^{(k)})$ of subcomplexes of C. By basic homological algebra, the spectral sequence associated to this filtration degenerates at E_1 if and only if for every k the inclusion $C_k/C_{k+1} \subset C/C_{k+1}$ is injective in cohomology, see e.g. [59, Thm. C.6.6].

Definition 6.2.4. The semiregularity maps of the curved DG-pair (A, I) and trace map $\mathrm{Tr}\colon A \to C$ are defined as:

$$\tau_k\colon H^2\left(\frac{A}{I}\right) \to H^{2+2k}\left(\frac{C_k}{C_{k+1}}\right), \quad \tau_k(x) = \frac{1}{k!}\mathrm{Tr}(\mathrm{At}(A, I)^k x).$$

The composition of τ_k with the natural morphism $H^{2+2k}(C_k/C_{k+1}) \to H^{2+2k}(C/C_{k+1})$ is induced by the morphism of complexes

$$\sigma_k^1\colon \frac{A}{I} \to \frac{C}{C_{k+1}}[2k], \quad \sigma_k^1(x) = \frac{1}{k!}\mathrm{Tr}(R^k x).$$

Considering C/C_{k+1} as a DG-Lie algebra with trivial bracket, we can immediately see that σ_k^1 is a morphism of DG-Lie algebras for $k = 0$, while for $k > 0$ the main result of [4], Corollary 4.1.10, can be stated as follows:

Theorem 6.2.5. *In the above situation, the map σ_k^1 is the linear component of an L_∞-morphism $\sigma_k\colon A/I \quad C/C_{k+1}[2k]$. In particular, σ_k^1 annihilates obstructions for the deformation functor associated to the DG-Lie algebra A/I.*

6.3 Lie algebroid connections

Let X be a smooth separated scheme of finite type over a field K of characteristic 0. We denote by Θ_X its tangent sheaf and by Ω^k_X, $k \geq 0$, the sheaves of differential forms.

Unless otherwise specified we write $\otimes$ for the tensor product over $\mathcal{O}_X$, in particular for two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we have $\mathcal{F} \otimes \mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

A Lie algebroid is a DG-Lie algebroid (Definition 5.1.1) concentrated in degree zero; however in this chapter we also require every Lie algebroid to be locally free.

Definition 6.3.1. A **Lie algebroid** over X is the data of $(\mathcal{L}, [-,-], a)$ where:

- $\mathcal{L}$ is a locally free coherent sheaf of $\mathcal{O}_X$-modules;
- $[-,-]$ is a K-linear Lie bracket on $\mathcal{L}$;
- $a\colon \mathcal{L} \to \Theta_X$ is a morphism of sheaves of $\mathcal{O}_X$-modules, called the **anchor map**, commuting with the brackets;
- finally, we require the Leibniz rule to hold

$$[l, fm] = a(l)(f)m + f[l,m], \qquad \forall\, l, m \in \mathcal{L}, f \in \mathcal{O}_X.$$

Example 6.3.2. As seen in Example 5.1.2, the tangent sheaf $\mathcal{L} = \Theta_X$, with anchor map equal to the identity, is a Lie algebroid. The trivial sheaf $\mathcal{L} = 0$ is also a Lie algebroid. A Lie algebroid over $\operatorname{Spec} \mathsf{K}$ is exactly a Lie algebra over the field K. Every sheaf of Lie algebras with $\mathcal{O}_X$-linear bracket can be considered as a Lie algebroid over X with trivial anchor map.

Example 6.3.3 (see [41] and Example 5.1.6)**.** Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module, then the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol has a natural structure of Lie algebroid. Since Θ_X is the sheaf of K-linear derivations of $\mathcal{O}_X$, we can introduce the sheaf

$$P(\Theta_X, \mathcal{E}) = \{(\theta, \phi) \in \Theta_X \times \mathcal{E}nd_{\mathsf{K}}(\mathcal{E}) \mid \phi(fe) = f\phi(e) + \theta(f)e,\ f \in \mathcal{O}_X,\ e \in \mathcal{E}\}.$$

Denoting by $a\colon P(\Theta_X, \mathcal{E}) \to \Theta_X$ the projection on the first factor, we have an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \to P(\Theta_X, \mathcal{E}) \xrightarrow{a} \Theta_X \to 0$$

and it is immediate to check that $P(\Theta_X, \mathcal{E})$ is a Lie algebroid with anchor map a. Moreover, the map $P(\Theta_X, \mathcal{E}) \to \mathcal{E}nd_{\mathsf{K}}(\mathcal{E})$, $(\theta, \phi) \mapsto \phi$, is injective and its image is the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol.

The **de Rham algebra** of $\mathcal{L}$ is defined as the sheaf of commutative graded algebras

$$\Omega\ (\mathcal{L}) = \bigoplus_{k \geq 0}^{\operatorname{rank} \mathcal{L}} \Omega^k(\mathcal{L}), \qquad \Omega^k(\mathcal{L}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}[1]\ \ ^k, \mathcal{O}_X),$$

equipped with the convolution product. Notice that $\mathcal{L}[1]$ is just $\mathcal{L}$ considered as a graded sheaf concentrated in degree -1, hence $\Omega\ (\mathcal{L})$ is a locally free graded sheaf with $\Omega^k(\mathcal{L})$ in degree k. By definition the convolution product is the dual of the coproduct Δ on the graded symmetric algebra $S(\mathcal{L}[1]) = \bigoplus_k \mathcal{L}[1]\ \ ^k$, defined by

$$\Delta(l_1, \ldots, l_n) = \sum_{a=0}^{n} \sum_{\sigma\ \ S(a,n-a)} \varepsilon(\sigma)(l_{\sigma(1)}, \ldots, l_{\sigma(a)}) \otimes (l_{\sigma(a+1)}, \ldots, l_{\sigma(n)}),$$

where $\varepsilon(\sigma)$ is the Koszul sign (Definition 1.2.8) and $S(a, n-a)$ is the subset of shuffles (Definition 1.2.9). More concretely, for $\omega \in \Omega^k(\mathcal{L})$ and $\eta \in \Omega^j(\mathcal{L})$ we have

$$(\omega\eta)(l_1, \ldots, l_{k+j})\ \ \sum_{\sigma\ \ S(k,j)} (-1)^{\sigma} \omega(l_{\sigma(1)}, \ldots, l_{\sigma(k)}) \eta(l_{\sigma(k+1)}, \ldots, l_{\sigma(k+j)}).$$

Notice that the **contraction product**

$$\mathcal{L} \times \Omega^{k+1}(\mathcal{L}) \xrightarrow{\lrcorner} \Omega^{k}(\mathcal{L}), \qquad (l \lrcorner \omega)(l_1, \ldots, l_k) = \omega(l, l_1, \ldots, l_k),$$

is $\mathcal{O}_X$-bilinear and satisfies the Koszul identity $l \lrcorner(\omega\eta) = (l \lrcorner \omega)\eta + (-1)^{\overline{\omega}}\omega(l \lrcorner \eta)$.

More generally, if $\mathcal{C}$ is a sheaf of graded associative $\mathcal{O}_X$-algebras, the same holds for

$$\Omega(\mathcal{L}, \mathcal{C}) := \Omega(\mathcal{L}) \otimes \mathcal{C} = \bigoplus_{k \geq 0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}[1]^{k}, \mathcal{C}).$$

The **de Rham differential** of $\mathcal{L}$, denoted by $d_{\mathcal{L}} \colon \Omega^k(\mathcal{L}) \to \Omega^{k+1}(\mathcal{L})$, is defined by the formula (see e.g. [53]):

$$\begin{aligned} d_{\mathcal{L}}(\omega)(l_0, \ldots, l_k) = & \sum_{i=0}^{n} (-1)^i a(l_i)(\omega(l_0, \ldots, \widehat{l_i}, \ldots, l_k)) \\ & + \sum_{i<j} (-1)^{i+j} \omega([l_i, l_j], l_0, \ldots, \widehat{l_i}, \ldots, \widehat{l_j}, \ldots, l_k). \end{aligned}$$

In particular for $\omega \in \Omega^0(\mathcal{L}) = \mathcal{O}_X$ we have $l \lrcorner d_{\mathcal{L}}(\omega) = d_{\mathcal{L}}(\omega)(l) = a(l)(\omega)$, for every $l \in \mathcal{L}$. By definition $\Omega^k(\Theta_X)[k] = \Omega^k_X$ is the sheaf of k-differential forms on X and the global formula for the exterior derivative implies that d_{Θ} is the usual de Rham differential.

For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we denote $\Omega(\mathcal{L}, \mathcal{F}) = \Omega(\mathcal{L}) \otimes \mathcal{F}$ and by

$$\Omega(\mathcal{L}) \times \Omega(\mathcal{L}, \mathcal{F}) \xrightarrow{\cdot} \Omega(\mathcal{L}, \mathcal{F}) \ : \quad \eta \cdot (\sum_i \mu_i \otimes e_i) = \sum_i \eta\mu_i \otimes e_i, \quad \mu_i \in \Omega(\mathcal{L}),\ e_i \in \mathcal{F},$$

$$\mathcal{L} \times \Omega(\mathcal{L}, \mathcal{F}) \xrightarrow{\lrcorner} \Omega(\mathcal{L}, \mathcal{F}) \ : \quad l \lrcorner (\sum_i \mu_i \otimes e_i) = \sum_i l \lrcorner \mu_i \otimes e_i, \quad \mu_i \in \Omega(\mathcal{L}),\ e_i \in \mathcal{F}.$$

Definition 6.3.4. Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, an $\mathcal{L}$**-connection** ∇ on $\mathcal{F}$ is a K-linear morphism of graded sheaves of degree 1

$$\nabla \colon \mathcal{F} \to \Omega^1(\mathcal{L}, \mathcal{F}) = \Omega^1(\mathcal{L}) \otimes \mathcal{F},$$

such that

$$\nabla(fe) = d_{\mathcal{L}}(f) \cdot e + f\nabla(e), \qquad \forall\, f \in \mathcal{O}_X,\ e \in \mathcal{F}.$$

As in the usual case, every $\mathcal{L}$-connection ∇ admits a unique extension to K-linear morphism of graded sheaves of $\mathcal{O}_X$-modules of degree 1

$$\nabla \colon \Omega(\mathcal{L}, \mathcal{F}) \to \Omega(\mathcal{L}, \mathcal{F})$$

such that

$$\nabla(f \cdot e) = d_{\mathcal{L}}(f) \cdot e + (-1)^{\overline{f}} f \cdot \nabla(e), \qquad \forall\, f \in \Omega(\mathcal{L}),\ e \in \Omega(\mathcal{L}, \mathcal{F}),$$

and the connection is called flat if $\nabla^2 = 0$.

Remark 6.3.5. Since the contraction product $\lrcorner \colon \mathcal{L} \times \Omega^1(\mathcal{L}) \to \mathcal{O}_X$ is nondegenerate, every K-linear morphism of sheaves $\nabla \colon \mathcal{F} \to \Omega^1(\mathcal{L}, \mathcal{F})$ is completely determined by the morphism of $\mathcal{O}_X$-modules

$$\mathcal{L} \to \mathcal{H}om_{\mathsf{K}}(\mathcal{F}, \mathcal{F}), \quad l \mapsto \nabla_l : \qquad \nabla_l(e) = l \lrcorner \nabla(e)\,.$$

It is straightforward to verify that ∇ is a connection if and only if

$$\nabla_l(fe) = a(l)(f)e + f\nabla_l(e), \qquad \forall\, f \in \mathcal{O}_X,\ l \in \mathcal{L},\ e \in \mathcal{F}.$$

A simple computation shows that the **curvature** of ∇ is given by the formula

$$\nabla^2(l, m)(e) = \nabla_l \nabla_m(e) - \nabla_m \nabla_l(e) - \nabla_{[l,m]}(e), \qquad \forall\, l, m \in \mathcal{L}, e \in \mathcal{F}.$$

For instance, if $\mathcal{F}$ is locally free and $\mathcal{L} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ (with trivial anchor map), then the natural inclusion $\mathcal{L} \to \mathcal{E}nd_{\mathsf{K}}(\mathcal{F})$ is a flat connection.

Since $\mathcal{L}$ is locally free we have natural isomorphisms

$$\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega\ (\mathcal{L},\mathcal{F})) = \mathcal{H}om_{\Omega\ (\mathcal{L})}(\Omega\ (\mathcal{L},\mathcal{F}), \Omega\ (\mathcal{L},\mathcal{F}))$$

and, therefore, a natural identification of $\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$ with the subset of morphisms of graded sheaves $f\colon \Omega\ (\mathcal{L},\mathcal{F}) \to \Omega\ (\mathcal{L},\mathcal{F})$ such that $f(\alpha\cdot\beta) = (-1)^{\overline{f}\,\overline{\alpha}}\alpha\cdot f(\beta)$ for every $\alpha\in\Omega\ (\mathcal{L})$, $\beta\in\Omega\ (\mathcal{L},\mathcal{F})$.

The following lemma is a completely straightforward generalisation of well-known facts about connections and curvature.

Lemma 6.3.6. *Let* $\nabla\colon \Omega\ (\mathcal{L},\mathcal{F}) \to \Omega\ (\mathcal{L},\mathcal{F})$ *be a connection, then* $\nabla^2 \in \Omega^2(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$ *and* $[\nabla, f] \in \Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$ *for every* $f\in\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}))$.

In particular, $(\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})), d=[\nabla,-], \nabla^2)$ *is a properly defined sheaf of curved DG-algebras over* X.

If in addition $\mathcal{F}$ admits a locally free resolution, then the trace map $\mathrm{Tr}\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \to \mathcal{O}_X$, which is a morphism of sheaves of Lie algebras, is properly defined. By an analogous calculation to that of Lemma 4.5.6, its extension

$$\mathrm{Tr}\colon \Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})) \to \Omega\ (\mathcal{L}), \quad \mathrm{Tr}(\omega\cdot f) = \omega\cdot\mathrm{Tr}(f),\ \omega\in\Omega\ (\mathcal{L}),\ f\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}), \tag{6.3.1}$$

is a trace map in the sense of Definition 4.1.9.

Definition 6.3.7. An **$\mathcal{L}$-module** is a pair $(\mathcal{F},\nabla)$ consisting of a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and a flat $\mathcal{L}$-connection ∇ on $\mathcal{F}$. An $\mathcal{L}$-module $(\mathcal{F},\nabla)$ is said to be coherent (resp.: torsion free, locally free) if $\mathcal{F}$ is coherent (resp.: torsion free, locally free) as an $\mathcal{O}_X$-module.

Example 6.3.8. Every $\mathcal{O}_X$-module has a unique structure of module over the trivial Lie algebroid $\mathcal{L}=0$.

Example 6.3.9. For every Lie algebroid $\mathcal{L}$, the pair $(\mathcal{O}_X, d_{\mathcal{L}})$ is an $\mathcal{L}$-module. More generally every choice of a basis on a free $\mathcal{O}_X$-module gives an $\mathcal{L}$-module structure.

Every $\mathcal{L}$-connection ∇ on a locally free $\mathcal{O}_X$-module $\mathcal{F}$ naturally induces $\mathcal{L}$-connections on the associated sheaves $\mathcal{F}\ $, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$, $\mathcal{F}\ ^k$ etc.. If $\mathcal{F}$ is an $\mathcal{L}$-module, then also $\mathcal{F}\ $, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$, $\mathcal{F}\ ^k$ etc. are $\mathcal{L}$-modules in a natural way.

Example 6.3.10. Let (X,π) be a smooth Poisson variety, and denote by $\{-,-\}$ the Poisson bracket on the sheaf of functions $\mathcal{O}_X$. The cotangent sheaf Ω^1_X of holomorphic differential 1-forms on X has an induced structure of holomorphic Lie algebroid with the anchor $a(df) := \{f,-\}$ and the bracket $[df,dg] := d\{f,g\}$ for all $f,g\in\mathcal{O}_X$ (this defines a and $[-,-]$ completely since Ω^1_X is generated by exact forms as an $\mathcal{O}_X$-module), see e.g. [25] for more details. An Ω^1_X-module is the same as a coherent sheaf $\mathcal{E}$ together with a sheaf of Poisson modules structure on the sections of $\mathcal{E}$. Namely, continuing to denote by $\{-,-\}$ the Poisson bracket on $\mathcal{E}$, the associated connection is defined by

$$\nabla\colon \Omega^1_X \to \mathcal{H}om_{\mathsf{K}}(\mathcal{E},\mathcal{E}), \qquad df\mapsto\nabla_{df}, \qquad \nabla_{df}e := \{f,e\} \quad \forall f\in\mathcal{O}_X,\ e\in\mathcal{E}.$$

The fact that ∇ is an Ω^1_X-connection on $\mathcal{E}$ is equivalent to the Poisson identities

$$\{f,ge\} = \{f,g\}e + g\{f,e\}, \qquad \{fg,e\} = f\{g,e\} + g\{f,e\},$$

while the flatness of ∇ is equivalent to the Jacobi identity

$$\{\{f,g\},e\} = \{f,\{g,e\}\} - \{g,\{f,e\}\}.$$

Definition 6.3.11. Let $\mathcal{L}$ be a Lie algebroid over X. The hypercohomology of the complex $(\Omega\ (\mathcal{L}), d_{\mathcal{L}})$ is called the **Lie algebroid cohomology of** $\mathcal{L}$, and it is denoted by $\mathsf{H}\ (\mathcal{L})$,

For an $\mathcal{L}$-module $(\mathcal{F}, \nabla)$ the complex $(\Omega\ (\mathcal{L}, \mathcal{F}), \nabla)$ is called the standard complex of $(\mathcal{F}, \nabla)$ and its hypercohomology, denoted by $\mathsf{H}\ (\mathcal{L}; \mathcal{F})$, is called the **Lie algebroid cohomology of** $\mathcal{L}$ **with coefficients in** $\mathcal{F}$.

Notice that $\mathsf{H}\ (\mathcal{L}) = \mathsf{H}\ (\mathcal{L}; \mathcal{O}_X)$, where $\mathcal{O}_X$ carries the $\mathcal{L}$-module structure of Example 6.3.9. The notion of standard complex is borrowed from [53], while for Lie algebroid cohomology we follow the notation of [1, 14].

Example 6.3.12. The Lie algebroid cohomology of the tangent sheaf Θ_X is the de Rham cohomology of X. The Lie algebroid cohomology of a Lie algebroid $\mathfrak{g}$ over Spec K is the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$.

6.4 Infinitesimal deformations of locally free $\mathcal{L}$-modules

In this section we describe a DG-Lie algebra controlling the infinitesimal deformations of a locally free $\mathcal{L}$-module, via the Thom–Whitney totalisation construction described in Section 1.4.

Let $\mathcal{L}$ be a Lie algebroid over X and let $(\mathcal{E}, \nabla)$ be an $\mathcal{L}$-module, with $\mathcal{E}$ locally free as an $\mathcal{O}_X$-module. Let B be an Artin local K-algebra with residue field K. We denote by $X_B = X \times \mathrm{Spec}(B)$, by $p_X \colon X \times \mathrm{Spec}(B) \to X$ the projection onto the first factor, and by $\imath_X : X \to X \times \mathrm{Spec}(B)$ the inclusion induced by $B \to B/\mathfrak{m}_B = \mathsf{K}$. Notice that the pull-back sheaf $p_X \mathcal{L} = \mathcal{L} \otimes_{\mathsf{K}} B$ has a natural structure of Lie algebroid over X_B, with the Lie bracket extending B-bilinearly the one on $\mathcal{L}$. Moreover, it is easy to check that a $p_X\mathcal{L}$-module $\mathcal{F}$ on X_B restricts to an $\mathcal{L}$-module $\imath_X \mathcal{F}$ on the central fibre X.

Definition 6.4.1. A **deformation of the** $\mathcal{L}$**-module** $(\mathcal{E}, \nabla)$ over $\mathrm{Spec}(B)$ consists of the data of a deformation $\mathcal{E}_B$ of $\mathcal{E}$ over X_B and a $p_X\mathcal{L}$-module structure

$$\nabla_B \colon \mathcal{E}_B \to \Omega^1(p_X\mathcal{L}, \mathcal{E}_B) = \Omega^1(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{E}_B$$

such that the restriction $\imath_X \mathcal{E}_B$ to X, with the naturally induced $\mathcal{L}$-module structure, coincides with $(\mathcal{E}, \nabla)$. An isomorphism of deformations $(\mathcal{E}_B, \nabla_B) \to (\mathcal{E}_B, \nabla_B)$ is an isomorphism of deformations of sheaves $\phi \colon \mathcal{E}_B \to \mathcal{E}_B$ such that $\phi\nabla_B = \nabla_B\phi$.

We want to describe a DG-Lie algebra controlling the infinitesimal deformations of $(\mathcal{E}, \nabla)$. Since the $\mathcal{L}$-connection ∇ is flat, by Lemma 6.3.6 $(\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), d = [\nabla, -])$ is a sheaf of locally free DG-algebras, which gives rise to a sheaf of locally free DG-Lie algebras

$$(\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), d = [\nabla, -], [-, -]).$$

Theorem 6.4.2. *In the above situation, for every a ne open cover* $\mathcal{U} = \{U_i\}$*, the DG-Lie algebra* $\mathrm{Tot}(\mathcal{U}, \Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$ *controls the infinitesimal deformations of* $(\mathcal{E}, \nabla)$*. In particular* $\mathsf{H}^1(\mathcal{L}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ *is the space of first order deformations and* $\mathsf{H}^2(\mathcal{L}; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ *is an obstruction space.*

Proof. This result is probably well known to experts, at least in the case $\mathcal{L} = \Theta_X$, cf. [33, Thm. 6.8], and follows easily from Hinich's theorem on descent of Deligne groupoids. According to [36], it is sufficient to check that locally the Deligne groupoid of $\Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is equivalent to the groupoid of deformations of $(\mathcal{E}, \nabla)$.

In order to check this, it is not restrictive to assume X affine. Given an Artin ring B as above, up to isomorphism every deformation of $\mathcal{E}$ is trivial, i.e. $\mathcal{E}_B = \mathcal{E} \otimes_{\mathsf{K}} B$ and $\mathcal{H}om_{\mathcal{O}_{X_B}}(\mathcal{E}_B, \mathcal{E}_B) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \otimes_{\mathsf{K}} B$. Denoting by $\nabla_0 \colon \mathcal{E}_B \to \Omega^1(p_X\mathcal{L}, \mathcal{E}_B) = \Omega^1(\mathcal{L}, \mathcal{E}) \otimes_{\mathsf{K}} B$ the natural B-linear extension of ∇, every deformation of ∇ over B is of the form $\nabla_0 + x$, with

$x \in \Gamma(X, \Omega^1(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))) \otimes \mathfrak{m}_B$, and the flatness condition $(\nabla_0 + x)^2 = 0$ is exactly the Maurer–Cartan equation $dx + \frac{1}{2}[x,x] = 0$.

To conclude the proof we only need to show that two solutions of the Maurer–Cartan equation x, y are gauge equivalent if and only if there exists an isomorphism of deformations $\phi\colon \mathcal{E}_B \to \mathcal{E}_B$ such that $\phi(\nabla_0 + x)\phi^{-1} = \nabla_0 + y$. Every ϕ as above is of the form $\phi = e^a$, with $a \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \otimes \mathfrak{m}_B$, and then the condition $\phi(\nabla_0 + x)\phi^{-1} = \nabla_0 + y$ is equivalent to

$$\nabla_0 + y = e^{[a,-]}(\nabla_0 + x) = \nabla_0 + x + \sum_{n=0}^{\infty} \frac{[a,-]^n}{(n+1)!}([a,x] - da),$$

which is the same as $y = e^a * x$, where $*$ denotes the gauge action. □

Remark 6.4.3. One can consider a different deformation problem, namely the deformation of pairs (bundle, $\mathcal{L}$-connection) without requiring the vanishing of the curvature. Then the same argument as above shows that this deformation problem is controlled by the DG-Lie algebra $\operatorname{Tot}(\mathcal{U}, \Omega^{\le 1}(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$, while it is well known that $\operatorname{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ controls the deformations of $\mathcal{E}$ [24].

6.5 Lie pairs

Definition 6.5.1. A **Lie pair** $(\mathcal{L}, \mathcal{A})$ of Lie algebroids over X is a pair consisting of a Lie algebroid $\mathcal{L}$ over X and a Lie subalgebroid $\mathcal{A} \subset \mathcal{L}$ such that the quotient sheaf $\mathcal{L}/\mathcal{A}$ is locally free.

Let $(\mathcal{L}, \mathcal{A})$ be a Lie pair. Since $\mathcal{L}/\mathcal{A}$ is assumed locally free we have a surjective restriction map $\varrho\colon \Omega(\mathcal{L}) \to \Omega(\mathcal{A})$, which is a morphism of sheaves of commutative differential graded algebras. The powers of its kernel give a finite decreasing filtration of differential graded ideal sheaves

$$\Omega(\mathcal{L}) = \mathcal{G}_0 \supset \mathcal{G}_1 = \ker(\varrho) \supset \cdots \mathcal{G}_r = (\ker(\varrho))^{(r)} \supset \cdots .$$

If we forget the de Rham differential, we can immediately see that $\mathcal{G}_p$ is the image of the morphism of graded $\mathcal{O}_X$-modules

$$\bigwedge^p (\mathcal{L}/\mathcal{A}) \ [-p] \otimes \Omega(\mathcal{L}) \to \Omega(\mathcal{L}),$$

and we have natural isomorphisms of graded sheaves

$$\frac{\mathcal{G}_p}{\mathcal{G}_{p+1}}[p] \cong \bigwedge^p (\mathcal{L}/\mathcal{A}) \ \otimes \Omega(\mathcal{A}). \tag{6.5.1}$$

In particular, $\mathcal{G}_p^i \neq 0$ only for pairs (i,p) such that $p \le i \le \operatorname{rank}\mathcal{L}$ and $p \le \operatorname{rank}\mathcal{L} - \operatorname{rank}\mathcal{A}$. For instance, whenever $i = 2$ we have $\mathcal{G}_0^2 = \Omega^2(\mathcal{L})$, $\mathcal{G}_3^2 = 0$,

$$\mathcal{G}_1^2 = \{\phi \in \Omega^2(\mathcal{L}) \mid \phi(a,b) = 0 \ \forall a,b \in \mathcal{A}\},$$
$$\mathcal{G}_2^2 = \{\phi \in \Omega^2(\mathcal{L}) \mid \phi(a,l) = 0 \ \forall a \in \mathcal{A},\ l \in \mathcal{L}\}.$$

Recall that $\mathsf{H}(\mathcal{L}) = \mathsf{H}(X, \Omega(\mathcal{L}))$ denotes the Lie algebroid cohomology of $\mathcal{L}$, as in Definition 6.3.11.

Definition 6.5.2. In the above notation, the filtration $\Omega(\mathcal{L}) = \mathcal{G}_0 \supset \mathcal{G}_1 \cdots$ is called the **Leray filtration** of the Lie pair $(\mathcal{L}, \mathcal{A})$. We shall call the associated spectral sequence in hypercohomology

$$E_1^{p,q} = \mathsf{H}^q\left(X, \mathcal{G}_p/\mathcal{G}_{p+1}[p]\right) \Rightarrow \mathsf{H}^{p+q}(\mathcal{L})$$

the *Leray spectral sequence* of the Lie pair $(\mathcal{L}, \mathcal{A})$.

The name Leray filtration is motivated by Example 6.5.4 below. Notice however that for the Lie pair $(\Theta_X, 0)$ the Leray filtration coincides with the Hodge filtration on differential forms.

Given an $\mathcal{A}$-module $(\mathcal{E},\nabla)$, we can also define a filtration $\mathcal{G}_r(\mathcal{E}) = \mathcal{G}_r \otimes \mathcal{E}$ of the graded sheaf $\Omega\ (\mathcal{L},\mathcal{E})$; equivalently, $\mathcal{G}_r(\mathcal{E})$ may be defined as the image of the multiplication map

$$\mathcal{G}_r \otimes \Omega\ (\mathcal{L},\mathcal{E}) \to \Omega\ (\mathcal{L},\mathcal{E}).$$

If ∇ is an $\mathcal{L}$-connection on $\mathcal{E}$ extending ∇, then by Leibniz rule the filtration $\mathcal{G}_r(\mathcal{E})$ is preserved by ∇ and we can immediately see that the maps induced on the quotients $\mathcal{G}_r(\mathcal{E})/\mathcal{G}_{r+1}(\mathcal{E})$ are independent from ∇ and square-zero operators. Notice also that the curvature of ∇ belongs to $\mathcal{G}_2^2(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ if and only if $[\nabla_l, \nabla_a] = \nabla_{[l,a]}$ for every $l \in \mathcal{L}$ and $a \in \mathcal{A}$.

Since ∇ always admits extensions locally (see Remark 6.7.3 below), for every r there is a properly defined structure of differential graded sheaf on $\mathcal{G}_r(\mathcal{E})/\mathcal{G}_{r+1}(\mathcal{E})$.

It is interesting to point out that the groups $E_1^{p,q} = \mathrm{H}^q\left(X, \mathcal{G}_p/\mathcal{G}_{p+1}[p]\right)$, and more generally the hypercohomology groups of $\mathcal{G}_r(\mathcal{E})/\mathcal{G}_{r+1}(\mathcal{E})$, are cohomology groups of $\mathcal{A}$ with coefficients in suitable $\mathcal{A}$-modules. In fact, there is a canonical $\mathcal{A}$-module structure on the quotient sheaf $\mathcal{L}/\mathcal{A}$ given by the **Bott connection**: denoting by $\pi\colon \mathcal{L} \to \mathcal{L}/\mathcal{A}$ the projection, the latter is defined by the formula

$$\nabla_a \pi(b) = \pi([a,b]), \qquad \forall\, a \in \mathcal{A},\ b \in \mathcal{L}.$$

Therefore, there is a canonical $\mathcal{A}$-module structure on $\bigwedge^r (\mathcal{L}/\mathcal{A})$ for every r.

Lemma 6.5.3. *Let $(\mathcal{L},\mathcal{A})$ be a Lie pair and let $\mathcal{E}$ be an $\mathcal{A}$-module. Then for every $r \geq 1$, the di erential graded sheaf $\dfrac{\mathcal{G}_r(\mathcal{E})}{\mathcal{G}_{r+1}(\mathcal{E})}[r]$ is isomorphic to the standard complex of the $\mathcal{A}$-module $\bigwedge^r (\mathcal{L}/\mathcal{A}) \otimes \mathcal{E}$. In particular, the Leray spectral sequence of the pair $(\mathcal{L},\mathcal{A})$ is*

$$E_1^{p,q} = \mathrm{H}^q\left(\mathcal{A}; \textstyle\bigwedge^p (\mathcal{L}/\mathcal{A})\ \right).$$

Proof. For every $r \geq 1$, consider the isomorphism of graded sheaves $\varphi\colon \frac{\mathcal{G}_r}{\mathcal{G}_{r+1}}[r] \to \bigwedge^r (\mathcal{L}/\mathcal{A}) \otimes \Omega\ (\mathcal{A})$ of (6.5.1). We begin by showing that this is an isomorphism of complexes, where the differential on the left is induced by $d_{\mathcal{L}}$, and the differential on the right is given by the dual connection to the Bott connection.

Denote by ∇^B the Bott connection on $\mathcal{L}/\mathcal{A}$, and by $\nabla^{B,}$ the induced connection on $\bigwedge^r(\mathcal{L}/\mathcal{A})$ for every $r \geq 0$. We denote by $a_{\mathcal{L}}$ and $a_{\mathcal{A}}$ the anchor maps of $\mathcal{L}$ and $\mathcal{A}$ respectively. Finally, denote by j the inclusion $j\colon \left(\frac{\mathcal{L}}{\mathcal{A}}\right)\ [-1] \to \Omega^1(\mathcal{L})$, and by π the projection $\pi\colon \mathcal{L} \to \frac{\mathcal{L}}{\mathcal{A}}$, so that for $m \in \mathcal{L}$ and $\eta \in \left(\frac{\mathcal{L}}{\mathcal{A}}\right)\ [-1]$ one has that $m \mathsf{y} j(\eta) = (j(\eta))(m) = \eta(\pi(m)) = \pi(m)\mathsf{y}\eta$. For every $\eta \in \mathcal{G}_r/\mathcal{G}_{r+1}[r]$, we prove that

$$\varphi(d_{\mathcal{L}}\eta) = \nabla^{B,}\ \varphi(\eta).$$

Consider $\omega \in \mathcal{G}_1/\mathcal{G}_2[1] \cong (\mathcal{L}/\mathcal{A})\ \otimes \Omega\ (\mathcal{A})$ of degree zero, so that ω belongs to $\mathcal{G}_1^1/\mathcal{G}_2^1[1] = \mathcal{G}_1^1[1] \cong (\mathcal{L}/\mathcal{A})$. Then $d_{\mathcal{L}}\omega$ belongs to $\mathcal{G}_1^2[1]$, but we consider its projection to $\frac{\mathcal{G}_1^2}{\mathcal{G}_2^2}[1] \cong \left(\frac{\mathcal{L}}{\mathcal{A}}\right)\ \otimes \Omega^1(\mathcal{A})$. Hence we calculate it on $b \in \mathcal{A}$ and $\pi(l) \in \frac{\mathcal{L}}{\mathcal{A}}$, obtaining

$$\begin{aligned} d_{\mathcal{L}}\omega(b,\pi(l)) &= a_{\mathcal{L}}(b)(j(\omega)(l)) - a_{\mathcal{L}}(l)(j(\omega)(b)) - j(\omega)([b,l]) \\ &= a_{\mathcal{A}}(b)(\omega(\pi(l))) - a_{\mathcal{L}}(l)(\omega(\pi(b))) - \omega(\pi([b,l])) \\ &= a_{\mathcal{A}}(b)(\omega(\pi(l))) - \omega(\pi([b,l])), \end{aligned}$$

since $\pi(b) = 0$. The connection $\nabla^{B,}$ for $\omega \in \left(\frac{\mathcal{L}}{\mathcal{A}}\right)$, $b \in \mathcal{A}$ and $\pi(l) \in \frac{\mathcal{L}}{\mathcal{A}}$ is given by

$$\pi(l)\mathsf{y}\nabla_b^{B,}\ \omega = d_{\mathcal{L}}(\pi(l)\mathsf{y}\omega)(b) - (\nabla_b^B\pi(l))\mathsf{y}\omega = a_{\mathcal{L}}(b)(\pi(l)\mathsf{y}\omega) - (\pi([b,l]))\mathsf{y}\omega$$

$$= a_{\mathcal{A}}(b)(\omega(\pi(l))) - \omega(\pi([b,l])),$$

therefore $d_{\mathcal{L}}\omega = \nabla^{B,}\ \omega$.

Consider now $\eta \in \frac{\mathcal{G}_r}{\mathcal{G}_{r+1}}[r]$ of degree $k - r \geq 0$, which we can assume to be of the form $\eta = \omega_1 \cdots \omega_k$, with $\omega_i \in \Omega^1(\mathcal{L})[1]$ for $i = 1, \dots, r$ such that $\varrho(\omega_1) = \cdots = \varrho(\omega_r) = 0$ and $\omega_j \in \Omega^1(\mathcal{L})$ for $j = r+1, \dots, k$ such that $\varrho(\omega_{r+1}), \dots, \varrho(\omega_k) \neq 0$.

In the isomorphism $\varphi\colon \frac{\mathcal{G}_r}{\mathcal{G}_{r+1}}[r] \to \bigwedge^r \left(\frac{\mathcal{L}}{\mathcal{A}}\right) \ \otimes \Omega\ (\mathcal{A})$, the form $\eta = \omega_1 \cdots \omega_k$ is sent to $\varphi(\eta) = \omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k)$. Then we have

$$\begin{aligned}
\nabla^{B,}\ (\varphi(\eta)) &= \nabla^{B,}\ (\omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k)) \\
&= \sum_{i=1}^{r} \omega_1 \cdots \nabla^{B,}\ (\omega_i) \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k) \\
&\quad + \sum_{i=r+1}^{k} (-1)^{i-1-r} \omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots d_{\mathcal{A}}(\varrho(\omega_i)) \cdots \varrho(\omega_k) \\
&= \sum_{i=1}^{r} \omega_1 \cdots \nabla^{B,}\ (\omega_i) \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k) \\
&\quad + \sum_{i=r+1}^{k} (-1)^{i-1-r} \omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(d_{\mathcal{L}}(\omega_i)) \cdots \varrho(\omega_k) \\
&= \sum_{i=1}^{r} \omega_1 \cdots d_{\mathcal{L}}(\omega_i) \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(\omega_k) \\
&\quad + \sum_{i=r+1}^{k} (-1)^{i-1-r} \omega_1 \cdots \omega_r \otimes \varrho(\omega_{r+1}) \cdots \varrho(d_{\mathcal{L}}(\omega_i)) \cdots \varrho(\omega_k) \\
&= (\mathrm{Id} \otimes \varrho) \left(\sum_{i=1}^{r} \omega_1 \cdots d_{\mathcal{L}}(\omega_i) \cdots \omega_k + \sum_{i=r+1}^{k} (-1)^{i-1-r} \omega_1 \cdots d_{\mathcal{L}}(\omega_i) \cdots \omega_k \right) \\
&= \varphi(d_{\mathcal{L}}(\omega_1 \cdots \omega_k)) = \varphi(d_{\mathcal{L}}(\eta)).
\end{aligned}$$

For every $r \geq 1$, it follows by (6.5.1) and by the definition of $\mathcal{G}_r(\mathcal{E})$ that there is an isomorphism of graded sheaves

$$\varphi \otimes \mathrm{Id}_{\mathcal{E}}\colon \frac{\mathcal{G}_r(\mathcal{E})}{\mathcal{G}_{r+1}(\mathcal{E})}[r] \to \bigwedge^r (\mathcal{L}/\mathcal{A}) \ \ \otimes \Omega\ (\mathcal{A}) \otimes \mathcal{E}.$$

Denote by ∇ the flat $\mathcal{A}$-connection on $\mathcal{E}$, and by ∇ a local extension of ∇ to an $\mathcal{L}$-connection on $\mathcal{E}$, which is such that $(\varrho \otimes \mathrm{Id})\nabla\ = \nabla$ and which induces a differential on $\mathcal{G}_r(\mathcal{E})/\mathcal{G}_{r+1}(\mathcal{E})[r]$.

Take now $\eta \otimes e \in \mathcal{G}_r(\mathcal{E})[r] = (\mathcal{G}_r \otimes \mathcal{E})[r]$, then $\nabla\ (\eta \otimes e) = d_{\mathcal{L}}\eta \otimes e + (-1)^{\overline{\eta}} \eta \otimes \nabla\ (e)$, and

$$\begin{aligned}
(\varphi \otimes \mathrm{Id}_{\mathcal{E}})(\nabla\ (\eta \otimes e)) &= \varphi(d_{\mathcal{L}}\eta) \otimes e + (-1)^{\overline{\eta}} \varphi(\eta) \otimes (\varphi \otimes \mathrm{Id}_{\mathcal{E}})\nabla\ (e) \\
&= \nabla^{B,}\ (\varphi(\eta)) \otimes e + (-1)^{\overline{\eta}} \varphi(\eta) \otimes (\varphi \otimes \mathrm{Id}_{\mathcal{E}})\nabla\ (e).
\end{aligned}$$

Since

$$(\nabla^{B,}\ \otimes \nabla)((\varphi \otimes \mathrm{Id}_{\mathcal{E}})(\eta \otimes e)) = (\nabla^{B,}\ \otimes \nabla)(\varphi(\eta) \otimes e) = \nabla^{B,}\ (\varphi(\eta)) \otimes e + (-1)^{\overline{\eta}} \varphi(\eta) \otimes \nabla(e),$$

it remains only to show that $(\varphi \otimes \mathrm{Id}_{\mathcal{E}})\nabla\ (e) = \nabla(e)$ for every $e \in \mathcal{E}$, which follows by the definition of φ and by the fact that $(\varrho \otimes \mathrm{Id})\nabla\ = \nabla$, since ∇ is a local extension of ∇. □

Example 6.5.4. Let $f\colon X \to Y$ be a smooth morphism of irreducible smooth schemes. Then a Lie pair on X is given by (Θ_X, Θ_f), where $\Theta_f = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ is the subsheaf of relative vector fields: since f is smooth there exists an exact sequence of sheaves

$$0 \to \Theta_f \to \Theta_X \to f\ \ \Theta_Y \to 0.$$

In this case $\Omega(\mathcal{L}) = \Omega_X$ is the usual de Rham complex of X, while $\Omega(\mathcal{A}) = \Omega_{X/Y}$ is the relative de Rham complex and the filtration $\mathcal{G}_r$ is the algebraic analogue of the holomorphic Leray filtration, see [75, 17.2], [77, 2.16].

Since the relative de Rham differential is $f^{-1}\mathcal{O}_Y$-linear and $\mathcal{G}_1$ is the ideal sheaf generated by $f^{-1}\Omega^1_Y$, for every r we have a natural isomorphism of differential graded sheaves

$$\frac{\mathcal{G}_r}{\mathcal{G}_{r+1}} \cong f^{-1}\Omega^r_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$$

and therefore the first page of the Leray spectral sequence is

$$E_1^p = \mathsf{H}\left(X, \mathcal{G}_p/\mathcal{G}_{p+1}\right) = \mathsf{H}\left(X, f^{-1}\Omega^p_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}\right) = \mathsf{H}\left(Y, \Omega^p_Y \otimes_{\mathcal{O}_Y} Rf\,\Omega_{X/Y}\right).$$

It is an easy consequence of Deligne's results on Hodge theory that if X and Y are complex projective manifolds, then the Leray spectral sequence of the Lie pair (Θ_X, Θ_f) degenerates at E_1. In fact, by Hodge decomposition we have

$$Rf\,\Omega_{X/Y} = \oplus_q R^q f\,\Omega_{X/Y}[-q] \simeq \oplus_q \mathcal{O}_Y \otimes_{\mathsf{C}} R^q f\,\mathsf{C}[-q],$$

and then $E_1^p = \oplus_q H(Y, \Omega^p_Y \otimes_{\mathsf{C}} R^q f\,\mathsf{C})[-p-q]$. Since $R^q f\,\mathsf{C}$ is a local system with real structure and Y is compact Kähler, according to [77, 2.11] (see also [33, 8.5]), the cohomology of $\Omega^p_Y \otimes_{\mathsf{C}} R^q f\,\mathsf{C}$ is a direct summand of the cohomology of $R^q f\,\mathsf{C}$. Since the (topological) Leray spectral sequence of $Rf\,\mathsf{C}$ degenerates at E_2 [19, 2.6.2], we have that E_1^p is a direct summand of $\mathsf{H}(Y, Rf\,\mathsf{C}) = H(X, \mathsf{C}) = \mathsf{H}(X, \Omega_X)$.

For every locally free sheaf $\mathcal{E}$ on Y its pull-back $f\,\mathcal{E} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E}$ has a natural structure of Θ_f-module with connection

$$\nabla_\eta(g \otimes e) = \eta(g) \otimes e.$$

More generally, every Θ_f-module can be interpreted, as in [10], as a locally free sheaf on X which is endowed with a connection relative to f that is flat.

6.6 Reduced Atiyah classes

For every Lie algebroid $\mathcal{L}$ and every $\mathcal{O}_X$-module $\mathcal{F}$ we define the sheaf of $\mathcal{O}_X$-modules

$$P(\mathcal{L}, \mathcal{F}) = \{(l, \phi) \in \mathcal{L} \times \mathcal{H}om_{\mathsf{K}}(\mathcal{F}, \mathcal{F}) \mid \phi(fe) = f\phi(e) + a(l)(f)e,\ f \in \mathcal{O}_X,\ e \in \mathcal{F}\}.$$

If $\mathcal{F}$ is coherent then also $P(\mathcal{L}, \mathcal{F})$ is coherent. This has been proved in [41, Prop. 5.1] in the case $\mathcal{L} = \Theta_X$ (see also Example 5.1.6), while for the general case it is sufficient to observe that $P(\mathcal{L}, \mathcal{F}) = P(\Theta_X, \mathcal{F}) \times_{\Theta_X} \mathcal{L}$.

Denoting by $p\colon P(\mathcal{L}, \mathcal{F}) \to \mathcal{L}$ the projection on the first factor, we have two exact sequences of (graded) $\mathcal{O}_X$-modules

$$\begin{gathered}0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to P(\mathcal{L}, \mathcal{F}) \xrightarrow{p} \mathcal{L}, \\ 0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{F}) \xrightarrow{p} \Omega^1(\mathcal{L}) \otimes \mathcal{L} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})[-1],\end{gathered} \tag{6.6.1}$$

where the second sequence is obtained by applying the exact functor $\Omega^1(\mathcal{L}) \otimes -$ to the first. Now and in the sequel, we will consider $\mathrm{Id}_{\mathcal{L}}$ as a global section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})[-1]$, a graded sheaf concentrated in degree 1.

Lemma 6.6.1. *In the above setup, there exists a natural bijection between the set of $\mathcal{L}$-connections on $\mathcal{F}$ and global sections $D \in \Gamma(X, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{F}))$ such that $p(D) = \mathrm{Id}_{\mathcal{L}}$.*

Proof. Let $l_1, \dots, l_r$ be a local frame of $\mathcal{L}$ with dual frame $\phi_1, \dots, \phi_r \in \Omega^1(\mathcal{L})$. Every K-linear morphism $\nabla\colon \mathcal{F} \to \Omega^1(\mathcal{L}, \mathcal{F})$ can be written locally as $\nabla = \sum_{i=1}^r \phi_i \cdot D_i$, with $D_i \in \mathcal{H}om_{\mathsf{K}}(\mathcal{F}, \mathcal{F})$. By definition, ∇ is a connection if and only if for every $f \in \mathcal{O}_X$, $e \in \mathcal{F}$, and every i we have

$$D_i(fe) = l_i \lrcorner \nabla(fe) = a(l_i)(f)e + fD_i(e)$$

and this is equivalent to the fact that $\sum_{i=1}^r \phi_i \otimes (l_i, D_i) \in \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{F})$. □

Lemma 6.6.2. *If $\mathcal{F}$ is a locally free sheaf, then the morphism $p\colon P(\mathcal{L}, \mathcal{F}) \to \mathcal{L}$ is surjective.*

Proof. We show this locally, with a proof similar to [41, Lemma 3.1]. Let R be a K-algebra, let $(L, [-,-], a)$ be a Lie algebroid over R with anchor map $a\colon L \to \mathrm{Der}_{\mathsf{K}}(R, R)$, and let F be a free R-module with basis $\{e_i\}$. We set

$$P(L, F) = \{(l, \phi) \in L \times \mathrm{Hom}_{\mathsf{K}}(F, F) \mid \phi(re) = r\phi(e) + a(l)(r)e,\ \forall r \in R,\ e \in F\},$$

and show that the projection $p\colon P(L, F) \to L$ is surjective. For every $x \in L$, consider the derivation $a(x) \in \mathrm{Der}_{\mathsf{K}}(R, R)$, and set

$$w(\sum_i r_i e_i) := \sum_i a(x)(r_i)e_i, \quad r_i \in R.$$

Then the pair (x, w) belongs to $P(L, F)$. □

Assume now that $\mathcal{F}$ is a locally free sheaf, so that the morphism $p\colon P(\mathcal{L}, \mathcal{F}) \to \mathcal{L}$ is surjective and we have an exact sequence of locally free graded sheaves of $\mathcal{O}_X$-modules

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}) \xrightarrow{p} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})[-1] \to 0.$$

We can rewrite the above short exact sequence of graded sheaves concentrated in degree 1 as a sequence of sheaves in degree 0:

$$0 \to \Omega^1(\mathcal{L})[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \Omega^1(\mathcal{L})[1] \otimes P(\mathcal{L}, \mathcal{E}) \xrightarrow{p} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to 0.$$

By Lemma 6.6.1, there exists an $\mathcal{L}$-connection on $\mathcal{E}$ if and only if the identity on $\mathcal{L}$ lifts to a global section of $\Omega^1(\mathcal{L})[1] \otimes P(\mathcal{L}, \mathcal{E})$. Writing

$$\mathrm{At}_{\mathcal{L}}(\mathcal{E}) = \partial(\mathrm{Id}_{\mathcal{L}}) \in H^1(X, \Omega^1(\mathcal{L})[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = \mathrm{Ext}^1_X(\mathcal{L} \otimes \mathcal{E}, \mathcal{E}),$$

where ∂ is the connecting morphism in the cohomology long exact sequence, we have that $\mathrm{At}_{\mathcal{L}}(\mathcal{E}) = 0$ if and only if there exists an $\mathcal{L}$-connection on $\mathcal{E}$.

Equivalently, we can define $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ as the extension class of the short exact sequence

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to Q(\mathcal{L}, \mathcal{E}) \xrightarrow{p} \mathcal{O}_X[-1] \to 0\,.$$

where, by definition, $Q(\mathcal{L}, \mathcal{E}) = p^{-1}(\mathcal{O}_X[-1] \cdot \mathrm{Id}_{\mathcal{L}})$. More explicitly, in a local frame $l_1, \dots, l_r$ of $\mathcal{L}$, with dual frame $\phi_1, \dots, \phi_r \in \Omega^1(\mathcal{L})$, the elements of $Q(\mathcal{L}, \mathcal{E})$ are those of $\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E})$ of the form $\sum_{i=1}^r \phi_i \otimes (fl_i, D_i)$ for some $f \in \mathcal{O}_X$.

Let now $(\mathcal{L}, \mathcal{A})$ be a Lie pair on X. Given an $\mathcal{A}$-connection $\nabla\colon \mathcal{E} \to \Omega^1(\mathcal{A}, \mathcal{E})$ on $\mathcal{E}$ locally free it makes sense to ask whether ∇ lifts to an $\mathcal{L}$-connection or not. We prove that the solution to this problem is completely determined by an obstruction

$$\partial(\nabla) \in \mathrm{Ext}^1_X\left(\frac{\mathcal{L}}{\mathcal{A}} \otimes \mathcal{E}, \mathcal{E}\right) = \mathrm{Ext}^1_X\left(\mathcal{E}, \mathcal{E} \otimes \mathcal{G}^1_1[1]\right).$$

It is possible to prove, by applying the results of [41, Section 3] to an injective resolution, that the same holds also if $\mathcal{E}$ is not locally free; however we don't need this result.

The case $\mathcal{A} = 0$ has been already considered. Suppose $\mathcal{A} \neq 0$, then we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \arrow[r] \arrow[d, "\alpha"] & Q(\mathcal{L},\mathcal{E}) \arrow[r, "p"] \arrow[d, "\beta"] & \mathcal{O}_X[-1] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1(\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \arrow[r] & Q(\mathcal{A},\mathcal{E}) \arrow[r, "p"] & \mathcal{O}_X[-1] \arrow[r] & 0
\end{tikzcd}$$

where α, β are the natural restriction maps. In a local frame $l_1, \ldots, l_r$ of $\mathcal{L}$, with dual frame $\phi_1, \ldots, \phi_r \in \Omega^1(\mathcal{L})$ and such that $l_1, \ldots, l_s$ is a local frame for $\mathcal{A}$, we have

$$\alpha(\sum_{i=1}^{r} \phi_i \otimes g_i) = \sum_{i=1}^{s} \phi_i \otimes g_i, \quad \beta(\sum_{i=1}^{r} \phi_i \otimes (fl_i, D_i)) = \sum_{i=1}^{s} \phi_i \otimes (fl_i, D_i)\,.$$

Since α and β are surjective, by the snake lemma we have an exact sequence

$$0 \to \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to Q(\mathcal{L},\mathcal{E}) \xrightarrow{\beta} Q(\mathcal{A},\mathcal{E}) \to 0, \tag{6.6.2}$$

since $\mathcal{G}_1^1$ is by definition the kernel of the surjective map $\Omega^1(\mathcal{L}) \to \Omega^1(\mathcal{A})$. For simplicity we can rewrite the above short exact sequence of graded sheaves living in degree 1 as a short exact sequence of sheaves in degree 0:

$$0 \to \mathcal{G}_1^1[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to Q(\mathcal{L},\mathcal{E})[1] \xrightarrow{\beta} Q(\mathcal{A},\mathcal{E})[1] \to 0.$$

Then the $\mathcal{A}$-connection ∇ is an element of $H^0(Q(\mathcal{A},\mathcal{E})[1])$ such that $p(\nabla) = 1$ and the element

$$\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E},\nabla) := \partial(\nabla) \in H^1\left(X, \mathcal{G}_1^1[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\right) = \mathrm{Ext}^1_X\left(\mathcal{E}, \mathcal{E} \otimes \mathcal{G}_1^1[1]\right),$$

is the obstruction to lifting ∇ to an $\mathcal{L}$-connection. We will call this the **reduced Atiyah class** of $(\mathcal{E},\nabla)$.

6.7 Simplicial $\mathcal{L}$-connections

In this section, similarly to Section 5.1, we define simplicial $\mathcal{L}$-connections for a Lie algebroid $\mathcal{L}$, and simplicial extensions of an $\mathcal{A}$-connection for a Lie pair $(\mathcal{L},\mathcal{A})$. We prove that the adjoint operator of a simplicial $\mathcal{L}$-connection on a locally free sheaf $\mathcal{E}$ induces a curved DG-algebra structure on $\mathrm{Tot}(\mathcal{U}, \Omega\ (\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$. In the case of a Lie pair $(\mathcal{L},\mathcal{A})$ and of a simplicial extension of a flat $\mathcal{A}$-connection ∇ on $\mathcal{E}$, we obtain the data of a curved DG-pair. Simplicial connections allow us to give representatives of the classes $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ and $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E},\nabla)$, and a representative of the obstruction to extending a flat $\mathcal{A}$-connection on $\mathcal{E}$ to a $\mathcal{L}$-connection on $\mathcal{E}$ with curvature in $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})$.

Let $\mathcal{L}$ be a Lie algebroid on X and $\mathcal{E}$ a locally free sheaf. We have seen that $\mathcal{L}$-connections on $\mathcal{E}$ exist locally but in general it does not exist any globally defined connection. However we can define a weaker notion of connection, which always exists and equally gives a significative example of curved DG-algebra.

In the notation of Section 6.3, consider the short exact sequence

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \to \Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E}) \xrightarrow{p} \Omega^1(\mathcal{L}) \otimes \mathcal{L} \to 0\,, \tag{6.7.1}$$

and recall that by Lemma 6.6.1 an $\mathcal{L}$-connection on $\mathcal{E}$ is a global section D of $\Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E})$ such that $p(D) = \mathrm{Id}_{\mathcal{L}}$, where $\mathrm{Id}_{\mathcal{L}}$ is considered as a global section of $\Omega^1(\mathcal{L}) \otimes \mathcal{L}$. Fix an affine open cover $\mathcal{U} = \{U_i\}$ of X; by the exactness of the Thom–Whitney totalisation functor one obtains a short exact sequence of DG-vector spaces

$$0 \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E})) \longrightarrow \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes P(\mathscr{L}, \mathscr{E})) \xrightarrow{p} \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{L}) \longrightarrow 0.$$

Because of the natural inclusion (1.4.2) of global sections in the totalisation, we can consider $\mathrm{Id}_{\mathscr{L}}$ as an element of $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{L})$.

Definition 6.7.1. A simplicial $\mathscr{L}$-connection on $\mathscr{E}$ is a lifting ∇ in $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes P(\mathscr{L}, \mathscr{E}))$ of $\mathrm{Id}_{\mathscr{L}}$ in $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{L})$.

It is clear that a simplicial $\mathscr{L}$-connection on $\mathscr{E}$ always exists.

In the case of a Lie pair $(\mathscr{L}, \mathscr{A})$ and of an $\mathscr{A}$-connection $\nabla^{\mathscr{A}}$ on the locally free sheaf $\mathscr{E}$, we can define an analogous notion of simplicial $\mathscr{L}$-connection extending $\nabla^{\mathscr{A}}$. It is not restrictive to assume $\mathscr{A} \neq 0$; then the exact sequence of locally free graded sheaves (6.6.2)

$$0 \to \mathscr{G}_1^1 \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}) \to Q(\mathscr{L}, \mathscr{E}) \xrightarrow{\beta} Q(\mathscr{A}, \mathscr{E}) \to 0$$

induces the short exact sequence of DG-vector spaces

$$0 \to \mathrm{Tot}\left(\mathcal{U}, \mathscr{G}_1^1 \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E})\right) \to \mathrm{Tot}(\mathcal{U}, Q(\mathscr{L}, \mathscr{E})) \xrightarrow{\beta} \mathrm{Tot}(\mathcal{U}, Q(\mathscr{A}, \mathscr{E})) \to 0. \qquad (6.7.2)$$

We have already observed that an $\mathscr{A}$-connection $\nabla^{\mathscr{A}}$ on $\mathscr{E}$ is a global section of $Q(\mathscr{A}, \mathscr{E})$ such that $p(\nabla^{\mathscr{A}}) = 1$, where $p \colon Q(\mathscr{A}, \mathscr{E}) \to \mathcal{O}_X[-1]$ is induced by the map p of (6.7.1). By the inclusion of global sections in the totalisation, $\nabla^{\mathscr{A}}$ belongs to $\mathrm{Tot}(\mathcal{U}, Q(\mathscr{A}, \mathscr{E}))$.

Definition 6.7.2. By a simplicial extension of an $\mathscr{A}$-connection $\nabla^{\mathscr{A}}$ on $\mathscr{E}$ we mean a lifting ∇ in $\mathrm{Tot}(\mathcal{U}, Q(\mathscr{L}, \mathscr{E}))$ of $\nabla^{\mathscr{A}}$ in $\mathrm{Tot}(\mathcal{U}, Q(\mathscr{A}, \mathscr{E}))$.

Remark 6.7.3. Notice that the exact sequence (6.6.2) implies that a local extension of an $\mathscr{A}$-connection to an $\mathscr{L}$-connection always exists.

Since maps on the totalisation are induced locally, a similar argument to that of Lemma 6.5.3 shows that every simplicial extension ∇ of a flat $\mathscr{A}$-connection $\nabla^{\mathscr{A}}$ on $\mathscr{E}$ induces a differential on the complex $\mathrm{Tot}(\mathcal{U}, \mathscr{G}_r(\mathscr{E})/\mathscr{G}_{r+1}(\mathscr{E})[r])$. We then have that $H\ (\mathrm{Tot}(\mathcal{U}, \mathscr{G}_r(\mathscr{E})/\mathscr{G}_{r+1}(\mathscr{E})[r])) \cong$ $\mathsf{H}\ (X, \mathscr{G}_r(\mathscr{E})/\mathscr{G}_{r+1}(\mathscr{E})[r])$ is isomorphic to the Lie algebroid cohomology of $\mathscr{A}$ with coefficients in the $\mathscr{A}$-module $\bigwedge^r(\mathscr{L}/\mathscr{A})\ \otimes \mathscr{E}$, again by Lemma 6.5.3.

Lemma 6.7.4. *For a Lie algebroid $\mathscr{L}$ and a simplicial $\mathscr{L}$-connection ∇ on $\mathscr{E}$, the cohomology class of $d_{\mathsf{Tot}}\nabla$ in $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))$ is the obstruction $\mathrm{At}_{\mathscr{L}}(\mathscr{E})$ to the existence of an $\mathscr{L}$-connection on $\mathscr{E}$.*

For a Lie pair $(\mathscr{L}, \mathscr{A})$ and a simplicial extension ∇ of an $\mathscr{A}$-connection $\nabla^{\mathscr{A}}$ on $\mathscr{E}$, the cohomology class of $d_{\mathsf{Tot}}\nabla$ in $\mathrm{Tot}(\mathcal{U}, \mathscr{G}_1^1 \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))$ is the obstruction $\overline{\mathrm{At}}_{\mathscr{L}/\mathscr{A}}(\mathscr{E}, \nabla^{\mathscr{A}})$ to the extension of $\nabla^{\mathscr{A}}$ to an $\mathscr{L}$-connection.

Proof. According to Example 1.4.5 we have natural isomorphisms

$$H^0(\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes P(\mathscr{L}, \mathscr{E}))) = \Gamma(X, \Omega^1(\mathscr{L}) \otimes P(\mathscr{L}, \mathscr{E})),$$
$$H^0(\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))) = \Gamma(X, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E})).$$

Consider first the case of a simplicial $\mathscr{L}$-connection ∇ on $\mathscr{E}$; notice that $d_{\mathsf{Tot}}\nabla$ belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))$, because $p(d_{\mathsf{Tot}}\nabla) = d_{\mathsf{Tot}}p(\nabla) = d_{\mathsf{Tot}}\,\mathrm{Id}_{\mathscr{L}} = 0$, since $\mathrm{Id}_{\mathscr{L}}$ is a global section. If there exists an $\mathscr{L}$-connection ∇ on $\mathscr{E}$ it belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes P(\mathscr{L}, \mathscr{E}))$ by the inclusion of global sections in the totalisation, and one has that $d_{\mathsf{Tot}}\nabla\ = 0$. Then for any simplicial connection ∇, the difference $\nabla - \nabla$ belongs to $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))$ and $d_{\mathsf{Tot}}(\nabla - \nabla\ ) = d_{\mathsf{Tot}}\nabla$, so that $d_{\mathsf{Tot}}\nabla$ is trivial in the cohomology of $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))$. Conversely, if $d_{\mathsf{Tot}}\nabla = d_{\mathsf{Tot}}\varphi$, with $\varphi \in \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{E}))$, then $\nabla - \varphi$ is a global $\mathscr{L}$-connection on $\mathscr{E}$.

In the case of a Lie pair $(\mathcal{L},\mathcal{A})$ and a simplicial extension ∇ of an $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ on $\mathcal{E}$, notice that $d_{\mathsf{Tot}}\nabla$ belongs to $\mathrm{Tot}\,(\mathcal{U},\mathcal{G}_1^1\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$: in fact, $\beta(d_{\mathsf{Tot}}\nabla)=d_{\mathsf{Tot}}\beta(\nabla)=d_{\mathsf{Tot}}\nabla^{\mathcal{A}}=0$, because $\nabla^{\mathcal{A}}$ is a global section. If $\nabla^{\mathcal{A}}$ extends to an $\mathcal{L}$-connection there exists ∇ in $\Gamma(X,Q(\mathcal{L},\mathcal{E}))$ with $\beta(\nabla)=\nabla^{\mathcal{A}}$, which is such that $d_{\mathsf{Tot}}\nabla=0$ in $\mathrm{Tot}(\mathcal{U},Q(\mathcal{L},\mathcal{E}))$, because it is a global section. Then for every simplicial connection ∇ lifting $\nabla^{\mathcal{A}}$, $\nabla-\nabla$ belongs to the kernel of β, which is $\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^1\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, and $d_{\mathsf{Tot}}(\nabla-\nabla)=d_{\mathsf{Tot}}\nabla$, so that $d_{\mathsf{Tot}}\nabla$ is trivial in cohomology. Vice versa, if $d_{\mathsf{Tot}}\nabla=d_{\mathsf{Tot}}\phi$ is trivial in the cohomology of $\mathrm{Tot}(\mathcal{U},\mathcal{G}_1^1\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$, it is easy to see that $\nabla-\phi$ is a connection lifting $\nabla^{\mathcal{A}}$. $\square$

A simplicial $\mathcal{L}$-connection on a locally free sheaf $\mathcal{E}$ induces a curved DG-algebra structure on the DG-vector space $\mathrm{Tot}(\mathcal{U},\Omega\ (\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$. To see this, the first step is the construction of an adjoint operator for the simplicial connection, which is done via the following lemma.

Lemma 6.7.5. *In the above situation, the $\mathcal{O}_X$-bilinear map*

$$[-,-]\colon(\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))\times\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})\to\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}),$$

$$[\eta\otimes(l,v),g]=\eta\otimes[v,g],\qquad \eta\in\Omega^1(\mathcal{L}),\ (l,v)\in P(\mathcal{L},\mathcal{E}),\ g\in\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}).$$

is well defined.

Proof. This follows immediately from Lemma 5.1.9. $\square$

The bracket defined in Lemma 6.7.5 induces a graded Lie bracket on the totalisation

$$[-,-]\colon\ \mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))\times\mathrm{Tot}(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})),$$

which allows to define the adjoint operator to a simplicial $\mathcal{L}$-connection ∇ on $\mathcal{E}$:

$$d\ :=[\nabla,-]\colon\ \mathrm{Tot}(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).\tag{6.7.3}$$

Recall that since $\Omega\ (\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ is a sheaf of graded algebras and the Tot functor preserves multiplicative structures, $\mathrm{Tot}(\mathcal{U},\Omega\ (\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})))$ is a differential graded algebra, with differential denoted by d_{Tot}.

Lemma 6.7.6. *The adjoint operator*

$$d\ =[\nabla,-]\colon\ \mathrm{Tot}(\mathcal{U},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$$

extends for every $i\geq 0$ to a K-linear operator

$$d\ :\ \mathrm{Tot}(\mathcal{U},\Omega^i(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))\to\mathrm{Tot}(\mathcal{U},\Omega^{i+1}(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})).$$

Then $(\mathrm{Tot}(\mathcal{U},\Omega\ (\mathcal{L},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))),d_{\mathsf{Tot}}+d\ )$ is a curved DG-algebra with curvature

$$d_{\mathsf{Tot}}\nabla+C,$$

with $d_{\mathsf{Tot}}\nabla\in\mathrm{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ and $C\in\mathrm{Tot}(\mathcal{U},\Omega^2(\mathcal{L})\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$ such that $d^2\ =[C,-]$.

Proof. Consider first the case of a germ of an $\mathcal{L}$-connection, i.e., an element Y of $\Gamma(V,\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))$ such that $p(Y)=\mathrm{Id}_{\mathcal{L}}|_V$, for some open set $V\subset X$. As usual, Y extends uniquely to a K-linear morphism of degree 1

$$Y\colon\Omega\ (\mathcal{L},\mathcal{E})|_V\to\Omega\ (\mathcal{L},\mathcal{E})|_V$$

such that

$$Y(\eta\otimes e)=d_{\mathcal{L}}(\eta)\otimes e+(-1)^{\overline{\eta}}\eta\otimes Y(e).$$

for all $\eta \in \Omega\ (\mathcal{L})|_V, e \in \mathcal{E}|_V$. It is easy to see that the map Y^2 is $\mathcal{O}_X$-linear, so it can be identified with a section of $\Omega^2(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V$.

One can define an adjoint operator

$$d_Y := [Y, -]\colon \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})|_V \to \Omega^1(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V,$$

which can be extended for all $i \geq 0$ to an operator

$$d_Y \colon \Omega^i(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V \to \Omega^{i+1}(\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))|_V$$

by setting

$$d_Y(\eta \otimes f) := d_{\mathcal{L}}(\eta) \otimes f + (-1)^{\overline{\eta}} \eta \otimes [Y, f], \tag{6.7.4}$$

where $[Y, f]$ denotes the Lie bracket of Lemma 6.7.5.

As in the classical case, one can see that

$$d_Y^2(\eta \otimes f) = [Y^2, \eta \otimes f] \tag{6.7.5}$$

for all $\eta \in \Omega\ (\mathcal{L})|_V$ and $f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})|_V$.

Let now ∇ be a simplicial $\mathcal{L}$-connection on $\mathcal{E}$, namely an element of $\mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E}))$ such that $p(\nabla) = \mathrm{Id}_{\mathcal{L}} \in \mathrm{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \mathcal{L})$. Then for every $i \geq 0$ the extension of the operator $d\ \ = [\nabla, -]$, defined in (6.7.3), to an operator

$$d\ \ = [\nabla, -]\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^i(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})) \to \mathrm{Tot}(\mathcal{U}, \Omega^{i+1}(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))$$

can be defined by using the map induced by (6.7.4) on the totalisation, and one obtains a degree one operator

$$d\ \ = [\nabla, -]\colon\ \mathrm{Tot}(\mathcal{U}, \Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))) \to \mathrm{Tot}(\mathcal{U}, \Omega\ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}))).$$

In detail, let $\nabla = (D_n)$ with $D_n \in A_n \otimes \prod_{i_0,\dots,i_n} (\Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E}))(U_{i_0,\dots,i_n})$ such that $p(D_n) = 1 \otimes (\mathrm{Id}_{\mathcal{L}}|_{U_{i_0,\dots,i_n}})$ for every $n \geq 0$. Since maps on the totalisation are defined componentwise, it is enough to define the bracket

$$[D_n, \phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n})],$$

for $\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n})$ in $A_n \otimes \prod_{i_0,\dots,i_n} (\Omega^i(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{E})(U_{i_0,\dots,i_n})$. Let

$$D_n = \sum_j \eta_{j,n} \otimes (t_{j,i_0,\dots,i_n}), \quad \eta_{j,n} \in A_n, \quad t_{j,i_0,\dots,i_n} \in (\Omega^1(\mathcal{L}) \otimes P(\mathcal{L},\mathcal{E}))(U_{i_0,\dots,i_n}); \tag{6.7.6}$$

then the bracket can be defined as

$$\begin{aligned}
&[D_n, \phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n})] = [\sum_j \eta_{j,n} \otimes (t_{j,i_0,\dots,i_n}), \phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n})] = \\
&\quad p(D_n)(\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n}))+ \\
&\quad (-1)^{\overline{\phi_n}+\overline{\omega_{i_0,\dots,i_n}}} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_0,\dots,i_n} \otimes [t_{j,i_0,\dots,i_n}, f_{i_0,\dots,i_n}]) = \\
&\quad (1 \otimes (\mathrm{Id}_{\mathcal{L}}|_{U_{i_0,\dots,i_n}}))(\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n}))+ \\
&\quad (-1)^{\overline{\phi_n}+\overline{\omega_{i_0,\dots,i_n}}} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_0,\dots,i_n} \otimes [t_{j,i_0,\dots,i_n}, f_{i_0,\dots,i_n}]) = \\
&\quad (-1)^{\overline{\phi_n}} \phi_n \otimes (d_{\mathcal{L}} \omega_{i_0,\dots,i_n} \otimes f_{i_0,\dots,i_n})+ \\
&\quad (-1)^{\overline{\phi_n}+\overline{\omega_{i_0,\dots,i_n}}} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_0,\dots,i_n} \otimes [t_{j,i_0,\dots,i_n}, f_{i_0,\dots,i_n}]),
\end{aligned}$$

where the bracket $[t_{j,i_0,\dots,i_n}, f_{i_0,\dots,i_n}]$ is induced by the one of Lemma 6.7.5.

For every $i \geq 0$ the simplicial $\mathscr{L}$-connection ∇ also induces a map

$$\nabla\colon \operatorname{Tot}(\mathcal{U}, \Omega^i(\mathscr{L}) \otimes \mathscr{E}) \to \operatorname{Tot}(\mathcal{U}, \Omega^{i+1}(\mathscr{L}) \otimes \mathscr{E})$$

which allows to define a degree one operator $\nabla\colon \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L}, \mathscr{E})) \to \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L}, \mathscr{E}))$. In fact, let $\nabla = (D_n)$ as in (6.7.6), and consider $\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes e_{i_0,\dots,i_n})$ in $A_n \otimes \prod_{i_0,\dots,i_n} (\Omega^1(\mathscr{L}) \otimes \mathscr{E})(U_{i_0,\dots,i_n})$. Then the operator can be defined as

$$\begin{aligned}
D_n(\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes e_{i_0,\dots,i_n})) &= (\sum_j \eta_{j,n} \otimes (t_{j,i_0,\dots,i_n}))(\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes e_{i_0,\dots,i_n})) \\
&= p(D_n)(\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes e_{i_0,\dots,i_n})) + \\
&\quad + (-1)^{\overline{\phi_n} + \overline{\omega_{i_0,\dots,i_n}}} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_0 \dots i_n} \otimes t_{i_0,\dots,i_n}(e_{i_0,\dots,i_n})) \\
&= (1 \otimes (\operatorname{Id}_{\mathscr{L}} |_{U_{i_0,\dots,i_n}}))(\phi_n \otimes (\omega_{i_0,\dots,i_n} \otimes e_{i_0,\dots,i_n})) + \\
&\quad + (-1)^{\overline{\phi_n} + \overline{\omega_{i_0,\dots,i_n}}} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_0 \dots i_n} \otimes t_{i_0,\dots,i_n}(e_{i_0,\dots,i_n})) \\
&= (-1)^{\overline{\phi_n}} (\phi_n \otimes (d_{\mathscr{L}} \omega_{i_0,\dots,i_n} \otimes e_{i_0,\dots,i_n}) \\
&\quad + (-1)^{\overline{\omega_{i_0,\dots,i_n}}} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_0 \dots i_n} \otimes t_{i_0,\dots,i_n}(e_{i_0,\dots,i_n}))).
\end{aligned}$$

Since all the maps considered on the totalisation are induced by the ones defined locally on the complexes of sheaves, for $d\ \ = [\nabla, -]$ one has that, by (6.7.5),

$$d^2\ \ = [C, -], \quad C \in \operatorname{Tot}(\mathcal{U}, \Omega^2(\mathscr{L}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}))).$$

Then $d_{\mathsf{Tot}} + d\ \ $ is a degree one derivation of $\operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L}) \otimes \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}))$, with square

$$(d_{\mathsf{Tot}} + d\ \ )^2 = d_{\mathsf{Tot}}^2 + d_{\mathsf{Tot}}[\nabla, -] + [\nabla, d_{\mathsf{Tot}} -] + d^2\ \ = [d_{\mathsf{Tot}}\nabla, -] + [C, -] = [d_{\mathsf{Tot}}\nabla + C, -],$$

so the curvature is $d_{\mathsf{Tot}}\nabla + C$. We have already seen in Lemma 6.7.4 that $d_{\mathsf{Tot}}\nabla$ belongs to $\operatorname{Tot}(\mathcal{U}, \Omega^1(\mathscr{L}) \otimes \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}))$.

The last thing to prove is that $(d_{\mathsf{Tot}} + d\ \ )(d_{\mathsf{Tot}}\nabla + C) = 0$. One has that

$$(d_{\mathsf{Tot}} + d\ \ )(d_{\mathsf{Tot}}\nabla + C) = d_{\mathsf{Tot}}^2\nabla + d\ \ d_{\mathsf{Tot}}\nabla + d_{\mathsf{Tot}}C + d\ \ C = d\ \ d_{\mathsf{Tot}}\nabla + d_{\mathsf{Tot}}C.$$

Then

$$d\ \ d_{\mathsf{Tot}}\nabla = [\nabla, d_{\mathsf{Tot}}\nabla] = -[d_{\mathsf{Tot}}\nabla, \nabla] = -\frac{1}{2} d_{\mathsf{Tot}}[\nabla, \nabla] = -d_{\mathsf{Tot}}C,$$

so that $(d_{\mathsf{Tot}} + d\ \ )(d_{\mathsf{Tot}}\nabla + C) = 0$. $\square$

In the case of a Lie pair $(\mathscr{L}, \mathscr{A})$ and a locally free sheaf $\mathscr{E}$, the natural surjective restriction maps

$$\varrho\colon \Omega\ (\mathscr{L}) \to \Omega\ (\mathscr{A}), \qquad \varrho \otimes \operatorname{Id}\colon \Omega\ (\mathscr{L}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})) \to \Omega\ (\mathscr{A}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})),$$

induce morphisms on the totalisation

$$\varrho\colon \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L})) \to \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{A})),$$

$$\varrho \otimes \operatorname{Id}\colon \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}))) \to \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{A}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}))),$$

whose kernels define bilateral ideals

$$\operatorname{Tot}(\mathcal{U}, \mathscr{G}_1) = \ker(\varrho) \subset \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L})),$$

$$\operatorname{Tot}(\mathcal{U}, \mathscr{G}_1 \otimes \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})) = \ker(\varrho \otimes \operatorname{Id}) \subset \operatorname{Tot}(\mathcal{U}, \Omega\ (\mathscr{L}, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}))).$$

Lemma 6.7.7. *Let $(\mathcal{E}, \nabla^{\mathcal{A}})$ be a locally free $\mathcal{A}$-module, and let ∇ be a simplicial extension of $\nabla^{\mathcal{A}}$ to an $\mathcal{L}$-connection. Then $I := \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is a curved ideal of the curved DG-algebra*

$$(\mathrm{Tot}(\mathcal{U}, \Omega\ \ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))), d_{\mathsf{Tot}} + d\ \ , d_{\mathsf{Tot}}\nabla + C),$$

where C, the curvature of the simplicial connection ∇, belongs to $\mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ and $d_{\mathsf{Tot}}\nabla$ belongs to $\mathrm{Tot}\left(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})\right)$.

Proof. It is clear that the ideal $I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is d_{Tot}-closed. Let x be an element of I, so that $(\varrho \otimes \mathrm{Id})(x) = 0$, then

$$(\varrho \otimes \mathrm{Id})(d\ \ x) = d_{\ \ \mathcal{A}}(\varrho \otimes \mathrm{Id})(x) = 0,$$

so I is also d -closed. Since the $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ is flat, the curvature C of ∇ belongs to $\mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \subset I$, which is the kernel of the surjective map

$$\varrho \otimes \mathrm{Id}\colon\ \mathrm{Tot}(\mathcal{U}, \Omega^2(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \to \mathrm{Tot}(\mathcal{U}, \Omega^2(\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})).$$

By Lemma 6.7.4, $d_{\mathsf{Tot}}\nabla$ belongs to $\mathrm{Tot}\left(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})\right)$, therefore it belongs to the ideal I. ☐

For the ideal $I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ we have that

$$I^{(n)} = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_n \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})). \tag{6.7.7}$$

In fact, the inclusion $I^{(n)} \subset \mathrm{Tot}(\mathcal{U}, \mathcal{G}_n \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is clear. For the other one, it suffices to notice that the multiplication map $\underbrace{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_1}_{n} \to \mathcal{G}_n$ is surjective on all affine open sets.

According to Definition 4.1.4, the Atiyah cocycle of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega\ \ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$$

is the class of the curvature $R = d_{\mathsf{Tot}}\nabla + C$ in

$$\frac{I}{I^{(2)}} = \mathrm{Tot}\left(\mathcal{U}, \frac{\mathcal{G}_1}{\mathcal{G}_2} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})\right).$$

Theorem 6.7.8. *Given a Lie pair $(\mathcal{L}, \mathcal{A})$ and a locally free $\mathcal{A}$-module $(\mathcal{E}, \nabla^{\mathcal{A}})$, the Atiyah class $\mathrm{At}(A, I)$ of the curved DG-pair*

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega\ \ (\mathcal{L}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$$

does not depend on the choice of the simplicial $\mathcal{L}$-connection extending $\nabla^{\mathcal{A}}$. Moreover, it is the obstruction to the existence of a $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^{\mathcal{A}}$ with curvature in $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$.

Proof. The difference of two simplicial extensions ∇ and ∇ of the $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ belongs to the ideal I. In fact, considering the short exact sequence (6.7.2),

$$0 \to \mathrm{Tot}\left(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})\right) \to \mathrm{Tot}(\mathcal{U}, Q(\mathcal{L}, \mathcal{E})) \xrightarrow{\beta} \mathrm{Tot}(\mathcal{U}, Q(\mathcal{A}, \mathcal{E})) \to 0,$$

we have that $\beta(\nabla - \nabla\ ) = \nabla^{\mathcal{A}} - \nabla^{\mathcal{A}} = 0$ and therefore, writing $\phi := \nabla - \nabla$, we have $\phi \in \mathrm{Tot}\left(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})\right) \subset I$. Then $d\ \ = d\ \ + [\phi, -]$ and the first claim follows from Lemma 6.2.3.

Next, we show that the Atiyah class $\mathrm{At}(A, I)$ of the curved DG-pair is the obstruction to the existence of a $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^{\mathcal{A}}$, with curvature in $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$. By Lemma 6.2.3, $\mathrm{At}(A, I)$ is the obstruction to existence of $x \in I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ of degree 1 such that $R + (d_{\mathsf{Tot}} + d\ \ )x$ belongs to $I^{(2)} = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$. Assume that there exists such x, and notice that by degree reasons it belongs to $\mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$, since $\mathcal{G}_1^0 = 0$. Then, since $\mathcal{G}_2^1 = 0$,

$$\begin{aligned} d_{\mathsf{Tot}}\nabla + d_{\mathsf{Tot}}x &\in I^{(2)} \cap \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = 0 \\ C + d\ \ x &\in I^{(2)} \cap \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), \end{aligned}$$

and by the first equation $\nabla + x$ is a global $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^{\mathcal{A}}$.

We denote by $R_x = d_{\mathsf{Tot}}(\nabla + x) + C_x = C_x$ the curvature of the curved DG-algebra $(A, d_{\mathsf{Tot}} + d\ \ _{+x})$. Then

$$R_x = R + (d_{\mathsf{Tot}} + d\ \ )x + \frac{1}{2}[x,x] = d_{\mathsf{Tot}}\nabla + C + d_{\mathsf{Tot}}x + d\ \ x + \frac{1}{2}[x,x] = C + d\ \ x + \frac{1}{2}[x,x],$$

so that the curvature of $\nabla + x$ is equal to $C_x = C + d\ \ x + \frac{1}{2}[x,x]$, which belongs to $\mathrm{Tot}(\mathcal{U}, \mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$. Finally, since $d\ \ _{+x}(C_x) = 0$, one has that

$$0 = (d_{\mathsf{Tot}} + d\ \ _{+x})(R_x) = (d_{\mathsf{Tot}} + d\ \ _{+x})(C_x) = d_{\mathsf{Tot}}C_x,$$

and C_x is a global section of $\mathcal{G}_2^2 \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$.

The converse is clear. □

By the above, the Atiyah class $\mathrm{At}(A, I)$ of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega\ \ (\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))$$

is well-defined:

$$\mathrm{At}(A, I) \in \mathsf{H}^2\left(X, \frac{\mathcal{G}_1}{\mathcal{G}_2} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

Definition 6.7.9. In the above situation, via the isomorphisms of Lemma 6.5.3, we call

$$\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) := \mathrm{At}(A, I) \in \mathsf{H}^1\left(\mathcal{A}; (\mathcal{L}/\mathcal{A})\ \ \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

the $(\mathcal{L}, \mathcal{A})$-Atiyah class of $\mathcal{E}$.

Remark 6.7.10. Recalling that $\mathcal{G}_2^1 = 0$, the morphism of graded sheaves $t\colon \frac{\mathcal{G}_1}{\mathcal{G}_2} \to \mathcal{G}_1^1$ with kernel $\frac{\mathcal{G}_1^{\geq 2}}{\mathcal{G}_2}$ induces a morphism of DG-vector spaces

$$t\colon \mathrm{Tot}\left(\mathcal{U}, \frac{\mathcal{G}_1}{\mathcal{G}_2} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) \to \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})),$$

which sends the class of $R = d_{\mathsf{Tot}}\nabla + C$ to $d_{\mathsf{Tot}}\nabla$. The reduced Atiyah class $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$ is then the image of the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega\ \ (\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))$$

via the map induced by t in hypercohomology

$$\begin{gathered} t\colon \mathsf{H}\ \ \left(X, \frac{\mathcal{G}_1}{\mathcal{G}_2} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right) \to \mathsf{H}\ \ (X, \mathcal{G}_1^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \\ \mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \mapsto \overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}}). \end{gathered}$$

In particular if $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is trivial, then so is $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$.

If we consider the Lie pair $(\mathcal{L}, 0)$, both the obstructions $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ and $\overline{\mathrm{At}}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}, \nabla^{\mathcal{A}})$ reduce to the obstruction $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ to the existence of an $\mathcal{L}$-connection on $\mathcal{E}$.

Corollary 6.7.11. *Let $(\mathcal{L}, \mathcal{A})$ be a Lie pair on X such that there exists an $\mathcal{O}_X$-linear projection $p\colon \mathcal{L} \to \mathcal{A}$ which commutes with anchor maps and with adjoint Lie actions of $\mathcal{A}$. Then for every $\mathcal{A}$-module $\mathcal{E}$ the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ is trivial.*

Proof. The assumption that $p\colon \mathcal{L}\to\mathcal{A}$ commutes with adjoint Lie actions of $\mathcal{A}$ means that $p([x,y])=[x,p(y)]$ for every $x\in\mathcal{A}$ and $y\in\mathcal{L}$.

Let $\nabla\colon \mathcal{A}\to\mathcal{E}nd_{\mathsf{K}}(\mathcal{E})$ be a flat $\mathcal{A}$-connection on $\mathcal{E}$. The existence of an $\mathcal{O}_X$-linear projection $p\colon\mathcal{L}\to\mathcal{A}$ commuting with anchor maps ensures that the composition $\widetilde{\nabla}:=\nabla p\colon\mathcal{L}\to\mathcal{E}nd_{\mathsf{K}}(\mathcal{E})$ is a connection. In fact, for $l\in\mathcal{L}$, $f\in\mathcal{O}_X$ and $e\in\mathcal{E}$,

$$\widetilde{\nabla}_l(fe)=\nabla_{p(l)}(fe)=a_{\mathcal{A}}(p(l))(f)e+f\nabla_{p(l)}(e)=a_{\mathcal{L}}(l)(f)e+f\widetilde{\nabla}_l(e).$$

For every $a\in\mathcal{A}$ and every $l\in\mathcal{L}$ we have

$$[\widetilde{\nabla}_a,\widetilde{\nabla}_l]=[\nabla_a,\nabla_{p(l)}]=\nabla_{[a,p(l)]}=\nabla_{p[a,l]}=\widetilde{\nabla}_{[a,l]},$$

and this implies that the curvature of $\widetilde{\nabla}$ belongs to $\mathcal{G}_2^2\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$, so that by Theorem 6.7.8 the Atiyah class of $\mathcal{E}$ is trivial. □

Notice that Corollary 6.7.11 applies in particular in the case $X=\operatorname{Spec}(\mathsf{K})$ and $\mathcal{A}$ a semisimple Lie algebra. On the other hand, the Examples 2.10 and 2.11 of [17] give explicit situations where X is a single point and the Atiyah class does not vanish.

6.8 Semiregularity maps and obstructions

Let $(\mathcal{L},\mathcal{A})$ be a Lie pair on a smooth separated scheme X of finite type over a field K of characteristic 0. Given a locally free $\mathcal{A}$-module $(\mathcal{E},\nabla^{\mathcal{A}})$ we introduced the Atiyah class

$$\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})\in\mathsf{H}^1(\mathcal{A};(\mathcal{L}/\mathcal{A})\ \ \otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})),$$

which is the ***primary obstruction*** to the extension of the $\mathcal{A}$-connection $\nabla^{\mathcal{A}}$ to a flat $\mathcal{L}$-connection; more precisely the Atiyah class is a complete obstruction to the extension of $\nabla^{\mathcal{A}}$ to an $\mathcal{L}$-connection with curvature in $\mathcal{G}_2^2\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

Taking exterior cup products in $\mathcal{A}$-cohomology it makes sense to consider the exterior powers

$$\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k\in\mathsf{H}^k\left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right)$$

together with the morphisms of graded vector spaces

$$\mathsf{H}\ \ (\mathcal{A};\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))\to\mathsf{H}\ \ \left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right)[k]\to\mathsf{H}\ \ \left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \right)[k],$$

$$x\mapsto\frac{1}{k!}\operatorname{Tr}(\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x).$$

The following definition is a clear natural extension of the definition of semiregularity maps for coherent sheaves of Section 3.4, [16].

Definition 6.8.1. In the above situation, for every $k\geq 0$ the map

$$\tau_k\colon\mathsf{H}^2(\mathcal{A};\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))\to\mathsf{H}^{2+k}\left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \right),\qquad\tau_k(x)=\frac{1}{k!}\operatorname{Tr}(\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),$$

is called the k**-semiregularity map of the** $\mathcal{A}$**-module** $(\mathcal{E},\nabla^{\mathcal{A}})$, with respect to the Lie pair $(\mathcal{L},\mathcal{A})$.

If $\mathcal{G}$ is the Leray filtration of the Lie pair $(\mathcal{L},\mathcal{A})$ we have proved in Lemma 6.5.3 that there exist canonical isomorphisms $\mathsf{H}^{2+k}\left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \right)\cong\mathsf{H}^{2+2k}\left(X,\mathcal{G}_k/\mathcal{G}_{k+1}\right)$ and therefore there exist natural maps

$$i_k\colon\mathsf{H}^{2+k}\left(\mathcal{A};\bigwedge^k(\mathcal{L}/\mathcal{A})\ \ \right)\to\mathsf{H}^{2+2k}\left(X,\frac{\Omega\ \ (\mathcal{L})}{\mathcal{G}_{k+1}}\right),$$

which are injective whenever the Leray spectral sequence degenerates at E_1.

We are now ready to apply the abstract general results of [4] to our situation in order to obtain the following result.

Theorem 6.8.2. *Let $(\mathcal{L},\mathcal{A})$ be a Lie pair on a smooth separated scheme X of finite type over a field K of characteristic 0. Given a locally free $\mathcal{A}$-module $(\mathcal{E},\nabla^{\mathcal{A}})$, for every $k\geq 0$ the composite map*

$$i_k\tau_k\colon \mathsf{H}^2\left(\mathcal{A};\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right)\to \mathsf{H}^{2+2k}\left(X,\frac{\Omega\ (\mathcal{L})}{\mathcal{G}_{k+1}}\right)$$

annihilates every obstruction to deformations of $(\mathcal{E},\nabla^{\mathcal{A}})$ as an $\mathcal{A}$-module. In particular, if the Leray spectral sequence of the Lie pair $(\mathcal{L},\mathcal{A})$ degenerates at E_1, then every semiregularity map annihilates obstructions.

Proof. We take an affine cover $\mathcal{U}$ of X and we choose a simplicial connection $\nabla\in \operatorname{Tot}(\mathcal{U},\Omega^1(\mathcal{L})\otimes P(\mathcal{L},\mathcal{E}))$ extending $\nabla^{\mathcal{A}}$. By Lemma 6.7.7, the ideal $I:=\operatorname{Tot}(\mathcal{U},\mathcal{G}_1\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ is a curved ideal of the curved DG-algebra

$$A:=(\operatorname{Tot}(\mathcal{U},\Omega\ (\mathcal{L})\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})),d_{\mathsf{Tot}}+d\ \ ,d_{\mathsf{Tot}}\nabla+C),$$

so that the quotient

$$B:=A/I=\operatorname{Tot}(\mathcal{U},\Omega\ (\mathcal{A})\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$$

is a non-curved DG-Lie algebra, with differential given by $d_{\mathsf{Tot}}+d_{\ \mathcal{A}}$. This is precisely the DG-Lie algebra controlling deformations of $\mathcal{E}$ as an $\mathcal{A}$-module of Theorem 6.4.2.

The trace morphism

$$\operatorname{Tr}\colon \Omega\ (\mathcal{L},\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))\to\Omega\ (\mathcal{L})$$

of (6.3.1) induces

$$\operatorname{Tr}\colon\ \operatorname{Tot}(\mathcal{U},\Omega\ (\mathcal{L},\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))\to\operatorname{Tot}(\mathcal{U},\Omega\ (\mathcal{L})),$$

which is a trace map as in Definition 4.1.9. It is plain that

$$\operatorname{Tr}(\operatorname{Tot}(\mathcal{U},\mathcal{G}_k\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})))\subset\operatorname{Tot}(\mathcal{U},\mathcal{G}_k),$$

for every $k\geq 0$. Finally, according to (6.7.7) and the exactness properties of Tot, for every $i\leq j$ we have

$$\frac{I^{(i)}}{I^{(j)}}=\frac{\operatorname{Tot}(\mathcal{U},\mathcal{G}_i\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))}{\operatorname{Tot}(\mathcal{U},\mathcal{G}_j\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))}=\operatorname{Tot}\left(\mathcal{U},\frac{\mathcal{G}_i}{\mathcal{G}_j}\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

Now, by Theorem 6.2.5, there exists an L_∞ morphism between DG-Lie algebras

$$\sigma^k\colon\ \operatorname{Tot}(\mathcal{U},\Omega\ (\mathcal{A})\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))\qquad\operatorname{Tot}\left(\mathcal{U},\frac{\Omega\ (\mathcal{L})}{\mathcal{G}_{k+1}}[2k]\right)$$

whose linear component is given by

$$\sigma^k_1\colon\ \operatorname{Tot}(\mathcal{U},\Omega\ (\mathcal{A})\otimes\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))\to\operatorname{Tot}\left(\mathcal{U},\frac{\Omega\ (\mathcal{L})}{\mathcal{G}_{k+1}}[2k]\right),\qquad \sigma^k_1(x)=\frac{1}{k!}\operatorname{Tr}(R^kx),$$

where $R=d_{\mathsf{Tot}}\nabla+C$ denotes the curvature of the DG-algebra A.

In cohomology the above maps σ^k_1 may be written as

$$\sigma^k_1\colon \mathsf{H}^2(\mathcal{A};\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))\to\mathsf{H}^{2k+2}\left(X,\frac{\Omega\ (\mathcal{L})}{\mathcal{G}_{k+1}}\right),\qquad\sigma^k_1(x)=\frac{1}{k!}\operatorname{Tr}(\operatorname{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^kx),$$

and then $\sigma^k_1=i_k\tau_k$.

Then the theorem is a consequence of the fact that the DG-Lie algebra $\operatorname{Tot}\left(\mathcal{U},\frac{\Omega\ (\mathcal{L})}{\mathcal{G}_{k+1}}[2k]\right)$ is abelian and then, by Lemma 2.4.6 and Remark 2.4.8, every obstruction of the deformation functor associated to the DG-Lie algebra B is annihilated by the maps σ^k_1.

$\square$

Remark 6.8.3. The induced map in hypercohomology σ_1^k depends only on the $\mathcal{A}$-module $(\mathcal{E}, \nabla^{\mathcal{A}})$ and not on the choice of a simplicial $\mathcal{L}$-connection ∇ extending $\nabla^{\mathcal{A}}$. In fact, σ_1^k depends only on the Atiyah class $\mathrm{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})$ of the curved DG-pair

$$(A = \mathrm{Tot}(\mathcal{U}, \Omega\ (\mathcal{L}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))), I = \mathrm{Tot}(\mathcal{U}, \mathcal{G}_1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))),$$

which we proved in Theorem 6.7.8 does not depend on the choice of ∇.

Appendix A

Some commutative and homological algebra

This appendix contains some useful results from commutative and homological algebra which are used in Section 2.5. The first section contains two basic facts about change of rings, the second section is about flat modules over Artin local rings, and the third contains a lemma about complexes of injective or projective modules.

A.1 Change of rings

Lemma A.1.1. *Let $\varphi\colon R \to S$ be a homomorphism of rings such that S is a flat R-module. Then every flat S-module N is also flat as an R-module.*

Proof [61, 3.B]. For any R-module M, $M \otimes_R N \cong M \otimes_R (S \otimes_S N) \cong (M \otimes_R S) \otimes_S N$, so the functor $- \otimes_R N$ is the composition of two exact functors $- \otimes_R S$ and $- \otimes_S N$, therefore it is exact. □

Lemma A.1.2. *Let $\varphi\colon R \to S$ be a morphism of commutative rings. Then the functor $- \otimes_R S$ sends projective R-modules to projective S-modules.*

Proof. Let P be a projective R-module, then there exists an R-module Q such that $P \oplus Q$ is free over R. Then $(P \otimes_R S) \oplus (Q \otimes_R S)$ is a free S-module, so that $P \otimes_R S$ is projective over S. □

A.2 Flatness and relations

Let A be an Artin local K-algebra with residue field K.

Definition A.2.1. The length of an Artin local ring A with maximal ideal $\mathfrak{m}$ and residue field K is

$$l(A) := \sum_{n \geq 0} \dim_{\mathsf{K}} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}.$$

Notice that if $l(A) = 1$, then $A = \mathsf{K}$.

Lemma A.2.2. *Let M be an A-module, if $M \otimes_A \mathsf{K} = 0$ then $M = 0$.*

Proof. By induction on $l(A)$: if $l(A) = 1$ then $A = \mathsf{K}$ and there is nothing to prove.

If $l(A) > 1$, there exists $n \in \mathsf{N}$ such that $\mathfrak{m}^n \neq 0$ and $\mathfrak{m}^{n+1} = 0$. Take $t \in \mathfrak{m}^n$, $t \neq 0$ and set $B := A/(t)$. There is a short exact sequence

$$0 \longrightarrow \mathsf{K} \xrightarrow{\cdot t} A \longrightarrow B \longrightarrow 0,$$

and tensoring with M we obtain

$$0 = \mathsf{K} \otimes_A M \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow 0.$$

Since $l(B) = l(A) - 1$ and $(M \otimes_A B) \otimes_B \mathsf{K} = M \otimes_A \mathsf{K} = 0$ by inductive hypothesis we obtain $M \otimes_A B = 0$. Then by the above diagram $M = 0$. □

Lemma A.2.3. *For every A-module M there exists a surjective morphism $f\colon F \to M$ such that F is a free module and the induced map $F \otimes_A \mathsf{K} \to M \otimes_A \mathsf{K}$ is an isomorphism.*

Proof. Since $M \to M \otimes_A \mathsf{K}$ is surjective it is sufficient to choose a subset $S \subset M$ inducing a basis of the K-vector space $M \otimes_A \mathsf{K}$ and define F as the free module generated by S. The surjectivity of f comes from the exact sequence

$$F \otimes_A \mathsf{K} \to M \otimes_A \mathsf{K} \to \operatorname{Coker}(f) \otimes_A \mathsf{K} \to 0$$

and by Lemma A.2.2. □

The following is a special case of the local flatness criterion [61, 20.C].

Lemma A.2.4. *For an A-module M the following conditions are equivalent:*

1. *M is free.*
2. *M is flat.*
3. $\operatorname{Tor}_1^A(M, \mathsf{K}) = 0$.

Proof. The only non-trivial assertion is 3) $\Rightarrow$ 1). Assume $\operatorname{Tor}_1^A(M, \mathsf{K}) = 0$ and let F be a free module such there exists a surjective morphism $\alpha\colon F \to M$ inducing an isomorphism $F \otimes_A \mathsf{K} \cong M \otimes_A \mathsf{K}$, as in Lemma A.2.3. Denoting by K the kernel of α, from the Tor long exact sequence of $0 \to K \to F \to M \to 0$ we obtain $K \otimes_A \mathsf{K} \cong \operatorname{Tor}_1^A(M, \mathsf{K}) = 0$, so that $K = 0$ by Lemma A.2.2. □

Lemma A.2.5. *Let $h\colon P \to L$ be a morphism of A-modules, $A \in \mathbf{Art}_{\mathsf{K}}$, and let $\bar{h}\colon P \otimes_A \mathsf{K} \to L \otimes_A \mathsf{K}$ denote its reduction.*

1. *If $\bar{h}$ is surjective, then h is surjective.*
2. *If $\bar{h}$ is injective and P and L are flat, then h is injective.*

Proof. For the first item, the proof is the same as the proof of Lemma A.2.3.

For the second item, let now $\bar{h}$ be injective, we prove that h in injective by induction on $l(A)$. If $l(A) = 1$, then $A = \mathsf{K}$ and there is nothing to prove. If $l(A) > 1$ there is a short exact sequence

$$0 \longrightarrow \mathsf{K} \longrightarrow A \longrightarrow B \longrightarrow 0$$

with $l(B) < l(A)$. Since

$$P \otimes_A \mathsf{K} = (P \otimes_A B) \otimes_B \mathsf{K} \xrightarrow{\bar{h}} L \otimes_A \mathsf{K} = (L \otimes_A B) \otimes_B \mathsf{K}$$

is injective, by inductive hypothesis the map $\tilde{h}\colon P \otimes_A B \to L \otimes_A B$ is also injective. Since P and L are both flat, we obtain the following diagram, where the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & P \otimes_A \mathsf{K} \arrow[r] \arrow[d, hook, "\bar{h}"'] & P \arrow[r] \arrow[d, "h"] & P \otimes_A B \arrow[r] \arrow[d, hook, "\tilde{h}"] & 0 \\
0 \arrow[r] & L \otimes_A \mathsf{K} \arrow[r] & L \arrow[r] & L \otimes_A B \arrow[r] & 0.
\end{tikzcd}$$

By the Five Lemma, h is injective. □

Corollary A.2.6. *Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules with N flat. Then:*

1. *$M \otimes_A \mathsf{K} \to N \otimes_A \mathsf{K}$ injective $\implies$ P flat.*

2. *P flat $\implies$ M flat and $M \otimes_A \mathsf{K} \to N \otimes_A \mathsf{K}$ injective.*

Proof. Take the associated long $\operatorname{Tor}^A(-, \mathsf{K})$ exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Tor}_2^A(M,\mathsf{K}) \arrow[r] & \operatorname{Tor}_2^A(N,\mathsf{K}) = 0 \arrow[r] & \operatorname{Tor}_2^A(P,\mathsf{K}) \arrow[dll] & \\
 & \operatorname{Tor}_1^A(M,\mathsf{K}) \arrow[r] & \operatorname{Tor}_1^A(N,\mathsf{K}) = 0 \arrow[r] & \operatorname{Tor}_1^A(P,\mathsf{K}) \arrow[dll] & \\
 & M \otimes_A \mathsf{K} \arrow[r] & N \otimes_A \mathsf{K} \arrow[r] & P \otimes_A \mathsf{K} \arrow[r] & 0
\end{tikzcd}$$

and apply Lemma A.2.4. □

Corollary A.2.7. *Let*

$$P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0 \tag{A.2.1}$$

be a complex of A-modules such that:

1. *P, Q, R are flat.*

2. *$Q \xrightarrow{g} R \xrightarrow{h} M \to 0$ is exact.*

3. *$P \otimes_A \mathsf{K} \xrightarrow{\bar{f}} Q \otimes_A \mathsf{K} \xrightarrow{\bar{g}} R \otimes_A \mathsf{K} \xrightarrow{\bar{h}} M \otimes_A \mathsf{K} \to 0$ is exact.*

Then M is flat and the sequence (A.2.1) is exact.

Proof. Denote by $H = \ker h = \operatorname{Im} g$ and $g = \phi\eta$, where $\phi \colon H \to R$ is the inclusion and $\eta \colon Q \to H$; by assumption we have an exact diagram

$$\begin{tikzcd}
P \otimes_A \mathsf{K} \arrow[r, "\bar{f}"] & Q \otimes_A \mathsf{K} \arrow[rr, "\bar{g}"] \arrow[dr, "\bar{\eta}"'] & & R \otimes_A \mathsf{K} \arrow[r, "\bar{h}"] & M \otimes_A \mathsf{K} \arrow[r] & 0. \\
 & & H \otimes_A \mathsf{K} \arrow[ur, "\bar{\phi}"'] \arrow[dr] & & & \\
 & & & 0 & &
\end{tikzcd}$$

It is easy to see that $\bar{\phi}$ is injective: let $x \in H \otimes_A \mathsf{K}$ such that $\bar{\phi}(x) = 0$, by surjectivity of $\bar{\eta}$ there exists $y \in Q \otimes_A \mathsf{K}$ such that $\bar{\eta}(y) = x$. Then $\bar{g}(y) = \bar{\phi}\bar{\eta}(y) = 0$, so that there exists $z \in P \otimes_A \mathsf{K}$ such that $\bar{f}(z) = y$. Then $x = \bar{\eta}(y) = \bar{\eta}\bar{f}(z) = 0$. According to Corollary A.2.6, H and M are therefore flat A-modules.

Denoting $L = \ker g$ we have, again by Corollary A.2.6 and by the flatness of H and Q, that also L is flat and $L \otimes_A \mathsf{K} \to Q \otimes_A \mathsf{K}$ injective. This implies that $P \otimes_A \mathsf{K} \to L \otimes_A \mathsf{K}$ is surjective, so that $P \to L$ is surjective, by Lemma A.2.5. □

Corollary A.2.8. *Let A be an Artin local ring, $n \geq 2$, and let*

$$\cdots \longrightarrow P^{-n} \xrightarrow{d_n} \cdots \longrightarrow P^{-1} \xrightarrow{d_1} P^0 \xrightarrow{d_0} M \longrightarrow 0$$

be an exact sequence of A-modules such that each P^i is flat over A and tensoring by $- \otimes_A \mathsf{K}$ we obtain an exact sequence. Then M and $\operatorname{Im}(d_n) = \operatorname{Ker}(d_{n+1})$ are flat over A.

Proof. Induction on n and Corollary A.2.7. □

Lemma A.2.9. *Let $A \in \mathbf{Art}_{\mathsf{K}}$ and let R be a unitary commutative K-algebra. Let G be an $R \otimes A$-module which is flat over A, and let $E \ \to G$ be a projective resolution of $R \otimes A$-modules. Then $E \ \ \otimes_A \mathsf{K} \to G \otimes_A \mathsf{K}$ is a projective resolution of R-modules.*

Proof. For every i, E^i is projective over $R \otimes A$, so it is flat over $R \otimes A$. Consider the ring homomorphism $A \to R \otimes A$, which makes $R \otimes A$ into a flat A-module, because $R \otimes A \otimes_A - \cong R \otimes -$ is an exact functor. Therefore, by Lemma A.1.1, every E^i is flat over A. Consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker} p \longrightarrow E^0 \xrightarrow{\ p\ } G \longrightarrow 0,$$

where both G and E^0 are flat over A. By Corollary A.2.6, $\operatorname{Ker} p$ is flat and the map $\operatorname{Ker} p \otimes_A \mathsf{K} \to E^0 \otimes_A \mathsf{K}$ is injective, so that the sequence

$$0 \longrightarrow \operatorname{Ker} p \otimes_A \mathsf{K} \longrightarrow E^0 \otimes_A \mathsf{K} \longrightarrow G \otimes_A \mathsf{K} \longrightarrow 0$$

is exact. We can iterate this procedure, by considering the short exact sequence

$$0 \longrightarrow \operatorname{Ker} q \longrightarrow E^{-1} \xrightarrow{\ q\ } \operatorname{Ker} p \longrightarrow 0,$$

because both E^{-1} and $\operatorname{Ker} p$ are flat over A. In the end, we obtain an exact sequence

$$\cdots \longrightarrow E^{-2} \otimes_A \mathsf{K} \longrightarrow E^{-1} \otimes_A \mathsf{K} \longrightarrow E^0 \otimes_A \mathsf{K} \longrightarrow G \otimes_A \mathsf{K} \longrightarrow 0$$

where every $E^i \otimes_A \mathsf{K}$ is a projective R-module, by Lemma A.1.2. □

Corollary A.2.10. *Let $A \in \mathbf{Art}_{\mathsf{K}}$ and let R be a unitary commutative K-algebra. Let I be an injective R-module and G an $R \otimes A$-module which is flat over A. Then $\operatorname{Ext}^k_{R \ A}(G, I) = 0$ for every $k \geq 1$.*

Proof. Let $E \ \to G$ be a projective resolution of G as an $R \otimes A$-module, so that $\operatorname{Ext}^k_{R \ A}(G, I) = H^k(\operatorname{Hom}_{R \ A}(E \ , I))$. There is an isomorphism $\operatorname{Hom}_{R \ A}(E \ , I) \cong \operatorname{Hom}_R(E \ \ \otimes_A \mathsf{K}, I)$, and by Lemma A.2.9 $E \ \ \otimes_A \mathsf{K}$ is a R-projective resolution of $G \otimes_A \mathsf{K}$. Then

$$\operatorname{Ext}^k_{R \ A}(G, I) = H^k(\operatorname{Hom}_{R \ A}(E \ , I)) \cong H^k(\operatorname{Hom}_R(E \ \ \otimes_A \mathsf{K}, I)) = \operatorname{Ext}^k_R(G \otimes_A \mathsf{K}, I) = 0,$$

for all $k \geq 1$, because I is R-injective. □

A.3 A result from homological algebra

The following useful result is taken from [29, III.5.24], where the lemma is proved in the hypothesis that the acyclic complex is bounded below (respectively above). The proof is basically the same in the unbounded case, which is the one needed in Section 2.5, and is reported here for the sake of completeness.

Lemma A.3.1. *Let $s \colon A \ \to I \ $ be a morphism of complexes of R-modules from an acyclic complex to a bounded below complex of injective modules, then s is homotopic to the zero map.*

Dually, let $t \colon P \ \to B \ $ be a morphism of complexes of R-modules from a bounded above complex of projective modules to an acyclic complex, then t is homotopic to the zero map.

Proof. The homotopy k between s and the zero morphism is constructed by induction starting from $k^0 \colon A^1 \to I^0$ and continuing to the right. It is clear that we can choose $k^i = 0$ for all $i < 0$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d] & A^0 \arrow[r, "d_A^0"] \arrow[d, "s^0"] & A^1 \arrow[r] \arrow[d, "s^1"] \arrow[dl, dashed, "k^0"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r, "d_I^0"'] & I^1 \arrow[r] & \cdots ;
\end{tikzcd}$$

since A is acyclic, $\operatorname{Im} d_A^{-1} = \ker d_A^0$, so that $d_A^0\colon \operatorname{Coker} d_A^{-1} \to A^1$ is injective. Since $s^0 d_A^{-1} = 0$, $s^0\colon \operatorname{Coker} d_A^{-1} \to I^0$ and we can consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Coker} d_A^{-1} \arrow[r, "d_A^0"] \arrow[d, "s^0"] & A^1 \arrow[dl, dashed, "k^0"] \\
& I^0, &
\end{tikzcd}$$

where k^0 exists because I^0 is an injective object.

Assume now we have constructed the homotopy up to k^{n-1}:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "s^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "s^n"'] \arrow[dl, "k^{n-1}"'] & A^{n+1} \arrow[r] \arrow[d, "s^{n+1}"] \arrow[dl, dashed, "k^n"'] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r, "d_I^{n-1}"'] & I^n \arrow[r, "d_I^n"'] & I^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

By inductive hypothesis $(s^n - d_I^{n-1} k^{n-1}) d_A^{n-1} = 0$, which means that as above we can consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Coker} d_A^{n-1} \arrow[r, "d_A^n"] \arrow[d, "s^n - d_I^{n-1} k^{n-1}"'] & A^{n+1} \arrow[dl, dashed, "k^n"] \\
& I^n, &
\end{tikzcd}$$

where $d_A^n\colon \operatorname{Coker} d_A^{n-1} \to A^{n+1}$ is injective because A is acyclic, and then k^n exists by the injectivity of I^n.

The proof for the dual statement is analogous. □

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