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A phase-field model for fracture in beams from asymptotic results in 2D elasticity

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Abstract. We propose a derivation of a damage model in slender structures, focusing on the particular case of a rod. The peculiarity of the model is that it takes into account the changes in rigidity of the body, distinguishing between bending, traction and the possible mixed interactions between the two. The approach is based on a matched asymptotic expansion, taking the recent work of Baldelli et al [1] as starting point. Choosing the slenderness of the rod as small parameter for the asymptotic expansion, we determine the first order at which a correction occurs with respect to the Saint-Venant solution of the elastic problem, due to the presence of a crack. The results highlight that the presence of a defect affects in different ways the bending and traction rigidities of the rod, and that a coupling between the two deformation modes might occur, depending on the geometry of the crack. Moreover, the derivation allows to explicitly calculate the coefficients of this correction, for any given depth of the crack, by means of a simple numerical procedure. Application to the classic three-point bending problem is considered in order to highlight the predictive capabilities of the model. These results suggest ways in which state of the art phase-field models (e.g. [2]) for damage could be refined. This work goes in the direction of developing phase-field models suitable for application to slender structures, where the use of reduced dimensional models has proved promising [3].

Introduction

In modeling of fracture and damage of structures, a lot of attention has been devoted to the case of slender structures. A particular regime of loading is that when bending and membrane stresses interact in non-trivial ways. This happens in some important applications, such as thin shells that have bending and membrane deformation coupling due to geometric constraints (Gauss compatibility) [4, 5]. Such structures are characterized by their slenderness and give way to interesting applications taking advantage of their shape morphing features [6, 7]. The question then arises as to how to properly model damage for such cases. Currently in damage models, the state of the art of the classical phase-field approach is to introduce a unique damage parameter, which modulates the change of all stiffness constants that appear in the definition of the elastic energy. For example, considering a 1D beam:

$$E_e = \frac{1}{2}\int_{\Omega}(1-\alpha)^2\,(A_0\varepsilon^2 + D_0\chi^2), \tag{1}$$

where ε and χ are the stretching and curvature measures, A_0 and D_0 the relative stiffness constants, and α is a scalar field describing the damaging of the structure. In this work we consider the case when different effects on the coefficients are possible, as well as coupling effects between stretching and bending contributions:

$$E_e = \frac{1}{2}\int_{\Omega}(A(\alpha)\varepsilon^2 + D(\alpha)\chi^2 + 2C(\alpha)\varepsilon\chi). \tag{2}$$

For sake of simplicity, we will focus on the case when the crack starts at the middle of a beam, on one of the two lateral faces. The main features of this model are that the progressive damaging of the material could affect the stiffnesses differently, and also that the presence of a crack is expected to couple bending and stretching effects, since the reduction in thickness of the cross section causes the neutral axis of bending to be displaced. In order to do this, we consider a case of particular interest, that of a slender beam subject to traction force and bending load at the bases. In the domain of small displacements, the base solution of the 1D elastic problem is that of Saint-Venant. The presence (or the possible nucleation) of a crack causes a modification of the base solution, in that at the middle section the axial displacement and section rotation will show a jump discontinuity. The assumption of slenderness allows us to use an asymptotic approach, expanding the base solution in powers of a small parameter.

In the following, we will give an outline of the asymptotic procedure. The final result will be the calculation of the first order of correction (in the small parameter) of the solution, with respect to the base Saint-Venant solution. This correction coincides with the jumps in axial displacement and rotation due to the crack. These additional deformations can be modeled with an additional compliance, or, inversely, with a decrease in the stiffnesses. These factors will determine the crack-dependent coefficients of the energy, as in the example of Eq. 2, and allow us to formulate our damage model.

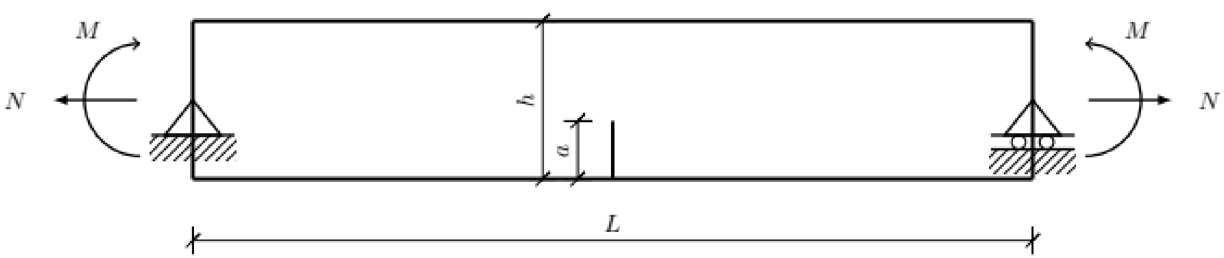

Figure 1. Schematic representation of the beam with a crack in the mid section, of the loading and other important geometric quantities.

Problem and non-dimensionalization

The solid domain is defined in the Cartesian space with coordinates x_1, x_2, x_3, and has large width, b, in the x_3 direction. In the other two directions, the domain of the beam occupies the rectangle $[-L/2\,,L/2] \times [-h/2\,,h/2]$, where $h \ll L \ll b$, and x_1 is in the direction of the axis, while x_2 is in the direction of the thickness. We introduce the dimensionless parameter $\eta = h/L$, and we will study the elasticity problem, in a small displacements setting, when the domain has a surface crack, with profile symmetric about the x_2 axis. The problem will be two dimensional, under the assumption of plane strain. The following non-dimensional coordinates are introduced: $\overline{x}_1 = x_1/L$ and $y = x_2/h$. We can then scale the displacement field

$$u(x) = Lu^{\eta}(\overline{x}_1, y), \tag{3}$$

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the strains and stress tensors

$$E(x) = \eta E^{\eta}(\bar{x}_1, y),$$
$$S(x) = \eta Y S^{\eta}(\bar{x}_1, y), \tag{4}$$

and the traction and bending load along the beam

$$M_{\alpha\beta}(x) = \eta^3 bYL^2 M^{\eta}_{\alpha\beta}(\bar{x}_1, y),$$
$$N(x) = \eta^2 bYLN^{\eta}(\bar{x}_1, y), \tag{5}$$

noticing that the powers of η that appear in Eq. 5 are such that the traction and bending load scale differently with respect to the small parameter: their relative scaling is such that their contribution to the stress will be of the same order in η in the expansion, allowing for the possibility of interactions between bending and traction in the solution.

Outer and inner asymptotic expansion

We suppose that, far from the boundaries of the crack in the middle of the domain, the displacements, stresses and strains admit the following asymptotic expansion in the small parameter:

$$u^{\eta} = u^0(\bar{x}_1, y) + \eta u^1(\bar{x}_1, y) + \eta^2 u^2(\bar{x}_1, y) + \eta^3 u^3(\bar{x}_1, y) + \cdots,$$
$$S^{\eta} = \eta^{-2} S^{-2}(\bar{x}_1, y) + \eta^{-1} S^{-1}(\bar{x}_1, y) + S^0(\bar{x}_1, y) + \eta S^1(\bar{x}_1, y) + \cdots, \tag{6}$$
$$E^{\eta} = \eta^{-2} E^{-2}(\bar{x}_1, y) + \eta^{-1} E^{-1}(\bar{x}_1, y) + E^0(\bar{x}_1, y) + \eta E^1(\bar{x}_1, y) + \cdots.$$

This coincides with the power expansion of the Saint-Venant solution, and is not valid in the proximity of the crack region, since a boundary layer is present. Another asymptotic expansion has to be derived, after a rescaling of the coordinates. Then, once the two expansions have been determined, it will be enforced that the two coincide in a transition region, where both are supposed to be valid. The rescaling consists of a stretching of the domain in the axial direction, rewriting the elastic problem in the variable

$$x = \frac{\bar{x}_1}{\eta}, \tag{7}$$

and the inner expansion:

$$u^{\eta} = v^0(x, y) + \eta v^1(x, y) + \eta^2 v^2(x, y) + \eta^3 v^3(x, y) + \cdots,$$
$$S^{\eta} = \eta^{-2} \sigma^{-2}(x, y) + \eta^{-1} \sigma^{-1}(x, y) + \sigma^0(x, y) + \eta \sigma^1(x, y) + \cdots, \tag{8}$$
$$E^{\eta} = \eta^{-2} \gamma^{-2}(x, y) + \eta^{-1} \gamma^{-1}(x, y) + \gamma^0(x, y) + \eta \gamma^1(x, y) + \cdots.$$

Matching and energy of the new beam model

The matching of the two asymptotic expansions requires that these coincide when the outer solution is evaluated at $\bar{x}_1 = 0$, and the inner one for x tending to infinity. This yields three equations, choosing to stop at the first order at which a correction is obtained with respect to the Saint-Venant solution:

$$\begin{gathered}\lim_{x\to\pm\infty} v^0(x,y) = u^0(0\pm,y),\\ \lim_{x\to\pm\infty} v^1(x,y) = u^1(0\pm,y) + \lim_{x\to\pm\infty} x u^0_{,1}(0\pm,y),\\ \lim_{x\to\pm\infty} v^2(x,y) = u^2(0\pm,y) + \lim_{x\to\pm\infty}\left(x u^1_{,1}(0\pm,y) + \tfrac{1}{2}x^2 u^0_{,11}(0\pm,y)\right),\end{gathered} \tag{9}$$

and from the solution of these, the jumps of the solution (which is allowed to possibly be discontinuous at $\bar{x}_1 = 0$) in terms of the external loads are calculated

$$\begin{aligned}\left[\left[U^1_{,1}\right]\right] &= C_t(1-\nu^2)N^0(0) + 12C_b(1-\nu^2)M^0_{11}(0),\\ \left[\left[U^2\right]\right] &= K_{t2}(1-\nu^2)N^0(0) + 12K_{b2}(1-\nu^2)M^0_{11}(0),\\ \left[\left[W^2\right]\right] &= K_{t1}(1-\nu^2)N^0(0) + 12K_{b1}(1-\nu^2)M^0_{11}(0).\end{aligned} \tag{10}$$

Here, W and U are the axial and transversal components of the displacements at a certain order. The coefficients that relate these jumps and the external loads are the additional compliance of the beam due to the presence of a boundary layer, which in turn is due to the crack of the domain. To get back to a continuous model, we assume that the calculated jumps occur as a linear distribution in a small region of width $\bar{l}_c = l_c/L$. This allows us to calculate the derivatives of the approximated jumps, and thus the strains and the compliance coefficients. These are given by:

$$\begin{pmatrix}\bar{a} & \bar{c}\\ \bar{c} & \bar{d}\end{pmatrix} = (1-\nu^2)\begin{pmatrix}\left(1+\frac{\eta K_{t1}}{\bar{l}_c}\right) & 12\frac{\eta K_{b1}}{\bar{l}_c}\\ \frac{\eta C_t}{\bar{l}_c} & 12\left(1+\frac{\eta C_b}{\bar{l}_c}\right)\end{pmatrix} \tag{11}$$

The energy of the beam, in the interval $\left[-\bar{l}_c/2\,,\bar{l}_c/2\right]$, can then be written as:

$$P^{\eta,\bar{l}_c} = -\frac{1}{2}\int_{-\frac{\bar{l}_c}{2}}^{\frac{\bar{l}_c}{2}}\begin{pmatrix}\bar{a} & \bar{c}\\ \bar{c} & \bar{d}\end{pmatrix}\begin{pmatrix}N^\eta_e\\ M^\eta_e\end{pmatrix}\cdot\begin{pmatrix}N^\eta_e\\ M^\eta_e\end{pmatrix}\mathrm{d}\bar{x}_1. \tag{12}$$

Far from the crack, instead, the energy is unchanged with respect to the base solution.

Discussion and conclusions

Using an asymptotic expansion, we calculated the perturbation of the solution of an elastic beam due to a crack. This perturbation is determined by 4 scalar coefficients, representing the additional deformations with respect to the non-cracked beam, due to traction and bending, or to the interaction of the two. This latter effect will be present if the non-diagonal coefficient of the stiffness (or compliance) matrix is non-zero. These coefficients have to be calculated numerically, for any value of the non-dimensional crack depth in the interval $(0,1)$. Once these are known, the problem is determined and one can solve it as a 1D damage model. The first thing to notice is that, from the numerical results, these coefficients are clearly different and have a different evolution when the crack propagates along the thickness of the structure. This confirms the starting idea that, in general, the effect of a crack on the bending and traction stiffnesses is different. Furthermore, the off-diagonal values of the stiffness (or compliance) matrix in general will be non-zero; that is, a coupling energy term between bending and traction will be generated by the presence of the crack. This opens up to interesting future directions of research: once the model is obtained, it can be applied to some cases of interest (e.g., the classic three-point bending problem), and compared with experimental data, if available. This initial work can then be extended to other cases where the interaction between traction and bending effects can be interesting, prominent among these, the study of thin shells [8-10], using asymptotic techniques which have been extensively studied [1, 11-14] in literature.

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