Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2023, Vol. 59, No. 2, 662–705
https://doi.org/10.1214/22-AIHP1269

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES

Stochastic homogenization of random walks on point processes

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Received 1 December 2020; revised 18 February 2022; accepted 21 March 2022

Abstract. We consider random walks on the support of a random purely atomic measure on $\mathbb{R}^d$ with random jump probability rates. The jump range can be unbounded. The purely atomic measure is reversible for the random walk and stationary for the action of the group $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$. By combining two-scale convergence and Palm theory for $\mathbb{G}$-stationary random measures and by developing a cut-off procedure, under suitable second moment conditions we prove for almost all environments the homogenization for the massive Poisson equation of the associated Markov generator. In addition, we obtain the quenched convergence of the L^2-Markov semigroup and resolvent of the diffusively rescaled random walk to the corresponding ones of the Brownian motion with covariance matrix $2D$. For symmetric jump rates, the above convergence plays a crucial role in the derivation of hydrodynamic limits when considering multiple random walks with site-exclusion or zero range interaction. We do not require any ellipticity assumption, neither non-degeneracy of the homogenized matrix D. Our results cover a large family of models, including e.g. random conductance models on $\mathbb{Z}^d$ and on general lattices (possibly with long conductances), Mott variable range hopping, simple random walks on Delaunay triangulations, simple random walks on supercritical percolation clusters.

Résumé. Nous considérons des marches aléatoires sur le support d'une mesure aléatoire purement atomique sur $\mathbb{R}^d$ avec taux de sauts aléatoires. Les sauts peuvent être arbitrairement longs. La mesure purement atomique est réversible pour la marche aléatoire et stationnaire pour l'action du groupe $\mathbb{G} = \mathbb{R}^d$ ou $\mathbb{G} = \mathbb{Z}^d$. En combinant la convergence à deux échelles et la théorie de Palm pour les mesures aléatoires $\mathbb{G}$-stationnaires et en développant une procédure de troncation, sous des conditions de moment d'ordre deux appropriées, nous prouvons pour presque tous les environnements l'homogénéisation pour l'équation de Poisson massive du générateur de Markov associé. De plus, nous obtenons la convergence du semi-groupe de Markov L^2 et de la résolvante de la marche aléatoire, après renormalisation diffusive, vers leur équivalent pour le mouvement brownien de matrice de covariance $2D$. Pour des taux de sauts symétriques, cette convergence joue un rôle crucial dans l'obtention de la limite hydrodynamique pour des modèles de marches multiples avec exclusion ou à portée nulle. Aucune hypothèse d'ellipticité, ou de non-dégénérescence de la matrice homogénéisée D, n'est nécessaire. Nos résultats couvrent une large classe de modèles, qui inclue notamment les modèles de conductances aléatoires sur $\mathbb{Z}^d$ et sur réseaux généraux (éventuellement à conductances longues), les modèles de sauts à distance variable de Mott, les marches aléatoires simples sur les triangulations de Delaunay et les marches aléatoires simples sur des amas de percolation surcritiques.

MSC2020 subject classifications: Primary 60K37; 35B27; 60H25; secondary 60G55

Keywords: Random measure; Point process; Palm distribution; Random walk in random environment; Stochastic homogenization; Two-scale convergence; Mott variable range hopping; Conductance model; Hydrodynamic limit

1. Introduction

Stochastic homogenization of random walks in random environment is a rich research field initiated in the Western school by Varadhan, Papanicolaou and coauthors and in the Russian school by Kozlov (cf. e.g. [2,4,14,23,24,30–36,38,39] and references therein). Hence, before describing our results, we illustrate some questions motivating the present work, thus providing a reading key to this article. A first question concerns hydrodynamic limits (HL's) of interacting particle systems in random environment. In [15,16] we proved the quenched HL for the simple exclusion process and the zero range process, respectively, on the supercritical percolation cluster with random conductances. The proof relies (between other) on a weak form of quenched convergence of the Markov semigroup and resolvent for a single random walk towards their counterparts for the Brownian motion. This convergence was obtained from the homogenization of the massive Poisson equation in [15]. It is then natural to ask if these results hold for a much larger class of models.

A second question motivating the present work concerns Mott's law, which predicts the anomalous conductivity decay at low temperature in amorphous solids with electron transport given by Mott variable range hopping [40]. A mean-field model is given by Mott random walk on a marked simple[1] point process in $\mathbb{R}^d$ [21]. By invoking Einstein's relation, Mott's law can be stated in terms of the dependence of the effective diffusion matrix $D(\beta)$ on the temperature β^{-1}. In [7,8] we proved a quenched invariance principle for Mott random walk under suitable conditions on the point process with a homogenization-type characterization of $D(\beta)$, while in [20,21] we proved bounds on $D(\beta)$ in agreement with Mott's law. As Mott's law has a large class of universality, it is natural to ask if a weak form of quenched CLT with the same matrix $D(\beta)$ holds for a much larger (compared to [7,8]) class of point processes. In the present work (cf. Theorems 4.1 and 4.4) we give a positive answer to the above questions. In our companion work [18] we then derive the quenched HL for simple exclusion processes in symmetric random environments. The zero range process will be treated in a future work.

Let us briefly describe our results. We consider here general random walks with state space given by the support of a given random purely atomic measure on $\mathbb{R}^d$, possibly contained in $\mathbb{Z}^d$ or a generic d-dimensional discrete lattice as e.g. the hexagonal one. Also the jump probability rates can be random. The above randomness of the environment is supposed to be stationary with respect to the action of the group $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$. We assume some second moment conditions and that the above random purely atomic measure is reversible for the random walk. Our first main result is then given by the quenched homogenization of the massive Poisson equation associated to the diffusively rescaled random walk, leading to an effective diffusive equation and a variational characterization of the homogenized matrix D (see Theorem 4.1). Our second main result concerns a quenched convergence of the Markov semigroup and resolvent of the diffusively rescaled random walk to their counterparts for the Brownian motion with covariance matrix $2D$ (see Theorem 4.4). This form of convergence includes a spatial "average" on the initial point of the random walk, and it is the proper form relevant to get the above quenched HL's. It can also be thought of as a weak form of quenched CLT. We point out that we do not require any ellipticity assumption and our results cover the case of degenerate homogenized matrix as well.

Our results cover a broad class of models. Just to list some examples: the random conductance models on $\mathbb{Z}^d$ and on general lattices (possibly with long conductances), Mott variable range hopping, simple random walks on Delaunay triangulations, simple random walks on supercritical percolation clusters with random conductances. To gain such a generality we have used the theory of $\mathbb{G}$-stationary random measures [26,27,29], thus allowing to have a common language for all models and to formulate the 2-scale ergodic properties of the environment in terms of Palm distributions. We have also used the method of 2-scale convergence ([1,2,37,44] and references therein).

Our proof of Theorem 4.1 is inspired by the strategy developed in [44], dealing with random differential operators on singular structures, but key technical obstructions due to possible arbitrarily long jumps (and therefore not present in [44]) have emerged. One main technical effort here has been to deal with 2-scale convergence using only L^2-concepts. Let us explain this issue. First, we observe that the standard gradient has to be replaced by an amorphous gradient, keeping knowledge of the variation of a function along all possible jumps. While the gradient of a regular function φ on $\mathbb{R}^d$ with compact support maintains the same properties, the amorphous gradient of φ is an irregular object, which cannot be bounded in uniform norm. Hence, the 2-scale method developed for random differential operators does not work properly, since several limits become now illegal. We have been able to overcome this difficulty by enhancing the standard method with suitable cut-off procedures (cf. Sections 15 and 17). Another main difference with [44] is the following. In order to prove that for almost any environment homogenization holds for all massive Poisson equations, we need to restrict to countable dense families of testing objects in the definitions of 2-scale convergence and not always, in our context and differently from [44], one can reduce for free to continuous objects when depending on the environment, as for the space of solenoidal forms (cf. Section 8). To overcome this obstruction, we have been able to deal only with countable families of L^2 environment-dependent testing objects. This has also the advantage to avoid topological assumptions on the probability space with exception of the separability of $L^2(\mathcal{P}_0)$, $\mathcal{P}_0$ being the Palm distribution of the random purely atomic measure. On the other hand, a special care has been necessary in defining the right class of testing objects (cf. Section 12).

We now give some comments on reference [24] containing results on spectral homogenization for the discrete Laplacian on $\mathbb{Z}^d$ with random conductances and unbounded length range. The conditions assumed in [24] are more complex, implying both Poincaré and Sobolev inequalities (on the other hand, the authors face with a different problem). Note that for the model in [24] the assumptions of our Theorem 4.1 and Theorem 4.4 reduce to Assumption 1.1-(a), (b) in [24]. In particular, we avoid the more sophisticated Assumptions 1.1-(c) and 1.2 in [24]. The conditions in [24] guarantee the uniform boundedness in L^∞-norm of the solution of the Poisson equation with Dirichlet boundary conditions there (cf. [24, Proposition 3.4]). As a consequence, the derivation of the structure result stated in Lemma 5.15 (and somehow corresponding to [44, Lemma 5.4] and to Proposition 18.1 here) is much simplified by the L^∞-boundedness of the

[1] We follow the terminology of [9–11], hence *simple* just means that points have unit multiplicity.

solutions and the obstruction mentioned above (solved by the cut-off procedure) does not emerge in [24]. Similarily, homogenization results for random walks on Delaunay triangulations have been obtained also in [28] under the condition that the diameters of the Voronoi cells are uniformly bounded both from below and from above (see Condition 1.2 in [28]). Our analysis does not need such uniform bounds. We also point out that some stronger assumptions in [24] imply the non-degeneracy of the homogenized matrix, and this property enters in their proof of the structure result given by [24, Lemma 5.15]. Finally, we remark that, when working with $\mathbb{Z}^d$ as in [24], the form of the Palm distribution is much simpler (compare (12) with (9) and (11)) and therefore several manipulations concerning square integrable forms become simple, differently from the general case (see e.g. Lemma 7.3).

Knowing the non-degeneracy of the diffusion matrix D gives important information on the effective homogenized equation. As possible techniques to prove the non-degeneracy we mention the use of lower Gaussian kernel bounds and the sublinearity of the corrector (cf. e.g. [4], [3, Section 6.1], [13, Prop. 2.5], [36]). Alternatively, one can use electrostatic arguments by combining the results of [19] for resistor networks on point processes with Remark 3.7 below (see [19, Section 3] and also [12] for a special case with a more complex construction of the resistor network). Further details are given in Section 5.

We conclude with some remarks on the quenched invariance principle of random walks on point processes. The results contained in this work, and in particular Proposition 18.1, allow to extend the analysis of the corrector in [36] to our general class of random walks on point processes, for the part concerning 2-scale convergence. Mainly the proof of tightness would then require a further analysis. We also mention [5,7,8,41] where the quenched invariance principle of random walks with long-range jumps is proved.

Outline of the paper. In Section 2 we describe our setting and main assumptions. In Section 3 we introduce the massive Poisson equation, the homogenized equation and discuss convergence types. In Section 4 we present our main results. In Section 5 we discuss some examples. In Section 6 we show that it is enough to consider the case $\mathbb{G} = \mathbb{R}^d$. In Section 7 we derive some key properties of the Palm distribution. In Sections 8, 9, 10 and 11 we study square integrable forms. In Section 12 we describe the set Ω_{typ} of typical environments appearing in Theorems 4.1 and 4.4. In Section 13 we discuss 2-scale convergence in our setting. In Section 14 we provide a roadmap for the proof of Theorem 4.1. In Sections 15 and 17 we develop the basis of our cut-off procedure. In Sections 16 and 18 we study bounded families of functions in the Hilbert space $H^1_{\omega,\varepsilon}$ (cf. Definition 3.2). Having developed all the necessary machinery, in Sections 19 and 20 we prove respectively Theorems 4.1 and 4.4. Finally, several technical facts have been collected in Appendixes A, B, C, D, E, F.

2. Setting

In this section we introduce the basic concepts in our modelisation: the group $\mathbb{G}$ ($\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$) acting on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and on the space $\mathbb{R}^d$; the random $\mathbb{G}$-stationary purely atomic measure μ_ω on $\mathbb{R}^d$ and the family of transition rates $r_{x,y}(\omega)$. We also list our main assumptions, given by (A1), ..., (A8) below. In Section 4 we will introduce assumption (A9) for Theorem 4.4.

First we fix some basic notation. We denote by $e_1, \dots, e_d$ the canonical basis of $\mathbb{R}^d$. We denote by $\ell(A)$ the Lebesgue measure of the Borel set $A \subset \mathbb{R}^d$. The standard scalar product of $a, b \in \mathbb{R}^d$ is denoted by $a \cdot b$. Given a measure ν, we denote by $\langle \cdot, \cdot \rangle_\nu$ the scalar product in $L^2(\nu)$. Given a topological space W, without further mention, W will be thought of as measurable space endowed with the σ-algebra $\mathcal{B}(W)$ of its Borel subsets.

2.1. *Actions of the group* $\mathbb{G}$

$\mathbb{G}$ will be the abelian group $\mathbb{R}^d$ or $\mathbb{Z}^d$, which are endowed with the standard Euclidean topology and the discrete topology, respectively.

• *Action of* $\mathbb{G}$ *on* $(\Omega, \mathcal{F}, \mathcal{P})$. The action of $\mathbb{G}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ is given by a family of measurable maps $(\theta_g)_{g \in \mathbb{G}}$ with $\theta_g : \Omega \to \Omega$ such that

(P1) $\theta_0 = \mathbb{1}$,
(P2) $\theta_g \circ \theta_{g'} = \theta_{g+g'}$ for all $g, g' \in \mathbb{G}$,
(P3) the map $\mathbb{G} \times \Omega \ni (g, \omega) \mapsto \theta_g \omega \in \Omega$ is measurable,
(P4) $\mathcal{P}$ is left invariant by the $\mathbb{G}$-action, i.e. $\mathcal{P} \circ \theta_g^{-1} = \mathcal{P}$ for all $g \in \mathbb{G}$.

The last property (P4) corresponds to the so-called $\mathbb{G}$-stationarity of $\mathcal{P}$. A subset $A \subset \Omega$ is called translation invariant if $\theta_g A = A$ for all $g \in \mathbb{G}$.

• *Action of* $\mathbb{G}$ *on* $\mathbb{R}^d$. The action $(\tau_g)_{g\in\mathbb{G}}$ of $\mathbb{G}$ on $\mathbb{R}^d$ is given by translations. More precisely, we assume that, for a basis $v_1, \dots, v_d$ in $\mathbb{R}^d$,

$$\tau_g x = x + g_1 v_1 + \cdots + g_d v_d \quad \forall x \in \mathbb{R}^d, g = (g_1, \dots, g_d) \in \mathbb{G}. \tag{1}$$

Equivalently, by thinking of g as a column vector, we can write

$$\tau_g x = x + Vg, \quad V := [v_1|v_2|\dots|v_d] \tag{2}$$

(V is the $d \times d$-matrix with columns $v_1, v_2, \dots, v_d$).

• *Orbits and representatives*. We set

$$\Delta := \{t_1 v_1 + \cdots + t_d v_d : (t_1, \dots, t_d) \in [0,1)^d\}. \tag{3}$$

Given $x \in \mathbb{R}^d$, the orbit of x is set $\{\tau_g x : g \in \mathbb{G}\}$.

If $\mathbb{G} = \mathbb{R}^d$, then the orbit of the origin of $\mathbb{R}^d$ equals $\mathbb{R}^d$ and we set

$$g(x) := g \quad \text{if } x = \tau_g 0. \tag{4}$$

When $V = \mathbb{I}$ (as in many applications), we have $\tau_g x = x + g$ and $g(x) = x$.

If $\mathbb{G} = \mathbb{Z}^d$, Δ is a set of orbit representatives for the action $(\tau_g)_{g\in\mathbb{G}}$. We introduce the functions $\beta : \mathbb{R}^d \to \Delta$ and $g : \mathbb{R}^d \to \mathbb{G}$ as follows:

$$x = \tau_g a \quad \text{and} \quad a \in \Delta \quad \Longrightarrow \quad \beta(x) := a, \quad g(x) := g. \tag{5}$$

• *Induced action of* $\mathbb{G}$ *on* $\mathcal{M}$. We denote by $\mathcal{M}$ the metric space of locally finite nonnegative measures on $\mathbb{R}^d$ [10, App. A2.6]. The action of $\mathbb{G}$ on $\mathbb{R}^d$ naturally induces an action of $\mathbb{G}$ on $\mathcal{M}$, which (with some abuse of notation) we still denote by $(\tau_g)_{g\in\mathbb{G}}$. In particular, $\tau_g : \mathcal{M} \to \mathcal{M}$ is given by

$$\tau_g \mathfrak{m}(A) := \mathfrak{m}(\tau_g A), \quad \forall A \in \mathcal{B}(\mathbb{R}^d). \tag{6}$$

Setting $\mathfrak{m}[f] := \int f \, d\mathfrak{m}$ for all $\mathfrak{m} \in \mathcal{M}$, we get $\tau_g \mathfrak{m}[f] = \int f(\tau_{-g} x) \, d\mathfrak{m}(x)$.

2.2. $\mathbb{G}$-*Stationary random measure* μ_ω

We suppose now to have a random locally finite nonnegative measure μ_ω on $\mathbb{R}^d$, i.e. a measurable map $\Omega \ni \omega \mapsto \mu_\omega \in \mathcal{M}$. We assume that μ_ω is a purely atomic (i.e. pure point) measure with locally finite support for any $\omega \in \Omega$. In particular, we have

$$\mu_\omega = \sum_{x \in \hat{\omega}} n_x(\omega)\delta_x, \qquad n_x(\omega) := \mu_\omega(\{x\}), \qquad \hat{\omega} := \{x \in \mathbb{R}^d : n_x(\omega) > 0\} \tag{7}$$

and $\hat{\omega}$ is a locally finite set. The map $\omega \mapsto \hat{\omega}$ then defines a simple point process. The fundamental relation between the above two actions of $\mathbb{G}$ and the random measure μ_ω is given by the assumption that μ_ω is $\mathbb{G}$-stationary (cf. [26, Section 2.4.2], [27, Eq. (21)]): $\mu_{\theta_g \omega} = \tau_g \mu_\omega$ for $\mathcal{P}$-a.a. $\omega \in \Omega$ and for all $g \in \mathbb{G}$.

2.3. *Palm distribution*

We introduce the Palm distribution $\mathcal{P}_0$ by distinguishing between two main cases and a special subcase. We will write $\mathbb{E}[\cdot]$ and $\mathbb{E}_0[\cdot]$ for the expectation w.r.t. $\mathcal{P}$ and $\mathcal{P}_0$, respectively.[2]

• *Case* $\mathbb{G} = \mathbb{R}^d$. The intensity of the random measure μ_ω is defined as

$$m := \mathbb{E}[\mu_\omega([0,1)^d)]. \tag{8}$$

[2] With some abuse, when f has a complex form, we will write $\mathbb{E}[f(\omega)]$ instead of $\mathbb{E}[f]$, and similarly for $\mathbb{E}_0[f(\omega)]$.

As stated below, m is assumed to be finite and positive. By the $\mathbb{G}$-stationarity of $\mathcal{P}$ we have $m\ell(U) = \mathbb{E}[\mu_\omega(U)]$ for any $U \in \mathcal{B}(\mathbb{R}^d)$. Then (see e.g. [11,26,29] and Appendixes B, C) the Palm distribution $\mathcal{P}_0$ is the probability measure on $(\Omega, \mathcal{F})$ such that, for any $U \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \ell(U) < \infty$,

$$\mathcal{P}_0(A) := \frac{1}{m\ell(U)} \int_\Omega d\mathcal{P}(\omega) \int_U d\mu_\omega(x) \mathbb{1}_A(\theta_{g(x)}\omega), \quad \forall A \in \mathcal{F}. \tag{9}$$

$\mathcal{P}_0$ has support inside the set $\Omega_0 := \{\omega \in \Omega : n_0(\omega) > 0\}$ (see Remark 2.1).

- *Case* $\mathbb{G} = \mathbb{Z}^d$. The intensity of the random measure μ_ω is defined as

$$m := \mathbb{E}\big[\mu_\omega(\Delta)\big]/\ell(\Delta). \tag{10}$$

By the $\mathbb{G}$-stationarity of $\mathcal{P}$, $m\ell(A) = \mathbb{E}[\mu_\omega(A)]$ for any $A \in \mathcal{B}(\mathbb{R}^d)$ which is an overlap of translated cells $\tau_g\Delta$ with $g \in \mathbb{G}$. As stated below, m is assumed to be finite and positive. Then (see Appendixes B, C) the Palm distribution $\mathcal{P}_0$ is the probability measure on $(\Omega \times \Delta, \mathcal{F} \otimes \mathcal{B}(\Delta))$ such that

$$\mathcal{P}_0(A) := \frac{1}{m\,\ell(\Delta)} \int_\Omega d\mathcal{P}(\omega) \int_\Delta d\mu_\omega(x) \mathbb{1}_A(\omega, x), \quad \forall A \in \mathcal{F} \otimes \mathcal{B}(\Delta). \tag{11}$$

$\mathcal{P}_0$ has support inside $\Omega_0 := \{(\omega, x) \in \Omega \times \Delta : n_x(\omega) > 0\}$ (see Remark 2.1).

- *Special discrete case:* $\mathbb{G} = \mathbb{Z}^d$, $V = \mathbb{I}$ *and* $\hat\omega \subset \mathbb{Z}^d$ $\forall \omega \in \Omega$. This is a subcase of the case $\mathbb{G} = \mathbb{Z}^d$ and in what follows we will call it simply *special discrete case*. As this case is very frequent in discrete probability, we discuss it apart pointing out some simplifications. As $\Delta = [0,1)^d$ intersects $\mathbb{Z}^d$ only at the origin, $\mathcal{P}_0$ (see case $\mathbb{G} = \mathbb{Z}^d$) is concentrated on $\{\omega \in \Omega : n_0(\omega) > 0\} \times \{0\}$. Hence we can think of $\mathcal{P}_0$ as a probability measure concentrated on the set $\Omega_0 := \{\omega \in \Omega : n_0(\omega) > 0\}$. Formulas (10) and (11) then read

$$m := \mathbb{E}[n_0], \qquad \mathcal{P}_0(A) := \mathbb{E}[n_0 \mathbb{1}_A]/\mathbb{E}[n_0] \quad \forall A \in \mathcal{F}. \tag{12}$$

In what follows, when treating the special discrete case, we will use the above identifications without explicit mention.

2.4. *Rate jumps and assumptions*

All objects introduced so far concern the environment and not the particle dynamics. The latter is encoded in the measurable function

$$r : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, x, y) \mapsto r_{x,y}(\omega) \in [0, +\infty). \tag{13}$$

As it will be clear below, only the value of $r_{x,y}(\omega)$ with $x \neq y$ in $\hat\omega$ will be relevant. Hence, without loss of generality, we take

$$r_{x,x}(\omega) \equiv 0, \qquad r_{x,y}(\omega) \equiv 0 \quad \forall \{x, y\} \not\subset \hat\omega. \tag{14}$$

By identifying the support $\hat\omega$ of μ_ω with the measure $\sum_{x \in \hat\omega} \delta_x$, we define the function $\lambda_k : \Omega_0 \to [0, +\infty]$ (for $k \in [0, \infty)$) as follows:

$$\begin{cases} \lambda_k(\omega) := \int_{\mathbb{R}^d} d\hat\omega(x) r_{0,x}(\omega) |x|^k \\ \Omega_0 = \{\omega \in \Omega : n_0(\omega) > 0\} \end{cases} \quad \text{Case } \mathbb{G} = \mathbb{R}^d \text{ and special discrete case,}$$
$$\begin{cases} \lambda_k(\omega, a) := \int_{\mathbb{R}^d} d\hat\omega(x) r_{a,x}(\omega) |x - a|^k \\ \Omega_0 := \{(\omega, x) \in \Omega \times \Delta : n_x(\omega) > 0\} \end{cases} \quad \text{Case } \mathbb{G} = \mathbb{Z}^d. \tag{15}$$

We list our assumptions (including the ones introduced above).

Assumptions. *We make the following assumptions where Ω_* is some translation invariant measurable subset of Ω with $\mathcal{P}(\Omega_*) = 1$:*

(A1) *$\mathcal{P}$ is stationary and ergodic w.r.t. the action $(\theta_g)_{g \in \mathbb{G}}$ of the group $\mathbb{G}$;*
(A2) *the $\mathcal{P}$-intensity m of the random measure μ_ω is finite and positive*;

(A3) *for all* $\omega \in \Omega_*$ *and for all* $g \neq g'$ *in* $\mathbb{G}$, *it holds* $\theta_g \omega \neq \theta_{g'}\omega$;
(A4) *for all* $\omega \in \Omega_*$, *for all* $g \in \mathbb{G}$ *and* $x, y \in \mathbb{R}^d$, *it holds*

$$\mu_{\theta_g \omega} = \tau_g \mu_\omega, \tag{16}$$

$$r_{x,y}(\theta_g \omega) = r_{\tau_g x, \tau_g y}(\omega); \tag{17}$$

(A5) *for all* $\omega \in \Omega_*$ *and for all* $x, y \in \hat{\omega}$, *it holds*

$$c_{x,y}(\omega) := n_x(\omega) r_{x,y}(\omega) = n_y(\omega) r_{y,x}(\omega); \tag{18}$$

(A6) *for all* $\omega \in \Omega_*$ *and for all* $x, y \in \hat{\omega}$, *there exists a path* $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ *such that* $x_i \in \hat{\omega}$ *and* $r_{x_i, x_{i+1}}(\omega) > 0$ *for all* $i = 0, 1, \dots, n-1$;
(A7) $\lambda_0, \lambda_2 \in L^1(\mathcal{P}_0)$;
(A8) $L^2(\mathcal{P}_0)$ *is separable.*

We conclude this section with some comments on the above assumptions. We observe that, by Zero-Infinity Dichotomy (see [11, Proposition 12.1.VI]) and Assumptions (A1), (A2), for $\mathcal{P}$-a.a. ω the support $\hat{\omega}$ of μ_ω is an infinite set.

For $k = 3, 4, 5, 6$ we call Ω_k the set of environments ω satisfying the properties stated in Assumption (Ak) (for example $\Omega_3 := \{\omega \in \Omega : \theta_g \omega \neq \theta_{g'}\omega \ \forall g \neq g' \text{ in } \mathbb{G}\}$). All Ω_k's are always translation invariant. If $\mathbb{G} = \mathbb{Z}^d$, they are also measurable. Therefore, for $\mathbb{G} = \mathbb{Z}^d$, we can simply take $\Omega_* := \bigcap_{k=3}^6 \Omega_k$, which is automatically a measurable and translation invariant set. When $\mathbb{G} = \mathbb{R}^d$ and, as common in applications, $\Omega_4 = \Omega_5 = \Omega_6 = \Omega$, we can prove that Ω_3 is measurable. Therefore, in the above case, we can simply take $\Omega_* := \Omega_3$, which is automatically a measurable and translation invariant set. The above proof and the discussion of further cases are provided in Appendix A.

We point out that (A3) is usually a rather superfluous assumption. Indeed, by free one can add some randomness enlarging Ω to assure (A3) (similarly to [12, Remark 4.2-(i)]). For example, if $(\Omega, \mathcal{F}, \mathcal{P})$ describes a random simple point process on $\mathbb{R}^d$ obtained by periodizing a random simple point process on $[0,1]^d$, then to gain (A3) it would be enough to mark points by i.i.d. random variables with non-degenerate distribution.

Considering the random walk X_t^ω introduced in Section 3.3 below, (A5) and (A6) correspond to reversibility of the measure μ_ω and to irreducibility for all $\omega \in \Omega_*$.

We observe that (A7) implies

$$\mathbb{E}_0[\lambda_1] \leq \mathbb{E}_0[\lambda_0] + \mathbb{E}_0[\lambda_2] < +\infty. \tag{19}$$

We point out that, by [6, Theorem 4.13], (A8) is fulfilled if $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is a separable measure space where $\mathcal{F}_0 := \{A \cap \Omega_0 : A \in \mathcal{F}\}$ (i.e. there is a countable family $\mathcal{G} \subset \mathcal{F}_0$ such that the σ-algebra $\mathcal{F}_0$ is generated by $\mathcal{G}$). For example, if Ω_0 is a separable metric space and $\mathcal{F}_0 = \mathcal{B}(\Omega_0)$ (which is valid if Ω is a separable metric space and $\mathcal{F} = \mathcal{B}(\Omega)$) then (cf. [6, p. 98]) $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is a separable measure space and (A8) is valid.

Remark 2.1. We report some useful identities restricting to the ω's which fulfill the properties in (A4) and (A5) (i.e. $\omega \in \Omega_4 \cap \Omega_5$ with the above notation). For all $a, b \in \mathbb{R}^d$ it holds $\tau_{g(a)} b = a + b$. For all $x \in \mathbb{R}^d$ and $g \in \mathbb{G}$ it holds $n_x(\theta_g \omega) = n_{\tau_g x}(\omega)$. Given $a, b \in \hat{\omega}$ we have $n_a(\omega) = n_0(\theta_{g(a)}\omega)$, $r_{a,b}(\omega) = r_{0,b-a}(\theta_{g(a)}\omega)$, $c_{a,b}(\omega) = c_{0,b-a}(\theta_{g(a)}\omega)$ and $\widehat{\theta_{g(a)}\omega} = \hat{\omega} - a$.

3. Massive Poisson equation, random walk, homogenized matrix, homogenized equation, convergence in $L^2(\mu_\omega^\varepsilon)$, $L^2(\nu_\omega^\varepsilon)$

Recall that we identify the support $\hat{\omega}$ of μ_ω with the measure $\sum_{x \in \hat{\omega}} \delta_x$.

3.1. *Measures* μ_ω^ε *and* ν_ω^ε

Given $\varepsilon > 0$ and $\omega \in \Omega$, we define μ_ω^ε as the measure on $\mathbb{R}^d$ (cf. (7))

$$\mu_\omega^\varepsilon := \sum_{x \in \hat{\omega}} \varepsilon^d n_x(\omega) \delta_{\varepsilon x}. \tag{20}$$

We write $\langle \cdot, \cdot \rangle_{\mu_\omega^\varepsilon}$ for the scalar product in $L^2(\mu_\omega^\varepsilon)$. By ergodicity, for $\mathcal{P}$-a.a. ω the measure μ_ω^ε converges vaguely to $m\,dx$, where m is the intensity of μ_ω. The above convergence is a special case of the following stronger ergodicity result, where the interplay between the microscale and macroscale emerges:

Proposition 3.1. *Let $f:\Omega_0\to\mathbb{R}$ be a measurable function with $\|f\|_{L^1(\mathcal{P}_0)}<\infty$. Then there exists a translation invariant measurable subset $\mathcal{A}[f]\subset\Omega$ such that $\mathcal{P}(\mathcal{A}[f])=1$ and such that, for any $\omega\in\mathcal{A}[f]$ and any $\varphi\in C_c(\mathbb{R}^d)$, it holds*

$$\lim_{\varepsilon\downarrow 0}\int d\mu_\omega^\varepsilon(x)\varphi(x)f(\theta_{g(x/\varepsilon)}\omega)=\int dx\, m\varphi(x)\cdot\mathbb{E}_0[f]. \tag{21}$$

Note that in (21) x/ε and x are respectively at the microscopic and macroscopic scale. The proof of Proposition 3.1 can be obtained by standard arguments from [42] (see also [17, App. B]).

We define ν_ω^ε as the measure on $\mathbb{R}^d\times\mathbb{R}^d$ given by (cf. Remark 2.1)

$$\nu_\omega^\varepsilon:=\varepsilon^d\int d\hat\omega(a)\int d\hat\omega(b)c_{a,b}(\omega)\delta_{(\varepsilon a,b-a)}=\int d\mu_\omega^\varepsilon(x)\int d(\widehat{\theta_{g(x/\varepsilon)}\omega})(z)r_{0,z}(\theta_{g(x/\varepsilon)}\omega)\delta_{(x,z)}. \tag{22}$$

We write $\langle\cdot,\cdot\rangle_{\nu_\omega^\varepsilon}$ for the scalar product in $L^2(\nu_\omega^\varepsilon)$.

3.2. *Microscopic gradient and space $H^1_{\omega,\varepsilon}$*

Given $\omega\in\Omega$ and a real function v whose domain contains $\varepsilon\hat\omega$, we define the *microscopic gradient* $\nabla_\varepsilon v$ as the function

$$\nabla_\varepsilon v(x,z)=\frac{v(x+\varepsilon z)-v(x)}{\varepsilon},\quad x \text{ and } x+\varepsilon z\in\varepsilon\hat\omega. \tag{23}$$

By Remark 2.1, given $x\in\varepsilon\hat\omega$, $x+\varepsilon z\in\varepsilon\hat\omega$ if and only if $z\in\widehat{\theta_{g(x/\varepsilon)}\omega}$.

Definition 3.2. We say that $v\in H^1_{\omega,\varepsilon}$ if $v\in L^2(\mu_\omega^\varepsilon)$ and $\nabla_\varepsilon v\in L^2(\nu_\omega^\varepsilon)$. Moreover, we endow the space $H^1_{\omega,\varepsilon}$ with the scalar product $\langle v,w\rangle_{H^1_{\omega,\varepsilon}}:=\langle v,w\rangle_{\mu_\omega^\varepsilon}+\langle\nabla v,\nabla w\rangle_{\nu_\omega^\varepsilon}$ and write $\|\cdot\|_{H^1_{\omega,\varepsilon}}$ for the norm in $H^1_{\omega,\varepsilon}$.

It is simple to check that $H^1_{\omega,\varepsilon}$ is a Hilbert space.

3.3. *Space $H^{1,\mathrm{f}}_{\omega,\varepsilon}$, massive Poisson equation, self-adjoint operator $\mathbb{L}^\varepsilon_\omega$, random walks X_t^ω and $\varepsilon X^\omega_{\varepsilon^{-2}t}$*

We introduce the set

$$\Omega_1:=\Big\{\omega\in\Omega: r_x(\omega):=\sum_{y\in\hat\omega}r_{x,y}(\omega)<\infty\ \forall x\in\hat\omega,\ c_{x,y}(\omega)=c_{y,x}(\omega)\ \forall x,y\in\hat\omega\Big\}. \tag{24}$$

Then the set $\Omega_1\subset\Omega$ is translation invariant and it holds $\mathcal{P}(\Omega_1=1)$ as can be checked at cost to reduce to the case $\mathbb{G}=\mathbb{R}^d$ by the method outlined in Section 6 and applying Corollary 7.2 below together with the bound $\mathbb{E}_0[\lambda_0]<\infty$. We restrict to $\omega\in\Omega_1$. We call $\mathcal{C}(\varepsilon\hat\omega)$ the set of real functions on $\varepsilon\hat\omega$ which are zero outside a finite set. Then $\mathcal{C}(\varepsilon\hat\omega)\subset H^1_{\omega,\varepsilon}$. Indeed, by symmetry of $c_{x,y}(\omega)$, for any $v:\varepsilon\hat\omega\to\mathbb{R}$ it holds

$$\varepsilon^2\int d\nu_\omega^\varepsilon(x,z)\nabla_\varepsilon v(x,z)^2\le 2\varepsilon^d\int d\hat\omega(x)\int d\hat\omega(y)c_{x,y}(\omega)\big[v(\varepsilon x)^2+v(\varepsilon y)^2\big]=4\varepsilon^d\int d\mu_\omega(x)r_x(\omega)v(\varepsilon x)^2. \tag{25}$$

When $v\in\mathcal{C}(\varepsilon\hat\omega)$, the last term is a finite sum (as $\hat\omega$ is locally finite and $\omega\in\Omega_1$).

Definition 3.3. Given $\omega\in\Omega_1$, the Hilbert space $H^{1,\mathrm{f}}_{\omega,\varepsilon}$ is defined as the closure of $\mathcal{C}(\varepsilon\hat\omega)$ inside the Hilbert space $H^1_{\omega,\varepsilon}$.

The index f in $H^{1,\mathrm{f}}_{\omega,\varepsilon}$ refers to finite support, as the functions in $\mathcal{C}(\varepsilon\hat\omega)$ are the ones with finite support on $\varepsilon\hat\omega$. The symmetric form $(f,h)\mapsto\frac12\langle\nabla_\varepsilon f,\nabla_\varepsilon h\rangle_{\nu_\omega^\varepsilon}$ with domain $H^{1,\mathrm{f}}_{\omega,\varepsilon}\subset L^2(\mu_\omega^\varepsilon)$ is a regular Dirichlet form with core $\mathcal{C}(\varepsilon\hat\omega)$ (consider [25, Example 1.2.5] with E, $q_{x,y}$, k_x, m^0, m there defined as $E:=\varepsilon\hat\omega$, $q_{x,y}:=\varepsilon^{-2}r_{x/\varepsilon,y/\varepsilon}$ for $x\neq y$ in $\varepsilon\hat\omega$, $q_{x,x}:=-\sum_{y:y\neq x}q_{x,y}$, $k_x:=0$, $m^0_x:=\varepsilon^d n_{x/\varepsilon}(\omega)$, $m:=m^0$). In particular, there exists a unique nonpositive definite self-adjoint operator $\mathbb{L}^\varepsilon_\omega$ in $L^2(\mu^\varepsilon_\omega)$ such that $H^{1,\mathrm{f}}_{\omega,\varepsilon}$ equals the domain of $\sqrt{-\mathbb{L}^\varepsilon_\omega}$ and $\frac12\langle\nabla_\varepsilon f,\nabla_\varepsilon f\rangle_{\nu_\omega^\varepsilon}=\|\sqrt{-\mathbb{L}^\varepsilon_\omega}f\|^2_{L^2(\mu^\varepsilon_\omega)}$ for any $f\in H^{1,\mathrm{f}}_{\omega,\varepsilon}$ (see [25, Theorem 1.3.1]). Note that, if $h\in\mathcal{D}(\mathbb{L}^\varepsilon_\omega)\subset\mathcal{D}(\sqrt{-\mathbb{L}^\varepsilon_\omega})=H^{1,\mathrm{f}}_{\omega,\varepsilon}$ and $f\in H^{1,\mathrm{f}}_{\omega,\varepsilon}$, we have

$$\big\langle f,-\mathbb{L}^\varepsilon_\omega h\big\rangle_{\mu^\varepsilon_\omega}=\frac{\varepsilon^{d-2}}{2}\int d\hat\omega(x)\int d\hat\omega(y)c_{x,y}(\omega)\big(f(\varepsilon y)-f(\varepsilon x)\big)\big(h(\varepsilon y)-h(\varepsilon x)\big)=\frac12\langle\nabla_\varepsilon f,\nabla_\varepsilon h\rangle_{\nu^\varepsilon_\omega}. \tag{26}$$

Identity (26) suggests a weak formulation of the equation $-\mathbb{L}^\varepsilon_\omega u+\lambda u=f$:

Definition 3.4. Let $\omega\in\Omega_1$. Given $f\in L^2(\mu^\varepsilon_\omega)$ and $\lambda>0$, a weak solution u of the equation

$$-\mathbb{L}^\varepsilon_\omega u+\lambda u=f \tag{27}$$

is a function $u\in H^{1,\mathrm{f}}_{\omega,\varepsilon}$ such that

$$\frac{1}{2}\langle\nabla_\varepsilon v,\nabla_\varepsilon u\rangle_{\nu^\varepsilon_\omega}+\lambda\langle v,u\rangle_{\mu^\varepsilon_\omega}=\langle v,f\rangle_{\mu^\varepsilon_\omega}\quad\forall v\in H^{1,\mathrm{f}}_{\omega,\varepsilon}. \tag{28}$$

By the Lax–Milgram theorem [6], given $f\in L^2(\mu^\varepsilon_\omega)$ the weak solution u of (27) exists and is unique. As any $\lambda>0$ belongs to the resolvent set of the nonpositive self-adjoint operator $\mathbb{L}^\varepsilon_\omega$, equation (27) is equivalent to $u=(-\mathbb{L}^\varepsilon_\omega+\lambda)^{-1}f$.

We point out that (cf. [25, Lemma 1.3.2 and Exercise 4.4.1]) the self-adjoint operator $\mathbb{L}^\varepsilon_\omega$ is the infinitesimal generator of the strongly continuous Markov semigroup in $L^2(\mu^\varepsilon_\omega)$ associated to the diffusively rescaled random walk $(\varepsilon X^\omega_{\varepsilon^{-2}t})_{t\geq 0}$, X^ω_t being the random walk on $\hat\omega$ with probability rate $r_{x,y}(\omega)$ for a jump from x to y in $\hat\omega$ (possibly with explosion). We can indeed show (cf. Appendix D) that, for $\mathcal{P}$-a.a. ω, explosion does not take place (one needs weaker assumptions for this result):

Lemma 3.5. *Assume* (A1), …, (A6), $\lambda_0\in L^1(\mathcal{P}_0)$ (*for some translation invariant measurable set* Ω_* *with* $\mathcal{P}(\Omega_*)=1$). *Then there exists a translation invariant measurable set* $\mathcal{A}\subset\Omega$ *with* $\mathcal{P}(\mathcal{A})=1$ *such that, for all* $\omega\in\mathcal{A}$, (i) $r_x(\omega):=\sum_{y\in\hat\omega}r_{x,y}(\omega)\in(0,+\infty)$ $\forall x\in\hat\omega$, (ii) *the continuous-time Markov chain on* $\hat\omega$ *starting at any* $x_0\in\hat\omega$, *with waiting time parameter* $r_x(\omega)$ *at* $x\in\hat\omega$ *and with probability* $r_{x,y}(\omega)/r_x(\omega)$ *for a jump from* x *to* y, *is non-explosive.*

3.4. *Homogenized matrix D and homogenized equation*

Definition 3.6. We define the *homogenized matrix* D as the unique $d\times d$ symmetric matrix such that:

- *Case* $\mathbb{G}=\mathbb{R}^d$ *and special discrete case*[3]

$$a\cdot Da=\inf_{f\in L^\infty(\mathcal{P}_0)}\frac{1}{2}\int_{\Omega_0}d\mathcal{P}_0(\omega)\int_{\mathbb{R}^d}d\hat\omega(x)r_{0,x}(\omega)\big(a\cdot x-\nabla f(\omega,x)\big)^2, \tag{29}$$

  for any $a\in\mathbb{R}^d$, where $\nabla f(\omega,x):=f(\theta_{g(x)}\omega)-f(\omega)$.
- *Case* $\mathbb{G}=\mathbb{Z}^d$

$$a\cdot Da=\inf_{f\in L^\infty(\mathcal{P}_0)}\frac{1}{2}\int_{\Omega\times\Delta}d\mathcal{P}_0(\omega,x)\int_{\mathbb{R}^d}d\hat\omega(y)r_{x,y}(\omega)\big(a\cdot(y-x)-\nabla f(\omega,x,y-x)\big)^2, \tag{30}$$

  for any $a\in\mathbb{R}^d$, where $\nabla f(\omega,x,y-x):=f(\theta_{g(y)}\omega,\beta(y))-f(\omega,x)$.

Since $\lambda_2\in L^1(\mathcal{P}_0)$ by (A7), $a\cdot Da$ is indeed finite for any $a\in\mathbb{R}^d$.

Remark 3.7. Consider the new random measure $\tilde\mu_\omega:=\hat\omega$ and the new rates $\tilde r_{x,y}(\omega):=c_{x,y}(\omega)$, under the assumption that the intensity $\tilde m$ of $\tilde\mu_\omega$ is finite and positive. One can then check that Assumptions (A1), …, (A8) imply the analogous assumptions in the new setting with $\tilde\mu_\omega$ and $\tilde r_{x,y}(\omega)$. Moreover, writing $\tilde{\mathcal{P}}_0$ and $\tilde D$ for the Palm distribution and the effective homogenized matrix in the new setting respectively, one easily gets that $m\,d\mathcal{P}_0(\omega)=\tilde m\,n_0(\omega)\,d\tilde{\mathcal{P}}_0(\omega)$ for $\mathbb{G}=\mathbb{R}^d$ and in the special discrete case, $m\,d\mathcal{P}_0(\omega,x)=\tilde m\,n_x(\omega)\,d\tilde{\mathcal{P}}_0(\omega,x)$ for $\mathbb{G}=\mathbb{Z}^d$ and $mD=\tilde m\tilde D$. As a consequence we get that $m/\tilde m=\tilde{\mathbb{E}}_0[n_0]$, $\tilde{\mathbb{E}}_0[\cdot]$ being the expectation w.r.t. $\tilde{\mathcal{P}}_0$.

Definition 3.8. We fix an orthonormal basis $\mathfrak{e}_1,\dots,\mathfrak{e}_d$ of eigenvectors of D (which is symmetric) and we let γ_i be the eigenvalue of $\mathfrak{e}_i$. At cost of a relabelling, $\gamma_1,\dots,\gamma_{d_*}$ are all positive and $\gamma_{d_*+1},\dots,\gamma_d$ are all zero. In particular, $d_*\in\{0,1,\dots,d\}$ and, if D is strictly positive, it holds $d_*=d$. Given a unit vector v, we write $\partial_v f$ for the weak derivative of f along the direction v (if $v=e_i$, then $\partial_v f$ is simply the standard weak derivative $\partial_i f$).

[3] In the special discrete case, $\int_{\mathbb{R}^d}d\hat\omega(x)$ can be replaced by a sum among $x\in\mathbb{Z}^d$.

Definition 3.9. We introduce the space $H^1_*(m\,dx)$ given by the functions $f \in L^2(m\,dx)$ such that the weak derivative $\partial_{\mathfrak{e}_i} f$ belongs to $L^2(m\,dx)$ $\forall i \in \{1, \dots, d_*\}$. We endow $H^1_*(m\,dx)$ with the scalar product $\langle f, h\rangle_{H^1_*(m\,dx)} := \langle f, h\rangle_{L^2(m\,dx)} + \sum_{i=1}^{d_*} \langle \partial_{\mathfrak{e}_i} f, \partial_{\mathfrak{e}_i} h\rangle_{L^2(m\,dx)}$. Moreover, given $f \in H^1_*(m\,dx)$, we set

$$\nabla_* f := \sum_{i=1}^{d_*} (\partial_{\mathfrak{e}_i} f)\mathfrak{e}_i \in L^2(m\,dx)^d. \tag{31}$$

We stress that, if $\mathfrak{e}_i = e_i$ for all $i = 1, \dots, d_*$ (as in many applications), then

$$\nabla_* f := (\partial_1 f, \dots, \partial_{d_*} f, 0, \dots, 0) \in L^2(m\,dx)^d. \tag{32}$$

We point out that $H^1_*(m\,dx)$ is an Hilbert space (adapt the standard proof for $H^1(dx)$). Moreover, $C_c^\infty(\mathbb{R}^d)$ is dense in $H^1_*(m\,dx)$ (adapt the arguments in the proof of [6, Thm. 9.2]).

We now move to the effective homogenized equation, where D denotes the homogenized matrix introduced in Definition 3.6.

Definition 3.10. Given $f \in L^2(m\,dx)$ and $\lambda > 0$, a weak solution u of the equation

$$-\nabla_* \cdot D\nabla_* u + \lambda u = f \tag{33}$$

is a function $u \in H^1_*(m\,dx)$ such that

$$\int D\nabla_* v(x) \cdot \nabla_* u(x)\,dx + \lambda \int v(x)u(x)\,dx = \int v(x) f(x)\,dx, \quad \forall v \in H^1_*(m\,dx). \tag{34}$$

Again, by the Lax–Milgram theorem, given $f \in L^2(m\,dx)$ the weak solution u of (27) exists and is unique.

3.5. *Weak and strong convergence for $L^2(\mu^\varepsilon_\omega)$ and $L^2(\nu^\varepsilon_\omega)$*

Definition 3.11. Fix $\omega \in \Omega$ and a family of ε-parametrized functions $v_\varepsilon \in L^2(\mu^\varepsilon_\omega)$. We say that the family $\{v_\varepsilon\}$ *converges weakly* to the function $v \in L^2(m\,dx)$, and write $v_\varepsilon \rightharpoonup v$, if

$$\limsup_{\varepsilon \downarrow 0} \|v_\varepsilon\|_{L^2(\mu^\varepsilon_\omega)} < +\infty \tag{35}$$

and

$$\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_\omega(x) v_\varepsilon(x)\varphi(x) = \int dx\, m v(x)\varphi(x) \tag{36}$$

for all $\varphi \in C_c(\mathbb{R}^d)$. We say that the family $\{v_\varepsilon\}$ *converges strongly* to $v \in L^2(m\,dx)$, and write $v_\varepsilon \to v$, if in addition to (35) it holds

$$\lim_{\varepsilon \downarrow 0} \int d\mu^\varepsilon_\omega(x) v_\varepsilon(x) g_\varepsilon(x) = \int dx\, m v(x) g(x), \tag{37}$$

for any family of functions $g_\varepsilon \in L^2(\mu^\varepsilon_\omega)$ weakly converging to $g \in L^2(m\,dx)$.

In general, when (35) is satisfied, one simply says that the family $\{v_\varepsilon\}$ is bounded.

Remark 3.12. One can prove (cf. [43, Prop. 1.1]) that $v_\varepsilon \to v$ if and only if $v_\varepsilon \rightharpoonup v$ and $\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} v_\varepsilon(x)^2\, d\mu^\varepsilon_\omega(x) = \int_{\mathbb{R}^d} v(x)^2 m\,dx$.

We introduce now a special form of convergence of microscopic gradients (note that the testing objects are gradients as well).

Definition 3.13. Fix $\omega \in \Omega$ and a family of ε-parametrized functions $v_\varepsilon \in L^2(\mu_\omega^\varepsilon)$. We say that the family $\{\nabla_\varepsilon v_\varepsilon\}$ *converges weakly* to the vector-valued function w belonging to the product space $L^2(m\,dx)^d$ and with values in $\mathbb{R}^d$, and write $\nabla_\varepsilon v_\varepsilon \rightharpoonup w$, if

$$\limsup_{\varepsilon \downarrow 0} \|\nabla_\varepsilon v_\varepsilon\|_{L^2(\nu_\omega^\varepsilon)} < +\infty \tag{38}$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2} \int d\nu_\omega^\varepsilon(x,z) \nabla_\varepsilon v_\varepsilon(x,z) \nabla_\varepsilon \varphi(x,z) = \int dx\, m D w(x) \cdot \nabla_* \varphi(x) \tag{39}$$

for all $\varphi \in C_c^1(\mathbb{R}^d)$. We say that family $\{\nabla_\varepsilon v_\varepsilon\}$ *converges strongly* to w as above, and write $\nabla_\varepsilon v_\varepsilon \to w$, if in addition to (38) it holds

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2} \int d\nu_\omega^\varepsilon(x,z) \nabla_\varepsilon v_\varepsilon(x,z) \nabla_\varepsilon g_\varepsilon(x,z) = \int dx\, m D w(x) \cdot \nabla_* g(x) \tag{40}$$

for any family of functions $g_\varepsilon \in L^2(\mu_\omega^\varepsilon)$ with $g_\varepsilon \rightharpoonup g \in L^2(m\,dx)$ such that $g_\varepsilon \in H^{1,\mathrm{f}}_{\omega,\varepsilon}$ and $g \in H^1_*(m\,dx)$.

Remark 3.14. Denoting by φ_ε the restriction of φ to $\varepsilon\hat{\omega}$, for all $\omega \in \Omega_1$ any $\varphi \in C_c(\mathbb{R}^d)$ has the property that $\varphi_\varepsilon \in \mathcal{C}(\varepsilon\hat{\omega}) \subset H^{1,\mathrm{f}}_{\omega,\varepsilon}$, as $\hat{\omega}$ is locally finite. Moreover, given $\omega \in \mathcal{A}[1]$ (cf. Proposition 3.1), by Remark 3.12 we get that $L^2(\mu_\omega^\varepsilon) \ni \varphi_\varepsilon \to \varphi \in L^2(m\,dx)$. In particular, for environments $\omega \in \Omega_1 \cap \mathcal{A}[1]$ (as the ones in Ω_{typ} appearing in Theorem 4.1), if $\nabla_\varepsilon v_\varepsilon \to w$ then $\nabla_\varepsilon v_\varepsilon \rightharpoonup w$.

4. Main results

We can now state our first main result (recall the definition of Ω_1 at the beginning of Section 3.3):

Theorem 4.1. *Let Assumptions* (A1), ..., (A8) *be satisfied. Then there exists a measurable subset* $\Omega_{\mathrm{typ}} \subset \Omega_1 \cap \Omega_* \subset \Omega$, *of so called* typical environments, *fulfilling the following properties*. Ω_{typ} *is translation invariant and* $\mathcal{P}(\Omega_{\mathrm{typ}}) = 1$. *Moreover, given* $\omega \in \Omega_{\mathrm{typ}}$, $\lambda > 0$, $f_\varepsilon \in L^2(\mu_\omega^\varepsilon)$ *and* $f \in L^2(m\,dx)$, *let* u_ε *and* u *be defined as the weak solutions*, *respectively in* $H^{1,\mathrm{f}}_{\omega,\varepsilon}$ *and* $H^1_*(m\,dx)$, *of the equations*

$$-\mathbb{L}_\omega^\varepsilon u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon, \tag{41}$$

$$-\nabla_* \cdot D \nabla_* u + \lambda u = f. \tag{42}$$

Then we have:

(i) Convergence of solutions (*cf. Definition* 3.11):

$$f_\varepsilon \rightharpoonup f \quad \Longrightarrow \quad u_\varepsilon \rightharpoonup u, \tag{43}$$

$$f_\varepsilon \to f \quad \Longrightarrow \quad u_\varepsilon \to u. \tag{44}$$

(ii) Convergence of flows (*cf. Definition* 3.13):

$$f_\varepsilon \rightharpoonup f \quad \Longrightarrow \quad \nabla_\varepsilon u_\varepsilon \rightharpoonup \nabla_* u, \tag{45}$$

$$f_\varepsilon \to f \quad \Longrightarrow \quad \nabla_\varepsilon u_\varepsilon \to \nabla_* u. \tag{46}$$

(iii) Convergence of energies:

$$f_\varepsilon \to f \quad \Longrightarrow \quad \frac{1}{2} \langle \nabla_\varepsilon u_\varepsilon, \nabla_\varepsilon u_\varepsilon \rangle_{\nu_\omega^\varepsilon} \to \int dx\, m \nabla_* u(x) \cdot D \nabla_* u(x). \tag{47}$$

Remark 4.2. Let $\omega \in \Omega_{\mathrm{typ}}$. Then, as $\Omega_{\mathrm{typ}} \subset \mathcal{A}[1]$ (cf. Section 12), by Remark 3.14 for any $f \in C_c(\mathbb{R}^d)$ it holds $L^2(\mu_\omega^\varepsilon) \ni f \to f \in L^2(m\,dx)$. By taking $f_\varepsilon := f$ and using (44), we get that $u_\varepsilon \to u$, where u_ε and u are defined as the weak solutions of (41) and (42), respectively.

We write $(P^\varepsilon_{\omega,t})_{t\geq 0}$ for the $L^2(\mu^\varepsilon_\omega)$-Markov semigroup associated to the random walk $(\varepsilon X^\omega_{\varepsilon^{-2}t})_{t\geq 0}$ on $\varepsilon\hat\omega$. In particular, $P^\varepsilon_{\omega,t} = e^{t\mathbb{L}^\varepsilon_\omega}$. Similarly we write $(P_t)_{t\geq 0}$ for the Markov semigroup on $L^2(m\,dx)$ associated to the (possibly degenerate) Brownian motion on $\mathbb{R}^d$ with diffusion matrix $2D$ given in Definition 3.6. Note that this Brownian motion is not degenerate when projected on $\mathrm{span}(\mathfrak{e}_1,\dots,\mathfrak{e}_{d_*})$, while no motion is present along $\mathrm{span}(\mathfrak{e}_{d_*+1},\dots,\mathfrak{e}_d)$. In particular, in the case $\mathfrak{e}_i = e_i$, writing $p_t(\cdot,\cdot)$ for the probability transition kernel of the Brownian motion on $\mathbb{R}^{d_*}$ with non-degenerate diffusion matrix $(2D_{i,j})_{1\leq i,j\leq d_*}$, it holds

$$P_t f\big(x',x''\big) = \int_{\mathbb{R}^{d_*}} p_t\big(x',y\big) f\big(y,x''\big)\,dy, \quad \big(x',x''\big)\in\mathbb{R}^{d_*}\times\mathbb{R}^{d-d_*}=\mathbb{R}^d. \tag{48}$$

Given $\lambda>0$ we write $R^\varepsilon_{\omega,\lambda}: L^2(\mu^\varepsilon_\omega)\to L^2(\mu^\varepsilon_\omega)$ for the λ-resolvent associated to the random walk $\varepsilon X^\omega_{\varepsilon^{-2}t}$, i.e. $R^\varepsilon_{\omega,\lambda} := (\lambda-\mathbb{L}^\varepsilon_\omega)^{-1} = \int_0^\infty e^{-\lambda s} P^\varepsilon_{\omega,s}\,ds$. We write $R_\lambda: L^2(m\,dx)\to L^2(m\,dx)$ for the λ-resolvent associated to the above Brownian motion on $\mathbb{R}^d$ with diffusion matrix $2D$. Note that (41) and (42) can be rewritten as $u_\varepsilon = R^\varepsilon_{\omega,\lambda} f_\varepsilon$ and $u = R_\lambda f$, respectively.

We now show several forms of semigroup and resolvent convergence, whose derivation uses Theorem 4.1 (they play a fundamental role in [18]). To this aim we introduce a new Assumption (recall definition (3) of Δ):

Assumption (A9). *At least one of the following properties is fulfilled*:

(i) *For $\mathcal{P}$-a.a. ω $\exists C(\omega)>0$ such that $\mu_\omega(\tau_k\Delta)\leq C(\omega)$ for all $k\in\mathbb{Z}^d$.*
(ii) *Setting $N_k(\omega) := \mu_\omega(\tau_k\Delta)$ for $k\in\mathbb{Z}^d$, for some $C_0\geq 0$ it holds $\mathbb{E}[N_0^2]<\infty$ and*

$$\big|\mathrm{Cov}(N_k,N_{k'})\big| \leq C_0\big|k-k'\big|^{-1} \tag{49}$$

for any $k\neq k'$ in $\mathbb{Z}^d$. More generally, we assume that, at cost to enlarge the probability space, one can define random variables $(N_k)_{k\in\mathbb{Z}^d}$ with $\mu_\omega(\tau_k\Delta)\leq N_k$, such that $\mathbb{E}[N_k]$, $\mathbb{E}[N_k^2]$ are bounded uniformly in k and such that (49) holds for all $k\neq k'$.

Remark 4.3. As follows from the proof of Theorem 4.4 below, when $\mathbb{G}=\mathbb{R}^d$ one can replace $\tau_k\Delta$ by $k+[0,1)^d$ as well. In general, for $\mathbb{G}=\mathbb{R}^d$, one can replace the cells $\{\tau_k\Delta\}_{k\in\mathbb{Z}^d}$ by the cells of any lattice partition of $\mathbb{R}^d$.

Theorem 4.4. *Let Assumptions* (A1), …, (A8) *be satisfied. Take $\omega\in\Omega_{\mathrm{typ}}$ and $f\in C_c(\mathbb{R}^d)$. Then for any $t\geq 0$ and $\lambda>0$, it holds*

$$L^2\big(\mu^\varepsilon_\omega\big) \ni P^\varepsilon_{\omega,t} f \to P_t f \in L^2(m\,dx), \tag{50}$$

$$L^2\big(\mu^\varepsilon_\omega\big) \ni R^\varepsilon_{\omega,\lambda} f \to R_\lambda f \in L^2(m\,dx). \tag{51}$$

Suppose in addition that Assumption (A9) *holds. Then there exists a translation invariant measurable set $\Omega_\sharp\subset\Omega$ with $\mathcal{P}(\Omega_\sharp)=1$ such that for any $\omega\in\Omega_\sharp\cap\Omega_{\mathrm{typ}}$, any $f\in C_c(\mathbb{R}^d)$, $\lambda>0$, $t\geq 0$ it holds*:

$$\lim_{\varepsilon\downarrow 0}\int \big|P^\varepsilon_{\omega,t} f(x) - P_t f(x)\big|^2\,d\mu^\varepsilon_\omega(x) = 0, \tag{52}$$

$$\lim_{\varepsilon\downarrow 0}\int \big|P^\varepsilon_{\omega,t} f(x) - P_t f(x)\big|\,d\mu^\varepsilon_\omega(x) = 0, \tag{53}$$

$$\lim_{\varepsilon\downarrow 0}\int \big|R^\varepsilon_{\omega,\lambda} f(x) - R_\lambda f(x)\big|^2\,d\mu^\varepsilon_\omega(x) = 0, \tag{54}$$

$$\lim_{\varepsilon\downarrow 0}\int \big|R^\varepsilon_{\omega,\lambda} f(x) - R_\lambda f(x)\big|\,d\mu^\varepsilon_\omega(x) = 0. \tag{55}$$

The proof of Theorem 4.4 is given in Section 20.

Remark 4.5. Assumption (A9) is used only to derive Lemma 20.1 in Section 20, which is applied in the proof of Theorem 4.4 only with $\psi(r) := 1/(1+r^{d+1})$.

5. Some examples

The class of reversible random walks in random environment is very large. We discuss just some popular examples. Below, when we state that assumptions (A3), …, (A6) are satisfied, we understand that this holds with ω in a suitable translation invariant measurable set $\Omega_* \subset \Omega$ with $\mathcal{P}(\Omega_*) = 1$.

5.1. *Nearest-neighbor random conductance model on $\mathbb{Z}^d$*

We take $\mathbb{G} := \mathbb{Z}^d$ and $V := \mathbb{I}$. Let $\mathbb{E}^d := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. We take $\Omega := (0, +\infty)^{\mathbb{E}^d}$ endowed with the product topology. We write $\omega = (\omega_b : b \in \mathbb{E}^d)$ for a generic element of Ω and we write $\omega_{x,y}$ instead of $\omega_{\{x,y\}}$. Note that $\omega_{x,y} = \omega_{y,x}$. The action $(\theta_x)_{x\in\mathbb{Z}^d}$ is the standard one: θ_x shifts the environment along the vector $-x$. We set $\mu_\omega := \sum_{x\in\mathbb{Z}^d} \delta_x$ and $r_{x,y}(\omega) := \omega_{x,y}$ if $\{x, y\} \in \mathbb{E}^d$ and $r_{x,y}(\omega) := 0$ otherwise. Then, given the environment ω, the random walk X^ω_t has state space $\mathbb{Z}^d$ and jumps from x to y, with $|x - y| = 1$, with rate $\omega_{x,y}$. Assumptions (A1), …, (A9) are satisfied whenever $\mathcal{P}$ is stationary and ergodic, it satisfies (A3) and $\mathbb{E}[\omega_{x,y}] < +\infty$ for all $|x - y| = 1$ (i.e., by stationarity, $\mathbb{E}[\omega_{0,e_i}] < +\infty$ for $i = 1, \dots, d$). Due to [4, Prop. 4.1], if in addition $\mathbb{E}[1/\omega_{0,e_i}] < +\infty$ for $i = 1, \dots, d$, then the matrix D is non-degenerate.

If one wants the version with waiting times of parameter 1, then one has to set $n_x(\omega) := \sum_{y:|x-y|=1} \omega_{x,y}$, $\mu_\omega := \sum_{x\in\mathbb{Z}^d} n_x(\omega)\delta_x$ and $r_{x,y}(\omega) := \omega_{x,y}/n_x(\omega)$. Then Assumptions (A1), …, (A8) are satisfied whenever $\mathcal{P}$ is stationary and ergodic, it satisfies (A3) and $\mathbb{E}[\omega_{x,y}] < +\infty$ for all $|x - y| = 1$ (use (12)). As in the previous case, D is non-degenerate if $\mathbb{E}[1/\omega_{0,e_i}] < +\infty$ for $i = 1, \dots, d$ (see Remark 3.7). To satisfy Assumption (A9) it is enough e.g. that the conductances $\omega_{x,y}$ are uniformly bounded or that the covariance between $\omega_{x,y}$ and $\omega_{x',y'}$ decays at least as the inverse of the distance between $\{x, y\}$ and $\{x', y'\}$.

5.2. *Random conductance model on $\mathbb{Z}^d$ with long conductances*

We take $\mathbb{G} := \mathbb{Z}^d$ and $V := \mathbb{I}$. We set $\mathbb{B}^d := \{\{x, y\} : x, y \in \mathbb{Z}^d, x \neq y\}$ and take $\Omega := (0, +\infty)^{\mathbb{B}^d}$ endowed with the product topology. We set $\omega_{x,y} := \omega_{\{x,y\}}$. The action $(\theta_x)_{x\in\mathbb{Z}^d}$ is the standard one. We take $\mu_\omega := \sum_{x\in\mathbb{Z}^d} \delta_x$, $r_{x,y}(\omega) := \omega_{x,y}$ if $\{x, y\} \in \mathbb{B}^d$ and $r_{x,y}(\omega) := 0$ otherwise. Then, given the environment ω, the random walk X^ω_t has state space $\mathbb{Z}^d$ and jumps from x to y with probability rate $\omega_{x,y}$. Assumptions (A1), …, (A9) are satisfied whenever $\mathcal{P}$ is stationary and ergodic, it satisfies (A3) and it satisfies $\mathbb{E}[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}|z|^2] < +\infty$ (which implies $\mathbb{E}[\sum_{z\in\mathbb{Z}^d} \omega_{0,z}] < +\infty$). Reasoning as in the proof of [4, Prop. 4.1], for all $a \in \mathbb{R}^d$ one can lower bound the scalar product $a \cdot Da$ by $C \sum_{x\in\mathbb{Z}^d} (a \cdot x)^2/\mathbb{E}[1/\omega_{0,x}]$ with $C > 0$. Hence, D is non-degenerate if the set $\{x \in \mathbb{Z}^d : \mathbb{E}[1/\omega_{0,x}] < +\infty\}$ is not contained in a subspace of $\mathbb{R}^d$ with dimension smaller than d.

5.3. *Random walk with random conductances on infinite clusters*

We take $\mathbb{G} := \mathbb{Z}^d$ and $V := \mathbb{I}$. Let $\mathbb{E}^d$ be as in Example 5.1. We take $\Omega := [0, +\infty)^{\mathbb{E}^d}$ with the product topology. The action $(\theta_x)_{x\in\mathbb{Z}^d}$ is the standard one. Let $\mathcal{P}$ be a probability measure on Ω stationary, ergodic and fulfilling (A3) for the above action. We assume that for $\mathcal{P}$-a.a. ω there exists a unique infinite connected component $\mathcal{C}(\omega) \subset \mathbb{Z}^d$ in the graph given by the edges $\{x, y\}$ in $\mathbb{E}^d$ with positive conductivity $\omega_{x,y}$.

We set $\mu_\omega := \sum_{x\in\mathcal{C}(\omega)} \delta_x$, $r_{x,y}(\omega) := \omega_{x,y}$ if $\{x, y\} \in \mathbb{E}^d$ and $r_{x,y}(\omega) := 0$ otherwise. Note that $n_x(\omega) = \mathbb{1}(x \in \mathcal{C}(\omega))$. Then the random walk X^ω_t has state space $\mathcal{C}(\omega)$ and jumps from x to y in $\mathcal{C}(\omega)$ (where $|x - y| = 1$) with probability rate $\omega_{x,y}$. Note that $d\mathcal{P}_0(\omega) = \mathbb{1}(0 \in \mathcal{C}(\omega))\, d\mathcal{P}(\omega)/\mathcal{P}(0 \in \mathcal{C}(\omega))$. If, in addition, $\mathcal{P}$ satisfies $\mathbb{E}[\omega_{x,y}] < +\infty$ for all $\{x, y\} \in \mathbb{E}^d$, then all Assumptions (A1), …, (A9) are satisfied (note that we need neither bounded conductances nor the non-degeneracy of the diffusion matrix, differently from [15]). This holds for example if $\mathcal{P}$ is the Bernoulli product probability on Ω such that $\mathcal{P}(\omega_{x,y} > 0) > p_c$, p_c being the bond percolation critical probability on $\mathbb{Z}^d$, and $\mathbb{E}[\omega_{x,y}] < +\infty$ for all $\{x, y\} \in \mathbb{E}^d$.

If interested to the modified version with waiting times of parameter 1, then we set $n_x(\omega) := \sum_{y:\{x,y\}\in\mathbb{E}^d} \omega_{x,y}\mathbb{1}(x \in \mathcal{C}(\omega))$, $\mu_\omega := \sum_{x\in\mathcal{C}(\omega)} n_x(\omega)\delta_x$ and $r_{x,y}(\omega) := \omega_{x,y}/n_x(\omega)$ if $\{x, y\} \in \mathbb{E}^d$ and $x, y \in \mathcal{C}(\omega)$, otherwise we set $r_{x,y}(\omega) := 0$. All assumptions (A1), …, (A8) are satisfied whenever $\mathbb{E}[\omega_{x,y}] < +\infty$ for $\{x, y\} \in \mathbb{E}^d$. For (A9) one can argue as in Example 5.1.

We refer e.g. to [3,13,36] for additional assumptions assuring the non-degeneracy of D.

5.4. *Mott random walk*

Mott random walk (see e.g. [7,8,20,21]) is a mean-field model for Mott variable range hopping in amorphous solids [40]. We take $\mathbb{G} := \mathbb{R}^d$ and $V := \mathbb{I}$. Ω is given by the space of marked simple counting measures with marks in $\mathbb{R}$. By identifying ω with its support, we have $\omega = \{(x_i, E_i)\}$ where $E_i \in \mathbb{R}$ and the set $\{x_i\}$ is locally finite. Ω is a metric space, being a subset of the metric space $\mathcal{N}$ of counting measures $\mu = \sum_i k_i \delta_{(x_i, E_i)}$, where $k_i \in \mathbb{N}$ and $\{(x_i, E_i)\}$ is a locally finite subset of $\mathbb{R}^d \times \mathbb{R}$. See [9, Eq. (A2.6.1) in App. A2.6] for the metric d associated to $\mathcal{N}$. One can prove that $(\mathcal{N}, d)$ is a Polish space having Ω as Borel subset [9, Cor. 7.1.IV, App. A2.6]. The action θ_x on Ω is given by $\theta_x\omega := \{(x_i - x, E_i)\}$ if $\omega = \{(x_i, E_i)\}$.

Given the environment $\omega = \{(x_i, E_i)\}$, to get Mott random walk we take $\mu_\omega := \sum_i \delta_{x_i}$ (hence $\hat\omega := \{x_i\}$ and $n_{x_i}(\omega) := 1$) and

$$r_{x_i,x_j}(\omega) := \exp\{-|x_i - x_j| - |E_{x_i}| - |E_{x_j}| - |E_{x_i} - E_{x_j}|\}, \quad x_i \neq x_j. \tag{56}$$

Hence, X_t^ω walks on $\{x_i\}$ with jump probability rates given by (56). Note that the properties in (A4), (A5), (A6) are automatically satisfied by all $\omega \in \Omega$ (i.e. $\Omega_4 = \Omega_5 = \Omega_6 = \Omega$ with the notation of Section 2.4). Suppose that $\mathcal{P}$ satisfies (A1), (A2) and $\mathcal{P}(\theta_g\omega \neq \theta_{g'}\omega\ \forall g \neq g' \text{ in } \mathbb{G}) = 1$ (as discussed in Section 2.4 the set $\{\omega : \theta_g\omega \neq \theta_{g'}\omega\ \forall g \neq g' \text{ in } \mathbb{G}\}$ is measurable). Then (A3) is satisfied by taking $\Omega_* = \Omega_3$ and requiring $\mathcal{P}(\Omega_*) = 1$. $\mathcal{P}_0$ is simply the standard Palm distribution associated to the marked simple point process with law $\mathcal{P}$ [11]. As the above space $\mathcal{N}$ is Polish and $\Omega_0 := \{\omega : 0 \in \hat\omega\}$ is a Borel subset of $\mathcal{N}$, Ω_0 is separable and therefore (A8) is satisfied (see the comment on (A8) in Section 2.4).

We claim that the bound $\mathbb{E}[|\hat\omega \cap [0,1]^d|^2] < \infty$ implies (A7). To prove our claim we observe that, by [21, Lemma 2], given a positive integer k it holds $\lambda_0 \in L^k(\mathcal{P}_0)$ if and only if $\mathbb{E}[|\hat\omega \cap [0,1]^d|^{k+1}] < \infty$. The proof provided there remains true when substituting λ_0 by any function f such that $|f(\omega)| \leq C \int d\hat\omega(x) e^{-c|x|}$ with $C, c > 0$. Hence we can take $f = \lambda_2$. Then the above bound on the second moment of $\hat\omega \cap [0,1]^d$ implies that $\lambda_0, \lambda_2 \in L^1(\mathcal{P}_0)$.

For (A9) we observe that $\mu_\omega(k + [0,1)^d)$ equals the number of points x_i in $k + [0,1)^d$. Hence, there are plenty of simple point processes satisfying (A9).

We refer to [7,8,21] for additional assumptions assuring the non-degeneracy of D.

5.5. *Simple random walk on Delaunay triangulation*

We take $\mathbb{G} := \mathbb{R}^d$ and $V := \mathbb{I}$. Ω is given by the space of simple counting measures on $\mathbb{R}^d$. We set $\mu_\omega := \omega$. We take $r_{x,y}(\omega) = \mathbb{1}(x \overset{\omega}{\sim} y)$, where $x \overset{\omega}{\sim} y$ means that x, y are adjacent in the ω-Delaunay triangulation. Then, given ω, the random walk X_t^ω is the simple random walk on the ω-Delaunay triangulation. By taking $\mathcal{P}$ stationary with finite intensity, $\mathcal{P}_0$ becomes the standard Palm distribution associated to the stationary simple point process on $\mathbb{R}^d$ with law $\mathcal{P}$ [11]. If for example $\mathcal{P}$ is a homogeneous Poisson point process, using the results in [41] it is simple to conclude that all Assumptions (A1), ..., (A9) are satisfied (for (A8) reason as for Mott random walk) and that D is non-degenerate.

More general cases, also with random conductances, will be discussed in [22].

5.6. *Nearest-neighbor random conductance models on lattices*

To have a concrete example let us consider the nearest-neighbor random conductance model on the hexagonal lattice $\mathcal{L} = (\mathcal{V}, \mathcal{E})$, partially drawn in Figure 1 (hexagons have edges of length one).

Let v_1, v_2 be the vectors $v_1 = (2\cos\frac{\pi}{6}, 0)$, $v_2 = (2\cos\frac{\pi}{6}\cos\frac{\pi}{3}, 2\cos\frac{\pi}{6}\sin\frac{\pi}{3})$. We take $\Omega := (0, +\infty)^{\mathcal{E}}$ endowed with the product topology and set $\omega_{x,y} := \omega_{\{x,y\}}$. Let $\mathbb{G} := \mathbb{Z}^d$ and let V be the matrix with columns v_1, v_2 respectively. The

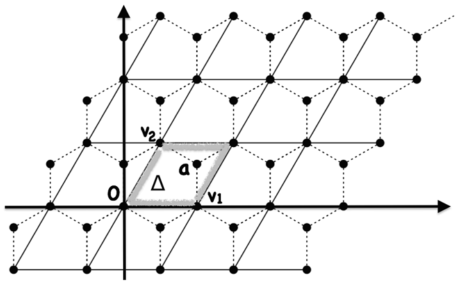

Fig. 1. Hexagonal lattice, fundamental cell Δ, basis $\{v_1, v_2\}$.

action of $(\theta_z)_{z\in\mathbb{Z}^d}$ and $(\tau_z)_{z\in\mathbb{Z}^d}$ of $\mathbb{Z}^d$ on Ω and $\mathbb{R}^d$, respectively, are given by

$$\theta_z\omega := \big(\omega_{x-Vz,y-Vz} : \{x,y\}\in\mathcal{E}\big) \quad \text{if } \omega = \big(\omega_{x,y} : \{x,y\}\in\mathcal{E}\big), z\in\mathbb{Z}^d,$$
$$\tau_z x := x + Vz \quad \text{for } x\in\mathbb{R}^d, z\in\mathbb{Z}^d.$$

Moreover, for any $\omega\in\Omega$, we set $\mu_\omega := \sum_{x\in\mathcal{V}}\delta_x$. The fundamental cell Δ is given by $\Delta = \{t_1v_1 + t_2v_2 : 0\le t_1, t_2 < 1\}$. Note that the set Ω_0 introduced after (11) equals $\Omega\times\{0,a\}$ and that, by (10), $m\ell(\Delta) = 2$. Hence (see (11)) $\mathcal{P}_0(d\omega, dx) = \mathcal{P}(d\omega)\otimes \mathrm{Av}_{u\in\{0,a\}}\delta_u(dx)$. Setting $r_{x,y} := \omega_{x,y}$ if $\{x,y\}\in\mathcal{E}$ and $r_{x,y} := 0$ otherwise, the random walk X_t^ω has state space $\mathcal{V}$ and jumps from x to a nearest-neighbor site y with probability rate $\omega_{x,y}$. If, for example, $\mathcal{P}$ is given by a Bernoulli product probability measure with $\mathbb{E}[\omega_{x,y}] < +\infty$ for all $\{x,y\}\in\mathcal{E}$, then all assumptions (A1), …, (A9) are satisfied.

6. From $\mathbb{Z}^d$-actions to $\mathbb{R}^d$-actions

Suppose $\mathbb{G} = \mathbb{Z}^d$ and call $\mathcal{S}[1]$ the setting described by $\mathbb{G}$, $(\Omega,\mathcal{F},\mathcal{P})$, $(\theta_g)_{g\in\mathbb{G}}$, $(\tau_g)_{g\in\mathbb{G}}$, μ_ω, $r_{x,y}(\omega)$, Ω_*. We now introduce a new setting $\mathcal{S}[2]$ described by new objects $\bar{\mathbb{G}} = \mathbb{R}^d$, $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathcal{P}})$, $(\bar\theta_g)_{g\in\bar{\mathbb{G}}}$, $(\bar\tau_g)_{g\in\bar{\mathbb{G}}}$, $\mu_{\bar\omega}$, $\bar r_{x,y}(\bar\omega)$, $\bar\Omega_*$ such that if $\mathcal{S}[1]$ satisfies the main assumptions (A1), …, (A8), then the same holds for $\mathcal{S}[2]$, and if the conclusion of Theorem 4.1 holds for $\mathcal{S}[2]$, then the same holds for $\mathcal{S}[1]$. As a consequence, to prove Theorem 4.1 it is enough to consider the case $\mathbb{G} = \mathbb{R}^d$. Many identities pointed out below will be proved in Appendix E.

We consider the extended probability space $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathcal{P}})$ defined as

$$\bar\Omega := \Omega\times\Delta, \qquad \bar{\mathcal{P}} := \ell(\Delta)^{-1}\mathcal{P}\otimes\ell, \qquad \bar{\mathcal{F}} := \mathcal{F}\otimes\mathcal{B}(\Delta).$$

We define the action $(\bar\theta_x)_{x\in\bar{\mathbb{G}}}$ of $\bar{\mathbb{G}} = \mathbb{R}^d$ on $\bar\Omega$ as

$$\bar\theta_x(\omega,a) = \big(\theta_{g(x+a)}\omega, \beta(x+a)\big). \tag{57}$$

One can easily check that $(\bar\theta_x)_{x\in\mathbb{R}^d}$ satisfies the properties analogous to (P1), …, (P4) in Section 2, when replacing $\mathbb{G}$ by $\bar{\mathbb{G}}$ (for the validity of (P4) concerning the $\bar{\mathbb{G}}$-stationary of $\bar{\mathcal{P}}$ see Lemma E.1 in Appendix E). Moreover, $\bar{\mathcal{P}}$ is ergodic w.r.t. the action of $(\bar\theta_x)_{x\in\bar{\mathbb{G}}}$ (cf. Lemma E.2). Hence, (A1) is fulfilled by $\mathcal{S}[2]$.

We define

$$\bar\Omega\ni\bar\omega\mapsto\mu_{\bar\omega}\in\mathcal{M}, \qquad \mu_{(\omega,a)}(\cdot) := \mu_\omega(\cdot + a). \tag{58}$$

The intensity $\bar m$ and m of the random measure $\mu_{\bar\omega}$ and μ_ω, respectively, coincide (cf. Lemma E.3). Hence, (A2) is fulfilled by $\mathcal{S}[2]$.

We set $\bar\Omega_* := \Omega_*\times\Delta$. It is simple to check that $\bar\Omega_*$ is a translation invariant measurable set with $\bar{\mathcal{P}}(\bar\Omega_*) = 1$ and that (A3) is fulfilled by $\mathcal{S}[2]$ for any $\bar\omega\in\bar\Omega_*$. The action $(\bar\tau_x)_{x\in\bar{\mathbb{G}}}$ on $\mathbb{R}^d$ is given by

$$\bar\tau_x z := z + x. \tag{59}$$

By writing $n_x(\bar\omega) = \mu_{\bar\omega}(\{x\})$ for $x\in\mathbb{R}^d$, the above definition (58) implies that

$$\bar\omega = (\omega,a) \quad\Longrightarrow\quad \hat{\bar\omega} = \hat\omega - a, \qquad n_x(\bar\omega) = n_{x+a}(\omega). \tag{60}$$

Then we have (cf. Lemma E.4)

$$\mu_{\bar\theta_x\bar\omega}(\cdot) = \mu_{\bar\omega}(\bar\tau_x\cdot) \quad \forall\bar\omega\in\bar\Omega_*, x\in\bar{\mathbb{G}}. \tag{61}$$

We define the measurable function

$$\bar r : \bar\Omega\times\mathbb{R}^d\times\mathbb{R}^d \ni (\bar\omega,x,y)\mapsto \bar r_{x,y}(\bar\omega)\in[0,+\infty) \tag{62}$$

as

$$\bar r_{x,y}(\omega,a) := r_{x+a,y+a}(\omega). \tag{63}$$

The analogous of (17) still holds for $\bar\omega\in\bar\Omega_*$ (cf. Lemma E.5). Note that, by (60) and (63), we have $\bar c_{x,y}(\omega,a) = c_{x+a,y+a}(\omega)$. Hence, (A4) and (A5) are fulfilled by $\mathcal{S}[2]$ for all $\bar\omega\in\bar\Omega_*$.

The Palm distributions $\mathcal{P}_0$, $\bar{\mathcal{P}}_0$ associated respectively to $\mathcal{P}$, $\bar{\mathcal{P}}$ coincide (cf. Lemma E.6). Hence, (A8) is trivially satisfied by $\mathcal{S}[2]$. Note moreover that $\bar\Omega_0 = \{(\omega, x) \in \bar\Omega : n_0(\omega, x) > 0\} = \{(\omega, x) \in \Omega \times \Delta : n_x(\omega) > 0\} = \Omega_0$.

Recall (15). We write λ_k and $\bar\lambda_k$ for the function corresponding to (15) in setting $\mathcal{S}[1]$ and $\mathcal{S}[2]$, respectively. Note that $\lambda_k, \bar\lambda_k$ are defined on the same set $\bar\Omega_0$. Given $\bar\omega = (\omega, a) \in \bar\Omega_0 = \Omega_0$, we have (using (60) and (63))

$$\bar\lambda_k(\bar\omega) = \int_{\mathbb{R}^d} d\hat{\bar\omega}(x)\bar r_{0,x}(\bar\omega)|x|^k = \int_{\mathbb{R}^d} d\hat\omega(y)\bar r_{0,y-a}(\bar\omega)|y-a|^k = \int_{\mathbb{R}^d} d\hat\omega(y) r_{a,y}(\omega)|y-a|^k = \lambda_k(\bar\omega). \tag{64}$$

In particular (A7) implies that $\bar\lambda_0, \bar\lambda_2 \in L^1(\bar{\mathcal{P}}_0)$. In conclusion we have: If (A1), ..., (A8) are satisfied by $\mathcal{S}[1]$, then (A1), ..., (A8) are satisfied by $\mathcal{S}[2]$. Finally, as the integral in the r.h.s. of (30) equals

$$\frac{1}{2}\int_{\bar\Omega} d\bar{\mathcal{P}}_0(\bar\omega) \int_{\mathbb{R}^d} d\hat{\bar\omega}(z)\bar r_{0,z}(\bar\omega)\big(a\cdot z - \big[f(\bar\theta_z\bar\omega) - f(\bar\omega)\big]\big)^2,$$

one can easily check that Theorem 4.1 for $\mathcal{S}[2]$ implies Theorem 4.1 for $\mathcal{S}[1]$.

Warning 6.1. Due to the above discussion it is enough to prove Theorem 4.1 only for $\mathbb{G} = \mathbb{R}^d$. Due to its relevance in discrete probability, we will also treat simultaneously the special discrete case. Moreover, to slightly simplify the notation, we will take $V = \mathbb{I}$, thus implying that $g(x) = x$. The reader interested to formulas with generic matrix V has only to replace θ_x by $\theta_{g(x)}$ in what follows. Indeed, the manipulations behind our formulas rely on Remark 2.1, which holds in the general case. In conclusion, from now on (with exception of Section 20 and the appendixes) and without further mention, we restrict to the case $\mathbb{G} = \mathbb{R}^d$ and to the special discrete case, we take $V = \mathbb{I}$ and we understand that Assumptions (A1), ..., (A8) are satisfied.

7. Some key properties of the Palm distribution $\mathcal{P}_0$

For the results of this section it would be enough to require (A1) and (A2) and, for Lemma 7.3, $\mathbb{E}_0[\lambda_0] < \infty$. Given a measurable set $A \subset \Omega$, we define

$$\tilde A := \{\omega \in \Omega : \theta_x\omega \in A \ \forall x \in \hat\omega\}. \tag{65}$$

Note that $\tilde A$ is translation invariant and measurable.

Lemma 7.1. *Given $A \subset \Omega$ measurable, the following facts are equivalent*: (i) $\mathcal{P}_0(A) = 1$; (ii) $\mathcal{P}(\tilde A) = 1$; (iii) $\mathcal{P}_0(\tilde A) = 1$. *Given a translation invariant measurable set $A \subset \Omega$, it holds $\mathcal{P}(A) = 1$ if and only if $\mathcal{P}_0(A) = 1$ and it holds $\mathcal{P}(A) = 0$ if and only if $\mathcal{P}_0(A) = 0$.*

Proof. We first prove the equivalence between (i), (ii) and (iii). By (9) for $\mathbb{G} = \mathbb{R}^d$ and (12) for the special discrete case, (ii) implies (i). If (i) holds, then we get (ii) by (8) and (12) and Campbell's identities (178) and (181) with $f(x, \omega) := (2\ell)^{-d}\mathbb{1}_{[-\ell,\ell]^d}(x)\mathbb{1}_A(\omega)$ and $\ell \in \mathbb{N}_+$. Note that $\tilde{\tilde A} = \tilde A$. Hence, by applying the equivalence between (i) and (ii) with A replaced by $\tilde A$, we get the equivalence between (ii) and (iii).

Now let A be translation invariant. To prove that $\mathcal{P}(A) = 1$ if and only if $\mathcal{P}_0(A) = 1$, it is enough to observe that $\tilde A = A$ and to apply the above equivalence between (ii) and (iii). To prove that $\mathcal{P}(A) = 0$ if and only if $\mathcal{P}_0(A) = 0$, it is enough to take the complement. □

As an immediate consequence of Lemma 7.1 we get:

Corollary 7.2. *Let $f \in L^1(\mathcal{P}_0)$. Let $B := \{\omega \in \Omega : |f(\theta_x\omega)| < +\infty \ \forall x \in \hat\omega\}$. Then B is translation invariant, $\mathcal{P}(B) = 1$ and $\mathcal{P}_0(B) = 1$.*

The following result generalizes [21, Lemma 1-(i)]:

Lemma 7.3. *Let $f : \Omega_0 \times \Omega_0 \to \mathbb{R}$ be a measurable function. Suppose that* (i) *at least one of the functions $\omega \mapsto \int d\hat\omega(x) r_{0,x}(\omega)|f(\omega, \theta_x\omega)|$ and $\omega \mapsto \int d\hat\omega(x) r_{0,x}(\omega)|f(\theta_x\omega, \omega)|$ is in $L^1(\mathcal{P}_0)$, or* (ii) *$f \geq 0$. Then it holds*

$$\mathbb{E}_0\Big[\int_{\mathbb{R}^d} d\hat\omega(x) r_{0,x}(\omega) f(\omega, \theta_x\omega)\Big] = \mathbb{E}_0\Big[\int_{\mathbb{R}^d} d\hat\omega(x) r_{0,x}(\omega) f(\theta_x\omega, \omega)\Big]. \tag{66}$$

We note that, by (12), in the special discrete case (66) reads

$$\sum_{x\in\mathbb{Z}^d}\mathbb{E}\big[c_{0,x}(\omega)f(\omega,\theta_x\omega)\big]=\sum_{x\in\mathbb{Z}^d}\mathbb{E}\big[c_{0,x}(\omega)f(\theta_x\omega,\omega)\big]. \tag{67}$$

Proof of Lemma 7.3. We first sketch the proof in the special discrete case, which is trivial. Given $\omega\in\Omega_0$, due to Remark 2.1 and (A5), $c_{0,x}(\omega)=c_{0,-x}(\theta_x\omega)$ $\mathcal{P}$-a.s. By the translation invariance of $\mathcal{P}$ we get $\mathbb{E}[c_{0,x}(\omega)|f|(\omega,\theta_x\omega)]=\mathbb{E}[c_{0,-x}(\omega)|f|(\theta_{-x}\omega,\omega)]$, and the same then must hold in Case (i) with f instead of $|f|$. This allows to conclude the proof of (67).

We move to the setting $\mathbb{G}=\mathbb{R}^d$. We start with Case (i) supposing first that both functions there are in $L^1(\mathcal{P}_0)$. We set $B(n):=[-n,n]^d$. Due to (9) and using also (A5) for (69) we get

$$\text{l.h.s. of (66)}=\frac{1}{m(2n)^d}\mathbb{E}\Big[\sum_{x\in\hat\omega\cap B(n)}\sum_{z\in\hat\omega}c_{x,z}(\omega)f(\theta_x\omega,\theta_z\omega)\Big], \tag{68}$$

$$\text{r.h.s. of (66)}=\frac{1}{m(2n)^d}\mathbb{E}\Big[\sum_{x\in\hat\omega}\sum_{z\in\hat\omega\cap B(n)}c_{x,z}(\omega)f(\theta_x\omega,\theta_z\omega)\Big]. \tag{69}$$

To prove that (68) equals (69) it is enough to show that

$$\lim_{n\to\infty}\frac{1}{m(2n)^d}\mathbb{E}\Big[\sum_{x\in\hat\omega\cap B(n)}\sum_{z\in\hat\omega\setminus B(n)}c_{x,z}(\omega)|f|(\theta_x\omega,\theta_z\omega)\Big]=0 \tag{70}$$

and that the same limit holds when summing among $x\in\hat\omega\setminus B(n)$ and $z\in\hat\omega\cap B(n)$. We prove (70), the other limit can be treated similarly. By (9)

$$\begin{aligned}&\frac{1}{m(2n)^d}\mathbb{E}\Big[\sum_{x\in\hat\omega\cap B(n)}\sum_{z\in\hat\omega\setminus B(n+\sqrt{n})}c_{x,z}(\omega)|f|(\theta_x\omega,\theta_z\omega)\Big]\\&\quad\le\frac{1}{m(2n)^d}\mathbb{E}\Big[\sum_{x\in\hat\omega\cap B(n)}\sum_{z\in\hat\omega:|z-x|_\infty\ge\sqrt{n}}c_{x,z}(\omega)|f|(\theta_x\omega,\theta_z\omega)\Big]\\&\quad=\mathbb{E}_0\Big[\int_{\mathbb{R}^d}d\hat\omega(x)r_{0,x}(\omega)|f|(\omega,\theta_x\omega)\mathbb{1}\big(|x|_\infty\ge\sqrt{n}\big)\Big].\end{aligned} \tag{71}$$

Due to our L^1-hypothesis and dominated convergence, the last member goes to zero as $n\to\infty$. Hence, it remains to prove (70) with "$z\in\hat\omega\setminus B(n)$" replaced by "$z\in\hat\omega\cap U(n)$" where $U(n):=B(n+\sqrt{n})\setminus B(n)$. To this aim, by (9) and (A5), we get

$$\mathbb{E}_0\Big[\int_{\mathbb{R}^d}d\hat\omega(x)r_{0,x}(\omega)|f|(\theta_x\omega,\omega)\Big]=\frac{1}{m\ell(U(n))}\mathbb{E}\Big[\sum_{x\in\hat\omega}\sum_{z\in\hat\omega\cap U(n)}c_{x,z}(\omega)|f|(\theta_x\omega,\theta_z\omega)\Big]. \tag{72}$$

By hypothesis, the above l.h.s. is finite. Hence (70) with "$z\in\hat\omega\cap U(n)$" follows by using (72) and that $\ell(U(n))/n^d\to 0$. This concludes the proof of (66) when both functions in Case (i) are in $L^1(\mathcal{P}_0)$. We interrupt with Case (i) and move to Case (ii) with $f\ge 0$. By the above result and since $\mathbb{E}_0[\lambda_0]<+\infty$, identity (66) holds with f replaced by $f\wedge n$. By taking the limit $n\to\infty$ and using monotone convergence, we get (66) for a generic $f\ge 0$. Let us come back to Case (i). Since (by Case (ii)) (66) holds with f replaced by $|f|$, we get that in Case (i) both functions there belong to $L^1(\mathcal{P}_0)$ as soon as at least one does. The conclusion then follows from our first result. □

8. Space of square integrable forms

We define ν as the Radon measure on $\Omega_0\times\mathbb{G}$ such that

$$\int d\nu(\omega,z)g(\omega,z)=\int d\mathcal{P}_0(\omega)\int d\hat\omega(z)r_{0,z}(\omega)g(\omega,z) \tag{73}$$

for any nonnegative measurable function $g(\omega,z)$.

Remark 8.1. When considering the special discrete case in the r.h.s. of (73) one can replace the integral $\int d\hat\omega(z)$ with the series $\sum_{z\in\mathbb{Z}^d}$ (recall that $r_{x,x}(\omega)=0$ and $r_{x,y}(\omega)=0$ if $\{x,y\}\not\subset\hat\omega$). A similar rewriting holds in the formulas presented in the rest of the paper.

We point out that, by Assumption (A7), ν has finite total mass: $\nu(\Omega\times\mathbb{R}^d)=\mathbb{E}_0[\lambda_0]<+\infty$. Elements of $L^2(\nu)$ are called *square integrable forms*.

Given a function $u:\Omega_0\to\mathbb{R}$ we define the function $\nabla u:\Omega_0\times\mathbb{G}\to\mathbb{R}$ as

$$\nabla u(\omega,z):=u(\theta_z\omega)-u(\omega). \tag{74}$$

Note that, by Lemma 7.1 with $A:=\{\omega\in\Omega_0:u(\omega)=f(\omega)\}$, if $u,f:\Omega_0\to\mathbb{R}$ are such that $u=f$ $\mathcal{P}_0$-a.s., then $\nabla u=\nabla f$ ν-a.s. In particular, if u is defined $\mathcal{P}_0$-a.s., then ∇u is well defined ν-a.s.

If u is bounded and measurable, then $\nabla u\in L^2(\nu)$. The subspace of *potential forms* $L^2_{\mathrm{pot}}(\nu)$ is defined as the following closure in $L^2(\nu)$:

$$L^2_{\mathrm{pot}}(\nu):=\overline{\{\nabla u: u \text{ is bounded and measurable}\}}.$$

The subspace of *solenoidal forms* $L^2_{\mathrm{sol}}(\nu)$ is defined as the orthogonal complement of $L^2_{\mathrm{pot}}(\nu)$ in $L^2(\nu)$.

8.1. *The subspace H^1_{env}*

We define

$$H^1_{\mathrm{env}}:=\{u\in L^2(\mathcal{P}_0):\nabla u\in L^2(\nu)\}. \tag{75}$$

We endow H^1_{env} with the norm $\|u\|_{H^1_{\mathrm{env}}}:=\|u\|_{L^2(\mathcal{P}_0)}+\|\nabla u\|_{L^2(\nu)}$. It is simple to check that H^1_{env} is a Hilbert space.

8.2. *Divergence*

Definition 8.2. Given a square integrable form $v\in L^2(\nu)$ we define its divergence $\operatorname{div} v\in L^1(\mathcal{P}_0)$ as

$$\operatorname{div} v(\omega)=\int d\hat\omega(z)r_{0,z}(\omega)\big(v(\omega,z)-v(\theta_z\omega,-z)\big). \tag{76}$$

By applying Lemma 7.3 with f such that $f(\omega,\theta_z\omega)=|v(\omega,z)|$ for $\mathcal{P}_0$-a.a. ω (such a function f exists by (A3) and Lemma 7.1) and by Schwarz inequality, one gets for any $v\in L^2(\nu)$ that

$$\int d\mathcal{P}_0(\omega)\int d\hat\omega(z)r_{0,z}(\omega)\big(\big|v(\omega,z)\big|+\big|v(\theta_z\omega,-z)\big|\big)=2\|v\|_{L^1(\nu)}\le 2\mathbb{E}_0[\lambda_0]^{1/2}\|v\|_{L^2(\nu)}<+\infty. \tag{77}$$

In particular, the definition of divergence is well posed and the map $L^2(\nu)\ni v\mapsto \operatorname{div} v\in L^1(\mathcal{P}_0)$ is continuous.

Lemma 8.3. *For any $v\in L^2(\nu)$ and any bounded and measurable function $u:\Omega_0\to\mathbb{R}$, it holds*

$$\int d\mathcal{P}_0(\omega)\operatorname{div} v(\omega)u(\omega)=-\int d\nu(\omega,z)v(\omega,z)\nabla u(\omega,z). \tag{78}$$

Proof. If is enough to apply Lemma 7.3 to $f(\omega,\omega')$ such that $f(\omega,\theta_z\omega)=v(\omega,z)u(\theta_z\omega)$ for $\mathcal{P}_0$-a.a. ω (such a function f exists by (A3) and Lemma 7.1) and observe that $f(\theta_z\omega,\omega)=v(\theta_z\omega,-z)u(\omega)$. □

Trivially, the above result implies the following:

Corollary 8.4. *Given a square integrable form $v\in L^2(\nu)$, we have that $v\in L^2_{\mathrm{sol}}(\nu)$ if and only if* $\operatorname{div} v=0$ *$\mathcal{P}_0$-a.s.*

Lemma 8.5. *If $\nabla u=0$ ν-a.e., then $u=$* constant *$\mathcal{P}_0$-a.s.*

Proof. We define $A := \{\omega \in \Omega_0 : u(\theta_z \omega) = u(\omega)\ \forall z \in \hat{\omega} \text{ with } r_{0,z}(\omega) > 0\}$. Hence (recall (65)) $\tilde{A} = \{\omega \in \Omega : u(\theta_y \omega) = u(\theta_z \omega)\ \forall y, z \in \hat{\omega} \text{ with } r_{z,y}(\omega) > 0\}$. The property that $\nabla u = 0$ ν-a.e. is equivalent to $\mathcal{P}_0(A) = 1$. By Lemma 7.1 we get that $\mathcal{P}(\tilde{A}) = 1$. Recall that the property in (A6) holds for $\omega \in \Omega_*$, Ω_* is translation invariant (as $\tilde{A}$) and $\mathcal{P}(\Omega_*) = 1$. Moreover, given $\omega \in \tilde{A} \cap \Omega_*$, there exists a constant $c(\omega)$ such that $u(\theta_y \omega) = c(\omega)$ for all $y \in \hat{\omega}$. Then we define $v(\omega) := c(\omega)$ if $\omega \in \tilde{A} \cap \Omega_*$ and $v(\omega) := 0$ if $\omega \notin \tilde{A} \cap \Omega_*$. As v is translation invariant and $\mathcal{P}$ is ergodic, there exists $c \in \mathbb{R}$ such that $\mathcal{P}(v = c) = 1$. By Lemma 7.1 we get $\mathcal{P}_0(v = c) = 1$. □

The proof of the following lemma is similar to the proof of [44, Lemma 2.5] (see also [17, App. B]). Recall (75).

Lemma 8.6. *Let $\zeta \in L^2(\mathcal{P}_0)$ be orthogonal to all functions $g \in L^2(\mathcal{P}_0)$ with $g = \mathrm{div}(\nabla u)$ for some $u \in H^1_{\mathrm{env}}$. Then $\zeta \in H^1_{\mathrm{env}}$ and $\nabla \zeta = 0$ in $L^2(\nu)$.*

By combining Lemma 8.5 and Lemma 8.6 we get:

Lemma 8.7. *The functions $g \in L^2(\mathcal{P}_0)$ of the form $g = \mathrm{div}\, v$ with $v \in L^2(\nu)$ are dense in $\{w \in L^2(\mathcal{P}_0) : \mathbb{E}_0[w] = 0\}$.*

Proof. Lemma 8.3 implies that $\mathbb{E}_0[g] = 0$ if $g = \mathrm{div}\, v$, $v \in L^2(\nu)$. Suppose that the claimed density fails. Then there exists $\zeta \in L^2(\mathcal{P}_0)$ different from zero with $\mathbb{E}_0[\zeta] = 0$ and such that $\mathbb{E}_0[\zeta g] = 0$ for any $g \in L^2(\mathcal{P}_0)$ of the form $g = \mathrm{div}\, v$ with $v \in L^2(\nu)$. By Lemma 8.6, we know that $\zeta \in H^1_{\mathrm{env}}$ and $\nabla \zeta = 0$ ν-a.s. By Lemma 8.5 we get that ζ is constant $\mathcal{P}_0$-a.s. Since $\mathbb{E}_0[\zeta] = 0$ it must be $\zeta = 0$ $\mathcal{P}_0$-a.s., which is absurd. □

9. The diffusion matrix *D* and the quadratic form *q*

Since $\lambda_2 \in L^1(\mathcal{P}_0)$ (see Assumption (A7)), given $a \in \mathbb{R}^d$ the form

$$u_a(\omega, z) := a \cdot z \tag{79}$$

is square integrable, i.e. it belongs to $L^2(\nu)$. We note that the symmetric diffusion matrix D defined in (29) satisfies, for any $a \in \mathbb{R}^d$,

$$q(a) := a \cdot Da = \inf_{v \in L^2_{\mathrm{pot}}(\nu)} \frac{1}{2} \int d\nu(\omega, x)\big(u_a(x) + v(\omega, x)\big)^2 = \inf_{v \in L^2_{\mathrm{pot}}(\nu)} \frac{1}{2} \|u_a + v\|^2_{L^2(\nu)} = \frac{1}{2} \big\|u_a + v^a\big\|^2_{L^2(\nu)}, \tag{80}$$

where $v^a = -\Pi u_a$ and $\Pi : L^2(\nu) \to L^2_{\mathrm{pot}}(\nu)$ denotes the orthogonal projection of $L^2(\nu)$ on $L^2_{\mathrm{pot}}(\nu)$. As a consequence, the map $\mathbb{R}^d \ni a \mapsto v^a \in L^2_{\mathrm{pot}}(\nu)$ is linear. Moreover, v^a is characterized by the property

$$v^a \in L^2_{\mathrm{pot}}(\nu), \qquad v^a + u_a \in L^2_{\mathrm{sol}}(\nu). \tag{81}$$

Hence we can write $a \cdot Da = \frac{1}{2}\|u_a + v^a\|^2_{L^2(\nu)} = \frac{1}{2}\langle u_a, u_a + v^a\rangle_\nu$. As the two symmetric bilinear forms $(a, b) \mapsto a \cdot Db$ and $(a, b) \mapsto \frac{1}{2}\int d\nu(\omega, z) a \cdot z(b \cdot z + v^b(\omega, z)) = \frac{1}{2}\int d\nu(u_a + v^a)(u_b + v^b)$ coincide on diagonal terms, we get

$$Da = \frac{1}{2} \int d\nu(\omega, z) z\big(a \cdot z + v^a(\omega, z)\big) \quad \forall a \in \mathbb{R}^d. \tag{82}$$

Let us come back to the quadratic form q on $\mathbb{R}^d$ defined in (80). By (80) its kernel $\mathrm{Ker}(q)$ is given by

$$\mathrm{Ker}(q) := \big\{a \in \mathbb{R}^d : q(a) = 0\big\} = \big\{a \in \mathbb{R}^d : u_a \in L^2_{\mathrm{pot}}(\nu)\big\}. \tag{83}$$

Lemma 9.1. *It holds*

$$\mathrm{Ker}(q)^\perp = \bigg\{\int d\nu(\omega, z) b(\omega, z) z \,:\, b \in L^2_{\mathrm{sol}}(\nu)\bigg\}. \tag{84}$$

Note that, since $\lambda_2 \in L^1(\mathcal{P}_0)$ by (A7), the integral in the r.h.s. of (84) is well defined. The above lemma corresponds to [44, Prop. 5.1].

Proof of Lemma 9.1. Let $b \in L^2_{\rm sol}(\nu)$ and $\eta_b := \int d\nu(\omega,z)b(\omega,z)z$. Then, given $a \in \mathbb{R}^d$, $a \cdot \eta_b = \langle u_a, b\rangle_\nu$. By (83), $a \in {\rm Ker}(q)$ if and only if $u_a \in L^2_{\rm pot}(\nu) = L^2_{\rm sol}(\nu)^\perp$. Therefore $a \in {\rm Ker}(q)$ if and only if $a \cdot \eta_b = 0$ for any $b \in L^2_{\rm sol}(\nu)$. □

Due to Lemma 9.1 and Definition 3.8 we have:

Corollary 9.2. ${\rm Span}\{\mathfrak{e}_1, \dots, \mathfrak{e}_{d_*}\} = \{\int d\nu(\omega,z)b(\omega,z)z : b \in L^2_{\rm sol}(\nu)\}$.

10. The contraction $b(\omega, z) \mapsto \hat{b}(\omega)$ and the set $\mathcal{A}_1[b]$

Definition 10.1. Let $b : \Omega_0 \times \mathbb{G} \to \mathbb{R}$ be a measurable function with $\|b\|_{L^1(\nu)} < +\infty$. We define the measurable function $r_b : \Omega_0 \to [0, +\infty]$ as

$$r_b(\omega) := \int d\hat{\omega}(z) r_{0,z}(\omega)\big|b(\omega,z)\big|, \tag{85}$$

the measurable function $\hat{b} : \Omega_0 \to \mathbb{R}$ as

$$\hat{b}(\omega) := \begin{cases} \int d\hat{\omega}(z) r_{0,z}(\omega) b(\omega,z) & \text{if } r_b(\omega) < +\infty, \\ 0 & \text{if } r_b(\omega) = +\infty, \end{cases} \tag{86}$$

and the measurable set $\mathcal{A}_1[b] := \{\omega \in \Omega_* : r_b(\theta_z\omega) < +\infty \ \forall z \in \hat{\omega}\}$.

Lemma 10.2. *Let $b : \Omega_0 \times \mathbb{G} \to \mathbb{R}$ be a measurable function with $\|b\|_{L^1(\nu)} < +\infty$. Then*

(i) $\|\hat{b}\|_{L^1(\mathcal{P}_0)} \le \|b\|_{L^1(\nu)} = \|r_b\|_{L^1(\mathcal{P}_0)}$ *and* $\mathbb{E}_0[\hat{b}] = \nu[b]$;
(ii) *given $\omega \in \mathcal{A}_1[b]$ and $\varphi \in C_c(\mathbb{R}^d)$, it holds*

$$\int d\mu^\varepsilon_\omega(x)\varphi(x)\hat{b}(\theta_{x/\varepsilon}\omega) = \int d\nu^\varepsilon_\omega(x,z)\varphi(x) b(\theta_{x/\varepsilon}\omega, z) \tag{87}$$

(*the series in the l.h.s. and in the r.h.s. are absolutely convergent*);
(iii) $\mathcal{P}(\mathcal{A}_1[b]) = \mathcal{P}_0(\mathcal{A}_1[b]) = 1$ *and $\mathcal{A}_1[b]$ is translation invariant.*

Proof. It is trivial to check Item (i) and Item (ii). We move to Item (iii). We have $\mathbb{E}_0[r_b] = \|b\|_{L^1(\nu)} < \infty$. This implies that $\mathcal{P}_0(\{\omega : r_b(\omega) < +\infty\}) = 1$ and therefore $\mathcal{P}(\mathcal{A}_1[b]) = \mathcal{P}_0(\mathcal{A}_1[b]) = 1$ by Lemma 7.1. The last property of $\mathcal{A}_1[b]$ follows immediately from the definition. □

We point out that, since ν has finite mass, $L^2(\nu) \subset L^1(\nu)$ and therefore Lemma 10.2 can be applied to b with $\|b\|_{L^2(\nu)} < +\infty$.

11. The transformation $b(\omega, z) \mapsto \tilde{b}(\omega, z)$

Definition 11.1. Given a measurable function $b : \Omega_0 \times \mathbb{G} \to \mathbb{R}$ we define $\tilde{b} : \Omega_0 \times \mathbb{G} \to \mathbb{R}$ as

$$\tilde{b}(\omega, z) := \begin{cases} b(\theta_z\omega, -z) & \text{if } z \in \hat{\omega}, \\ 0 & \text{otherwise.} \end{cases} \tag{88}$$

Lemma 11.2. *Given a measurable function $b : \Omega_0 \times \mathbb{G} \to \mathbb{R}$, it holds $\tilde{\tilde{b}}(\omega,z) = b(\omega,z)$ and $\|b\|_{L^p(\nu)} = \|\tilde{b}\|_{L^p(\nu)}$ for any p. If $b \in L^2(\nu)$, then ${\rm div}\,\tilde{b} = -{\rm div}\,b$.*

Proof. To get that $\|b\|_{L^p(\nu)} = \|\tilde{b}\|_{L^p(\nu)}$ it is enough to apply Lemma 7.3 with f such that $f(\omega, \theta_z\omega) = |b(\omega,z)|^p$ for $\mathcal{P}_0$-a.a. ω (to define f use (A3)). The other identities are trivial. □

Recall Definition 10.1.

Lemma 11.3. (i) *Let $b:\Omega_0\times\mathbb{G}\to[0,+\infty]$ and $\varphi,\psi:\mathbb{R}^d\to[0,+\infty]$ be measurable functions. Then, for each $\omega\in\Omega_*$, it holds*

$$\int d\nu^\varepsilon_\omega(x,z)\varphi(x)\psi(x+\varepsilon z)b(\theta_{x/\varepsilon}\omega,z)=\int d\nu^\varepsilon_\omega(x,z)\psi(x)\varphi(x+\varepsilon z)\tilde b(\theta_{x/\varepsilon}\omega,z). \tag{89}$$

(ii) *Let $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ be a measurable function and take $\omega\in\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]$. Given functions $\varphi,\psi:\mathbb{R}^d\to\mathbb{R}$ such that at least one between φ, ψ has compact support and the other is bounded, identity* (89) *is still valid. Given now φ with compact support and ψ bounded, it holds*

$$\int d\nu^\varepsilon_\omega(x,z)\nabla_\varepsilon\varphi(x,z)\psi(x+\varepsilon z)b(\theta_{x/\varepsilon}\omega,z)=-\int d\nu^\varepsilon_\omega(x,z)\nabla_\varepsilon\varphi(x,z)\psi(x)\tilde b(\theta_{x/\varepsilon}\omega,z). \tag{90}$$

Moreover, the above integrals in (89), (90) (*under the hypothesis of this Item* (*ii*)) *correspond to absolutely convergent series and are therefore well defined.*

Proof. We check (89) in Item (i). Since $c_{a,a'}(\omega)=c_{a',a}(\omega)$ and $b(\theta_a\omega,a'-a)=\tilde b(\theta_{a'}\omega,a-a')$ for all $a,a'\in\hat\omega$ (as $\omega\in\Omega_*$), we can write

$$\begin{aligned}\text{l.h.s. of (89)}&=\varepsilon^d\sum_{a\in\hat\omega}\sum_{a'\in\hat\omega}c_{a,a'}(\omega)\varphi(\varepsilon a)\psi\big(\varepsilon a'\big)b\big(\theta_a\omega,a'-a\big)\\&=\varepsilon^d\sum_{a'\in\hat\omega}\sum_{a\in\hat\omega}c_{a',a}(\omega)\psi\big(\varepsilon a'\big)\varphi(\varepsilon a)\tilde b\big(\theta_{a'}\omega,a-a'\big)=\text{r.h.s. of (89)}.\end{aligned}$$

The above identities hold also in Item (ii) since one deals indeed with absolutely convergent sum. For example, when φ has compact support and ψ is bounded, it is enough to observe that (as $\omega\in\mathcal{A}_1[b]$)

$$\int d\nu^\varepsilon_\omega(x,z)\big|\varphi(x)\big|\big|b(\theta_{x/\varepsilon}\omega,z)\big|\le\int d\mu^\varepsilon_\omega(x)\big|\varphi(x)\big|r_b(\theta_{x/\varepsilon}\omega)<+\infty. \tag{91}$$

We now prove (90). We have

$$\begin{aligned}\text{l.h.s. of (90)}&=\varepsilon^d\sum_{a\in\hat\omega}\sum_{a'\in\hat\omega}c_{a,a'}(\omega)\frac{\varphi(\varepsilon a')-\varphi(\varepsilon a)}{\varepsilon}\psi\big(\varepsilon a'\big)b\big(\theta_a\omega,a'-a\big)\\&=-\varepsilon^d\sum_{a'\in\hat\omega}\sum_{a\in\hat\omega}c_{a',a}(\omega)\frac{\varphi(\varepsilon a)-\varphi(\varepsilon a')}{\varepsilon}\psi\big(\varepsilon a'\big)\tilde b\big(\theta_{a'}\omega,a-a'\big)=\text{r.h.s. of (90)}.\end{aligned}$$

The above arrangements are indeed legal as one can easily prove that the above series are absolutely convergent. □

Definition 11.4. Let $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ be a measurable function. If $\omega\in\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]\cap\Omega_0$, we set $\operatorname{div}_*b(\omega):=\hat b(\omega)-\hat{\tilde b}(\omega)\in\mathbb{R}$.

Lemma 11.5. *Let $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ be a measurable function with $\|b\|_{L^2(\nu)}<+\infty$. Then $\mathcal{P}_0(\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b])=1$ and* $\operatorname{div}_*b=\operatorname{div}b$ *in $L^1(\mathcal{P}_0)$.*

Proof. By Lemma 11.2 we have $\|\tilde b\|_{L^2(\nu)}<\infty$. Hence, both b and $\tilde b$ are ν-integrable. By Lemma 10.2-(iii) we get that $\mathcal{P}_0(\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b])=1$. The identity $\operatorname{div}_*b=\operatorname{div}b$ in $L^1(\mathcal{P}_0)$ is trivial. □

Lemma 11.6. *Let $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ be a measurable function with $\|b\|_{L^2(\nu)}<+\infty$ and such that its class of equivalence in $L^2(\nu)$ belongs to $L^2_{\mathrm{sol}}(\nu)$. Let*

$$\mathcal{A}_d[b]:=\big\{\omega\in\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]:\operatorname{div}_*b(\theta_z\omega)=0\ \forall z\in\hat\omega\big\}. \tag{92}$$

Then $\mathcal{P}(\mathcal{A}_d[b])=1$ and $\mathcal{A}_d[b]$ is translation invariant.

Proof. By Corollary 8.4 and Lemma 11.5 we have $\mathcal{P}_0(A)=1$ where $A:=\{\omega\in\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]\cap\Omega_0:\operatorname{div}_* b(\omega)=0\}$. By Lemma 7.1 and (65), $\mathcal{P}(\tilde A)=1$. To get that $\mathcal{P}(\mathcal{A}_d[b])=1$ it is enough to observe that $\tilde A=\mathcal{A}_d[b]$. The translation invariance is trivial. □

Lemma 11.7. *Suppose that $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ is a measurable function with $\|b\|_{L^2(\nu)}<+\infty$. Take $\omega\in\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]$. Then for any $\varepsilon>0$ and any $u:\mathbb{R}^d\to\mathbb{R}$ with compact support it holds*

$$\int d\mu_\omega^\varepsilon(x)u(x)\operatorname{div}_* b(\theta_{x/\varepsilon}\omega)=-\varepsilon\int d\nu_\omega^\varepsilon(x,z)\nabla_\varepsilon u(x,z)b(\theta_{x/\varepsilon}\omega,z). \tag{93}$$

Proof. Note that $\theta_{x/\varepsilon}\omega$ in the l.h.s. of (93) belongs to $\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]\cap\Omega_0$. Hence, by Definition 11.4, we can write the l.h.s. of (93) as

$$\int d\nu_\omega^\varepsilon(x,z)u(x)b(\theta_{x/\varepsilon}\omega,z)-\int d\nu_\omega^\varepsilon(x,z)u(x)\tilde b(\theta_{x/\varepsilon}\omega,z). \tag{94}$$

Due to our assumptions we are dealing with absolutely convergent series, hence the above rearrangements are free. By applying (89) to the r.h.s. of (94) (see Item (ii) of Lemma 11.3), we can rewrite (94) as $\int d\nu_\omega^\varepsilon(x,z)b(\theta_{x/\varepsilon}\omega,z)[u(x)-u(x+\varepsilon z)]$ and this allows to conclude. □

12. Typical environments

We can now describe the set $\Omega_{\rm typ}$ of typical environments appearing in Theorem 4.1 and Theorem 4.4. We first fix some basic notation and observations, frequently used below. Given $M>0$ and $a\in\mathbb{R}$, we define $[a]_M$ as

$$[a]_M=M\mathbb{1}_{\{a>M\}}+a\mathbb{1}_{\{|a|\le M\}}-M\mathbb{1}_{\{a<-M\}}. \tag{95}$$

Given $a\ge b$, it holds $a-b\ge[a]_M-[b]_M\ge 0$. Hence, for any $a,b\in\mathbb{R}$, it holds

$$\big|[a]_M-[b]_M\big|\le|a-b|, \tag{96}$$

$$\big|[a-b]-\big[[a]_M-[b]_M\big]\big|\le|a-b|. \tag{97}$$

Recall that, due to Assumption (A8), the space $L^2(\mathcal{P}_0)$ is separable.

Lemma 12.1. *The space $L^2(\nu)$ is separable.*

Proof. By the separability of $L^2(\mathcal{P}_0)$ there exists a countable dense set $\{f_j\}$ in $L^2(\mathcal{P}_0)$. At cost to approximate, in $L^2(\mathcal{P}_0)$, f_j by $[f_j]_M$ as $M\to\infty$ (cf. (95)), we can suppose that f_j is bounded. Let $\{B_k\}$ be the countable family of closed balls in $\mathbb{R}^d$ with rational radius and with center in $\mathbb{Q}^d$. It is then trivial to check that the zero function is the only function in $L^2(\nu)$ orthogonal to all functions $f_j(\omega)\mathbb{1}_{B_k}(z)$ (which belong indeed to $L^2(\nu)$). □

In the construction of the functional sets presented below, we will use the separability of $L^2(\mathcal{P}_0)$ and $L^2(\nu)$ without further mention. The definition of these functional sets and the typical environments (cf. Definition 12.3) consists of a list of technical assumptions, which are necessary to justify several steps in the next sections (there, we will indicate explicitly which technical assumption we are using).

Recall the sets $\mathcal{A}_1[b]$ and $\mathcal{A}_d[b]$ introduced respectively in Definition 10.1 and (92). Recall Definition 11.1 of $\tilde b$. Recall (65). Recall the set $\mathcal{A}[f]$ in Proposition 3.1 for a measurable function $f:\Omega_0\to\mathbb{R}$ with $\|f\|_{L^1(\mathcal{P}_0)}<+\infty$.

Definition 12.2. Given a function $f:\Omega_0\to[0,+\infty]$ such that $\|f\|_{L^1(\mathcal{P}_0)}<+\infty$, we define $\mathcal{A}[f]$ as $\mathcal{A}[f_0]$, where $f_0:\Omega_0\to\mathbb{R}$ is defined as f on $\{f<+\infty\}$ and as 0 on $\{f=+\infty\}$.

⊙ *The functional sets $\mathcal{G}_1$, $\mathcal{H}_1$.* We fix a countable set $\mathcal{H}_1$ of measurable functions $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ such that $\|b\|_{L^2(\nu)}<+\infty$ for any $b\in\mathcal{H}_1$ and such that $\{\operatorname{div} b:b\in\mathcal{H}_1\}$ is a dense subset of $\{w\in L^2(\mathcal{P}_0):\mathbb{E}_0[w]=0\}$ when thought of as set of L^2-functions (recall Lemma 8.7). For each $b\in\mathcal{H}_1$ we define the measurable function $g_b:\Omega_0\to\mathbb{R}$ as

$$g_b(\omega):=\begin{cases}\operatorname{div}_* b(\omega) & \text{if } \omega\in\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b],\\ 0 & \text{otherwise.}\end{cases} \tag{98}$$

Note that by Lemma 11.5 $g_b=\operatorname{div} b$ $\mathcal{P}_0$-a.s. We set $\mathcal{G}_1:=\{g_b:b\in\mathcal{H}_1\}$.

⊙ *The functional sets* $\mathcal{G}_2$, $\mathcal{H}_2$, $\mathcal{H}_3$. We fix a countable set $\mathcal{G}_2$ of bounded measurable functions $g:\Omega_0\to\mathbb{R}$ such that the set $\{\nabla g: g\in\mathcal{G}_2\}$, thought in $L^2(\nu)$, is dense in $L^2_{\text{pot}}(\nu)$ (this is possible by the definition of $L^2_{\text{pot}}(\nu)$). We define $\mathcal{H}_2$ as the set of measurable functions $h:\Omega_0\times\mathbb{G}\to\mathbb{R}$ such that $h=\nabla g$ for some $g\in\mathcal{G}_2$. We define $\mathcal{H}_3$ as the set of measurable functions $h:\Omega_0\times\mathbb{G}\to\mathbb{R}$ such that $h(\omega,z)=g(\theta_z\omega)z_i$ for some $g\in\mathcal{G}_2$ and some direction $i=1,\dots,d$. Note that, since $\mathbb{E}_0[\lambda_2]<+\infty$ by (A7) and since g is bounded, $\|h\|_{L^2(\nu)}<+\infty$ for all $h\in\mathcal{H}_3$.

⊙ *The functional set* $\mathcal{W}$. We fix a countable set $\mathcal{W}$ of measurable functions $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ such that, thought of as subset of $L^2(\nu)$, $\mathcal{W}$ is dense in $L^2_{\text{sol}}(\nu)$. By Corollary 8.4 and Lemma 11.2, $\tilde b\in L^2_{\text{sol}}(\nu)$ for any $b\in L^2_{\text{sol}}(\nu)$. Hence, at cost to enlarge $\mathcal{W}$, we assume that $\tilde b\in\mathcal{W}$ for any $b\in\mathcal{W}$.

⊙ *The functional set* $\mathcal{G}$. We fix a countable set $\mathcal{G}$ of measurable functions $g:\Omega_0\to\mathbb{R}$ such that:

- $\|g\|_{L^2(\mathcal{P}_0)}<+\infty$ for any $g\in\mathcal{G}$ and $\mathcal{G}$ is dense in $L^2(\mathcal{P}_0)$ when thought of as a subset of $L^2(\mathcal{P}_0)$;
- $1\in\mathcal{G}$, $\mathcal{G}_1\subset\mathcal{G}$, $\mathcal{G}_2\subset\mathcal{G}$;
- $\lambda_0\wedge\sqrt{M}\in\mathcal{G}$ for any $M\in\mathbb{N}$;
- for each $b\in\mathcal{W}$, $M\in\mathbb{N}$, $i\in\{1,\dots,d\}$ and $\ell\in\mathbb{N}$, the function $[f]_\ell:\Omega_0\to\mathbb{R}$ where (cf. (95))

$$f(\omega):=\begin{cases}\int d\hat\omega(z)r_{0,z}(\omega)z_i\mathbb{1}_{\{|z|\le\ell\}}[b]_M(\omega,z) & \text{if } \int d\hat\omega(z)r_{0,z}(\omega)|z_i|<+\infty,\\ 0 & \text{otherwise}\end{cases}\tag{99}$$

  belongs to $\mathcal{G}$;
- at cost to enlarge $\mathcal{G}$ we assume that $[g]_M\in\mathcal{G}$ for any $g\in\mathcal{G}$ and $M\in\mathbb{N}$.

⊙ *The functional set* $\mathcal{H}$. We fix a countable set of measurable functions $b:\Omega_0\times\mathbb{G}\to\mathbb{R}$ such that

- $\|b\|_{L^2(\nu)}<+\infty$ for any $b\in\mathcal{H}$ and $\mathcal{H}$ is dense in $L^2(\nu)$ when thought of as a subset of $L^2(\nu)$;
- $\mathcal{H}_1\cup\mathcal{H}_2\cup\mathcal{H}_3\cup\mathcal{W}\subset\mathcal{H}$ and $1\in\mathcal{H}$;
- $\forall i=1,\dots,d$ the map $(\omega,z)\mapsto z_i$ is in $\mathcal{H}$ (recall: $\lambda_2\in L^1(\mathcal{P}_0)$ by (A7));
- at cost to enlarge $\mathcal{H}$ we assume that $[b]_M\in\mathcal{H}$ and that $\tilde b\in\mathcal{H}$ for any $b\in\mathcal{H}$ and $M\in\mathbb{N}$ ($\tilde b\in L^2(\nu)$ by Lemma 11.2).

Definition 12.3. We define Ω_{typ} as the intersection of the following sets:

- $\mathcal{A}[gg']$ for all $g,g'\in\mathcal{G}$ (recall that $1\in\mathcal{G}$);
- $\mathcal{A}_1[bb']\cap\mathcal{A}[\widehat{bb'}]$ for all $b,b'\in\mathcal{H}$ (recall that $1\in\mathcal{H}$, $\tilde b\in\mathcal{H}$ $\forall b\in\mathcal{H}$ and recall Lemma 10.2);
- $\tilde A\cap\mathcal{A}[\lambda_i]$ where $A:=\{\lambda_i<+\infty\}$ and $i=0,1,2$ (recall (A7) and (65));
- $\tilde A\cap\mathcal{A}[\lambda_0\mathbb{1}_{\{\lambda_0>\sqrt{M}\}}]$ with $A=\{\lambda_0<+\infty\}$ and $M\in\mathbb{N}$ (recall (A7));
- $\mathcal{A}_d[b]$ for all $b\in\mathcal{W}$ (recall (92));
- $\mathcal{A}[h_\ell]$ where $\ell\in\mathbb{N}$ and $h_\ell(\omega):=\int d\hat\omega(z)r_{0,z}(\omega)|z|^2\mathbb{1}_{\{|z|\ge\ell\}}$;
- $\mathcal{A}[f-[f]_\ell]$ where f varies among the functions (99) with $b\in\mathcal{W}$, $M\in\mathbb{N}$, $i\in\{1,\dots,d\}$ and $\ell\in\mathbb{N}$.

Remark 12.4. $\Omega_{\text{typ}}\subset\Omega_*\cap\Omega_1$ (see our Assumptions and (24) for the definition of Ω_* and Ω_1).

Proposition 12.5. *The above set* Ω_{typ} *is a translation invariant measurable subset of* Ω *such that* $\mathcal{P}(\Omega_{\text{typ}})=1$.

Proof. The claim follows from Proposition 3.1 for all sets of the form $\mathcal{A}[\cdot]$, from Lemma 10.2 for all sets of the form $\mathcal{A}_1[\cdot]$, from Lemma 11.6 for all sets of the form $\mathcal{A}_d[\cdot]$ and from Corollary 7.2 for all sets of the form $\tilde A$. □

13. 2-Scale convergence of $v_\varepsilon\in L^2(\mu^\varepsilon_{\tilde\omega})$ and of $w_\varepsilon\in L^2(\nu^\varepsilon_{\tilde\omega})$

In this section we recall the notion of 2-scale convergence in our context.

Definition 13.1. Fix $\tilde\omega\in\Omega_{\text{typ}}$, an ε-parametrized family $v_\varepsilon\in L^2(\mu^\varepsilon_{\tilde\omega})$ and a function $v\in L^2(m\,dx\times\mathcal{P}_0)$.

- We say that v_ε *is weakly* 2*-scale convergent to* v, and write $v_\varepsilon\stackrel{2}{\rightharpoonup}v$, if the family $\{v_\varepsilon\}$ is bounded, i.e. $\limsup_{\varepsilon\downarrow 0}\|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})}<+\infty$, and

$$\lim_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega)=\int d\mathcal{P}_0(\omega)\int dx\,m v(x,\omega)\varphi(x)g(\omega),\tag{100}$$

  for any $\varphi\in C_c(\mathbb{R}^d)$ and any $g\in\mathcal{G}$.

- We say that v_ε *is strongly* 2*-scale convergent to* v, and write $v_\varepsilon \xrightarrow{2} v$, if the family $\{v_\varepsilon\}$ is bounded and

$$\lim_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)u_\varepsilon(x)=\int d\mathcal{P}_0(\omega)\int dx\, m v(x,\omega)u(x,\omega) \tag{101}$$

whenever $u_\varepsilon \overset{2}{\rightharpoonup} u$.

Lemma 13.2. *Let $\tilde\omega\in\Omega_{\text{typ}}$. Then, for any $\varphi\in C_c(\mathbb{R}^d)$ and $g\in\mathcal{G}$, setting $v_\varepsilon(x):=\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega)$ it holds $L^2(\mu^\varepsilon_{\tilde\omega})\ni v_\varepsilon \overset{2}{\rightharpoonup} \varphi(x)g(\omega)\in L^2(m\,dx\times\mathcal{P}_0)$.*

Proof. Since $\tilde\omega\in\Omega_{\text{typ}}\subset\mathcal{A}[g]$, we get $\lim_{\varepsilon\downarrow 0}\|v_\varepsilon\|^2_{L^2(\mu^\varepsilon_{\tilde\omega})}=\int dx\, m\varphi(x)^2\mathbb{E}_0[g^2]$, hence $\{v_\varepsilon\}$ is bounded in $L^2(\mu^\varepsilon_{\tilde\omega})$. Since $g\in\mathcal{G}\subset L^2(\mathcal{P}_0)$, we have $\varphi(x)g(\omega)\in L^2(m\,dx\times\mathcal{P}_0)$. Take $\varphi_1\in C_c(\mathbb{R}^d)$ and $g_1\in\mathcal{G}$. Since $\tilde\omega\in\Omega_{\text{typ}}\subset\mathcal{A}[gg_1]$, it holds $\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)\varphi_1(x)g_1(\theta_{x/\varepsilon}\tilde\omega)\to\int dx\, m\varphi(x)\varphi_1(x)\mathbb{E}_0[gg_1]$. □

Lemma 13.3. *Given $\tilde\omega\in\Omega_{\text{typ}}$, if $v_\varepsilon \overset{2}{\rightharpoonup} v$ then*

$$\varliminf_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)^2\ge\int d\mathcal{P}_0(\omega)\int dx\, m v(x,\omega)^2. \tag{102}$$

The proof is similar to the proof of [43, Item (iii), p. 984] and therefore omitted: replace Φ in [43] with a linear combination of functions $\varphi(x)g(\omega)$ with $\varphi\in C_c(\mathbb{R}^d)$ and $g\in\mathcal{G}$, use the density of $\mathcal{G}$ in $L^2(\mathcal{P}_0)$ and the property that $\Omega_{\text{typ}}\subset\mathcal{A}[gg']$ for all $g,g'\in\mathcal{G}$ (cf. [17, Lemma 10.5]).

Using Lemmas 13.2 and 13.3 one gets the following characterization:

Lemma 13.4. *Given $\tilde\omega\in\Omega_{\text{typ}}$, $v_\varepsilon\xrightarrow{2} v$ if and only if $v_\varepsilon \overset{2}{\rightharpoonup} v$ and*

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)^2=\int_\Omega d\mathcal{P}_0(\omega)\int_{\mathbb{R}^d} dx\, m v(x,\omega)^2. \tag{103}$$

Proof. If $v_\varepsilon\to v$, then $v_\varepsilon \overset{2}{\rightharpoonup} v$ by Lemma 13.2. By then applying (101) with $u_\varepsilon:=v_\varepsilon$, we get (103). The opposite implication corresponds to [43, Item (iv), p. 984] and the proof there can be easily adapted to our setting due to Lemma 13.3. (cf. [17, Lemma 10.4]). □

Lemma 13.5. *Let $\tilde\omega\in\Omega_{\text{typ}}$. Then, given a bounded family of functions $v_\varepsilon\in L^2(\mu^\varepsilon_{\tilde\omega})$, there exists a sequence $\varepsilon_k\downarrow 0$ such that $v_{\varepsilon_k} \overset{2}{\rightharpoonup} v$ for some $v\in L^2(m\,dx\times\mathcal{P}_0)$ with $\|v\|_{L^2(m\,dx\times\mathcal{P}_0)}\le\limsup_{\varepsilon\downarrow 0}\|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})}$.*

The proof of the above lemma is similar to the proof of [43, Prop. 2.2], but in [43] some density in uniform norm is used. Since that density is absent here, we provide the proof in Appendix F to explain how to fill the gap.

Definition 13.6. Given $\tilde\omega\in\Omega_{\text{typ}}$, a family $w_\varepsilon\in L^2(\nu^\varepsilon_{\tilde\omega})$ and a function $w\in L^2(m\,dx\times d\nu)$, we say that w_ε *is weakly* 2*-scale convergent to* w, and write $w_\varepsilon \overset{2}{\rightharpoonup} w$, if $\{w_\varepsilon\}$ is bounded in $L^2(\nu^\varepsilon_{\tilde\omega})$, i.e. $\varlimsup_{\varepsilon\downarrow 0}\|w_\varepsilon\|_{L^2(\nu^\varepsilon_{\tilde\omega})}<+\infty$, and

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)w_\varepsilon(x,z)\varphi(x)b(\theta_{x/\varepsilon}\tilde\omega,z)=\int dx\, m\int d\nu(\omega,z)w(x,\omega,z)\varphi(x)b(\omega,z), \tag{104}$$

for any $\varphi\in C_c(\mathbb{R}^d)$ and any $b\in\mathcal{H}$.

Lemma 13.7. *Let $\tilde\omega\in\Omega_{\text{typ}}$. Then, given a bounded family of functions $w_\varepsilon\in L^2(\nu^\varepsilon_{\tilde\omega})$, there exists a sequence $\varepsilon_k\downarrow 0$ such that $w_{\varepsilon_k} \overset{2}{\rightharpoonup} w$ for some $w\in L^2(m\,dx\times\nu)$ with $\|w\|_{L^2(m\,dx\times\nu)}\le\limsup_{\varepsilon\downarrow 0}\|w_\varepsilon\|_{L^2(\nu^\varepsilon_{\tilde\omega})}$.*

We postpone a sketch of the proof of Lemma 13.7 to Appendix F.

14. Roadmap of the proof of Theorem 4.1

The main strategy to prove Theorem 4.1 is the same e.g. of the one in [44] to prove Theorem 6.1 there. Below we provide a list of the main steps to arrive to some key structure results concerning the solutions u_ε of (41) and allowing then to easily conclude the proof. The details are provided in the next sections.

We fix a typical environment $\tilde\omega \in \Omega_{\rm typ}$. The first step is to prove the following structure result: given a bounded family of functions u_ε in $H^1_{\tilde\omega,\varepsilon}$, along a subsequence ε_k it holds

$$L^2\big(\mu^\varepsilon_{\tilde\omega}\big) \ni u_\varepsilon \stackrel{2}{\rightharpoonup} u \in L^2(m\,dx \times \mathcal{P}_0),$$

$$L^2\big(\nu^\varepsilon_{\tilde\omega}\big) \ni \nabla u_\varepsilon(x,z) \stackrel{2}{\rightharpoonup} w(x,\omega,z) := \nabla_* u(x)\cdot z + u_1(x,\omega,z) \in L^2(m\,dx \times \nu),$$

for suitable functions u, u_1 with $u = u(x) \in H^1_*(m\,dx)$, $u_1 \in L^2(\mathbb{R}^d, L^2_{\rm pot}(\nu))$. We devote Sections 15, 16, 17 and 18 to the above result.

Suppose now that $\{f_\varepsilon\}$ is a bounded family in $L^2(\mu^\varepsilon_{\tilde\omega})$. Let $u_\varepsilon \in H^{1,{\rm f}}_{\tilde\omega,\varepsilon}$ be the weak solution of (41), i.e.

$$\frac{1}{2}\langle \nabla_\varepsilon v, \nabla_\varepsilon u_\varepsilon\rangle_{\nu^\varepsilon_{\tilde\omega}} + \langle v, u_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}} = \langle v, f_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}}, \quad \forall v \in H^{1,{\rm f}}_{\tilde\omega,\varepsilon}. \tag{105}$$

It is then standard to get from (105) that the family $\{u_\varepsilon\}$ is bounded in $H^1_{\tilde\omega,\varepsilon}$. Hence we can apply the above structure result.

The next step is to prove that, for dx-a.e. x, the map $(\omega,z) \mapsto w(x,\omega,z)$ belong to $L^2_{\rm sol}(\nu)$ (this is the first part of the proof of Claim 19.3 in Section 19 and relies on a suitable choice of v in (105)).

As $w(x,\omega,z) := \nabla_* u(x)\cdot z + u_1(x,\omega,z)$, $w(x,\cdot,\cdot) \in L^2_{\rm sol}(\nu)$ and $u_1(x,\cdot,\cdot) \in L^2_{\rm pot}(\nu)$ for dx-a.e. x, from (81) we get that $u_1(x,\cdot,\cdot) = v^a(\cdot,\cdot)$ with $a = \nabla_* u(x)$ for dx-a.e. x (i.e. $u_1(x,\cdot,\cdot)$ is the orthogonal projection onto $L^2_{\rm pot}(\nu)$ of the form $(\omega,z) \mapsto -\nabla_* u(x)\cdot z$). As a byproduct with (82) we then get that $\int d\nu(\omega,z) w(x,\omega,z) z = 2D\nabla_* u(x)$ for dx-a.e. x (see Claim 19.3). The effective homogenized matrix D has finally emerged.

From this point the conclusion of the proof of Theorem 4.1 becomes relatively simple and is detailed in Section 19 after the proof of Claim 19.3.

15. Cut-off for functions $v_\varepsilon \in L^2(\mu^\varepsilon_{\tilde\omega})$

$\mathbb{N}_+$ denotes the set of positive integers. Recall (95).

Lemma 15.1. *Let $\tilde\omega \in \Omega_{\rm typ}$ and let $\{v_\varepsilon\}$ be a family of functions such that $v_\varepsilon \in L^2(\mu^\varepsilon_{\tilde\omega})$ and $\limsup_{\varepsilon\downarrow 0} \|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} < +\infty$. Then there exist functions $v, v_M \in L^2(m\,dx \times \mathcal{P}_0)$ with M varying in $\mathbb{N}_+$ such that*

(i) $v_\varepsilon \stackrel{2}{\rightharpoonup} v$ *and* $[v_\varepsilon]_M \stackrel{2}{\rightharpoonup} v_M$ *for all* $M \in \mathbb{N}_+$, *along a sequence* $\varepsilon_k \downarrow 0$;
(ii) *for any* $\varphi \in C_c(\mathbb{R}^d)$ *and* $u \in \mathcal{G}$ *it holds*

$$\lim_{M\to\infty} \int dx\, m \int d\mathcal{P}_0(\omega) v_M(x,\omega)\varphi(x)u(\omega) = \int dx\, m \int d\mathcal{P}_0(\omega) v(x,\omega)\varphi(x)u(\omega). \tag{106}$$

Proof. Without loss, we assume that $\|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \le C_0 < +\infty$ for all ε. We set $v^\varepsilon_M := [v_\varepsilon]_M$. Since $\|v^\varepsilon_M\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \le \|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \le C_0$, Item (i) follows from Lemma 13.5 and a diagonal procedure.

Below the convergence $\varepsilon \downarrow 0$ is understood along the sequence $\{\varepsilon_k\}$. Let us define $F(\bar v, \bar\varphi, \bar u) := \int dx\, m \int \mathcal{P}_0(d\omega)\bar v(x,\omega)\bar\varphi(x)\bar u(\omega)$. Then Item (ii) corresponds to the limit

$$\lim_{M\to\infty} F(v_M,\varphi,u) = F(v,\varphi,u) \quad \forall \varphi \in C_c\big(\mathbb{R}^d\big), \forall u \in \mathcal{G}. \tag{107}$$

We fix functions φ, u as in (107) and set $u_k := [u]_k$ for all $k \in \mathbb{N}_+$. By definition of $\mathcal{G}$, we have $u_k \in \mathcal{G}$ for all k (see Section 12).

Claim 15.2. *For each $k, M \in \mathbb{N}_+$ it holds*

$$\big|F(v,\varphi,u) - F(v,\varphi,u_k)\big| \le C_0\|\varphi\|_{L^2(m\,dx)}\|u-u_k\|_{L^2(\mathcal{P}_0)}, \tag{108}$$

$$\big|F(v_M,\varphi,u) - F(v_M,\varphi,u_k)\big| \le C_0\|\varphi\|_{L^2(m\,dx)}\|u-u_k\|_{L^2(\mathcal{P}_0)}. \tag{109}$$

Proof of Claim 15.2. By Schwarz inequality

$$\big|F(v,\varphi,u)-F(v,\varphi,u_k)\big| \le \|v\|_{L^2(m\,dx\times\mathcal{P}_0)}\big\|\varphi(u-u_k)\big\|_{L^2(m\,dx\times\mathcal{P}_0)}.$$

To get (108) it is then enough to apply Lemma 13.3 (or Lemma 13.5) to bound $\|v\|_{L^2(m\,dx\times\mathcal{P}_0)}$ by C_0. The proof of (109) is identical. □

Claim 15.3. *For each $k, M \in \mathbb{N}_+$ it holds*

$$\big|F(v,\varphi,u_k)-F(v_M,\varphi,u_k)\big| \le (k/M)\|\varphi\|_\infty C_0^2. \tag{110}$$

Proof of Claim 15.3. We note that $(v_\varepsilon - v_M^\varepsilon)(x)=0$ if $|v_\varepsilon(x)|\le M$. Hence we can bound

$$\big|v_\varepsilon - v_M^\varepsilon\big|(x) = \big|v_\varepsilon - v_M^\varepsilon\big|(x)\mathbb{1}_{\{|v_\varepsilon(x)|>M\}} \le \big|v_\varepsilon - v_M^\varepsilon\big|(x)\frac{|v_\varepsilon(x)|}{M} \le \frac{v_\varepsilon(x)^2}{M}. \tag{111}$$

We observe that $F(v,\varphi,u_k)=\lim_{\varepsilon\downarrow 0}\int d\mu_{\tilde\omega}^\varepsilon(x)v_\varepsilon(x)\varphi(x)u_k(\theta_{x/\varepsilon}\tilde\omega)$, since $u_k\in\mathcal{G}$ and $v_\varepsilon \overset{2}{\rightharpoonup} v$. A similar representation holds for $F(v_M,\varphi,u_k)$. As a consequence, and using (111), we get

$$\begin{aligned}\big|F(v,\varphi,u_k)-F(v_M,\varphi,u_k)\big| &\le \overline{\lim_{\varepsilon\downarrow 0}}\int d\mu_{\tilde\omega}^\varepsilon(x)\big|\big(v_\varepsilon - v_M^\varepsilon\big)(x)\varphi(x)u_k(\theta_{x/\varepsilon}\tilde\omega)\big| \\ &\le (k/M)\|\varphi\|_\infty \overline{\lim_{\varepsilon\downarrow 0}}\int d\mu_{\tilde\omega}^\varepsilon(x)v_\varepsilon(x)^2 \le (k/M)\|\varphi\|_\infty C_0^2.\end{aligned}$$

□

We can finally conclude the proof of (107). Given $\varphi\in C_c(\mathbb{R}^d)$ and $u\in\mathcal{G}$, by applying Claims 15.2 and 15.3, we can bound

$$\begin{aligned}\big|F(v_M,\varphi,u)-F(v,\varphi,u)\big| &\le \big|F(v_M,\varphi,u)-F(v_M,\varphi,u_k)\big| \\ &\quad + \big|F(v_M,\varphi,u_k)-F(v,\varphi,u_k)\big| + \big|F(v,\varphi,u_k)-F(v,\varphi,u)\big| \\ &\le 2C_0\|\varphi\|_{L^2(m\,dx)}\|u-u_k\|_{L^2(\mathcal{P}_0)} + (k/M)\|\varphi\|_\infty C_0^2.\end{aligned} \tag{112}$$

(107) then follows by taking first the limit $M\to\infty$ and afterwards the limit $k\to\infty$, and using that $\lim_{k\to\infty}\|u-u_k\|_{L^2(\mathcal{P}_0)}=0$. □

16. Structure of the 2-scale weak limit of a bounded family in $H^1_{\tilde\omega,\varepsilon}$: Part I

It is simple to check the following Leibniz rule for the microscopic gradient:

$$\nabla_\varepsilon(fg)(x,z) = \nabla_\varepsilon f(x,z)g(x) + f(x+\varepsilon z)\nabla_\varepsilon g(x,z), \tag{113}$$

where $f,g:\varepsilon\hat\omega\to\mathbb{R}$.

The following Proposition 16.1 is related to [44, Lemma 5.3]. In the proof we will use a cut-off procedure based on Lemma 15.1 (see Remark 16.2 below).

Proposition 16.1. *Let $\tilde\omega\in\Omega_{\rm typ}$. Let $\{v_\varepsilon\}$ be a family of functions $v_\varepsilon\in H^1_{\tilde\omega,\varepsilon}$ satisfying*

$$\limsup_{\varepsilon\downarrow 0}\|v_\varepsilon\|_{L^2(\mu_{\tilde\omega}^\varepsilon)} < +\infty, \qquad \limsup_{\varepsilon\downarrow 0}\|\nabla_\varepsilon v_\varepsilon\|_{L^2(\nu_{\tilde\omega}^\varepsilon)} < +\infty. \tag{114}$$

Then, along a sequence $\varepsilon_k\downarrow 0$, we have that $v_\varepsilon \overset{2}{\rightharpoonup} v$, where $v\in L^2(m\,dx\times\mathcal{P}_0)$ does not depend on ω, i.e. for dx-a.e. $x\in\mathbb{R}^d$ the function $\omega\mapsto v(x,\omega)$ is constant $\mathcal{P}_0$-a.s.

Proof. Due to Lemma 13.5 we have that $v_\varepsilon \overset{2}{\rightharpoonup} v\in L^2(m\,dx\times\mathcal{P}_0)$ along a sequence $\varepsilon_k\downarrow 0$. Recall the definition of the functional sets $\mathcal{G}_1, \mathcal{H}_1$ given in Section 12. We claim that $\forall\varphi\in C_c^1(\mathbb{R}^d)$ and $\forall\psi\in\mathcal{G}_1$ it holds

$$\int dx\, m\int d\mathcal{P}_0(\omega)v(x,\omega)\varphi(x)\psi(\omega) = 0. \tag{115}$$

Before proving our claim, let us explain how it leads to the thesis. Since φ varies among $C_c^1(\mathbb{R}^d)$ while ψ varies in a countable set, (115) implies that, dx-a.e., $\int d\mathcal{P}_0(\omega)v(x,\omega)\psi(\omega)=0$ for any $\psi\in\mathcal{G}_1$. We conclude that, dx-a.e., $v(x,\cdot)$ is orthogonal in $L^2(\mathcal{P}_0)$ to $\{w\in L^2(\mathcal{P}_0):\mathbb{E}_0[w]=0\}$ (due to the density of $\mathcal{G}_1$), which is equivalent to the fact that $v(x,\omega)=\mathbb{E}_0[v(x,\cdot)]$ for $\mathcal{P}_0$-a.a. ω.

It now remains to prove (115). Since $\tilde\omega\in\Omega_{\mathrm{typ}}$ and due to (114), at cost to refine the sequence $\{\varepsilon_k\}$, Items (i) and (ii) of Lemma 15.1 hold (we keep the same notation of Lemma 15.1). Hence, in oder to prove (115), it is enough to prove for any M that, given $\varphi\in C_c^1(\mathbb{R}^d)$ and $\psi\in\mathcal{G}_1$,

$$\int dx\, m\int d\mathcal{P}_0(\omega)v_M(x,\omega)\varphi(x)\psi(\omega)=0. \tag{116}$$

We write $v_M^\varepsilon:=[v_\varepsilon]_M$. Since $|\nabla_\varepsilon v_M^\varepsilon|\le|\nabla_\varepsilon v_\varepsilon|$ (cf. (96)), by Lemma 13.7 (using (114)) and a diagonal procedure, at cost to refine the sequence $\{\varepsilon_k\}$ we have for any M that $\nabla_\varepsilon v_M^\varepsilon\overset{2}{\rightharpoonup}w_M\in L^2(m\,dx\times\nu)$, along the sequence $\{\varepsilon_k\}$. In what follows, we understand that the parameter ε varies in $\{\varepsilon_k\}$. Note in particular that, by (100) and since $\tilde\omega\in\Omega_{\mathrm{typ}}$ and $\psi\in\mathcal{G}_1\subset\mathcal{G}$,

$$\text{l.h.s. of (116)}=\lim_{\varepsilon\downarrow 0}\int d\mu_{\tilde\omega}^\varepsilon(x)v_M^\varepsilon(x)\varphi(x)\psi(\theta_{x/\varepsilon}\tilde\omega). \tag{117}$$

Let us write $\psi=g_b$ with $b\in\mathcal{H}_1$ (recall (98)). By Lemma 11.7, since $\tilde\omega\in\Omega_{\mathrm{typ}}\subset\mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]$, the r.h.s. of (117) equals the limit as $\varepsilon\downarrow 0$ of

$$-\varepsilon\int d\nu_{\tilde\omega}^\varepsilon(x,z)\nabla_\varepsilon\big(v_M^\varepsilon\varphi\big)(x,z)b(\theta_{x/\varepsilon}\tilde\omega,z)=-\varepsilon C_1(\varepsilon)-\varepsilon C_2(\varepsilon), \tag{118}$$

where (due to (113))

$$C_1(\varepsilon):=\int d\nu_{\tilde\omega}^\varepsilon(x,z)\nabla_\varepsilon v_M^\varepsilon(x,z)\varphi(\varepsilon x)b(\theta_{x/\varepsilon}\tilde\omega,z),$$
$$C_2(\varepsilon):=\int d\nu_{\tilde\omega}^\varepsilon(x,z)v_M^\varepsilon(x+\varepsilon z)\nabla_\varepsilon\varphi(x,z)b(\theta_{x/\varepsilon}\tilde\omega,z).$$

Due to (117) and (118), to get (116) we only need to show that $\lim_{\varepsilon\downarrow 0}\varepsilon C_1(\varepsilon)=0$ and $\lim_{\varepsilon\downarrow 0}\varepsilon C_2(\varepsilon)=0$. Since $\nabla_\varepsilon v_M^\varepsilon\overset{2}{\rightharpoonup}w_M$ and $b\in\mathcal{H}_1$, by (104) we have that

$$\lim_{\varepsilon\downarrow 0}C_1(\varepsilon)=\int dx\, m\int d\nu(\omega,z)w_M(x,\omega,z)\varphi(x)b(\omega,z), \tag{119}$$

which is finite, thus implying that $\lim_{\varepsilon\downarrow 0}\varepsilon C_1(\varepsilon)=0$.

We move to $C_2(\varepsilon)$. Let ℓ be such that $\varphi(x)=0$ if $|x|\ge\ell$. Fix $\phi\in C_c(\mathbb{R}^d)$ with values in $[0,1]$, such that $\phi(x)=1$ for $|x|\le\ell$ and $\phi(x)=0$ for $|x|\ge\ell+1$. Since $\nabla_\varepsilon\varphi(x,z)=0$ if $|x|\ge\ell$ and $|x+\varepsilon z|\ge\ell$, by the mean value theorem we conclude that

$$\big|\nabla_\varepsilon\varphi(x,z)\big|\le\|\nabla\varphi\|_\infty|z|\big(\phi(x)+\phi(x+\varepsilon z)\big). \tag{120}$$

We apply the above bound and Schwarz inequality to $C_2(\varepsilon)$ getting

$$\big|C_2(\varepsilon)\big|\le M\|\nabla\varphi\|_\infty\int d\nu_{\tilde\omega}^\varepsilon(x,z)|z|\,\big|b(\theta_{x/\varepsilon}\tilde\omega,z)\big|\big(\phi(x)+\phi(x+\varepsilon z)\big)\le M\|\nabla\varphi\|_\infty A_1(\varepsilon)^{1/2}A_2(\varepsilon)^{1/2}, \tag{121}$$

where (see below for explanations)

$$A_1(\varepsilon):=\int\nu_{\tilde\omega}^\varepsilon(x,z)|z|^2\big(\phi(x)+\phi(x+\varepsilon z)\big)=2\int\nu_{\tilde\omega}^\varepsilon(x,z)|z|^2\phi(x)^2,$$
$$A_2(\varepsilon):=\int\nu_{\tilde\omega}^\varepsilon(x,z)b(\theta_{x/\varepsilon}\tilde\omega,z)^2\big(\phi(x)+\phi(x+\varepsilon z)\big)=2\int\nu_{\tilde\omega}^\varepsilon(x,z)\big(b^2+\tilde b^2\big)(\theta_{x/\varepsilon}\tilde\omega,z)\phi(x)^2.$$

To get the second identities in the above formulas for $A_1(\varepsilon)$ and $A_2(\varepsilon)$ we have applied Lemma 11.3-(i) to the forms $(\omega,z)\mapsto|z|^2$ and $(\omega,z)\mapsto b^2(\omega,z)$.

We now write

$$A_1(\varepsilon) = 2\int d\mu^\varepsilon_{\tilde\omega}(x)\lambda_2(\theta_{x/\varepsilon}\tilde\omega)\phi(x)^2, \qquad A_2(\varepsilon) = 2\int d\mu^\varepsilon_{\tilde\omega}(x)\big(\widehat{\tilde b^2}+\widehat{b^2}\big)(\theta_{x/\varepsilon}\tilde\omega)\phi(x)^2.$$

For the second identity above we used that $\tilde\omega \in \Omega_{\rm typ} \in \mathcal{A}_1[\tilde b^2]\cap \mathcal{A}_1[b^2]$. By using respectively that $\omega \in \Omega_{\rm typ} \subset \tilde A \cap \mathcal{A}[\lambda_2]$ with $A := \{\lambda_2 < +\infty\}$ (recall Definition 12.2) and that $\omega \in \Omega_{\rm typ} \subset \mathcal{A}[\widehat{\tilde b^2}]\cap \mathcal{A}[\widehat{b^2}]\cap \mathcal{A}_1[\tilde b^2]\cap \mathcal{A}_1[b^2]$, we conclude that $A_1(\varepsilon)$ and $A_2(\varepsilon)$ have finite limits as $\varepsilon \downarrow 0$, thus implying (cf. (121)) that $\lim_{\varepsilon\downarrow 0}\varepsilon C_2(\varepsilon) = 0$. This concludes the proof of (116). □

Remark 16.2. We stress that the cut-off method developed in Lemma 15.1 has been essential to get (121) and go on. If one tries to prove (116) with v instead of v_M, then one would be stopped when trying to control $C_2(\varepsilon)$ as $\varepsilon \downarrow 0$.

17. Cut-off for gradients $\nabla_\varepsilon v_\varepsilon$

Lemma 17.1. *Let $\tilde\omega \in \Omega_{\rm typ}$ and let $\{v_\varepsilon\}$ be a family of functions with $v_\varepsilon \in H^1_{\tilde\omega,\varepsilon}$, satisfying (114). Then there exist functions $w, w_M \in L^2(m\,dx \times \nu)$, with M varying in $\mathbb{N}_+$, such that*

(i) *$\nabla_\varepsilon v_\varepsilon \stackrel{2}{\rightharpoonup} w$ and $\nabla_\varepsilon [v_\varepsilon]_M \stackrel{2}{\rightharpoonup} w_M$ for all $M \in \mathbb{N}_+$, along a sequence $\varepsilon_k \downarrow 0$;*
(ii) *for any $\varphi \in C^1_c(\mathbb{R}^d)$ and $b \in \mathcal{H}$ it holds*

$$\lim_{M\to\infty}\int dx\, m\int d\nu(\omega,z) w_M(x,\omega,z)\varphi(x)b(\omega,z) = \int dx\, m\int d\nu(\omega,z)w(x,\omega,z)\varphi(x)b(\omega,z). \tag{122}$$

Proof. At cost to restrict to ε small enough, we can assume that $\|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \le C_0$ and $\|\nabla_\varepsilon v_\varepsilon\|_{L^2(\nu^\varepsilon_{\tilde\omega})} \le C_0$ for some $C_0 < +\infty$ and all $\varepsilon > 0$. Due to (96), the same holds respectively for v^ε_M and $\nabla_\varepsilon v^\varepsilon_M$, for all $M \in \mathbb{N}_+$, where we have set $v^\varepsilon_M := [v_\varepsilon]_M$. In particular, by a diagonal procedure, due to Lemmas 13.5 and 13.7 along a sequence $\{\varepsilon_k\}$ we have that $v^\varepsilon_M \stackrel{2}{\rightharpoonup} v_M$, $v_\varepsilon \stackrel{2}{\rightharpoonup} v$, $\nabla_\varepsilon v^\varepsilon_M \stackrel{2}{\rightharpoonup} w_M$ and $\nabla_\varepsilon v_\varepsilon \stackrel{2}{\rightharpoonup} w$, where $v_M, v \in L^2(m\,dx \times \mathcal{P}_0)$, $w_M, w \in L^2(m\,dx\times\nu)$, simultaneously for all $M \in \mathbb{N}_+$. This proves in particular Item (i). We point out that we are not claiming that $v_M = [v]_M$, $w_M = [w]_M$. Moreover, from now on we restrict to ε belonging to the above special sequence without further mention.

We prove Item (ii). By extending the diagonal procedure we can assume that along the sequence $\{\varepsilon_k\}$ it holds $|v_\varepsilon| \stackrel{2}{\rightharpoonup} \tilde v$, as $\|\,|v_\varepsilon|\,\|_{L^2(\mu^\varepsilon_{\tilde\omega})} = \|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \le C_0$. We set $H(\bar w,\bar\varphi,\bar b) := \int dx\, m\int d\nu(\omega,z)\bar w(x,\omega,z)\bar\varphi(x)\bar b(\omega,z)$. Then (122) corresponds to the limit $\lim_{M\to\infty} H(w_M,\varphi,b) = H(w,\varphi,b)$. Here and below $b \in \mathcal{H}$ and $\varphi \in C^1_c(\mathbb{R}^d)$. Recall that $b_k := [b]_k \in \mathcal{H}$ for any $k \in \mathbb{N}_+$ (see Section 12).

Claim 17.2. *For each $k, M \in \mathbb{N}_+$ it holds*

$$\big|H(w,\varphi,b) - H(w,\varphi,b_k)\big| \le C_0\|\varphi\|_{L^2(m\,dx)}\|b-b_k\|_{L^2(\nu)}, \tag{123}$$

$$\big|H(w_M,\varphi,b) - H(w_M,\varphi,b_k)\big| \le C_0\|\varphi\|_{L^2(m\,dx)}\|b-b_k\|_{L^2(\nu)}. \tag{124}$$

We omit the proof of the above claim since it can be obtained by reasoning exactly as in the proof of Claim 15.2.

Claim 17.3. *For any $k \in \mathbb{N}_+$, it holds*

$$\lim_{M\uparrow\infty}\big|H(w,\varphi,b_k) - H(w_M,\varphi,b_k)\big| \le kC_*\big(M^{-1/2} + \mathbb{E}_0[\lambda_0 \mathbb{1}_{\{\lambda_0 \ge \sqrt{M}\}}]\big)^{1/2}, \tag{125}$$

where C_* is a constant depending only on C_0 and φ.

Proof of Claim 17.3. We note that $\nabla_\varepsilon v_\varepsilon(x,z) = \nabla_\varepsilon v^\varepsilon_M(x,z)$ if $|v_\varepsilon(x)| \le M$ and $|v_\varepsilon(x+\varepsilon z)| \le M$. Moreover, by (97), we have $|\nabla_\varepsilon v_\varepsilon - \nabla_\varepsilon v^\varepsilon_M| \le |\nabla_\varepsilon v_\varepsilon|$. Hence we can bound

$$\big|\nabla_\varepsilon v_\varepsilon - \nabla_\varepsilon v^\varepsilon_M\big|(x,z) \le |\nabla_\varepsilon v_\varepsilon|(x,z)\big(\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}} + \mathbb{1}_{\{|v_\varepsilon(x+\varepsilon z)|\ge M\}}\big). \tag{126}$$

Due to the above bound we can estimate

$$\begin{aligned}(127)\qquad &\left|H(w,\varphi,b_k)-H(w_M,\varphi,b_k)\right|=\left|\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\big(\nabla_\varepsilon v_\varepsilon-\nabla_\varepsilon v^\varepsilon_M\big)(x,z)\varphi(x)b_k(\theta_{x/\varepsilon}\tilde\omega,z)\right|\\ &\qquad\le k\,\overline{\lim_{\varepsilon\downarrow 0}}\int d\nu^\varepsilon_{\tilde\omega}(x,z)|\nabla_\varepsilon v_\varepsilon|(x,z)(\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}+\mathbb{1}_{\{|v_\varepsilon(x+\varepsilon z)|\ge M\}})\left|\varphi(x)\right|.\end{aligned}$$

Note that the identity in (127) follows from (104) since $b_k\in\mathcal{H}$ (recall that $\tilde\omega\in\Omega_{\mathrm{typ}}$, $\nabla_\varepsilon v^\varepsilon_M\overset{2}{\rightharpoonup}w_M$, $\nabla_\varepsilon v_\varepsilon\overset{2}{\rightharpoonup}w$). By Schwarz inequality we have

$$(128)\qquad \int d\nu^\varepsilon_{\tilde\omega}(x,z)|\nabla_\varepsilon v_\varepsilon|(x,z)\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}\left|\varphi(x)\right|\le C_0A(\varepsilon)^{1/2},$$

where

$$A(\varepsilon):=\int d\nu^\varepsilon_{\tilde\omega}(x,z)\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}\varphi(x)^2=\int d\mu^\varepsilon_{\tilde\omega}(x)\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}\varphi(x)^2\lambda_0(\theta_{x/\varepsilon}\tilde\omega).$$

As $\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}\le|v_\varepsilon(x)|/M$ we can bound $A(\varepsilon)\le A_1(\varepsilon)+A_2(\varepsilon)$ where

$$A_1(\varepsilon):=\frac{1}{M}\int d\mu^\varepsilon_{\tilde\omega}(x)\left|v_\varepsilon(x)\right|\varphi(x)^2(\lambda_0\wedge\sqrt{M})(\theta_{x/\varepsilon}\tilde\omega),\qquad A_2(\varepsilon):=\int d\mu^\varepsilon_{\tilde\omega}(x)\varphi(x)^2(\lambda_0\mathbb{1}_{\{\lambda_0>\sqrt{M}\}})(\theta_{x/\varepsilon}\tilde\omega).$$

As $\lambda_0\wedge\sqrt{M}\in\mathcal{G}$ and $|v_\varepsilon|\overset{2}{\rightharpoonup}\tilde v\in L^2(m\,dx\otimes\mathcal{P}_0)$, we have $\lim_{\varepsilon\downarrow 0}A_1(\varepsilon)=M^{-1}\int dx\,m\int d\mathcal{P}_0(\omega)\tilde v(x,\omega)\varphi(x)^2(\lambda_0\wedge\sqrt{M})(\omega)$. We recall (see Lemma 13.5) that $\|\tilde v\|_{L^2(m\,dx\otimes\mathcal{P}_0)}\le C_0$. Hence by Schwarz inequality we have

$$(129)\qquad \left|\int dx\,m\int d\mathcal{P}_0(\omega)\tilde v(x,\omega)\varphi(x)^2(\lambda_0\wedge\sqrt{M})(\omega)\right|\le C_0M^{1/2}\left\|\varphi^2\right\|_{L^2(m\,dx)}.$$

As a consequence we have $\lim_{\varepsilon\downarrow 0}A_1(\varepsilon)\le(C_0/M^{1/2})\|\varphi^2\|_{L^2(m\,dx)}$. As $\tilde\omega\in\Omega_{\mathrm{typ}}\subset\mathcal{A}[\lambda_0\mathbb{1}_{\{\lambda_0>\sqrt{M}\}}]\cap\tilde A$ with $A=\{\lambda_0<+\infty\}$, we have $\lim_{\varepsilon\downarrow 0}A_2(\varepsilon)=\int dx\,m\varphi(x)^2\mathbb{E}_0[\lambda_0\mathbb{1}_{\{\lambda_0>\sqrt{M}\}}]$. Since $A(\varepsilon)\le A_1(\varepsilon)+A_2(\varepsilon)$, we then get

$$(130)\qquad \limsup_{\varepsilon\downarrow 0}A(\varepsilon)\le\big(C_0/M^{1/2}\big)\left\|\varphi^2\right\|_{L^2(m\,dx)}+\int dx\,m\varphi(x)^2\mathbb{E}_0[\lambda_0\mathbb{1}_{\{\lambda_0>\sqrt{M}\}}].$$

Reasoning as above we have

$$(131)\qquad \int d\nu^\varepsilon_{\tilde\omega}(x,z)|\nabla_\varepsilon v_\varepsilon|(x,z)\mathbb{1}_{\{|v_\varepsilon(x+\varepsilon z)|\ge M\}}\left|\varphi(x)\right|\le C_0B(\varepsilon)^{1/2},$$

where (applying also (89) to the form $(\omega,z)\mapsto 1$)

$$B(\varepsilon):=\int d\nu^\varepsilon_{\tilde\omega}(x,z)\mathbb{1}_{\{|v_\varepsilon(x+\varepsilon z)|\ge M\}}\varphi(x)^2=\int d\nu^\varepsilon_{\tilde\omega}(x,z)\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}\varphi(x+\varepsilon z)^2.$$

We want to replace in the last expression the term $\varphi(x+\varepsilon z)^2$ with $\varphi(x)^2$ (note that this replacement would produce $A(\varepsilon)$). To estimate the error we observe that by (120) (we use the same notation here) we can bound

$$(132)\qquad \left|\varphi(x+\varepsilon z)^2-\varphi(x)^2\right|\le C\varepsilon|z|\big(\phi(x)+\phi(x+\varepsilon z)\big),$$

where $C=C(\varphi)$. Hence we have $B(\varepsilon)\le A(\varepsilon)+C\varepsilon B_1(\varepsilon)+C\varepsilon B_2(\varepsilon)$, where

$$B_1(\varepsilon):=\int d\nu^\varepsilon_{\tilde\omega}(x,z)\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}\phi(x)|z|\le\int d\mu^\varepsilon_{\tilde\omega}(x)\phi(x)\lambda_1(\theta_{x/\varepsilon}\tilde\omega),$$

$$B_2(\varepsilon):=\int d\nu^\varepsilon_{\tilde\omega}(x,z)\mathbb{1}_{\{|v_\varepsilon(x)|\ge M\}}|z|\phi(x+\varepsilon z)\le\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|\phi(x+\varepsilon z).$$

Note that by (89) we have

$$\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|\phi(x+\varepsilon z)=\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|\phi(x)=\int d\mu^\varepsilon_{\tilde\omega}(x)\phi(x)\lambda_1(\theta_{x/\varepsilon}\tilde\omega).$$

Hence $B_1(\varepsilon) + B_2(\varepsilon) \leq 2\int d\mu^\varepsilon_{\tilde\omega}(x)\phi(x)\lambda_1(\theta_{x/\varepsilon}\tilde\omega)$. As $\phi \in C_c(\mathbb{R}^d)$ and $\tilde\omega \in \Omega_{\rm typ} \subset \mathcal{A}[\lambda_1] \cap \tilde A$ with $A := \{\lambda_1 < +\infty\}$, we conclude that $\limsup_{\varepsilon\downarrow 0}(B_1(\varepsilon) + B_2(\varepsilon)) < +\infty$. By using that $B(\varepsilon) \leq A(\varepsilon) + C\varepsilon B_1(\varepsilon) + C\varepsilon B_2(\varepsilon)$, we get that $\limsup_{\varepsilon\downarrow 0} B(\varepsilon) \leq \limsup_{\varepsilon\downarrow 0} A(\varepsilon)$ and the latter has been estimated in (130). At this point, using also (127), (128) and (131), we get the claim. □

We can finally derive (122), i.e. that $\lim_{M\to\infty} H(w_M, \varphi, b) = H(w, \varphi, b)$. By using Claims 17.2 and 17.3 we have

$$\begin{aligned}\big|H(w_M,\varphi,b) - H(w,\varphi,b)\big| &\leq \big|H(w_M,\varphi,b) - H(w_M,\varphi,b_k)\big| + \big|H(w_M,\varphi,b_k) - H(w,\varphi,b_k)\big| \\ &\quad + \big|H(w,\varphi,b_k) - H(w,\varphi,b)\big| \\ &\leq C_0 C(\varphi)\|b-b_k\|_{L^2(\nu)} + kC_*\big(M^{-1/2} + \mathbb{E}_0[\lambda_0 \mathbb{1}_{\{\lambda_0 \geq \sqrt{M}\}}]\big)^{1/2}.\end{aligned}$$

At this point it is enough to take first the limit $M \to \infty$ and afterwards the limit $k \to \infty$ and to use that $\lim_{k\to\infty}\|b - b_k\|_{L^2(\nu)} = 0$. □

18. Structure of the 2-scale weak limit of a bounded family in $H^1_{\tilde\omega,\varepsilon}$: Part II

We point out that the next result is the analogous of [44, Lemma 5.4]. The proof relies also on a cut-off procedure based on Lemma 15.1 and Lemma 17.1 (see also Remark 18.2 below). Recall Definition 3.9.

Proposition 18.1. *Let $\tilde\omega \in \Omega_{\rm typ}$ and let $\{v_\varepsilon\}$ be a family of functions $v_\varepsilon \in H^1_{\tilde\omega,\varepsilon}$ satisfying* (114). *Then, along a sequence $\varepsilon_k \downarrow 0$, we have*:

(i) $L^2(\mu^\varepsilon_{\tilde\omega}) \ni v_\varepsilon \stackrel{2}{\rightharpoonup} v \in L^2(m\,dx \times \mathcal{P}_0)$, *where v does not depend on ω. Writing v simply as $v(x)$ we have that $v \in H^1_*(m\,dx)$;*

(ii) $L^2(\nu^\varepsilon_{\tilde\omega}) \ni \nabla v_\varepsilon(x,z) \stackrel{2}{\rightharpoonup} \nabla_* v(x)\cdot z + v_1(x,\omega,z) \in L^2(m\,dx \times \nu)$, *where $v_1 \in L^2(\mathbb{R}^d, L^2_{\rm pot}(\nu))$.*

The property $v_1 \in L^2(\mathbb{R}^d, L^2_{\rm pot}(\nu))$ means that for dx-almost every x in $\mathbb{R}^d$ the map $(\omega, z) \mapsto v_1(x,\omega,z)$ is a potential form, hence in $L^2_{\rm pot}(\nu)$, moreover the map $\mathbb{R}^d \ni x \to v_1(x,\cdot,\cdot) \in L^2_{\rm pot}(\nu)$ is measurable and

$$\int dx \big\|v_1(x,\cdot,\cdot)\big\|^2_{L_2(\nu)} = \int dx \int \nu(\omega,z) v_1(x,\omega,z)^2 < +\infty. \tag{133}$$

Proof of Proposition 18.1. At cost to restrict to ε small enough, we can assume that $\|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \leq C_0$ and $\|\nabla_\varepsilon v_\varepsilon\|_{L^2(\nu^\varepsilon_{\tilde\omega})} \leq C_0$ for some $C_0 < +\infty$ and all $\varepsilon > 0$. We can assume the same bounds for $v^\varepsilon_M := [v_\varepsilon]_M$. Along a sequence $\varepsilon_k \downarrow 0$ the 2-scale convergences in Item (i) of Lemma 15.1 and in Item (i) of Lemma 17.1 take place. We keep here the same notation. In particular, $v_\varepsilon \stackrel{2}{\rightharpoonup} v$, $v^\varepsilon_M \stackrel{2}{\rightharpoonup} v_M$, $\nabla_\varepsilon v_\varepsilon \stackrel{2}{\rightharpoonup} w$ and $\nabla_\varepsilon v^\varepsilon_M \stackrel{2}{\rightharpoonup} w_M$. By Lemmas 13.5 and 13.7 the norms $\|v_M\|_{L^2(m\,dx\times\mathcal{P}_0)}$, $\|v\|_{L^2(m\,dx\times\mathcal{P}_0)}$, $\|w_M\|_{L^2(m\,dx\times\nu)}$ and $\|w\|_{L^2(m\,dx\times\nu)}$ are upper bounded by C_0.

Due to Proposition 16.1 $v = v(x)$ and $v_M = v_M(x)$. We claim that for each solenoidal form $b \in L^2_{\rm sol}(\nu)$ and each function $\varphi \in C^2_c(\mathbb{R}^d)$, it holds

$$\int dx \varphi(x) \int d\nu(\omega,z) w(x,\omega,z) b(\omega,z) = -\int dx\, v(x)\nabla\varphi(x)\cdot\eta_b, \tag{134}$$

where $\eta_b := \int d\nu(\omega,z) z b(\omega,z)$. Note that η_b is well defined since both b and the map $(\omega,z) \mapsto z$ are in $L^2(\nu)$. Moreover, by applying Lemma 7.3 with f such that $f(\omega,\theta_z\omega) = zb(\omega,z)$ $\forall z \in \hat\omega$ for $\mathcal{P}_0$-a.a. ω (use (A3) to define f), we get that $\eta_b = -\eta_{\tilde b}$ (cf. Definition 11.1).

Before proving (134) we show how to conclude the proof of Proposition 18.1 starting with Item (i). Due to Corollary 9.2 for each $i = 1,\dots,d_*$ there exists $b_i \in L^2_{\rm sol}(\nu)$ such that $\eta_{b_i} = \mathfrak{e}_i$. Consider the measurable function

$$g_i(x) := \int d\nu(\omega,z) w(x,\omega,z) b_i(\omega,z), \quad 1 \leq i \leq d_*. \tag{135}$$

We have that $g_i \in L^2(dx)$ since, by Schwarz inequality,

$$\int g_i(x)^2 dx = \int dx \Big[\int d\nu(\omega,z) w(x,\omega,z) b_i(\omega,z)\Big]^2 \leq \|b_i\|^2_{L^2(\nu)} \int dx \int d\nu(\omega,z) w(x,\omega,z)^2 < \infty. \tag{136}$$

Moreover, by (134) we have that $\int dx\varphi(x)g_i(x) = -\int dx\, v(x)\partial_{\mathfrak{e}_i}\varphi(x)$ for $1 \le i \le d_*$. This proves that $v(x) \in H^1_*(m\,dx)$ and $\partial_{\mathfrak{e}_i} v(x) = g_i(x)$ for $1 \le i \le d_*$. This concludes the proof of Item (i).

We move to Item (ii) (always assuming (134)). By Item (i) and Corollary 9.2 implying that $\eta_b \in \operatorname{span}\{\mathfrak{e}_1, \dots, \mathfrak{e}_{d_*}\}$ for all $b \in L^2_{\mathrm{sol}}(\nu)$, we can replace the r.h.s. of (134) by $\int dx(\nabla_* v(x)\cdot\eta_b)\varphi(x)$. Hence (134) can be rewritten as

$$\int dx\varphi(x)\int d\nu(\omega,z)\big[w(x,\omega,z) - \nabla_* v(x)\cdot z\big]b(\omega,z) = 0. \tag{137}$$

By the arbitrariness of φ we conclude that dx-a.s.

$$\int d\nu(\omega,z)\big[w(x,\omega,z) - \nabla_* v(x)\cdot z\big]b(\omega,z) = 0, \quad \forall b \in L^2_{\mathrm{sol}}(\nu). \tag{138}$$

Let us now show that the map $w(x,\omega,z) - \nabla_* v(x)\cdot z$ belongs to $L^2(dx, L^2(\nu))$. Indeed, we have $\int dx\|w(x,\cdot,\cdot)\|^2_{L^2(\nu)} = \|w\|^2_{L^2(m\,dx\times d\nu)} < +\infty$ and also

$$\int dx\big\|\nabla_* v(x)\cdot z\big\|^2_{L^2(\nu)} \le \int dx\big|\nabla_* v(x)\big|^2\int d\nu(\omega,z)|z|^2 < \infty, \tag{139}$$

by Schwarz inequality and since $\nabla_* v \in L^2(dx)$ and $\mathbb{E}_0[\lambda_2] < \infty$.

As the map $w(x,\omega,z) - \nabla_* v(x)\cdot z$ belongs to $L^2(dx, L^2(\nu))$, for dx-a.e. x we have that the map $(\omega,z) \mapsto w(x,\omega,z) - \nabla_* v(x)\cdot z$ belongs to $L^2(\nu)$ and therefore, by (138), to $L^2_{\mathrm{pot}}(\nu)$. This concludes the proof of Item (ii).

It remains to prove (134). Here is a roadmap: (i) we reduce (134) to (140); (ii) we prove (144); (iii) by (144) we reduce (140) to (146); (iv) we prove (147); (v) by (147) we reduce (146) to (150); (vi) we prove (150).

Since both sides of (134) are continuous as functions of $b \in L^2_{\mathrm{sol}}(\nu)$, it is enough to prove it for $b \in \mathcal{W}$ (see Section 12). We apply Lemma 17.1-(ii) (recall that $b \in \mathcal{W} \subset \mathcal{H}$) to approximate the l.h.s. of (134) and Lemma 15.1-(ii) with $u := 1 \in \mathcal{G}$ to approximate the r.h.s. of (134). Then to prove (134) it is enough to show that

$$\int dx\, m\varphi(x)\int d\nu(\omega,z)w_M(x,\omega,z)b(\omega,z) = -\int dx\, m v_M(x)\nabla\varphi(x)\cdot\eta_b, \tag{140}$$

for any $\varphi \in C^2_c(\mathbb{R}^d)$, $b \in \mathcal{W}$ and $M \in \mathbb{N}_+$. From now on M is fixed.

Since $\tilde\omega \in \Omega_{\mathrm{typ}}$, $\nabla_\varepsilon v^\varepsilon_M \overset{2}{\rightharpoonup} w_M$ and $b \in \mathcal{W} \subset \mathcal{H}$ we can write (cf. (104))

$$\text{l.h.s. of (140)} = \lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon v^\varepsilon_M(x,z)\varphi(x)b(\theta_{x/\varepsilon}\tilde\omega, z). \tag{141}$$

Since $b \in L^2_{\mathrm{sol}}(\nu)$ and $\tilde\omega \in \Omega_{\mathrm{typ}} \subset \mathcal{A}_d[b]$ (cf. Lemmata 11.6 and 11.7), we get

$$\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon\big(v^\varepsilon_M\varphi\big)(x,z)b(\theta_{x/\varepsilon}\tilde\omega, z) = 0.$$

Using the above identity, (113) and finally (90) in Lemma 11.3 as $\tilde\omega \in \Omega_{\mathrm{typ}} \subset \mathcal{A}_1[b]\cap\mathcal{A}_1[\tilde b]$, we conclude that

$$\begin{aligned}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon v^\varepsilon_M(x,z)\varphi(x)b(\theta_{x/\varepsilon}\tilde\omega, z) &= -\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x+\varepsilon z)\nabla_\varepsilon\varphi(x,z)b(\theta_{x/\varepsilon}\tilde\omega, z)\\ &= \int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\nabla_\varepsilon\varphi(x,z)\tilde b(\theta_{x/\varepsilon}\tilde\omega, z).\end{aligned} \tag{142}$$

Up to now we have obtained that

$$\text{l.h.s. of (140)} = \lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\nabla_\varepsilon\varphi(x,z)\tilde b(\theta_{x/\varepsilon}\tilde\omega, z). \tag{143}$$

We now set $b_k := [b]_k$, $\tilde b_k := [\tilde b]_k = \widetilde{[b]_k} = \widetilde{b_k}$. We want to prove that

$$\varlimsup_{k\uparrow\infty}\varlimsup_{\varepsilon\downarrow 0}\left|\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\nabla_\varepsilon\varphi(x,z)(\tilde b - \tilde b_k)(\theta_{x/\varepsilon}\tilde\omega, z)\right| = 0. \tag{144}$$

To this aim let ℓ be such that $\varphi(x)=0$ if $|x|\geq \ell$. Fix $\phi\in C_c(\mathbb{R}^d)$ with values in [0, 1], such that $\phi(x)=1$ for $|x|\leq \ell$ and $\phi(x)=0$ for $|x|\geq \ell+1$. Using (120) and Schwarz inequality we can bound

$$\begin{aligned}&\left|\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\nabla_\varepsilon\varphi(x,z)(\tilde b-\tilde b_k)(\theta_{x/\varepsilon}\tilde\omega,z)\right|\\ &\quad\leq M\|\nabla\varphi\|_\infty\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|\big(\phi(x)+\phi(x+\varepsilon z)\big)|\tilde b-\tilde b_k|(\theta_{x/\varepsilon}\tilde\omega,z)\\ &\quad\leq M\|\nabla\varphi\|_\infty\big[2A(\varepsilon)\big]^{1/2}\big[B(\varepsilon,k)+C(\varepsilon,k)\big]^{1/2},\end{aligned}\tag{145}$$

where (using (89) in Lemma 11.3 for $A(\varepsilon)$ and $C(\varepsilon)$)

$$\begin{aligned}A(\varepsilon)&:=\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|^2\phi(x)=\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|^2\phi(x+\varepsilon z),\\ B(\varepsilon,k)&:=\int d\nu^\varepsilon_{\tilde\omega}(x,z)(\tilde b-\tilde b_k)^2(\theta_{x/\varepsilon}\tilde\omega,z)\phi(x),\\ C(\varepsilon,k)&:=\int d\nu^\varepsilon_{\tilde\omega}(x,z)(\tilde b-\tilde b_k)^2(\theta_{x/\varepsilon}\tilde\omega,z)\phi(x+\varepsilon z)=\int d\nu^\varepsilon_{\tilde\omega}(x,z)(b-b_k)^2(\theta_{x/\varepsilon}\tilde\omega,z)\phi(x).\end{aligned}$$

As $\Omega_{\rm typ}\subset\tilde A\cap\mathcal{A}[\lambda_2]$ with $A:=\{\lambda_2<+\infty\}$ (cf. Definition 12.2), $A(\varepsilon)=\int d\mu^\varepsilon_{\tilde\omega}(x)\phi(x)\lambda_2(\theta_{x/\varepsilon}\tilde\omega)$ has finite limit as $\varepsilon\downarrow 0$. Hence to get (144) we only need to show that $\lim_{k\uparrow\infty,\varepsilon\downarrow 0}B(\varepsilon,k)=\lim_{k\uparrow\infty,\varepsilon\downarrow 0}C(\varepsilon,k)=0$. Setting $d:=|\tilde b-\tilde b_k|$ we can write $B(\varepsilon,k)=\int d\mu^\varepsilon_{\tilde\omega}(x)\phi(x)\widehat{d^2}(\theta_{x/\varepsilon}\tilde\omega)$ as $\omega\in\Omega_{\rm typ}\subset\mathcal{A}_1[d^2]$ (recall that $\tilde b,\tilde b_k\in\mathcal{H}$). As in addition $\omega\in\Omega_{\rm typ}\subset\mathcal{A}[\widehat{d^2}]$, we conclude that $\lim_{\varepsilon\downarrow 0}B(\varepsilon,k)=\int dx\, m\phi(x)\|\tilde b-\tilde b_k\|^2_{L^2(\nu)}$. Similarly we get that $\lim_{\varepsilon\downarrow 0}C(\varepsilon,k)=\int dx\, m\phi(x)\|b-b_k\|^2_{L^2(\nu)}$. As the above limits go to zero as $k\to\infty$, we get (144).

Due to (143), (144) and since, by Schwarz inequality, $\lim_{k\to\infty}\eta_{\tilde b_k}=\eta_{\tilde b}=-\eta_b$, to prove (140) we only need to show, for fixed M,k, that

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\nabla_\varepsilon\varphi(x,z)\tilde b_k(\theta_{x/\varepsilon}\tilde\omega,z)=\int dx\, m v_M(x)\nabla\varphi(x)\cdot\eta_{\tilde b_k}.\tag{146}$$

To prove (146) we first show that

$$\lim_{\varepsilon\downarrow 0}\left|\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\big[\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z\big]\tilde b_k(\theta_{x/\varepsilon}\tilde\omega,z)\right|=0.\tag{147}$$

Since $\|v^\varepsilon_M\|_\infty\leq M$ and $\|\tilde b_k\|_\infty\leq k$, it is enough to show that

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\big|\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z\big|=0.\tag{148}$$

Since $\varphi\in C^2_c(\mathbb{R}^d)$, by Taylor expansion we have $\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z=\frac{1}{2}\sum_{i,j}\partial^2_{ij}\varphi(\zeta_\varepsilon(x,z))z_iz_j\varepsilon$, where $\zeta_\varepsilon(x,z)$ is a point between x and $x+\varepsilon z$. Moreover we note that $\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z=0$ if $|x|\geq\ell$ and $|x+\varepsilon z|\geq\ell$. All these observations imply that

$$\big|\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z\big|\leq\varepsilon C(\varphi)|z|^2\big(\phi(x)+\phi(x+\varepsilon z)\big).\tag{149}$$

Due to (89) we can write

$$\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|^2\phi(x+\varepsilon z)=\int d\nu^\varepsilon_{\tilde\omega}(x,z)|z|^2\phi(x)=\int d\mu^\varepsilon_{\tilde\omega}(x)\phi(x)\lambda_2(\theta_{x/\varepsilon}\tilde\omega).$$

As $\omega\in\Omega_{\rm typ}\subset\tilde A\cap\mathcal{A}[\lambda_2]$ with $A:=\{\lambda_2<+\infty\}$, we conclude that the above r.h.s. has a finite limit as $\varepsilon\downarrow 0$. Due to (149), we get (148) and hence (147).

Having (147), to get (146) it is enough to show that

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)v^\varepsilon_M(x)\nabla\varphi(x)\cdot z\tilde b_k(\theta_{x/\varepsilon}\tilde\omega,z)=\int dx\, m v_M(x)\nabla\varphi(x)\cdot\eta_{\tilde b_k}.\tag{150}$$

To this aim we observe that

$$\int d\nu^{\varepsilon}_{\tilde\omega}(x,z)v^{\varepsilon}_{M}(x)\partial_i\varphi(x)z_i\tilde b_k(\theta_{x/\varepsilon}\tilde\omega,z)=\int d\mu^{\varepsilon}_{\tilde\omega}(x)v^{\varepsilon}_{M}(x)\partial_i\varphi(x)u_k(\theta_{x/\varepsilon}\tilde\omega), \tag{151}$$

where $u_k(\omega):=\int d\hat\omega(z)r_{0,z}(\omega)z_i\tilde b_k(\omega,z)$. Note that $u_k\in L^1(\mathcal{P}_0)$ as $\lambda_1\in L^1(\mathcal{P}_0)$ by (19). Given $\ell\in\mathbb{N}$ we consider $u_k-[u_k]_\ell$ (cf. (95)). As $\tilde\omega\in\Omega_{\rm typ}\subset\tilde A\cap\mathcal{A}[|u_k-[u_k]_\ell|]$ where $A:=\{\lambda_1<+\infty\}$, we have

$$\begin{aligned}\left|\int d\mu^{\varepsilon}_{\tilde\omega}(x)v^{\varepsilon}_{M}(x)\partial_i\varphi(x)\big(u_k-[u_k]_\ell\big)(\theta_{x/\varepsilon}\tilde\omega)\right|&\le M\int d\mu^{\varepsilon}_{\tilde\omega}(x)\big|\partial_i\varphi(x)\big|\,\big|u_k-[u_k]_\ell\big|(\theta_{x/\varepsilon}\tilde\omega)\\&\overset{\varepsilon\downarrow0}{\to}M\int dx\,m\big|\partial_i\varphi(x)\big|\mathbb{E}_0\big[\big|u_k-[u_k]_\ell\big|\big].\end{aligned}\tag{152}$$

Note that the last expectation goes to zero when $\ell\uparrow\infty$ by dominated convergence as $\lambda_1\in L^1(\mathcal{P}_0)$ by (19). Since $\tilde\omega\in\Omega_{\rm typ}$, $v^{\varepsilon}_{M}\overset{2}{\rightharpoonup}v_M$ and $[u_k]_\ell\in\mathcal{G}$ (cf. (99) and recall that $\tilde b\in\mathcal{W}\ \forall b\in\mathcal{W}$), by (100) we conclude that

$$\begin{aligned}&\lim_{\varepsilon\downarrow0}\int d\mu^{\varepsilon}_{\tilde\omega}(x)v^{\varepsilon}_{M}(x)\partial_i\varphi(x)u_k(\theta_{x/\varepsilon}\tilde\omega)\\&\quad=\lim_{\ell\uparrow\infty}\lim_{\varepsilon\downarrow0}\int d\mu^{\varepsilon}_{\tilde\omega}(x)v^{\varepsilon}_{M}(x)\partial_i\varphi(x)[u_k]_\ell(\theta_{x/\varepsilon}\tilde\omega)=\lim_{\ell\uparrow\infty}\int dx\,mv_M(x)\partial_i\varphi(x)\mathbb{E}_0\big[[u_k]_\ell\big]\\&\quad=\int dx\,mv_M(x)\partial_i\varphi(x)\mathbb{E}_0[u_k]=\int dx\,mv_M(x)\partial_i\varphi(x)(\eta_{\tilde b_k}\cdot e_i).\end{aligned}$$

Our target (150) then follows from the above equation and (151). □

Remark 18.2. We stress that, without the cut-off trick allowing to move from (134) to (140), one would be blocked when trying to study the limit in the r.h.s. of (143) with v^{ε}_{M} replaced by v_ε.

19. Proof of Theorem 4.1

Without loss of generality, we prove Theorem 4.1 with $\lambda=1$ to simplify the notation. Due to Proposition 12.5 we only need to prove Items (i), (ii) and (iii). Some arguments below are taken from [44], others are intrinsic to the possible presence of long jumps. We start with two results (Lemmas 19.1 and 19.2) concerning the microscopic gradient $\nabla_\varepsilon\varphi$ for $\varphi\in C_c(\mathbb{R}^d)$.

Lemma 19.1. *Let $\omega\in\Omega_{\rm typ}$. Then $\overline{\lim}_{\varepsilon\downarrow0}\|\nabla_\varepsilon\varphi\|_{L^2(\nu^{\varepsilon}_{\omega})}<\infty$ for any $\varphi\in C^1_c(\mathbb{R}^d)$.*

Proof. Let ϕ be as in (120). By (120) and since $\omega\in\Omega_{\rm typ}$ (apply (89) with $b(\omega,z):=|z|^2$), we get

$$\begin{aligned}\|\nabla_\varepsilon\varphi\|^2_{L^2(\nu^{\varepsilon}_{\omega})}&\le C(\varphi)\int d\nu^{\varepsilon}_{\omega}(x,z)|z|^2\big(\phi(x)+\phi(x+\varepsilon z)\big)\\&=2C(\varphi)\int d\nu^{\varepsilon}_{\omega}(x,z)|z|^2\phi(x)=2C(\varphi)\int d\mu^{\varepsilon}_{\omega}(x)\phi(x)\lambda_2(\theta_{x/\varepsilon}\omega).\end{aligned}$$

The thesis then follows from Proposition 3.1 as $\omega\in\Omega_{\rm typ}\subset\tilde A\cap\mathcal{A}[\lambda_2]$ with $A:=\{\lambda_2<+\infty\}$ (recall Definition 12.2). □

Lemma 19.2. *Given $\omega\in\Omega_{\rm typ}$ and $\varphi\in C^2_c(\mathbb{R}^d)$ it holds*

$$\lim_{\varepsilon\downarrow0}\int d\nu^{\varepsilon}_{\omega}(x,z)\big[\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z\big]^2=0. \tag{153}$$

Proof. Let ℓ be such that $\varphi(x)=0$ if $|x|\ge\ell$. Let $\phi\in C_c(\mathbb{R}^d)$ be as in (120). The upper bound given by (120) with $\nabla_\varepsilon\varphi(x,z)$ replaced by $\nabla\varphi(x)\cdot z$ is also true. We will apply the above bounds for $|z|\ge\ell$. On the other hand, we apply (149) for $|z|<\ell$. As a result, we can bound

$$\int d\nu^{\varepsilon}_{\omega}(x,z)\big[\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z\big]^2\le C(\varphi)\big[A(\varepsilon,\ell)+B(\varepsilon,\ell)\big], \tag{154}$$

where (cf. (89) and the Definition 12.3 for h_ℓ)

$$A(\varepsilon,\ell) := \int d\nu^\varepsilon_\omega(x,z)|z|^2\big(\phi(x)+\phi(x+\varepsilon z)\big)\mathbb{1}_{\{|z|\geq \ell\}} = 2\int d\nu^\varepsilon_\omega(x,z)|z|^2\phi(x)\mathbb{1}_{\{|z|\geq \ell\}} = 2\int d\mu^\varepsilon_\omega(x)\phi(x)h_\ell(\theta_{x/\varepsilon}\omega),$$

$$B(\varepsilon,\ell) := \varepsilon^2\ell^4\int d\nu^\varepsilon_\omega(x,z)\big(\phi(x)+\phi(x+\varepsilon z)\big) = 2\varepsilon^2\ell^4\int d\nu^\varepsilon_\omega(x,z)\phi(x) = 2\varepsilon^2\ell^4\int d\mu^\varepsilon_\omega(x)\phi(x)\lambda_0(\theta_{x/\varepsilon}\omega).$$

Since $\omega\in\Omega_{\rm typ}\subset(\bigcap_{\ell\in\mathbb{N}}\mathcal{A}[h_\ell])\cap\tilde A$ with $A:=\{\lambda_2<+\infty\}$ (recall Definition 12.2), it holds $\lim_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_\omega(x)\phi(x)\times h_\ell(\theta_{x/\varepsilon}\omega)=\int dx\, m\phi(x)\mathbb{E}_0[h_\ell]$. By dominated convergence and (A7) we then get that $\lim_{\ell\uparrow\infty,\varepsilon\downarrow 0}A(\varepsilon,\ell)=0$. As $\omega\in\Omega_{\rm typ}\subset\tilde A\cap\mathcal{A}[\lambda_0]$ with $A=\{\lambda_0<+\infty\}$, the integral $\int d\mu^\varepsilon_\omega(x)\phi(x)\lambda_0(\theta_{x/\varepsilon}\omega)$ converges to $\int dx\, m\phi(x)\mathbb{E}_0[\lambda_0]$ as $\varepsilon\downarrow 0$. As a consequence, $\lim_{\varepsilon\downarrow 0}B(\varepsilon,\ell)=0$. Coming back to (154) we finally get (153). □

From now on we denote by $\tilde\omega$ the environment in $\Omega_{\rm typ}$ for which we want to prove Items (i), (ii) and (iii) of Theorem 4.1.

• *Convergence of solutions*. We start by proving Item (i).

We consider (43). We recall that the weak solution $u_\varepsilon\in H^{1,{\rm f}}_{\tilde\omega,\varepsilon}$ satisfies (cf. (28))

$$\frac{1}{2}\langle\nabla_\varepsilon v,\nabla_\varepsilon u_\varepsilon\rangle_{\nu^\varepsilon_{\tilde\omega}}+\langle v,u_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}}=\langle v,f_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}}\quad\forall v\in H^{1,{\rm f}}_{\tilde\omega,\varepsilon}. \tag{155}$$

Due to (155) with $v:=u_\varepsilon$ we get that $\|u_\varepsilon\|^2_{L^2(\mu^\varepsilon_{\tilde\omega})}\leq\langle u_\varepsilon,f_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}}$ and therefore $\|u_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})}\leq\|f_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})}$ by Schwarz inequality. Hence, by (155) it holds $\frac12\|\nabla_\varepsilon u_\varepsilon\|^2_{L^2(\nu^\varepsilon_{\tilde\omega})}\leq\|f_\varepsilon\|^2_{L^2(\mu^\varepsilon_{\tilde\omega})}$. Since $f_\varepsilon\rightharpoonup f$, the family $\{f_\varepsilon\}$ is bounded and therefore there exists $C>0$ such that, for ε small enough as we assume below,

$$\|u_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})}\leq C,\qquad\|\nabla_\varepsilon u_\varepsilon\|_{L^2(\nu^\varepsilon_{\tilde\omega})}\leq C. \tag{156}$$

Due to (156) and by Proposition 18.1, along a sequence $\varepsilon_k\downarrow 0$ we have:

(i) $u_\varepsilon\overset{2}{\rightharpoonup}u$, where u is of the form $u=u(x)$ and $u\in H^1_*(m\,dx)$;
(ii) $\nabla_\varepsilon u_\varepsilon(x,z)\overset{2}{\rightharpoonup}w(x,\omega,z):=\nabla_*u(x)\cdot z+u_1(x,\omega,z)$, $u_1\in L^2(\mathbb{R}^d,L^2_{\rm pot}(\nu))$.

Below, convergence for $\varepsilon\downarrow 0$ is understood along the sequence $\{\varepsilon_k\}$.

Claim 19.3. *For dx-a.e. $x\in\mathbb{R}^d$ it holds*

$$\int d\nu(\omega,z)w(x,\omega,z)z=2D\nabla_*u(x). \tag{157}$$

Proof of Claim 19.3. We apply (155) to the test function $v(x):=\varepsilon\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega)$, where $\varphi\in C^2_c(\mathbb{R}^d)$ and $g\in\mathcal{G}_2$ (cf. Section 12). Recall that $\mathcal{G}_2$ is given by bounded functions. Note that $v\in\mathcal{C}(\varepsilon\widehat{\tilde\omega})\subset H^{1,{\rm f}}_{\tilde\omega,\varepsilon}$.

Due to (113) we have

$$\nabla_\varepsilon v(x,z)=\varepsilon\nabla_\varepsilon\varphi(x,z)g(\theta_{z+x/\varepsilon}\tilde\omega)+\varphi(x)\nabla g(\theta_{x/\varepsilon}\tilde\omega,z). \tag{158}$$

In the above formula, the gradient ∇g is the one defined in (74). Note that both the expressions in the r.h.s. belongs to $L^2(\nu^\varepsilon_{\tilde\omega})$ (as this holds for the l.h.s., it is enough to check it for the first expression by using (25)). Due to (158), (155) can be rewritten as

$$\begin{aligned}&\frac{\varepsilon}{2}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon\varphi(x,z)g(\theta_{z+x/\varepsilon}\tilde\omega)\nabla_\varepsilon u_\varepsilon(x,z)+\frac12\int d\nu^\varepsilon_{\tilde\omega}(x,z)\varphi(x)\nabla g(\theta_{x/\varepsilon}\tilde\omega,z)\nabla_\varepsilon u_\varepsilon(x,z)\\&\quad+\varepsilon\int d\mu^\varepsilon_{\tilde\omega}(x)\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega)u_\varepsilon(x)=\varepsilon\int d\mu^\varepsilon_{\tilde\omega}(x)\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega)f_\varepsilon(x).\end{aligned} \tag{159}$$

Since the families of functions $\{u_\varepsilon(x)\}$, $\{f_\varepsilon(x)\}$, $\{\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega)\}$ are bounded families in $L^2(\mu^\varepsilon_{\tilde\omega})$, the expressions in the third line of (159) go to zero as $\varepsilon\downarrow 0$.

We now claim that

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon u_\varepsilon(x,z)\big[\nabla_\varepsilon\varphi(x,z)-\nabla\varphi(x)\cdot z\big]g(\theta_{z+x/\varepsilon}\tilde\omega)=0. \tag{160}$$

This follows by using that $\|g\|_\infty<+\infty$, applying Schwarz inequality and afterwards Lemma 19.2 (recall that $\|\nabla_\varepsilon u_\varepsilon\|_{L^2(\nu^\varepsilon_{\tilde\omega})}\leq C$ for ε small). The above limit (160), the 2-scale convergence $\nabla_\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} w$ and the fact that (104) holds for all functions in $\mathcal{H}_3\subset\mathcal{H}$ (cf. Section 12), imply that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon u_\varepsilon(x,z)\nabla_\varepsilon\varphi(x,z)g(\theta_{z+x/\varepsilon}\tilde\omega)&=\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon u_\varepsilon(x,z)\nabla\varphi(x)\cdot zg(\theta_{z+x/\varepsilon}\tilde\omega)\\&=\int dx\, m\int d\nu(\omega,z)w(x,\omega,z)\nabla\varphi(x)\cdot zg(\theta_z\omega).\end{aligned} \tag{161}$$

Due to (161) also the expression in the first line of (159) goes to zero as $\varepsilon\downarrow 0$. We conclude therefore that also the expression in the second line of (159) goes to zero as $\varepsilon\downarrow 0$. Hence

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon u_\varepsilon(x,z)\varphi(x)\nabla g(\theta_{x/\varepsilon}\tilde\omega,z)=0.$$

Due to the 2-scale convergence $\nabla_\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} w$ and since (104) holds for all gradients ∇g, $g\in\mathcal{G}_2$ (since $\mathcal{H}_2\subset\mathcal{H}$), we conclude that

$$\int dx\, m\varphi(x)\int d\nu(\omega,z)w(x,\omega,z)\nabla g(\omega,z)=0.$$

Since $\{\nabla g: g\in\mathcal{G}_2\}$ is dense in $L^2_{\mathrm{pot}}(\nu)$, the above identity implies that, for dx-a.e. x, the map $(\omega,z)\mapsto w(x,\omega,z)$ belongs to $L^2_{\mathrm{sol}}(\nu)$. On the other hand, we know that $w(x,\omega,z)=\nabla_* u(x)\cdot z+u_1(x,\omega,z)$, where $u_1\in L^2(\mathbb{R}^d,L^2_{\mathrm{pot}}(\nu))$. Hence, by (81), for dx-a.e. x we have that

$$u_1(x,\cdot,\cdot)=v^a(\cdot,\cdot),\quad a:=\nabla_* u(x).$$

As a consequence (using also (82)), for dx-a.e. x, we have

$$\int d\nu(\omega,z)w(x,\omega,z)z=\int d\nu(\omega,z)z\big[\nabla_* u(x)\cdot z+v^{\nabla_* u(x)}(\omega,z)\big]=2D\nabla_* u(x).$$

This concludes the proof of Claim 19.3. □

We now reapply (155) but with $v(x):=\varphi(x)\in C^2_c(\mathbb{R}^d)$ (note that $v\in\mathcal{C}(\widehat{\varepsilon\tilde\omega})\subset H^{1,\mathrm{f}}_{\tilde\omega,\varepsilon}$). We get

$$\frac12\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon\varphi(x,z)\nabla_\varepsilon u_\varepsilon(x,z)+\int d\mu^\varepsilon_{\tilde\omega}(x)\varphi(x)u_\varepsilon(x)=\int d\mu^\varepsilon_{\tilde\omega}(x)\varphi(x)f_\varepsilon(x). \tag{162}$$

Let us analyze the first term in (162). By (160) which holds also with $g\equiv 1$, the expression $\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon\varphi(x,z)\nabla_\varepsilon u_\varepsilon(x,z)$ equals $\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon u_\varepsilon(x,z)\nabla\varphi(x)\cdot z+o(1)$ as $\varepsilon\downarrow 0$. Since the function $(\omega,z)\mapsto z_i$ is in $\mathcal{H}$ and since $\tilde\omega\in\Omega_{\mathrm{typ}}$, by the 2-scale convergence $\nabla_\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} w$ we obtain that

$$\lim_{\varepsilon\downarrow 0}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\nabla_\varepsilon\varphi(x,z)\nabla_\varepsilon u_\varepsilon(x,z)=\int dx\, m\int d\nu(\omega,z)w(x,\omega,z)\nabla\varphi(x)\cdot z.$$

Since $u_\varepsilon \overset{2}{\rightharpoonup} u$ with $u=u(x)$, $1\in\mathcal{G}$ and $f_\varepsilon\rightharpoonup f$, by taking the limit $\varepsilon\downarrow 0$ in (162) we get

$$\frac12\int dx\, m\nabla\varphi(x)\cdot\int d\nu(\omega,z)w(x,\omega,z)z+\int dx\, m\varphi(x)u(x)=\int dx\, m\varphi(x)f(x). \tag{163}$$

Due to (157) the above identity reads

$$\int dx\,\nabla\varphi(x)\cdot D\nabla_* u(x)+\int dx\,\varphi(x)u(x)=\int dx\,\varphi(x)f(x), \tag{164}$$

i.e. u is a weak solution of (42) (recall that $C_c^\infty(\mathbb{R}^d)$ is dense in $H^1_*(m\,dx)$ and note that $\nabla\varphi$ in (164) can be replaced by $\nabla_*\varphi$ due to Definitions 3.8, 3.9). This concludes the proof of limit (43) as $\varepsilon \downarrow 0$ along the sequence $\{\varepsilon_k\}$. Since for each sequence $\{\varepsilon_n\}$ converging to zero we can extract a subsequence $\{\varepsilon_{n_k}\}$ for which Items (i) and (ii) stated after (156) hold and for which the limit (43) holds along $\{\varepsilon_{n_k}\}$, we get (43) as $\varepsilon \downarrow 0$.

It remains to prove (44). It is enough to apply the same arguments of [44, Proof of Thm. 6.1]. Since $f_\varepsilon \to f$ we have $f_\varepsilon \rightharpoonup f$ and therefore, by (43), we have $u_\varepsilon \rightharpoonup u$. This implies that $v_\varepsilon \rightharpoonup v$ (again by (43)), where v_ε and v are respectively the weak solution in $H^{1,\mathrm{f}}_{\tilde\omega,\varepsilon}$ and $H^1_*(m\,dx)$ of $-\mathbb{L}^\varepsilon_{\tilde\omega} v_\varepsilon + v_\varepsilon = u_\varepsilon$ and $-\nabla_*\cdot D\nabla_* v + v = u$. By taking the scalar product in the weak version of (41) with v_ε (as in (28)), the scalar product in the weak version of (42) with v (as in (34)), the scalar product in the weak version of $-\mathbb{L}^\varepsilon_{\tilde\omega} v_\varepsilon + v_\varepsilon = u_\varepsilon$ with $u_\varepsilon \in H^{1,\mathrm{f}}_{\tilde\omega,\varepsilon}$ and the scalar product in the weak version of $-\nabla_*\cdot D\nabla_* v + v = u$ with u and comparing the resulting expressions, we get

$$\langle u_\varepsilon, u_\varepsilon\rangle_{\mu^\varepsilon_\omega} = \langle v_\varepsilon, f_\varepsilon\rangle_{\mu^\varepsilon_\omega}, \qquad \int u(x)^2\,dx = \int f(x)v(x)\,dx. \tag{165}$$

Since $f_\varepsilon \to f$ and $v_\varepsilon \rightharpoonup v$ we get that $\langle v_\varepsilon, f_\varepsilon\rangle_{\mu^\varepsilon_\omega} \to \int v(x)f(x)m\,dx$. Hence, by (165), we conclude that $\lim_{\varepsilon\downarrow 0}\langle u_\varepsilon, u_\varepsilon\rangle_{\mu^\varepsilon_\omega} = \int u(x)^2 m\,dx$. The last limit and the weak convergence $u_\varepsilon \rightharpoonup u$ imply the strong convergence $u_\varepsilon \to u$ by Remark 3.12. This concludes the proof of (44) and therefore of Theorem 4.1-(i).

• *Convergence of flows*. We now prove (45) in Item (ii), i.e. $\nabla_\varepsilon u_\varepsilon \rightharpoonup \nabla_* u$. By (156) the analogous of bound (38) with $\nabla_\varepsilon u_\varepsilon$ is satisfied. Suppose that $f_\varepsilon \rightharpoonup f$. Take $\varphi \in C^1_c(\mathbb{R}^d)$, then $\langle \varphi, f_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}} \to \langle \varphi, f\rangle_{m\,dx}$. By Item (i) we know that $u_\varepsilon \rightharpoonup u$ and therefore $\langle \varphi, u_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}} \to \langle \varphi, u\rangle_{m\,dx}$. The above convergences and (155) with v given by φ restricted to $\varepsilon\widehat{\tilde\omega}$ (note that $v \in \mathcal{C}(\varepsilon\widehat{\tilde\omega}) \subset H^{1,\mathrm{f}}_{\tilde\omega,\varepsilon}$), we conclude that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2}\langle \nabla_\varepsilon\varphi, \nabla_\varepsilon u_\varepsilon\rangle_{\nu^\varepsilon_{\tilde\omega}} = \lim_{\varepsilon\downarrow 0}\bigl[\langle \varphi, f_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}} - \langle \varphi, u_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}}\bigr] = \langle \varphi, f-u\rangle_{m\,dx}.$$

Due to (42) and (34), the r.h.s. equals $\int dx\, m D(x)\nabla_*\varphi(x)\cdot\nabla_* u(x)$. This proves the analogous of (39) and therefore (45).

Take now $f_\varepsilon \to f$. Then, by (44), $u_\varepsilon \to u$. Reasoning as above we get that, given $g_\varepsilon \in H^{1,\mathrm{f}}_{\tilde\omega,\varepsilon}$ and $g \in H^1_*(m\,dx)$ with $L^2(\mu^\varepsilon_{\tilde\omega}) \ni g_\varepsilon \rightharpoonup g \in L^2(m\,dx)$, it holds

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2}\langle \nabla_\varepsilon g_\varepsilon, \nabla_\varepsilon u_\varepsilon\rangle_{\nu^\varepsilon_{\tilde\omega}} = \lim_{\varepsilon\downarrow 0}\bigl[\langle g_\varepsilon, f_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}} - \langle g_\varepsilon, u_\varepsilon\rangle_{\mu^\varepsilon_{\tilde\omega}}\bigr] = \langle g, f-u\rangle_{m\,dx}.$$

Since $g \in H^1_*(m\,dx)$, due to (42), the r.h.s. equals $\int dx\, m D(x)\nabla_* g(x)\cdot\nabla_* u(x)$. This proves (46).

• *Convergence of energies*. We prove Item (iii). Since $f_\varepsilon \to f$, we have $u_\varepsilon \to u$ by (44) and $\nabla_\varepsilon u_\varepsilon \to \nabla_* u$ by (46). It is enough to apply (40) with v_ε replaced by u_ε, w replaced by $\nabla_* u$, $g_\varepsilon := u_\varepsilon$ and $g := u$ and one gets (47).

20. Proof of Theorem 4.4

Limit (51) corresponds to Remark 4.2. Limit (50) follows from (51) and [44, Thm. 9.2]. To treat (52), (53), (54), (55) we claim that it is enough to prove them for a fixed f and for all $t \geq 0$, $\lambda > 0$ as ω varies in a translation invariant set of full $\mathcal{P}$-probability. To prove this claim, given $n \in \mathbb{N}$ we set $B_n := \{x \in \mathbb{R}^d : |x| \leq n\}$ and we restrict to $\omega \in \mathcal{A}[1]$ (cf. Proposition 3.1), thus implying that $\lim_{\varepsilon\downarrow 0}\mu^\varepsilon_\omega(B_n) = m\ell(B_n)$ (recall that $\mathcal{A}[1]$ is translation invariant and measurable and that $\mathcal{P}(\mathcal{A}[1]) = 1$). Then, given $f \in C_c(\mathbb{R}^d)$ with support in B_n, we have (since μ^ε_ω is reversible for $P^\varepsilon_{\omega,t}$)

$$\|P^\varepsilon_{\omega,t} f\|^2_{L^2(\mu^\varepsilon_\omega)} = \int d\mu^\varepsilon_\omega(x)\bigl(P^\varepsilon_{\omega,t} f\bigr)^2(x) \leq \int d\mu^\varepsilon_\omega(x) P^\varepsilon_{\omega,t} f^2(x) = \mu^\varepsilon_\omega\bigl(f^2\bigr) \leq \|f\|^2_\infty \mu^\varepsilon_\omega(B_n). \tag{166}$$

Hence $\|P^\varepsilon_{\omega,t} f\|_{L^2(\mu^\varepsilon_\omega)} \leq \|f\|_\infty \mu^\varepsilon_\omega(B_n)^{1/2}$. Similarly, $\|P^\varepsilon_{\omega,t} f\|_{L^1(\mu^\varepsilon_\omega)} \leq \|f\|_\infty \mu^\varepsilon_\omega(B_n)$. As $R^\varepsilon_{\omega,\lambda} = \int_0^\infty e^{-\lambda s} P^\varepsilon_{\omega,s}\,ds$, we also have $\|R^\varepsilon_{\omega,\lambda} f\|_{L^2(\mu^\varepsilon_\omega)} \leq \lambda^{-1}\|f\|_\infty \mu^\varepsilon_\omega(B_n)^{1/2}$ and $\|R^\varepsilon_{\omega,\lambda} f\|_{L^1(\mu^\varepsilon_\omega)} \leq \lambda^{-1}\|f\|_\infty \mu^\varepsilon_\omega(B_n)$. The same bounds hold for the Brownian motion with diffusion matrix $2D$ and for the measure $m\,dx$. Hence, by a density argument with functions in $C_c(\mathbb{R}^d)$, one gets our claim.

To prove (52), …, (55) for fixed f we need the following fact (proved at the end of the section):

Lemma 20.1. *Suppose that Assumption* (A9) *is satisfied. Fix a weakly decreasing function* $\psi : [0,+\infty) \to [0,+\infty)$ *such that* $\mathbb{R}^d \ni x \mapsto \psi(|x|) \in [0,\infty)$ *is Riemann integrable. Then $\mathcal{P}$-a.s. it holds*

$$\lim_{\ell\uparrow\infty} \overline{\lim_{\varepsilon\downarrow 0}} \int d\mu^\varepsilon_\omega(x)\psi\big(|x|\big)\mathbb{1}_{\{|x|\geq \ell\}} = 0. \tag{167}$$

We will apply the above lemma only with $\psi(r) := 1/(1+r^{d+1})$. By this choice it is then simple to check that (167) holds for ω varying in a translation invariant set, as $\int d\mu^\varepsilon_{\theta_g\omega}(x) f(x) = \int d\mu^\varepsilon_\omega(x) f(\tau_{-g}x)$ for all $g \in \mathbb{G}$ and $\omega \in \Omega_*$.

By the same arguments used to prove [15, Lemma 6.1], the above lemma implies the following for ω varying in a translation invariant set with full $\mathcal{P}$-probability: given $h \in C(\mathbb{R}^d)$ with $|h(x)| \leq C/(1+|x|^{d+1})$ for all $x \in \mathbb{R}^d$ and given $L^2(\mu^\varepsilon_\omega) \ni h^\varepsilon_\omega \to h \in L^2(m\,dx)$, it holds $\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d} |h^\varepsilon_\omega(x) - h(x)|^2 d\mu^\varepsilon_\omega(x) = 0$. Then, as $P_t f$ and $R_\lambda f$ decay exponentially fast, (50) and (51) imply, respectively, (52) and (54) by Lemma 20.1. It remains to derive (53) from (52) and to derive (55) from (54). We use some manipulations as in the proof of [15, Corollary 2.5]. We show the derivation of (55) (which is absent in [15]), since the derivation of (53) is similar. To this aim, without loss of generality, we restrict to $f \geq 0$. For any $n \in \mathbb{N}$ we can bound

$$\big\|R^\varepsilon_{\omega,\lambda} f - R_\lambda f\big\|_{L^1(\mu^\varepsilon_\omega)} \leq \big\|\big(R^\varepsilon_{\omega,\lambda} f\big)\mathbb{1}_{B_n^c}\big\|_{L^1(\mu^\varepsilon_\omega)} + \big\|(R_\lambda f)\mathbb{1}_{B_n^c}\big\|_{L^1(\mu^\varepsilon_\omega)} + \mu^\varepsilon_\omega(B_n)^{\frac{1}{2}}\big\|R^\varepsilon_{\omega,\lambda} f - R_\lambda f\big\|_{L^2(\mu^\varepsilon_\omega)}. \tag{168}$$

We restrict to $\omega \in \mathcal{A}[1]$ ($\mathcal{A}[1]$ is a translation invariant measurable set with $\mathcal{P}(\mathcal{A}[1]) = 1$) and ω satisfying (167) with $\psi(r) := 1/(1+r^{d+1})$. Hence it holds $\lim_{\varepsilon\downarrow 0}\mu^\varepsilon_\omega(B_n) = m\ell(B_n)$. Then, by (54), the last addendum in the r.h.s. of (168) goes to zero as $\varepsilon \downarrow 0$. Let us move to the second addendum in the r.h.s. of (168). As $R_\lambda f$ decays exponentially (hence $R_\lambda f \leq C\psi$), by Lemma 20.1 it holds $\lim_{n\uparrow\infty}\lim_{\varepsilon\downarrow 0}\|(R_\lambda f)\mathbb{1}_{B_n^c}\|_{L^1(\mu^\varepsilon_\omega)} = 0$.

Let us finally move to the first addendum in the r.h.s. of (168). Since $R^\varepsilon_{\omega,\lambda} f \geq 0$ we can write

$$\big\|\big(R^\varepsilon_{\omega,\lambda} f\big)\mathbb{1}_{B_n^c}\big\|_{L^1(\mu^\varepsilon_\omega)} = \big\|R^\varepsilon_{\omega,\lambda} f\big\|_{L^1(\mu^\varepsilon_\omega)} - \big\|\big(R^\varepsilon_{\omega,\lambda} f\big)\mathbb{1}_{B_n}\big\|_{L^1(\mu^\varepsilon_\omega)}. \tag{169}$$

We claim that

$$\lim_{\varepsilon\downarrow 0}\big\|R^\varepsilon_{\omega,\lambda} f\big\|_{L^1(\mu^\varepsilon_\omega)} = \|R_\lambda f\|_{L^1(m\,dx)}. \tag{170}$$

To prove our claim we observe that, as $R^\varepsilon_{\omega,\lambda} = \int_0^\infty e^{-\lambda s} P^\varepsilon_{\omega,s}\, ds$ and $f \geq 0$, it holds $R^\varepsilon_{\omega,\lambda} f \geq 0$ and therefore

$$\big\|R^\varepsilon_{\omega,\lambda} f\big\|_{L^1(\mu^\varepsilon_\omega)} = \int_0^\infty ds\, e^{-\lambda s}\int d\mu^\varepsilon_\omega(x) P^\varepsilon_{\omega,s} f(x) = \int_0^\infty ds\, e^{-\lambda s}\mu^\varepsilon_\omega(f) = \frac{\mu^\varepsilon_\omega(f)}{\lambda}.$$

As $\omega \in \mathcal{A}[1]$ we have $\mu^\varepsilon_\omega(f) \to \int dx\, m f(x)$. On the other hand, arguing as above, we get $\|R_\lambda f\|_{L^1(m\,dx)} = \lambda^{-1}\int dx\, m f(x)$. By combining the above observations we get (170).

Now we claim that

$$\lim_{\varepsilon\downarrow 0}\big\|\big(R^\varepsilon_{\omega,\lambda} f\big)\mathbb{1}_{B_n}\big\|_{L^1(\mu^\varepsilon_\omega)} = \big\|(R_\lambda f)\mathbb{1}_{B_n}\big\|_{L^1(m\,dx)}. \tag{171}$$

Indeed, by Schwarz inequality, (54) and since $\omega \in \mathcal{A}[1]$, $\|(R^\varepsilon_{\omega,\lambda} f)\mathbb{1}_{B_n}\|_{L^1(\mu^\varepsilon_\omega)} - \|(R_\lambda f)\mathbb{1}_{B_n}\|_{L^1(\mu^\varepsilon_\omega)} = \langle R^\varepsilon_{\omega,\lambda} f - R_\lambda f, \mathbb{1}_{B_n}\rangle_{L^2(\mu^\varepsilon_\omega)}$ goes to 0 as $\varepsilon \downarrow 0$. On the other hand, as $\omega \in \mathcal{A}[1]$ and by upper and lower bounding $(R_\lambda f)\mathbb{1}_{B_n}$ with non-negative functions $\varphi \in C_c(\mathbb{R}^d)$, we get that $\|(R_\lambda f)\mathbb{1}_{B_n}\|_{L^1(\mu^\varepsilon_\omega)} \to \|(R_\lambda f)\mathbb{1}_{B_n}\|_{L^1(m\,dx)}$ as $\varepsilon \downarrow 0$, thus allowing to prove our claim (171).

By combining (169), (170) and (171) we get that $\|(R^\varepsilon_{\omega,\lambda} f)\mathbb{1}_{B_n^c}\|_{L^1(\mu^\varepsilon_\omega)}$ goes to $\|(R_\lambda f)\mathbb{1}_{B_n^c}\|_{L^1(m\,dx)}$ as $\varepsilon \downarrow 0$. By taking then the limit $n \uparrow \infty$ we get that $\lim_{n\uparrow\infty}\lim_{\varepsilon\downarrow 0}\|(R^\varepsilon_{\omega,\lambda} f)\mathbb{1}_{B_n^c}\|_{L^1(\mu^\varepsilon_\omega)} = 0$.

Proof of Lemma 20.1. To simplify the notation, we take $V = \mathbb{I}$ in (2), thus implying that $\tau_k\Delta = k + [0,1)^d$ (the arguments below can be easily adapted to the general case). Always to simplify the notation, we prove a slightly different version of (167), the method can be easily adapted to (167). In particular, we now prove that $\mathcal{P}$-a.s. it holds

$$\lim_{\ell\uparrow\infty} \overline{\lim_{\varepsilon\downarrow 0}} X_{\varepsilon,\ell} = 0 \quad \text{where } X_{\varepsilon,\ell}(\omega) := \varepsilon^d \sum_{k\in\mathbb{Z}^d : |k|\geq \ell/\varepsilon} \psi\big(|\varepsilon k|\big) N_k. \tag{172}$$

Trivially Item (i) in Assumption (A9) implies (172). Let us suppose that Item (ii) is satisfied (we restrict to the first case in Item (ii), the second more general case can be treated similarly). Given $\varepsilon \in (0,1)$ let $r = r(\varepsilon)$ be the positive integer of the form 2^a, $a \in \mathbb{N}$, such that $r^{-1} \leq \varepsilon < 2r^{-1}$. Then, since ψ is weakly decreasing,

$$X_{\varepsilon,\ell}(\omega) \leq 2^d Y_{r,\ell}(\omega) \quad \text{where } Y_{r,\ell}(\omega) := r^{-d} \sum_{k\in\mathbb{Z}^d : |k|\geq r\ell/2} \psi\big(|k/r|\big) N_k. \tag{173}$$

In particular, to get (172) it is enough to show that, $\mathcal{P}$-a.s, $\lim_{\ell\uparrow\infty} \overline{\lim}_{r\uparrow\infty} Y_{r,\ell} = 0$, where r varies in $\Gamma := \{2^0, 2^1, 2^2, \dots\}$. From now on we understand that $r \in \Gamma$. Since $\mathbb{E}[N_k] = m$ and since $\psi(|x|)$ is Riemann integrable, we have

$$\lim_{r\uparrow\infty} \mathbb{E}[Y_{r,\ell}] = z_\ell := m \int \psi\big(|x|\big) \mathbb{1}_{\{|x|\geq \ell/2\}}\, dx < \infty. \tag{174}$$

We now estimate the variance of $Y_{r,\ell}$. Due to the stationarity of $\mathcal{P}$ and since $\mathbb{E}[N_0^2] < +\infty$, it holds $\sup_{k\in\mathbb{Z}^d} \operatorname{Var}(N_k) < +\infty$. By Condition (ii) we have, for some fixed constant $C_1 > 0$,

$$\operatorname{Var}(Y_{r,\ell}) \leq C_1 r^{-2d} \sum_{\substack{k\in\mathbb{Z}^d:\\ |k|\geq r\ell/2}} \sum_{\substack{k'\in\mathbb{Z}^d:\\ |k'|\geq r\ell/2}} \big[\big|k-k'\big|^{-1} \mathbb{1}_{k\neq k'} + \mathbb{1}_{k=k'}\big] \psi\big(|k/r|\big) \psi\big(\big|k'/r\big|\big) =: I_0(r,\ell) + I_1(r,\ell) + I_2(r,\ell),$$

where $I_0(r,\ell)$, $I_1(r,\ell)$ and $I_2(r,\ell)$ denote the contribution from addenda as above respectively with (a) $k = k'$, (b) $|k-k'| \geq r$ and (c) $1 \leq |k-k'| < r$. Then we have

$$\lim_{r\uparrow\infty} r^d I_0(r,\ell) = C_1 \int_{|x|\geq\ell/2} \psi\big(|x|\big)^2\, dx < +\infty, \tag{175}$$

$$\lim_{r\uparrow\infty} r I_1(r,\ell) = C_1 \int_{|x|\geq\ell/2} dx \int_{|y|\geq\ell/2} dy \frac{\mathbb{1}_{\{|x-y|\geq 1\}}}{|x-y|} \psi\big(|x|\big)\psi\big(|y|\big) < +\infty. \tag{176}$$

To control $I_2(r,\ell)$ we observe that, for $r \geq 2$,

$$\sum_{\substack{v\in\mathbb{Z}^d:\\ 1\leq\|v\|_\infty\leq cr}} \|v\|_\infty^{-1} \leq C' \sum_{n=1}^{cr} n^{d-2} \leq \begin{cases} C'' r^{d-1} & \text{if } d \geq 2, \\ C'' \ln r & \text{if } d = 1. \end{cases}$$

The above bound implies for r large that

$$I_2(r,\ell) \leq C_1 \|\psi\|_\infty r^{-2d} \sum_{\substack{k\in\mathbb{Z}^d:\\ |k|\geq r\ell/2}} \psi\big(|k/r|\big) \sum_{\substack{k'\in\mathbb{Z}^d:\\ 1\leq|k-k'|\leq r}} \big|k-k'\big|^{-1} \leq C_2 r^{-1} \ln r \int_{|x|\geq\ell/2} \psi\big(|x|\big)\, dx. \tag{177}$$

Due to (175), (176) and (177), $\operatorname{Var}(Y_{r,\ell}) \leq C_3(\ell) r^{-1} \ln r$ for $r \geq C_4(\ell)$. Now we write explicitly $r = 2^j$. By Markov's inequality, we have for $j \geq C_5(\ell)$ that

$$\mathcal{P}\big(\big|Y_{2^j,\ell} - \mathbb{E}[Y_{2^j,\ell}]\big| \geq 1/j\big) \leq j^2 \operatorname{Var}(Y_{2^j,\ell}) \leq C_3(\ell) j^2 \ln\big(2^j\big) 2^{-j}.$$

Since the last term is summable among j, by Borel–Cantelli lemma we conclude that, for $\mathcal{P}$-a.a. ω, $|Y_{2^j,\ell}(\omega) - \mathbb{E}[Y_{2^j,\ell}]| \leq 1/j$ for all $\ell \geq 1$ and $j \geq C_6(\ell,\omega)$. This proves that, $\mathcal{P}$-a.s., $\lim_{r\uparrow\infty, r\in\Gamma} Y_{r,\ell} = z_\ell$ (cf. (174)). Since $\lim_{\ell\uparrow\infty} z_\ell = 0$, we get that $\lim_{\ell\uparrow\infty} \lim_{r\uparrow\infty, r\in\Gamma} Y_{r,\ell} = 0$, $\mathcal{P}$-a.s. □

Appendix A: Further comments on assumptions (A3), …, (A6)

In this appendix we extend our comments concerning the choice of the set Ω_* in our main assumptions when $\mathbb{G} = \mathbb{R}^d$. We recall that all sets Ω_k are translation invariant.

We first point out that Ω_5 and Ω_6 are always measurable. Indeed the points of the simple point process $\hat{\omega}$ can be enumerated as $x_1(\hat{\omega}), x_2(\hat{\omega}), \dots$ in a measurable way by ordering the points according to their distance from the origin and, in case of more points at the same distance, ordering these points in lexicographic order (see [9, p. 480] for details).

By using the above measurable functions $x_1(\hat\omega), x_2(\hat\omega), \dots$, one can easily express Ω_5 and Ω_6 as countable intersection of measurable sets, thus leading to the measurability of Ω_5 and Ω_6. As a consequence, in (A5) and (A6), one could replace "$\forall \omega \in \Omega_*$" by "for $\mathcal{P}$-a.a. ω".

We now claim that Ω_3 is measurable if (16) holds for all $\omega \in \Omega$ and $g \in \mathbb{G}$. To prove our claim we observe that $\theta_g\omega = \theta_{g'}\omega$ for some $g \neq g'$ in $\mathbb{G}$ if and only if $\theta_g\omega = \omega$ for some $g \in \mathbb{G}\setminus\{0\}$. The last property implies that $\mu_{\theta_g\omega} = \mu_\omega$ for some $g \in \mathbb{G}\setminus\{0\}$ and therefore, by (16), that $\tau_g\mu_\omega = \mu_\omega$ for some $g \in \mathbb{G}\setminus\{0\}$. If $\tau_g\mu_\omega = \mu_\omega$, then $\tau_{-g}\hat\omega = \hat\omega$ and therefore $g = V^{-1}(x-y)$ for some $x, y \in \hat\omega$. Hence, when (16) holds for all $\omega \in \Omega$, then Ω_3 is given by the measurable set $\bigcap_{m\neq n}\{\omega \in \Omega : \theta_{V^{-1}(x_m(\hat\omega)-x_n(\hat\omega))}\omega \neq \omega\} = \{\omega \in \Omega : \theta_{V^{-1}x}\omega \neq \theta_{V^{-1}y}\omega \ \forall x \neq y \text{ in } \hat\omega\}$. This concludes the proof of our claim. As a consequence, if (16) holds for all $\omega \in \Omega$ and $g \in \mathbb{G}$, in (A3) we can replace "$\forall \omega \in \Omega_*$" by "for $\mathcal{P}$-a.a. ω".

Appendix B: Campbell's identity

In this appendix we consider the general context (instead of the context of Warning 6.1) and we recall Campell's identity for the Palm distribution $\mathcal{P}_0$. For what follows, it is enough to require only (A1) and (A2). Recall definition (3) of Δ and that $\Omega_0 := \{\omega \in \Omega : n_0(\omega) > 0\}$ when $\mathbb{G} = \mathbb{R}^d$ and in the special discrete case, and that $\Omega_0 := \{(\omega, x) \in \Omega \times \Delta : n_x(\omega) > 0\}$ when $\mathbb{G} = \mathbb{Z}^d$.

• *Case* $\mathbb{G} = \mathbb{R}^d$. For any measurable function $f : \mathbb{R}^d \times \Omega_0 \to [0, +\infty)$ it holds

$$\int_{\mathbb{R}^d} dx \int_{\Omega_0} d\mathcal{P}_0(\omega) f(x,\omega) = \frac{1}{m\ell(\Delta)} \int_\Omega d\mathcal{P}(\omega) \int_{\mathbb{R}^d} d\mu_\omega(x) f\big(g(x), \theta_{g(x)}\omega\big) \tag{178}$$

(cf. [26, Eq. (4.11)] together with Appendix C below, cf. [11, Thm. 13.2.III]). As $g(x) = V^{-1}x$ by (4), by taking $f(x,\omega) := \mathbb{1}_{V^{-1}U}(x)\mathbb{1}_A(\omega)$ Campbell's identity (178) reduces to (9). Moreover, note that, when $V = \mathbb{I}$, from (178) we recover the more common Campbell's formula

$$\int_{\mathbb{R}^d} dx \int_{\Omega_0} d\mathcal{P}_0(\omega) f(x,\omega) = \frac{1}{m} \int_\Omega d\mathcal{P}(\omega) \int_{\mathbb{R}^d} d\mu_\omega(x) f(x, \theta_x\omega) \quad \text{if } V = \mathbb{I}. \tag{179}$$

• *Case* $\mathbb{G} = \mathbb{Z}^d$. For any measurable function $f : \mathbb{Z}^d \times \Omega_0 \to [0, +\infty)$ it holds

$$\begin{aligned}\sum_{g\in\mathbb{G}} \int_{\Omega_0} d\mathcal{P}_0(\omega,x) f(g,\omega,x) &= \frac{1}{m\,\ell(\Delta)} \int_\Omega d\mathcal{P}(\omega) \int_{\mathbb{R}^d} d\mu_\omega(x) f\big(g(x), \theta_{g(x)}\omega, \beta(x)\big) \\ &= \frac{1}{m\,\ell(\Delta)} \sum_{g\in\mathbb{G}} \int_\Omega d\mathcal{P}(\omega) \int_{\tau_g\Delta} d\mu_\omega(x) f(g, \theta_g\omega, \tau_{-g}x)\end{aligned} \tag{180}$$

(cf. [26, Eq. (4.11)] together with Appendix C below). Note that, Campbell's identity (180) with $f(g,\omega,x) := \delta_{0,g}\mathbb{1}_A(\omega,x)$ reduces to (11).

• *Special discrete case*. As discussed in Section 2.3 we think of $\mathcal{P}_0$ as a probability measure on $\Omega_0 = \{\omega \in \Omega : n_0(\omega) > 0\}$. Then, due to (180),

$$\sum_{g\in\mathbb{Z}^d} \int_{\Omega_0} d\mathcal{P}_0(\omega) f(g,\omega) = \frac{1}{\mathbb{E}[n_0]} \sum_{g\in\mathbb{G}} \int_\Omega d\mathcal{P}(\omega) n_g(\omega) f(g, \theta_g\omega), \tag{181}$$

for any measurable function $f : \mathbb{Z}^d \times \Omega_0 \to [0, +\infty)$.

Appendix C: Sign choices and cumulative Palm measure

In this appendix we consider the general context (instead of the context of Warning 6.1) and explain how to derive our formulas concerning the Palm distribution $\mathcal{P}_0$ from the present literature. Our main reference is given by [26]. When the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is such that Ω is a Borel space and $\mathcal{F} = \mathcal{B}(\Omega)$, then one can refer as well to the theory of Palm pairs to get our formulas (cf. [26, Theorem 4.10], [27,29]). For what follows, it is enough to require (A1) and (A2).

Our action $(\theta_g)_{g\in\mathbb{G}}$ on Ω is related to the action $(\theta_g^{\mathrm{Ge}})_{g\in\mathbb{G}}$ on Ω in [26] by the identity $\theta_g^{\mathrm{Ge}} = \theta_{-g}$ for all $g \in \mathbb{G}$ (we have added here the supfix "Ge" in order to distinguish the two actions). As a consequence, when applying some formulas from [26] sign changes are necessary. By setting $\tau_g\mathfrak{m}(A) := \mathfrak{m}(\tau_g A)$ in (6), we have followed the convention used in [11, Section 12.1] and [44, p. 23]. By setting $\mu_{\theta_g\omega}=\tau_g\mu_\omega$ in (16), we have followed the convention of [44, p. 23]. We stress that in e.g. [26,27], $\tau_g\mathfrak{m}(A)$ is defined as $\mathfrak{m}(\tau_{-g}A)$. On the other hand, the fundamental relation (16) is valid also in [26] (see Eq. (2.27) in [26]). Indeed, as already observed, $\theta_g^{\rm Ge}=\theta_{-g}$.

The Palm distribution $\mathcal{P}_0$ introduced in Section 2.3 is the normalized version of the cumulative Palm measure $\mathbb{Q}$ [26, Thm. 4.1, Def. 4.2]. We clarify the notation there. As Haar measure λ of $\mathbb{G}$ we take the Lebesgue measure dx if $\mathbb{G}=\mathbb{R}^d$ and the counting measure on $\mathbb{G}$ if $\mathbb{G}=\mathbb{Z}^d$. As a set of orbit representatives $\mathcal{O}$ for the action of $\mathbb{G}$ on $\mathbb{R}^d$, we take $\mathcal{O}=\{0\}$ if $\mathbb{G}=\mathbb{R}^d$ and $\mathcal{O}=\Delta$ if $\mathbb{G}=\mathbb{Z}^d$ (cf. (3)). Given $x\in\mathbb{R}^d$, we define $\beta(x):=a$ and $g(x):=g$ if $x=\tau_g a$ with $a\in\mathcal{O}$. Note that this definition incorporates both (4) and (5) and that $\beta(x)\equiv 0$ when $\mathbb{G}=\mathbb{R}^d$. Given $x\in\mathbb{R}^d$, as in [26] we define $\pi_x:\mathbb{G}\to\mathbb{R}^d$ as $\pi_x(g):=\tau_g x$ (cf. [26, Section 2.2.2]) and we introduce the measure μ_x on $\mathbb{R}^d$ as $\mu_x:=\lambda\circ\pi_x^{-1}$ (cf. [26, Section 2.2.3]). In particular, $\mu_x(dy)=\ell(\Delta)^{-1}dy$ if $\mathbb{G}=\mathbb{R}^d$, and μ_x is the counting measure on $\{\tau_g x: g\in\mathbb{Z}^d\}$ if $\mathbb{G}=\mathbb{Z}^d$. Given $x,y\in\mathbb{R}^d$, we define the measure $\kappa_{x,y}$ on $\mathbb{G}$ as follows: if y does not belong to the $\mathbb{G}$-orbit of x, then $\kappa_{x,y}$ is the zero measure; if y belongs to the $\mathbb{G}$-orbit of x, then $\kappa_{x,y}$ is the Dirac atomic measure δ_g, g being the unique element of $\mathbb{G}$ such that $\tau_g x=y$, i.e. $g=V^{-1}(y-x)$. Given a measurable set $A\subset\mathbb{G}$, the map $\mathbb{R}^d\times\mathbb{R}^d\ni(x,y)\mapsto\kappa_{x,y}(A)=\mathbb{1}(V^{-1}(y-x)\in A)\in\{0,1\}$ is measurable. Hence $\kappa_{x,y}$ is a kernel from $\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{G}$ (cf. [26, Section 2.1.2]). Note moreover that for any measurable function $f:\mathbb{R}^d\times\mathbb{G}\to[0,+\infty)$ and for any $x\in\mathbb{R}^d$ it holds

$$\int_{\mathbb{G}} f(\tau_g x,g)d\lambda(g)=\int_{\mathbb{R}^d}d\mu_x(y)\int_{\mathbb{G}}d\kappa_{x,y}(g)f(y,g). \tag{182}$$

The above identity is equivalent to [26, Eq. (3.1)] with the notation there (take $S:=\mathbb{R}^d$ and $gx:=\tau_g x$ there). Due to these observations, it is simple to check that the kernel $\kappa_{x,y}$ satisfies the properties listed in [26, Thm. 3.1] (note that the map θ_g appearing in Item (i) of [26, Thm. 3.1] refers to the action of $\mathbb{G}$ on $\mathbb{G}$ itself, hence it is the map $\mathbb{G}\ni g'\mapsto g+g'\in\mathbb{G}$). Hence, $\kappa_{x,y}$ is the kernel entering in [26, Thm. 4.1]. To apply now [26, Eq. (4.4)] we observe that $\kappa_{\beta(x),x}=\delta_{g(x)}$ for all $x\in\mathbb{R}^d$. It remains to fix the function $w:\mathbb{R}^d\to[0,+\infty)$ appearing there. Defining $w:=\mathbb{1}_\Delta$, due to the above description of μ_x and $\mathcal{O}$ we have that $\mu_x(w)=1$ for any $x\in\mathcal{O}$. We can finally reformulate [26, Eq. (4.4)] in our context: for each measurable $B\subset\Omega\times\mathcal{O}$ the measure $\mathbb{Q}$ is given by

$$\mathbb{Q}(B)=\int_\Omega d\mathcal{P}(\omega)\int_{\mathbb{R}^d}\mu_\omega(dx)\int_{\mathbb{G}}\kappa_{\beta(x),x}(dg)\mathbb{1}_B\big(\theta_g\omega,\beta(x)\big)w(x), \tag{183}$$

which reads

$$\mathbb{Q}(B)=\int_\Omega d\mathcal{P}(\omega)\int_\Delta\mu_\omega(dx)\mathbb{1}_B\big(\theta_{g(x)}\omega,\beta(x)\big). \tag{184}$$

Since $\mathbb{Q}(\Omega\times\mathcal{O})=\mathbb{E}[\mu_\omega(\Delta)]=m\ell(\Delta)$, we get that $\mathcal{P}_0:=\mathbb{Q}(\Omega\times\mathcal{O})^{-1}\mathbb{Q}=(m\ell(\Delta))^{-1}\mathbb{Q}$. When $\mathbb{G}=\mathbb{R}^d$, as $\mathcal{O}=\{0\}$ we identify $\Omega\times\mathcal{O}$ with Ω. Using the stationarity of $\mathcal{P}$ and that $\beta(x)=0$, (184) becomes (9) for $U=\Delta$. When $\mathbb{G}=\mathbb{Z}^d$, as $\mathcal{O}=\Delta$ we have $\beta(x)=x$ and $g(x)=0$ for any $x\in\Delta$, hence (184) becomes (11).

Appendix D: Proof of Lemma 3.5

In this appendix we consider the general context (instead of the context of Warning 6.1). Due to the discussion in Section 6, to prove Lemma 3.5 when $\mathbb{G}=\mathbb{Z}^d$ it is enough to prove the analogous claim for the random walk $\bar{X}_t$ with rates (63) in the setting $\mathcal{S}[2]$ introduced in Section 6. As a consequence, from now on we restrict to the case $\mathbb{G}=\mathbb{R}^d$ and to the special discrete case. Moreover, to simplify the notation we take $V=\mathbb{I}$ (in the general case it would be enough to use Remark 2.1). Recall the notation introduced in (65).

We set $A_1:=\{\omega\in\Omega_0: 0<\lambda_0(\omega)<\infty\}$ and define $\tilde{A}_1$ according to (65). We point out that $\mathcal{P}_0(\lambda_0>0)=1$ due (A6) and the property that $|\hat{\omega}|=\infty$ for $\mathcal{P}$-a.a. ω and therefore for $\mathcal{P}_0$-a.a. ω by Lemma 7.1 (see the comments on the main assumptions before Remark 2.1). Using that $\mathbb{E}_0[\lambda_0]<\infty$, as done for Corollary 7.2, we get that $\mathcal{P}_0(\tilde{A}_1)=1$. Since $r_x(\omega)=\lambda_0(\theta_x\omega)$, for each $\omega\in\tilde{A}_1$ it holds $r_x(\omega)\in(0,+\infty)$ for all $x\in\hat{\omega}$.

Consider now the translation invariant measurable set Ω_*. By assumption $\mathcal{P}(\Omega_*)=1$ and $\theta_x\omega\neq\theta_y\omega$ for all $x\neq y$ in $\mathbb{G}$ and $\omega\in\Omega_*$. By Lemma 7.1, we get $\mathcal{P}_0(\Omega_*)=1$. Note that, given $\omega,\omega'\in\Omega_*\cap\Omega_0$, if $\omega'=\theta_x\omega$ for some $x\in\mathbb{G}$, then x is unique and in this case we define $r(\omega,\omega'):=r_{0,x}(\omega)$, otherwise we define $r(\omega,\omega'):=0$.

We set $A_2:=\tilde{A}_1\cap\Omega_*\cap\Omega_0$. Due to the above properties, we have $\mathcal{P}_0(A_2)=1$. Moreover, given $\omega_*\in A_2$ we can introduce the discrete time Markov chain $(\omega_n)_{n\in\mathbb{N}}$ with state space A_2, initial configuration ω_* and jump probabilities $p(\omega,\omega'):=r(\omega,\omega')/\lambda_0(\omega)$. We write P_{ω_*} for its law and E_{ω_*} for the associated expectation. P_{ω_*} is a probability measure on $A_2^{\mathbb{N}}$.

We now introduce the distribution $d\mathcal{Q}_0(\omega) := \mathbb{E}_0[\lambda_0]^{-1}\lambda_0(\omega)\, d\mathcal{P}_0(\omega)$ on A_2. We claim that $\mathcal{Q}_0$ is reversible (hence stationary) and ergodic for the discrete-time Markov chain $(\omega_n)_{n\geq 0}$. To get reversibility we observe that, by Lemma 7.3 and since $\mathbb{E}_0[\lambda_0] < \infty$,

$$\mathbb{E}_0\left[\int_{\mathbb{R}^d} d\hat{\omega}(x) r(\omega, \theta_x\omega) f(\omega) h(\theta_x\omega)\right] = \mathbb{E}_0\left[\int_{\mathbb{R}^d} d\hat{\omega}(x) r(\omega, \theta_x\omega) h(\omega) f(\theta_x\omega)\right],$$

for any bounded measurable functions $f, h : \Omega_0 \to \mathbb{R}$. Let us now prove the ergodicity of $\mathcal{Q}_0$. To this aim let $B \subset A_2$ be a set left invariant by the Markov chain. Due to (A6) we get $B = \Omega_0 \cap \tilde{B}$ (cf. (65)). Since $\tilde{B}$ is translation invariant and $\mathcal{P}$ is ergodic by (A1), we get that $\mathcal{P}(\tilde{B}) \in \{0, 1\}$. By Lemma 7.1 we conclude that $\mathcal{P}_0(B) = \mathcal{P}_0(\tilde{B} \cap \Omega_0) = \mathcal{P}_0(\tilde{B}) = \mathcal{P}(\tilde{B}) \in \{0, 1\}$.

By the ergodicity of $\mathcal{Q}_0$ and since $\int_{A_2} d\mathcal{Q}_0(\omega)\lambda_0^{-1}(\omega) = \mathbb{E}_0[\lambda_0]^{-1}$, we have

$$\int_{A_2} d\mathcal{Q}_0(\omega) P_\omega\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \frac{1}{\lambda_0(\omega_n)} = \frac{1}{\mathbb{E}_0[\lambda_0]}\right) = 1. \tag{185}$$

Let $A_3 := \{\omega \in A_2 : \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{\lambda_0(\omega_n)} = +\infty \ P_\omega\text{-a.s.}\}$. By (185) and since $\mathbb{E}_0[\lambda_0] < \infty$, we have $\mathcal{Q}_0(A_3) = 1$. As $\lambda_0 > 0$ $\mathcal{P}_0$-a.s., $\mathcal{P}_0$ and $\mathcal{Q}_0$ are mutually absolutely continuous, hence we conclude that $\mathcal{P}_0(A_3) = 1$.

Then for any $\omega_* \in A_3$ we can define the continuous-time Markov chain $(\omega_t)_{t\geq 0}$ starting at ω_* obtained by a random time-change from the Markov chain $(\omega_n)_{n\in\mathbb{N}}$ starting at ω_*, imposing that the waiting time at ω is an exponential random variable with parameter $\lambda_0(\omega)$. Note in particular that ω_t is defined for all $t \geq 0$. Given $\omega \in \tilde{A}_3$ and $x_0 \in \hat{\omega}$ let $(\omega_t)_{t\geq 0}$ be the above continuous-time Markov chain starting now at $\theta_{x_0}\omega$. For $t \geq 0$ we set $X_t^\omega := x_0 + \sum_{s\in[0,t]:\omega_{s-}\neq\omega_s} F(\omega_{s-}, \omega_s)$, where $F(\omega, \omega') := x$ if $\omega' = \theta_x\omega$. Then X_t^ω coincides with the random walk described in Lemma 3.5. Setting $\mathcal{A} := \tilde{A}_3$ and using Lemma 7.1, the above construction implies the content of Lemma 3.5.

Appendix E: Technical facts concerning Section 6

In this appendix we consider the general context (instead of the context of Warning 6.1). Below $\bar{\mathbb{E}}[\cdot]$ denotes the expectation w.r.t. $\bar{\mathcal{P}}$.

Lemma E.1. *$\bar{\mathcal{P}}$ is $\bar{\mathbb{G}}$-stationary.*

Proof. Let $f : \bar{\Omega} \to [0, +\infty)$ be a measurable function and let $x \in \mathbb{R}^d$. Then we get (see below for some comments)

$$\begin{aligned}
\ell(\Delta)\bar{\mathbb{E}}[f \circ \bar{\theta}_x] &= \mathbb{E}\left[\int_\Delta da\, f\big(\bar{\theta}_x(\omega, a)\big)\right] = \mathbb{E}\left[\int_\Delta da\, f\big(\theta_{g(x+a)}\omega, \beta(x+a)\big)\right] = \mathbb{E}\left[\int_{\Delta+x} da\, f\big(\theta_{g(a)}\omega, \beta(a)\big)\right] \\
&= \sum_{g\in\mathbb{G}} \int_{(\Delta+x)\cap\tau_g\Delta} da\, \mathbb{E}\big[f(\theta_g\omega, a - Vg)\big] = \sum_{g\in\mathbb{G}} \int_{(\Delta+x)\cap\tau_g\Delta} da\, \mathbb{E}\big[f(\omega, a - Vg)\big] \\
&= \sum_{g\in\mathbb{G}} \int_{(\tau_{-g}(\Delta+x))\cap\Delta} du\, \mathbb{E}\big[f(\omega, u)\big] = \int_\Delta du\, \mathbb{E}\big[f(\omega, u)\big] = \ell(\Delta)\bar{\mathbb{E}}[f].
\end{aligned}$$

For the forth identity, we point out that if $a = \tau_g y = y + Vg$ with $y \in \Delta$, then $\beta(a) = y = a - Vg$. The firth identity follows from the $\mathbb{G}$-stationarity of $\mathcal{P}$. For the seventh identity observe that $\mathbb{R}^d = \bigsqcup_{g\in\mathbb{G}} (\tau_{-g}(\Delta + x))$. $\square$

Lemma E.2. *$\bar{\mathcal{P}}$ is ergodic w.r.t. the action $(\bar{\theta}_g)_{g\in\bar{\mathbb{G}}}$.*

Proof. Let $f : \bar{\Omega} \to \mathbb{R}$ be a measurable function such that $f(\bar{\theta}_x\bar{\omega}) = f(\bar{\omega})$ for any $x \in \bar{\mathbb{G}}$ and any $\bar{\omega} \in \bar{\Omega}$. We need to prove that f is constant $\bar{\mathcal{P}}$-a.s. By (57) the invariance of f reads $f(\theta_{g(x+a)}\omega, \beta(x + a)) = f(\omega, a)$ for all $(\omega, a) \in \bar{\Omega}$ and all $x \in \mathbb{R}^d$. Fixed $(\omega, a) \in \bar{\Omega}$, given $a' \in \Delta$ we take $x := a' - a$. As $x + a = a' \in \Delta$ we have $\beta(x + a) = a'$ and $g(x + a) = 0$. In particular, the invariance of f implies that $f(\omega, a') = f(\omega, a)$ for any $\omega \in \Omega$ and $a, a' \in \Delta$. When $x = Vg$ for some $g \in \mathbb{G}$ and $a \in \Delta$, we have $g(x + a) = g$ and $\beta(x + a) = a$. The invariance of f then implies that $f(\theta_g\omega, a) = f(\omega, a)$ for all $g \in \mathbb{G}$, $\omega \in \Omega$ and $a \in \Delta$. As $\mathcal{P}$ is ergodic w.r.t. the action $(\theta_g)_{g\in\mathbb{G}}$, given $a \in \Delta$ we conclude that $\exists c \in \mathbb{R}$ such that for $f(\omega, a) = c$ for $\mathcal{P}$-a.a. ω. By combining this identity with the fact that $f(\omega, a') = f(\omega, a)$ for any $\omega \in \Omega$ and $a, a' \in \Delta$, we conclude that $f \equiv c$ $\bar{\mathcal{P}}$-a.s. $\square$

Lemma E.3. *The intensities $\bar m$ and m of the random measure $\mu_{\bar\omega}$ and μ_ω, respectively, coincide.*

Proof. We have (see comments below)

$$\begin{aligned}\bar m\,\ell(\Delta)^2 &= \bar{\mathbb{E}}\big[\mu_{\bar\omega}(\Delta)\big]\ell(\Delta)=\mathbb{E}\bigg[\int_\Delta da\,\mu_\omega(\Delta+a)\bigg]\\ &=\sum_{g\in\mathbb{G}}\int_\Delta da\,\mathbb{E}\big[\mu_\omega\big((\Delta+a)\cap\tau_g\Delta\big)\big]=\sum_{g\in\mathbb{G}}\int_\Delta da\,\mathbb{E}\big[\mu_{\theta_{-g}\omega}\big((\Delta+a)\cap\tau_g\Delta\big)\big]\\ &=\sum_{g\in\mathbb{G}}\int_\Delta da\,\mathbb{E}\big[\mu_\omega\big(\tau_{-g}(\Delta+a)\cap\Delta\big)\big]=\int_\Delta da\,\mathbb{E}\big[\mu_\omega(\Delta)\big]=m\ell(\Delta)^2.\end{aligned}$$

In the third identity we used that $\mathbb{R}^d=\bigsqcup_{g\in\mathbb{G}}\tau_g\Delta$. In the forth identity we used the $\mathbb{G}$-stationarity of $\mathcal{P}$. In the fifth identity we used (16). In the sixth identity we used that $\mathbb{R}^d=\bigsqcup_{g\in\mathbb{G}}\tau_{-g}(\Delta+a)$. □

Lemma E.4. *For all $\bar\omega\in\bar\Omega_*$ it holds $\mu_{\bar\theta_x\bar\omega}(\cdot)=\mu_{\bar\omega}(\bar\tau_x\cdot)$ for any $x\in\bar{\mathbb{G}}$.*

Proof. Let $\bar\omega=(\omega,a)$ and $A\in\mathcal{B}(\mathbb{R}^d)$. We have (16). We apply (in order) (57), (58), (16), (2) to get: $\mu_{\bar\theta_x\bar\omega}(A)=\mu_{(\theta_{g(x+a)}\omega,\beta(x+a))}(A)=\mu_{\theta_{g(x+a)}\omega}(A+\beta(x+a))=\mu_\omega(\tau_{g(x+a)}(A+\beta(x+a))=\mu_\omega(A+\beta(x+a)+Vg(x+a))$. Since $\forall u\in\mathbb{R}^d$ we have $u=\beta(u)+Vg(u)$, $\mu_\omega(A+\beta(x+a)+Vg(x+a))$ equals $\mu_\omega(A+x+a)=\mu_{(\omega,a)}(A+x)=\mu_{\bar\omega}(A+x)$. (59) then allows to conclude. □

Lemma E.5. *For all $\bar\omega\in\bar\Omega_*$ it holds $\bar r_{x,y}(\bar\theta_z\bar\omega)=\bar r_{\bar\tau_z x,\bar\tau_z y}(\bar\omega)$ for any $x,y\in\mathbb{R}^d$, $z\in\bar{\mathbb{G}}$.*

Proof. Let $\bar\omega=(\omega,a)$. We have (17). By (57) and (63), we have

$$\bar r_{x,y}(\bar\theta_z\bar\omega)=\bar r_{x,y}\big(\theta_{g(z+a)}\omega,\beta(z+a)\big)=r_{x+\beta(z+a),y+\beta(z+a)}(\theta_{g(z+a)}\omega).$$

Note that, for any $u\in\mathbb{R}^d$, $\tau_{g(z+a)}(u+\beta(z+a))=u+\beta(z+a)+Vg(z+a)=u+z+a$. As a byproduct of this observation and (17), $r_{x+\beta(z+a),y+\beta(z+a)}(\theta_{g(z+a)}\omega)$ can be rewritten as $r_{x+z+a,y+z+a}(\omega)$. Hence, $\bar r_{x,y}(\bar\theta_z\bar\omega)=r_{x+z+a,y+z+a}(\omega)$. On the other hand, by (59) and (63), $\bar r_{\bar\tau_z x,\bar\tau_z y}(\bar\omega)=\bar r_{\bar\tau_z x,\bar\tau_z y}(\omega,a)=r_{\bar\tau_z x+a,\bar\tau_z y+a}(\omega)=r_{x+z+a,y+z+a}(\omega)$. □

Lemma E.6. *The Palm distributions $\mathcal{P}_0$ and $\bar{\mathcal{P}}_0$ associated respectively to $\mathcal{P}$ and $\bar{\mathcal{P}}$ coincide.*

Proof. Let $A\in\bar{\mathcal{F}}$. Due to (59) the function $\bar g$ analogous to (4) for $\mathcal{S}[2]$ is the identity map. Hence, by (9) and since $\bar m=m$, we have

$$\begin{aligned}\bar{\mathcal{P}}_0(A)&=\frac{1}{m\ell(\Delta)^2}\mathbb{E}\bigg[\int_\Delta da\int_\Delta d\mu_{(\omega,a)}(x)\mathbb{1}_A\big(\theta_{g(x+a)}\omega,\beta(x+a)\big)\bigg]\\ &=\frac{1}{m\ell(\Delta)^2}\mathbb{E}\bigg[\int_\Delta da\int_{\Delta+a} d\mu_\omega(x)\mathbb{1}_A\big(\theta_{g(x)}\omega,\beta(x)\big)\bigg]\\ &=\frac{1}{m\ell(\Delta)^2}\sum_{g\in\mathbb{G}}\int_\Delta da\,\mathbb{E}\bigg[\int_{\Delta+a} d\mu_\omega(x)\mathbb{1}_{\{g(x)=g\}}\mathbb{1}_A\big(\theta_g\omega,\beta(x)\big)\bigg]\\ &=\frac{1}{m\ell(\Delta)^2}\sum_{g\in\mathbb{G}}\int_\Delta da\,\mathbb{E}\bigg[\int_{\Delta+a} d\mu_{\theta_{-g}\omega}(x)\mathbb{1}_{\{g(x)=g\}}\mathbb{1}_A\big(\omega,\beta(x)\big)\bigg].\end{aligned}\tag{186}$$

Note that in the last identity we have used the $\mathbb{G}$-stationarity of $\mathcal{P}$. For all $\omega\in\Omega_*$ we can write (see comments below)

$$\begin{aligned}\int_{\Delta+a}d\mu_{\theta_{-g}\omega}(x)\mathbb{1}_{\{g(x)=g\}}\mathbb{1}_A\big(\omega,\beta(x)\big)&=\int_{\Delta+a}d(\tau_{-g}\mu_\omega)(x)\mathbb{1}_{\{g(x)=g\}}\mathbb{1}_A\big(\omega,\beta(x)\big)\\ &=\int_{\tau_{-g}(\Delta+a)}d\mu_\omega(x)\mathbb{1}_{\{x\in\Delta\}}\mathbb{1}_A\big(\omega,\beta(\tau_g x)\big)\\ &=\int_{(\Delta+a-Vg)\cap\Delta}d\mu_\omega(x)\mathbb{1}_A(\omega,x).\end{aligned}\tag{187}$$

Above, for the first identity we used (16). For the second one we used that $\tau_{-g}\mathfrak{m}[f] = \int f(\tau_g x)\,d\mathfrak{m}(x)$ observing that $g(\tau_g x) = g$ if and only $x \in \Delta$. For the third one, we used that $\beta(\tau_g x) = x$ for any $x \in \Delta$. As a byproduct of (186) and (187) we get (see comments below)

$$
\begin{aligned}
\bar{\mathcal{P}}_0(A) &= \frac{1}{m\ell(\Delta)^2}\sum_{g\in\mathbb{G}}\int_\Delta da\,\mathbb{E}\Big[\int_{(\Delta+a-Vg)\cap\Delta} d\mu_\omega(x)\mathbb{1}_A(\omega,x)\Big]\\
&= \frac{1}{m\ell(\Delta)^2}\int_\Delta da\,\mathbb{E}\Big[\int_\Delta d\mu_\omega(x)\mathbb{1}_A(\omega,x)\Big] = \frac{1}{m\ell(\Delta)}\mathbb{E}\Big[\int_\Delta d\mu_\omega(x)\mathbb{1}_A(\omega,x)\Big] = \mathcal{P}_0(A).
\end{aligned}
\tag{188}
$$

Above, in the second identity we have used that $\mathbb{R}^d = \bigsqcup_{g\in\mathbb{G}}(\Delta + a - Vg)$, while the last identity follows from (11). $\square$

Appendix F: Proof of Lemma 13.5 and 13.7

F.1. Proof of Lemma 13.5

Since $\{v_\varepsilon\}$ is bounded in $L^2(\mu^\varepsilon_{\tilde\omega})$, there exist C, ε_0 such that $\|v_\varepsilon\|_{L^2(\mu^\varepsilon_{\tilde\omega})} \le C$ for $\varepsilon \le \varepsilon_0$. We fix a countable set $\mathcal{V} \subset C_c(\mathbb{R}^d)$ such that $\mathcal{V}$ is dense in $L^2(m\,dx)$. We call $\mathcal{L}$ the family of functions Φ of the form $\Phi(x,\omega) = \sum_{i=1}^r a_i\varphi_i(x)g_i(\omega)$, where $r \in \mathbb{N}_+$, $g_i \in \mathcal{G}$, $\varphi_i \in \mathcal{V}$ and $a_i \in \mathbb{Q}$. $\mathcal{L}$ is a dense subset of $L^2(m\,dx \times \mathcal{P}_0)$ as $\mathcal{G}$ is dense in $L^2(\mathcal{P}_0)$. By Schwarz inequality we have

$$
\Big|\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)\Phi(x,\theta_{x/\varepsilon}\tilde\omega)\Big| \le C\Big[\int d\mu^\varepsilon_{\tilde\omega}(x)\Phi(x,\theta_{x/\varepsilon}\tilde\omega)^2\Big]^{1/2}.
\tag{189}
$$

Since $\tilde\omega \in \Omega_{\text{typ}} \subset \mathcal{A}[gg']$ for all $g, g' \in \mathcal{G}$, we have

$$
\lim_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)\Phi(x,\theta_{x/\varepsilon}\tilde\omega)^2 = \sum_i\sum_j a_i a_j\int dx\,m\,\varphi_i(x)\varphi_j(x)\mathbb{E}_0[g_i g_j] = \|\Phi\|^2_{L^2(m\,dx\times\mathcal{P}_0)}.
\tag{190}
$$

Due to (190) we get that the integral in l.h.s. of (189) admits a convergent subsequence. Since $\mathcal{L}$ is countable, by a diagonal procedure we can extract a subsequence $\varepsilon_k \downarrow 0$ such that the limit $F(\Phi) := \lim_{k\to\infty}\int d\mu^{\varepsilon_k}_{\tilde\omega}(x)v_{\varepsilon_k}(x)\Phi(x,\theta_{x/\varepsilon_k}\tilde\omega)$ exists for any $\Phi \in \mathcal{L}$ and it satisfies $|F(\Phi)| \le C\|\Phi\|_{L^2(m\,dx\times\mathcal{P}_0)}$ by (189) and (190). Since $\mathcal{L}$ is a dense subset of $L^2(m\,dx \times \mathcal{P}_0)$, by Riesz's representation theorem there exists a unique $v \in L^2(m\,dx \times \mathcal{P}_0)$ such that $F(\Phi) = \int d\mathcal{P}_0(\omega)\int dx\,m\,\Phi(x,\omega)v(x,\omega)$ for any $\Phi \in \mathcal{L}$. We also get $\|v\|_{L^2(m\,dx\times\mathcal{P}_0)} \le C$. As $\Phi(x,\omega) := \varphi(x)g(\omega)$ – with $\varphi \in \mathcal{V}$ and $b \in \mathcal{G}$ – belongs to $\mathcal{L}$, we get that (100) is satisfied along the subsequence $\{\varepsilon_k\}$ for any $\varphi \in \mathcal{V}$, $b \in \mathcal{G}$. It remains to show that we can indeed take $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$. To this aim we observe that we can take $\mathcal{V}$ fulfilling the following properties: (i) for each $N \in \mathbb{N}_+$ $\mathcal{V}$ contains a function $\phi_N \in C_c(\mathbb{R}^d)$ with values in $[0,1]$ and equal to 1 on $[-N,N]^d$; (ii) each $\varphi \in C_c(\mathbb{R}^d)$ can be approximated in uniform norm by functions $\psi_n \in \mathcal{V}$ such that ψ_n has support inside $[-N,N]^d$, where $N = N(\varphi)$ is the minimal integer for which φ has support inside $[-N,N]^d$. By Schwarz inequality and the boundedness of $\{v_\varepsilon\}$ we can bound $|\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)[\varphi(x) - \psi_n]g(\theta_{x/\varepsilon}\tilde\omega)|^2$ by $\le C^2\|\varphi - \psi_n(x)\|^2_\infty\int d\mu^\varepsilon_{\tilde\omega}(x)\phi_N(x)g(\theta_{x/\varepsilon}\tilde\omega)^2$. Since $\tilde\omega \in \Omega_{\text{typ}} \subset \mathcal{A}[g^2]$ for all $g \in \mathcal{G}$, the last integral converges as $\varepsilon \downarrow 0$ to $(C')^2 := \int dx\,m\phi_N(x)\mathbb{E}_0[g^2]$. In particular, using also that $\psi_n \in \mathcal{V}$, along the subsequence $\{\varepsilon_k\}$ we have

$$
\begin{aligned}
\varlimsup_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega) &\le CC'\|\varphi - \psi_n\|_\infty + \varlimsup_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)\psi_n(x)g(\theta_{x/\varepsilon}\tilde\omega)\\
&= CC'\|\varphi - \psi_n\|_\infty + \int d\mathcal{P}_0(\omega)\int dx\,m\,v(x,\omega)\psi_n(x)g(\omega).
\end{aligned}
\tag{191}
$$

We now take the limit $n \to \infty$. By dominated convergence we conclude that, along the subsequence $\{\varepsilon_k\}$,

$$
\varlimsup_{\varepsilon\downarrow 0}\int d\mu^\varepsilon_{\tilde\omega}(x)v_\varepsilon(x)\varphi(x)g(\theta_{x/\varepsilon}\tilde\omega) \le \int d\mathcal{P}_0(\omega)\int dx\,m\,v(x,\omega)\varphi(x)g(\omega).
\tag{192}
$$

A similar result holds with the liminf, thus implying that (100) holds along the subsequence $\{\varepsilon_k\}$ for any $\varphi \in C_c(\mathbb{R}^d)$ and $g \in \mathcal{G}$.

F.2. Proof of Lemma 13.7

The proof of Lemma 13.7 is similar to the proof of Lemma 13.5. We only give some comments on some new steps. One has to replace $L^2(m\,dx \times \mathcal{P}_0)$ with $L^2(m\,dx \times \nu)$. Now $\mathcal{L}$ is the family of functions Φ of the form $\Phi(x,\omega,z) = \sum_{i=1}^r a_i \varphi_i(x) b_i(\omega,z)$, where $r \in \mathbb{N}_+$, $b_i \in \mathcal{H}$, $\varphi \in \mathcal{V}$ and $a_i \in \mathbb{Q}$ ($\mathcal{V}$ is as in the proof of Lemma 13.5). Due to Lemma 10.2 and since $\tilde\omega \in \Omega_{\mathrm{typ}} \subset \mathcal{A}_1[bb']$ for all $b, b' \in \mathcal{H}$, we can write

$$\begin{aligned}\int d\nu^\varepsilon_{\tilde\omega}(x,z)\Phi(x,\theta_{x/\varepsilon}\tilde\omega,z)^2 &= \sum_i\sum_j a_i a_j \int d\nu^\varepsilon_{\tilde\omega}(x,z)\varphi_i(x)\varphi_j(x) b_i(\theta_{x/\varepsilon}\tilde\omega,z) b_j(\theta_{x/\varepsilon}\tilde\omega,z) \\ &= \sum_i\sum_j a_i a_j \int d\mu^\varepsilon_{\tilde\omega}(x)\varphi_i(x)\varphi_j(x)\widehat{b_i b_j}(\theta_{x/\varepsilon}\tilde\omega).\end{aligned} \tag{193}$$

Since $\tilde\omega \in \Omega_{\mathrm{typ}} \subset \mathcal{A}[\widehat{bb'}]$ for all $b, b' \in \mathcal{H}$, as $\varepsilon \downarrow 0$ the above r.h.s. converges to $\sum_i\sum_j a_i a_j \int_{\mathbb{R}^d} dx\, m\, \varphi_i(x)\varphi_j(x) \times \mathbb{E}_0[\widehat{b_i b_j}] = \|\Phi\|^2_{L^2(m\,dx\times\nu)}$. At this point we can proceed as in the proof of Lemma 13.5 (recalling that $\mathcal{H}$ is dense in $L^2(\nu)$).

Acknowledgements

I thank Günter Last for useful comments on the literature concerning random measures, Pierre Mathieu for useful comments on a preliminary version of this preprint and on the minimality of the assumptions and Andrey Piatnitski for useful discussions on 2-scale convergence. This work comes from the remains of the manuscript [17], which has its own story. I thank my family for the support along that story.

Funding

This work has been partially supported by the ERC Starting Grant 680275 MALIG.

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