Evolutionary dynamics of populations structured by dietary diversity and starvation : cross-diffusion systems

Dynamique évolutive des populations structurées par la diversité alimentaire: systèmes de diffusion croisée

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Contents

Résumé en Français

Cette thèse porte sur l'analyse de problèmes paraboliques non linéaires apparaissants dans des phénomènes biologiques, écologiques et plus généralement de dynamique des populations. Plus précisément, nous étudions l'existence, la régularité et la stabilité des solutions d'équations aux dérivées partielles (EDPs) qui décrivent l'évolution de deux espèces qui diffusent dans un environnement homogène et interagissent entre elles. Les EDPs que nous considérons sont fortement couplées, c'est-à-dire couplées par des termes du second ordre du point de vue des dérivées, et le système qu'elles forment appartient à une classe de systèmes non linéaires de réaction-diffusion, appelés *systèmes de diffusion croisée*.

En 1979, Shigesada, Kawasaki et Teramoto proposent dans [81] un modèle de diffusion croisée décrivant l'interaction entre deux populations. Le système s'écrit comme suit,

$$\begin{cases} \partial_t u - \Delta(u(d_u + d_{11}u + d_{12}v)) = u(r_u - r_{11}u - r_{12}v), & \text{sur } (0,+\infty)\times\Omega, \\ \partial_t v - \Delta(v(d_v + d_{21}u + d_{22}v)) = v(r_v - r_{21}u - r_{22}v), & \text{sur } (0,+\infty)\times\Omega, \end{cases} \tag{SKT}$$

où dorénavant Ω est un domaine ouvert, borné et regulier (classe C^2) de $\mathbb{R}^N, N \in \mathbb{N}$, et les inconnues $u = u(t,x)$, $v = v(t,x)$ sont deux quantités positives qui modélisent les densités de deux populations, avec les coefficients de diffusion d_u, d_v, d_{ij}, $i, j = 1,2$ et les coefficients de réaction r_u, r_v, r_{ij}, strictement positifs. Plus précisément, r_u, r_v représentent les taux de croissance intrinsèques linéaires des deux espèces et $r_{ij}, i, j = 1,2$ représentent les coefficients de compétition. En particulier, r_{ii} est le taux de compétition intra-spécifique, c'est-à-dire le taux des interactions négatives entre les individus de la même espèce, tandis que $r_{ij}, i \neq j$ correspond au taux de compétition inter-spécifique qui gère les interactions négatives entre les individus d'espèces différentes. Les termes de diffusion non linéaires dans (SKT) décrivent les mouvements des individus dans l'environnement Ω. En particulier, on reconnait les termes de diffusion linéaires $\Delta(d_u u)$, $\Delta(d_v v)$ qui modélisent la diffusion intrinsèque des individus dans le domain et les termes de diffusion non linéaires $\Delta(u(d_{11}u + d_{12}v))$, $\Delta(v(d_{21}u + d_{22}v))$ qui quantifient l'effet répulsif de l'interférence mutuelle. Plus précisément, les termes $d_{11}u^2, d_{22}v^2$ sont dits d'*autodiffusion* et les termes non diagonaux $d_{12}uv, d_{21}uv$ sont dits de *diffusion croisée*. Un système de diffusion croisée (SKT) est dit triangulaire lorsque $d_{21} = 0$, et on remarque que si la matrice de diffusion non linéaire est nulle, c'est-à-dire $d_{ij} = 0$, $i, j = 1,2$, alors on retrouve un système de réaction-diffusion usuel de type Lotka-Volterra.

Le système (SKT) a été introduit pour résoudre un problème de modélisation lié à l'instabilité de Turing [85] et à la formation de pattern, par conséquence [5, 15, 46]. Néanmois, ce système s'est avéré être un objet mathématique extrémement riche pour lequel plusieures questions mathématiques de nature différente se posent. Une liste non exaustive concerne: existence locale et globale en temps des solutions, régularité (ou borne L^∞) et unicité des solutions, stabilité des équilibres et comportement asymptotique.

Dans cette thèse, nous nous intéressons à ces questions pour une classe de systèmes de diffusion croisée triangulaires, issus de problèmes de modélisation en dynamique des populations, différents de ceux conduisant à (SKT), et liés à des questions de diversification de régime alimentaire des individus. Dans les paragraphes qui suivent, nous résumons brièvement le contenu de chaque chapitre, les résultats obtenus et les techniques employées.

- Dans le *Chapitre 1*, nous étudions l'existence de solutions faibles et la stabilité linéaire d'un système triangulaire de diffusion croisée avec les conditions de Neumann homogènes au bord. Pour ce qui concerne le résultat d'existence, on montre de manière rigoureuse le passage d'un système de réaction-diffusion de type Lotka-Volterra (système mésoscopique) vers un système de diffusion croisée (système macroscopique), obtenu comme limite de réaction rapide. Le système mésoscopique modélise la compétition de deux espèces, où une espèce a un régime alimentaire plus diversifié que l'autre. À la limite, on trouve un système de diffusion croisée de type *starvation driven*. Les outils principaux utilisés pour passer rigoureusement à la limite incluent des estimations a priori, données par l'analyse d'une fonctionnelle d'entropie, et un argument de compacité. De plus, nous étudions la stabilité linéaire des équilibres homogènes en espace des systèmes macroscopique et mésoscopique et nous excluons la possibilité de l'apparition d'une instabilité de Turing et de la formation de patterns. En particulier, nous étudions la relation à la limite entre la stabilité linéaire de l'état d'équilibre de coexistence à l'échelle mésoscopique et macroscopique. Des simulations numériques sont également réalisées pour compléter les résultats théoriques. Le contenu du *Chapitre 1* est le résultat d'une collaboration avec L. Corrias, H. Dietert et Y.-J. Kim et il a été publié dans *Journal of Mathematical Biology* sous le titre de *Evolution of dietary diversity and a starvation driven cross-diffusion system as its singular limit* [12].

- Dans le *Chapitre 2*, on montre l'existence des solutions faibles pour une vaste classe de systèmes triangulaires de diffusion croisée avec les conditions de Neumann homogènes au bord, en utilisant la dérivation mésoscopique, de manière similaire à celle du *Chapitre 1*. Nous introduisons la généralisation naturelle du système mésoscopique étudié dans le *Chapitre 1* et nous obtenons à la limite une classe plus vaste de systèmes de diffusion croisée triangulaires de type *starvation driven*. L'ingrédient principal d'analyse est une famille de fonctionnelles d'entropie qui inclut la fonctionnelle d'entropie utilisée dans le *Chapitre 1*. Afin d'avoir une compacité suffisante et ensuite de passer à la limite, il suffit de considérer les premiers éléments de la famille de la fonctionnelle d'entropie. Cependant, afin d'étudier la régularité de la solution, on peut étudier l'évolution de l'entropie pour tout élément de la famille, en améliorant ainsi les estimations a priori, à l'aide d'un argument de type bootstrap. La régularité des solutions fait l'objet d'un travail à venir pour lequel nous renvoyons au *Chapitre 4* pour plus de détails.

- Dans le *Chapitre 3*, nous étudions l'existence, l'unicité et la régularité des solutions fortes pour une classe générale de systèmes de diffusion croisée triangulaires avec les conditions de Neumann homogènes au bord. Le terme *solutions fortes* signifie que les équations du système sont satisfaites presque partout. La méthode utilisée pour montrer le résultat d'existence diffère de celle employée dans les *Chapitres 1, 2*, où le système de diffusion croisée a été dérivé par limite mésoscopique. La stratégie présentée dans le *Chapitre 3* est d'introduire un changement de variable approprié, en utilisant de manière cruciale les propriétés de la fonction de diffusion du système de diffusion croisée. Ce changement de variables conduit à un système parabolique sous forme *non divergence*. Par conséquent, les méthodes analytiques classiques, telles que les arguments de régularisation et de point fixe, nous permettent de montrer l'existence d'une solution forte et le caractère borné, $L^\infty((0,T)\times\Omega)$ pour tout $T > 0$, des solutions si la dimension de l'espace est $N \leq 3$. Enfin, nous prouvons l'unicité de la solution forte, un résultat de stabilité *forte-faible* et un résultat d'unicité *forte-faible*, à condition que $N \leq 2$.

Notations

We denote

N	spatial dimension,
$x \cdot y$	the Euclidean scalar product in $\mathbb{R}^N$ between the vectors $x, y \in \mathbb{R}^N$,
Ω	a bounded, open and sufficiently smooth domain of $\mathbb{R}^N$,
$\bar{\Omega}$	the closure of Ω,
$\|\Omega\|$	the Lebesgue measure of Ω,
Ω_T	the set $(0, T) \times \Omega$, with $T > 0$,
∂_i	derivative of first order with respect to the $i-$th variable,
∂_{ij}	derivative of second order with respect to the $i-$th and $j-$th variables,
$C^k(\Omega)$	set of functions with continuous $k-$th derivatives on Ω, with $k \in \mathbb{N}$,
$L^p(\Omega)$	set of functions with $p-$power Lebesgue integrability in Ω, with $p \in [1, +\infty)$,
$L^\infty(\Omega)$	set of essentially bounded functions on Ω,
$W^{k,p}(\Omega)$	set of functions with the first $k-$th weak derivatives in $L^p(\Omega)$ and $k \in \mathbb{N}$,
$H^k(\Omega)$	Hilbert space $W^{k,2}(\Omega)$, with $k \in \mathbb{N}$,
$L^p(0, T; X)$	set of L^p functions on $(0, T)$ with values in X and $p \in [1, \infty]$,
$W^{k,p}(0, T; X)$	set of $W^{k,p}$ functions on $(0, T)$ with values in X, with $k \in \mathbb{N}$ and $p \in [1, \infty]$,
$L^p(\Omega_T)$	$L^p(0, T; L^p(\Omega))$ with $p \in [1, \infty]$,
X_+	set of functions $u \in X$ such that $u \geq 0$ almost everywhere.

Introduction

This thesis aims to present some recent results and advances in the analysis of nonlinear parabolic problems arising in biology and ecology. More precisely, we study the existence, regularity and stability of solutions to partial differential equations (PDEs), describing the evolution of two species that diffuse in a homogeneous environment and interact with each other. The PDEs we consider are strongly coupled, i.e. coupled through second-order derivative terms, and the system they give rise to belongs to a class of nonlinear reaction-diffusion systems, called cross-diffusion systems.

Introduction to population dynamics

Mathematical modeling represents an essential support to investigate biological and ecological phenomena. By combining experimental data and theoretical analysis, several natural events may be described. From a modeling point of view, many mathematical models have been proposed to describe natural processes, in order to predict the outcomes. In particular, we are interested in population dynamics that focus on interactions of individuals or concentrations, depending on biological and environmental conditions (cfr. [70, 72]). In population dynamics, competition among individuals of the same or different species is fundamental, especially if the ecosystems admit a limited amount of resources. A classical system describing interactions has been simultaneously formulated by A. J. Lotka and V. Volterra in 1925, in terms of ordinary differential equations (ODEs). This model describes the evolution of two populations $u = u(t)$, $v = v(t)$ in competitive interactions and at this level, spatial movements of individuals are neglected. Hence, the model writes as below,

$$\begin{cases} \partial_t u = u(r_u - r_{11}u - r_{12}v), & t > 0, \\ \partial_t v = v(r_v - r_{21}u - r_{22}v), & t > 0, \end{cases} \tag{LV}$$

with the coefficients $r_u, r_v, r_{ij} > 0$, $i, j = 1, 2$. More precisely, r_u, r_v are the linear intrinsic growth rates whereas $r_{ij}, i, j = 1, 2$ represent the competitive coefficients. In particular, r_{ii} is the intra-specific competition rate, i.e. the rate of the negative interactions between individuals of the same species, while $r_{ij}, i \neq j$ corresponds to the inter-specific competition rate. We point out that the nature of the inter-specific interactions in (LV) changes if the sign of the inter-specific rates is changed [7]. Indeed, by considering the more general

Lotka-Volterra reaction functions below, for all $u, v \geq 0$ and $\theta_u, \theta_v \in \{-1, 0, +1\}$, we get

$$\begin{aligned} f_{\theta_u}(u,v) &:= u(r_u - r_{11}u + \theta_u r_{12}v), \\ g_{\theta_v}(u,v) &:= v(r_v + \theta_v r_{21}u - r_{22}v), \end{aligned}$$

with $r_u, r_v, r_{ij} > 0$, $i, j = 1, 2$, so that different kinds of interactions can be modeled.

- $\theta_u\theta_v = 1$. This condition includes the case $\theta_u = \theta_v = -1$, giving the reaction functions in (LV) and modeling the competition between the two species. The fitness of both species (i.e. the reproductive success) is lowered by the presence of other species. Conversely, the case $\theta_u = \theta_v = 1$ models a mutualistic relationship, i.e. a reciprocal altruism where both populations benefit.

- $\theta_u\theta_v = -1$. This condition models predator-prey or parasitism interactions, meaning that only one of the two species gains benefits from the interactions with the other population. In particular, if $\theta_u = 1, \theta_v = -1$ (resp. $\theta_u = -1, \theta_v = 1$) then interactions are advantageous for u (resp. v) and harmful for v (resp. u).

- $\theta_u\theta_v = 0$. This condition includes the case $\theta_u = \theta_v = 0$, describing a neutral relationship, which means that individuals of different species don't interact with each other and the associated equations are uncoupled by the reaction functions. Otherwise, if only one between θ_u, θ_v is zero, then one species does not take advantage and is not harmed.

A similar analysis can be performed for the intra-specific interactions, depending on the sign of the intra-specific rates r_{11}, r_{22}.

We conclude the introduction of system (LV) by pointing out that it is possible to predict the mutual exclusion or the coexistence of the species, depending on the parameter values. The analysis concerns the nature of the equilibria of (LV) that we list below

$$(u_1, v_1) = (0,0), \qquad (u_2, v_2) = \left(\frac{r_u}{r_{11}}, 0\right), \qquad (u_3, v_3) = \left(0, \frac{r_v}{r_{22}}\right), \tag{0.0.1}$$

$$(u^*, v^*) = \left(\frac{r_u r_{22} - r_v r_{12}}{r_{11}r_{22} - r_{12}r_{21}}, \frac{r_v r_{11} - r_u r_{21}}{r_{11}r_{22} - r_{12}r_{21}}\right). \tag{0.0.2}$$

Thus, system (LV) admits the equilibrium of total extinction (u_1, v_1), partial extinction $(u_i, v_i)_{i=1,2}$ and the equilibrim of coexistence (u^*, v^*). We outline that the coexistence equilibrium is biologically meaningful (i.e. $u^* > 0$, $v^* > 0$) only in two cases.

- *Weak inter-specific competition.*

$$\frac{r_{12}}{r_{22}} < \frac{r_u}{r_v} < \frac{r_{11}}{r_{21}}. \tag{0.0.3}$$

  Biologically speaking, the inter-specific competition is weaker than the intra-specific one. In this case, one can prove that all the trivial or semi-trivial equilibria in (0.0.1) are linearly unstable while the coexistence steady state in (0.0.2) is linearly stable.

- *Strong inter-specific competition.*

$$\frac{r_{11}}{r_{21}} < \frac{r_u}{r_v} < \frac{r_{12}}{r_{22}}. \tag{0.0.4}$$

Biologically speaking, the inter-specific competition is stronger than the intra-specific one. Under this condition, one can prove that all the trivial or semi-trivial equilibria in (0.0.1) are linearly stable while the coexistence steady state in (0.0.2) is linearly unstable.

Furthermore, experimental observations show that the mutual interference between individuals generates internal pressure that can affect the movements of species. Therefore, it's natural to consider the motility of individuals, i.e. their capacity to diffuse in space. Biological observations point out the phenomenon of segregation in population dynamics, which occurs between species in competition that try to avoid each other. In particular, spatial segregation has been observed in birds [46], mammals [5], [76], amphibians [15], [45], fishes and insects. Segregation may lead to pattern formation and is related to linear stability analysis of spatially homogeneous equilibria of reaction-diffusion systems. In 1952, Alan Turing first obtained pattern formation [85] in the context of morphogenesis. The analysis yielded various (and sometimes unexpected) conclusions: firstly, a minimum of two interacting chemicals are required to create pattern formation. Then, the diffusion effect in a reactive chemical system can lead to a destabilization effect, unlike the common stabilizing role of diffusion which reduces the spatial variations of a concentration field. Thus, the diffusion-induced instability may give rise to structural growth at a particular wavelength. This provides a possible mechanism for producing patterns like animal stripes. The final insight concerns the diffusion coefficients (motility) that generally need to be substantially different to lead to pattern formations.

From now on, we call *linear diffusion* quantities such as Δu_i, appearing in the i-th equation of a system in the unknowns $u_i, i = 1, \dots, m$, $m \in \mathbb{N}$. Conversely, we call *nonlinear diffusion* quantities such as $\Delta(u_i^\alpha u_j^\beta)$, with $i \neq j$ and $\alpha, \beta > 0$ and Δu_i^α with $\alpha \neq 1$ and $i, j = 1, \dots, m$, appearing in the $i-$th equation. In order to investigate pattern formation, we introduce the simplest reaction-diffusion model of two populations in interaction, that is frequently used to describe experimental situations in evolutionary dynamics. It writes as follows

$$\begin{cases} \partial_t u - d_u \Delta u = f(u, v), & \text{in } (0, +\infty) \times \Omega, \\ \partial_t v - d_v \Delta v = g(u, v), & \text{in } (0, +\infty) \times \Omega, \end{cases} \tag{0.0.5}$$

where from now on Ω is a bounded, smooth and open set $\Omega \subset \mathbb{R}^N, N \geq 1$, modeling the ecosystem. By considering in (0.0.5) the reaction functions f, g as the Lotka-Volterra type

$$\begin{aligned} f(u, v) &= u(r_u - r_{11} u - r_{12} v), \\ g(u, v) &= v(r_v - r_{21} u - r_{22} v), \end{aligned} \tag{0.0.6}$$

we will refer to (0.0.5), (0.0.6) as the Lotka-Volterra reaction-diffusion system. System (0.0.5), (0.0.6) has been widely studied [28, 61, 77]. It is worth noticing that Turing instability does not occur, implying no segregation of species. In terms of stability of the spatially homogeneous steady states: whatever the diffusion and reaction coefficients are, all spatially homogeneous equilibria, which are linearly stable for the homogeneous system of (0.0.5), (0.0.6) (that is the system without diffusion), remain linearly stable by considering the linear diffusion effect in a convex domain [56]. Hence, system (0.0.5), (0.0.6) does not reproduce natural ecosystems like patches of land in which one population is dominant with respect to another and viceversa. More generally, it's not obvious to get Turing instability with linear diffusion and two reacting populations with quadratic reaction structure as (0.0.6).

In order to have a model with linear diffusion that reproduces pattern formation, one can consider the Beddington-DeAngelis functional responses for predator-prey model [4, 26] or cubic reaction terms, than often appear in chemistry models [70] or for chemotactic model [44, 88].

A cross-diffusion system: the SKT model

In order to get Turing instability, and pattern formation as a consequence, Shigesada, Kawasaki and Teramoto in 1979 had the intuition to modify the structure of the diffusion terms in (0.0.5), (0.0.6), rather than the reaction functions. Hence, a cross-diffusion system appeared in literature [81] to solve a modeling issue related to the segregation of species. It writes as below

$$\begin{cases} \partial_t u - \Delta(u(d_u + d_{11}u + d_{12}v)) = u(r_u - r_{11}u - r_{12}v), & \text{in } (0,+\infty)\times\Omega, \\ \partial_t v - \Delta(v(d_v + d_{21}u + d_{22}v)) = v(r_v - r_{21}u - r_{22}v), & \text{in } (0,+\infty)\times\Omega, \end{cases} \tag{SKT}$$

where the unknowns $u = u(t,x)$, $v = v(t,x)$ are two nonnegative quantities, modeling the densities of two populations, and all the coefficients d_u, d_v, r_u, r_v, d_{ij}, r_{ij}, $i,j = 1,2$ are strictly positive. In addition, we endow the system (SKT) with the Neumann boundary conditions below,

$$\nabla u \cdot \sigma = \nabla v \cdot \sigma = 0, \qquad \text{in } (0,+\infty)\times\partial\Omega, \tag{0.0.7}$$

where $\sigma = \sigma(x)$ stands for the outward unit normal vector on the boundary $\partial\Omega$ at point x. From the biological point of view, the boundary condition (0.0.7) models an isolated ecosystem. The nonlinear diffusion terms in the l.h.s. of (SKT) describe the spatial movement of individuals in the external environment Ω. More precisely, the linear terms $\Delta(d_u u)$ and $\Delta(d_v v)$ represent the intrinsic diffusion growth with rates d_u, d_v, while the nonlinear terms $\Delta(u(d_{11}u + d_{12}v))$, $\Delta(v(d_{21}u + d_{22}v))$ indicate the repulsive effect of the mutual interference. In particular, we refer to $d_{11}u^2, d_{22}v^2$ as the *self-diffusion* terms and we refer to the non-diagonal terms $d_{12}uv, d_{21}uv$ as the *cross-diffusion* terms. We denote by $D = (d_{ij})_{i,j=1,2}$ the matrix of the diffusion coefficients,

$$D := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Then, if the matrix has a lower or upper triangular structure, i.e. $d_{12} = 0$ or $d_{21} = 0$, respectively, then we say that system (SKT) is triangular (see the following section for more details). Moreover, if all the non-diagonal entries are zero, i.e. $D = \text{diag}\{d_{ii}\}$, then only self diffusion terms appear in (SKT) and the equations are uncoupled by the diffusion terms. Finally, if all entries are zero $d_{ij} = 0, i,j = 1,2$, system (SKT) reduces to the parabolic reaction-diffusion system with linear diffusion (0.0.5).

System (SKT) describes a typical biological mechanism which lead to Turing instability so that we refer to this phenomenon as *cross-diffusion induced instability* [9, 36, 42, 50, 84]. The structure of system (SKT) models the capacity of individuals to measure the density of the same or of the other species and to increase their diffusion rate if the measured density is large. Indeed in (SKT), the diffusion rate corresponding to u, i.e. the quantity $(d_u + d_{11}u + d_{12}v)$, is a function of u, v, depending on d_{11}, d_{12}. The diffusion coefficient d_{11} (resp. d_{12}) measures the amount of extra diffusion rate if there are more individuals of species u (resp. v) around. The same argument holds for the population v with the diffusion function $(d_v + d_{21}u + d_{22}v)$.

In order to understand the dynamic modelled by the cross-diffusion terms in (SKT), M. Ida, M. Mimura and H. Ninomiya proposed in [50] an interpretation of the cross-diffusion terms in the triangular case of (SKT). We will detail this interpretation in the following section after a bref discussion on the multiscale description for ecological systems.

Multiscale description for ecological models

The hallmarked events in ecology concern the distribution of organisms in the environment and the natural life processes such as adaptation, reproduction and interactions that can lead to patterns in ecosystem processes. Moreover, living organisms are organized on a multiple and interconnected scale. Therefore, multiscale modeling is a fundamental tool to describe the spatio-temporal complexity, where the term scale refers to the unit of space and time used to measure the ecological mechanism [47, 67]. The identification of a scale structure also implies the existence of a hierarchical organization of the system. Therefore in any complex system, descriptions at different spatial or temporal levels contribute to an overall understanding of the system's behavior. In particular, since the primary entities involved in population dynamics are organisms, then multiscale methods describe the ecosystem from the level of the single individual's trajectories to the dynamic of the entire environment. More precisely, we want to detail the microscopic, mesoscopic and macroscopic approaches. Each formulation provides different information, depending on the scale at which the ecosystem is analysed, thus all the different multiscale descriptions are not unrelated.

- The microscopic formulation outlines the dynamic of single components or a finite number of particles. Both the probabilistic tools and the analytic approach are used for the microscopic derivation (ex. stochastic processes or mean field limits, lattice models or reaction-kinetics ODEs).

- The mesoscopic description is the intermediate scale and concerns the dynamic of groups of individuals. The focus is on the evolution of the density of each group and it is used not only to understand the macroscopic limit and how it is achieved but to describe intrinsic phenomena that the macroscopic level does not outline. The typical modeling approach is by reaction-diffusion equations.

- The macroscopic scale focuses on the total mass of the system or on the average of variables of the mesoscopic description. It is formulated by reaction-diffusion equations.

Cross-diffusion systems may be derived at a microscopic scale by random-walk lattice models [73]. See [40, 52] for probabilistic approach. In the following subsection, we detail the mesoscopic description of the cross-diffusion system (SKT) in the triangular case, introduced by M. Ida, M. Mimura and H. Ninomiya in [50]. They formally approximated the triangular (SKT) system at the mesoscopic scale by a reaction-diffusion system with linear diffusion.

A mesoscopic model related to the triangular SKT model

In [50], M. Ida, M. Mimura and H. Ninomiya proposed a formal derivation at the mesoscopic scale of the cross-diffusion system (SKT) in the triangular case below,

$$\begin{cases} \partial_t u - \Delta(u(d_u + d_{12}v)) = u(r_u - r_{11}u - r_{12}v), & \text{in } (0,+\infty)\times\Omega, \\ \partial_t v - d_v\Delta v = v(r_v - r_{21}u - r_{22}v), & \text{in } (0,+\infty)\times\Omega, \end{cases} \tag{0.0.8}$$

where $u = u(t,x), v = v(t,x)$ are the species of two populations living in the space domain $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$. In order to understand the interpretation proposed in [50], we consider the simpler case of the evolution of a single species $n = n(t,x)$ that diffuse in a heterogeneous environement. Thus, we consider the following problem,

$$\begin{cases} \partial_t n = \Delta\left((d_u + \mu P(x))\, n\right), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \nabla n \cdot \sigma = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ n(0,x) = n_0(x) & x \in \Omega, \end{cases} \tag{0.0.9}$$

where no reaction terms are taken into account. The quantity $d_u + \mu P(x)$ models the motility of n in a neighborhood of $x \in \Omega$, with the coefficients $d_u, \mu > 0$, and $P := P(x)$ in $[0,1]$ models the local external pressure. More precisely, we assume that the place x is more unfavourable for the population of density n where $P(x)$ is larger. By the identity

$$(d_u + \mu P(x))n = d_u(1 - P(x))n + (d_u + \mu)P(x)n,$$

it is natural to decompose n as below

$$n(t,x) = n_a(t,x) + n_b(t,x), \qquad \text{for all } (t,x) \text{ in } \mathbb{R}_+ \times \Omega,$$

with

$$n_a(t,x) := (1 - P(x))n(t,x) \qquad \text{and} \qquad n_b(t,x) := P(x)n(t,x).$$

For simplicity of notations, we denote $n_a = n_a(t,x)$ and $n_b = n_b(t,x)$, therefore the first equation of (0.0.9) becomes

$$\partial_t\,(n_a + n_b) = \Delta(d_u\, n_a + (d_u + \mu)n_b).$$

The above equation suggests that the individuals of density n are divided into two subpopulations n_a, n_b with the low motility d_u and the hight motility $d_u + \mu$, respectively. In other words, each individual of density n has two different states: a less active state with motility d_u and a more stressed state with highter motility $d_u + \mu$. Therefore, we now construct a system for n_a and n_b according to the previous argument. We fix a small parameter $\varepsilon > 0$, then locally around x, each individual can switch from the state a to b with probability $p_{a\to b}(x)$, in average time $\varepsilon\tau_{a\to b}$. Conversely, we refer to $p_{b\to a}(x)$ as the probability of the passage from the state b to a in a transition time $\varepsilon\tau_{b\to a}$. Biologically speaking, if x is an unfavorable place then $p_{a\to b}(x)$ is higher than $p_{b\to a}(x)$ so that locally around x, individuals of density n_a tend to migrate towards more suitable areas of the space domain. Then, by defining the following quantities for all $x \in \Omega$,

$$h(x) := \frac{p_{a\to b}(x)}{\varepsilon\tau_{a\to b}} \qquad \text{and} \qquad k(x) := \frac{p_{b\to a}(x)}{\varepsilon\tau_{b\to a}},$$

and assuming that the state transition is much faster than the random diffusion of n_a and n_b (i.e. $\varepsilon \ll d_u$), the dynamic becomes

$$\begin{cases} \partial_t n_a = d_u \Delta n_a + \frac{1}{\varepsilon}[k(x)n_b - h(x)n_a], \\ \partial_t n_b = (d_u + \mu)\Delta n_b - \frac{1}{\varepsilon}[k(x)n_b - h(x)n_a]. \end{cases} \tag{0.0.10}$$

By conveniently choosing the transition functions h and k, then system (0.0.10) formally approximates (0.0.9). Indeed, by computing the equation satisfied by $n := n_a + n_b$, we end up with

$$\partial_t n = d_u \Delta n + \mu \Delta n_b.$$

Thus, taking the limit in (0.0.10), as $\varepsilon \to 0$, formally one can expect,

$$\begin{cases} h(x)n_a(\cdot\,, x) = k(x)n_b(\cdot\,, x)\,, \\ n(\cdot, x) = n_a(\cdot, x) + n_b(\cdot, x), \end{cases}$$

implying

$$n_a = \frac{k(x)}{h(x) + k(x)} n, \qquad n_b = \frac{h(x)}{h(x) + k(x)} n.$$

Then, the equation satisfied by n becomes

$$\partial_t n = \Delta\left(\left(d_u + \mu \frac{h(x)}{h(x) + k(x)}\right) n\right),$$

where the ratio $\frac{h(x)}{h(x) + k(x)}$ models the external pressure $P(x)$ at the point x.

By analogy with the limiting procedure used to approximate the system (0.0.9), we end up with a formal approximation of (0.0.8). We firstly remark that the environment Ω of the reaction diffusion system (0.0.8) is homogeneous and the external pressure $P(x)$ in (0.0.9) is now depending on the presence of individuals of species v. We assume that the population u is split into two substates u_a and u_b. Each individual converts its own state to the other depending on the spatial distribution of the competitor v. We additionally assume that the density v is bounded, i.e. there exists a constant $M > 0$ such that

$$0 \le v(t, x) \le M, \qquad \text{a.e. in } (0, +\infty) \times \Omega. \tag{0.0.11}$$

It is worth noticing that since in the triangular cross-diffusion system (0.0.8) the density v satisfies a heat equation, then the maximum principal gives an L^∞ bound for v (assuming L^∞ initial data), so that condition (0.0.11) is not restrictive. Then, identity below holds true,

$$(d_u + d_{21}v)u = d_u\left(1 - \frac{v}{M}\right)u + (d_u + d_{12}M)\frac{v}{M}u = d_u\, u_a + (d_u + d_{12}M)u_B,$$

with

$$u_a := \left(1 - \frac{v}{M}\right)u, \qquad u_b := \frac{v}{M}u \qquad \text{and} \qquad u = u_a + u_b. \tag{0.0.12}$$

Using the previous limiting argument, we describe the dynamic of two species, u^ε and v^ε where u^ε in two sub-populations, u_a^ε and u_b^ε, associated to two states class: a moderately mobile type and an highly mobile one, with the corresponding diffusive coefficients d_u and

$d_u + d_{12}M$. Therefore, taking into account the interpretation proposed in [50], the natural approximation of (0.0.8) at the mesoscopic scale is the following

$$\begin{cases} \partial_t u_a^\varepsilon = d_u \Delta u_a^\varepsilon + u_a^\varepsilon (r_u - r_{11}(u_a^\varepsilon + u_b^\varepsilon) - r_{12} v^\varepsilon) + \dfrac{1}{\varepsilon}\left(h(v^\varepsilon) u_b^\varepsilon - k(v^\varepsilon) u_a^\varepsilon\right), \\ \partial_t u_b^\varepsilon = (d_u + d_{12}M) \Delta u_b^\varepsilon + u_b^\varepsilon (r_u - r_{11}(u_a^\varepsilon + u_b^\varepsilon) - r_{12} v^\varepsilon) - \dfrac{1}{\varepsilon}\left(h(v^\varepsilon) u_b^\varepsilon - k(v^\varepsilon) u_a^\varepsilon\right), \\ \partial_t v^\varepsilon = d_v \Delta v^\varepsilon + v^\varepsilon (r_v - r_{21}(u_a^\varepsilon + u_b^\varepsilon) - r_{22} v^\varepsilon). \end{cases} \tag{0.0.13}$$

Then, taking $\varepsilon \to 0$ and using the decomposition $u^\varepsilon := u_a^\varepsilon + u_b^\varepsilon$, we formally obtain

$$u_a = \frac{k(v)}{h(v) + k(v)} u, \qquad u_b = \frac{h(v)}{h(v) + k(v)} u,$$

and thus

$$\begin{cases} \partial_t u = \Delta\left(\left(d_u + d_{12}M \dfrac{h(v)}{h(v) + k(v)}\right) u\right) + (r_u - r_{11} u - r_{12} v)\, u, \\ \partial_t v = d_v \Delta v + (r_v - r_{21}\, u - r_{22}\, v)\, v, \end{cases}$$

that corresponds to (0.0.8) if h and k satisfy

$$\frac{h(v)}{h(v) + k(v)} M = v.$$

For example, the following choice is admissible,

$$h(v) = \frac{v}{M}, \qquad k(v) = 1 - \frac{v}{M}. \tag{0.0.14}$$

Following this interpretation, a certain number of cross-diffusion systems are obtained as limit of a mesoscopic reaction-diffusion system. For example see [25] for the space dimension $N = 1$ and [37]. See also [36] for a class of non triangular cross-diffusion systems with Beddington–DeAngelis functional response.

State of the art

Reaction cross-diffusion models arise in many different fields of physics, such as respiratory airways, chemical reactors or gaseous mixtures [8], in subfields of medicine, for that we refer to [33] for a chemotaxis model of multiple sclerosis and to [51, 53] for tumor growth models, and in many biological contests such as cell migration in tissues and chemosensitive movements [74, 75] or population dynamics of multiple species [50] . This class of nonlinear systems is also involved in various biological processes, such as the transport of ions in cells [13] (volume filling model) and several applications in population dynamics. As already mentioned, (SKT) was introduced for a modelling problem linked to the Turing instability. However, (SKT) proved to be an extremely rich mathematical object from which many natural questions arise, including existence, uniqueness and regularity of solutions, stability of equilibria and asymptotic behavior.

The analysis of cross-diffusion systems is delicate and sometimes intricate. One of the main difficulties is the strongly-coupled structure of the equations so that standard parabolic theory such as the maximum principle or the classical regularity results generally fail. Moreover, as previously mentioned, the unknowns of this class of systems represent concentrations, population densities or quantities that are typically expected to be nonnegative or even bounded. Therefore, the nonnegativity and the L^∞ boundedness of the solutions may represent a challenging issue.

A fundamental theory for strongly coupled systems was developped by H. Amann in [2, 3]. He showed the local in time existence of weak solutions in a certain Sobolev space ($W^{1,p}$) and proposed a criterion for their global in time extension, based on the boundedness of the L^∞ and Hölder norms. Then, the main difficulty relies in proving the bounds on the solutions in suitable Sobolev spaces to prevent blowups. In this direction, several different methods have been employed to get the L^∞ boundedness of solutions. We refer to the Moser-type or Alikakos-type method [1] in which the L^∞ bondedness is obtained as the limit for $p \to +\infty$ of the $p-$uniform L^p-norms control. We also mention the work of T. Lepoutre *et al* in [60], where they obtained the existence of bounded solutions using spatially regularization arguments and Hölder theory.

Concerning the existence result for the specific system (SKT), a certain number of results are shown under restrictive hypothesis on the space dimension N, on the diffusion coefficients in the nontriangular case ($d_{12}, d_{21} \neq 0$) or on the initial data. As first, Kim [54] proved the existence of global solutions assuming $N = 1$, $d_u = d_v$ and without self-diffusion ($d_{11} = d_{22} = 0$). For $N = 2$, Yagi proved in [86, 87] the existence of global solutions under the assumptions on the cross-diffusion coefficients $0 < d_{12} \leq 8d_{11}$ and $0 < d_{21} \leq 8d_{22}$. This condition was weakened in [49] with $d_{12}d_{21} < 64d_{11}d_{22}$ or $0 < d_{12}d_{21} = 64d_{11}d_{22}$. For $N = 2$, Lou, Ni and Wu published in [64] the existence of global solution for the system (SKT) with $d_{12} > 0$ and $d_{21} = 0$. For higher space dimension, we refer to Deuring in [38] for the existence of global (in time) solutions, provided that d_{12}, d_{21} are small coefficients depending on the initial data, and to Choi in [22, 23], assuming smooth initial data.

An important progress for the existence of global in time solutions was made by Chen and Jüngel in [17] (see also [16]). Indeed, they proved the existence of an entropy functional for the system (SKT) (that is a Lyapunov functional if the reaction terms are neglected), without any restrictions on the diffusion coefficients in the nontriangular case ($d_{12}, d_{21} \neq 0$). The entropy functional writes as follows for all $u, v \geq 0$,

$$J(u, v) := d_{21} \int_\Omega (u \ln u - u + 1)dx + d_{12} \int_\Omega (v \ln v - v + 1)dx. \tag{0.0.15}$$

The analysis of the evolution of J along the solution to (SKT) yields a priori estimates to construct global in time weak solutions. This entropy structure was shown to be robust enough to treat a generalization of the (SKT) system in [34]. Afterwards, many other works deeply investigate the relation between the structure of systems involving cross-diffusion terms and the existence of an entropy functional [19, 29, 59].

A fundamental tool, coming out of the reaction-diffusion theory and often referred to in the literature as the *Duality Lemma*, has been set up by M. Pierre and D. Schmitt in [78], giving an $L^2(\Omega_T)$ *a priori* estimates on the solution. It is typically used to construct weak solutions to reaction cross-diffusion systems where the diffusivity function depend on the species [32]. For example of application, see [14, 34, 37, 59] and [83] for a class of triangular cross-diffusion systems with possible self-diffusion.

In the following section, we present an overview of our results about existence of weak and strong solutions, regularity and uniqueness, that enlarge the analysis of the class of cross-diffusion systems. However, many questions on this class of nonlinear systems are still open, some of which are subjects of forthcoming works (see *Chapter 4*).

Overview

In this section, we present an overview of the thesis by summarizing our main contributions and outlining the results in the literature that have been more relevant for our work.

- In *Chapter 1*, we study the existence of weak solutions and the linear stability of a triangular cross-diffusion system. For the existence result, we rigorously prove the passage from a Lotka-Volterra reaction-diffusion system (mesoscopic system) towards a cross-diffusion system (macroscopic system) at the fast reaction limit. The mesoscopic system models the competition of two species, when one species has a more diverse diet than the other one. The resulting limit gives a cross-diffusion system of a so called starvation-driven type. The main tools used to rigorously pass to the limit consist in a priori estimates, given by the analysis of an entropy functional, and in compactness arguments. Moreover, we investigate the linear stability of spatially homogeneous equilibria of the macroscopic system and the mesoscopic one and we rule out the possibility of Turing instability. In particular, we investigate the relationship at the limit between the linear stability of the coexistence steady state at the mesoscopic and macroscopic scale. Numerical simulations are also performed to complement the abstract results. *Chapter 1* is the result of a collaboration with L. Corrias, H. Dietert and Y.-J. Kim and was published in *Journal of Mathematical Biology* under the title *Evolution of dietary diversity and a starvation driven cross-diffusion system as its singular limit* [12].

- In *Chapter 2*, we prove the existence of weak solutions for a general class of triangular cross-diffusion systems, using a mesoscopic derivation, similarly as in *Chapter 1*. We study a natural generalisation of the mesoscopic system introduced in *Chapter 1* and we obtain a wider class of triangular cross-diffusion systems of a starvation-driven type at the fast reaction limit. The main tool used consists in studying a family of entropy functionals that includes the one used in *Chapter 1*. In order to have enough compactness and then to pass to the limit, it is sufficient to consider a subfamily of the family of the entropy functionals. However, in order to investigate the regularity of the solution, one can study the evolution of the entropy for all the family, thus improving the entropy a priori estimates by a bootstrap argument. The regularity of the solutions is the object of a forthcoming work. We refer to *Chapter 4* for more details.

- In *Chapter 3*, we study the existence, uniqueness and regularity of strong solutions for a general class of triangular cross-diffusion systems. The term *strong* means that the equations of the system are satisfied almost everywhere. The method used to prove the existence result is different from that employed in *Chapter 1, 2*, where we obtained the cross-diffusion system as the limit of a mesoscopic system. Here, the main idea is to introduce an appropriate change of variable that strongly uses the properties of the diffusity function of the cross-diffusion system and that gives rise to a system in a *non divergence* form. Classical analytic methods, such as regularization and fixed point arguments, allow us to prove the existence of strong solutions. Moreover, the $L^\infty(\Omega_T)$ boundedness of the solutions is proved if the space dimension $N \leq 3$, and the uniqueness holds, provided that $N \leq 2$. We conclude by showing a *weak-strong* stability and a *weak-strong* uniqueness result.

Chapter 1

Evolution of dietary diversity and a starvation driven cross-diffusion system as its singular limit

1.1 Introduction

We consider a semilinear reaction-diffusion system that models a competition dynamics when two species have partially different diets. The population densities of the two species are denoted by $u = u(t, x)$ and $v = v(t, x)$. The species u has a more diverse diet and is divided into two substates $u_a = u_a(t, x)$ and $u_b = u_b(t, x)$ so that $u = u_a + u_b$. The system is parametrized by a small parameter $\varepsilon > 0$ and written as

$$\begin{cases} \partial_t u_a^\varepsilon = d_a \Delta u_a^\varepsilon + f_a(u_a^\varepsilon) + \dfrac{1}{\varepsilon} Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon), & \text{in } (0, +\infty) \times \Omega, \\ \partial_t u_b^\varepsilon = d_b \Delta u_b^\varepsilon + f_b(u_b^\varepsilon, v^\varepsilon) - \dfrac{1}{\varepsilon} Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon), & \text{in } (0, +\infty) \times \Omega, \\ \partial_t v^\varepsilon = d_v \Delta v^\varepsilon + f_v(u_b^\varepsilon, v^\varepsilon), & \text{in } (0, +\infty) \times \Omega, \end{cases} \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with a smooth boundary, and d_a, d_b and d_v are diffusivities for the three populations. The unknown solutions depend on the parameter ε and we denote it explicitly if needed. The above system is complemented with nonnegative initial data

$$u_a^\varepsilon(0, x) = u_a^{\text{in}}(x), \quad u_b^\varepsilon(0, x) = u_b^{\text{in}}(x), \quad v^\varepsilon(0, x) = v^{\text{in}}(x), \quad x \in \Omega, \tag{1.1.2}$$

and zero flux boundary conditions,

$$d_a \nabla u_a^\varepsilon \cdot \sigma = d_b \nabla u_b^\varepsilon \cdot \sigma = d_v \nabla v^\varepsilon \cdot \sigma = 0, \quad \text{on } (0, +\infty) \times \partial\Omega, \tag{1.1.3}$$

where σ denotes the outward unit normal vector on the boundary $\partial\Omega$.

In this chapter, we want to see the effect of diet diversity in a competition context using the system and the emergence of cross-diffusion triggered by such a difference through its singular limit, as $\varepsilon \to 0$. The competition dynamics is given in the reaction terms. The reaction terms of order one are given by

$$\begin{aligned} f_a(u_a) &:= \eta_a u_a\Big(1 - \frac{u_a}{a}\Big), \\ f_b(u_b, v) &:= \eta_b u_b\Big(1 - \frac{u_b + v}{b}\Big), \\ f_v(u_b, v) &:= \eta_v v\Big(1 - \frac{u_b + v}{b}\Big), \end{aligned} \tag{1.1.4}$$

where $a, b > 0$ are carrying capacities supported by two different groups of resources and η_a, η_b, and $\eta_v > 0$ are the intrinsic growth rates of u_a, u_b, and v, respectively. The competition of the two species, u and v, is for the resource b. However, the species u has a diverse diet and can survive by consuming the other resource a without competition. In order to model such a competition using a Lotka-Volterra type system, the species u is divided into two substates u_a and u_b, depending on their diets. In the above reaction terms, u_a takes a logistic equation type reaction, and u_b and v take Lotka-Volterra competition equations type reactions as given in (1.1.4). Since competition exists only partially for the species u, the competition is weak to u. However, the species v competes with u for all of its resources and hence the competition is not weak in general and the competition result may depend on the parameter ε (see Sections 1.4 and 1.5).

The individuals of the species u may freely change the type of food depending on the availability, which is modeled by the fast reaction term of order ε^{-1},

$$\frac{1}{\varepsilon}\, Q(u_a, u_b, v) := \frac{1}{\varepsilon}\Big[\phi\Big(\frac{u_b + v}{b}\Big)\, u_b - \psi\Big(\frac{u_a}{a}\Big)\, u_a\Big], \qquad \varepsilon > 0\,. \tag{1.1.5}$$

In this reaction term, $\varepsilon^{-1}\,\phi\Big(\dfrac{u_b + v}{b}\Big)$ is the conversion rate for individuals in the state u_b which switch to the other state u_a, and $\varepsilon^{-1}\,\psi\Big(\dfrac{u_a}{a}\Big)$ is the conversion rate in the other direction. The conversion rate $\phi(\frac{u_b+v}{b})$ is assumed as a function of the starvation measure $\frac{u_b+v}{b}$ for the populations u_b and v. If the resource b dwindles or the population $u_b + v$ increases, the resource b becomes scarce relatively, and more individuals of population u_b will convert to u_a and consume the other resource a. Hence, we assume that ϕ is an increasing function of the starvation measure (see [55] for more discussion on the starvation measure). In the same way, the conversion rate ψ is a function of the starvation measure $\frac{u_a}{a}$ for the population u_a and is assumed to be increasing. For this reason, it makes sense to call the conversion dynamics given by (1.1.5) a starvation-driven conversion, which eventually results in the starvation-driven cross-diffusion after taking the limit as $\varepsilon \to 0$ (see [21, 24]). More specifically, we assume the following starvation-driven conversion hypothesis

- ϕ and ψ in (1.1.5) are increasing functions belonging to $C^1([0, +\infty))$; in addition, there exist strictly positive constants δ_ψ, δ_ϕ, $M_{\phi'}$, and $M_{\psi'}$ such that, for all $x \geq 0$,

$$\psi(x) \geq \delta_\psi > 0, \quad \phi(x) \geq \delta_\phi > 0, \quad \phi'(x) \leq M_{\phi'} \quad \text{and} \quad \psi'(x) \leq M_{\psi'}. \tag{H1}$$

The main result of this chapter is that, as $\varepsilon \to 0$, the (unique) solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ of the initial boundary value problem (1.1.1) – (1.1.5) converges to a limit (u_a, u_b, v) and this limit is a weak solution to the reaction cross-diffusion system

$$\begin{cases} \partial_t u = \Delta(d_a u_a + d_b u_b) + f_a(u_a) + f_b(u_b, v), & \text{in } (0, +\infty) \times \Omega, \\ \partial_t v = d_v \Delta v + f_v(u_b, v), & \text{in } (0, +\infty) \times \Omega, \end{cases} \tag{1.1.6}$$

where u_a and u_b are (uniquely) determined by the nonlinear system

$$\begin{cases} u_a + u_b = u, \\ Q(u_a, u_b, v) = 0, \end{cases} \tag{1.1.7}$$

complemented by the initial data,

$$u(0,x) = u^{\text{in}}(x) := u_a^{\text{in}}(x) + u_b^{\text{in}}(x), \quad v(0,x) = v^{\text{in}}(x), \quad x \in \Omega, \tag{1.1.8}$$

and the zero flux boundary condition,

$$\nabla(d_a u_a + d_b u_b) \cdot \sigma = d_v \nabla v \cdot \sigma = 0, \qquad \text{in } (0,+\infty) \times \partial\Omega. \tag{1.1.9}$$

Note that the zero flux boundary conditions in (1.1.3) are equivalent to the homogeneous Neumann boundary conditions,

$$\nabla u_a^\varepsilon \cdot \sigma = \nabla u_b^\varepsilon \cdot \sigma = \nabla v^\varepsilon \cdot \sigma = 0, \quad \text{on } (0,+\infty) \times \partial\Omega,$$

(see [50] for similar diffusion operator for a single species with two phenotypes). However, after taking the singular limit, we obtain the zero flux boundary conditions (1.1.9), but not the homogeneous Neumann boundary conditions. Moreover, the initial data (1.1.8) satisfy

$$u_a^{\text{in}},\, u_b^{\text{in}},\, v^{\text{in}} \in C^2(\bar{\Omega}) \qquad \text{and} \qquad \nabla u_a^{\text{in}} \cdot \sigma = \nabla u_b^{\text{in}} \cdot \sigma = \nabla v^{\text{in}} \cdot \sigma = 0, \quad \text{on } \partial\Omega. \tag{H2}$$

If $d_a = d_b$, the diffusion for the species u given in (1.1.6) is the homogeneous linear diffusion. However, the diffusivity of a species usually depends on its food (or prey) and $d_a \neq d_b$ in general. In that case ($d_a \neq d_b$), the diffusion for the total population in (1.1.6) contains cross-diffusion dynamics depending on the distribution of the three populations groups, u_a, u_b and v, through the relations in (1.1.7). This explains the starvation-driven diffusion for the specific case of the chapter, a concept formally introduced by Cho and Kim [20]. Funaki *et al.* [41] derived a macroscopic cross-diffusion model from a system of two phenotypes and a signaling chemical in the context of chemotaxis.

The proof of convergence as $\varepsilon \to 0$ is rigorously obtained via *a priori* estimates for $u_a^\varepsilon, u_b^\varepsilon$, and v^ε. The main tool is the energy (or entropy) functional

$$\mathcal{E}(u_a, u_b, v) := \int_\Omega h_a(u_a)\, dx + \int_\Omega h_b(u_b, v)\, dx, \tag{1.1.10}$$

where

$$h_a(u_a) := \int_0^{u_a} \psi\Big(\frac{z}{a}\Big) z\, dz, \quad \text{and} \quad h_b(u_b, v) := \int_0^{u_b} \phi\Big(\frac{z+v}{b}\Big) z\, dz. \tag{1.1.11}$$

Notice here that the assumption (H1) implies that h_a is positive, increasing, and convex, and that h_b is positive, increasing in both variables, and convex with respect to the first variable. We refer to [27] and [37] for the use of such entropies in the context of triangular cross-diffusion systems (that is, systems in which only one of the two equations includes a cross-diffusion term). For more general systems, we refer to [17, 18, 30, 35, 52] among other works.

Then, by invoking the *Aubin-Lions Lemma*, we pass to the limit along a subsequence and conclude that the limit is a very weak solution to (1.1.6) – (1.1.9), in the sense of *Theorem 1.2.1*.

Remark 1.1.
The regularity of the initial data (H2) guarantees the existence of classical solutions $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ to the system (1.1.1) - (1.1.5), for any fixed $\varepsilon > 0$. Furthermore, (H2) ensures the boundedness of $\mathcal{E}$ at $t = 0$ and allows us to get the ε−uniform estimates for $\partial_t v^\varepsilon, \partial_{ij} v^\varepsilon, \nabla v^\varepsilon, i, j = 1, \dots N$, thanks to the parabolic maximal regularity (see *Lemma 1.3.1*). On the other hand, the initial data $u_a^{\text{in}}, u_b^{\text{in}}, v^{\text{in}}$ for the reaction diffusion system (1.1.1) do not satisfy a priori the nonlinear equation $Q(u_a^{\text{in}}, u_b^{\text{in}}, v^{\text{in}}) = 0$ in (1.1.7). Thus, the appearance of an initial layer is expected (see also Section 1.5).

We conclude this introduction proposing a formal derivation of (1.1.1) out of a microscopic system.

The rest of the chapter is organised as follows. *Section 1.2* is devoted to the statement of the existence result. In *Section 1.3.1*, we prove a priori estimates, which are the preliminary ingredients for the proof of the existence result, obtained in *Section 1.3.2*. The chapter concludes with the existence and linear stability analysis of trivial and non-trivial spatially homogeneous steady states, in *Section* 1.4 and *Appendix* A.2, with a particular emphasis put on the coexistence state. Some numerical tests in *Section 1.5* illustrate the linear stability analysis.

1.1.1 Formal derivation of the reaction-diffusion system with fast switching

We explain here how the mesoscopic scale model (1.1.1) is obtained at a formal level from a microscopic scale model in which the resources inducing the competition explicitly appear. Consider

$$\begin{cases} \partial_t s_1 = \dfrac{1}{\delta}\Big[r_1 s_1\Big(1 - \dfrac{s_1}{A_1}\Big) - p_1 s_1 U_1\Big], \\ \partial_t s_2 = \dfrac{1}{\delta}\Big[r_2 s_2\Big(1 - \dfrac{s_2}{A_2}\Big) - p_2 s_2 U_2 - p_V s_2 V\Big], \\ \partial_t U_1 = D_1 \Delta U_1 + k_1 p_1 s_1 U_1 + \dfrac{1}{\varepsilon}\Big[\Phi\Big(\dfrac{p_2 U_2 + p_V V}{s_2}\Big)U_2 - \Psi\Big(\dfrac{p_1 U_1}{s_1}\Big)U_1\Big], \\ \partial_t U_2 = D_2 \Delta U_2 + k_2 p_2 s_2 U_2 - \dfrac{1}{\varepsilon}\Big[\Phi\Big(\dfrac{p_2 U_2 + p_V V}{s_2}\Big)U_2 - \Psi\Big(\dfrac{p_1 U_1}{s_1}\Big)U_1\Big], \\ \partial_t V = D_V \Delta V + k_V p_V s_2 V, \end{cases} \tag{1.1.12}$$

where $\delta > 0$ is the microscopic reaction time scale and ε is the mesoscopic one (hence $\delta \ll \varepsilon \ll 1$). These equations describe the time evolution of a small ecosystem with two prey population densities (or vegetal resources), s_1 and s_2, and two predator population densities (or harvesters of the vegetal resources), U and V. Moreover, the population U is composed of two subpopulations U_1 and U_2 depending on the prey they consume, i.e., s_1 and s_2, respectively. The prey species s_i follows the logistic dynamics with a carrying capacity A_i and an intrinsic growth rate r_i. The predator species consume a certain amount of preys which is proportional to the prey density with proportionality factors p_1, p_2 and p_V. The harvested prey mass is converted to the predator mass with conversion rates k_1, k_2 and k_V. The subpopulations U_1 and U_2 convert to each other depending on the availability of the prey. The two functions Φ and Ψ are the conversion rates which are respectively increasing functions of the starvation measures $\frac{p_2 U_2 + p_V V}{s_2}$ and $\frac{p_1 U_1}{s_1}$. The other species V consumes only the second prey s_2. Hence, the active competition is only between V and U_2, while U_1

competes with V passively (via conversion). Finally, since the dispersal rate of a predator species usually depends on the nature of its prey, $D_1 \neq D_2$ in general.

Remark 1.2. If the heterogeneity of prey densities s_1 and s_2 is considered, one needs to add diffusion terms in the first two equations of (1.1.12) in order to include random migration of prey species. However, the lack of the diffusion terms does not affect the formal derivation of the mesoscopic system since we take $\delta \to 0$ anyway.

Next, we make the asymptotic approximation as $\delta \to 0$ with fixed $\varepsilon > 0$ for the prey (or resources) densities s_1, s_2, and formally obtain a mesoscopic scale model. First, we have

$$s_1\Big(r_1 - \frac{r_1 s_1}{A_1} - p_1 U_1\Big) = 0 \implies s_1 = 0 \text{ or } s_1 = A_1\Big(1 - \frac{p_1 U_1}{r_1}\Big),$$

and

$$s_2\Big[r_2(1 - \frac{s_2}{A_2}) - p_2 U_2 - p_V V\Big] = 0 \implies s_2 = 0 \text{ or } s_2 = A_2\Big(1 - \frac{p_2 U_2 + p_V V}{r_2}\Big).$$

Only the nontrivial case, $s_1 \neq 0 \neq s_2$, is meaningful (since $s_1 = 0$ and $s_2 = 0$ correspond to unstable equilibria), and we obtain two relations

$$\frac{p_1 U_1}{s_1} = \frac{r_1}{s_1} - \frac{r_1}{A_1} \quad \text{and} \quad \frac{p_2 U_2 + p_V V}{s_2} = \frac{r_2}{s_2} - \frac{r_2}{A_2}.$$

Therefore, the last three equations in (1.1.12) turn into

$$\begin{cases} \partial_t U_1 = D_1 \Delta U_1 + A_1 k_1 p_1 U_1 (1 - \frac{p_1 U_1}{r_1}) + \dfrac{1}{\varepsilon}[\Phi U_2 - \Psi U_1], \\ \partial_t U_2 = D_2 \Delta U_2 + A_2 k_2 p_2 U_2 (1 - \frac{p_2 U_2 + p_V V}{r_2}) - \dfrac{1}{\varepsilon}[\Phi U_2 - \Psi U_1], \\ \partial_t V = D_V \Delta V + A_2 k_V p_V V (1 - \frac{p_2 U_2 + p_V V}{r_2}), \end{cases} \tag{1.1.13}$$

where the conversion rates Φ and Ψ read as

$$\Phi = \Phi\Big(\frac{r_2}{s_2} - \frac{r_2}{A_2}\Big) \quad \text{and} \quad \Psi = \Psi\Big(\frac{r_1}{s_1} - \frac{r_1}{A_1}\Big),$$

and the Lotka-Volterra reaction dynamics of competition type naturally appears.

Now we consider the relationship between the variables in (1.1.1) and in (1.1.13). First, we define

$$u_a^\varepsilon := U_1, \; u_b^\varepsilon := U_2, \; v^\varepsilon := \frac{p_V}{p_2} V,$$

and keep the same diffusivity coefficients

$$d_a := D_1, \; d_b := D_2, \; d_v := D_V.$$

Then, the coefficients in the Lotka-Volterra type competition dynamics, f_a, f_b and f_v, are given as

$$\eta_a := p_1 A_1 k_1, \quad \eta_b := p_2 A_2 k_2, \quad \eta_v := p_V A_2 k_V, \quad a := \frac{r_1}{p_1}, \quad b := \frac{r_2}{p_2}. \tag{1.1.14}$$

Finally, the mesoscopic conversion rates are given as

$$\phi(x) := \Phi\Big(\frac{r_2}{A_2}\frac{x}{1-x}\Big), \quad \psi(x) := \Psi\Big(\frac{r_1}{A_1}\frac{x}{1-x}\Big). \tag{1.1.15}$$

After replacing the previous coefficients with the new ones, the system (1.1.13) becomes our system (1.1.1).

Remark 1.3.
(i) The conversion rates of the microscopic model, Φ and Ψ, are functions of the starvation measures $\frac{p_2U_2+p_VV}{s_2}$ and $\frac{p_1U_1}{s_1}$, instead of simply $\frac{U_2+V}{s_2}$ and $\frac{U_1}{s_1}$, in order to take into account the difference in the harvesting rates p_2 and p_V.
(ii) The mesoscopic conversion rates ϕ and ψ in (1.1.15) are increasing functions, since Φ and Ψ are chosen to be increasing functions.
(iii) It is worth noticing that the carrying capacities a and b for the predator species are proportional to the growth rates r_i's of the prey species and that the prey carrying capacities A_i's are also involved in deciding ϕ and ψ (see (1.1.14) and (1.1.15)).
(iv) The macroscopic system reduces to the classical Lotka-Volterra system of competition type with linear diffusion, whenever the conversion rates ϕ and ψ are both constant.

1.2 Statement of the main result

Before stating our main result in *Theorem* 1.2.1 below, we observe that, thanks to hypothesis (H1), the function

$$q(u_b; u, v) := Q(u-u_b, u_b, v) = \phi\Big(\frac{u_b+v}{b}\Big)u_b - \psi\Big(\frac{u-u_b}{a}\Big)(u-u_b)\,, \tag{1.2.1}$$

defined for $u_b \in [0, u]$, satisfies for given $u \geq 0, v \geq 0$

$$\partial_{u_b} q(u_b, u, v) = \phi\Big(\frac{u_b+v}{b}\Big) + \frac{u_b}{b}\phi'\Big(\frac{u_b+v}{b}\Big) + \psi\Big(\frac{u-u_b}{a}\Big) + \frac{u-u_b}{a}\psi'\Big(\frac{u-u_b}{a}\Big) > 0,$$

and for $u > 0, v \geq 0$

$$q(0; u, v) < 0\,, \qquad q(u; u, v) > 0\,.$$

Hence, for any given $(u, v) \in \mathbb{R}_+^2$, there exists a unique $u_b^*(u, v) \in [0, u]$ zero of q, and thus a unique solution to the nonlinear system (1.1.7) is well defined. Furthermore, the implicit function theorem guarantees the continuity (and even the C^1 character) of u_b^* with respect to (u, v).

Theorem 1.2.1.
Let Ω be a smooth bounded domain of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$. We assume (H1) *and* (H2)*. Then, the unique positive classical solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ of* (1.1.1) - (1.1.5) *converges for a.e. $(t, x) \in (0, +\infty) \times \Omega$ (up to extraction of a subsequence) towards a nonnegative triplet (u_a^*, u_b^*, v), as $\varepsilon \to 0$. Moreover, for a.e. $(t, x) \in (0, +\infty) \times \Omega$, the pair of function (u_a^*, u_b^*) is the unique solution to the nonlinear system* (1.1.7)*, corresponding to $u := u_a^* + u_b^*$ and v. Furthermore, (u, v) is a very weak solution to the macroscopic system* (1.1.6) – (1.1.9)*, in the sense that, for all test functions $\xi_1, \xi_2 \in C_c^2([0, +\infty) \times \bar{\Omega})$, with $\nabla\xi_1 \cdot \sigma = \nabla\xi_2 \cdot \sigma = 0$ on $[0, +\infty) \times \partial\Omega$, it holds*

$$-\int_0^{+\infty}\int_\Omega (\partial_t\xi_1)u\,dxdt - \int_\Omega \xi_1(0,\cdot)u^{in}dx - \int_0^{+\infty}\int_\Omega \Delta\xi_1(d_a u_a^* + d_b u_b^*)dxdt$$
$$= \int_0^{+\infty}\int_\Omega \xi_1(f_a(u_a^*) + f_b(u_b^*, v))\,dx\,dt\,, \tag{1.2.2}$$

and

$$-\int_0^{+\infty}\int_\Omega (\partial_t\xi_2)\, v\, dxdt - \int_\Omega \xi_2(0,\cdot)\, v^{in}\, dx - d_v \int_0^{+\infty}\int_\Omega \Delta\xi_2\, v\, dxdt$$
$$= \int_0^{+\infty}\int_\Omega \xi_2\, f_v(u_b^*, v)\, dxdt\,. \tag{1.2.3}$$

Finally, the following regularity holds true, for all $T > 0$

(i) $u \in L^q(\Omega_T)$ *for* $q = 2 + \frac{2}{N}$ *if* $N \geq 3$, $q < 3$ *if* $N = 2$ *and* $q = 3$ *if* $N = 1$, $|\nabla u| \in L^2(\Omega_T)$*;*

(ii) $v \in L^\infty(\Omega_T)$, $|\nabla v| \in L^{2q}(\Omega_T)$*;* $\partial_{x_i,x_j} v\,, \partial_t v \in L^q(\Omega_T)$, $i, j = 1, \ldots, \mathrm{N}$, *for the same previous q.*

1.3 Proof of the main Theorem

We first observe that for any $\varepsilon > 0$, there exists a unique global strong (for $t > 0$) solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ to system (1.1.1) – (1.1.3), under the assumption on the initial data of *Theorem 1.2.1*. We refer to *Proposition 2.5.1* in *Chapter 2* for the proof (see also [31, 80] for similar results).

1.3.1 A priori estimates

In this section, we shall obtain *a priori* estimates on the subpopulation densities $u_a^\varepsilon, u_b^\varepsilon$, on the total population densities $u^\varepsilon := u_a^\varepsilon + u_b^\varepsilon$ and v^ε, and on $Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$. More specifically, we take advantage of the triangular structure of the system that gives us *a priori* estimates on the density v^ε and its derivatives (see *Lemma* 1.3.1). The reaction functions f_a and f_b of competition type allow us to control the total mass $\int_\Omega u^\varepsilon(t)\, dx$, and to get an $L^2(\Omega_T)$ estimate on u^ε (see *Lemma* 1.3.2). The latter will be employed in *Lemma* 1.3.3 to obtain estimates on ∇u_a^ε, ∇u_b^ε and $Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$, through the use of the energy functional (1.1.10), (1.1.11). In addition, the triplet $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ will be shown to have finite energy $\mathcal{E}(T)$ as well, for all $T > 0$.

Hereafter, all constants C and C_T are strictly positive and may depend on Ω, the initial data $u_a^{\mathrm{in}}, u_b^{\mathrm{in}}, v^{\mathrm{in}}$, the coefficients in system (1.1.1), the transition functions ϕ, ψ and on T, but never on ε. They may change also from line to line in the computations.

Lemma 1.3.1.
Let $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ *be the positive classical solution to* (1.1.1) - (1.1.5)*. Then, the following statements hold true*

(i) there exists a constant $C > 0$ *such that for all* $\varepsilon > 0$

$$\|v^\varepsilon\|_{L^\infty((0,+\infty)\times\Omega)} \leqslant C\,; \tag{1.3.1}$$

(ii) for all $q \in (1, +\infty)$ *there exists a constant* $C > 0$ *depending on* q, v^{in}, Ω*, such that, for all* $\varepsilon > 0$, $T > 0$ *and all* $i, j = 1, .., \mathrm{N}$,

$$\|\partial_t v^\varepsilon\|_{L^q(\Omega_T)} + \|\partial_{x_i,x_j} v^\varepsilon\|_{L^q(\Omega_T)} \leqslant C(1 + \|u_b^\varepsilon\|_{L^q(\Omega_T)})\,; \tag{1.3.2}$$

(iii) for all $q \in (1, +\infty)$ there exists a constant $C > 0$ depending on q, v^{in}, N, Ω, such that, for all $\varepsilon > 0$ and all $T > 0$,

$$\|\nabla v^\varepsilon\|^{2q}_{L^{2q}(\Omega_T)} \le C\left(1 + T + \|u^\varepsilon_b\|^q_{L^q(\Omega_T)}\right). \tag{1.3.3}$$

Remark 1.4.
In the sequel, the value of q in (1.3.2), (1.3.3) will be first chosen equal to 2 (see *Lemma* 1.3.2), and then to a different number after *Corollary* 1.3.4.

Proof.
It is easily seen that

$$0 \leqslant v^\varepsilon(t,x) \leqslant K := \max\{\|v^{\text{in}}\|_{L^\infty(\Omega)}\,;\, b\}, \quad \text{for a.e.} \quad (t,x) \in (0,+\infty)\times\Omega. \tag{1.3.4}$$

Indeed, by the existence result of strong solution for (1.1.1), we know that the nonnegativity of v^ε is preserved in time. Concerning the upper bound in (1.3.4), it is obtained by multiplying the equation for v^ε in (1.1.1) by $(v^\varepsilon - K)^+ := \max\{0, v^\varepsilon - K\}$ and integrating over Ω, to obtain for all $t > 0$,

$$\int_\Omega (v^\varepsilon(t) - K)^2_+\,dx \le \int_\Omega (v^{\text{in},\varepsilon} - K)^2_+\,dx = 0\,.$$

Next, by the maximal regularity property of the heat equation (see *Section A.1*), for all $q \in (1,+\infty)$ there exists a strictly positive constant C, which depends on q, v^{in} and Ω, such that for all $i, j = 1, .., \text{N}$,

$$\begin{aligned}\|\partial_t v^\varepsilon\|_{L^q(\Omega_T)} + \|\partial_{x_i,x_j} v^\varepsilon\|_{L^q(\Omega_T)} &\le C(1 + \|f_v(u^\varepsilon_b, v^\varepsilon)\|_{L^q(\Omega_T)})\\ &\le C(1 + \|u^\varepsilon_b\|_{L^q(\Omega_T)}),\end{aligned} \tag{1.3.5}$$

so that estimate (1.3.2) holds. Then, thanks to the Gagliardo-Nirenberg inequality [71], for all $q \in (1,+\infty)$, there exists a strictly positive constant C, depending on q, N, Ω such that, for all $t > 0$ and $i = 1, \ldots \text{N}$, we have

$$\|\partial_{x_i} v^\varepsilon(t)\|_{L^{2q}(\Omega)} \le C \sum_{j=1}^{\text{N}} \|\partial_{x_i,x_j} v^\varepsilon(t)\|^{1/2}_{L^q(\Omega)}\, \|v^\varepsilon(t)\|^{1/2}_{L^\infty(\Omega)} + C\|v^\varepsilon(t)\|_{L^\infty(\Omega)}\,.$$

Integrating the above inequality over $(0, T)$ and using (1.3.1) and (1.3.5), we get estimate (1.3.3). □

Lemma 1.3.2.
Let $(u^\varepsilon_a, u^\varepsilon_b, v^\varepsilon)$ be the positive classical solution to (1.1.1) - (1.1.5). Then, for all $T > 0$, there exists $C_T > 0$ such that for all $\varepsilon > 0$ the following estimates hold:

$$\sup_{t\in[0,T]} \int_\Omega (u^\varepsilon_a + u^\varepsilon_b)(t)\,dx \le C_T \qquad \textit{and} \qquad \|u^\varepsilon_a + u^\varepsilon_b\|_{L^2(\Omega_T)} \le C_T\,. \tag{1.3.6}$$

Proof.
Adding the first two equations in (1.1.1) and using the positivity of $u^\varepsilon_a, u^\varepsilon_b, v^\varepsilon$, we get

$$\partial_t(u^\varepsilon_a + u^\varepsilon_b) \le d_a\Delta u^\varepsilon_a + d_b\Delta u^\varepsilon_b + \eta_a u^\varepsilon_a\Big(1 - \frac{u^\varepsilon_a}{a}\Big) + \eta_b u^\varepsilon_b\Big(1 - \frac{u^\varepsilon_b}{b}\Big) \tag{1.3.7}$$

$$\le d_a\Delta u^\varepsilon_a + d_b\Delta u^\varepsilon_b + \frac{1}{4}(a\eta_a + b\eta_b). \tag{1.3.8}$$

Then, integrating (1.3.8) over Ω, the inequality becomes

$$\frac{d}{dt}\int_\Omega (u_a^\varepsilon + u_b^\varepsilon)(t)\,dx \le C\,,$$

implying, for all t in $[0,T]$, that

$$\|u_a^\varepsilon(t) + u_b^\varepsilon(t)\|_{L^1(\Omega)} \le \|u_a^{\text{in}} + u_b^{\text{in}}\|_{L^1(\Omega)} + C\,T\,. \tag{1.3.9}$$

In order to obtain the $L^2(\Omega_T)$ estimate for $u_a^\varepsilon + u_b^\varepsilon$, we integrate inequality (1.3.7) first over Ω and then over $(0,t)$, for $t \in (0,T)$, to obtain

$$\begin{aligned}\int_\Omega (u_a^\varepsilon + u_b^\varepsilon)(t)\,dx + \frac{\eta_a}{a}\int_{\Omega_t} (u_a^\varepsilon)^2\,dx\,dt + \frac{\eta_b}{b}\int_{\Omega_t} (u_b^\varepsilon)^2\,dx\,dt&\\ \le \|u_a^{\text{in}} + u_b^{\text{in}}\|_{L^1(\Omega)} + C\|u_a^\varepsilon + u_b^\varepsilon\|_{L^1(\Omega_T)}\,.&\end{aligned}$$

The second estimate in (1.3.6) follows, using the first one. □

Lemma 1.3.3.
Let $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ be the positive classical solution to (1.1.1) - (1.1.5). Then, for all $T > 0$, there exists $C_T > 0$ such that, for all $\varepsilon > 0$, the global solution to (1.1.1) satisfies

$$\mathcal{E}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T) + C\|\nabla u_a^\varepsilon\|^2_{L^2(\Omega_T)} + C\|\nabla u_b^\varepsilon\|^2_{L^2(\Omega_T)} + \frac{1}{\varepsilon}\|Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\|^2_{L^2(\Omega_T)} \le C_T\,. \tag{1.3.10}$$

Proof.
We shall analyse the evolution of $\mathcal{E}$, along the trajectories of the solution to (1.1.1). Thus, from the first equation in (1.1.1) and assumption (H1), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega h_a(u_a^\varepsilon)\,dx &= \int_\Omega (\partial_t u_a^\varepsilon)\,u_a^\varepsilon\,\psi\Big(\frac{u_a^\varepsilon}{a}\Big)\,dx\\ &= -\,d_a\int_\Omega \Big[\psi\Big(\frac{u_a^\varepsilon}{a}\Big) + \frac{u_a^\varepsilon}{a}\,\psi'\Big(\frac{u_a^\varepsilon}{a}\Big)\Big]|\nabla u_a^\varepsilon|^2\,dx\\ &\quad + \int_\Omega u_a^\varepsilon\,f_a(u_a^\varepsilon)\,\psi\Big(\frac{u_a^\varepsilon}{a}\Big)\,dx + \frac{1}{\varepsilon}\int_\Omega u_a^\varepsilon\,\psi\Big(\frac{u_a^\varepsilon}{a}\Big)\,Q_\varepsilon\,dx\\ &\le -\,d_a\delta_\psi\int_\Omega |\nabla u_a^\varepsilon|^2\,dx\\ &\quad + C\int_\Omega (u_a^\varepsilon)^2\Big(1 - \frac{u_a^\varepsilon}{a}\Big)\mathbb{1}_{\{u_a^\varepsilon\le a\}}dx + \frac{1}{\varepsilon}\int_\Omega u_a^\varepsilon\psi\Big(\frac{u_a^\varepsilon}{a}\Big)Q^\varepsilon dx.\end{aligned} \tag{1.3.11}$$

Concerning the second term in the energy (1.1.10), we see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega h_b(u_b^\varepsilon, v^\varepsilon)\,dx &= \int_\Omega (\partial_t u_b^\varepsilon)\,u_b^\varepsilon\,\phi\Big(\frac{u_b^\varepsilon + v^\varepsilon}{b}\Big)dx + \int_\Omega (\partial_t v^\varepsilon)\partial_2 h_b(u_b^\varepsilon, v^\varepsilon)\,dx\\ &=: I_1 + I_2\,.\end{aligned} \tag{1.3.12}$$

Using the second equation in (1.1.1), I_1 rewrites as follows

$$\begin{aligned}I_1 \le &-d_b\int_\Omega |\nabla u_b^\varepsilon|^2\left[\phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) + \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)\right]dx\\ &- d_b\int_\Omega \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)\nabla u_b^\varepsilon\cdot\nabla v^\varepsilon\,dx\\ &+ C\int_\Omega (u_b^\varepsilon)^2\left(1 - \frac{u_b^\varepsilon + v^\varepsilon}{b}\right)\mathbb{1}_{\{u_b^\varepsilon + v^\varepsilon\le b\}}\,dx - \frac{1}{\varepsilon}\int_\Omega u_b^\varepsilon\,\phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)Q^\varepsilon\,dx\,.\end{aligned} \tag{1.3.13}$$

On the other hand, observing that

$$\partial_2 h_b(u_b, v) = \int_0^{u_b} \frac{z}{b}\,\phi'\left(\frac{z+v}{b}\right) dz = u_b \phi\left(\frac{u_b+v}{b}\right) - \int_0^{u_b} \phi\left(\frac{z+v}{b}\right) dz\,, \tag{1.3.14}$$

by the positivity of ϕ', $\partial_2 h_b$ is positive as well and

$$\begin{aligned}\int_\Omega \partial_2 h_b(u_b^\varepsilon, v^\varepsilon)\, f_v(u_b^\varepsilon, v^\varepsilon)\, dx &\le \eta_v \int_\Omega \partial_2 h_b(u_b^\varepsilon, v^\varepsilon)\, v^\varepsilon \left(1 - \frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \mathbb{1}_{\{u_b^\varepsilon + v^\varepsilon \le b\}}\, dx \\ &\le \eta_v \int_\Omega u_b^\varepsilon \phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) v^\varepsilon \left(1 - \frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \mathbb{1}_{\{u_b^\varepsilon + v^\varepsilon \le b\}}\, dx\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} I_2 \le &- d_v \int_\Omega \partial_{22} h_b(u_b^\varepsilon, v^\varepsilon) |\nabla v^\varepsilon|^2\, dx - d_v \int_\Omega \partial_{21} h_b(u_b^\varepsilon, v^\varepsilon) \nabla u_b^\varepsilon \cdot \nabla v^\varepsilon\, dx \\ &+ \eta_v \int_\Omega u_b^\varepsilon \phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) v^\varepsilon \left(1 - \frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \mathbb{1}_{\{u_b^\varepsilon + v^\varepsilon \le b\}}\, dx\,. \end{aligned} \tag{1.3.15}$$

Computing from (1.3.14)

$$\partial_{21} h_b(u_b, v) = \frac{u_b}{b}\,\phi'\left(\frac{u_b+v}{b}\right),$$

and plugging estimates (1.3.13) and (1.3.15) into (1.3.12), we end up with the estimate

$$\begin{aligned} \frac{d}{dt}\int_\Omega h_b(u_b^\varepsilon, v^\varepsilon)\, dx \le &- d_b \int_\Omega \left[\phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) + \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)\right] |\nabla u_b^\varepsilon|^2\, dx \\ &- d_v \int_\Omega \partial_{22} h_b(u_b^\varepsilon, v^\varepsilon) |\nabla v^\varepsilon|^2\, dx \\ &- (d_b + d_v) \int_\Omega \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \nabla u_b^\varepsilon \cdot \nabla v^\varepsilon\, dx \\ &+ C \int_\Omega (u_b^\varepsilon)^2 \left(1 - \frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \mathbb{1}_{\{u_b^\varepsilon + v^\varepsilon \le b\}}\, dx \\ &+ \eta_v \int_\Omega u_b^\varepsilon \phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) v^\varepsilon \left(1 - \frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \mathbb{1}_{\{u_b^\varepsilon + v^\varepsilon \le b\}}\, dx \\ &- \frac{1}{\varepsilon} \int_\Omega u_b^\varepsilon\, \phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) Q^\varepsilon\, dx\,. \end{aligned} \tag{1.3.16}$$

Next, using the positivity of ϕ' again, we estimate the third term in (1.3.16) with a weight $\eta > 0$ as

$$\begin{aligned} &- (d_b + d_v) \int_\Omega \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) \nabla u_b^\varepsilon \cdot \nabla v^\varepsilon\, dx \\ &\le (d_b + d_v)\frac{\eta}{2} \int_\Omega \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) |\nabla u_b^\varepsilon|^2\, dx + \frac{d_b + d_v}{2\eta} \int_\Omega \frac{u_b^\varepsilon}{b}\,\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) |\nabla v^\varepsilon|^2\, dx. \end{aligned}$$

Thus, choosing $\eta \in (0, 2d_b(d_b + d_v)^{-1})$, gives $C(\eta) := (d_b - (d_b + d_v)\frac{\eta}{2}) > 0$, and inequality (1.3.16) becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega h_b(u_b^\varepsilon, v^\varepsilon)\,dx \leq &- d_b\delta_\phi \int_\Omega |\nabla u_b^\varepsilon|^2\,dx - d_v \int_\Omega \partial_{22}h_b(u_b^\varepsilon, v^\varepsilon)|\nabla v^\varepsilon|^2\,dx \\ &- C(\eta)\int_\Omega \frac{u_b^\varepsilon}{b}\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)|\nabla u_b^\varepsilon|^2\,dx \\ &+ \frac{(d_b + d_v)}{2\eta}\int_\Omega \frac{u_b^\varepsilon}{b}\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)|\nabla v^\varepsilon|^2\,dx \\ &+ C - \frac{1}{\varepsilon}\int_\Omega u_b^\varepsilon\,\phi\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right) Q^\varepsilon\,dx\,.\end{aligned} \tag{1.3.17}$$

Finally, by assumption (H1), the derivative

$$\partial_{22}h_b(u_b, v) = \frac{u_b}{b}\phi'\left(\frac{u_b + v}{b}\right) - \left[\phi\left(\frac{u_b + v}{b}\right) - \phi\left(\frac{v}{b}\right)\right] = \int_v^{u_b+v}\left[\phi'\left(\frac{u_b + v}{b}\right) - \phi'\left(\frac{z}{b}\right)\right]\frac{dz}{b},$$

satisfies

$$|\partial_{22}h_b(u_b, v)| \leq 2\,M_{\phi'}\frac{u_b}{b}\,.$$

Therefore, adding (1.3.11) and (1.3.17), and using the boundedness of ϕ' again, we arrive at the following estimate for the time derivative of the energy

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u_a^\varepsilon(t), u_b^\varepsilon(t), v^\varepsilon(t)) \leq &- d_a\delta_\psi \int_\Omega |\nabla u_a^\varepsilon|^2\,dx - d_b\delta_\phi \int_\Omega |\nabla u_b^\varepsilon|^2\,dx \\ &- C(\eta)\int_\Omega \frac{u_b^\varepsilon}{b}\phi'\left(\frac{u_b^\varepsilon + v^\varepsilon}{b}\right)|\nabla u_b^\varepsilon|^2\,dx \\ &+ C\|u_a^\varepsilon + u_b^\varepsilon\|_{L^2(\Omega)}\|\nabla v^\varepsilon\|^2_{L^4(\Omega)} - \frac{1}{\varepsilon}\int_\Omega (Q^\varepsilon)^2 dx + C.\end{aligned} \tag{1.3.18}$$

Integrating in time over $[0,\ T]$ the latter inequality, estimate (1.3.10) is proved by the means of *Lemma* 1.3.1 (with $q = 2$), *Lemma* 1.3.2 and the boundedness of the initial energy. □

We conclude this section by giving improved estimates from interpolation arguments.

Corollary 1.3.4.
Let $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ be the positive classical solution to (1.1.1) - (1.1.5)*. Then, for all $T > 0$, the following estimates hold*

$$\|u_a^\varepsilon + u_b^\varepsilon\|_{L^2\left(0,T;\,H^1(\Omega)\right)} \leqslant C_T\,, \tag{1.3.19}$$

and

$$\|u_a^\varepsilon + u_b^\varepsilon\|_{L^q(\Omega_T)} \leqslant C_T\,, \tag{1.3.20}$$

where

$$q := \begin{cases} 2 + 2/\mathrm{N} & \textit{if } \mathrm{N} > 2\,; \\ 3, & \textit{if } \mathrm{N} = 1, \end{cases} \tag{1.3.21}$$

and $q < 3$, if $\mathrm{N} = 2$.

Proof.
The following argument is performed for the subpopulation u_a^ε. It can be applied similarly to u_b^ε and thus to $u_a^\varepsilon + u_b^\varepsilon$.

Lemmas 1.3.2 and 1.3.3 give that u_a^ε is bounded in $L^2(0,T;\ H^1(\Omega))$. Thus, by the Sobolev embedding theorem, we have that u_a^ε is bounded in $L^2(0,T;\ L^{N^*}(\Omega))$, with $N^* = \frac{2N}{N-2}$ if $N > 2$, $N^* \in [2,+\infty)$ if $N = 2$ and $N^* = \infty$ if $N = 1$. Since we also know that u_a^ε is bounded in $L^\infty(0,T;\ L^1(\Omega))$, by interpolation we obtain that u_a^ε is bounded in $L^q(\Omega_T)$, with q as in (1.3.21). □

Remark 1.5.
At this point, using *Lemma* 1.3.1 again, we see that $\partial_t v^\varepsilon$ and $\nabla\nabla v^\varepsilon$ are bounded in $L^q(\Omega_T)$.

1.3.2 End of the proof of the main result

End of the proof of Theorem 1.2.1.

The proof is divided in four steps and uses compactness to identify limits along subsequences. The first and the second one focus on the identification of the limit (as $\varepsilon \to 0$) of the densities v^ε and $u^\varepsilon = u_a^\varepsilon + u_b^\varepsilon$, a.e. in $[0,T]\times\Omega$, respectively. In the third step we obtain the a.e. convergence of the subpopulation densities $u_a^\varepsilon, u_b^\varepsilon$ and we identify the obtained limit as the unique solution to the nonlinear system (1.1.7). The convergence argument is also extended globally in time by a diagonal argument. Finally, the proof is concluded in the fourth step, taking the limit as ε tends to zero, in the very weak formulation of the system satisfied by $u^\varepsilon = u_a^\varepsilon + u_b^\varepsilon$ and v^ε.

First step. Let $T > 0$ be arbitrarily fixed. Thanks to the control of the density v^ε given in *Lemma* 1.3.1 and to the boundedness of $u_a^\varepsilon + u_b^\varepsilon$ in $L^2(\Omega_T)$ obtained in *Lemma* 1.3.2, we have that $(v^\varepsilon)_\varepsilon$ is bounded in $L^4(0,T;W^{1,4}(\Omega))$ and $(\partial_t v^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;L^2(\Omega))$. Therefore, by applying Rellich's Theorem, there exists a subsequence, still denoted v^ε, and $v \in L^4(\Omega_T)$ such that, as $\varepsilon \to 0$,

$$v^\varepsilon(t,x) \longrightarrow v(t,x)\,, \qquad \text{a.e. in } \Omega_T\,. \tag{1.3.22}$$

Moreover,

$$\nabla v^\varepsilon \rightharpoonup \nabla v \qquad \text{in } L^4(\Omega_T), \tag{1.3.23}$$

and due to *Lemma* 1.3.1 again, v is nonnegative and belongs to $L^\infty(\Omega_T)$, while ∇v lies in $L^4(\Omega_T)$.

Second step. We rewrite the parabolic equation satisfied by the density $u^\varepsilon = u_a^\varepsilon + u_b^\varepsilon$ as

$$\partial_t u^\varepsilon = \Delta(d_a\, u_a^\varepsilon + d_b\, u_b^\varepsilon) + f_a(u_a^\varepsilon) + f_b(u_b^\varepsilon, v^\varepsilon)\,. \tag{1.3.24}$$

Thanks to *Corollary* 1.3.4, we see that $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$ and in $L^{2+2\delta}(\Omega_T)$ for some $\delta > 0$, so that the reaction term in (1.3.24) is uniformly bounded in $L^{1+\delta}(\Omega_T)$. Then $(\partial_t(u_a^\varepsilon + u_b^\varepsilon))_\varepsilon$ is uniformly bounded in $L^{1+\delta}(0,T;W^{-1,1+\delta}(\Omega))$. Thus, Aubin-Lions' Lemma (cf. [69]) yields a subsequence (still denoted u^ε), and a function $u \geq 0$, $u \in L^2(\Omega_T)$, such that, as $\varepsilon \to 0$,

$$u^\varepsilon(t,x) = u_a^\varepsilon(t,x) + u_b^\varepsilon(t,x) \longrightarrow u(t,x)\,, \qquad \text{a. e. in } \Omega_T\,, \tag{1.3.25}$$

where the nonnegativity of u follows from that of u^ε. Furthermore,

$$\nabla u^\varepsilon \rightharpoonup \nabla u \qquad \text{in } L^2(\Omega_T)\,, \tag{1.3.26}$$

and

$$\begin{aligned} \|u\|_{L^2(\Omega_T)} &= \lim_{\varepsilon\to 0} \|u_a^\varepsilon + u_b^\varepsilon\|_{L^2(\Omega_T)} \leqslant C_T\,,\\ \|\nabla u\|_{L^2(\Omega_T)} &\le \liminf_{\varepsilon\to 0} \|\nabla u^\varepsilon\|_{L^2(\Omega_T)} \leqslant C_T\,. \end{aligned}$$

Third step. The energy estimate (1.3.10) yields the estimate

$$\Big\|\phi\Big(\frac{u_b^\varepsilon + v^\varepsilon}{b}\Big)u_b^\varepsilon - \psi\Big(\frac{u_a^\varepsilon}{a}\Big)u_a^\varepsilon\Big\|_{L^2(\Omega_T)} \le \sqrt{\varepsilon}\,C_T\,. \tag{1.3.27}$$

Therefore, $Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ converges to zero in $L^2(\Omega_T)$, as $\varepsilon \to 0$, and (up to extraction of a subsequence)

$$\phi\Big(\frac{u_b^\varepsilon + v^\varepsilon}{b}\Big)u_b^\varepsilon - \psi\Big(\frac{u_a^\varepsilon}{a}\Big)u_a^\varepsilon \longrightarrow 0, \qquad \text{a.e. in } \Omega_T\,. \tag{1.3.28}$$

It remains to prove the existence of the a.e. limit of subsequences of $(u_a^\varepsilon)_\varepsilon$, $(u_b^\varepsilon)_\varepsilon$ and to verify that this limit is the unique solution to (1.1.7), a.e. in Ω_T, corresponding to the functions u and v obtained in (1.3.25) and (1.3.22), respectively.

Let $(u_a^*(u^\varepsilon, v^\varepsilon), u_b^*(u^\varepsilon, v^\varepsilon))$ be the unique solution to (1.1.7), corresponding to $(u^\varepsilon, v^\varepsilon)$. Then, using the function q defined in (1.2.1), we get

$$\begin{aligned} Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) &= Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) - Q(u_a^*(u^\varepsilon, v^\varepsilon), u_b^*(u^\varepsilon, v^\varepsilon), v^\varepsilon)\\ &= q(u_b^\varepsilon; u^\varepsilon, v^\varepsilon) - q(u_b^*(u^\varepsilon, v^\varepsilon); u^\varepsilon, v^\varepsilon)\\ &= \partial_{u_b} q(\zeta; u^\varepsilon, v^\varepsilon)(u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon))\,, \end{aligned}$$

with $\zeta \in \Big(\min\{u_b^\varepsilon,\, u_b^*(u^\varepsilon, v^\varepsilon)\}, \max\{u_b^\varepsilon,\, u_b^*(u^\varepsilon, v^\varepsilon)\}\Big)$. Thus, thanks to hypothesis (H1) we obtain

$$|Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)| \ge (\delta_\phi + \delta_\psi)|u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)|\,.$$

Thus by (1.3.28), $|u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)| \to 0$ as $\varepsilon \to 0$, a.e. in Ω_T. Finally, the proved convergences (1.3.22) and (1.3.25) and the continuity of u_b^*, with respect to its arguments, yield the desired result, i.e.

$$u_b^\varepsilon \to u_b^*(u, v)\,, \quad u_a^\varepsilon = u^\varepsilon - u_b^\varepsilon \to u_a^*(u, v)\,, \quad \varepsilon \to 0\,, \qquad \text{a.e. in } \Omega_T.$$

To conclude, let us remark that all the a.e. convergence results obtained so far have been performed on $[0, T]$, for any arbitrary $T > 0$. Since $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ is defined on $[0, +\infty)$, by extracting subsequences, these arguments can be replicated in the time intervals $[0, 2T]$, $[0, 3T]$ and so on. Then by Cantor's diagonal argument, the convergences (1.3.22), (1.3.25) and (1.3.28), and the convergence of the pair $(u_a^\varepsilon, u_b^\varepsilon)$ towards the solution to (1.1.7) are verified a.e. in $(0, +\infty) \times \Omega$.

Fourth step. We shall prove now that (u, v) is a weak solution to (1.1.6), in the sense of Theorem 1.2.1. For this purpose, let us consider two test functions ξ_1, ξ_2 in C_c^2, satisfying

$\nabla\xi_1 \cdot \sigma = \nabla\xi_2 \cdot \sigma = 0$, on $[0,T] \times \partial\Omega$. Multiplying the equation satisfied by $u_a^\varepsilon + u_b^\varepsilon$ by ξ_1 and the third equation of (1.1.1) by ξ_2 and integrating over $(0,+\infty) \times \Omega$, we get,

$$\begin{aligned} &-\int_0^\infty \int_\Omega (\partial_t\xi_1)\,(u_a^\varepsilon + u_b^\varepsilon)\,dx\,dt - \int_\Omega \xi_1(0)\,(u_a^{\mathrm{in},\varepsilon} + u_b^{\mathrm{in},\varepsilon})\,dx \\ &= \int_0^\infty \int_\Omega \Delta\xi_1\,(\,d_a u_a^\varepsilon + d_b u_b^\varepsilon\,)\,dx\,dt + \int_0^\infty \int_\Omega \xi_1 (f_a(u_a^\varepsilon) + f_b(u_b^\varepsilon, v^\varepsilon)) dx\,dt\,, \end{aligned} \tag{1.3.29}$$

and

$$\begin{aligned} -\int_0^\infty \int_\Omega (\partial_t\xi_2\,)\,v^\varepsilon\,dx\,dt - \int_\Omega \xi_2(0)\,v^{\mathrm{in},\varepsilon}\,dx \\ = d_v \int_0^\infty \int_\Omega \Delta\xi_2\,v^\varepsilon\,dx\,dt + \int_0^\infty \int_\Omega \xi_2\,f_v(u_b^\varepsilon, v^\varepsilon)\,dx\,dt\,. \end{aligned} \tag{1.3.30}$$

Concerning the equation (1.3.29), the convergence results obtained in the previous steps and the estimates in (1.3.6) allow us to pass to the limit as $\varepsilon \to 0$, in all the terms of the equation, using Lebesgue's dominated convergence theorem, thus obtaining (1.2.2).

The same conclusion holds for equation (1.3.30). Indeed, the boundedness of v^ε and its convergence (1.3.22), together with the estimates in (1.3.6), allow us to pass to the limit in all terms of (1.3.30), using Lebesgue's dominated convergence theorem again, thus obtaining (1.2.3). The proof of *Theorem* 1.2.1 is now completed. □

1.4 Linear stability analysis

In this section, we investigate the linear stability of spatially homogeneous steady states of the macroscopic system (1.1.6) – (1.1.9), with reaction and fast reaction functions given by (1.1.4) and (1.1.5), respectively. We shall also see the relationship between the linear stability of the coexistence steady state at the mesoscopic and macroscopic scale, as $\varepsilon \to 0$.

Let ψ and ϕ be conversion rates satisfying assumption (H1). We introduce the following few notations for later use,

$$\psi_1 = \psi(1), \qquad \phi_1 = \phi(1)\,,$$

and the parameter providing a criterion for the linear stability (see *Theorem 1.4.1* and *Proposition 1.4.2*),

$$\alpha := \frac{\psi_1}{\phi_1}\,\frac{a}{b} > 0\,. \tag{1.4.1}$$

The pair $(\bar{u}, \bar{v}) \in \mathbb{R}_+^2$ is a spatially homogeneous steady state of the macroscopic system if and only if $\bar{u} = \bar{u}_a + \bar{u}_b$ and the triplet $(\bar{u}_a, \bar{u}_b, \bar{v})$ satisfy the nonlinear system

$$f_a(\bar{u}_a) + f_b(\bar{u}_b, \bar{v}) = f_v(\bar{u}_b, \bar{v}) = Q(\bar{u}_a, \bar{u}_b, \bar{v}) = 0\,. \tag{1.4.2}$$

Extinction of u. From $Q(\bar{u}_a, \bar{u}_b, \bar{v}) = 0$ and the strict positivity of ϕ and ψ, we see that $\bar{u}_a = 0$ if and only if $\bar{u}_b = 0$: no extinction of a single subpopulation of the species u is admitted. Thus, for $\bar{u}_a = \bar{u}_b = 0$, we obtain the trivial and semi-trivial steady states

$$(\bar{u}_1, \bar{v}_1) = (0,0) \qquad \text{and} \qquad (\bar{u}_2, \bar{v}_2) = (0,b)\,, \tag{1.4.3}$$

corresponding to the total extinction of the two species in the ecosystem and to a partial extinction, respectively.

Survival of u and extinction of v. The other steady states with $\bar{u}_a \neq 0$ and $\bar{u}_b \neq 0$ are of main interest. The first case is with $\bar{v} = 0$. Denoting $\bar{u}_a = \lambda\, a$ and $\bar{u}_b = \sigma\, b$, $\lambda, \sigma > 0$, system (1.4.2) reduces to

$$\eta_a a\, \lambda(1-\lambda) + \eta_b b\, \sigma(1-\sigma) = 0\,, \qquad \frac{\lambda\psi(\lambda)}{\sigma\phi(\sigma)} = \frac{b}{a}\,. \tag{1.4.4}$$

Such a semi-trivial state always exists but the uniqueness is non-trivial. Indeed, the second equation in (1.4.4) can be written equivalently as

$$\frac{\sigma\phi(\sigma)}{\phi_1} = \alpha\, \frac{\lambda\psi(\lambda)}{\psi_1}\,. \tag{1.4.5}$$

Due to assumption (H1), the functions $\Lambda(\lambda) := \lambda\psi(\lambda)/\psi_1$ and $\Sigma(\sigma) := \sigma\phi(\sigma)/\phi_1$ are strictly increasing functions from 0 to $+\infty$. Hence, for every $\lambda > 0$ there exists a unique $\sigma(\lambda) > 0$ solving (1.4.5) and given by

$$\sigma(\lambda) = \Sigma^{-1}(\alpha\Lambda(\lambda))\,. \tag{1.4.6}$$

Plugging (1.4.6) into the left hand side equation in (1.4.4), the stationary states correspond to the zeros of the function F below

$$F(\lambda) := \eta_a a\, \lambda(1-\lambda) + \eta_b b\, \sigma(\lambda)(1-\sigma(\lambda))\,. \tag{1.4.7}$$

Furthermore, by the competition structure, it follows that F is positive for small enough λ and $F(\lambda) \to -\infty$ as $\lambda \to +\infty$. Thus, the macroscopic system (1.1.4) – (1.1.7) admits at least one semi-trivial equilibrium

$$(\bar{u}_3, \bar{v}_3) = (a\lambda + b\sigma, 0)\,, \tag{1.4.8}$$

solution to system (1.4.4), with $\sigma = \sigma(\lambda)$ uniquely determined by (1.4.6). Moreover, if the equilibrium is unique, F is decreasing around the corresponding λ, i.e. $F'(\lambda) < 0$.

In general it is possible to have several semi-trivial states of type (1.4.8). As an example, take

$$a = b = 1, \quad \eta_a = 0.2, \quad \eta_b = 1, \quad \phi \equiv 1, \quad \psi(x) = \begin{cases} 0.1 & \text{if } x \leq 1.6, \\ 0.3 & \text{otherwise.} \end{cases} \tag{1.4.9}$$

The corresponding $F(\lambda)$ is shown in Figure 1.1, from where we see that there exist three semi-trivial states.

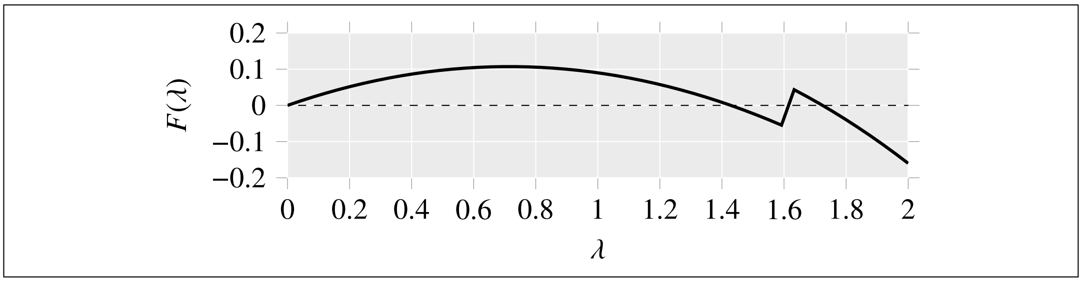


Figure 1.1: Reaction term $F(\lambda)$ for the example (1.4.9).

We will discuss the uniqueness issue in *Proposition* 1.4.2, where a sufficient condition for uniqueness of (1.4.8) is given, and *Proposition* 1.4.3, where we exhibit a family of conversion rates functions ϕ, ψ for which uniqueness of (1.4.8) holds true.

Coexistence of u and v. Finally, if $\bar{u}_a \neq 0$, $\bar{u}_b \neq 0$, $\bar{v} \neq 0$, from $f_v(\bar{u}_b, \bar{v}) = 0$ we get $\bar{u}_b + \bar{v} = b$ and thus $\bar{u}_a = a$. Then, from $Q(\bar{u}_a, \bar{u}_b, \bar{v}) = 0$ and the definition of α it follows that $\bar{u}_b = b\alpha$. Therefore, system (1.4.2) has a unique totally nontrivial solution given by

$$(\bar{u}_4, \bar{v}_4) = (a + b\alpha, b(1-\alpha)), \tag{1.4.10}$$

provided that $\alpha < 1$.

We shall see in the following subsection (see *Theorem* 1.4.1) that the stationary states (1.4.3) are unstable, so that the total extinction of the species u never occurs. The species u always survives and its coexistence with the species v is conditioned by the switching strategy that the subpopulations u_a and u_b adopt when both resources run out, quantified through the parameter α. Indeed, the coexistence occurs if the switch from the state u_b to the state u_a is faster than the opposite switch, i.e. $\alpha < 1$. On the other hand, v goes extinct only if $\alpha > 1$.

The relationship between the linear stability of the mesoscopic and macroscopic coexistence steady states, as $\varepsilon \to 0$, is seen in *Subsection 1.4.3*.

1.4.1 Linear stability analysis for the cross-diffusion system

Let us introduce the partial starvation measures

$$\lambda = \frac{\bar{u}_a}{a} \geq 0, \qquad \sigma = \frac{\bar{u}_b}{b} \geq 0, \qquad \delta = \frac{\bar{v}}{b} \in \{0, 1-\sigma\},$$

so that each of the above steady states can be identified with the triplet $(\lambda, \sigma, \delta)$ and written as

$$\bar{P} = (\bar{u}, \bar{v}) = (\lambda a + \sigma b, \delta b). \tag{1.4.11}$$

Linearizing around $\bar{P}$ the ODEs system associated to (1.1.4) – (1.1.7), in the sense of small perturbation τ, $|\tau| \ll 1$, i.e.

$$\begin{aligned} u_a &= \bar{u}_a + \tau\,\tilde{u}_a \quad \text{and} \quad u_b = \bar{u}_b + \tau\,\tilde{u}_b, \\ u &= u_a + u_b = (\bar{u}_a + \bar{u}_b) + \tau(\tilde{u}_a + \tilde{u}_b) = \bar{u} + \tau\,\tilde{u}, \\ v &= \bar{v} + \tau\tilde{v}, \end{aligned} \tag{1.4.12}$$

we obtain

$$\begin{cases} \dot{\tilde{u}} = \eta_a(1-2\lambda)\tilde{u}_a + \eta_b(1-2\sigma-\delta)\tilde{u}_b - \eta_b\sigma\,\tilde{v} + o(1), \\ \dot{\tilde{v}} = -\eta_v\delta\,\tilde{u}_b + \eta_v(1-\sigma-2\delta)\tilde{v} + o(1). \end{cases} \tag{1.4.13}$$

Moreover, from the linearization of $Q(u_a, u_b, v)$ around $(\bar{u}_a, \bar{u}_b, \bar{v})$, we have

$$\partial_1\bar{Q}\,\tilde{u}_a + \partial_2\bar{Q}\,\tilde{u}_b + \partial_3\bar{Q}\,\tilde{v} + o(1) = 0, \tag{1.4.14}$$

where $\partial_j\bar{Q} = \partial_j Q(\bar{u}_a, \bar{u}_b, \bar{v})$ and

$$\begin{aligned} \partial_1\bar{Q} &= -\psi(\lambda) - \lambda\psi'(\lambda) =: -\beta(\lambda) < 0, \\ \partial_2\bar{Q} &= \phi(\sigma+\delta) + \sigma\phi'(\sigma+\delta) =: \gamma(\sigma,\delta) > 0, \\ \partial_3\bar{Q} &= \sigma\phi'(\sigma+\delta) =: \theta(\sigma,\delta) > 0. \end{aligned} \tag{1.4.15}$$

Using $\tilde{u} = \tilde{u}_a + \tilde{u}_b$, from (1.4.14) we obtain $\tilde{u}_a$ and $\tilde{u}_b$ in terms of $\tilde{u}$ and $\tilde{v}$ as follows

$$\tilde{u}_a = \frac{1}{r}\,\gamma(\sigma,\delta)\,\tilde{u} + \frac{1}{r}\,\theta(\sigma,\delta)\,\tilde{v} + o(1)\,, \qquad \tilde{u}_b = \frac{1}{r}\beta(\lambda)\,\tilde{u} - \frac{1}{r}\,\theta(\sigma,\delta)\,\tilde{v} + o(1)\,, \tag{1.4.16}$$

where $r = r(\lambda,\sigma,\delta) := \partial_2\bar{Q} - \partial_1\bar{Q} = \beta(\lambda) + \gamma(\sigma,\delta) > 0$. Thus, system (1.4.13) becomes

$$\dot{\tilde{w}} = \bar{M}\,\tilde{w} + o(1), \qquad \tilde{w} := \begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix},$$

and the matrix $\bar{M} = M(\bar{P})$ has the following entries

$$\begin{aligned}
M_{11}(\bar{P}) &= \frac{\eta_a}{r}(1-2\lambda)\gamma(\sigma,\delta) + \frac{\eta_b}{r}(1-2\sigma-\delta)\beta(\lambda)\,,\\
M_{12}(\bar{P}) &= \frac{\eta_a}{r}(1-2\lambda)\theta(\sigma,\delta) - \frac{\eta_b}{r}(1-2\sigma-\delta)\theta(\sigma,\delta) - \eta_b\sigma\,,\\
M_{21}(\bar{P}) &= -\frac{\eta_v}{r}\delta\,\beta(\lambda)\,,\\
M_{22}(\bar{P}) &= \frac{\eta_v}{r}\delta\,\theta(\sigma,\delta) + \eta_v(1-\sigma-2\delta)\,.
\end{aligned} \tag{1.4.17}$$

Next, for u_a and u_b as in (1.4.12), using (1.4.16) again, the linearization of the cross diffusion operator in (1.1.6) reads as

$$\Delta(d_a u_a + d_b u_b) = \tau\Big(d_a\frac{\gamma(\sigma,\delta)}{r} + d_b\frac{\beta(\lambda)}{r}\Big)\Delta\tilde{u} + \tau(d_a - d_b)\frac{\theta(\sigma,\delta)}{r}\Delta\tilde{v} + o(\tau),$$

and the linearized cross-diffusion macroscopic system writes

$$\partial_t\tilde{w} = \bar{J}\Delta\tilde{w} + \bar{M}\tilde{w} + o(1)\,,$$

with

$$\bar{J} := \begin{bmatrix} d_a\frac{\gamma(\sigma,\delta)}{r} + d_b\frac{\beta(\lambda)}{r} & (d_a-d_b)\frac{\theta(\sigma,\delta)}{r}\\ 0 & d_v\end{bmatrix}.$$

Finally, denoting $\{\lambda_n\}_n$ the eigenvalues sequence associated to the operator $-\Delta$ with the Neumann boundary condition ($0 = \lambda_0 < \lambda_1 \le ... \le \lambda_n \le ...$), the matrix to be analyzed for the stability of the macroscopic system is $N = -\lambda_n\bar{J} + \bar{M}$, i.e.

$$N := \begin{bmatrix} -\frac{1}{r}(d_a\,\gamma + d_b\beta)\lambda_n + M_{11} & -\frac{1}{r}(d_a-d_b)\theta\,\lambda_n + M_{12}\\ M_{21} & -d_v\lambda_n + M_{22}\end{bmatrix}. \tag{1.4.18}$$

We are now ready to prove the following stability result.

Theorem 1.4.1.
Let ψ and ϕ be conversion rates satisfying assumption (H1) *and $\alpha > 0$ defined as in* (1.4.1). *Then, the following holds true.*

(i) The trivial and semi-trivial steady states $(\bar{u}_1,\bar{v}_1) = (0,0)$ and $(\bar{u}_2,\bar{v}_2) = (0,b)$ are linearly unstable.

(ii) The family of semi-trivial steady states $(\bar{u}_3, \bar{v}_3) = (a\lambda + b\sigma, 0)$ *satisfies*

$$\sigma = \lambda = 1\,, \qquad \text{if } \alpha = 1\,, \tag{1.4.19}$$

$$0 < \sigma < 1 < \lambda < \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{b\eta_b}{a\eta_a}}\,, \qquad \text{if } \alpha < 1\,, \tag{1.4.20}$$

and the swapped relation

$$0 < \lambda < 1 < \sigma < \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{a\eta_a}{b\eta_b}}\,, \qquad \text{if } \alpha > 1\,. \tag{1.4.21}$$

Furthermore, they are linearly unstable if $\alpha \leq 1$*, and if* $\alpha > 1$*, they are linearly stable if and only if the function* F *in* (1.4.7) *is strictly decreasing around* λ*, i.e.* $F'(\lambda) < 0$*.*

(iii) If $\alpha < 1$*, there exists a unique strictly positive steady state given by* $(\bar{u}_4, \bar{v}_4) = (a + b\alpha, b(1 - \alpha))$ *and it is linearly stable.*

Proof.
(i) From (1.4.17) and (1.4.15), we have

$$M(0,0) = \operatorname{diag}\Big\{\frac{\eta_a\phi(0) + \eta_b\psi(0)}{\phi(0) + \psi(0)},\, \eta_v\Big\} \qquad \text{and} \qquad M(0,b) = \begin{bmatrix} \frac{\eta_a\phi_1}{\phi_1 + \psi(0)} & 0 \\ -\frac{\eta_v\psi(0)}{\phi_1 + \psi(0)} & -\eta_v \end{bmatrix},$$

implying that the steady states $(0,0)$ and $(0,b)$ are linearly unstable, both for the macroscopic system and for the associated diffusion-less one, because of the zero eigenvalue of the Laplacian.

(ii) In order to proceed with the investigation of the family of steady states $(\bar{u}_3, \bar{v}_3) = (a\lambda + b\sigma, 0)$, let us observe that from the first equation in (1.4.4), we have

$$(1 - \lambda)(1 - \sigma) < 0 \quad \text{or} \quad \lambda = \sigma = 1\,. \tag{1.4.22}$$

Thus, according to the value of α, we get from (1.4.5): if $\alpha > 1$, then $\lambda \in (0,1)$ and $\sigma > 1$, i.e. $\bar{u}_a < a$ and $\bar{u}_b > b$; if $\alpha < 1$, then $\lambda > 1$ and $\sigma \in (0,1)$, i.e. $\bar{u}_a > a$ and $\bar{u}_b < b$; if $\alpha = 1$, then $\lambda = \sigma = 1$ giving the optimal selection case $\bar{u}_a = a, \bar{u}_b = b$.

Next, let us rewrite the left equation in (1.4.4) as

$$\sigma(1 - \sigma) = \frac{\eta_a a}{\eta_b b}\lambda(\lambda - 1) =: K(\lambda)\,. \tag{1.4.23}$$

If $\alpha > 1$, as $\lambda \in (0,1)$, it follows that $K(\frac{1}{2}) \leq K(\lambda) < 0$ and σ is upper bounded by the positive root of the above equation with $\lambda = \frac{1}{2}$. Hence, (1.4.21) follows. If $\alpha < 1$, swapping the role between λ and σ, we obtain (1.4.20).

Furthermore, the entries (1.4.17) of the matrix $M(\bar{P}) = M(a\lambda + b\sigma, 0)$ are now

$$\begin{aligned}
M_{11}(\bar{P}) &= \eta_a(1 - 2\lambda)\frac{\gamma}{r} + \eta_b(1 - 2\sigma)\frac{\beta}{r}\,,\\
M_{12}(\bar{P}) &= (\eta_a(1 - 2\lambda) - \eta_b(1 - 2\sigma))\frac{\theta}{r} - \eta_b\,\sigma\,,\\
M_{21}(\bar{P}) &= 0\,,\\
M_{22}(\bar{P}) &= \eta_v(1 - \sigma)\,.
\end{aligned}$$

As $M_{21} = 0$, the steady state is linearly stable for the diffusionless macroscopic system if and only if : $M_{11} < 0$ and $M_{22} < 0$. Hence, $\sigma > 1$ is a necessary condition for the linear stability, and it holds only if $\alpha > 1$.

In the case $\alpha = 1$, giving the optimal selection case $\lambda = \sigma = 1$, $M(a+b, 0)$ has a zero eigenvalue, so that the equilibrium is a non hyperbolic equilibrium. Adding the contribution of the cross diffusion term, it does not change the nature of the equilibrium because of the zero eigenvalue of the Laplacian.

Let $\alpha > 1$. The steady states under consideration satisfy $Q(\lambda a, \sigma(\lambda) b, 0) = 0$, where $\sigma(\lambda)$ is defined in (1.4.6). Taking the derivative with respect to λ and using (1.4.15), we obtain

$$a\partial_1 Q(\lambda a, \sigma(\lambda) b, 0) + b\,\sigma'(\lambda)\partial_2 Q(\lambda a, \sigma(\lambda) b, 0) = -\beta(\lambda) a + \gamma(\sigma(\lambda), 0) b\,\sigma'(\lambda) = 0\,.$$

Thus

$$\sigma'(\lambda) = \frac{a}{b}\frac{\beta(\lambda)}{\gamma(\sigma(\lambda), 0)}\,.$$

Plugging $\sigma'(\lambda)$ into the derivative of F

$$F'(\lambda) = \eta_a a(1 - 2\lambda) + \eta_b b\,\sigma'(\lambda)(1 - 2\sigma(\lambda))\,, \tag{1.4.24}$$

we now find

$$F'(\lambda) = \eta_a a(1 - 2\lambda) + \eta_b a\frac{\beta(\lambda)}{\gamma(\sigma(\lambda), 0)}(1 - 2\sigma(\lambda)) = \frac{a}{\gamma(\sigma(\lambda), 0)}\, r M_{11}(\bar{P}).$$

Hence, M_{11} is negative if and only if $F'(\lambda)$ is negative, which implies (*ii*) for the diffusionless macroscopic system and for the cross-diffusion one.

(iii) Let $\alpha < 1$. Since now $(\lambda, \sigma, \delta) = (1, \alpha, 1 - \alpha)$, from (1.4.17), we have

$$M(\bar{u}_4, \bar{v}_4) = -\frac{1}{r}\begin{bmatrix} \eta_a\gamma + \eta_b\alpha\beta & \eta_a\theta + \eta_b\alpha(r - \theta) \\ \eta_v(1 - \alpha)\beta & \eta_v(1 - \alpha)(r - \theta) \end{bmatrix}. \tag{1.4.25}$$

As $r - \theta > 0$, it holds

$$\mathrm{tr} M < 0\,. \tag{1.4.26}$$

By $r = \beta + \gamma$ and $\gamma - \theta = \phi_1$, we have

$$\begin{aligned} \det M =& \frac{\eta_v(1 - \alpha)}{r^2}[(\eta_a\gamma + \eta_b\alpha\beta)(r - \theta) - \eta_a\theta\beta - \eta_b\alpha\beta(r - \theta)] \\ =& \frac{\eta_a\eta_v(1 - \alpha)}{r^2}[\gamma(r - \theta) - \theta\beta] = \frac{\eta_a\eta_v(1 - \alpha)}{r}\,\phi_1 > 0, \end{aligned} \tag{1.4.27}$$

i.e. the equilibrium $(\bar{u}_4, \bar{v}_4)$ is stable for the diffusionless macroscopic system.

The expression form (1.4.25) for M implies for N, by (1.4.18), that

$$\mathrm{tr}\, N < 0\,,$$

and

$$\det N = A\lambda_n^2 + B\lambda_n + C\,,$$

with

$$A := d_v \frac{d_a\gamma + d_b\beta}{r} > 0,$$
$$B := \frac{(d_a - d_b)\theta}{r} M_{21} - \frac{d_a\gamma + d_b\beta}{r} M_{22} - d_v M_{11}, \tag{1.4.28}$$
$$C := \det M > 0.$$

Furthermore, using the definition of r and the strict negativity of all the entries of $M(\bar{u}_4, \bar{v}_4)$, we find for B in (1.4.28)

$$\begin{aligned} B &= -(d_a - d_b)\frac{\eta_v\theta\beta(1-\alpha)}{r^2} + (d_a\gamma + d_b\beta)\frac{\eta_v(r-\theta)(1-\alpha)}{r^2} - d_v M_{11} \\ &= \frac{\eta_v(1-\alpha)}{r^2}\Big(- d_a\theta\beta + d_a r\gamma - d_a\theta\gamma + d_b r\beta\Big) - d_v M_{11} \\ &= \frac{\eta_v(1-\alpha)}{r}(d_a\phi_1 + d_b\beta) - d_v M_{11} > 0, \end{aligned}$$

which implies that $\det N > 0$, for all $n \in \mathbb{N}$. Therefore, the equilibrium $(\bar{u}_4, \bar{v}_4)$ remains linearly stable by adding the cross-diffusion terms. □

1.4.2 Uniqueness semi-trivial states with extinction of v

One possibility to ensure uniqueness of the steady state $(\bar{u}_3, \bar{v}_3) = (a\lambda + b\sigma, 0)$ is to impose, in the case $\alpha > 1$, that the net flux of the individuals of the species u goes from the state u_b to the state u_a, when the population u_b reached the capacity of its resource and the population u_a has only halved the capacity of its resource. When $\alpha < 1$, the opposite switching mechanism has to be imposed. A precise version is the following.

Proposition 1.4.2.
Consider $\Lambda(\lambda) = \lambda\psi(\lambda)/\psi_1$ and $\Sigma(\sigma) = \sigma\phi(\sigma)/\phi_1$, with ϕ, ψ satisfying assumption (H1). Assume that

$$\alpha\Lambda(1/2) \le 1\,, \qquad \text{if} \qquad \alpha > 1\,, \tag{1.4.29}$$

and

$$\alpha^{-1}\Sigma(1/2) \le 1\,, \qquad \text{if} \qquad \alpha < 1\,. \tag{1.4.30}$$

Then, there exists a unique solution to (1.4.4). Furthermore, the corresponding steady state (1.4.8) is linearly stable if $\alpha > 1$, and unstable if $\alpha < 1$.

Proof.
Let $\alpha > 1$. For the proof recall the function $\lambda \mapsto \sigma(\lambda)$ from (1.4.6). Then, $\sigma(0) = 0$, while the increasing behavior of Λ and Σ together with condition (1.4.29) imply that, for $\lambda \in (0, 1/2]$,

$$\sigma(\lambda) \le \Sigma^{-1}(\alpha\Lambda(1/2)) \le \Sigma^{-1}(1) = 1\,.$$

Hence, for $\lambda \in (0, 1/2]$, the function F from (1.4.7) is strictly positive.

Now, let $\bar{\lambda}$ be the smallest zero of F, so that $(a\bar{\lambda} + b\sigma(\bar{\lambda}), 0)$ is one of the steady states under consideration. By the above argument $\bar{\lambda} > 1/2$, and by *Theorem* 1.4.1, $\alpha > 1$ implies that $\sigma(\bar{\lambda}) > 1$. Therefore, the monotonicity of $\lambda \mapsto \sigma(\lambda)$ again implies that $\sigma(\lambda) > 1$, for any $\lambda \ge \bar{\lambda}$.

Finally, we find from (1.4.24) that $F'(\lambda) < 0$, for all $\lambda \geq \bar{\lambda}$. Hence there exists a unique stationary state and the claimed stability follows from *Theorem* 1.4.1.

The case $\alpha < 1$ follows changing the role between the variables λ and σ and between the functions Λ and Σ, i.e. defining $\lambda(\sigma) := \Lambda^{-1}(\alpha^{-1}\Sigma(\sigma))$ and analyzing the behavior of $G(\sigma) := \eta_a a\lambda(\sigma)(1-\lambda(\sigma)) + \eta_b b\sigma(1-\sigma)$, instead of $F(\lambda)$. The claimed instability follows again by *Theorem* 1.4.1. □

Conditions (1.4.29) and (1.4.30) can be rephrased in terms of the ratio $\frac{b}{a}$, respectively as

$$\frac{\frac{1}{2}\psi(\frac{1}{2})}{\phi_1} \leq \frac{b}{a} < \frac{\psi_1}{\phi_1} \qquad \text{and} \qquad \frac{\psi_1}{\phi_1} < \frac{b}{a} \leq \frac{\psi_1}{\frac{1}{2}\phi(\frac{1}{2})}\,.$$

They are not necessary necessary conditions. Indeed, we provide below a family of conversion rates ψ, ϕ, for which the uniqueness of the stationary states (1.4.8) holds true, whatever is $\frac{b}{a}$. For that family of conversion rates, some numerical test are shown in *Section* 1.5.

Since the population densities u_a and u_b are of the same species, it is natural to expect that the conversion dynamics from u_a to u_b is similar to that from u_b to u_a. So, in order to be consistent with the modelling considerations in *Subsection* 1.1.1, (see (1.1.15)), we choose

$$\psi(x) = \omega_1\phi(\omega_2 x), \qquad \omega_1 > 0\,, \omega_2 \geq 0\,, \tag{1.4.31}$$

and we prove the following.

Proposition 1.4.3.
Consider ψ as in (1.4.31) *and*

$$\phi(x) = \theta_1 x + \theta_2, \qquad \theta_1 \geq 0\,, \theta_2 > 0. \tag{1.4.32}$$

Then, there exists a unique stationary state $(\bar{u}_3, \bar{v}_3) = (a\lambda + b\sigma, 0)$. It is linearly stable if $\frac{b}{a} < \omega_1\phi(\omega_2)/\phi_1$, and unstable otherwise.

Proof.
Let $\sigma(\lambda)$ be as in (1.4.6). As observed previously, the stationary states (1.4.8) corresponds to the zeros of the function $F(\lambda)$ in (1.4.7). Taking the second derivative of F, gives

$$F''(\lambda) = b\,\eta_b[\sigma''(\lambda) - 2(\sigma'(\lambda))^2 - 2\sigma(\lambda)\sigma''(\lambda)] - 2a\,\eta_a\,. \tag{1.4.33}$$

By (1.4.32) and (1.4.31), we have

$$\frac{\sigma\phi(\sigma)}{\phi_1} = \bar{\theta}\sigma^2 + (1-\bar{\theta})\sigma\,, \qquad \bar{\theta} = \frac{\theta_1}{\theta_1+\theta_2}\,,$$

and

$$\frac{\lambda\psi(\lambda)}{\psi_1} = \bar{\omega}\lambda^2 + (1-\bar{\omega})\lambda\,, \qquad \bar{\omega} = \frac{\omega_2\theta_1}{\omega_2\theta_1+\theta_2}\,.$$

Hence, equation (1.4.5) reads as

$$\bar{\theta}\sigma^2(\lambda) + (1-\bar{\theta})\sigma(\lambda) = \alpha[\bar{\omega}\lambda^2 + (1-\bar{\omega})\lambda] =: W(\lambda)\,, \tag{1.4.34}$$

and

$$\sigma(\lambda) = \frac{\bar{\theta}-1}{2\bar{\theta}} + \frac{1}{2\bar{\theta}}[(\bar{\theta}-1)^2 + 4\bar{\theta}\,W(\lambda)]^{\frac{1}{2}}\,.$$

Furthermore, differentiating twice (1.4.34) with respect to λ, we obtain the identity

$$2(\sigma'(\lambda))^2 + 2\sigma(\lambda)\sigma''(\lambda) = 2\alpha\frac{\bar{\omega}}{\bar{\theta}} + (1 - \frac{1}{\bar{\theta}})\sigma''(\lambda)\,.$$

Plugging the latter into (1.4.33), we end up with

$$F''(\lambda) = \frac{b\,\eta_b}{\bar{\theta}}\sigma''(\lambda) - (2\alpha\frac{\bar{\omega}}{\bar{\theta}}b\,\eta_b + 2a\,\eta_a)\,.$$

Finally, observing that $W'^2 - 2W\,W'' = \alpha^2(1-\bar{\omega})^2$, we compute

$$\sigma''(\lambda) = \left(\frac{W'(\lambda)}{[(\bar{\theta}-1)^2 + 4\bar{\theta}\,W(\lambda)]^{\frac{1}{2}}}\right)' = \frac{W''[(\bar{\theta}-1)^2 + 4\bar{\theta}\,W] - 2\bar{\theta}W'^2}{[(\bar{\theta}-1)^2 + 4\bar{\theta}\,W]^{\frac{3}{2}}}$$

$$= \frac{2\alpha\bar{\omega}(\bar{\theta}-1)^2 - 2\bar{\theta}(W'^2 - 2W\,W'')}{[(\bar{\theta}-1)^2 + 4\bar{\theta}\,W]^{\frac{3}{2}}} = 2\alpha\frac{\bar{\omega}(1-\bar{\theta})^2 - \alpha\bar{\theta}(1-\bar{\omega})^2}{[(\bar{\theta}-1)^2 + 4\bar{\theta}\,W(\lambda)]^{\frac{3}{2}}}\,.$$

If $\bar{\omega}(1-\bar{\theta})^2 - \alpha\bar{\theta}(1-\bar{\omega})^2 \leq 0$, the function F is strictly concave and therefore has a unique zero. If $\bar{\omega}(1-\bar{\theta})^2 - \alpha\bar{\theta}(1-\bar{\omega})^2 > 0$, then $\sigma''(\lambda)$ is a strictly positive decreasing function that converge to 0 as $\lambda \to +\infty$, and consequently F has at most one inflection point and a unique zero. Moreover, F is decreasing around its unique zero. So that it gives a stable stationary point if $\alpha > 1$. □

1.4.3 Linear stability analysis for the mesoscopic system

A triple $(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, \bar{v}^\varepsilon)$ is a homogeneous stationary solutions to the mesoscopic scale problem (1.1.1) if and only if

$$f_a(\bar{u}_a^\varepsilon) + \frac{1}{\varepsilon}Q(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = f_b(\bar{u}_b^\varepsilon, \bar{v}^\varepsilon) - \frac{1}{\varepsilon}Q(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = f_v(\bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = 0.$$

If $\bar{v}^\varepsilon = 0$, then either $\bar{u}_a^\varepsilon = \bar{u}_b^\varepsilon = 0$, which gives the totally trivial steady state corresponding to the trivial macroscopic one $(\bar{u}_1, \bar{v}_1)$, or $\bar{u}_a^\varepsilon \neq 0$ and $\bar{u}_b^\varepsilon \neq 0$. In the second case the triplet $(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, 0)$ satisfies the system

$$\begin{cases} \eta_a\bar{u}_a^\varepsilon(1 - \dfrac{\bar{u}_a^\varepsilon}{a}) + \dfrac{1}{\varepsilon}[\phi(\dfrac{\bar{u}_b^\varepsilon}{b})\,\bar{u}_b^\varepsilon - \psi(\dfrac{\bar{u}_a^\varepsilon}{a})\,\bar{u}_a^\varepsilon] = 0, \\ \eta_b\bar{u}_b^\varepsilon(1 - \dfrac{\bar{u}_b^\varepsilon}{b}) - \dfrac{1}{\varepsilon}[\phi(\dfrac{\bar{u}_b^\varepsilon}{b})\,\bar{u}_b^\varepsilon - \psi(\dfrac{\bar{u}_a^\varepsilon}{a})\,\bar{u}_a^\varepsilon] = 0, \end{cases}$$

it can be non unique, as in the macroscopic case, and it converges to a macroscopic equilibrium $(\bar{u}_3, \bar{v}_3)$, in the limit $\varepsilon \to 0$.

If $\bar{v}^\varepsilon \neq 0$, then from $f_v(u_b, v) = 0$ we have $\bar{u}_b^\varepsilon + \bar{v}^\varepsilon = b$. Hence, for all $\varepsilon > 0$, $f_b(\bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = Q(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = 0$ and we obtain the two stationary states $(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = (0, 0, b)$ and

$$(\bar{u}_a^\varepsilon, \bar{u}_b^\varepsilon, \bar{v}^\varepsilon) = (a, b\alpha, b(1-\alpha))\,, \tag{1.4.35}$$

provided that $\alpha < 1$. These equilibria do not depend on $\varepsilon > 0$, so that we shall drop the ε exponent in the sequel. In the limit $\varepsilon \to 0$, they correspond to the linearly unstable equilibrium $(\bar{u}_2, \bar{v}_2)$ and to the positive linearly stable equilibrium $(\bar{u}_4, \bar{v}_4)$, respectively.

Hereafter, we focus on the totally nontrivial spatially homogeneous steady (1.4.35), and we see that, for all $\varepsilon > 0$, it is also stable for the mesoscopic system (1.1.1) and the corresponding ODEs system. Indeed, setting

$$u_a^\varepsilon = \bar{u}_a + \tau\tilde{u}_a^\varepsilon \qquad u_b^\varepsilon = \bar{u}_b + \tau\tilde{u}_b^\varepsilon, \qquad v^\varepsilon = \bar{v} + \tau\tilde{v}^\varepsilon, \qquad |\tau| \ll 1,$$

the linearization of (1.1.1) around $(\bar{u}_a, \bar{u}_b, \bar{v})$ writes as

$$\partial_t\tilde{w}^\varepsilon = \mathrm{diag}\{d_a, d_b, d_v\}\Delta\tilde{w}^\varepsilon + M^\varepsilon\tilde{w}^\varepsilon + o(1), \qquad \tilde{w}^\varepsilon := (\tilde{u}_a^\varepsilon, \tilde{u}_b^\varepsilon, \tilde{v}^\varepsilon)^{\mathrm{T}},$$

with

$$M^\varepsilon := \begin{bmatrix} -\eta_a + \frac{1}{\varepsilon}\partial_1\bar{Q} & \frac{1}{\varepsilon}\partial_2\bar{Q} & \frac{1}{\varepsilon}\partial_3\bar{Q} \\ -\frac{1}{\varepsilon}\partial_1\bar{Q} & -\eta_b\alpha - \frac{1}{\varepsilon}\partial_2\bar{Q} & -\eta_b\alpha - \frac{1}{\varepsilon}\partial_3\bar{Q} \\ 0 & -\eta_v(1-\alpha) & -\eta_v(1-\alpha) \end{bmatrix}.$$

Again, we need to analyse the stability of the matrix M^ε above and N^ε below

$$N^\varepsilon := -\lambda_n\mathrm{diag}\{d_a, d_b, d_v\} + M^\varepsilon,$$

i.e.

$$N^\varepsilon = \begin{bmatrix} -d_a\lambda_n - \eta_a + \frac{1}{\varepsilon}\partial_1\bar{Q} & \frac{1}{\varepsilon}\partial_2\bar{Q} & \frac{1}{\varepsilon}\partial_3\bar{Q} \\ -\frac{1}{\varepsilon}\partial_1\bar{Q} & -d_b\lambda_n - \eta_b\alpha - \frac{1}{\varepsilon}\partial_2\bar{Q} & -\eta_b\alpha - \frac{1}{\varepsilon}\partial_3\bar{Q} \\ 0 & -\eta_v(1-\alpha) & -d_v\lambda_n - \eta_v(1-\alpha) \end{bmatrix}.$$

For that, we apply the Routh-Hurwitz criterion [66] and we obtain the result below, proved in *Appendix* A.2.

Proposition 1.4.4.
Under the assumption $\alpha < 1$, for all $\varepsilon > 0$ and $\lambda_n \geq 0$, the matrices M^ε and N^ε are stable, i.e. all their eigenvalues have negative real part.

To complete the analysis, we shall see below how the previous linear stability property is preserved in the limit as $\varepsilon \to 0$. Indeed, two eigenvalues of N^ε converge to those of N, while the third one goes to $-\infty$.

Let us denote

$$D^\varepsilon(\mu) := N^\varepsilon - \mu I_3,$$

where I_3 stands for the 3×3 identity matrix. The goal of the computations below is to compute $\det D^\varepsilon(\mu)$, also denoted by $|D^\varepsilon|$, (see also [50]).

First, adding the second row of D^ε to the first one, we get

$$|D^\varepsilon| = \begin{vmatrix} -(d_a\lambda_n + \eta_a + \mu) & -(d_b\lambda_n + \eta_b\alpha + \mu) & -\eta_b\alpha \\ -\frac{1}{\varepsilon}\partial_1\bar{Q} & -(d_b\lambda_n + \eta_b\alpha + \mu) - \frac{1}{\varepsilon}\partial_2\bar{Q} & -\eta_b\alpha - \frac{1}{\varepsilon}\partial_3\bar{Q} \\ 0 & -\eta_v(1-\alpha) & -(d_v\lambda_n + \eta_v(1-\alpha) + \mu) \end{vmatrix}.$$

Then, using the identity

$$\partial_3\bar{Q} + (\partial_1\bar{Q} - \partial_2\bar{Q})\frac{\alpha\phi_1'}{r} = 0\,,$$

and adding to the third column the difference between the first and the second column, both multiplied by $\frac{\alpha\phi_1'}{r}$, we obtain

$$|D^\varepsilon| = \begin{vmatrix} -(d_a\lambda_n + \eta_a + \mu) & -(d_b\lambda_n + \eta_b\alpha + \mu) & N_{12} \\ -\frac{1}{\varepsilon}\partial_1\bar{Q} & -(d_b\lambda_n + \eta_b\alpha + \mu) - \frac{1}{\varepsilon}\partial_2\bar{Q} & d_{23} \\ 0 & -\eta_v(1-\alpha) & N_{22}-\mu \end{vmatrix},$$

with

$$d_{23} := (d_b\lambda_n + \eta_b\alpha + \mu)\frac{\alpha\phi_1'}{r} - \eta_b\,\alpha.$$

Furthermore, using

$$\partial_1\bar{Q}(\alpha\phi_1' + \phi_1) + \partial_2\bar{Q}\beta = 0\,,$$

and adding the second column, multiplied by $\frac{\beta}{r}$, to the first one, multiplied by $\frac{\alpha\phi_1'+\phi_1}{r}$, we get

$$(1-\frac{\beta}{r})|D^\varepsilon| = \begin{vmatrix} N_{11}-\mu & -(d_b\lambda_n + \eta_b\alpha + \mu) & N_{12} \\ -(d_b\lambda_n + \eta_b\alpha + \mu)\frac{\beta}{r} & -(d_b\lambda_n + \eta_b\alpha + \mu) - \frac{1}{\varepsilon}\partial_2\bar{Q} & d_{23} \\ N_{21} & -\eta_v(1-\alpha) & N_{22}-\mu \end{vmatrix}.$$

Finally, subtracting the first column to the second one, multiplied by $\frac{\beta}{r}$, we have

$$\frac{\beta}{r}(1-\frac{\beta}{r})|D^\varepsilon| = \begin{vmatrix} N_{11}-\mu & d_{12} & N_{12} \\ d_{21} & -\frac{1}{\varepsilon}\frac{\beta}{r}\partial_2\bar{Q} & d_{23} \\ N_{21} & 0 & N_{22}-\mu \end{vmatrix}, \tag{1.4.36}$$

with

$$d_{12} := \mu\Big(1-\frac{\beta}{r}\Big) - (d_b\lambda_n + \eta_b\alpha)\frac{\beta}{r} - N_{11},$$
$$d_{21} := -(d_b\lambda_n + \eta_b\alpha + \mu)\frac{\beta}{r}\,.$$

Thus, (1.4.36) rewrites as

$$\frac{\beta}{r}(1-\frac{\beta}{r})\ \det(D^\varepsilon(\mu)) = -\frac{1}{\varepsilon}\beta\Big(1-\frac{\beta}{r}\Big)\ \det(N-\mu\, I_2) + R(\mu)\,,$$

where

$$R(\mu) = -\frac{\beta}{r}\Big(1-\frac{\beta}{r}\Big)\mu^3 + p(\mu)\,,$$

with $p(\mu)$ a polynomial function of degree two that does not depend on ε. Consequently

$$\det(D^\varepsilon(\mu)) = -\mu^3 - \frac{r}{\varepsilon}\det(N-\mu\, I_2) + \frac{r^2}{\beta(r-\beta)}p(\mu)\,, \tag{1.4.37}$$

with

$$\det(N-\mu\, I_2) = \mu^2 - (trN)\mu + \det N\,. \tag{1.4.38}$$

Let γ_i, $i = 1, 2$ denote the eigenvalues of N and let μ_i^ε denote the eigenvalues of N^ε, $i = 1, 2, 3$. It has been shown that $\Re(\gamma_i) < 0$ and $\Re(\mu_i^\varepsilon) < 0$. Moreover, observe that μ_i^ε is a root of (1.4.37) if and only if it is a root of

$$-\varepsilon\mu^3 - r\det(N - \mu I_2) + \varepsilon\frac{r^2}{\beta(r-\beta)}p(\mu)\,. \tag{1.4.39}$$

Plugging in (1.4.39) the simple asymptotic expansion in ε of $\mu_i^\varepsilon = \nu_0^i + \varepsilon\nu_1^i + \varepsilon^2\nu_2^i + \cdots$, the zero order terms gives $-r\det(N - \nu_0^i I_2) = 0$. Therefore,

$$\mu_i^\varepsilon = \gamma_i + O(\varepsilon)\,, \qquad i = 1, 2\,, \tag{1.4.40}$$

and

$$\begin{aligned}\mu_1^\varepsilon + \mu_2^\varepsilon &= \operatorname{tr} N + O(\varepsilon),\\ \mu_1^\varepsilon\mu_2^\varepsilon &= \det N + O(\varepsilon)\,.\end{aligned}$$

On the other hand, writing $\det(D^\varepsilon(\mu)) = -(\mu-\mu_1^\varepsilon)(\mu-\mu_2^\varepsilon)(\mu-\mu_3^\varepsilon)$, from (1.4.37), (1.4.38), we deduce the identities below

$$\begin{aligned}\mu_1^\varepsilon + \mu_2^\varepsilon + \mu_3^\varepsilon &= -\frac{r}{\varepsilon} + O(1),\\ \mu_1^\varepsilon\mu_2^\varepsilon + \mu_3^\varepsilon(\mu_1^\varepsilon + \mu_2^\varepsilon) &= -\frac{r}{\varepsilon}\operatorname{tr} N + O(1),\\ \mu_1^\varepsilon\mu_2^\varepsilon\mu_3^\varepsilon &= -\frac{r}{\varepsilon}\det N + O(1)\,,\end{aligned}$$

so that,

$$\mu_3^\varepsilon = -\frac{r}{\varepsilon} + O(1)\,.$$

1.5 Numerical simulations

For the numerical simulations, we consider the linear conversion rates

$$\phi(x) = x + \delta \qquad \text{and} \qquad \psi(x) = \theta x + \gamma\,, \tag{1.5.1}$$

with $\delta = 0.5$, $\theta = 5$ and $\gamma = 1$, together with the growth rates

$$\eta_a = 3, \qquad \eta_b = 2, \qquad \eta_v = 40\,. \tag{1.5.2}$$

Depending on the choice of a and b, we consider two cases: the v extinction case

$$a = 1.5, \quad b = 6, \quad \Rightarrow \alpha = 1, \tag{1.5.3}$$

and the coexistence case

$$a = 1.5, \quad b = 8, \quad \Rightarrow \alpha < 1. \tag{1.5.4}$$

In the case of the ODE system associated to the mesoscopic system (1.1.1) with (1.1.4) and (1.5.1), the numerical solution is illustrated in Figure 1.2 ($\alpha = 1$) and Figure 1.4 ($\alpha < 1$). The expected initial layer for the subpopulations u_a^ε and u_b^ε can be observed in Figure 1.3 and 1.5 (see Remark 1.1).

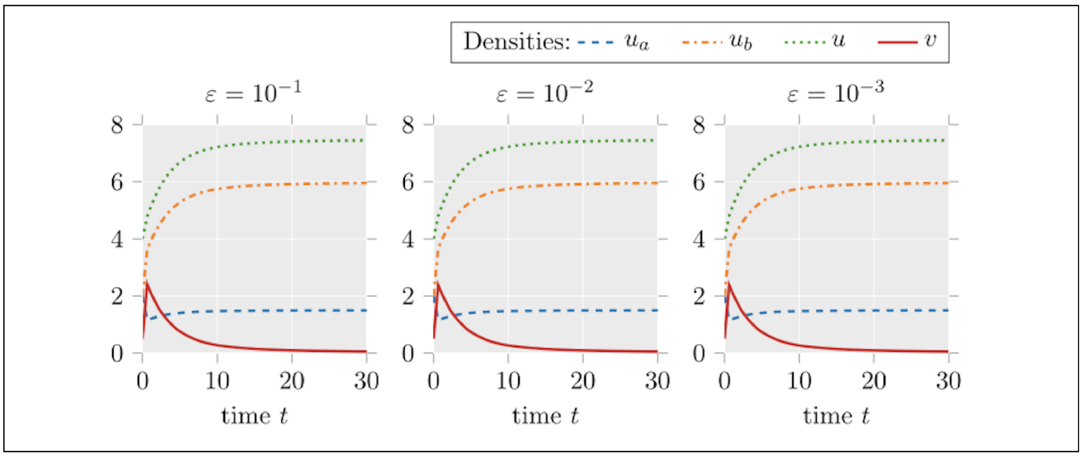


Figure 1.2: $\alpha = 1$. Solution to the mesoscopic ODE system with parameters given in (1.5.1),(1.5.2) and (1.5.3), for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ (from left to right), with extinction of v^ε, and convergence of $u^\varepsilon = u_a^\varepsilon + u_b^\varepsilon$ towards $a + b$. Here the maximal time is $T = 30$.

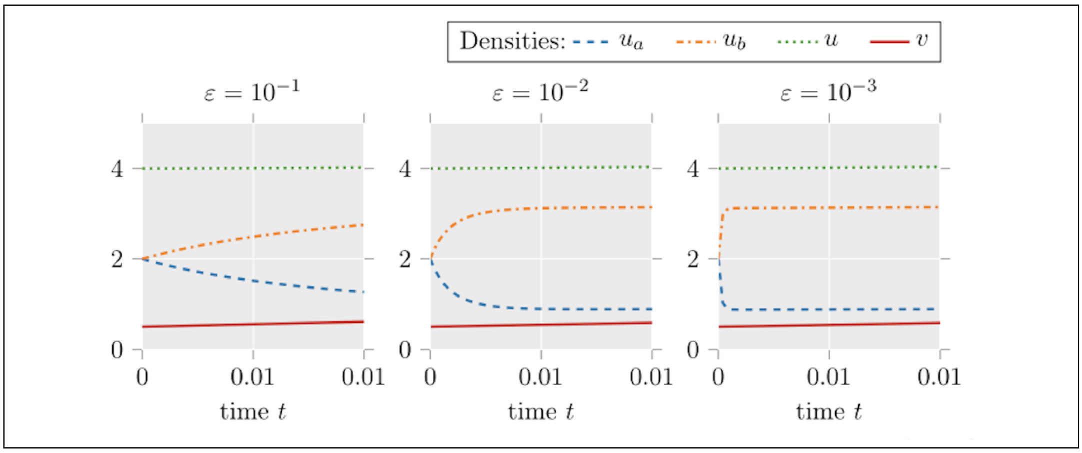


Figure 1.3: $\alpha = 1$. Zoom of the solution in Figure 1.2 in a right neighbourhood of $t = 0$ for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ (from left to right).

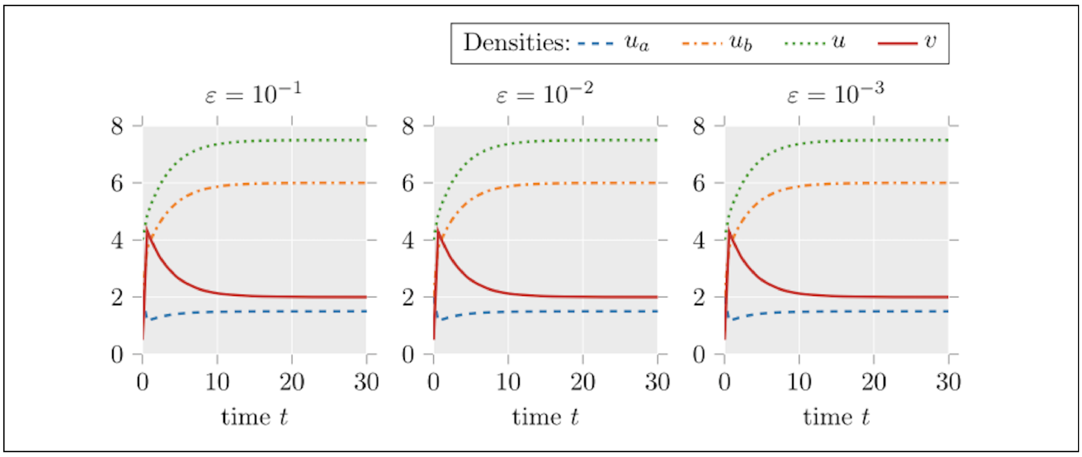


Figure 1.4: $\alpha < 1$. Solution to the mesoscopic ODE system with parameters given in (1.5.1),(1.5.2) and (1.5.4), for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ (from left to right), with convergence of $(u^\varepsilon, v^\varepsilon) = (u_a^\varepsilon + u_b^\varepsilon, v^\varepsilon)$ towards $(a + b\alpha, b(1-\alpha))$. Here the maximal time is $T = 30$.

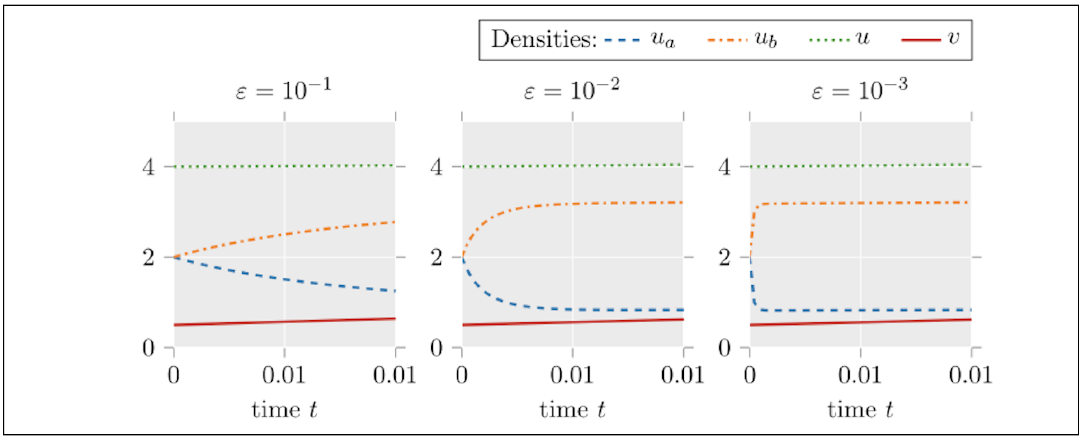


Figure 1.5: $\alpha < 1$. Zoom of the solution in Figure 1.4 in a right neighbourhood of $t = 0$ for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ (from left to right).

The effect of the spatial dispersal of the species by diffusion is shown in Figure 1.6 ($\alpha = 1$) and Figure 1.7 ($\alpha < 1$) below, in the case of the one dimensional spatial domain $[0, 1]$. Additionally, we provide a video in the supplements. All the parameters are kept as in the previous computations and the diffusion coefficients are

$$d_a = 2, \qquad d_b = 0.1, \qquad d_v = 0.1\,,$$

and the initial conditions has been chosen as

$$u_a^{\text{in}}(x) = \cos(4\pi x) + 4\,, \qquad u_b^{\text{in}}(x) = (x-1)\sin(4\pi x^2) + 2\,,$$
$$v^{\text{in}}(x) = \cos(4\pi x) + \cos(2\pi x) + 2.5\,.$$

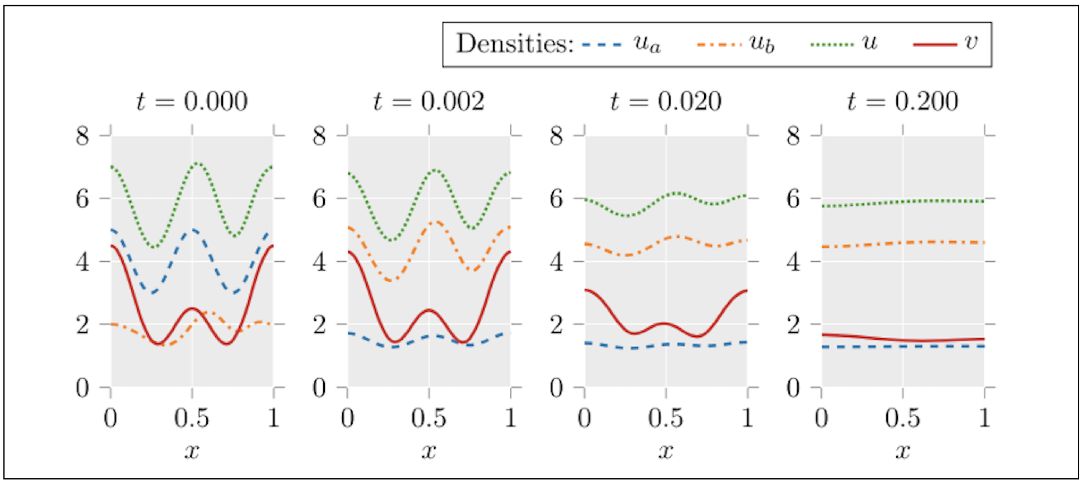


Figure 1.6: $\alpha = 1$. Solution to the mesoscopic PDE system (1.1.1) in the extinction case.

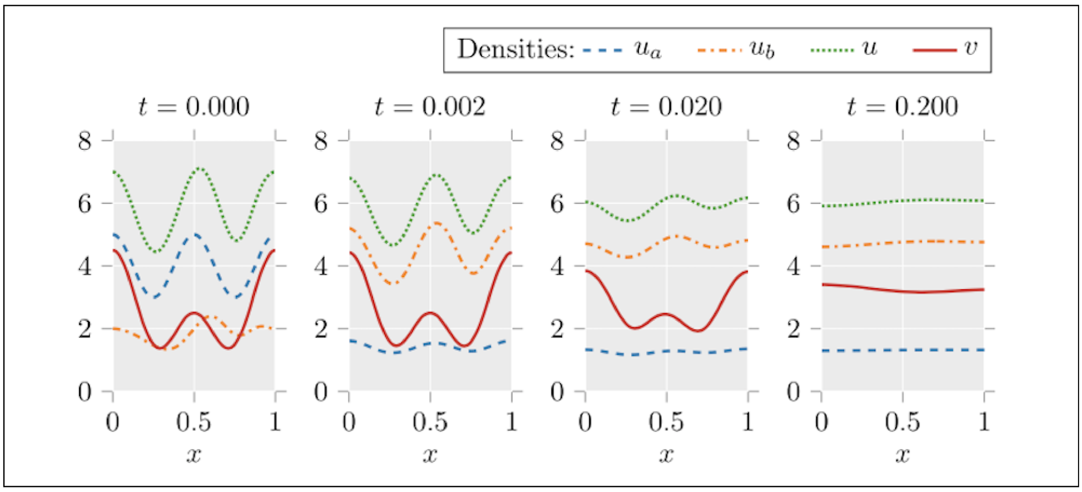


Figure 1.7: $\alpha < 1$. Solution to the mesoscopic PDE system (1.1.1) in the coexistence case.

Chapter 2

Triangular cross-diffusion systems driven by intra-specific survival strategy

2.1 Introduction

In this chapter we consider the following reaction-diffusion system with fast reaction

$$\begin{cases} \partial_t u_a^\varepsilon - d_a \Delta u_a^\varepsilon = f_a(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) + \frac{1}{\varepsilon} Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon), & \text{in } (0,T)\times\Omega, \\ \partial_t u_b^\varepsilon - d_b \Delta u_b^\varepsilon = f_b(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) - \frac{1}{\varepsilon} Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon), & \text{in } (0,T)\times\Omega, \\ \partial_t v^\varepsilon - d_v \Delta v^\varepsilon = f_v(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon), & \text{in } (0,T)\times\Omega, \end{cases} \tag{2.1.1}$$

where $d_a, d_b, d_v > 0$, $d_a \neq d_b$, and the reaction functions are given for all $u_a, u_b, v \geq 0$ by

$$\begin{aligned} f_a(u_a, u_b, v) &:= \eta_a u_a\Big(1 - a u_a - c v\Big) - \gamma_a u_a u_b, \\ f_b(u_a, u_b, v) &:= \eta_b u_b\Big(1 - b u_b - d v\Big) - \gamma_b u_a u_b, \\ f_v(u_a, u_b, v) &:= \eta_v' v\Big(1 - a u_a - c v\Big) + \eta_v'' v\Big(1 - b u_b - d v\Big), \end{aligned} \tag{2.1.2}$$

with $a, b > 0, c, d \in \mathbb{R}_+$, $\eta_a, \eta_b > 0$, $\eta_v', \eta_v'', \gamma_a, \gamma_b \in \mathbb{R}_+$, $(c\eta_v', d\eta_v'') \neq (0,0)$ and

$$Q(u_a, u_b, v) := \phi(b u_b + d v)\, u_b - \psi(a u_a + c v)\, u_a. \tag{2.1.3}$$

In addition, Ω is a smooth bounded open set of $\mathbb{R}^N$, $N \geq 1$, and system (2.1.1) - (2.1.3) is supplemented with homogeneous Neumann boundary conditions

$$\nabla u_a^\varepsilon \cdot \sigma = \nabla u_b^\varepsilon \cdot \sigma = \nabla v^\varepsilon \cdot \sigma = 0, \qquad \text{on } (0,T)\times\partial\Omega, \tag{2.1.4}$$

and the initial conditions

$$\begin{aligned} u_a^\varepsilon(0,x) &= u_a^{\varepsilon,\text{in}}(x) := u_a^{\text{in}}(x) \geq C_{0,a} > 0, \qquad x \in \Omega, \\ u_b^\varepsilon(0,x) &= u_b^{\varepsilon,\text{in}}(x) := u_b^{\text{in}}(x) \geq C_{0,b} > 0, \qquad x \in \Omega, \end{aligned} \tag{2.1.5}$$

with the constants $C_{0,a}$, $C_{0,b} > 0$ not depending on ε, and

$$v^\varepsilon(0,x) = v^{\varepsilon,\text{in}}(x) := v^{\text{in}}(x) \geq 0, \quad x \in \Omega. \tag{2.1.6}$$

Remark 2.1.
The strict positivity of $u_a^\varepsilon(0,\cdot)$, $u_b^\varepsilon(0,\cdot)$ will be crucial to obtain strict positive dentities u_a^ε, u_b^ε, for any fixed $\varepsilon > 0$, and thus get the a priori energy estimates (see *Lemma* (2.7.1)).

The mesoscopic system (2.1.1) is the natural generalisation of the mesoscopic system introduced and analysed in *Chapter 1*. It is worth noticing that in the model considered in *Chapter 1*, only u_b was in direct competition with v and no intra-specific competition in the population u was taken into account. In other words in *Chapter 1*, we considered the reaction terms (2.1.2) with $c = \gamma_a = \gamma_b = \eta'_v = 0$ and $b = d$.

As we did in *Chapter 1*, we can interpret u_a and u_b as two sub-populations of a population of density $u = u_a + u_b$ in competition with the population of density v. The individuals of the populations of densities u_a and u_b switch between them. The conversion function is given by (2.1.3) and $\varepsilon > 0$ is the average time for the sub-populations conversion into each other. In addition, we consider here the conversion functions ϕ, ψ

$$\psi(x) := (A + x)^\alpha, \qquad \phi(x) := (B + x)^\beta, \qquad x \geq 0, \tag{H1}$$

and we assume

$$A > 0,\ B \geq 0 \qquad \text{and} \qquad 0 < \alpha \leq \beta. \tag{H2}$$

We observe that ψ, ϕ satisfy more general hypothesis than in *Chapter 1*. Unlike *Chapter 1*, we assume only ψ to be lower bounded by a strictly positive constant and we don't force ψ, ϕ to have bounded first derivatives.

Remark 2.2. Due to the symmetry of the mesoscopic system (2.1.1) - (2.1.3), i.e. due to the interchangeable role of $u_a^\varepsilon, u_b^\varepsilon$, it is possible to replace (H2) with

$$A \geq 0,\ B > 0 \qquad \text{and} \qquad 0 < \beta \leq \alpha.$$

The main result of this chapter (see *Theorem* 2.5.2) is that, as $\varepsilon \to 0$, the (unique) solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ of the initial boundary value problem (2.1.1) - (2.1.6) converges to a limit (u_a, u_b, v) and this limit is a weak solution to the class of macroscopic cross-diffusion systems, given by

$$\begin{cases} \partial_t u - \Delta(A(u,v)) = F_u(u,v), & \text{in } (0,+\infty)\times\Omega, \\ \partial_t v - \Delta(d_v v) = F_v(u,v), & \text{in } (0,+\infty)\times\Omega, \end{cases} \tag{2.1.7}$$

with

$$A(u,v) := d_a u_a^*(u,v) + d_b u_b^*(u,v), \tag{2.1.8}$$

and where u_a^*, u_b^* are two positive maps from $\mathbb{R}_+^2$ to $\mathbb{R}_+$ such that, for all $(u,v) \in \mathbb{R}_+^2$, the pair $(u_a^*(u,v), u_b^*(u,v))$ is the unique solution to the nonlinear system

$$\begin{cases} u_a + u_b = u, \\ Q(u_a, u_b, v) = \phi(bu_b + dv)\, u_b - \psi(au_a + cv)\, u_a = 0, \end{cases} \tag{2.1.9}$$

with ϕ, ψ as in (H1), (H2) (see *Section 2.3* for the uniqueness of $(u_a^*(u,v), u_b^*(u,v))$ and further properties). Furthermore, the reaction terms F_u and F_v are

$$\begin{aligned} F_u(u,v) &:= f_u(u_a^*(u,v), u_b^*(u,v), v), \\ F_v(u,v) &:= f_v(u_a^*(u,v), u_b^*(u,v), v), \end{aligned} \tag{2.1.10}$$

with

$$f_u(u_a, u_b, v) := f_a(u_a, u_b, v) + f_b(u_a, u_b, v), \tag{2.1.11}$$

and f_a, f_b, f_v defined in (2.1.2). Finally, system (2.1.7) is supplemented with the no-flux boundary conditions

$$\nabla A(u, v) \cdot \sigma = \nabla v \cdot \sigma = 0, \qquad \text{on } (0, T) \times \partial\Omega, \tag{2.1.12}$$

and the initial data

$$\begin{aligned} &u(0, x) = u^{\text{in}}(x) = u_a^{\text{in}}(x) + u_b^{\text{in}}(x) \geq C_{0,a} + C_{0,b} > 0, && x \in \Omega, \\ &v(0, x) = v^{\text{in}}(x) \geq 0, && x \in \Omega. \end{aligned} \tag{2.1.13}$$

The proof of convergence as $\varepsilon \to 0$ is rigorously obtained via *a priori estimates* for $u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon$, satisfying (2.1.1) - (2.1.6). The main tool is the family of energy (or entropy) functionals that we will introduce in the following section (see *Lemma 2.7.1* for the a priori-estimates). Then, by invoking the Aubin-Lions Lemma, we pass to the limit along a subsequence and we conclude by verifying that the limit is a very weak solution to (2.1.7) – (2.1.13) (see (2.5.1), (2.5.2)).

The rest of the chapter is organised as follows: in *Section 2.2*, we introduce a family of energy functionals, in *Section* 2.3, we outline some properties of the macroscopic system (2.1.7) - (2.1.11) and in *Section 2.4*, we give a formal derivation of (2.1.1) - (2.1.3), out of a microscopic system. *Section 2.5* is devoted to the statement of the existence results: the existence of solutions to the mesoscopic system (2.1.1) - (2.1.6) and the existence of solutions to the macroscopic system (2.1.7) - (2.1.13). *Sections 2.6, 2.7* aim to prove the $\varepsilon-$uniform a priori estimates. More precisely, in *Section 2.6* we show some basic a priori estimates and in *Section 2.7* we prove the energy estimates. We conclude the chapter with the proof of the existence results: in *Section 2.8* we prove the existence to the mesoscopic system and in *Section 2.9* we prove the existence to the macroscopic system.

2.2 A family of energy functionals

We introduce the following family of energy (or entropy) functionals

$$\mathcal{E}_p(u_a, u_b, v) := \int_\Omega h_{a,p}(u_a, v)\, dx + \int_\Omega h_{b,p}(u_b, v)\, dx, \qquad p \geq 1, \tag{2.2.1}$$

with the energy densities $h_{a,p}$ and $h_{b,p}$ defined as

$$\begin{aligned} h_{a,p}(u_a, v) &:= \int_0^{u_a} \psi^{p-1}(az + cv) z^{p-1} dz, \\ h_{b,p}(u_b, v) &:= \int_0^{u_b} \phi^{p-1}(bz + dv) z^{p-1} dz. \end{aligned} \tag{2.2.2}$$

Assumptions (H1), (H2) imply that $h_{a,p}$, $h_{b,p}$ are positive and increasing functions which are convex with respect to the first variable. The evolutionary analysis of the energy functional, along the solution to the mesoscopic system (2.1.1) - (2.1.6), is fundamental to get the suitable a priori estimates. In order to do that, we introduce the additional hypothesis concerning the values of α,β in (H1), (H2) and the regularity of the initial data (2.1.5), (2.1.6),

$$0 < \alpha \leq \beta \leq \min\left\{\frac{6}{N}, (\sqrt{7}+2)\alpha + \sqrt{7}+1\right\}, \tag{H3}$$

$$u_a^{\text{in}},\, u_b^{\text{in}},\, v^{\text{in}} \in C^2(\bar{\Omega}) \qquad \text{and} \qquad \nabla u_a^{\text{in}} \cdot \sigma = \nabla u_b^{\text{in}} \cdot \sigma = \nabla v^{\text{in}} \cdot \sigma = 0, \quad \text{on } \partial\Omega. \tag{H4}$$

Remark 2.3.
The regularity of the initial data (H4) guarantees to obtain the existence of classical solutions $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ to the system (2.1.1) - (2.1.6), for any fixed $\varepsilon > 0$. Furthermore, (H4) allows us to get the $\varepsilon-$uniform estimates for $\partial_t v^\varepsilon, \partial_{ij} v^\varepsilon, \nabla v^\varepsilon, i, j = 1, \dots N$, thanks to the parabolic maximal regularity (see *Lemma 2.6.2*). Finally, it is worth noticing that (H4) ensures the boundedness of $\mathcal{E}_p$ at the initial time.

In order to have enough compactness from the energy functional and thus take the $\varepsilon-$limit, it is sufficient to consider the following values of p

$$p = 1, \qquad p = 1 + \frac{1}{1+\beta}, \qquad p = 1 + \frac{1}{1+\alpha}, \qquad p = 2. \tag{2.2.3}$$

In the sequel, we outline the contribution that each energy $\mathcal{E}_p$ with p in (2.2.3) gives in terms of regularity for u_a^ε and u_b^ε. Let us denote $\mathcal{F} := (f_a, f_b, f_v)^T$ and define the total energy density for all $p \geq 1$

$$h_p(u_a, u_b, v) := h_{a,p}(u_a, v) + h_{b,p}(u_b, v). \tag{2.2.4}$$

The variation of $\mathcal{E}_p$ along the solutions to (2.1.1) is given by

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_p(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) &= \frac{d}{dt}\int_\Omega h_p(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\, dx \\
&= \int_\Omega (\partial_1 h_p \partial_t u_a^\varepsilon + \partial_2 h_p \partial_t u_b^\varepsilon + \partial_3 h_p \partial_t v^\varepsilon) dx \\
&= \int_\Omega (d_a \partial_1 h_p \Delta u_a^\varepsilon + d_b \partial_2 h_p \Delta u_b^\varepsilon + d_v \partial_3 h_p \Delta v^\varepsilon) dx && (2.2.5) \\
&\quad + \int_\Omega \nabla h_p \cdot \mathcal{F}\, dx + \frac{1}{\varepsilon}\int_\Omega (\partial_1 h_p - \partial_2 h_p) Q\, dx && (2.2.6) \\
&=: I_{diff}^p + I_{rea}^p + I_{fast}^p, && (2.2.7)
\end{aligned}$$

where for simplicity, we neglect the dependence on $u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon$ in $h_{a,p}$, $h_{b,p}$, and h_p.

These entropy functionals are reminiscent of the functionals introduced in [37] to analyse the triangular SKT system (SKT) without self-diffusion (i.e $d_{12} > 0$ and $d_{11} = d_{22} = d_{21} = 0$). The interest in these functionals is twofold: firstly, $\mathcal{E}_p$ is not the sum of functionals of the single densities of the system ([17], [52]). Secondly, they allow to easily handle the contribution from the fast reaction term. Indeed, for all nonnegative ϕ and ψ and for all $p \geq 1$, we have

$$\begin{aligned}
I_{fast}^p &:= \frac{1}{\varepsilon}\int_\Omega (\partial_1 h_p - \partial_2 h_p) Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon,)\, dx \\
&= -\frac{1}{\varepsilon}\int_\Omega [(\phi(b u_b^\varepsilon + d v^\varepsilon) u_b^\varepsilon)^{p-1} - (\psi(a u_a^\varepsilon + c v^\varepsilon) u_a^\varepsilon)^{p-1}] \cdot \\
&\qquad\qquad [\phi(b u_b^\varepsilon + d v^\varepsilon) u_b^\varepsilon - \psi(a u_a^\varepsilon + c v^\varepsilon) u_a^\varepsilon]\, dx \leq 0, && (2.2.8)
\end{aligned}$$

thanks to the following elementary inequality for $p > 1$,

$$(x - y)(x^{p-1} - y^{p-1}) \geq 0, \qquad \text{for all } x, y \in \mathbb{R}_+.$$

Therefore, by (2.2.8) we have from (2.2.7)

$$\frac{d}{dt}\mathcal{E}_p(u_a^\varepsilon(t), u_b^\varepsilon(t), v^\varepsilon(t)) \leq I_{diff}^p + I_{rea}^p.$$

Moreover, according to the values of p in (2.2.3), we obtain the results below.

(i) $p = 1$. The total energy density h_p reads as

$$h_1(u_a, u_b, v) = u_a + u_b.$$

So that $I_{diff}^1 = I_{fast}^1 = 0$ and (2.2.7) reduces to

$$\frac{d}{dt}\int_\Omega (u_a^\varepsilon + u_b^\varepsilon)dx = \int_\Omega f_a(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\,dx + \int_\Omega f_b(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\,dx.$$

Then, we get the uniform control on the densities $u_a^\varepsilon, u_b^\varepsilon$ in the Lebesgue spaces $L^\infty(0, T; L^1(\Omega))$ and $L^2(\Omega_T)$, for any $T > 0$, using the quadratic structure of the reaction functions in (2.1.2) (see *Lemma 2.6.3*).

(ii) $p = p_\beta := 1 + \frac{1}{\beta+1} > 1$. This case allows us to prove the $L^2(\Omega_T)$ boundedness of ∇u_b^ε and the $L^3(\Omega_T)$ control of u_b^ε, thus improving the obtained regularity in the case $p = 1$.

(iii) $p = p_\alpha := 1 + \frac{1}{\alpha+1} > 1$. This case allows us to prove the $L^2(\Omega_T)$ boundedness of ∇u_a^ε and the $L^3(\Omega_T)$ control of u_a^ε, thus improving the obtained regularity in the case $p = 1$.

(iv) $p = 2$. This case is crucial to get compactness for the fast reaction $\frac{1}{\varepsilon}Q$. Indeed if $p = 2$, (2.2.8) reads as

$$I_{fast}^2 = -\frac{1}{\varepsilon}\int_\Omega Q^2(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\,dx. \tag{2.2.9}$$

The latter identity will allow us to prove that $\frac{1}{\varepsilon}\|Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\|_{L^2(\Omega_T)}^2$ is estimated by a constant not depending on ε, using the obtained regularity in the previous cases *(i) - (iii)*.

We outline that assumptions (H2), (H3) are crucially used to get the energy estimates in *Lemma 2.7.1*. In particular, $A > 0$ in (H2) and the upper bounds for α and β in (H3) are fundamental to handle the diffusion term I_{diff}^p in (2.2.7) but not necessary to estimate the I_{rea}^p term in (2.2.7). More precisely, the upper bound $\beta \leq (\sqrt{7} + 2)\alpha + \sqrt{7} + 1$ allows us to handle the diffusion terms when $p = p_\alpha$, whereas $\alpha, \beta \leq 6$ (consequence of $\alpha, \beta \leq 6/N$) enables to control the diffusion terms when $p = 2$.

To conclude, we observe that we can consider all values of $p \geq 1$ in (2.2.1) to get further ε-uniform a priori estimates, by performing a bootstrap argument and improving the regularity of the solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$. This is the subject of a forthcoming work (see *Chapter 4*).

2.3 Properties of the cross-diffusion

The aim of this section is to outline the main properties of the cross-diffusion term in (2.1.7) - (2.1.9). Firstly, we observe that by (H1), (H2), the nonlinear system (2.1.9) admits a unique nonnegative solution $(u_a^*(u,v), u_b^*(u,v))$, for any $u, v \geq 0$. Indeed, if $u = 0$, we have from (2.1.9), for all $v \geq 0$,

$$\begin{cases} u_a + u_b = 0, \\ Q(u_a, u_b, v) = 0, \end{cases}$$

giving the trivial solution $u_a^*(u,v) = u_b^*(u,v) = 0$. Otherwise, if $u > 0$, we consider the function of the u_b variable

$$q(u_b; u, v) := Q(u - u_b, u_b, v) = \phi\,(bu_b + dv)\,u_b - \psi\,(a(u - u_b) + cv)\,(u - u_b). \tag{2.3.1}$$

Thanks to the strictly increasing character of ϕ and ψ, the function q is strictly increasing in u_b and satisfies

$$q(0; u, v) < 0\,, \qquad q(u; u, v) > 0.$$

Therefore, for any given $v \geq 0$, there exists a unique zero $u_b \in (0, u)$ of $q(u_b; u, v)$ and thus a unique solution to the nonlinear system (2.1.9).

Furthermore, the solution $(u_a^*(u,v), u_b^*(u,v))$ can be rewritten as

$$u_a^*(u,v) := r_a^*(u,v)u \qquad \text{and} \qquad u_b^*(u,v) := r_b^*(u,v)u, \tag{2.3.2}$$

with

$$r_a^*(u,v), r_b^*(u,v) \in (0,1), \qquad r_a^*(u,v) + r_b^*(u,v) = 1. \tag{2.3.3}$$

More precisely, from the nonlinear system (2.1.9) we have

$$\begin{cases} u_a = u - u_b, \\ \phi(bu_b + dv)(u - u_a) - \psi(au_a + cv)u_a = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} u_a = u - u_b, \\ u_a = \dfrac{\phi(bu_b + dv)}{\phi(bu_b + dv) + \psi(au_a + cv)} u, \end{cases}$$

thus giving

$$u_a^*(u,v) := \frac{\phi^*}{\phi^* + \psi^*} u \qquad \text{and} \qquad u_b^*(u,v) := \frac{\psi^*}{\phi^* + \psi^*} u, \tag{2.3.4}$$

with

$$\psi^* = \psi(au_a^*(u,v) + cv)\,, \qquad \text{and} \qquad \phi^* = \phi(bu_b^*(u,v) + dv).$$

Therefore, we obtain from (2.3.2), (2.3.4)

$$r_a^*(u,v) := \frac{\phi^*}{\phi^* + \psi^*} \in (0,1), \qquad r_b^*(u,v) := \frac{\psi^*}{\phi^* + \psi^*} \in (0,1)\,, \qquad u, v > 0. \tag{2.3.5}$$

We conclude by showing an explicit example of the triangular cross-diffusion system (2.1.7) - (2.1.12) with the conversion functions ϕ, ψ defined below for all $x > 0$

$$\phi(x) = x, \qquad \psi(x) = A + x, \qquad \text{with } A > 0,$$

corresponding to (H1), (H2) with $\alpha = \beta = 1$ and $B = 0$. Therefore, we compute from the nonlinear system (2.1.9)

$$(bu_b + dv)u_b - (A + a(u - u_b) + cv)(u - u_b) = 0,$$

i.e.

$$(b - a)u_b^2 + (dv + A + 2au + cv)u_b - (A + au + cv)u = 0.$$

In the particular case $a = b$, the computations above give the unique solution $(u_a^*(u, v), u_b^*(u, v))$

$$u_b^*(u, v) = \frac{A + au + cv}{A + 2au + (d + c)v}u \qquad \text{and} \qquad u_a^*(u, v) = \frac{au + dv}{A + 2au + (d + c)v}u.$$

2.4 Formal derivation from a microscopic fast switching mechanism

This paragraph aims to propose a formal derivation of the mesoscopic scale model (2.1.1) - (2.1.3), starting from a microscopic scale model where the ressources appear explicitly and induce the competition. We consider the following system

$$\begin{cases} \partial_t s_1 = \dfrac{1}{\delta}s_1\Big(r_1 - a_1 s_1 - p_1 u_1 - p_v^1 v\Big), \\ \partial_t s_2 = \dfrac{1}{\delta}s_2\Big(r_2 - a_2 s_2 - p_2 u_2 - p_v^2 v\Big), \\ \partial_t u_1 = d_1 \Delta u_1 + k_1 s_1 p_1 u_1 - \gamma_1 u_1 u_2 + \dfrac{1}{\varepsilon}\Big[\Phi\Big(\dfrac{p_2 u_2 + p_v^2\, v}{s_2}\Big)u_2 - \Psi\Big(\dfrac{p_1 u_1 + p_v^1\, v}{s_1}\Big)u_1\Big], \\ \partial_t u_2 = d_2 \Delta u_2 + k_2 s_2 p_2 u_2 - \gamma_2 u_1 u_2 - \dfrac{1}{\varepsilon}\Big[\Phi\Big(\dfrac{p_2 u_2 + p_v^2\, v}{s_2}\Big)u_2 - \Psi\Big(\dfrac{p_1 u_1 + p_v^1\, v}{s_1}\Big)u_1\Big], \\ \partial_t v = d_v \Delta v + k_v^1 s_1 p_v^1 v + k_v^2 s_2 p_v^2 v, \end{cases} \tag{2.4.1}$$

where $\delta > 0$ is the microscopic reaction time scale and ε is the mesoscopic one (hence $\delta \ll \varepsilon \ll 1$). Next, we make the asymptotic approximation $\delta \to 0$ with fixed $\varepsilon > 0$ for the prey/ resources densities s_1, s_2. Since, only the nontrivial case, i.e. $s_1 \neq 0 \neq s_2$, is meaningful (as the trivial cases $s_1 = 0$ and $s_2 = 0$ correspond to unstable equilibria), we obtain at a formal level

$$s_i = (r_i - p_i u_i - p_v^i v)a_i^{-1}, \qquad i = 1, 2.$$

Hence, the last three equations in (2.4.1) turn into

$$\begin{cases} \partial_t u_1 = d_1 \Delta u_1 + \frac{k_1 p_1}{a_1} u_1(r_1 - p_1 u_1 - p_v^1 v) - \gamma_1 u_1 u_2 + \frac{1}{\varepsilon}Q(u_1, u_2, v), \\ \partial_t u_2 = d_2 \Delta u_2 + \frac{k_2 p_2}{a_2} u_2(r_2 - p_2 u_2 - p_v^2 v) - \gamma_2 u_1 u_2 - \frac{1}{\varepsilon}Q(u_1, u_2, v), \\ \partial_t v = d_v \Delta v + \frac{k_v^1 p_v^1}{a_1} v(r_1 - p_1 u_1 - p_v^1 v) + \frac{k_v^2 p_v^2}{a_2} v(r_2 - p_2 u_2 - p_v^2 v), \end{cases}$$

where

$$Q(u_1, u_2, v) = \Phi\left(a_2\left(\frac{p_2 u_2 + p_v^2 v}{r_2 - (p_2 u_2 + p_v^2 v)}\right)\right)u_2 - \Psi\left(a_1\left(\frac{p_1 u_1 + p_v^1 v}{r_1 - (p_1 u_1 + p_v^1 v)}\right)\right)u_1.$$

Finally, renaming the variables u_1 and u_2 as u_a and u_b respectively, the diffusion coefficients d_1 and d_2 as d_a and d_b respectively, the intra-specific coefficients γ_1 and γ_2 as γ_a and γ_b respectively, defining the constants

$$\eta_a := \frac{k_1 p_1 r_1}{a_1}, \qquad a := \frac{p_1}{r_1}, \qquad c := \frac{p_v^1}{r_1},$$

$$\eta_b := \frac{k_2 p_2 r_2}{a_2}, \qquad b := \frac{p_2}{r_2}, \qquad d := \frac{p_v^2}{r_2},$$

$$\eta_v' := \frac{k_v^1 p_v^1 r_1}{a_1}, \qquad \eta_v'' := \frac{k_v^2 p_v^2 r_2}{a_2},$$

and the conversion rate functions

$$\phi(x) = \Phi\Big(a_2 \frac{x}{1-x}\Big), \qquad \psi(x) = \Psi\Big(a_1 \frac{x}{1-x}\Big),$$

we end up with the mesoscopic system (2.1.1) - (2.1.3).

2.5 Statement of the existence results

This section is devoted to the statement of the main existence results.

Proposition 2.5.1.
Let Ω be a smooth bounded domain of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$. We assume (H1) - (H4). Then, for any fixed $\varepsilon > 0$, the mesoscopic system (2.1.1) - (2.1.6) admits a unique classical positive (for each component) solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$.

Theorem 2.5.2.
Let Ω be a smooth bounded domain of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$. We assume (H1) - (H4). Then, the unique classical positive solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ of (2.1.1) - (2.1.13) converges for a.e. $(t,x) \in (0,+\infty) \times \Omega$ (up to extraction of a subsequence) towards a nonnegative triplet (u_a^*, u_b^*, v), as $\varepsilon \to 0$. Moreover, for a.e. $(t,x) \in (0,+\infty) \times \Omega$, the pair of function (u_a^*, u_b^*) is the unique solution to the nonlinear system (2.1.9), corresponding to $u := u_a^* + u_b^*$ and v. Furthermore, (u,v) is a very weak solution to the macroscopic system (2.1.7) – (2.1.13), in the sense that, for all test functions $\xi_1, \xi_2 \in C_c^2([0,+\infty) \times \bar{\Omega})$, with $\nabla\xi_1 \cdot \sigma = \nabla\xi_2 \cdot \sigma = 0$ on $[0,+\infty) \times \partial\Omega$, it holds

$$- \int_0^{+\infty} \int_\Omega (\partial_t \xi_1) u \, dx dt - \int_\Omega \xi_1(0,\cdot) u^{in} dx - \int_0^{+\infty} \int_\Omega \Delta\xi_1 \Big(d_a u_a^*(u,v) + d_b u_b^*(u,v)\Big) dx dt$$
$$= \int_0^{+\infty} \int_\Omega \xi_1 F_u(u,v) \, dx \, dt, \tag{2.5.1}$$

and

$$- \int_0^{+\infty} \int_\Omega (\partial_t \xi_2) \, v \, dx dt - \int_\Omega \xi_2(0,\cdot) \, v^{in} \, dx - d_v \int_0^{+\infty} \int_\Omega \Delta\xi_2 \, v \, dx dt$$
$$= \int_0^{+\infty} \int_\Omega \xi_2 \, F_v(u,v) \, dx dt. \tag{2.5.2}$$

Finally, the following regularity holds true, for all $T > 0$

(i) $u \in L^{\infty}(0,T; L^{2+\alpha}(\Omega)) \cap L^{3+\alpha}(\Omega_T)$, $|\nabla u| \in L^2(\Omega_T)$;

(ii) $v \in L^{\infty}(\Omega_T)$, $|\nabla v| \in L^{2(3+\alpha)}(\Omega_T)$; $\partial_{x_i,x_j} v\,, \partial_t v \in L^{3+\alpha}(\Omega_T)$, $i, j = 1, \ldots, \mathrm{N}$.

We observe that all terms in (2.5.1), (2.5.2) are well-defined thanks to the regularity of the solution (u, v). In particular, as the subpopulation densities u_a^*, u_b^* satisfy (2.3.2), (2.3.3), we have that u_a^*, u_b^* belong to $L^{3+\alpha}(\Omega_T)$. Moreover, the logistic structure of the reaction functions F_u, F_v involves at most quadratic nonlinearities, so that the integrals in (2.5.1), (2.5.2), containing F_u, F_v are well-defined.

2.6 Basic a priori-estimates

In this section we shall obtain *a priori* estimates for the subpopulation densities $u_a^\varepsilon, u_b^\varepsilon$ and for the total population densities $u^\varepsilon := u_a^\varepsilon + u_b^\varepsilon$ and v^ε. More specifically, we take advantage of the triangular structure of the system that allows us to prove *a priori* estimates for the density v^ε and its derivatives (see *Lemmas* 2.6.1, 2.6.2). The reaction functions f_a and f_b of competition type allow us to control the total mass $\int_\Omega u^\varepsilon(t)\,dx$, and to get an $L^2(\Omega_T)$ estimate on u^ε (see *Lemma* 2.6.3). The basic estimates shown in this section will be crucially used to get the energy a priori estimates in *Section 2.7*.

Hereafter, all constants C and C_T are strictly positive and may depend on Ω, the initial data $u_a^{\mathrm{in}}, u_b^{\mathrm{in}}, v^{\mathrm{in}}$, the coefficients in system (2.1.1), the transition functions ϕ, ψ and on T, but never on ε. They may change also from line to line in the computations. However, we can also introduce strictly positive constant that depends on its explicit argument: for example $C(\alpha, \beta, p)$ is a strictly positive constant depending on α, β, p.

In the following proofs, we shall drop the ε index for the sake of simplicity and we denote

$$\overline{\eta} := \max\{\eta_a, \eta_b\}, \qquad \eta := \min\{a\eta_a, b\eta_b\}, \tag{2.6.1}$$
$$\eta_v := \eta_v' + \eta_v'', \qquad r_v := c\,\eta_v' + d\,\eta_v''. \tag{2.6.2}$$

We observe that the hypothesis on the reaction coefficients imply $\bar{\eta}, \eta, \eta_v, r_v > 0$.

Lemma 2.6.1.
Let $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ *be the positive classical solution to* (2.1.1) - (2.1.6)*. Then, the following estimates hold true, for all* $p \in (1, \infty)$ *and for all* $\varepsilon > 0$,

$$\|v^\varepsilon\|_{L^\infty(0,+\infty;L^p(\Omega))} \le \max\left\{\|v^{in}\|_{L^p(\Omega)}, \frac{\eta_v}{r_v}|\Omega|^{\frac{1}{p}}\right\} =: K_p, \tag{2.6.3}$$

and

$$\|v^\varepsilon\|_{L^\infty((0,+\infty)\times\Omega)} \le \max\left\{\|v^{in}\|_{L^\infty(\Omega)}, \frac{\eta_v}{r_v}\right\} =: K_\infty. \tag{2.6.4}$$

Proof.
By multiplying the equation for v in (2.1.1) by $(v)^{p-1}$ with $p > 1$, integrating over Ω and

using $u_a, u_b > 0$, $v \geq 0$, we get

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega v^p\,dx &= d_v \int_\Omega v^{p-1}\Delta v dx + \int_\Omega v^{p-1} f_v(u_a,u_b,v)dx \\
&= -d_v(p-1)\int_\Omega v^{p-2}|\nabla v|^2 dx \\
&\quad + \eta_v' \int_\Omega v^p(1-au_a-cv)dx + \eta_v''\int_\Omega v^p(1-bu_b-dv)dx \\
&\leq -d_v\frac{4(p-1)}{p^2}\int_\Omega |\nabla(v^{p/2})|^2\,dx + \eta_v\int_\Omega v^p\,dx - r_v\int_\Omega v^{p+1}\,dx, \qquad (2.6.5)
\end{aligned}$$

using the definitions (2.6.2). Neglecting the first intergral in the r.h.s. of (2.6.5) and using in the last integral the Hölder's inequality below,

$$\int_\Omega v^p\,dx \leq \left(\int_\Omega v^{p+1}\,dx\right)^{\frac{p}{p+1}} |\Omega|^{\frac{1}{p+1}},$$

i.e.

$$-\int_\Omega v^{p+1}dx \leq -\frac{1}{|\Omega|^{\frac{1}{p}}}\left(\int_\Omega v^p dx\right)^{1+\frac{1}{p}},$$

we end up with

$$\frac{d}{dt}\int_\Omega v^p\,dx \leq \eta_v\,p\int_\Omega v^p\,dx - \frac{r_v\,p}{|\Omega|^{1/p}}\left(\int_\Omega v^p\,dx\right)^{1+\frac{1}{p}},$$

which can be rewritten as a differential inequality in the unknown $y(t) := \|v(t)\|^p_{L^p(\Omega)}$. Thus, we have for all $t > 0$

$$\frac{d}{dt}y(t) \leq \eta_v p y(t) - \frac{r_v p}{|\Omega|^{\frac{1}{p}}} y(t)^{1+\frac{1}{p}},$$

that gives by integrating over $[0,t]$,

$$y^{\frac{1}{p}}(t) \leq \frac{\eta_v}{r_v}|\Omega|^{\frac{1}{p}} \frac{y^{\frac{1}{p}}(0)}{y^{\frac{1}{p}}(0) + \left(\frac{\eta_v}{r_v}|\Omega|^{\frac{1}{p}} - y^{\frac{1}{p}}(0)\right)e^{-\eta_v t}} \leq \max\left\{y^{\frac{1}{p}}(0), \frac{\eta_v}{r_v}|\Omega|^{\frac{1}{p}}\right\} = K_p,$$

i.e. for all $t > 0$

$$\|v(t)\|_{L^p(\Omega)} \leq \max\left\{\|v^{\text{in}}\|_{L^p(\Omega)}, \frac{\eta_v}{r_v}|\Omega|^{\frac{1}{p}}\right\} = K_p, \qquad (2.6.6)$$

implying (2.6.3). Taking $p \to +\infty$ in (2.6.6) and the supremum in time, we get (2.6.4).

□

Lemma 2.6.2 (Maximal regularity for v^ε)**.**
Let $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ be the positive classical solution to (2.1.1) - (2.1.6). Then, the following statements hold true

(i) for all $p \in (1,+\infty)$ there exists a constant $C > 0$ depending on p, v^{in}, Ω, such that, for all $\varepsilon > 0$, $T > 0$ and all $i, j = 1, .., \mathrm{N}$,

$$\|\partial_t v^\varepsilon\|_{L^p(\Omega_T)} + \|\partial_{x_i,x_j} v^\varepsilon\|_{L^p(\Omega_T)} \leqslant C(1 + \|u_a^\varepsilon + u_b^\varepsilon\|_{L^p(\Omega_T)})\,; \qquad (2.6.7)$$

(ii) *for all* $p \in (1, +\infty)$ *there exists a constant* $C > 0$ *depending on* p, v^{in}, N, Ω*, such that, for all* $\varepsilon > 0$ *and all* $T > 0$*,*

$$\|\nabla v^\varepsilon\|^{2p}_{L^{2p}(\Omega_T)} \leq C(1 + T + \|u^\varepsilon_a + u^\varepsilon_b\|^p_{L^p(\Omega_T)}). \tag{2.6.8}$$

For the proof of *Lemma 2.6.2* we refer to the proof of *Lemma 1.3.1*.

Lemma 2.6.3.
Let $(u^\varepsilon_a, u^\varepsilon_b, v^\varepsilon)$ *be the positive classical solution to* (2.1.1) - (2.1.6)*. Then, the following estimates hold true for all* $\varepsilon > 0$ *and* $T > 0$*,*

$$\sup_{t>0} \|u^\varepsilon_a(t) + u^\varepsilon_b(t)\|_{L^1(\Omega)} \leq \max\left\{\|u^{in}_a + u^{in}_b\|_{L^1(\Omega)}, 2|\Omega|\overline{\eta}\eta^{-1}\right\} := \mathcal{K}, \tag{2.6.9}$$

and

$$\|u^\varepsilon_a\|^2_{L^2(\Omega_T)} + \|u^\varepsilon_b\|^2_{L^2(\Omega_T)} \leq \eta^{-1}\|u^{in}_a + u^{in}_b\|_{L^1(\Omega)} + \overline{\eta}\eta^{-1}\,\mathcal{K}\,T. \tag{2.6.10}$$

Proof.
We integrate over Ω the equation satisfied by $u_a + u_b$ in (2.1.1). Then, using the definition (2.6.1), the boundary condition (2.1.4), the competition structure of the reaction functions f_a, f_b and the nonnegativity of u_a, u_b, v, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u_a + u_b)dx \\
&\qquad = \int_\Omega (d_a\Delta u_a + d_b\Delta u_b)\,dx + \int_\Omega (f_a(u_a, u_b, v) + f_b(u_a, u_b, v))\,dx \\
&\qquad = \int_\Omega (\eta_a u_a(1 - au_a - cv) - \gamma_a u_a u_b)\,dx + \int_\Omega (\eta_b u_b(1 - bu_b - dv) - \gamma_b u_a u_b)\,dx \\
&\qquad \leq \overline{\eta}\int_\Omega (u_a + u_b)dx - \eta\int_\Omega \left(u_a^2 + u_b^2\right)dx \qquad\qquad (2.6.11)\\
&\qquad \leq \overline{\eta}\int_\Omega (u_a + u_b)dx - \frac{\eta}{2}\int_\Omega \left(u_a^2 + u_b^2\right)dx - \eta\int_\Omega u_a u_b dx \\
&\qquad = \overline{\eta}\int_\Omega (u_a + u_b)dx - \frac{\eta}{2}\int_\Omega (u_a + u_b)^2 dx, \qquad\qquad (2.6.12)
\end{aligned}$$

by the inequality $2xy \leq x^2 + y^2$ for all $x, y \in \mathbb{R}_+$. Then, we apply to the second integral in (2.6.12) the Cauchy-Schwarz inequality below

$$\|u_a + u_b\|^2_{L^1(\Omega)} \leq |\Omega|\|u_a + u_b\|^2_{L^2(\Omega)},$$

to get

$$\frac{d}{dt}\int_\Omega (u_a + u_b)dx \leq \overline{\eta}\|u_a + u_b\|_{L^1(\Omega)} - \frac{\eta}{2|\Omega|}\|u_a + u_b\|^2_{L^1(\Omega)},$$

that can be rewritten in terms of a differential inequality for $y(t) = \|u_a(t) + u_b(t)\|_{L^1(\Omega)}$. Thus, integrating the above inequality over $[0, t]$ for all $t > 0$

$$y(t) \leq \frac{y(0)}{\frac{\eta}{2|\Omega|\overline{\eta}}y(0) + \left(1 - \frac{\eta}{2|\Omega|\overline{\eta}}y(0)\right)e^{-\overline{\eta}\,t}} \leq \max\left\{y(0), \frac{2|\Omega|\overline{\eta}}{\eta}\right\},$$

giving the uniform estimate (2.6.9).

In order to prove (2.6.10), we integrate (2.6.11) over $(0, T)$ and we obtain

$$\eta \left(\|u_a\|^2_{L^2(\Omega_T)} + \|u_b\|^2_{L^2(\Omega_T)} \right) \le \|u_a^{\text{in}} + u_b^{\text{in}}\|_{L^1(\Omega)} + \bar{\eta} \int_{(0,T)} \|(u_a + u_b)(t)\|_{L^1(\Omega)} dt$$
$$\le \|u_a^{\text{in}} + u_b^{\text{in}}\|_{L^1(\Omega)} + \bar{\eta} T \mathcal{K},$$

where we use in the last inequality the estimate (2.6.9). Thus, we conclude. □

2.7 Energy estimates

In this section, we will obtain the energy a priori estimates by studying the evolution of the energy functional (2.2.1), (2.2.2) in the following three cases

$$p = p_\beta := 1 + \frac{1}{1+\beta} \in (1,2), \qquad p = p_\alpha := 1 + \frac{1}{1+\alpha} \in (1,2), \qquad p = 2, \tag{2.7.1}$$

with $p_\beta \le p_\alpha$, by assumption (H2). The case $p = 1$ has been analysed in *Lemma 2.6.3*. For each value of p in (2.7.1), we will show $\varepsilon-$uniform a priori estimates by using the estimates obtained in the previous step, i.e. in the case $p = p_\beta$ we will use the a priori estimates of *Lemmas 2.6.1 - 2.6.3*, the estimates for $p = p_\alpha$ are obtained thanks to the estimates shown for $p = p_\beta$ and finally, in the case $p = 2$, we will use the obtained estimates in the cases $p = p_\beta$ and $p = p_\alpha$.

Before stating the main result of this section (see *Energy Lemma 2.7.1* below), we introduce the following quantities for all $p \ge 1$,

$$q(p) := \alpha(p-1) + p \ge 1 \qquad \text{and} \qquad r(p) := \beta(p-1) + p \ge 1, \tag{2.7.2}$$

so that

$$q(p_\beta) = \frac{\alpha+1}{\beta+1} + 1 \in (1,2] \qquad \text{and} \qquad r(p_\beta) = 2, \tag{2.7.3}$$
$$q(p_\alpha) = 2 \qquad \text{and} \qquad r(p_\alpha) = \frac{\beta+1}{\alpha+1} + 1 \ge 2, \tag{2.7.4}$$
$$q(2) = \alpha + 2 \qquad \text{and} \qquad r(2) = \beta + 2, \tag{2.7.5}$$

and the following relation holds

$$q(p_\beta) - 1 = \frac{1}{r(p_\alpha) - 1}. \tag{2.7.6}$$

Lemma 2.7.1 (Energy Lemma)**.**
Let $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ be the positive classical solution to (2.1.1) - (2.1.6). Then, there exists $C_T > 0$, such that the global strong solution to (2.1.1) - (2.1.6) satisfies, for all $\varepsilon > 0$,

$$\mathcal{E}_{p_\beta}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T) + C\left(\|\nabla u_b^\varepsilon\|^2_{L^2(\Omega_T)} + \|u_b^\varepsilon\|^3_{L^3(\Omega_T)} + \|u_a^\varepsilon\|^{q(p_\beta)+1}_{L^{q(p_\beta)+1}(\Omega)} \right)$$
$$\le \mathcal{E}_{p_\beta}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T, \tag{2.7.7}$$

$$\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T) + C\left(\|\nabla u_a^\varepsilon\|^2_{L^2(\Omega_T)} + \|u_a^\varepsilon\|^3_{L^3(\Omega_T)} + \|u_b^\varepsilon\|^{r(p_\alpha)+1}_{L^{r(p_\alpha)+1}(\Omega_T)} \right)$$
$$\le \mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T, \tag{2.7.8}$$

and

$$\begin{aligned}\mathcal{E}_2(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T) &+ C\left(\|\nabla(u_a^\varepsilon)^{\alpha/2+1}\|^2_{L^2(\Omega_T)} + \|\nabla(u_b^\varepsilon)^{\beta/2+1}\|^2_{L^2(\Omega_T)}\right)\\ &+ C\left(\|u_a^\varepsilon\|^{3+\alpha}_{L^{3+\alpha}(\Omega_T)} + \|u_b^\varepsilon\|^{3+\beta}_{L^{3+\beta}(\Omega_T)}\right) + \frac{1}{\varepsilon}\|Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\|^2_{L^2(\Omega_T)}\\ &\leq \mathcal{E}_2(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T. \end{aligned} \tag{2.7.9}$$

Before starting the proof, we shall point out some comments and introduce some quantities that will be useful in the sequel. Firstly, we introduce the affine functions

$$\theta(z, v) := A + az + cv, \qquad \omega(z, v) := B + bz + dv. \tag{2.7.10}$$

By (H1) and (2.7.10), the total energy density (2.2.1) – (2.2.4) reads now as

$$\begin{aligned} h_p(u_a, u_b, v) &= h_{a,p}(u_a, v) + h_{b,p}(u_b, v)\\ &= \int_0^{u_a} \theta(z, v)^{\alpha(p-1)} z^{p-1} dz + \int_0^{u_b} \omega(z, v)^{\beta(p-1)} z^{p-1} dz. \end{aligned} \tag{2.7.11}$$

Thus, neglecting A, B, v in (2.7.10) and (2.7.11), we obtain for all $p \geq 1$,

$$\begin{aligned} h_p(u_a, u_b, v) &\geq \int_0^{u_a} (az)^{\alpha(p-1)} z^{p-1} dz + \int_0^{u_b} (bz)^{\beta(p-1)} z^{p-1} dz\\ &\geq C(a, b, p, \alpha, \beta)\left(u_a^{q(p)} + u_b^{r(p)}\right). \end{aligned}$$

Hence, any uniform in time upper bound of the energy $\mathcal{E}_p(t)$ over $(0, T)$ implies the boundedness of

$$\|u_a\|_{L^\infty(0,T;L^{q(p)}(\Omega))} + \|u_b\|_{L^\infty(0,T;L^{r(p)}(\Omega))}.$$

It is worth noticing that, thanks to the choice of ϕ and ψ in (H1), (H2), the integrals in (2.7.11), defining the densities $h_{a,p}(u_a, v)$ and $h_{b,p}(u_b, v)$, are finite for all $p \geq 1$. The same holds true for

$$\nabla h_p(u_a, u_b, v) = \left(\partial_1 h_{a,p}(u_a, v), \partial_1 h_{b,p}(u_b, v), \partial_2 h_{a,p}(u_a, v) + \partial_2 h_{b,p}(u_a, v)\right),$$

where

$$\begin{aligned} \partial_1 h_{a,p}(u_a, v) &= \theta(u_a, v)^{\alpha(p-1)} u_a^{p-1},\\ \partial_1 h_{b,p}(u_b, v) &= \omega(u_b, v)^{\beta(p-1)} u_b^{p-1}, \end{aligned} \tag{2.7.12}$$

and

$$\begin{aligned} \partial_2 h_{a,p}(u_a, v) &= c\alpha(p-1)\int_0^{u_a} \theta(z, v)^{\alpha(p-1)-1} z^{p-1} dz,\\ \partial_2 h_{b,p}(u_b, v) &= d\beta(p-1)\int_0^{u_b} \omega(z, v)^{\beta(p-1)-1} z^{p-1} dz. \end{aligned} \tag{2.7.13}$$

Concerning the Hessian of h_p, we have

$$\text{Hess}(h_p) = \begin{pmatrix} \partial_{11}h_p & 0 & \partial_{13}h_p\\ 0 & \partial_{22}h_p & \partial_{23}h_p\\ \partial_{31}h_p & \partial_{32}h_p & \partial_{33}h_p \end{pmatrix}$$

$$= \begin{pmatrix} \partial_{11}h_{a,p} & 0 & \partial_{12}h_{a,p}\\ 0 & 0 & 0\\ \partial_{21}h_{a,p} & 0 & \partial_{22}h_{a,p} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0\\ 0 & \partial_{11}h_{b,p} & \partial_{12}h_{b,p}\\ 0 & \partial_{21}h_{b,p} & \partial_{22}h_{b,p} \end{pmatrix},$$

where

$$\begin{aligned}\partial_{11}h_{a,p}(u_a,v) &= a\alpha(p-1)\theta(u_a,v)^{\alpha(p-1)-1}u_a^{p-1} + (p-1)\theta(u_a,v)^{\alpha(p-1)}u_a^{p-2},\\ \partial_{12}h_{a,p}(u_a,v) &= \partial_{21}h_{a,p}(u_a,v) = c\alpha(p-1)\theta(u_a,v)^{\alpha(p-1)-1}u_a^{p-1},\end{aligned} \tag{2.7.14}$$

and

$$\begin{aligned}\partial_{11}h_{b,p}(u_b,v) &= b\beta(p-1)\omega(u_b,v)^{\beta(p-1)-1}u_b^{p-1} + (p-1)\omega(u_b,v)^{\beta(p-1)}u_b^{p-2},\\ \partial_{12}h_{b,p}(u_b,v) &= \partial_{21}h_{b,p}(u_b,v) = d\beta(p-1)\omega(u_b,v)^{\beta(p-1)-1}u_b^{p-1},\end{aligned} \tag{2.7.15}$$

and finally

$$\begin{aligned}\partial_{22}h_{a,p}(u_a,v) &= c^2\alpha(p-1)(\alpha(p-1)-1)\int_0^{u_a}\theta(z,v)^{\alpha(p-1)-2}z^{p-1}dz,\\ \partial_{22}h_{b,p}(u_b,v) &= d^2\beta(p-1)(\beta(p-1)-1)\int_0^{u_b}\omega(z,v)^{\beta(p-1)-2}z^{p-1}dz.\end{aligned} \tag{2.7.16}$$

The derivatives $\partial_{22}h_{a,p}$, $\partial_{22}h_{b,p}$ in (2.7.16) are well-defined if and only if $p > p_\alpha$ and $p > p_\beta$, respectively. Therefore, in order to estimate the evolution of the energy $\mathcal{E}_p$ in (2.2.7), with p in (2.7.1), along the solution to (2.1.1) – (2.1.6), we will handle separately the first critical case $p = p_\beta$, the second critical case $p = p_\alpha$ and the super-critical case $p = 2 > p_\alpha$.

For the reader's convenience, we rewrite here the decomposition (2.2.5) - (2.2.7) of the evolution equation of the energy functional as following

$$\frac{d}{dt}\mathcal{E}_p(u_a^\varepsilon,u_b^\varepsilon,v^\varepsilon) = \int_\Omega (d_a\partial_1 h_p\Delta u_a^\varepsilon + d_b\partial_2 h_p\Delta u_b^\varepsilon + d_v\partial_3 h_p\Delta v^\varepsilon)dx \tag{2.7.17}$$

$$+ \int_\Omega \nabla h_p\cdot\mathcal{F}\,dx \tag{2.7.18}$$

$$+ \frac{1}{\varepsilon}\int_\Omega(\partial_1 h_p - \partial_2 h_p)Q\,dx \tag{2.7.19}$$

$$=: I^p_{diff} + I^p_{rea} + I^p_{fast}. \tag{2.7.20}$$

The following two subsections are devoted to estimate the reaction terms I^p_{rea} and the diffusion terms I^p_{diff}, respectively, depending on the value of p in (2.7.1). For the sake of simplicity, we will drop the ε index.

We conclude with the definitions below, for $l, m \in \mathbb{R}$,

$$\mathcal{X}_l(m) := \begin{cases}1, & \text{if } m \ge l,\\ 0, & \text{if } m < l.\end{cases} \tag{2.7.21}$$

2.7.1 A priori estimate for the reaction terms

This paragraph is devoted to estimate the reaction term $I^p_{rea} := \int_\Omega \nabla h_p\cdot\mathcal{F}\,dx$ in (2.7.18) with p in (2.7.1). In order to do that, we use the competition form of the reaction term $\mathcal{F}$ and the following technical lemma, whose proof is given in *Section A.3*.

Lemma 2.7.2.
For any $\gamma > 0$ and $\delta \in (0,1)$, there exists $C(\gamma,\delta) > 0$ such that for all $\lambda > 0, \eta \ge 0$, it holds

$$\eta^\gamma(\lambda-\eta) \le C(\gamma,\delta)\lambda^{\gamma+1} - \delta\eta^{\gamma+1}. \tag{2.7.22}$$

Moreover, the best constant in (2.7.22) *is*

$$C(\gamma,\delta)=\frac{1}{\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{1-\delta}\right)^{\gamma}.$$

Hereafter, we will consider $\delta=\frac{1}{2}$ when applying *Lemma 2.7.2* and we will denote $C_L:=C(\gamma,\frac{1}{2})$.

Recalling that $\mathcal{F}:=(f_a,f_b,f_v)^T$, we compute from (2.7.18)

$$\begin{aligned}I^p_{rea}&=\int_\Omega \partial_1 h_{a,p}(u_a,v)f_a(u_a,u_b,v)\,dx+\int_\Omega \partial_1 h_{b,p}(u_b,v)f_b(u_a,u_b,v)\,dx\\&\quad+\int_\Omega \partial_2 h_{a,p}(u_a,v)f_v(u_a,u_b,v)\,dx+\int_\Omega \partial_2 h_{b,p}(u_b,v)f_v(u_a,u_b,v)\,dx\\&:=J^p_1+J^p_2+J^p_3+J^p_4.\end{aligned}\tag{2.7.23}$$

By (2.7.12) and recalling the reaction functions defined in (2.1.2), we get

$$\begin{aligned}J^p_1&:=\int_\Omega \theta(u_a,v)^{\alpha(p-1)}u_a^{p-1}\,f_a(u_a,u_b,v)\,dx\\&\le\eta_a\int_\Omega \theta(u_a,v)^{\alpha(p-1)}u_a^{p-1}\,u_a(1+A-\theta(u_a,v))\,dx\\&=\eta_a\int_\Omega u_a^p\Big(\theta(u_a,v)^{\alpha(p-1)}(1+A-\theta(u_a,v))\Big)\,dx.\end{aligned}\tag{2.7.24}$$

Applying the inequality (2.7.22) in (2.7.24), using $\theta(u_a,v)\ge au_a$ and the definition of $q(p)$ in (2.7.2), we obtain

$$\begin{aligned}J^p_1&\le\eta_a\int_\Omega u_a^p\left(C_L(1+A)^{\alpha(p-1)+1}-\frac{1}{2}\,\theta(u_a,v)^{\alpha(p-1)+1}\right)\,dx\\&\le\eta_aC_L(1+A)^{\alpha(p-1)+1}\int_\Omega u_a^p\,dx-\frac{1}{2}\eta_a\,a^{\alpha(p-1)+1}\int_\Omega u_a^{q(p)+1}\,dx.\end{aligned}\tag{2.7.25}$$

Similarly, we have for J^p_2

$$\begin{aligned}J^p_2&:=\int_\Omega \omega(u_b,v)^{\beta(p-1)}u_b^{p-1}\,f_b(u_a,u_b,v)\,dx\\&\le\eta_bC_L(1+B)^{\beta(p-1)+1}\int_\Omega u_b^p\,dx-\frac{1}{2}\eta_b\,b^{\beta(p-1)+1}\int_\Omega u_b^{r(p)+1}\,dx.\end{aligned}\tag{2.7.26}$$

The terms J^p_3 and J^p_4 in (2.7.23) will be estimated in the critical cases $p=p_\beta$, $p=p_\alpha$ and in the super-critical case $p=2$ in the next three paragraphs, respectively.

- I_{rea}^{p} **in the critical case** $p = p_\beta$

Using $\theta(z,v) \geq az$ and recalling that $\eta_v = \eta_v' + \eta_v''$, we get

$$\begin{aligned}
J_3^{p_\beta} &:= c\alpha(p_\beta - 1)\int_\Omega\left(\int_0^{u_a}\theta(z,v)^{\alpha(p_\beta-1)-1}z^{p_\beta-1}dz\right)f_v(u_a,u_b,v)\,dx\\
&\leq (\eta_v' + \eta_v'')\frac{c\alpha}{\beta+1}\int_\Omega v\left(\int_0^{u_a}\theta(z,v)^{\frac{\alpha}{\beta+1}-1}z^{\frac{1}{\beta+1}}dz\right)dx\\
&\leq \eta_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\frac{\alpha}{\beta+1}\int_\Omega v\left(\int_0^{u_a}\left(\frac{z}{z^{\beta+1-\alpha}}\right)^{\frac{1}{\beta+1}}dz\right)dx\\
&= \eta_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\frac{\alpha}{\beta+1}\frac{\beta+1}{\alpha+1}\int_\Omega u_a^{\frac{\alpha+1}{\beta+1}}v\,dx\\
&= \eta_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\frac{\alpha}{\alpha+1}\int_\Omega u_a^{\frac{\alpha+1}{\beta+1}}v\,dx.
\end{aligned}\tag{2.7.27}$$

Then, using the inequality

$$x^\gamma \leq 1 + x, \qquad \forall x \geq 0 \quad \text{and} \quad 0 < \gamma \leq 1, \tag{2.7.28}$$

in (2.7.27) with $\gamma = \frac{\alpha+1}{\beta+1}$, we end up with

$$J_3^{p_\beta} \leq \eta_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\|v\|_{L^\infty(\Omega)}(1 + \|u_a\|_{L^1(\Omega)}). \tag{2.7.29}$$

Concerning the term $J_4^{p_\beta}$, using $\omega(z,v) \geq bz$, we get

$$\begin{aligned}
J_4^{p_\beta} &:= d\beta(p_\beta - 1)\int_\Omega\left(\int_0^{u_b}\omega(z,v)^{\beta(p_\beta-1)-1}z^{p_\beta-1}dz\right)f_v(u_a,u_b,v)dx\\
&\leq (\eta_v' + \eta_v'')\frac{d\beta}{\beta+1}\int_\Omega v\left(\int_0^{u_b}\omega(z,v)^{\frac{\beta}{\beta+1}-1}z^{\frac{1}{\beta+1}}dz\right)dx\\
&\leq \eta_v\frac{d}{b^{\frac{1}{\beta+1}}}\int_\Omega v\left(\int_0^{u_b}\omega(z,v)^{\frac{\beta}{\beta+1}-1}\omega(z,v)^{\frac{1}{\beta+1}}dz\right)dx\\
&= \eta_v\frac{d}{b^{\frac{1}{\beta+1}}}\int_\Omega vu_b\,dx\\
&\leq \eta_v\frac{d}{b^{\frac{1}{\beta+1}}}\|v\|_{L^\infty(\Omega)}\|u_b\|_{L^1(\Omega)}.
\end{aligned}\tag{2.7.30}$$

Therefore, taking $p = p_\beta$ in (2.7.25), (2.7.26), recalling that $r(p_\beta) + 1 = 3$ and gathering with (2.7.29), (2.7.30), we obtain

$$\begin{aligned}
I_{rea}^{p_\beta} &\leq \eta_a C_L(1+A)^{\frac{\alpha}{\beta+1}+1}\|u_a\|_{L^{p_\beta}(\Omega)}^{p_\beta} - \frac{1}{2}\eta_a\,a^{\frac{\alpha}{\beta+1}+1}\|u_a\|_{L^{q(p_\beta)+1}(\Omega)}^{q(p_\beta)+1}\\
&+ \eta_b C_L(1+B)^{\frac{\beta}{\beta+1}+1}\|u_b\|_{L^{p_\beta}(\Omega)}^{p_\beta} - \frac{1}{2}\eta_b b^{\frac{\beta}{\beta+1}+1}\|u_b\|_{L^3(\Omega)}^3\\
&+ \eta_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\|v\|_{L^\infty(\Omega)}(1 + \|u_a\|_{L^1(\Omega)}) + \eta_v\frac{d}{b^{\frac{1}{\beta+1}}}\|v\|_{L^\infty(\Omega)}\|u_b\|_{L^1(\Omega)}.
\end{aligned}\tag{2.7.31}$$

- I^p_{rea} **in the critical case** $p = p_\alpha$

We estimate $J_3^{p_\alpha}$ similarly as we did for $J_4^{p_\beta}$ in (2.7.29), so that we have

$$\begin{aligned} J_3^{p_\alpha} &:= c\alpha(p_\alpha - 1)\int_\Omega \left(\int_0^{u_a} \theta(z,v)^{\alpha(p_\alpha-1)-1} z^{p_\alpha-1}dz\right) f_v(u_a,u_b,v)dx \\ &\le \eta_v c\frac{\alpha}{\alpha+1}\int_\Omega v\left(\int_0^{u_a}\theta(z,v)^{-\frac{1}{\alpha+1}} z^{\frac{1}{\alpha+1}}dz\right)dx \\ &\le \eta_v \frac{c}{a^{\frac{1}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|u_a\|_{L^1(\Omega)}. \end{aligned} \tag{2.7.32}$$

Concerning $J_4^{p_\alpha}$, we get

$$\begin{aligned} J_4^{p_\alpha} &:= d\beta(p_\alpha - 1)\int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\beta(p_\alpha-1)-1} z^{p_\alpha-1}dz\right) f_v(u_a,u_b,v)dx \\ &\le d\eta_v \frac{\beta}{\alpha+1}\int_\Omega v\left(\int_0^{u_b}\omega(z,v)^{\frac{\beta}{\alpha+1}-1} z^{\frac{1}{\alpha+1}}dz\right)dx. \end{aligned} \tag{2.7.33}$$

To continue, we need to distiguish the cases $\beta < 1+\alpha$ and $\beta \ge 1+\alpha$, i.e. $\frac{\beta}{\alpha+1} - 1 < 0$ and $\frac{\beta}{\alpha+1} - 1 \ge 0$, respectively.

Case $\beta < 1+\alpha$. Using $\omega(z,v) \ge bz$, we compute

$$\int_0^{u_b}\omega(z,v)^{\frac{\beta}{\alpha+1}-1} z^{\frac{1}{\alpha+1}}dz = \int_0^{u_b}\frac{z^{\frac{1}{\alpha+1}}}{\omega(z,v)^{1-\frac{\beta}{\alpha+1}}}dz \le \frac{1}{b^{1-\frac{\beta}{\alpha+1}}}\int_0^{u_b} z^{\frac{\beta+1}{\alpha+1}-1}dz \le \frac{1}{b^{1-\frac{\beta}{\alpha+1}}}u_b^{\frac{\beta+1}{\alpha+1}}.$$

Therefore, recalling (2.7.4), $J_4^{p_\alpha}$ is estimated as follows

$$\begin{aligned} J_4^{p_\alpha} &\le \frac{d\eta_v}{b^{1-\frac{\beta}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|u_b\|^{\frac{\beta+1}{\alpha+1}}_{L^{\frac{\beta+1}{\alpha+1}}(\Omega)} \\ &= \frac{d\eta_v}{b^{1-\frac{\beta}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|u_b\|^{r(p_\alpha)-1}_{L^{r(p_\alpha)-1}(\Omega)}. \end{aligned} \tag{2.7.34}$$

Case $\beta \ge 1+\alpha$. Using again $\omega(z,v) \ge bz$ and recalling (2.7.4), we compute from (2.7.33)

$$\begin{aligned} J_4^{p_\alpha} &\le \eta_v \frac{d}{b^{\frac{1}{\alpha+1}}}\frac{\beta}{\alpha+1}\int_\Omega v\left(\int_0^{u_b}\omega(z,v)^{\frac{\beta+1}{\alpha+1}-1}dz\right)dx \\ &\le \eta_v\frac{d}{b^{1+\frac{1}{\alpha+1}}}\frac{\beta}{\alpha+1}\frac{\alpha+1}{\beta+1}\int_\Omega v\omega(u_b,v)^{\frac{\beta+1}{\alpha+1}}dx \\ &\le \eta_v\frac{d}{b^{1+\frac{1}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|\omega(u_b,v)\|^{r(p_\alpha)-1}_{L^{r(p_\alpha)-1}(\Omega)}. \end{aligned}$$

Since $r(p_\alpha) - 1 \ge 1$, we can apply to $\|\omega(u_b,v)\|^{r(p_\alpha)-1}_{L^{r(p_\alpha)-1}(\Omega)}$ in the above estimate the inequality

$$(x+y)^p \le C_J\left(x^p + y^p\right), \qquad \forall\ x,y \in \mathbb{R}^+ \text{ and } \ p \ge 1, \tag{2.7.35}$$

with $C_J > 0$, to get

$$J_4^{p_\alpha} \le \eta_v C_J\frac{d}{b^{1+\frac{1}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\left(\|B + dv\|^{r(p_\alpha)-1}_{L^{r(p_\alpha)-1}(\Omega)} + b^{r(p_\alpha)-1}\|u_b\|^{r(p_\alpha)-1}_{L^{r(p_\alpha)-1}(\Omega)}\right). \tag{2.7.36}$$

By gathering (2.7.34), (2.7.36) and using (2.7.21), we estimate $J_4^{p_\alpha}$ as follows

$$J_4^{p_\alpha} \leq (1-\mathcal{X}_{\alpha+1}(\beta))\,\frac{d\eta_v}{b^{1-\frac{\beta}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|u_b\|_{L^{r(p_\alpha)-1}(\Omega)}^{r(p_\alpha)-1}$$
$$+\mathcal{X}_{\alpha+1}(\beta)\eta_v C_J\frac{d}{b^{1+\frac{1}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\left(\|B+dv\|_{L^{r(p_\alpha)-1}(\Omega)}^{r(p_\alpha)-1}+b^{r(p_\alpha)-1}\|u_b\|_{L^{r(p_\alpha)-1}(\Omega)}^{r(p_\alpha)-1}\right). \quad (2.7.37)$$

Finally, taking $p=p_\alpha$ in (2.7.25), (2.7.26), recalling that $q(p_\alpha)+1=3$ and gathering with (2.7.32), (2.7.37), we obtain

$$I_{rea}^{p_\alpha} \leq \eta_a C_L(1+A)^{\frac{\alpha}{\alpha+1}+1}\|u_a\|_{L^{p_\alpha}(\Omega)}^{p_\alpha}-\frac{1}{2}\eta_a a^{\frac{\alpha}{\alpha+1}+1}\|u_a\|_{L^3(\Omega)}^3$$
$$+\eta_b C_L(1+B)^{\frac{\beta}{\alpha+1}+1}\|u_b\|_{L^{p_\alpha}(\Omega)}^{p_\alpha}-\frac{1}{2}\eta_b b^{\frac{\beta}{\alpha+1}+1}\|u_b\|_{L^{r(p_\alpha)+1}(\Omega)}^{r(p_\alpha)+1}$$
$$+\eta_v\frac{c}{a^{\frac{1}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|u_a\|_{L^1(\Omega)}$$
$$+(1-\mathcal{X}_{\alpha+1}(\beta))\,\frac{d\eta_v}{b^{1-\frac{\beta}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\|u_b\|_{L^{r(p_\alpha)-1}(\Omega)}^{r(p_\alpha)-1}$$
$$+\mathcal{X}_{\alpha+1}(\beta)\eta_v C_J\frac{d}{b^{1+\frac{1}{\alpha+1}}}\|v\|_{L^\infty(\Omega)}\left(\|B+dv\|_{L^{r(p_\alpha)-1}(\Omega)}^{r(p_\alpha)-1}+b^{r(p_\alpha)-1}\|u_b\|_{L^{r(p_\alpha)-1}(\Omega)}^{r(p_\alpha)-1}\right). \quad (2.7.38)$$

- **I_{rea}^p in the super-critical case** $p=2$

Concerning J_3^2, we get

$$J_3^2 := c\alpha\int_\Omega\left(\int_0^{u_a}\theta(z,v)^{\alpha-1}z\,dz\right)f_v(u_a,u_b,v)dx$$
$$\leq c\alpha\eta_v\int_\Omega v\left(\int_0^{u_a}\theta(z,v)^{\alpha-1}z\,dz\right)dx. \quad (2.7.39)$$

Thus, similarly as for $J_4^{p_\alpha}$ in (2.7.33), we distiguish the cases $\alpha<1$ and $\alpha\geq 1$.

Case $\alpha<1$. Using $\theta(z,v)\geq az$, we compute

$$\int_0^{u_a}\theta(z,v)^{\alpha-1}z\,dz=\int_0^{u_a}\frac{z}{\theta(z,v)^{1-\alpha}}dz\leq\frac{1}{a^{1-\alpha}}\int_0^{u_a}z^\alpha dz=\frac{1}{a^{1-\alpha}}\frac{u_a^{\alpha+1}}{\alpha+1}.$$

Therefore, recalling (2.7.5), J_3^2 is estimated as follows

$$J_3^2\leq\frac{c}{a^{1-\alpha}}\eta_v\|v\|_{L^\infty(\Omega)}\|u_a\|_{L^{\alpha+1}(\Omega)}^{\alpha+1}$$
$$=\frac{c}{a^{1-\alpha}}\eta_v\|v\|_{L^\infty(\Omega)}\|u_a\|_{L^{q(2)-1}(\Omega)}^{q(2)-1}. \quad (2.7.40)$$

Case $\alpha\geq 1$. Using $\theta(z,v)\geq az$ again, we compute

$$J_3^2\leq\frac{c}{a}\alpha\eta_v\int_\Omega v\left(\int_0^{u_a}\theta(z,v)^\alpha dz\right)dx$$
$$\leq c\eta_v\frac{\alpha}{\alpha+1}\int_\Omega v\,\theta(u_a,v)^{\alpha+1}dx$$
$$\leq c\eta_v\|v\|_{L^\infty(\Omega)}\|\theta(u_a,v)\|_{L^{q(2)-1}(\Omega)}^{q(2)-1}$$
$$\leq c\eta_v C_J\|v\|_{L^\infty(\Omega)}\left(\|A+cv\|_{L^{q(2)-1}(\Omega)}^{q(2)-1}+a^{\alpha+1}\|u_a\|_{L^{q(2)-1}(\Omega)}^{q(2)-1}\right), \quad (2.7.41)$$

where we applied again (2.7.35) in the last estimate.

By gathering (2.7.40), (2.7.41) and using (2.7.21), we estimate J_3^2 as follows,

$$\begin{aligned} J_3^2 \le\ & (1-\mathcal{X}_1(\alpha))\,\frac{c}{a^{1-\alpha}}\eta_v\|v\|_{L^\infty(\Omega)}\|u_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega)} \\ & + \mathcal{X}_1(\alpha)c\eta_v C_J\|v\|_{L^\infty(\Omega)}\left(\|A+cv\|^{q(2)-1}_{L^{q(2)-1}(\Omega)} + a^{\alpha+1}\|u_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega)}\right). \end{aligned} \tag{2.7.42}$$

Concerning J_4^2, we have

$$\begin{aligned} J_4^2 &:= d\beta\int_\Omega\left(\int_0^{u_b}\omega(z,v)^{\beta-1}z\,dz\right)f_v(u_a,u_b,v)dx \\ &\le d\beta\eta_v\int_\Omega v\left(\int_0^{u_b}\omega(z,v)^{\beta-1}z\,dz\right)dx. \end{aligned}$$

By analogy with J_3^2 in (2.7.39), we apply to J_4^2 the same tools used to get the estimate (2.7.42). Distinguishing the cases $\beta < 1$ and $\beta \ge 1$, we obtain

$$\begin{aligned} J_4^2 \le\ & (1-\mathcal{X}_1(\beta))\,\frac{d}{b^{1-\beta}}\eta_v\|v\|_{L^\infty(\Omega)}\|u_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega)} \\ & + \mathcal{X}_1(\beta)d\eta_v C_J\|v\|_{L^\infty(\Omega)}\left(\|B+dv\|^{r(2)-1}_{L^{r(2)-1}(\Omega)} + b^{\beta+1}\|u_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega)}\right). \end{aligned} \tag{2.7.43}$$

Finally, taking $p = 2$ in (2.7.25), (2.7.26) and gathering with (2.7.42), (2.7.43), we obtain

$$\begin{aligned} I^2_{rea} \le\ & \eta_a C_L(1+A)^{\alpha+1}\|u_a\|^2_{L^2(\Omega)} - \frac{1}{2}\eta_a a^{\alpha+1}\|u_a\|^{q(2)+1}_{L^{q(2)+1}(\Omega)} \\ & + \eta_b C_L(1+B)^{\beta+1}\|u_b\|^2_{L^2(\Omega)} - \frac{1}{2}\eta_b b^{\beta+1}\|u_b\|^{r(2)+1}_{L^{r(2)+1}(\Omega)} \\ & + (1-\mathcal{X}_1(\alpha))\,\frac{c}{a^{1-\alpha}}\eta_v\|v\|_{L^\infty(\Omega)}\|u_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega)} \\ & + \mathcal{X}_1(\alpha)c\eta_v C_J\|v\|_{L^\infty(\Omega)}\left(\|A+cv\|^{q(2)-1}_{L^{q(2)-1}(\Omega)} + a^{\alpha+1}\|u_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega)}\right) \\ & + (1-\mathcal{X}_1(\beta))\,\frac{d}{b^{1-\beta}}\eta_v\|v\|_{L^\infty(\Omega)}\|u_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega)} \\ & + \mathcal{X}_1(\beta)d\eta_v C_J\|v\|_{L^\infty(\Omega)}\left(\|B+dv\|^{r(2)-1}_{L^{r(2)-1}(\Omega)} + b^{\beta+1}\|u_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega)}\right). \end{aligned} \tag{2.7.44}$$

2.7.2 A priori estimate for the diffusion terms

The estimate of the diffusion term I^p_{diff} in (2.7.17) will be done in such a way that we obtain the L^2 control on the gradient of u_b^ε, in the case $p = p_\beta$, and the L^2 control on the gradient of u_a^ε, in the case $p = p_\alpha$. It is worth noticing that we will use the assumption $A > 0$ to handle the diffusion term I^p_{diff}, in the case $p = p_\beta$, more precisely to estimate the term $K_1^{p_\beta}$ in (2.7.47).

- **I^p_{diff} in the critical case** $p = p_\beta$

From (2.7.17), we write $I^{p_\beta}_{diff}$ as

$$
\begin{aligned}
I^{p_\beta}_{diff} &= \int_\Omega (d_a \partial_1 h_{p_\beta} \Delta u_a + d_b \partial_2 h_{p_\beta} \Delta u_b + d_v \partial_3 h_{p_\beta} \Delta v) dx \\
&= -d_a \int_\Omega \partial_{11} h_{a,p_\beta} |\nabla u_a|^2 dx - d_a \int_\Omega \partial_{12} h_{a,p_\beta} \nabla u_a \cdot \nabla v dx \\
&\quad - d_b \int_\Omega \partial_{11} h_{b,p_\beta} |\nabla u_b|^2 dx - d_b \int_\Omega \partial_{12} h_{b,p_\beta} \nabla u_b \cdot \nabla v dx \\
&\quad + d_v \int_\Omega \partial_2 h_{a,p_\beta} \Delta v \, dx \\
&\quad + d_v \int_\Omega \partial_2 h_{b,p_\beta} \Delta v \, dx \\
&:= K_1^{p_\beta} + K_2^{p_\beta} + K_3^{p_\beta} + K_4^{p_\beta},
\end{aligned}
\tag{2.7.45}
$$

where the derivatives of the energy dentities $h_{a,p}$, $h_{b,p}$ are defined in (2.7.12) - (2.7.15).

We compute for $K_1^{p_\beta}$

$$
\begin{aligned}
K_1^{p_\beta} &:= -d_a a \, \alpha(p_\beta - 1) \int_\Omega \theta(u_a, v)^{\alpha(p_\beta-1)-1} u_a^{p_\beta-1} |\nabla u_a|^2 \, dx \\
&\quad - d_a(p_\beta - 1) \int_\Omega \theta(u_a, v)^{\alpha(p_\beta-1)} u_a^{p_\beta-2} |\nabla u_a|^2 dx \\
&\quad - d_a c \alpha(p_\beta - 1) \int_\Omega \theta(u_a, v)^{\alpha(p_\beta-1)-1} u_a^{p_\beta-1} \nabla u_a \cdot \nabla v \, dx,
\end{aligned}
$$

and we apply Young's inequality in the third integral to obtain

$$
\begin{aligned}
K_1^{p_\beta} &\leq -\frac{d_a a}{2} \alpha(p_\beta - 1) \int_\Omega \theta(u_a, v)^{\alpha(p_\beta-1)-1} u_a^{p_\beta-1} |\nabla u_a|^2 \, dx \\
&\quad - d_a(p_\beta - 1) \int_\Omega \theta(u_a, v)^{\alpha(p_\beta-1)} u_a^{p_\beta-2} |\nabla u_a|^2 dx \\
&\quad + d_a \frac{c^2}{2a} \alpha(p_\beta - 1) \int_\Omega \theta(u_a, v)^{\alpha(p_\beta-1)-1} u_a^{p_\beta-1} |\nabla v|^2 \, dx \\
&= -\frac{d_a \, a}{2} \frac{\alpha}{\beta + 1} \int_\Omega \theta(u_a, v)^{\frac{\alpha}{\beta+1}-1} u_a^{\frac{1}{\beta+1}} |\nabla u_a|^2 \, dx \\
&\quad - \frac{d_a}{a^{p_\beta-2}} \frac{1}{\beta + 1} \int_\Omega \theta(u_a, v)^{\frac{\alpha}{\beta+1}} (a u_a)^{\frac{1}{\beta+1}-1} |\nabla u_a|^2 dx \\
&\quad + d_a \frac{c^2}{2a^{p_\beta}} \frac{\alpha}{\beta + 1} \int_\Omega \theta(u_a, v)^{\frac{\alpha}{\beta+1}-1} (a u_a)^{\frac{1}{\beta+1}} |\nabla v|^2 \, dx.
\end{aligned}
\tag{2.7.46}
$$

By neglecting the first and second integral in (2.7.46), observing that $\frac{\alpha}{\beta+1} - 1 < 0$ and using the inequality $\theta(u_a, v) \geq A + a u_a$, we arrive at

$$
K_1^{p_\beta} \leq \frac{d_a}{2} \frac{c^2}{a^{p_\beta}} \int_\Omega \left(\frac{a u_a}{(A + a u_a)^{\beta-\alpha+1}} \right)^{\frac{1}{\beta+1}} |\nabla v|^2 dx. \tag{2.7.47}
$$

Finally, the assumption $A > 0$ implies

$$
K_1^{p_\beta} \leq \frac{d_a}{2} \frac{c^2}{a^{p_\beta}} C_A \int_\Omega |\nabla v|^2 dx, \tag{2.7.48}
$$

with the constant $C_A > 0$ depending on $A, \alpha, \beta > 0$.

Concerning $K_2^{p_\beta}$, we get

$$\begin{aligned}K_2^{p_\beta} &:= -d_b b\beta(p_\beta - 1)\int_\Omega \omega(u_b, v)^{\beta(p_\beta-1)-1} u_b^{p_\beta-1}|\nabla u_b|^2 dx\\ &\quad - d_b(p_\beta - 1)\int_\Omega \omega(u_b, v)^{\beta(p_\beta-1)} u_b^{p_\beta-2}|\nabla u_b|^2 dx\\ &\quad - d_b d\beta(p_\beta - 1)\int_\Omega \omega(u_b, v)^{\beta(p_\beta-1)-1} u_b^{p_\beta-1}\nabla u_b\cdot\nabla v dx.\end{aligned}$$

Proceeding as we did for $K_1^{p_\beta}$ in (2.7.46), we obtain

$$\begin{aligned}K_2^{p_\beta} &\le -\frac{d_b b}{2}\beta(p_\beta - 1)\int_\Omega \omega(u_b, v)^{\beta(p_\beta-1)-1} u_b^{p_\beta-1}|\nabla u_b|^2 dx\\ &\quad - d_b(p_\beta - 1)\int_\Omega \omega(u_b, v)^{\beta(p_\beta-1)} u_b^{p_\beta-2}|\nabla u_b|^2 dx\\ &\quad + \frac{d_b}{2}\frac{d^2}{b}\beta(p_\beta - 1)\int_\Omega \omega(u_b, v)^{\beta(p_\beta-1)-1} u_b^{p_\beta-1}|\nabla v|^2 dx.\end{aligned} \tag{2.7.49}$$

Neglecting the first integral in (2.7.49) and using $\omega(u_b, v) \ge bu_b$ in the other two, together with the identity

$$\beta(p_\beta - 1) + p_\beta - 2 = \frac{\beta}{\beta+1} + \frac{1}{\beta+1} - 1 = 0,$$

we get

$$K_2^{p_\beta} \le -d_b\frac{b^{\frac{\beta}{\beta+1}}}{\beta+1}\int_\Omega |\nabla u_b|^2 dx + \frac{d_b}{2}\frac{d^2}{b^{1+\frac{1}{\beta+1}}}\int_\Omega |\nabla v|^2 dx. \tag{2.7.50}$$

Now, we estimate the terms $K_3^{p_\beta}$ and $K_4^{p_\beta}$ defined in (2.7.45). Thus, we compute

$$\begin{aligned}K_3^{p_\beta} &:= d_v c\alpha(p_\beta - 1)\int_\Omega\left(\int_0^{u_a}\theta(z, v)^{\alpha(p_\beta-1)-1} z^{p_\beta-1} dz\right)\Delta v dx\\ &= d_v c\frac{\alpha}{\beta+1}\int_\Omega\left(\int_0^{u_a}\theta(z, v)^{\frac{\alpha}{\beta+1}-1} z^{\frac{1}{\beta+1}} dz\right)\Delta v dx.\end{aligned}$$

Similarly as we did to estimate $J_3^{p_\beta}$ in (2.7.29), using $\theta(z, v) \ge au_a$, we get

$$\begin{aligned}K_3^{p_\beta} &\le d_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\frac{\alpha}{\beta+1}\int_\Omega\left(\int_0^{u_a}\left(\frac{z}{z^{\beta+1-\alpha}}\right)^{\frac{1}{\beta+1}} dz\right)|\Delta v| dx\\ &= d_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\frac{\alpha}{\beta+1}\int_\Omega\left(\int_0^{u_a} z^{\frac{\alpha-\beta}{\beta+1}} dz\right)|\Delta v| dx\\ &= d_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\frac{\alpha}{\beta+1}\frac{1}{\frac{\alpha+1}{\beta+1}}\int_\Omega u_a^{\frac{\alpha+1}{\beta+1}}|\Delta v| dx.\end{aligned}$$

Using the inequality (2.7.28) with $\gamma = \frac{\alpha+1}{\beta+1}$ together with Hölder's inequality, we end up with

$$K_3^{p_\beta} \le d_v\frac{c}{a^{1-\frac{\alpha}{\beta+1}}}\|1 + u_a\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)}. \tag{2.7.51}$$

Concerning $K_4^{p_\beta}$, using $\omega(z,v) \geq bz$ it holds

$$
\begin{aligned}
K_4^{p_\beta} &:= d_v d\beta(p_\beta - 1) \int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\beta(p_\beta-1)-1} z^{p_\beta-1} dz \right) \Delta v dx \\
&\leq d_v \frac{d}{b^{\frac{1}{\beta+1}}} \frac{\beta}{\beta+1} \int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\frac{\beta}{\beta+1}-1} (bz)^{\frac{1}{\beta+1}} dz \right) |\Delta v| dx \\
&\leq d_v \frac{d}{b^{\frac{1}{\beta+1}}} \int_\Omega u_b |\Delta v| dx \\
&\leq d_v \frac{d}{b^{\frac{1}{\beta+1}}} \|u_b\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)}.
\end{aligned} \tag{2.7.52}
$$

Finally, by gathering (2.7.48), (2.7.50), (2.7.51), (2.7.52), we end up with

$$
\begin{aligned}
I_{diff}^{p_\beta} \leq & \frac{1}{2} \left(d_a \frac{c^2}{a^{p_\beta}} C_A + d_b \frac{d^2}{b^{p_\beta}} \right) \|\nabla v\|_{L^2(\Omega)}^2 - d_b \frac{b^{\frac{\beta}{\beta+1}}}{\beta+1} \|\nabla u_b\|_{L^2(\Omega)}^2 \\
& + d_v \left(\frac{c}{a^{1-\frac{\alpha}{\beta+1}}} + \frac{d}{b^{\frac{1}{\beta+1}}} \right) \Big(\|1 + u_a\|_{L^2(\Omega)} + \|u_b\|_{L^2(\Omega)} \Big) \|\Delta v\|_{L^2(\Omega)}.
\end{aligned} \tag{2.7.53}
$$

- **I_{diff}^p in the critical case $p = p_\alpha$**

From (2.7.17), we write $I_{diff}^{p_\alpha}$ as

$$
\begin{aligned}
I_{diff}^{p_\alpha} &= -d_a \int_\Omega \partial_{11} h_{a,p_\alpha} |\nabla u_a|^2 dx - d_a \int_\Omega \partial_{12} h_{a,p_\alpha} \nabla u_a \cdot \nabla v dx \\
&\quad - d_b \int_\Omega \partial_{11} h_{b,p_\alpha} |\nabla u_b|^2 dx - (d_b + d_v) \int_\Omega \partial_{12} h_{b,p_\alpha} \nabla u_b \cdot \nabla v dx \\
&\quad + d_v \int_\Omega \partial_2 h_{a,p_\alpha} \Delta v \, dx \\
&\quad - d_v \int_\Omega \partial_{22} h_{b,p_\alpha} |\nabla v|^2 \, dx \\
&:= K_1^{p_\alpha} + K_2^{p_\alpha} + K_3^{p_\alpha} + K_4^{p_\alpha}.
\end{aligned} \tag{2.7.54}
$$

Concerning $K_1^{p_\alpha}$, it holds

$$
\begin{aligned}
K_1^{p_\alpha} &:= -d_a a\, \alpha(p_\alpha - 1) \int_\Omega \theta(u_a,v)^{\alpha(p_\alpha-1)-1} u_a^{p_\alpha-1} |\nabla u_a|^2 \, dx \\
&\quad - d_a(p_\alpha - 1) \int_\Omega \theta(u_a,v)^{\alpha(p_\alpha-1)} u_a^{p_\alpha-2} |\nabla u_a|^2 dx \\
&\quad - d_a c\, \alpha(p_\alpha - 1) \int_\Omega \theta(u_a,v)^{\alpha(p_\alpha-1)-1} u_a^{p_\alpha-1} \nabla u_a \cdot \nabla v \, dx.
\end{aligned}
$$

Similarly as we did for $K_2^{p_\beta}$, we use Young's inequality to get

$$\begin{aligned}
K_1^{p_\alpha} \leq & -\frac{d_a\,a}{2}\alpha(p_\alpha - 1)\int_\Omega \theta(u_a,v)^{\alpha(p_\alpha-1)-1}u_a^{p_\alpha-1}|\nabla u_a|^2\,dx \\
& -\frac{d_a}{a^{p_\alpha-2}}(p_\alpha-1)\int_\Omega \theta(u_a,v)^{\alpha(p_\alpha-1)}(au_a)^{p_\alpha-2}|\nabla u_a|^2 dx \\
& + d_a\frac{c^2}{2a^{p_\alpha}}\alpha(p_\alpha-1)\int_\Omega \theta(u_a,v)^{\alpha(p_\alpha-1)-1}(au_a)^{p_\alpha-1}|\nabla v|^2\,dx \\
= & -\frac{d_a\,a}{2}\frac{\alpha}{\alpha+1}\int_\Omega \theta(u_a,v)^{\frac{\alpha}{\alpha+1}-1}u_a^{\frac{1}{\alpha+1}}|\nabla u_a|^2\,dx \\
& -\frac{d_a}{a^{p_\alpha-2}}\frac{1}{\alpha+1}\int_\Omega \theta(u_a,v)^{\frac{\alpha}{\alpha+1}}(au_a)^{-\frac{\alpha}{\alpha+1}}|\nabla u_a|^2 dx \\
& + d_a\frac{c^2}{2a^{p_\alpha}}\frac{\alpha}{\alpha+1}\int_\Omega \theta(u_a,v)^{\frac{\alpha}{\alpha+1}-1}(au_a)^{\frac{1}{\alpha+1}}|\nabla v|^2\,dx.
\end{aligned}$$

Thus, neglecting the first integral and using $\theta(u_a,v)\geq au_a$ in the second and third integrals, we obtain

$$K_1^{p_\alpha} \leq -d_a a^{\frac{\alpha}{\alpha+1}}\frac{1}{\alpha+1}\int_\Omega |\nabla u_a|^2 dx + \frac{d_a}{2}\frac{c^2}{a^{p_\alpha}}\int_\Omega |\nabla v|^2 dx. \tag{2.7.55}$$

Now, we estimate the term $K_2^{p_\alpha}$ in (2.7.54) by applying Young's inequality to get

$$\begin{aligned}
K_2^{p_\alpha} := & -d_b b\beta(p_\alpha-1)\int_\Omega \omega(u_b,v)^{\beta(p_\alpha-1)-1}u_b^{p_\alpha-1}|\nabla u_b|^2 dx \\
& - d_b(p_\alpha-1)\int_\Omega \omega(u_b,v)^{\beta(p_\alpha-1)}u_b^{p_\alpha-2}|\nabla u_b|^2 dx \\
& - (d_b+d_v)d\beta(p_\alpha-1)\int_\Omega \omega(u_b,v)^{\beta(p_\alpha-1)-1}u_b^{p_\alpha-1}\nabla u_b\cdot\nabla v dx \\
\leq & -\frac{d_b}{2}\frac{b\beta}{\alpha+1}\int_\Omega \omega(u_b,v)^{\beta(p_\alpha-1)-1}u_b^{p_\alpha-1}|\nabla u_b|^2 dx \\
& -\frac{d_b}{\alpha+1}\int_\Omega \omega(u_b,v)^{\beta(p_\alpha-1)}u_b^{p_\alpha-2}|\nabla u_b|^2 dx \\
& + \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b}\frac{\beta}{\alpha+1}\int_\Omega \omega(u_b,v)^{\beta(p_\alpha-1)-1}u_b^{p_\alpha-1}|\nabla v|^2 dx.
\end{aligned}$$

Neglecting the first and the second integral we obtain

$$K_2^{p_\alpha} \leq \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b}\frac{\beta}{\alpha+1}\int_\Omega \omega(u_b,v)^{\frac{\beta}{\alpha+1}-1}u_b^{\frac{1}{\alpha+1}}|\nabla v|^2 dx. \tag{2.7.56}$$

Like for $J_4^{p_\alpha}$ in (2.7.33), we estimate $K_2^{p_\alpha}$ by distinguishing the cases $\beta < 1+\alpha$ and $\beta \geq 1+\alpha$, i.e. $\frac{\beta}{\alpha+1}-1<0$ and $\frac{\beta}{\alpha+1}-1\geq 0$, respectively.

Case $\beta < \alpha+1$. Using $\omega(u_b,v)\geq bu_b$, we compute

$$K_2^{p_\alpha} \leq \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\frac{\beta}{\alpha+1}}}\int_\Omega u_b^{\frac{\beta-\alpha}{\alpha+1}}|\nabla v|^2 dx,$$

and by the inequality (2.7.28) with $\gamma = \frac{\beta-\alpha}{\alpha+1}$, we end up with

$$K_2^{p_\alpha} \leq \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\frac{\beta}{\alpha+1}}}\int_\Omega (1+u_b)|\nabla v|^2 dx$$
$$\leq \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\frac{\beta}{\alpha+1}}}\|1+u_b\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}. \tag{2.7.57}$$

Case $\beta \geq \alpha+1$. Using $bu_b \leq \omega(u_b, v)$, we get from (2.7.56)

$$K_2^{p_\alpha} \leq \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\int_\Omega \omega(u_b,v)^{\frac{\beta-\alpha}{\alpha+1}}|\nabla v|^2 dx.$$

The main tool used in the case $\beta < \alpha+1$ was the inequality (2.7.28) with $\gamma < 1$. However, it holds that $\frac{\beta-\alpha}{\alpha+1} < 1$ if and only if $\beta < 2\alpha+1$. This suggests to distinguish the following two cases

$$\beta < 2\alpha+1 \qquad \text{and} \qquad \beta \geq 2\alpha+1.$$

Using the inequality (2.7.28) in the case $\beta < 2\alpha+1$ and using (2.7.21), we end up with

$$K_2^{p_\alpha} \leq \mathcal{X}_{\alpha+1}(\beta)\,(1-\mathcal{X}_{2\alpha+1}(\beta))\,\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\|1+\omega(u_b,v)\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}$$
$$+\mathcal{X}_{2\alpha+1}(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\int_\Omega \omega(u_b,v)^{\frac{\beta-\alpha}{\alpha+1}}|\nabla v|^2 dx. \tag{2.7.58}$$

By gathering the estimates (2.7.57), (2.7.58), we obtain

$$K_2^{p_\alpha} \leq (1-\mathcal{X}_{\alpha+1}(\beta))\,\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\frac{\beta}{\alpha+1}}}\|1+u_b\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}$$
$$+\mathcal{X}_{\alpha+1}(\beta)\,(1-\mathcal{X}_{2\alpha+1}(\beta))\,\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\|1+\omega(u_b,v)\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}$$
$$+\mathcal{X}_{2\alpha+1}(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\int_\Omega \omega(u_b,v)^{\frac{\beta-\alpha}{\alpha+1}}|\nabla v|^2 dx. \tag{2.7.59}$$

Now, we estimate the term $K_3^{p_\alpha}$ in (2.7.54).

$$K_3^{p_\alpha} := d_v c\alpha(p_\alpha-1)\int_\Omega\left(\int_0^{u_a}\theta(z,v)^{\alpha(p_\alpha-1)-1}z^{p_\alpha-1}dz\right)\Delta v dx$$
$$= d_v\frac{c}{a^{\frac{1}{\alpha+1}}}\frac{\alpha}{\alpha+1}\int_\Omega\left(\int_0^{u_a}\theta(z,v)^{\frac{\alpha}{\alpha+1}-1}(az)^{\frac{1}{\alpha+1}}dz\right)\Delta v dx$$
$$\leq d_v\frac{c}{a^{\frac{1}{\alpha+1}}}\int_\Omega\left(\int_0^{u_a}\left(\frac{az}{\theta(z,v)}\right)^{\frac{1}{\alpha+1}}dz\right)|\Delta v| dx$$
$$\leq d_v\frac{c}{a^{\frac{1}{\alpha+1}}}\int_\Omega u_a|\Delta v| dx$$
$$\leq \frac{d_v}{2}\frac{c}{a^{\frac{1}{\alpha+1}}}\left(\|u_a\|^2_{L^2(\Omega)}+\|\Delta v\|^2_{L^2(\Omega)}\right). \tag{2.7.60}$$

Concerning the term $K_4^{p_\alpha}$, we have

$$K_4^{p_\alpha} := -d_v d^2 \beta(p_\alpha - 1)\left(\beta(p_\alpha - 1) - 1\right) \int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\beta(p_\alpha-1)-2} z^{p_\alpha - 1} dz \right) |\nabla v|^2 dx$$
$$= -d_v d^2 \frac{\beta}{\alpha+1} \left(\frac{\beta}{\alpha+1} - 1 \right) \int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\frac{\beta}{\alpha+1}-2} z^{\frac{1}{\alpha+1}} dz \right) |\nabla v|^2 dx.$$

Observing that if $\frac{\beta}{\alpha+1} - 1 \geq 0$, $K_4^{p_\alpha}$ is negative, we estimate $K_4^{p_\alpha}$ only when $\beta < \alpha + 1$. Recalling that $r(p_\alpha) > 2$ and definition (2.7.21), we get

$$\begin{aligned}
K_4^{p_\alpha} &\leq (1 - \mathcal{X}_{\alpha+1}(\beta))\, d_v d^2 \frac{\beta}{\alpha+1}\Big(1 - \frac{\beta}{\alpha+1}\Big) \int_\Omega \left(\int_0^{u_b} \left(\frac{z}{\omega(z,v)^{2(\alpha+1)-\beta}} \right)^{\frac{1}{\alpha+1}} dz \right) |\nabla v|^2 dx \\
&\leq (1 - \mathcal{X}_{\alpha+1}(\beta))\, \frac{d_v d^2}{b^{2-\frac{\beta}{\alpha+1}}} \int_\Omega \left(\int_0^{u_b} z^{\frac{\beta+1}{\alpha+1}-2} dz \right) |\nabla v|^2 dx \\
&= (1 - \mathcal{X}_{\alpha+1}(\beta))\, \frac{d_v d^2}{b^{2-\frac{\beta}{\alpha+1}}} \int_\Omega \left(\int_0^{u_b} z^{r(p_\alpha)-3} dz \right) |\nabla v|^2 dx \\
&= (1 - \mathcal{X}_{\alpha+1}(\beta))\, \frac{d_v d^2}{b^{2-\frac{\beta}{\alpha+1}}} \frac{1}{r(p_\alpha)-2} \int_\Omega u_b^{r(p_\alpha)-2} |\nabla v|^2 dx \\
&\leq (1 - \mathcal{X}_{\alpha+1}(\beta))\, \frac{d_v d^2}{2b^{2-\frac{\beta}{\alpha+1}}} \left(\int_\Omega u_b^{2(r(p_\alpha)-2)} dx + \int_\Omega |\nabla v|^4 dx \right),
\end{aligned} \tag{2.7.61}$$

where we used Young's inequality in the last estimate.

Finally by gathering (2.7.55), (2.7.59), (2.7.60), (2.7.61), we obtain

$$\begin{aligned}
I_{diff}^{p_\alpha} \leq& -d_a a^{\frac{\alpha}{\alpha+1}} \frac{1}{\alpha+1} \|\nabla u_a\|^2_{L^2(\Omega)} + \frac{d_a}{2} \frac{c^2}{a^{p_\alpha}} \|\nabla v\|^2_{L^2(\Omega)} \\
&+ (1 - \mathcal{X}_{\alpha+1}(\beta))\, \frac{(d_b + d_v)^2}{2d_b} \frac{d^2}{b^{2-\frac{\beta}{\alpha+1}}} \|1 + u_b\|_{L^2(\Omega)} \|\nabla v\|^2_{L^4(\Omega)} \\
&+ \mathcal{X}_{\alpha+1}(\beta)\,(1 - \mathcal{X}_{2\alpha+1}(\beta))\, \frac{(d_b + d_v)^2}{2d_b} \frac{d^2}{b^{p_\alpha}} \frac{\beta}{\alpha+1} \|1 + \omega(u_b, v)\|_{L^2(\Omega)} \|\nabla v\|^2_{L^4(\Omega)} \\
&+ \mathcal{X}_{2\alpha+1}(\beta) \frac{(d_b + d_v)^2}{2d_b} \frac{d^2}{b^{p_\alpha}} \frac{\beta}{\alpha+1} \int_\Omega \omega(u_b, v)^{\frac{\beta-\alpha}{\alpha+1}} |\nabla v|^2 dx \\
&+ \frac{d_v}{2} \frac{c}{a^{\frac{1}{\alpha+1}}} \left(\|u_a\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right) \\
&+ (1 - \mathcal{X}_{\alpha+1}(\beta))\, \frac{d_v d^2}{2b^{2-\frac{\beta}{\alpha+1}}} \left(\|u_b\|^{2(r(p_\alpha)-2)}_{L^{2(r(p_\alpha)-2)}(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)} \right).
\end{aligned} \tag{2.7.62}$$

- I_{diff}^p **in the super-critical case** $p = 2$

As we are considering now the super-critical case $p = 2$, the hessian of h_p in (2.7.14), (2.7.15) is well defined and we write I^2_{diff} in (2.7.17) as

$$\begin{aligned}
I^2_{diff} &= -d_a \int_\Omega \partial_{11} h_{a,2} |\nabla u_a|^2 dx - (d_a + d_v) \int_\Omega \partial_{12} h_{a,2} \nabla u_a \cdot \nabla v dx \\
&\quad - d_b \int_\Omega \partial_{11} h_{b,2} |\nabla u_b|^2 dx - (d_b + dv) \int_\Omega \partial_{21} h_{b,2} \nabla u_b \cdot \nabla v dx \\
&\quad - d_v \int_\Omega \partial_{22} h_{a,2} |\nabla v|^2 \, dx \\
&\quad - d_v \int_\Omega \partial_{22} h_{b,2} |\nabla v|^2 \, dx \\
&=: K^2_1 + K^2_2 + K^2_3 + K^2_4.
\end{aligned} \tag{2.7.63}$$

In order to estimate the term K^2_1 in (2.7.63), we apply Young's inequality to get

$$\begin{aligned}
K^2_1 &:= -d_a a\alpha \int_\Omega \theta(u_a, v)^{\alpha-1} u_a |\nabla u_a|^2 dx - d_a \int_\Omega \theta(u_a, v)^\alpha |\nabla u_a|^2 dx \\
&\quad - (d_a + d_v) c\alpha \int_\Omega \theta(u_a, v)^{\alpha-1} u_a \nabla u_a \cdot \nabla v dx \\
&\leq -\frac{d_a}{2} a\alpha \int_\Omega \theta(u_a, v)^{\alpha-1} u_a |\nabla u_a|^2 dx - d_a \int_\Omega \theta(u_a, v)^\alpha |\nabla u_a|^2 dx \\
&\quad + \frac{(d_a + d_v)^2}{2d_a} \frac{c^2 \alpha}{a} \int_\Omega \theta(u_a, v)^{\alpha-1} u_a |\nabla v|^2 \, dx.
\end{aligned} \tag{2.7.64}$$

Neglecting the first integral and using $\theta(u_a, v) > au_a$ in the second integral, we obtain

$$K^2_1 \leq -d_a a^\alpha \int_\Omega u_a^\alpha |\nabla u_a|^2 dx + \frac{(d_a + d_v)^2}{2d_a} \frac{c^2 \alpha}{a} \int_\Omega \theta(u_a, v)^{\alpha-1} u_a |\nabla v|^2 dx. \tag{2.7.65}$$

Then recalling that $q(2) = \alpha + 2$, we use the identity below in the first integral of (2.7.65)

$$u_a^\alpha |\nabla u_a|^2 = \left| u_a^{\frac{q(2)-2}{2}} \nabla u_a \right|^2 = \frac{4}{q(2)^2} \left| \nabla (u_a^{q(2)/2}) \right|^2 = \frac{4}{(\alpha + 2)^2} \left| \nabla (u_a^{q(2)/2}) \right|^2, \tag{2.7.66}$$

to get

$$K^2_1 \leq -\frac{4 d_a a^\alpha}{(\alpha + 2)^2} \int_\Omega \left| \nabla (u_a^{q(2)/2}) \right|^2 dx + \frac{(d_a + d_v)^2}{2d_a} \frac{c^2 \alpha}{a} \int_\Omega \theta(u_a, v)^{\alpha-1} u_a |\nabla v|^2 \, dx. \tag{2.7.67}$$

Now, we estimate the second intergral in (2.7.67) with the tools used to handle the term $K^{p_\alpha}_2$ in (2.7.56). We distinguish the cases $\alpha < 1$ and $\alpha \geq 1$.

Case $\alpha < 1$. Using $\theta(u_a, v) \geq au_a$ and the inequality (2.7.28) with $\gamma = \alpha$, we compute

$$\begin{aligned}
&\frac{(d_a + d_v)^2}{2d_a} \frac{c^2 \alpha}{a} \int_\Omega \frac{u_a}{\theta(u_a, v)^{1-\alpha}} |\nabla v|^2 \, dx \\
&\qquad \leq \frac{(d_a + d_v)^2}{2d_a} \frac{c^2}{a^{2-\alpha}} \int_\Omega u_a^\alpha |\nabla v|^2 \, dx \\
&\qquad \leq \frac{(d_a + d_v)^2}{2d_a} \frac{c^2}{a^{2-\alpha}} \int_\Omega (1 + u_a) |\nabla v|^2 dx \\
&\qquad \leq \frac{(d_a + d_v)^2}{4d_a} \frac{c^2}{a^{2-\alpha}} \left(\|1 + u_a\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)} \right).
\end{aligned}$$

Case $\alpha \geq 1$. Using $\theta(u_a, v) \geq a u_a$, we get

$$\frac{(d_a+d_v)^2}{2d_a}\frac{c^2\alpha}{a}\int_\Omega \theta(u_a,v)^{\alpha-1}u_a|\nabla v|^2\,dx \leq \frac{(d_a+d_v)^2}{2d_a}\frac{c^2\alpha}{a^2}\int_\Omega \theta(u_a,v)^{\alpha}|\nabla v|^2\,dx.$$

Therefore, we end up with the following estimate for K_1^2,

$$\begin{aligned} K_1^2 \leq & -\frac{4d_a a^\alpha}{(\alpha+2)^2}\|\nabla(u_a^{q(2)/2})\|^2_{L^2(\Omega)} \\ & + \frac{(d_a+d_v)^2}{2d_a}\frac{c^2}{a^{2-\alpha}}(1-\mathcal{X}_1(\alpha))\left(\|1+u_a\|^2_{L^2(\Omega)}+\|\nabla v\|^4_{L^4(\Omega)}\right) \\ & + \frac{(d_a+d_v)^2}{2d_a}\frac{c^2\alpha}{a^2}\mathcal{X}_1(\alpha)\int_\Omega \theta(u_a,v)^\alpha|\nabla v|^2\,dx. \end{aligned} \tag{2.7.68}$$

From (2.7.63), we compute K_2^2 similarly as we did for K_1^2 in (2.7.64),

$$\begin{aligned} K_2^2 :=& -d_b b\beta\int_\Omega \omega(u_b,v)^{\beta-1}u_b|\nabla u_b|^2dx - d_b\int_\Omega \omega(u_b,v)^\beta|\nabla u_b|^2dx \\ & -(d_b+d_v)d\beta\int_\Omega \omega(u_b,v)^{\beta-1}u_b\nabla u_b\cdot\nabla v dx \\ \leq & -\frac{d_b}{2}b\beta\int_\Omega \omega(u_b,v)^{\beta-1}u_b|\nabla u_b|^2dx - d_b\int_\Omega \omega(u_b,v)^\beta|\nabla u_b|^2dx \\ & + \frac{(d_b+d_v)^2}{2d_b}\frac{d^2\beta}{b}\int_\Omega \omega(u_b,v)^{\beta-1}u_b|\nabla v|^2dx \\ \leq & -d_b b^\beta\int_\Omega u_b^\beta|\nabla u_b|^2dx + \frac{(d_b+d_v)^2}{2d_b}\frac{d^2\beta}{b}\int_\Omega \omega(u_b,v)^{\beta-1}u_b|\nabla v|^2dx, \end{aligned} \tag{2.7.69}$$

where in the last inequality, we neglected the first term in (2.7.69) and we use $\omega(u_b, v) \geq b u_b$ in the second term. Then, by analogy with (2.7.65), (2.7.66) we rewrite the first term in the last inequality as follows

$$K_2^2 \leq -\frac{4d_b b^\beta}{(\beta+2)^2}\int_\Omega |\nabla(u_b^{r(2)/2})|^2dx + \frac{(d_b+d_v)^2}{2d_b}\frac{d^2\beta}{b}\int_\Omega \omega(u_b,v)^{\beta-1}u_b|\nabla v|^2dx.$$

Similarly as we did in (2.7.67), we distinguish the cases $\beta < 1$ and $\beta \geq 1$. Therefore, by analogy with (2.7.68) we end up with

$$\begin{aligned} K_2^2 \leq & -\frac{4d_b b^\beta}{(\beta+2)^2}\|\nabla(u_b^{r(2)/2})\|^2_{L^2(\Omega)} \\ & + \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\beta}}(1-\mathcal{X}_1(\beta))\left(\|1+u_b\|^2_{L^2(\Omega)}+\|\nabla v\|^4_{L^4(\Omega)}\right) \\ & + \frac{(d_b+d_v)^2}{2d_b}\frac{d^2\beta}{b^2}\mathcal{X}_1(\beta)\int_\Omega \omega(u_b,v)^\beta|\nabla v|^2\,dx. \end{aligned} \tag{2.7.70}$$

Now, we estimate the terms K_3^2 and K_4^2 defined in (2.7.63). We observe that $\partial_{22}h_{a,2}, \partial_{22}h_{b,2}$ have a sign which changes depending on the value of α, β. More precisely, by (2.7.16) it holds $\partial_{22}h_{a,2} < 0$ (resp. $\partial_{22}h_{b,2} < 0$) if and only if $\alpha < 1$ (resp. $\beta < 1$), so that we need to estimate K_3^2 (resp. K_4^2) when $\alpha < 1$ (resp. $\beta < 1$), otherwise $K_3^2 \leq 0$ (resp. $K_4^2 \leq 0$).

Using $\theta(z,v) \geq az$ and the inequality (2.7.28) with $\gamma = \alpha$, we have

$$\begin{aligned}
K_3^2 &:= -d_v c^2 \alpha(\alpha-1) \int_\Omega \left(\int_0^{u_a} \theta(z,v)^{\alpha-2} z \, dz \right) |\nabla v|^2 dx \\
&\leq (1-\mathcal{X}_1(\alpha)) d_v \frac{c^2}{a} \alpha(1-\alpha) \int_\Omega \left(\int_0^{u_a} \theta(z,v)^{\alpha-2} (az) \, dz \right) |\nabla v|^2 dx \\
&\leq (1-\mathcal{X}_1(\alpha)) d_v \frac{c^2}{a} \alpha(1-\alpha) \int_\Omega \left(\int_0^{u_a} \theta(z,v)^{\alpha-1} \, dz \right) |\nabla v|^2 dx \\
&\leq (1-\mathcal{X}_1(\alpha)) d_v \frac{c^2}{a^2} (1-\alpha) \int_\Omega \theta(u_a,v)^{\alpha} |\nabla v|^2 dx \\
&\leq (1-\mathcal{X}_1(\alpha)) d_v \frac{c^2}{a^2} (1-\alpha) \int_\Omega (1+\theta(u_a,v)) |\nabla v|^2 dx \\
&\leq (1-\mathcal{X}_1(\alpha)) d_v \frac{c^2}{2a^2} (1-\alpha) (\|1+\theta(u_a,v)\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)}).
\end{aligned} \tag{2.7.71}$$

Concerning K_4^2, it holds

$$\begin{aligned}
K_4^2 &:= -d_v d^2 \beta(\beta-1) \int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\beta-2} z \, dz \right) |\nabla v|^2 dx \\
&= d_v \frac{d^2}{b} \beta(1-\beta) \int_\Omega \left(\int_0^{u_b} \omega(z,v)^{\beta-2} (bz) \, dz \right) |\nabla v|^2 dx.
\end{aligned}$$

Thus, similarly as we did for K_3^2, we obtain

$$K_4^2 \leq (1-\mathcal{X}_1(\beta)) d_v \frac{d^2}{2b^2} (1-\beta) (\|1+\omega(u_b,v)\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)}). \tag{2.7.72}$$

Finally, by gathering (2.7.68), (2.7.70), (2.7.71), (2.7.72), we end up with

$$\begin{aligned}
I^2_{diff} \leq &-\frac{4 d_a a^{\alpha}}{(\alpha+2)^2} \|\nabla(u_a^{q(2)/2})\|_{L^2(\Omega)} - \frac{4 d_b b^{\beta}}{(\beta+2)^2} \|\nabla(u_b^{r(2)/2})\|^2_{L^2(\Omega)} \\
&+ \frac{(d_a+d_v)^2}{2d_a} \frac{c^2}{a^{2-\alpha}} (1-\mathcal{X}_1(\alpha)) \left(\|1+u_a\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)} \right) \\
&+ \frac{(d_a+d_v)^2}{2d_a} \frac{c^2 \alpha}{a^2} \mathcal{X}_1(\alpha) \int_\Omega \theta(u_a,v)^{\alpha} |\nabla v|^2 \, dx \\
&+ \frac{(d_b+d_v)^2}{2d_b} \frac{d^2}{b^{2-\beta}} (1-\mathcal{X}_1(\beta)) \left(\|1+u_b\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)} \right) \\
&+ \frac{(d_b+d_v)^2}{2d_b} \frac{d^2 \beta}{b^2} \mathcal{X}_1(\beta) \int_\Omega \omega(u_b,v)^{\beta} |\nabla v|^2 \, dx \\
&+ (1-\mathcal{X}_1(\alpha)) d_v \frac{c^2}{2a^2} (1-\alpha) (\|1+\theta(u_a,v)\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)}) \\
&+ (1-\mathcal{X}_1(\beta)) d_v \frac{d^2}{2b^2} (1-\beta) (\|1+\omega(u_b,v)\|^2_{L^2(\Omega)} + \|\nabla v\|^4_{L^4(\Omega)}).
\end{aligned} \tag{2.7.73}$$

2.7.3 Proof of the energy estimates

Hereafter, we combine the obtained estimates in *Subsections 2.7.1, 2.7.2*, together with the basic estimates shown in *Section 2.6*, in order to control the evolution of the energy functional $\mathcal{E}_p$ in (2.7.20) when $p = p_\beta$, $p = p_\alpha$, $p = 2$. From now on, we restore the index ε.

- **The critical case** $p = p_\beta$

We put into (2.7.20) the estimate for the reaction term $I^{p_\beta}_{rea}$ in (2.7.31) and the estimate for the diffusion term $I^{p_\beta}_{diff}$ in (2.7.53). Thus, recalling that $I^{p}_{fast} \le 0$ (see (2.2.8)) and renaming the constants, we obtain for all $t \in (0,T)$

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_{p_\beta}(u^\varepsilon_a,u^\varepsilon_b,v^\varepsilon)(t) &+ \|u^\varepsilon_a(t)\|^{q(p_\beta)+1}_{L^{q(p_\beta)+1}(\Omega)} + \|u^\varepsilon_b(t)\|^3_{L^3(\Omega)} + \|\nabla u^\varepsilon_b(t)\|^2_{L^2(\Omega)} \\
&\le C\left(\|u^\varepsilon_a(t)\|^{p_\beta}_{L^{p_\beta}(\Omega)} + \|u^\varepsilon_b(t)\|^{p_\beta}_{L^{p_\beta}(\Omega)} + \|v^\varepsilon(t)\|_{L^\infty(\Omega)}(1+\|(u^\varepsilon_a+u^\varepsilon_b)(t)\|_{L^1(\Omega)})\right) \\
&\quad + C\left(\|\nabla v^\varepsilon(t)\|^2_{L^2(\Omega)} + (\|u^\varepsilon_a(t)\|_{L^2(\Omega)} + \|u^\varepsilon_b(t)\|_{L^2(\Omega)})\|\Delta v^\varepsilon(t)\|_{L^2(\Omega)}\right),
\end{aligned}
$$

where the constant $C > 0$ depends on the diffusion and reaction coefficients, on α,β but not on ε. Therefore, recalling that $p_\beta < 2$, integrating over $t \in (0,T)$ and using the estimates in *Lemmas 2.6.1 - 2.6.3*, together with the assumption on the initial data (H4), we end up with (2.7.7).

- **The critical case** $p = p_\alpha$

Now, we integrate in time over $(0,T)$ the inequality (2.7.62), we use $r(p_\alpha) - 2 = \frac{\beta-\alpha}{\alpha+1}$ and the Hölder inequality when necessary, to obtain

$$
\begin{aligned}
\int_{(0,T)} I^{p_\alpha}_{diff}dt \le\; & -d_a a^{\frac{\alpha}{\alpha+1}}\frac{1}{\alpha+1}\|\nabla u^\varepsilon_a\|^2_{L^2(\Omega_T)} + \frac{d_a}{2}\frac{c^2}{a^{p_\alpha}}\|\nabla v^\varepsilon\|^2_{L^2(\Omega_T)} \\
& + (1-\mathcal{X}_{\alpha+1}(\beta))\,\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\frac{\beta}{\alpha+1}}}\|1+u^\varepsilon_b\|_{L^2(\Omega_T)}\|\nabla v^\varepsilon\|^2_{L^4(\Omega_T)} \\
& + \mathcal{X}_{\alpha+1}(\beta)\,(1-\mathcal{X}_{2\alpha+1}(\beta))\,\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\|1+\omega(u^\varepsilon_b,v^\varepsilon)\|_{L^2(\Omega_T)}\|\nabla v^\varepsilon\|^2_{L^4(\Omega_T)} \\
& + \mathcal{X}_{2\alpha+1}(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\int_{\Omega_T}\omega(u^\varepsilon_b,v^\varepsilon)^{r(p_\alpha)-2}|\nabla v^\varepsilon|^2dxdt \\
& + \frac{d_v}{2}\frac{c}{a^{\frac{1}{\alpha+1}}}\left(\|u^\varepsilon_a\|^2_{L^2(\Omega_T)} + \|\Delta v^\varepsilon\|^2_{L^2(\Omega_T)}\right) \\
& + (1-\mathcal{X}_{\alpha+1}(\beta))\,\frac{d_v d^2}{2b^{2-\frac{\beta}{\alpha+1}}}\left(\|u^\varepsilon_b\|^{2(r(p_\alpha)-2)}_{L^{2(r(p_\alpha)-2)}(\Omega_T)} + \|\nabla v^\varepsilon\|^4_{L^4(\Omega_T)}\right).
\end{aligned}
\tag{2.7.74}
$$

Using in (2.7.74) that $2(r(p_\alpha)-2) = \frac{2(\beta-\alpha)}{\alpha+1} < 2$ if $\beta < \alpha+1$, we have

$$
(1-\mathcal{X}_{\alpha+1}(\beta))\,\|u^\varepsilon_b\|_{L^{2(r(p_\alpha)-2)}(\Omega_T)} \le (1-\mathcal{X}_{\alpha+1}(\beta))\,C\|u^\varepsilon_b\|_{L^2(\Omega_T)}.
$$

Then, together with the estimates in *Lemmas 2.6.1 - 2.6.3*, we end up with

$$
\begin{aligned}
\int_{(0,T)} I^{p_\alpha}_{diff}dt \le\; & -d_a a^{\frac{\alpha}{\alpha+1}}\frac{1}{\alpha+1}\|\nabla u^\varepsilon_a\|^2_{L^2(\Omega_T)} + C_T \\
& + \mathcal{X}_{2\alpha+1}(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\int_{\Omega_T}\omega(u^\varepsilon_b,v^\varepsilon)^{r(p_\alpha)-2}|\nabla v^\varepsilon|^2dxdt.
\end{aligned}
\tag{2.7.75}
$$

Similarly, for the reaction term estimate (2.7.38), we have

$$
\begin{aligned}
\int_{(0,T)} I_{rea}^{p_\alpha} dt \leq\ & \eta_a C_L (1+A)^{\alpha(p_\alpha-1)+1} \|u_a^\varepsilon\|_{L^{p_\alpha}(\Omega_T)}^{p_\alpha} - \frac{1}{2}\eta_a a^{\alpha(p_\alpha-1)+1}\|u_a^\varepsilon\|_{L^3(\Omega_T)}^3 \\
& + \eta_b C_L (1+B)^{\beta(p_\alpha-1)+1} \|u_b^\varepsilon\|_{L^{p_\alpha}(\Omega_T)}^{p_\alpha} - \frac{1}{2}\eta_b b^{\beta(p_\alpha-1)+1}\|u_b^\varepsilon\|_{L^{r(p_\alpha)+1}(\Omega_T)}^{r(p_\alpha)+1} \\
& + \eta_v \frac{c}{a^{\frac{1}{\alpha+1}}} \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u_a^\varepsilon\|_{L^1(\Omega_T)} \\
& + (1-\mathcal{X}_{\alpha+1}(\beta)) \frac{d\eta_v}{b^{1-\frac{\beta}{\alpha+1}}} \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u_b^\varepsilon\|_{L^{r(p_\alpha)-1}(\Omega_T)}^{r(p_\alpha)-1} \\
& + \mathcal{X}_{\alpha+1}(\beta)\eta_v C_J \frac{d}{b^{1+\frac{1}{\alpha+1}}} \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|B + dv^\varepsilon\|_{L^{r(p_\alpha)-1}(\Omega_T)}^{r(p_\alpha)-1} \\
& + \mathcal{X}_{\alpha+1}(\beta)\eta_v C_J b^{\frac{\beta}{\alpha+1}-1} d \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u_b^\varepsilon\|_{L^{r(p_\alpha)-1}(\Omega_T)}^{r(p_\alpha)-1}.
\end{aligned}
$$

Using that $r(p_\alpha) - 1 = \frac{\beta+1}{\alpha+1} < 2$, if $\beta < \alpha + 1$, we have

$$
(1-\mathcal{X}_{\alpha+1}(\beta)) \|u_b^\varepsilon\|_{L^{r(p_\alpha)-1}(\Omega_T)} \leq (1-\mathcal{X}_{\alpha+1}(\beta))\, C \|u_b^\varepsilon\|_{L^2(\Omega_T)}.
$$

Thus, using that $p_\alpha < 2$ and gathering with the estimates in *Lemmas 2.6.1 - 2.6.3*, we end up with

$$
\begin{aligned}
\int_{(0,T)} I_{rea}^{p_\alpha} dt \leq\ & -\frac{1}{2}\eta_a a^{\alpha(p_\alpha-1)+1}\|u_a^\varepsilon\|_{L^3(\Omega_T)}^3 - \frac{1}{2}\eta_b b^{\beta(p_\alpha-1)+1}\|u_b^\varepsilon\|_{L^{r(p_\alpha)+1}(\Omega_T)}^{r(p_\alpha)+1} \\
& + C_T + \mathcal{X}_{\alpha+1}(\beta)\eta_v C_J b^{\frac{\beta}{\alpha+1}-1} d \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u_b^\varepsilon\|_{L^{r(p_\alpha)-1}(\Omega_T)}^{r(p_\alpha)-1}.
\end{aligned} \tag{2.7.76}
$$

We take $p = p_\alpha$ in (2.7.20) and we use the estimates (2.7.75), (2.7.76), together with $I_{fast}^{p_\alpha} \leq 0$ (see (2.2.8)). Then, integrating (2.7.20) in time over $(0, T)$, we obtain

$$
\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T) + C\left(\|\nabla u_a^\varepsilon\|_{L^2(\Omega_T)}^2 + \|u_a^\varepsilon\|_{L^3(\Omega_T)}^3\right) + \frac{1}{2}\eta_b b^{\beta(p_\alpha-1)+1}\|u_b^\varepsilon\|_{L^{r(p_\alpha)+1}(\Omega_T)}^{r(p_\alpha)+1} \tag{2.7.77}
$$

$$
\leq \mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T
$$

$$
+ \mathcal{X}_{2\alpha+1}(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{r(p_\alpha)-2}|\nabla v^\varepsilon|^2 dxdt \tag{2.7.78}
$$

$$
+ \mathcal{X}_{\alpha+1}(\beta)\eta_v C_J b^{\frac{\beta}{\alpha+1}-1} d \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u_b^\varepsilon\|_{L^{r(p_\alpha)-1}(\Omega_T)}^{r(p_\alpha)-1}, \tag{2.7.79}
$$

where the constant $C > 0$ in (2.7.77) depends on the diffusion and reaction coefficients, on α,β but not on ε.

It remains to estimate the terms in (2.7.78) and (2.7.79).

In order to handle the term in (2.7.78) where $\beta \geq 2\alpha + 1$, we rename the constant as

$$
R_1 := \frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{p_\alpha}}\frac{\beta}{\alpha+1}.
$$

For a pair of conjugate exponents (m, m') to be chosen, we have by Young's inequality

$$
\begin{aligned}
R_1 & \int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{r(p_\alpha)-2}|\nabla v^\varepsilon|^2 dxdt \\
& \leq \frac{1}{m}\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{m(r(p_\alpha)-2)} dxdt + \frac{1}{m'} R_1^{m'} \int_{\Omega_T} |\nabla v^\varepsilon|^{2m'} dxdt.
\end{aligned} \tag{2.7.80}
$$

As $\beta \geq 2\alpha + 1$, we have that $r(p_\alpha) - 2 = \frac{\beta-\alpha}{\alpha+1} \geq 1$. Then, we can choose m and m' such that

$$1 \leq m(r(p_\alpha) - 2) \leq r(p_\alpha) + 1 \qquad \text{and} \qquad 1 \leq m' \leq q(p_\beta) + 1,$$

i.e. using $m' = \frac{m}{m-1}$

$$\frac{1 + q(p_\beta)}{q(p_\beta)} \leq m \leq \frac{r(p_\alpha) + 1}{r(p_\alpha) - 2}. \tag{2.7.81}$$

It is worth noticing that by assumption (H3), condition (2.7.81) is not empty. Indeed, by (2.7.6), condition (2.7.81) is not empty if and only if

$$\frac{1 + q(p_\beta)}{q(p_\beta)}(r(p_\alpha) - 2) = \frac{1 + q(p_\beta)}{q(p_\beta)}\Big(\frac{1}{q(p_\beta) - 1} - 1\Big) \leq r(p_\alpha) + 1 = \frac{1}{q(p_\beta) - 1} + 2,$$

and the above inequality rewrites as

$$3q^2(p_\beta) - 2q(p_\beta) - 2 \geq 0 \qquad \Longleftrightarrow \qquad q(p_\beta) \geq \frac{1 + \sqrt{7}}{3}.$$

Therefore, using the definition of $q(p_\beta)$ in (2.7.3), the above condition holds if and only if

$$(\sqrt{7} - 2)\beta \leq 3\alpha + 5 - \sqrt{7} \quad \Longleftrightarrow \quad \beta \leq (\sqrt{7} + 2)\alpha + \sqrt{7} + 1,$$

which is guaranteed by (H3).

Hereafter, for the sake of simplicity, we denote $r_\alpha := r(p_\alpha)$. Then, in order to estimate the first term in (2.7.80), we first apply the inequality (2.7.35) and we get

$$\frac{1}{m}\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{m(r_\alpha-2)} dxdt \leq \frac{C_J}{m}\|B + dv^\varepsilon\|^{m(r_\alpha-2)}_{L^{m(r_\alpha-2)}(\Omega_T)} + \frac{C_J b^{m(r_\alpha-2)}}{m}\|u_b^\varepsilon\|^{m(r_\alpha-2)}_{L^{m(r_\alpha-2)}(\Omega_T)}. \tag{2.7.82}$$

Furthermore, since $1 \leq m(r_\alpha - 2) \leq r_\alpha + 1$, we use the inequality

$$x^\gamma = x^\gamma \mathbb{1}_{\{x\leq C\}} + x^\gamma \mathbb{1}_{\{x>C\}} \leq C^{\gamma-1}x + C^{\gamma-\lambda}x^\lambda, \tag{2.7.83}$$

for all $x \geq 0$, $C > 0$ and $1 \leq \gamma \leq \lambda$. Thus, for any $\sigma > 0$ to be chosen later as small as nedeed, there exists a constant C_σ, not depending on u_b^ε, such that

$$\frac{C_J b^{m(r_\alpha-2)}}{m}\|u_b^\varepsilon\|^{m(r_\alpha-2)}_{L^{m(r_\alpha-2)}(\Omega_T)} \leq C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)} + \sigma\|u_b^\varepsilon\|^{r_\alpha+1}_{L^{r_\alpha+1}(\Omega_T)}.$$

Therefore, (2.7.82) becomes

$$\begin{aligned}\frac{1}{m}\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{m(r_\alpha-2)} dxdt \leq{}& \frac{C_J}{m}\|B + dv^\varepsilon\|^{m(r_\alpha-2)}_{L^{m(r_\alpha-2)}(\Omega_T)} + C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)} \\ &+ \sigma\|u_b^\varepsilon\|^{r_\alpha+1}_{L^{r_\alpha+1}(\Omega_T)}.\end{aligned} \tag{2.7.84}$$

Next, as $m' \leq q(p_\beta) + 1 \leq 3$, we can use *Lemma 2.6.2* and the estimates obtained in the previous step (see (2.7.7)), in order to get, for the second term in (2.7.80),

$$\begin{aligned}\frac{1}{m'}R_1^{m'}\int_{\Omega_T} |\nabla v^\varepsilon|^{2m'} dxdt &\leq C(m', N, R_1)\Big(1 + \|u_a^\varepsilon + u_b^\varepsilon\|^{m'}_{L^{m'}(\Omega_T)}\Big) + C(m', R_1)T \\ &\leq C(m', N, R_1, T, \Omega)\Big(1 + \|u_a^\varepsilon\|^{m'}_{L^{m'}(\Omega_T)} + \|u_b^\varepsilon\|^{m'}_{L^{m'}(\Omega_T)}\Big) + C(m', R_1)T \\ &\leq C(m', N, R_1, T, \Omega)\Big(\mathcal{E}_{p_\beta}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T\Big).\end{aligned} \tag{2.7.85}$$

Finally, by gathering (2.7.80), (2.7.84), (2.7.85) and renaming the constants, the term in (2.7.78) is estimated as below

$$\begin{aligned}\mathcal{X}_{2\alpha+1}(\beta)R_1\int_{\Omega_T}\omega(u_b^\varepsilon,v^\varepsilon)^{r_\alpha-2}|\nabla v^\varepsilon|^2dxdt\\ \leq \mathcal{X}_{2\alpha+1}(\beta)C_T\left(\mathcal{E}_{p_\beta}(u_a^\varepsilon,u_b^\varepsilon,v^\varepsilon)(0)+1\right)\\ +\mathcal{X}_{2\alpha+1}(\beta)\left(\frac{C_J}{m}\|B+dv^\varepsilon\|^{m(r_\alpha-2)}_{L^{m(r_\alpha-2)}(\Omega_T)}+C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)}\right)\\ +\mathcal{X}_{2\alpha+1}(\beta)\sigma\|u_b^\varepsilon\|^{r_\alpha+1}_{L^{r_\alpha+1}(\Omega_T)},\end{aligned} \tag{2.7.86}$$

for any $\sigma>0$.

Now, we estimate the term in (2.7.79). As the term is positive if and only if $\beta\geq\alpha+1$ and that in this case $1<r_\alpha-1=\frac{\beta+1}{\alpha+1}\leq r_\alpha+1$, we can proceed as before, i.e.

$$\mathcal{X}_{\alpha+1}(\beta)\eta_vC_Jb^{\frac{\beta}{\alpha+1}-1}d\|v^\varepsilon\|_{L^\infty(\Omega_T)}\|u_b^\varepsilon\|^{r_\alpha-1}_{L^{r_\alpha-1}(\Omega_T)}\leq\mathcal{X}_{\alpha+1}(\beta)\left(C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)}+\sigma\|u_b^\varepsilon\|^{r_\alpha+1}_{L^{r_\alpha+1}(\Omega_T)}\right), \tag{2.7.87}$$

where the constant C_σ depends on σ, $\|v^\varepsilon\|_{L^\infty(\Omega_T)}$ but not on u_b^ε.

Finally, putting (2.7.86) and (2.7.87) into (2.7.77) - (2.7.79), we end up with

$$\begin{aligned}\mathcal{E}_{p_\alpha}(u_a^\varepsilon,u_b^\varepsilon,v^\varepsilon)(T)+C\left(\|\nabla u_a^\varepsilon\|^2_{L^2(\Omega_T)}+\|u_a^\varepsilon\|^3_{L^3(\Omega_T)}\right)\\ +\left(\frac{1}{2}\eta_bb^{\beta(p_\alpha-1)+1}-\sigma\Big(\mathcal{X}_{\alpha+1}(\beta)+\mathcal{X}_{2\alpha+1}(\beta)\Big)\right)\|u_b^\varepsilon\|^{r(p_\alpha)+1}_{L^{r(p_\alpha)+1}(\Omega_T)}\\ \leq\mathcal{E}_{p_\alpha}(u_a^\varepsilon,u_b^\varepsilon,v^\varepsilon)(0)+C_T+\mathcal{X}_{2\alpha+1}(\beta)C_T\left(\mathcal{E}_{p_\beta}(u_a^\varepsilon,u_b^\varepsilon,v^\varepsilon)(0)+1\right)\\ +\mathcal{X}_{2\alpha+1}(\beta)\frac{C_J}{m}\|B+v^\varepsilon\|^{m(r(p_\alpha)-2)}_{L^{m(r(p_\alpha)-2)}(\Omega_T)}\\ +\Big(\mathcal{X}_{\alpha+1}(\beta)+\mathcal{X}_{2\alpha+1}(\beta)\Big)C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)}.\end{aligned}$$

Therefore, taking $\sigma>0$ such that

$$0\leq\sigma\Big(\mathcal{X}_{\alpha+1}(\beta)+\mathcal{X}_{2\alpha+1}(\beta)\Big)<\frac{1}{2}\eta_bb^{\beta(p_\alpha-1)+1},$$

and using the estimates of *Lemmas 2.6.1 - 2.6.3*, we get (2.7.8).

- **The super-critical case** $p=2$

Now, we integrate in time over $(0,T)$ the inequality (2.7.73) and we use $\alpha=q(2)-2$ and

$\beta = r(2) - 2$, to obtain

$$\int_{(0,T)} I^2_{diff} dt \leq -\frac{4d_a a^\alpha}{(\alpha+2)^2}\|\nabla((u^\varepsilon_a)^{q(2)/2})\|^2_{L^2(\Omega_T)} - \frac{4d_b b^\beta}{(\beta+2)^2}\|\nabla((u^\varepsilon_b)^{r(2)/2})\|^2_{L^2(\Omega_T)}$$

$$+ (1-\mathcal{X}_1(\alpha))\frac{(d_a+d_v)^2}{2d_a}\frac{c^2}{a^{2-\alpha}}\left(\|1+u^\varepsilon_a\|^2_{L^2(\Omega_T)} + \|\nabla v^\varepsilon\|^4_{L^4(\Omega_T)}\right) \tag{2.7.88}$$

$$+ \mathcal{X}_1(\alpha)\frac{(d_a+d_v)^2}{2d_a}\frac{c^2\alpha}{a^2}\int_{\Omega_T}\theta(u^\varepsilon_a, v^\varepsilon)^{q(2)-2}|\nabla v^\varepsilon|^2\, dxdt$$

$$+ (1-\mathcal{X}_1(\beta))\frac{(d_b+d_v)^2}{2d_b}\frac{d^2}{b^{2-\beta}}\left(\|1+u^\varepsilon_b\|^2_{L^2(\Omega_T)} + \|\nabla v^\varepsilon\|^4_{L^4(\Omega_T)}\right) \tag{2.7.89}$$

$$+ \mathcal{X}_1(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2\beta}{b^2}\int_{\Omega_T}\omega(u^\varepsilon_b, v^\varepsilon)^{r(2)-2}|\nabla v^\varepsilon|^2\, dxdt$$

$$+ (1-\mathcal{X}_1(\alpha))\frac{d_v c^2}{2a^2}(1-\alpha)(\|1+\theta(u^\varepsilon_a, v^\varepsilon)\|^2_{L^2(\Omega_T)} + \|\nabla v^\varepsilon\|^4_{L^4(\Omega_T)}) \tag{2.7.90}$$

$$+ (1-\mathcal{X}_1(\beta))\frac{d_v d^2}{2b^2}(1-\beta)(\|1+\omega(u^\varepsilon_b, v^\varepsilon)\|^2_{L^2(\Omega_T)} + \|\nabla v^\varepsilon\|^4_{L^4(\Omega_T)}). \tag{2.7.91}$$

Using in (2.7.88) - (2.7.91) the estimates in *Lemmas 2.6.1 - 2.6.3*, we end up with

$$\int_{(0,T)} I^2_{diff} dt \leq -\frac{4d_a a^\alpha}{(\alpha+2)^2}\|\nabla((u^\varepsilon_a)^{q(2)/2})\|^2_{L^2(\Omega_T)} - \frac{4d_b b^\beta}{(\beta+2)^2}\|\nabla((u^\varepsilon_b)^{r(2)/2})\|^2_{L^2(\Omega_T)} + C_T$$

$$+ \mathcal{X}_1(\alpha)\frac{(d_a+d_v)^2}{2d_a}\frac{c^2\alpha}{a^2}\int_{\Omega_T}\theta(u^\varepsilon_a, v^\varepsilon)^{q(2)-2}|\nabla v^\varepsilon|^2\, dxdt$$

$$+ \mathcal{X}_1(\beta)\frac{(d_b+d_v)^2}{2d_b}\frac{d^2\beta}{b^2}\int_{\Omega_T}\omega(u^\varepsilon_b, v^\varepsilon)^{r(2)-2}|\nabla v^\varepsilon|^2\, dxdt. \tag{2.7.92}$$

Similarly, for the reaction term estimate in (2.7.44), we use the Hölder inequality when necessary, to obtain

$$\int_{(0,T)} I^2_{rea} dt \leq \eta_a C_L(1+A)^{\alpha+1}\|u^\varepsilon_a\|^2_{L^2(\Omega_T)} - \frac{1}{2}\eta_a a^{\alpha+1}\|u^\varepsilon_a\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)}$$

$$+ \eta_b C_L(1+B)^{\beta+1}\|u^\varepsilon_b\|^2_{L^2(\Omega_T)} - \frac{1}{2}\eta_b b^{\beta+1}\|u^\varepsilon_b\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)}$$

$$+ (1-\mathcal{X}_1(\alpha))\,\frac{c}{a^{1-\alpha}}\eta_v\|v^\varepsilon\|_{L^\infty(\Omega_T)}\|u^\varepsilon_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega_T)}$$

$$+ \mathcal{X}_1(\alpha) c\eta_v C_J\|v^\varepsilon\|_{L^\infty(\Omega_T)}\left(\|A+cv^\varepsilon\|^{q(2)-1}_{L^{q(2)-1}(\Omega_T)} + a^{\alpha+1}\|u^\varepsilon_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega_T)}\right)$$

$$+ (1-\mathcal{X}_1(\beta))\,\frac{d}{b^{1-\beta}}\eta_v\|v^\varepsilon\|_{L^\infty(\Omega_T)}\|u^\varepsilon_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega_T)}$$

$$+ \mathcal{X}_1(\beta) d\eta_v C_J\|v^\varepsilon\|_{L^\infty(\Omega_T)}\left(\|B+dv^\varepsilon\|^{r(2)-1}_{L^{r(2)-1}(\Omega_T)} + b^{\beta+1}\|u^\varepsilon_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega_T)}\right). \tag{2.7.93}$$

As $q(2)-1 = \alpha+1 < 2$, if $\alpha < 1$, and $r(2)-1 = \beta+1 < 2$, if $\beta < 1$, we have

$$(1-\mathcal{X}_1(\alpha))\,\|u^\varepsilon_a\|_{L^{q(2)-1}(\Omega_T)} \leq (1-\mathcal{X}_1(\alpha))\,C\|u^\varepsilon_a\|_{L^2(\Omega_T)},$$

and

$$(1-\mathcal{X}_1(\beta))\,\|u^\varepsilon_b\|_{L^{r(2)-1}(\Omega_T)} \leq (1-\mathcal{X}_1(\beta))\,C\|u^\varepsilon_b\|_{L^2(\Omega_T)}.$$

Thus, using in (2.7.93) the estimates in *Lemmas 2.6.1 - 2.6.3* , we end up with

$$\int_{(0,T)} I^2_{rea} dt \leq -\frac{1}{2}\eta_a a^{\alpha+1} \|u^\varepsilon_a\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)} - \frac{1}{2}\eta_b b^{\beta+1} \|u^\varepsilon_b\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)} + C_T$$
$$+ X_1(\alpha) c a^{\alpha+1} \eta_v C_J \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u^\varepsilon_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega_T)}$$
$$+ X_1(\beta) d b^{\beta+1} \eta_v C_J \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u^\varepsilon_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega_T)}. \tag{2.7.94}$$

We take $p = 2$ in (2.7.20) and we use the estimates (2.7.92), (2.7.94) and (2.2.9). Then, integrating (2.7.20) in time over $(0, T)$, we obtain

$$\mathcal{E}_2(u^\varepsilon_a, u^\varepsilon_b, v^\varepsilon)(T) + C\left(\|\nabla((u^\varepsilon_a)^{q(2)/2})\|^2_{L^2(\Omega_T)} + \|\nabla((u^\varepsilon_b)^{r(2)/2})\|^2_{L^2(\Omega_T)}\right) \tag{2.7.95}$$
$$+ \frac{1}{\varepsilon}\int_{\Omega_T} Q^2(u^\varepsilon_a, u^\varepsilon_b, v^\varepsilon) dx dt$$
$$+ \frac{1}{2}\eta_a a^{\alpha+1} \|u^\varepsilon_a\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)} + \frac{1}{2}\eta_b b^{\beta+1} \|u^\varepsilon_b\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)} \tag{2.7.96}$$
$$\leq \mathcal{E}_2(u^\varepsilon_a, u^\varepsilon_b, v^\varepsilon)(0) + C_T$$
$$+ X_1(\alpha)\frac{(d_a + d_v)^2}{2d_a}\frac{c^2\alpha}{a^2}\int_{\Omega_T} \theta(u^\varepsilon_a, v^\varepsilon)^{q(2)-2} |\nabla v^\varepsilon|^2 \, dx dt \tag{2.7.97}$$
$$+ X_1(\beta)\frac{(d_b + d_v)^2}{2d_b}\frac{d^2\beta}{b^2}\int_{\Omega_T} \omega(u^\varepsilon_b, v^\varepsilon)^{r(2)-2} |\nabla v^\varepsilon|^2 \, dx dt \tag{2.7.98}$$
$$+ X_1(\alpha) c a^{\alpha+1} \eta_v C_J \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u^\varepsilon_a\|^{q(2)-1}_{L^{q(2)-1}(\Omega_T)} \tag{2.7.99}$$
$$+ X_1(\beta) d b^{\beta+1} \eta_v C_J \|v^\varepsilon\|_{L^\infty(\Omega_T)} \|u^\varepsilon_b\|^{r(2)-1}_{L^{r(2)-1}(\Omega_T)}, \tag{2.7.100}$$

where the constant $C > 0$ in (2.7.95) depends on the diffusion and reaction coefficients, on α, β but not on ε.

It remains to estimate the terms (2.7.97) - (2.7.100).

In order to estimate the term in (2.7.97), where $\alpha \geq 1$, by analogy with (2.7.78) we first rename the constant

$$R_2 := \frac{(d_a + d_v)^2}{2d_a}\frac{c^2\alpha}{a^2}.$$

Then, for a pair of conjugate exponents (m, m') to be chosen, we have by Young's inequality

$$R_2 \int_{\Omega_T} \theta(u^\varepsilon_a, v^\varepsilon)^{q(2)-2} |\nabla v^\varepsilon|^2 \, dx dt$$
$$\leq \frac{1}{m}\int_{\Omega_T} \theta(u^\varepsilon_a, v^\varepsilon)^{m(q(2)-2)} dx dt + \frac{1}{m'} R_2^{m'} \int_{\Omega_T} |\nabla v^\varepsilon|^{2m'} dx dt. \tag{2.7.101}$$

As $\alpha \geq 1$, we have that $q(2) - 2 = \alpha \geq 1$. Then, we can choose m and m' such that

$$1 \leq m(q(2) - 2) \leq q(2) + 1 \qquad \text{and} \qquad 1 \leq m' \leq 3,$$

i.e. using $m' = \frac{m}{m-1}$

$$\frac{3}{2} \leq m \leq \frac{\alpha + 3}{\alpha}. \tag{2.7.102}$$

It is worth noticing that thanks to the assumption $\alpha \leq \frac{6}{N}$ in (H3), condition (2.7.102) is not empty.

In order to estimate the first integral in (2.7.101), recalling that $1 \le m(q(2)-2) \le q(2)+1$, we first use the inequality (2.7.35) and then the inequality (2.7.83), to get

$$\frac{1}{m}\int_{\Omega_T} \theta(u_a^\varepsilon, v^\varepsilon)^{m(q(2)-2)} dxdt \le \frac{C_J}{m}\|A + cv^\varepsilon\|^{m(q(2)-2)}_{L^{m(q(2)-2)}(\Omega_T)} + C_\sigma \|u_a^\varepsilon\|_{L^1(\Omega_T)} + \sigma \|u_a^\varepsilon\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)}, \tag{2.7.103}$$

where the constant C_σ does not depend on u_a^ε.

Next, as $m' \le 3$, we can use *Lemma 2.6.2* and the estimates of the previous steps (see (2.7.7), (2.7.8)), in order to get, for the last term in (2.7.101),

$$\begin{aligned}
\frac{1}{m'}R_2^{m'} \int_{\Omega_T} |\nabla v^\varepsilon|^{2m'} dxdt &\le C(m', N, R_2)\left(1 + \|u_a^\varepsilon + u_b^\varepsilon\|^{m'}_{L^{m'}(\Omega_T)}\right) + C(m', R_2)T \\
&\le C(m', N, R_2, T, \Omega)\left(1 + \|u_a^\varepsilon\|^{m'}_{L^{m'}(\Omega_T)} + \|u_b^\varepsilon\|^{m'}_{L^{m'}(\Omega_T)}\right) + C(m', R_2)T \\
&\le C(m', N, R_2, T, \Omega)\left(\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T\right).
\end{aligned} \tag{2.7.104}$$

Finally, by gathering (2.7.101), (2.7.103), (2.7.104) and renaming the constants, the term in (2.7.97) is estimated as below

$$\begin{aligned}
&\mathcal{X}_1(\alpha) R_2 \int_{\Omega_T} \theta(u_a^\varepsilon, v^\varepsilon)^{q(2)-2} |\nabla v^\varepsilon|^2 \, dxdt \\
&\qquad \le \mathcal{X}_1(\alpha) C_T \left(\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + 1\right) \\
&\qquad\quad + \mathcal{X}_1(\alpha)\left(\frac{C_J}{m}\|A + cv^\varepsilon\|^{m(q(2)-2)}_{L^{m(q(2)-2)}(\Omega_T)} + C_\sigma \|u_a^\varepsilon\|_{L^1(\Omega_T)}\right) \\
&\qquad\quad + \mathcal{X}_1(\alpha)\sigma \|u_a^\varepsilon\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)},
\end{aligned} \tag{2.7.105}$$

for any $\sigma > 0$.

Now, we estimate the term in (2.7.98) where $\beta \ge 1$. By analogy with the term in (2.7.97), we apply the same tools used to get (2.7.105). Thus, we define

$$R_3 := \frac{(d_b + d_v)^2}{2d_b}\frac{d^2\beta}{b^2}.$$

For a pair of conjugate exponents (m, m') to be chosen, we have by Young's inequality

$$\begin{aligned}
&R_3 \int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{r(2)-2} |\nabla v^\varepsilon|^2 \, dxdt \\
&\qquad \le \frac{1}{m}\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{m(r(2)-2)} dxdt + \frac{1}{m'}R_3^{m'} \int_{\Omega_T} |\nabla v^\varepsilon|^{2m'} dxdt.
\end{aligned} \tag{2.7.106}$$

As $\beta \ge 1$, we have that $r(2) - 2 = \beta \ge 1$. Then, we can choose m and m' such that

$$1 \le m(r(2) - 2) \le r(2) + 1 \qquad \text{and} \qquad 1 \le m' \le 3,$$

i.e. using $m' = \frac{m}{m-1}$

$$\frac{3}{2} \le m \le \frac{\beta + 3}{\beta}. \tag{2.7.107}$$

It is worth noticing that thanks to the assumption $\beta \le \frac{6}{N}$ in (H3), condition (2.7.107) is not empty.

Furthermore, as $1 \leq m(r(2)-2) \leq r(2)+1$, using the same tools used to get (2.7.103) and skipping the details, we end up with

$$\frac{1}{m}\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{m(r(2)-2)} dxdt \leq \frac{C_J}{m}\|B + dv^\varepsilon\|^{m(r(2)-2)}_{L^{m(r(2)-2)}(\Omega_T)} + C_\sigma \|u_b^\varepsilon\|_{L^1(\Omega_T)} + \sigma\|u_b^\varepsilon\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)}, \tag{2.7.108}$$

for any $\sigma > 0$ and with the constant $C_\sigma > 0$ not depending on u_b^ε.

Next, as $m' \leq 3$ and by analogy with (2.7.104), we can use *Lemma 2.6.2* and the estimates of the previous step, in order to get for the last term in (2.7.106)

$$\frac{1}{m'}R_3^{m'}\int_{\Omega_T} |\nabla v^\varepsilon|^{2m'} dxdt \leq C(m', N, R_3, T, \Omega)\left(\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T\right). \tag{2.7.109}$$

Finally, by gathering (2.7.106), (2.7.108), (2.7.109) and renaming the constants, the term in (2.7.98) is estimated as below

$$\begin{aligned} \mathcal{X}_1(\beta)R_3\int_{\Omega_T} \omega(u_b^\varepsilon, v^\varepsilon)^{r(2)-2}|\nabla v^\varepsilon|^2\, dxdt &\leq \mathcal{X}_1(\beta)C_T\left(\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + 1\right) \\ &\quad + \mathcal{X}_1(\beta)\left(\frac{C_J}{m}\|B + dv^\varepsilon\|^{m(r(2)-2)}_{L^{m(r(2)-2)}(\Omega_T)} + C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)}\right) \\ &\quad + \mathcal{X}_1(\beta)\sigma\|u_b^\varepsilon\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)}, \end{aligned} \tag{2.7.110}$$

for any $\sigma > 0$.

The estimates of the integrals in (2.7.99), (2.7.100) go similarly as before since $q(2)-1 = \alpha + 1 \geq 2$, if $\alpha \geq 1$, and $r(2) - 1 = \beta + 1 \geq 2$, if $\beta \geq 1$. Therefore, skipping the details, for any $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$\mathcal{X}_1(\alpha)ca^{\alpha+1}\eta_v C_J\|v^\varepsilon\|_{L^\infty(\Omega_T)}\|u_a^\varepsilon\|^{q(2)-1}_{L^{q(2)-1}(\Omega_T)} \leq \mathcal{X}_1(\alpha)\left(C_\sigma\|u_a^\varepsilon\|_{L^1(\Omega_T)} + \sigma\|u_a^\varepsilon\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)}\right), \tag{2.7.111}$$

and

$$\mathcal{X}_1(\beta)db^{\beta+1}\eta_v C_J\|v^\varepsilon\|_{L^\infty(\Omega_T)}\|u_b^\varepsilon\|^{r(2)-1}_{L^{r(2)-1}(\Omega_T)} \leq \mathcal{X}_1(\beta)\left(C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)} + \sigma\|u_b^\varepsilon\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)}\right), \tag{2.7.112}$$

where the constant C_σ depends on σ, $\|v^\varepsilon\|_{L^\infty(\Omega_T)}$ but not on u_a^ε and u_b^ε.

Finally, by putting (2.7.105), (2.7.110), (2.7.111), (2.7.112) into (2.7.96) - (2.7.100), we end up with

$$\begin{aligned} \mathcal{E}_2(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T) &+ C\left(\|\nabla((u_a^\varepsilon)^{q(2)/2})\|^2_{L^2(\Omega_T)} + \|\nabla((u_b^\varepsilon)^{r(2)/2})\|^2_{L^2(\Omega_T)}\right) \\ &+ \left(\frac{1}{2}\eta_a a^{\alpha+1} - 2\sigma\mathcal{X}_1(\alpha)\right)\|u_a^\varepsilon\|^{q(2)+1}_{L^{q(2)+1}(\Omega_T)} \\ &+ \left(\frac{1}{2}\eta_b b^{\beta+1} - 2\sigma\mathcal{X}_1(\beta)\right)\|u_b^\varepsilon\|^{r(2)+1}_{L^{r(2)+1}(\Omega_T)} + \frac{1}{\varepsilon}\|Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)\|^2_{L^2(\Omega_T)} \\ &\leq \mathcal{E}_2(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + C_T\left(\mathcal{X}_1(\alpha) + \mathcal{X}_1(\beta)\right)\left(\mathcal{E}_{p_\alpha}(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(0) + 1\right) \\ &\quad + \mathcal{X}_1(\alpha)\left(\frac{C_J}{m}\|A + cv^\varepsilon\|^{m(q(2)-2)}_{L^{m(q(2)-2)}(\Omega_T)} + 2C_\sigma\|u_a^\varepsilon\|_{L^1(\Omega_T)}\right) \\ &\quad + \mathcal{X}_1(\beta)\left(\frac{C_J}{m}\|B + dv^\varepsilon\|^{m(r(2)-2)}_{L^{m(r(2)-2)}(\Omega_T)} + 2C_\sigma\|u_b^\varepsilon\|_{L^1(\Omega_T)}\right). \end{aligned}$$

with C, $C_\sigma > 0$. Thus, taking $\sigma > 0$ such that

$$0 \le \sigma X_1(\alpha) < \frac{1}{4}\eta_a a^{\alpha+1} \quad \text{and} \quad 0 \le \sigma X_1(\beta) < \frac{1}{4}\eta_b b^{\beta+1},$$

i.e.

$$0 < \sigma \le \frac{1}{4}\min\left\{\eta_a a^{\alpha+1}, \eta_b b^{\beta+1}\right\},$$

and using the estimates of *Lemmas 2.6.1 - 2.6.3*, we get (2.7.9).

2.8 The existence result to the mesoscopic system

Proof of Proposition 2.5.1.

We first start by showing that for any fixed $\varepsilon > 0$, the mesoscopic system (2.1.1) - (2.1.6) has a very weak nonnegative (for each component) solution. Hereafter, we will drop the subscript ε for readability.

In order to prove that, we introduce for any $\delta > 0$ the approximated system

$$\begin{cases} \partial_t u_a^\delta - d_a\,\Delta_x u_a^\delta = f_a(u_a^\delta, u_b^\delta, v^\delta) + \frac{1}{\varepsilon}\,Q_\delta(u_a^\delta, u_b^\delta, v^\delta), & \text{in } (0,+\infty)\times\Omega, \\ \partial_t u_b^\delta - d_b\,\Delta_x u_b^\delta = f_b(u_a^\delta, u_b^\delta, v^\delta) - \frac{1}{\varepsilon}\,Q_\delta(u_a^\delta, u_b^\delta, v^\delta), & \text{in } (0,+\infty)\times\Omega, \\ \partial_t v^\delta - d_v\,\Delta_x v^\delta = f_v(u_a^\delta, u_b^\delta, v^\delta), & \text{in } (0,+\infty)\times\Omega, \end{cases} \tag{2.8.1}$$

with homogeneous Neumann boundary conditions (2.1.4) and initial data (2.1.5), (2.1.6). Here, the reaction functions are the same as those in (2.1.2) whereas Q_δ is defined by the formula

$$Q_\delta(u_a^\delta, u_b^\delta, v^\delta) := \frac{\phi(bu_b^\delta + dv^\delta)\,u_b^\delta}{1 + \delta\,\phi(bu_b^\delta + dv^\delta)\,u_b^\delta} - \frac{\psi(au_a^\delta + cv^\delta)\,u_a^\delta}{1 + \delta\,\psi(au_a^\delta + cv^\delta)\,u_a^\delta}, \tag{2.8.2}$$

and is therefore bounded when $\delta > 0$ is fixed. As a consequence, taking into account that the reaction functions f_a, f_b, f_v are upper bounded over $\mathbb{R}^3_+$, there exists a positive (for each component) classical solution to (2.8.1), (2.8.2), when $\delta > 0$ is fixed.

Next, we observe that it is possible to reproduce the a priori estimates in *Sections 2.6, 2.7*, when Q is replaced by Q^δ. Indeed, one can first check that by *Lemma 2.6.1*, it holds

$$\|v^\delta\|_{L^\infty((0,+\infty)\times\Omega)} \le K_\infty.$$

Then, using the definitions (2.2.1), (2.2.2) for $\mathcal{E}_p$ and $h_{a,p}$, $h_{b,p}$, h_p, the computation of the time derivative of $\mathcal{E}_p$ along the solutions $(u_a^\delta, u_b^\delta, v^\delta)$ to (2.8.1), (2.8.2) is almost identical to the one considered in (2.7.17) - (2.7.19), and gives the following result

$$\frac{d}{dt}\mathcal{E}_p(u_a^\delta, u_b^\delta, v^\delta) = I^p_{diff} + I^p_{rea} + I^p_{fast,\delta},$$

with I^p_{diff} and I^p_{rea} as in (2.7.17) and (2.7.18), respectively, while $I^p_{fast,\delta}$ is given by

$$\begin{aligned} I^p_{fast,\delta} &= \frac{1}{\varepsilon}\int_\Omega (\partial_1 h_p - \partial_2 h_p)\,Q_\delta dx \\ &= -\frac{1}{\varepsilon}\int_\Omega \Big([\phi(bu_b^\delta + dv^\delta)\,u_b^\delta]^{p-1} - [\psi(au_a^\delta + cv^\delta)\,u_a^\delta]^{p-1}\Big) \\ &\quad \times\left(\frac{\phi(bu_b^\delta + dv^\delta)\,u_b^\delta}{1 + \delta\,\phi(bu_b^\delta + dv^\delta)\,u_b^\delta} - \frac{\psi(au_a^\delta + cv^\delta)\,u_a^\delta}{1 + \delta\,\psi(au_a^\delta + cv^\delta)\,u_a^\delta}\right) dx. \end{aligned}$$

Moreover, as both functions $x \mapsto x^{p-1}$ and $x \mapsto \frac{x}{1+\delta x}$ are increasing on $\mathbb{R}_+$, it holds, for all $\delta > 0$ and $p \geq 1$,

$$I^p_{fast,\delta} \leq 0\,.$$

As a consequence, we see that the proof of *Energy Lemma 2.7.1* can be reproduced without any changes, yielding the estimates for any $T > 0$

$$\|u_a^\delta\|_{L^{3+\alpha}(\Omega_T)} \leq C_T, \qquad \|u_b^\delta\|_{L^{3+\beta}(\Omega_T)} \leq C_T, \tag{2.8.3}$$

where C_T does not depend on δ. Therefore, up to the extraction of subsequences, we see that, as $\delta \to 0$,

$$\begin{aligned} u_a^\delta &\rightharpoonup u_a\,, \quad \text{weakly in} \quad L^{3+\alpha}(\Omega_T), \\ u_b^\delta &\rightharpoonup u_b\,, \quad \text{weakly in} \quad L^{3+\beta}(\Omega_T), \\ v^\delta &\rightharpoonup v\,, \quad \text{weakly}^* \text{ in} \quad L^\infty(\Omega_T). \end{aligned} \tag{2.8.4}$$

Furthermore, we observe that thanks to the growing behavior of ϕ and ψ (see (H1)), there exists $C > 0$, not depending on $\delta > 0$, such that

$$\frac{\phi(bu_b^\delta + dv^\delta)\,u_b^\delta}{1 + \delta\,\phi(bu_b^\delta + dv^\delta)\,u_b^\delta} + \frac{\psi(au_a^\delta + cv^\delta)\,u_a^\delta}{1 + \delta\,\psi(au_a^\delta + cv^\delta)\,u_a^\delta} \leq C\Big(1 + (u_a^\delta)^{1+\alpha} + (u_b^\delta)^{1+\beta}\Big). \tag{2.8.5}$$

Thus, the estimate (2.8.3) gives

$$\|Q_\delta(u_a^\delta, u_b^\delta, v^\delta)\|_{L^{\frac{3+\beta}{1+\beta}}(\Omega_T)} \leq C_T. \tag{2.8.6}$$

As a consequence, thanks to the properties of the heat equation, we see that, for all $T > 0$, u_a^δ, u_b^δ and v^δ converge in fact a.e. on Ω_T, and that $(u_a^\delta, u_b^\delta, v^\delta)$ is nonnegative (for each component).

Using the estimates (2.8.3), (2.8.6), we get on the one hand

$$f_a(u_a^\delta, u_b^\delta, v^\delta) \to f_a(u_a, u_b, v) \quad \text{and} \quad f_b(u_a^\delta, u_b^\delta, v^\delta) \to f_b(u_a, u_b, v) \quad \text{in} \quad L^1(\Omega_T)\,, \tag{2.8.7}$$

and on the other hand

$$Q_\delta(u_a^\delta, u_b^\delta, v^\delta) \to Q(u_a, u_b, v) \quad \text{in } L^1(\Omega_T)\,. \tag{2.8.8}$$

Finally, since the linear parabolic operator passes to the limit (in the sense of distributions), we can pass to the limit as $\delta \to 0$ in all the terms of the (very weak form of the) approximated system (2.8.1), (2.8.2) and get a very weak solution to the mesoscopic system (2.1.1) - (2.1.6).

We now show, by a bootstrap technique, that this very weak solution is in fact a classical solution. Indeed, we first observe that, thanks to the growing behavior (2.8.5) of Q, if the solution satisfies, for $r \geq 1 + \beta$,

$$\|u_a^\delta\|_{L^r(\Omega_T)} + \|u_b^\delta\|_{L^r(\Omega_T)} \leq C_T\,,$$

one has that

$$\|Q(u_a, u_b, v)\|_{L^{\frac{r}{1+\beta}}(\Omega_T)} \leq C_T.$$

Then, thanks to the semigroup properties of the heat equation (cf. [33]), it holds that $u_a^\delta, u_b^\delta \in L^s(\Omega_T)$ for any $s \geq 1$ such that $\frac{1}{s} > \frac{1+\beta}{r} - \frac{2}{2+N}$ (note that the reaction term does not lead to extra constraints since it is bounded from above).

We can now perform the bootstrap, starting from the estimate (2.8.3) (for the first step of the bootstrap, we use moreover the fact that $\frac{3+\alpha}{1+\alpha} \geq \frac{3+\beta}{1+\beta}$, when $\alpha \leq \beta$). Defining by induction

$$S_{n+1} := (1+\beta)\, S_n - \frac{2}{2+N}, \quad S_0 := \frac{1}{3+\beta},$$

we see that S_n becomes strictly negative (for the first time) for some $n > 0$ under the condition $\beta \leq 6/N$, guaranteed by (H3). Stopping the induction for this n, we get that $u_a^\delta, u_b^\delta \in L^\infty(\Omega_T)$. Extra regularity leading to unique positive (for all components) classical solutions is then easily deduced.

In order to conclude, we observe that all the convergence results obtained so far have been performed on $[0,T]$, for any arbitrary $T > 0$. Since $(u_a^\delta, u_b^\delta, v^\delta)$ is defined on $[0,+\infty)$, by extracting subsequences, these arguments can be replicated in the time intervals $[0,2T]$, $[0,3T]$, and so on. Then by Cantor's diagonal argument, the convergences (2.8.4), (2.8.7), (2.8.8) are verified in $(0,+\infty)\times\Omega$. □

2.9 The existence result to the macroscopic system

Proof of Theorem 2.5.2.
The proof is divided in four steps and uses compactness arguments to identify the limits along subsequences. The first and second step focus on the identification of the limit (as $\varepsilon \to 0$) of the densities v^ε and $u^\varepsilon := u_a^\varepsilon + u_b^\varepsilon$, a.e. in Ω_T, respectively. In the third step we obtain the a.e. convergence of the subpopulation densities $u_a^\varepsilon, u_b^\varepsilon$ and we identify the obtained limit as the unique solution to the nonlinear system (2.1.9). Thus, the obtained convergence result is extended globally in time by a diagonal argument. Finally, in the fourth step we take the limit as ε tends to zero in the very weak formulation of the system satisfied by $u^\varepsilon = u_a^\varepsilon + u_b^\varepsilon$ and v^ε.

First step. Let $T > 0$ be arbitrarily fixed. Thanks to the control of the density v^ε given in *Lemmas 2.6.1, 2.6.2* and to the boundedness of $u_a^\varepsilon + u_b^\varepsilon$ in $L^2(\Omega_T)$ obtained in *Lemma 2.6.3*, we have that $(v^\varepsilon)_\varepsilon$ is bounded in $L^4(0,T;W^{1,4}(\Omega))$ and $(\partial_t v^\varepsilon)_\varepsilon$ is bounded in $L^2((0,T);L^2(\Omega))$. Therefore, Rellich's Theorem implies the existence of a subsequence, still denoted v^ε, and $v \in L^4(\Omega_T)$ such that, as $\varepsilon \to 0$,

$$v^\varepsilon(t,x) \longrightarrow v(t,x)\,, \qquad \text{a.e. in } \Omega_T\,. \tag{2.9.1}$$

Moreover, we have

$$\nabla v^\varepsilon \rightharpoonup \nabla v \qquad \text{weakly in } L^4(\Omega_T),$$

and by *Lemmas 2.6.1, 2.6.2* again, v is nonnegative and belongs to $L^\infty(\Omega_T)$, whereas ∇v lies in $L^4(\Omega_T)$.

Second step. The parabolic equation satisfied by the density $u^\varepsilon := u_a^\varepsilon + u_b^\varepsilon$ is

$$\partial_t u^\varepsilon = \Delta(d_a\, u_a^\varepsilon + d_b\, u_b^\varepsilon) + f_a(u_a^\varepsilon) + f_b(u_b^\varepsilon, v^\varepsilon)\,. \tag{2.9.2}$$

Thanks to *Lemma 2.7.1*, we have that $(u_a^\varepsilon)_\varepsilon$, $(u_b^\varepsilon)_\varepsilon$ are uniformly bounded in $L^2(0,T;H^1(\Omega))\cap L^3(\Omega_T)$, so that the reaction term in (2.9.2) is uniformly bounded in $L^{3/2}(\Omega_T)$. Then $(\partial_t(u_a^\varepsilon + u_b^\varepsilon))_\varepsilon$ is uniformly bounded in $L^{3/2}(0,T;W^{-1,3/2}(\Omega))$. Thus, Aubin-Lions' lemma (cf. [69]) yields a subsequence (still denoted u^ε), and a function $u \geq 0$, u bounded in $L^3(\Omega_T)$, such that, as $\varepsilon \to 0$,

$$u^\varepsilon(t,x) = u_a^\varepsilon(t,x) + u_b^\varepsilon(t,x) \longrightarrow u(t,x)\,, \qquad \text{a. e. in } \Omega_T\,, \tag{2.9.3}$$

where the nonnegativity of u follows from that of u^ε. Furthermore, we have

$$\nabla u^\varepsilon \rightharpoonup \nabla u \qquad \text{weakly in } L^2(\Omega_T).$$

Third step. The energy estimate (2.7.9) yields the estimate

$$\left\| \phi(bu_b^\varepsilon + dv^\varepsilon)u_b^\varepsilon - \psi(au_a^\varepsilon + cv^\varepsilon)u_a^\varepsilon \right\|_{L^2(\Omega_T)} \leq \sqrt{\varepsilon}\, C_T\,.$$

Therefore, $Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ converges to zero in $L^2(\Omega_T)$, as $\varepsilon \to 0$, and (up to extraction of a subsequence)

$$Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) = \phi(bu_b^\varepsilon + dv^\varepsilon)u_b^\varepsilon - \psi(au_a^\varepsilon + cv^\varepsilon)u_a^\varepsilon \longrightarrow 0, \qquad \text{a.e. in } \Omega_T. \tag{2.9.4}$$

It remains to prove the existence of the a.e. limit of subsequences of $(u_a^\varepsilon)_\varepsilon, (u_b^\varepsilon)_\varepsilon$ and to verify that this limit is the unique solution to (2.1.9), a.e. in Ω_T, corresponding to the functions u and v obtained in (2.9.3) and (2.9.1), respectively.

Let $(u_a^*(u^\varepsilon, v^\varepsilon), u_b^*(u^\varepsilon, v^\varepsilon))$ be the unique solution to (2.1.9), corresponding to $(u^\varepsilon, v^\varepsilon)$. Then, we have

$$\begin{aligned} Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) &= Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) - Q(u_a^*(u^\varepsilon, v^\varepsilon), u_b^*(u^\varepsilon, v^\varepsilon), v^\varepsilon) \\ &=: I_1^\varepsilon + I_2^\varepsilon, \end{aligned}$$

where

$$I_1^\varepsilon = \phi(bu_b^\varepsilon + dv^\varepsilon)u_b^\varepsilon - \phi(bu_b^*(u^\varepsilon, v^\varepsilon) + dv^\varepsilon)u_b^*(u^\varepsilon, v^\varepsilon),$$

and

$$I_2^\varepsilon = \psi(au_a^*(u^\varepsilon, v^\varepsilon) + cv^\varepsilon)u_a^*(u^\varepsilon, v^\varepsilon) - \psi(au_a^\varepsilon + cv^\varepsilon)u_a^\varepsilon.$$

Then, whatever is $B \geq 0$ and $v^\varepsilon \geq 0$, there exists $C(u_b^\varepsilon, u^\varepsilon, v^\varepsilon) > 0$ such that

$$I_1^\varepsilon = C(u_b^\varepsilon, u^\varepsilon, v^\varepsilon)(u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)). \tag{2.9.5}$$

Concerning I_2^ε, as $A > 0$, we have for all $u_a, v \geq 0$

$$\partial_{u_a}(\psi(au_a + cv)u_a) = \psi(au_a + cv) + \psi'(au_a + cv)au_a \geq A^\alpha > 0,$$

so that, whatever is $v^\varepsilon \geq 0$, and for some $\zeta \geq 0$,

$$\begin{aligned} I_2^\varepsilon &= \Big(\partial_{u_a}(\psi(au_a + cv)u_a)\Big)(\zeta, v^\varepsilon)\,(u_a^*(u^\varepsilon, v^\varepsilon) - u_a^\varepsilon) \\ &= \Big(\partial_{u_a}(\psi(au_a + cv)u_a)\Big)(\zeta, v^\varepsilon)\Big(u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)\Big). \end{aligned} \tag{2.9.6}$$

Thus, by (2.9.5), (2.9.6) we have

$$
\begin{aligned}
|Q(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)| &= |I_1^\varepsilon + I_2^\varepsilon| \\
&= \Big(C(u_b^\varepsilon, u^\varepsilon, v^\varepsilon) + \Big(\partial_{u_a}(\psi(au_a + cv)u_a)\Big)(\zeta, v^\varepsilon)\Big)\,|u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)| \\
&\geq A^\alpha |u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)|.
\end{aligned}
$$

Therefore, by (2.9.4) we get $|u_b^\varepsilon - u_b^*(u^\varepsilon, v^\varepsilon)| \to 0$ as $\varepsilon \to 0$, a.e. in Ω_T. Finally, the proved convergence (2.9.3) and (2.9.1) and the continuity of $u_b^*(u, v)$ with respect to its arguments, yields the desired result, i.e.

$$
u_b^\varepsilon \to u_b^*(u,v)\,, \quad u_a^\varepsilon = u^\varepsilon - u_b^\varepsilon \to u_a^*(u,v)\,, \quad \varepsilon \to 0\,, \qquad \text{a.e. in } \Omega_T.
$$

In order to conclude, we observe that all the a.e. convergence results obtained so far have been performed on $[0, T]$, for any arbitrary $T > 0$. Since $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ is defined on $[0, +\infty)$, by extracting subsequences, these arguments can be replicated in the time intervals $[0, 2T]$, $[0, 3T]$, and so on. Then by Cantor's diagonal argument, the convergences (2.9.1), (2.9.3) and (2.9.4), and the convergence of the pair $(u_a^\varepsilon, u_b^\varepsilon)$ towards the solution to (2.1.9) are verified a.e. in $(0, +\infty) \times \Omega$.

Fourth step. We will prove now that (u, v) is a very weak solution to (2.1.7) - (2.1.6), in the sense of *Theorem 2.5.2*. For this purpose, let us consider two test functions ξ_1, ξ_2 in $C_c^2([0, +\infty) \times \bar{\Omega})$, satisfying $\nabla\xi_1 \cdot \sigma = \nabla\xi_2 \cdot \sigma = 0$, on $[0, \infty) \times \partial\Omega$. Multiplying by ξ_1 the equation satisfied by $u_a^\varepsilon + u_b^\varepsilon$ and the third equation of (2.1.1) by ξ_2 and integrating over $(0, +\infty) \times \Omega$, we get

$$
\begin{aligned}
&- \int_0^\infty \int_\Omega (\partial_t \xi_1)\,(u_a^\varepsilon + u_b^\varepsilon)\,dx\,dt - \int_\Omega \xi_1(0)\,(u_a^{\text{in}} + u_b^{\text{in}})\,dx = \\
&\int_0^\infty \int_\Omega \Delta\xi_1\,(\,d_a u_a^\varepsilon + d_b u_b^\varepsilon\,)\,dx\,dt + \int_0^\infty \int_\Omega \xi_1 (f_a(u_a^\varepsilon, v^\varepsilon) + f_b(u_b^\varepsilon, v^\varepsilon))dx\,dt\,,
\end{aligned}
\tag{2.9.7}
$$

and

$$
\begin{aligned}
&- \int_0^\infty \int_\Omega (\partial_t \xi_2\,)\,v^\varepsilon\,dx\,dt - \int_\Omega \xi_2(0)\,v^{\text{in}}\,dx = \\
&\qquad d_v \int_0^\infty \int_\Omega \Delta\xi_2\,v^\varepsilon\,dx\,dt + \int_0^\infty \int_\Omega \xi_2\,f_v(u_b^\varepsilon, v^\varepsilon)\,dx\,dt\,.
\end{aligned}
\tag{2.9.8}
$$

Concerning the equation (2.9.7), the convergence results obtained in the previous steps and the estimates of *Lemma 2.6.3* allow us to pass to the limit as $\varepsilon \to 0$, in all the terms of the equation, using Lebesgue's dominated convergence theorem, thus obtaining (2.5.1).

The same conclusion holds for equation (2.9.8). Indeed, the boundedness of v^ε and its convergence (2.9.1), together with the estimates in *Lemma 2.6.3*, allow us to pass to the limit in all terms of (2.9.8), using Lebesgue's dominated convergence theorem again, thus obtaining (2.5.2).

Finally, using the weak lower semicontinuity property of the L^p norm for $p \in (1, +\infty]$, the obtained limit v verifies the estimates shown in *Lemmas 2.6.1, 2.6.2*, whereas the estimates in *Lemma 2.7.1* give the announced regularity of $u := u_a^*(u, v) + u_b^*(u, v)$. More precisely, in (2.7.9), the uniform control of $\mathcal{E}_2(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)(T)$ gives the boundedness of u in

$L^\infty(0,T;L^{2+\alpha}(\Omega))$ while the one of $\|u_a^\varepsilon\|_{L^{3+\alpha}(\Omega_T)}$ and of $\|u_b^\varepsilon\|_{L^{3+\beta}(\Omega_T)}$ gives the boundedness of u in $L^{3+\alpha}(\Omega_T)$, since $3+\alpha = \min\{3+\alpha, 3+\beta\}$. Moreover, estimates (2.7.7) and (2.7.8) imply that ∇u lies in $L^2(\Omega_T)$. Therefore, we conclude that u, v satisfy (2.1.7) - (2.1.13), in the sense of *Theorem 2.5.2*.

□

CHAPTER 3

Existence and uniqueness of strong solutions for a class of triangular cross-diffusion systems

3.1 Introduction and main result

This chapter is devoted to the analysis of a class of triangular cross-diffusion systems where the unknowns $u = u(t, x)$ and $v = v(t, x)$ represent the densities of two populations, the nonlinear diffusion term is modelled by the diffusivity function A and the reaction functions generalize the Lotka-Volterra competition interactions that we considered in *Chapter 1, 2*. From now on, we denote $\Omega_T := (0, T) \times \Omega$ where $T > 0$ is fixed and Ω is a smooth (say of class C^∞) bounded open set of $\mathbb{R}^N$, $N \geq 1$. The system writes as

$$\begin{cases} \partial_t u = \Delta(A(u, v)) + uf(u, v), & \text{on } \Omega_T, \\ \partial_t v = d_v \Delta v + vg(u, v), & \text{on } \Omega_T, \end{cases} \tag{3.1.1}$$

where $d_v > 0$ is a diffusion coefficient. We endow the system (3.1.1) with the zero flux boundary conditions

$$\nabla(A(u, v)) \cdot \sigma = \nabla v \cdot \sigma = 0, \qquad \text{on } (0, T) \times \partial\Omega, \tag{3.1.2}$$

and the nonnegative initial data

$$u(0, x) = u_{\text{in}}(x) \geq 0, \qquad v(0, x) = v_{\text{in}}(x) \geq 0, \qquad x \in \Omega. \tag{3.1.3}$$

The hypothesis that we present below concern the regularity and the monotonicity of A and the regularity of the reaction functions. The diffusivity function A is such that

$$A \in C^2(\mathbb{R}_+^2, \mathbb{R}_+) \quad \text{and} \quad A(0, v) = 0, \qquad \text{for all } v \geq 0, \tag{D1}$$

there exist $a_0, a_1, a_2 > 0$ such that for all $u, v \geq 0$,

$$0 < a_0 \leq \partial_1 A(u, v) \leq a_1 \qquad \text{and} \qquad |\partial_2 A(u, v)| \leq a_2, \tag{D2}$$

and there exists $a_3 > 0$ such that for all $u, v \geq 0$,

$$|\partial_{12} A(u, v)| \leq a_3. \tag{D3}$$

Remark 3.1.
We observe that (D1) gives for all $u, v \geq 0$

$$A(u,v) = \int_0^1 \partial_1 A(\theta u, v) u \, d\theta,$$

implying that there exists a function $B \in C^1(\mathbb{R}_+^2, \mathbb{R}_+)$ such that for all $u, v \geq 0$,

$$A(u,v) = uB(u,v) \qquad \text{with} \qquad 0 < a_0 \leq B(u,v) \leq a_1, \tag{3.1.4}$$

thanks to (D2). In addition, assumption (D3) implies for all $u, v \geq 0$,

$$|\partial_2 B(u,v)| \leq a_3. \tag{3.1.5}$$

The functions f, g are $C^1(\mathbb{R}_+^2)$ and there exist the constants $C_f, C_g, C_g' > 0$ such that for all $u, v \geq 0$

$$\begin{aligned} -C_f(1+u+v) &\leq f(u,v) \leq C_f, \\ -C_g(1+u+v) &\leq g(u,v) \leq C_g, \\ |\partial_1 g(u,v)|, |\partial_2 g(u,v)| &\leq C_g'. \end{aligned} \tag{R1}$$

This chapter aims to prove the existence, regularity and uniqueness of strong solutions of the system (3.1.1) - (3.1.3), where we refer to *strong solutions* as a class of solutions satisfying the equations in (3.1.1) a.e. in Ω_T and where the boundary and initial conditions hold in the sense of traces. Our strategy for showing the existence result (see *Theorem 3.1.1*) consists in introducing a convenient change of variable that strongly uses the monotonicity of A in (D2) and gives rise to a parabolic system in a non divergence form.

Finally, we state the main existence result for (3.1.1) - (3.1.3), in *Theorem 3.1.1* below. In order to explain the idea of proof, we will give some heuristic computations in the remaining *Subsections 3.1.1, 3.1.2, 3.1.3*. The rest of the chapter is structured as follows: *Sections 3.2 - 3.6* are devoted to the existence result and a more detailed organization is given at the end of this section. In *Section 3.7*, we prove a uniqueness result of strong solutions, provided that the space dimension $N \leq 2$. We conclude by showing a result of weak-strong stability and weak strong uniqueness, in *Section 3.8*.

Theorem 3.1.1.
Let $N \geq 1$. We assume (D1), (D2), (D3), (R1) and the initial data $u_{in} \in (L^4 \cap H^1)(\Omega)$, $v_{in} \in (L^\infty \cap H^3)(\Omega)$ compatible with Neumann boundary condition. Then, for all $T > 0$ there exists a strong nonnegative solution (u, v) of (3.1.1) - (3.1.3), in the sense that

(i) $u \in L^\infty(0,T; L^4(\Omega))$, $\partial_{x_i} u \in L^\infty(0,T; L^2(\Omega))$, $\partial_t u, \partial_{x_i,x_j} A(u,v) \in L^2(\Omega_T)$, $i, j = 1, \ldots N$,

(ii) $v \in L^\infty(\Omega_T)$, $\partial_t v \in L^4(\Omega_T)$, $\partial_{x_i,x_j} v \in L^4(\Omega_T)$, $\partial_{x_i} v \in L^8(\Omega_T)$, $i, j = 1, \ldots N$, *and* $\partial_t v \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$,

(iii) the unknowns (u,v) satisfy the equations of (3.1.1) a.e., and the trace of (u,v) on $[0,T] \times \partial\Omega$ satisfies a.e. the Neumann boundary conditions (note that since $\partial_{x_i,x_j} A(u,v) \in L^2(\Omega_T)$ and $\partial_{x_i,x_j} v \in L^4(\Omega_T)$, the quantities $\nabla A(u,v)$ and ∇v are well defined in $L^2(\Omega_T)$ and thus a.e on $[0,T] \times \partial\Omega$). Moreover the initial condition (3.1.3) is satisfied by the trace of (u,v) on $\{0\} \times \Omega$ (note that this trace exists since $\partial_t u \in L^2(\Omega_T)$ and $\partial_t v \in L^4(\Omega_T)$).

Moreover, if $N \leq 3$ it holds

$$u \in L^\infty(\Omega_T), \qquad \partial_{x_i} u \in L^4(\Omega_T), \qquad \partial_t v \in L^2(0,T;L^\infty(\Omega)).$$

3.1.1 A truncated-regularized system

In order to prove *Theorem 3.1.1*, we introduce a truncated-regularized system that, at this level, formally converges to (3.1.1), (3.1.3). For a given function $\omega = \omega(u,v)$ and a given $M > 0$, we define the truncated function

$$\omega_M(u,v) := \begin{cases} \omega(u,v), & \text{if } u < M, \\ \omega(M,v), & \text{if } u \geq M. \end{cases}$$

For given functions $\rho = \rho(t)$ defined on $[0,T]$ and $z = z(x)$ defined on Ω, we also define the extended functions

$$\check{\rho}(t) := \begin{cases} \rho(t), & \text{if } t \in [0,T], \\ \rho(0), & \text{if } t < 0, \\ \rho(T), & \text{if } t > T, \end{cases} \qquad \text{and} \qquad \widetilde{z}(x) := \begin{cases} z(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases} \tag{3.1.6}$$

Therefore, the truncated-regularized system writes as

$$\begin{cases} \partial_t u_{\varepsilon,M} = \Delta(A(u_{\varepsilon,M}, v_{\varepsilon,M})) + u_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M}), & \text{on } \Omega_T, \\ \partial_t v_{\varepsilon,M} = d_v \Delta v_{\varepsilon,M} + v_{\varepsilon,M} g_{\varepsilon,M}(u_{\varepsilon,M}, v_{\varepsilon,M}), & \text{on } \Omega_T, \\ \nabla(A(u_{\varepsilon,M}, v_{\varepsilon,M})) \cdot \sigma = \nabla v_{\varepsilon,M} \cdot \sigma = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.1.7}$$

where we define for a.e. $(t,x) \in \mathbb{R}^{N+1}$ and for all $\varepsilon > 0$

$$g_{\varepsilon,M}(u_{\varepsilon,M}(t,x), v_{\varepsilon,M}(t,x)) := (g_M(u_{\varepsilon,M}\widetilde{\check{(t,x)}, v_{\varepsilon,M}}(t,x))) *_{t,x} \varphi_\varepsilon, \tag{3.1.8}$$

where $*_{t,x}$ stands for the convolution operation in time and space variables and $(\varphi_\varepsilon)_{\varepsilon>0}$ is a family of standard mollifiers on $\mathbb{R}^{N+1}$. Moreover, we complete the system (3.1.7), (3.1.8) with the regularized initial conditions

$$\begin{aligned} u_{\varepsilon,M}(0,x) &= u_{\text{in},\varepsilon}(x) = (\widetilde{u_{\text{in}}} *_x \psi_\varepsilon)(x), \\ v_{\varepsilon,M}(0,x) &= v_{\text{in},\varepsilon}(x) = (\widetilde{v_{\text{in}}} *_x \psi_\varepsilon)(x), \qquad \forall\, x \in \Omega, \end{aligned} \tag{3.1.9}$$

where $*_x$ stands for the convolution operation in the space variable and $(\psi_\varepsilon)_{\varepsilon>0}$ is a family of standard mollifiers on $\mathbb{R}^N$.

It is worth noticing that the regularization and truncation only affect the reaction part in (3.1.7) and the initial conditions (3.1.9). In particular, we truncate the functions f and g only with respect to u, while no truncation w.r.t. the v unknown is needed, by the properties coming from the triangular structure of the system (3.1.7). Indeed, since $v_{\varepsilon,M}$ satisfies a linear heat equation, we find that the $L^\infty-$boundedness of the initial datum $v_{\text{in},\varepsilon}$ is preserved in time for a.e. t in $(0,T)$ (see (3.5.1)).

The main difficulty in showing the existence of solutions to the system (3.1.7) is the presence of the nonlinear diffusion term in the equation for $u_{\varepsilon,M}$. Our strategy consists in proving the existence for an auxiliary system in a non divergence form that is equivalent to (3.1.7) - (3.1.9). The auxiliary system will be introduced in the following subsection, using a convenient change of variable.

3.1.2 The change of variable: an auxiliary system in *non divergence* form

The aim of this paragraph is to introduce a parabolic system which is, at this level, formally equivalent to (3.1.7) - (3.1.9). The idea consists in replacing the first equation of (3.1.7) with the equation satisfied by the nonlinearity A. By assumption (D2), we can define the reciprocal U of A with respect to the first variable, that is for a given $v \geq 0$

$$a = A(u, v) \qquad \Longleftrightarrow \qquad u = U(a, v). \tag{3.1.10}$$

Using the change of variable (3.1.10), we can rewrite (3.1.1), (3.1.2) as a system in non divergence form, satisfied by the pair of unknowns (a, v). Indeed, it holds

$$\begin{aligned}\partial_t a &= \partial_1 A(u, v)\, \partial_t u + \partial_2 A(u, v)\, \partial_t v\\ &= \partial_1 A(u, v)\Big(\Delta(A(u, v)) + U(a, v) f(U(a, v), v)\Big) + \partial_2 A(u, v)\partial_t v,\end{aligned}$$

so that (3.1.1), (3.1.2) is formally equivalent to

$$\begin{cases}\partial_t a = \partial_1 A(U(a, v), v)\Delta a \\ \qquad\quad +\partial_1 A(U(a, v), v) U(a, v) f(U(a, v), v) + \partial_2 A(U(a, v), v)\partial_t v, & \text{on } \Omega_T,\\ \partial_t v = d_v \Delta v + v g(U(a, v), v), & \text{on } \Omega_T,\\ \nabla a \cdot \sigma = \nabla v \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega,\end{cases}$$

with $a_{\text{in}} := A(u_{\text{in}}, v_{\text{in}})$.

Then, for any fixed $\varepsilon, M > 0$, by defining

$$A_{\varepsilon,M} := A(u_{\varepsilon,M}, v_{\varepsilon,M}), \tag{3.1.11}$$

and using (3.1.10), we introduce the truncated system in non divergence form that is, at this level, formally equivalent to (3.1.7) - (3.1.9)

$$\begin{cases}\partial_t a_{\varepsilon,M} = \mu(a_{\varepsilon,M}, v_{\varepsilon,M})\Delta a_{\varepsilon,M} + a_{\varepsilon,M} s_M(a_{\varepsilon,M}, v_{\varepsilon,M}, \partial_t v_{\varepsilon,M}), & \text{on } \Omega_T,\\ \partial_t v_{\varepsilon,M} = d_v \Delta v_{\varepsilon,M} + v_{\varepsilon,M} g_{\varepsilon,M}(U(a_{\varepsilon,M}, v_{\varepsilon,M}), v_{\varepsilon,M}), & \text{on } \Omega_T,\\ \nabla a_{\varepsilon,M} \cdot \sigma = \nabla v_{\varepsilon,M} \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega,\end{cases} \tag{3.1.12}$$

with

$$\mu(a, v) := \partial_1 A(U(a, v), v), \tag{3.1.13}$$

and for all $a \geq 0$

$$s_M(a, v, \partial_t v) := \frac{U(a, v)}{a}\left[f_M(U(a, v), v)\partial_1 A(U(a, v), v) + \partial_2 B(U(a, v), v)\partial_t v\right], \tag{3.1.14}$$

with B defined in (3.1.4). The system (3.1.12) is completed with the intial data

$$\begin{aligned} a_{\varepsilon,M}(0, x) &= a_{\text{in},\varepsilon}(x) = A\left(u_{\text{in},\varepsilon}(x), v_{\text{in},\varepsilon}(x)\right), && x \in \Omega,\\ v_{\varepsilon,M}(0, x) &= v_{\text{in},\varepsilon}(x) = (\widetilde{v_{\text{in}}} *_x \psi_\varepsilon)(x), && x \in \Omega.\end{aligned} \tag{3.1.15}$$

By (D1), (D2) we observe that from (3.1.13), it holds, for all $a, v \geq 0$,

$$\mu \in C(\mathbb{R}_+^2) \qquad \text{and} \qquad 0 < a_0 \leq \mu(a, v) \leq a_1, \tag{3.1.16}$$

and by the L^∞−boundedness of B in (3.1.4), we get for all $a > 0$ and $v \geq 0$

$$0 < \frac{1}{a_1} \leq \frac{U(a,v)}{a} \leq \frac{1}{a_0}. \tag{3.1.17}$$

Moreover by (D1), the implicit function theorem guarantees the C^1 character of U with for all $a, v \geq 0$

$$\frac{1}{a_1} \leq \partial_1 U(a,v) = \Big(\partial_1 A\big(U(a,v),v\big)\Big)^{-1} = \frac{1}{\mu(a,v)} \leq \frac{1}{a_0}, \tag{3.1.18}$$

using (3.1.13), (3.1.16), and

$$\partial_2 U(a,v) = -\frac{\partial_2 A\big(U(a,v),v\big)}{\partial_1 A\big(U(a,v),v\big)} = -\frac{\partial_2 A\big(U(a,v),v\big)}{\mu(a,v)}, \tag{3.1.19}$$

with

$$|\partial_2 U(a,v)| \leq \frac{a_2}{a_0}.$$

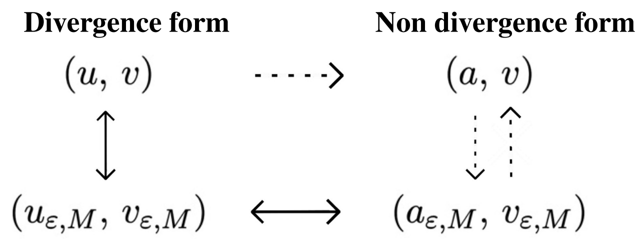
Remark 3.2.
We observe that the function $(a,v) \mapsto \frac{U(a,v)}{a}$ is $C^0(\mathbb{R}^2_+)$. Indeed from (3.1.10), the implicit function theorem gives $(a,v) \mapsto U(a,v)$ in $C^1(\mathbb{R}^2_+)$ by the regularity of A in (D1), and thus $(a,v) \mapsto \frac{U(a,v)}{a}$ in C^1 for all $a > 0$ and $v \geq 0$. We conclude by proving that the function $(a,v) \mapsto \frac{U(a,v)}{a}$ can be continuously extended at $a = 0$, for any $v \geq 0$. Indeed, for any $v \geq 0$ it holds

$$U(a,v) = \int_0^1 \partial_1 U(\theta a, v) a \, d\theta = a \int_0^1 \partial_1 U(\theta a, v) d\theta,$$

where $(a,v) \mapsto \int_0^1 \partial_1 U(\theta a, v) \, d\theta$ clearly belongs to $C^0(\mathbb{R}^2_+)$.

The strategy of proof of the existence of solutions to (3.1.12) is based on a fixed point argument. In order to do that, we introduce in the following subsection an additional regularization to (3.1.12), concerning the diffusion coefficient μ. We conclude the paragraph with the following remark.

Remark 3.3.
The order of the truncation-regularization procedures, used to construct the approximating system (3.1.12) - (3.1.15) on the pair of unknowns $(a_{\varepsilon,M}, v_{\varepsilon,M})$, is fundamental. We outline it in the figure below.



Starting from the cross-diffusion system (3.1.1) in the unknowns u, v and following the continuous arrows in the figure above, we first introduce the truncated-regularized system (3.1.7) in $(u_{\varepsilon,M}, v_{\varepsilon,M})$ and then we use the change of variables (3.1.10) to get $(a_{\varepsilon,M}, v_{\varepsilon,M})$ satisfying (3.1.12). Indeed, this is the approximation used in order to prove the existence of (u, v) : firstly we show the existence of $(a_{\varepsilon,M}, v_{\varepsilon,M})$, then the equivalence with the system satisfied by $(u_{\varepsilon,M}, v_{\varepsilon,M})$ (see *Lemma 3.4.2*) and finally we take the limit as $\varepsilon \to 0$ and $M \to +\infty$. On the other hand from the system satisfied by (u, v), following the dotted arrows we could first use the change of variables and then regularise and truncate the obtained system. However, it is not easy to justify the passage back from the (a, v) formulation of the system to the (u, v) formulation of the system, and further estimates for u.

3.1.3 Regularization of the system in *non divergence* form

Let $\varepsilon, M > 0$ be fixed, we introduce the approximating system below for all $\delta > 0$ (we only indicate the dependence of the unknowns a, v with respect to δ since ε and M are fixed),

$$\begin{cases} \partial_t a_\delta = (\mu(a_\delta, v_\delta) *_x \varphi_\delta)\Delta a_\delta + a_\delta s_M(a_\delta, v_\delta, \partial_t v_\delta), & \text{on } \Omega_T, \\ \partial_t v_\delta = d_v \Delta v_\delta + v_\delta\, g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta), & \text{on } \Omega_T, \\ \nabla a_\delta \cdot \sigma = \nabla v_\delta \cdot \sigma = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.1.20}$$

where, slightly abusing notations, $\mu(a_\delta, v_\delta)$ is identified to its extension by 0 defined on $[0, T] \times \mathbb{R}^N$. Then we denote $(\varphi_\delta)_\delta$ as a standard mollifier. Finally, s_M, $g_{\varepsilon,M}$ are defined in (3.1.14), (3.1.8) respectively and the nonnegative initial data are defined as in (3.1.15)

$$a_\delta(0, x) = a_{\text{in},\varepsilon}(x) \geq 0, \qquad v_\delta(0, x) = v_{\text{in},\varepsilon}(x) \geq 0. \tag{3.1.21}$$

Sections 3.2 - 3.6 aim to show the existence result, stated in *Theorem 3.1.1*, and are organized as follows: in *Section 3.2*, we prove the existence of solutions to system (3.1.20), (3.1.21) for any δ, ε, $M > 0$ fixed (see *Proposition 3.2.1*). In *Section 3.3*, we take the limit as $\delta \to 0$, for any fixed $\varepsilon, M > 0$ and thus we prove the existence for system (3.1.12) - (3.1.15) (see *Proposition 3.3.1*). *Section 3.4* is devoted to the proof of the equivalence between the system in *non divergence* form (3.1.12) - (3.1.15) and the original (regularized) system (3.1.7) - (3.1.9). We conclude with some $\varepsilon, M-$uniform a priori estimates in *Section 3.5* and with the proof of *Theorem 3.1.1* in *Section 3.6*, by taking the limit as $\varepsilon \to 0, M \to +\infty$.

3.2 Existence for the regularized system in *non divergence* form

The goal of this section is to prove the existence of solutions to (3.1.20), (3.1.21). In all this section, the parameters $\varepsilon, M > 0$ are fixed so that we explicitly indicate only the dependence with respect to δ of the unknowns (a, v). Thus, we state the following result

Proposition 3.2.1 (Existence of (a_δ, v_δ))**.**
We assume (D1)*,* (D2)*,* (D3)*,* (R1) *and we consider the initial data* (3.1.21)*. Then, for any* $\delta > 0$ *there exists a solution* $(a_\delta, v_\delta) : \Omega_T \to \mathbb{R}^2_+$ *satisfying the system* (3.1.20)*,* (3.1.21) *in the sense that*

(i) there exist two constants $C(T), C(M, T) > 0$ *independent on* δ *(and* ε*) such that*

$$\|v_\delta\|_{L^\infty(\Omega_T)} \leq C(T) \qquad \textit{and} \qquad \|a_\delta\|_{L^\infty(\Omega_T)} \leq C(M, T), \tag{3.2.1}$$

(ii) there exist two strictly positive constants $C_1(\varepsilon, M, T)$*,* $C_2(\varepsilon, M, T)$ *independent on* δ*, such that for all* $i, j = 1, \ldots, N$ *and* $\delta > 0$

$$\|\partial_t v_\delta\|_{L^\infty(\Omega_T)} + \|\partial_{x_i x_j} v_\delta\|_{L^\infty(\Omega_T)} + \|\partial_{x_i} v_\delta\|_{L^\infty(\Omega_T)} \leq C_1(\varepsilon, M, T), \tag{3.2.2}$$

and

$$\|\partial_t a_\delta\|_{L^2(\Omega_T)} + \|\partial_{x_i x_j} a_\delta\|_{L^2(\Omega_T)} + \|\partial_{x_i} a_\delta\|_{L^4(\Omega_T)} \leq C_2(\varepsilon, M, T), \tag{3.2.3}$$

(iii) the unknowns (a_δ, v_δ) *satisfy the first two equations of* (3.1.20)*, a.e. in* Ω_T*. Moreover, the boundary conditions in* (3.1.20) *and the initial conditions* (3.1.21) *hold in the sense of traces.*

The key ingredients of the proof of *Proposition 3.2.1* are Schauder's fixed point theorem [39] and the following existence result for a linear parabolic equation in a *non divergence* form, whose proof is presented in *Section A.4*.

Proposition 3.2.2.
We consider the following linear parabolic problem

$$\begin{cases} \partial_t b - \gamma(t,x)\Delta b = r(t,x)b, & \text{on } \Omega_T, \\ \nabla b \cdot \sigma = 0, & \text{on } (0,T)\times\partial\Omega, \\ b(0,x) = b_{in}(x) \geq 0, & \text{on } \Omega, \end{cases} \tag{3.2.4}$$

where

(i) $\gamma : \Omega_T \to \mathbb{R}_+$, γ *lies in* $W^{1,\infty}(\Omega_T)$ *and there exist two constants* $\gamma_0, \gamma_1 > 0$ *s.t.*

$$0 < \gamma_0 \leq \gamma(t,x) \leq \gamma_1, \quad \text{a.e. in } \Omega_T, \tag{3.2.5}$$

(ii) $r : \Omega_T \to \mathbb{R}$, r *lies in* $L^2(\Omega_T)$ *and there exists a constant* $R > 0$ *s.t.*

$$r(t,x) \leq R \quad \text{a.e. in } \Omega_T, \tag{3.2.6}$$

(iii) $b_{in} : \Omega \to \mathbb{R}_+$ *is s.t*

$$b_{in} \in (L^\infty \cap H^1)(\Omega). \tag{3.2.7}$$

Then, there exists a nonnegative solution b in the sense that

(i) for all $t \in (0,T)$, b *satisfies*

$$\|b(t,\cdot)\|_{L^\infty(\Omega)} \leq \|b_{in}\|_{L^\infty(\Omega)}\, e^{\int_0^t \sup_{x\in\Omega} r(s,\cdot)ds}, \tag{3.2.8}$$

so that

$$\|b\|_{L^\infty(\Omega_T)} \leq \|b_{in}\|_{L^\infty(\Omega)} e^{RT}, \tag{3.2.9}$$

(ii) there exist two constants $C_1, C_2 > 0$ *such that*

$$\|\partial_t b\|^2_{L^2(\Omega_T)} + \|\nabla b\|^2_{L^\infty(0,T;L^2(\Omega))} + \gamma_0\|\Delta b\|^2_{L^2(\Omega_T)} \leq C_1\|\nabla b_{in}\|^2_{L^2(\Omega)} + C_2\|rb\|^2_{L^2(\Omega_T)}, \tag{3.2.10}$$

with

$$C_1 := 2\frac{\gamma_1^2}{\gamma_0} + 1 \qquad \text{and} \qquad C_2 := 2\Big(\frac{\gamma_1}{\gamma_0}\Big)^2 + \frac{1}{\gamma_0} + 2,$$

(iii) *the unknown b satisfies the first equation of* (3.2.4) *a.e. in* Ω_T, *while the Neumann boundary and initial conditions in* (3.2.4) *are verified in the sense of traces.*

Proof of Proposition 3.2.1.
Let $\varepsilon > 0$ and $M > 0$ fixed. We prove the existence to (3.1.20), (3.1.21) by applying Schauder's fixed point theorem. In this regard, we introduce the Banach space

$$E := L^\infty(0,T;L^2(\Omega)),$$

and its closed bounded convex subset

$$B_Q := B_{L^\infty(\Omega_T)}(0,Q)_+ = \{w \in E \text{ s.t. } 0 \le w \le Q\},$$

where $Q > 0$ is a constant to be determined later. Then, we consider the map

$$\Phi : (a_\delta, v_\delta) \in E^2 \to (\bar{a}_\delta, \bar{v}_\delta), \tag{3.2.11}$$

where $\bar{v}_\delta$ satisfies

$$\begin{cases} \partial_t \bar{v}_\delta = d_v \Delta \bar{v}_\delta + \bar{v}_\delta g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta), & \text{in } \Omega_T, \\ \nabla \bar{v}_\delta \cdot \sigma = 0, & \text{in } (0,T) \times \partial\Omega, \\ \bar{v}_\delta(0,x) = v_{\text{in},\varepsilon} \ge 0, & \text{in } \Omega, \end{cases} \tag{3.2.12}$$

and $\bar{a}_\delta$ solves

$$\begin{cases} \partial_t \bar{a}_\delta = (\mu(a_\delta, \bar{v}_\delta) *_x \varphi_\delta) \Delta \bar{a}_\delta + \bar{a}_\delta s_M(a_\delta, \bar{v}_\delta, \partial_t \bar{v}_\delta), & \text{in } \Omega_T \\ \nabla \bar{a}_\delta \cdot \sigma = 0, & \text{in } (0,T) \times \partial\Omega, \\ \bar{a}_\delta(0,x) = a_{\text{in},\varepsilon} \ge 0, & \text{in } \Omega. \end{cases} \tag{3.2.13}$$

We shall underline the key role of the truncations and the regularizations in (3.2.12), (3.2.13) that allow to satisfy the assumptions of Schauder's fixed point theorem. Indeed, the truncation-regularization of the reaction term in (3.2.12) is fundamental to improve the regularity of the solution $\bar{v}_\delta$ of the linear parabolic equation, namely to prove the $L^\infty-$boundedness of $\partial_t \bar{v}_\delta$ (see *Lemma 3.2.4*). This regularity result is crucial to get the $L^\infty-$boundedness of s_M which in turn implies the $L^\infty-$bound of $\bar{a}_\delta$ in (3.2.27) (note that s_M depends on $\partial_t \bar{v}_\delta$). Finally, the $L^\infty-$boundedness of $\bar{a}_\delta$ and the regularization of μ are strongly used to prove the continuity of the map Φ in *Subsection 3.2.3*.

The rest of the proof aims to show that the map Φ satisfies the hypothesis of Schauder's fixed point theorem (see *Subsections 3.2.1, 3.2.2, 3.2.3*).

3.2.1 Well-posedness of the map Φ

In this paragraph, we prove that the map Φ is well defined and satisfies the property below.

Lemma 3.2.3.
Let the map Φ *be defined by* (3.2.11) - (3.2.13). *There exists a constant* $Q > 0$ *depending on* ε, M, T, C_g *such that the following inclusion holds,*

$$\Phi(B_Q \times B_Q) \subset B_Q \times B_Q.$$

• The equation satisfied by $\bar{v}_\delta$

By *Proposition 3.2.2* the problem (3.2.12), being a linear heat equation, admits a unique solution so that $\bar{v}_\delta$ is well-defined. Moreover, the nonnegativity of $\bar{v}_\delta$ follows from the nonnegativity of the initial datum $v_{\text{in},\varepsilon}$ (see (A.5.1)). Finally $\bar{v}_\delta$ satisfies the following estimates

(i) similarly as in *Subsection A.4.2*, it holds for all $\delta > 0$

$$\|\bar{v}_\delta\|_{L^\infty(\Omega_T)} \le e^{C_g T}\|v_{\text{in}}\|_{L^\infty(\Omega)}, \tag{3.2.14}$$

(ii) for all $p \in (1, +\infty)$ there exists a constant $C(p, M, T) > 0$ such that for all $\delta, \varepsilon > 0$ and $i, j = 1, \dots, N$ [58]

$$\begin{aligned} \|\partial_t \bar{v}_\delta\|_{L^p(\Omega_T)} + \|\partial_{x_i,x_j}\bar{v}_\delta\|_{L^p(\Omega_T)} &\le C(p,T)\big(\|\bar{v}_\delta\, g_{\varepsilon,M}\|_{L^p(\Omega_T)} + \|v_{\text{in},\varepsilon}\|_{L^p(\Omega)}\big) \\ &\le C(p,M,T)(1 + \|v_{\text{in}}\|_{L^\infty(\Omega)}), \end{aligned} \tag{3.2.15}$$

using the boundedness of $g_{\varepsilon,M}$ and (3.2.14),

(iii) by the Gagliardo-Nirenberg inequality [71] and (3.2.14), (3.2.15), for all $p \in (1, +\infty)$ there exist $C(p, M) > 0$ such that for all $\delta, \varepsilon > 0$

$$\|\nabla \bar{v}_\delta\|^p_{L^p(\Omega)} \le C(p,M)\Big(\|\nabla\nabla\bar{v}_\delta\|^{p/2}_{L^p(\Omega)}\|\bar{v}_\delta\|^{p/2}_{L^p(\Omega)} + \|\bar{v}_\delta\|^p_{L^p(\Omega)}\Big).$$

Thus, by integrating in time over $(0, T)$ we get

$$\begin{aligned} \|\nabla \bar{v}_\delta\|^p_{L^p(\Omega_T)} &\le C(p,M)\Big(\|\nabla\nabla\bar{v}_\delta\|^{p/2}_{L^p(\Omega_T)}\|\bar{v}_\delta\|^{p/2}_{L^\infty(0,T,L^p(\Omega))} + \|\bar{v}_\delta\|^p_{L^p(\Omega_T)}\Big) \\ &\le C(p,M,T). \end{aligned} \tag{3.2.16}$$

We conclude with the following regularity result.

Lemma 3.2.4.
Let $\bar{v}_\delta$ be the solution to (3.2.12), satisfying (3.2.14) - (3.2.16). There exists a strictly positive constant $C(\varepsilon, M, T)$ not depending on δ, such that for all $p \in (1, +\infty)$ it holds for all $i, j = 1, \dots, N$,

$$\|\partial_t^2 \bar{v}_\delta\|_{L^p(\Omega_T)} + \|\partial_{x_i}\partial_t\bar{v}_\delta\|_{L^p(\Omega_T)} + \|\partial_t\bar{v}_\delta\|_{L^\infty(\Omega_T)} \le C(\varepsilon, M, T), \tag{3.2.17}$$

and

$$\|\partial_{x_i}\bar{v}_\delta\|_{L^\infty(\Omega_T)} + \|\partial_{x_i x_j}\bar{v}_\delta\|_{L^\infty(\Omega_T)} \le C(\varepsilon, M, T). \tag{3.2.18}$$

Proof.
The key ingredient of the proof is the following continuous Sobolev embedding [39]

$$W^{k,p}(\Omega_T) \subset C^{k-1-[\frac{N+1}{p}],\gamma}(\overline{\Omega_T}), \tag{3.2.19}$$

with $k \in \mathbb{N}$, $k > \frac{N+1}{p}$ and γ such that

$$\gamma = \begin{cases} [\frac{N+1}{p}] + 1 - \frac{N+1}{p}, & \text{if } \frac{N+1}{p} \notin \mathbb{N}, \\ \text{any } \gamma \text{ in } (0,1), & \text{if } \frac{N+1}{p} \in \mathbb{N}. \end{cases} \tag{3.2.20}$$

In order to prove the estimate (3.2.17), we recall the definition (3.1.8) and we differentiate (weakly) in time the equation satisfied by $\bar{v}_\delta$ in (3.2.12), to get

$$\partial_t^2 \bar{v}_\delta - d_v \partial_t \Delta \bar{v}_\delta = (\partial_t \bar{v}_\delta) g_{\varepsilon,M} + \bar{v}_\delta (\check{\widetilde{g_M}} *_{t,x} \partial_t \varphi_\varepsilon). \tag{3.2.21}$$

Thus by assumption (R1) and inequalities (3.2.14), (3.2.15), the r.h.s. of the above equation is bounded in $L^p(\Omega_T)$ for all $p \in [1, \infty)$, uniformly in δ. Then, thanks to the maximal regularity and the Gagliardo-Nirenberg inequality, we see that there exists a constant $C(\varepsilon, M, T) > 0$ s.t. for all $i, j = 1, \ldots, N$

$$\|\partial_t^2 \bar{v}_\delta\|_{L^p(\Omega_T)} + \|\partial_t \partial_{x_i} \bar{v}_\delta\|_{L^p(\Omega_T)} + \|\partial_t \partial_{x_i x_j} \bar{v}_\delta\|_{L^p(\Omega_T)} \le C(\varepsilon, M, T), \tag{3.2.22}$$

for all $p \in (1, +\infty)$, where $C(\varepsilon, M, T)$ does not depend on δ. Therefore, $\partial_t \bar{v}_\delta$ is bounded in $W^{1,p}(\Omega_T)$ and thus in $L^\infty(\Omega_T)$, using (3.2.19), (3.2.20) with $k = 1$.

In order to prove (3.2.18), we show that $\bar{v}_\delta$ is bounded in $W^{3,p}(\Omega_T)$ for all $p \in [1, +\infty)$, for any fixed $\varepsilon, M > 0$ and then we conclude using (3.2.19), (3.2.20) with $k = 3$. By differentiating (weakly) in time (3.2.21) and recalling (3.1.8), we obtain

$$\partial_t^3 \bar{v}_\delta - d_v \partial_t^2 \Delta \bar{v}_\delta = (\partial_t^2 \bar{v}_\delta)\, g_{\varepsilon,M} + 2\partial_t \bar{v}_\delta (\check{\widetilde{g_M}} *_{t,x} \partial_t \varphi_\varepsilon) + \bar{v}_\delta (\check{\widetilde{g_M}} *_{t,x} \partial_t^2 \varphi_\varepsilon). \tag{3.2.23}$$

Then, we use (3.2.14), (3.2.15), (3.2.22) to prove that the r.h.s. is bounded in $L^p(\Omega_T)$ for all $p \in (1, +\infty)$, uniformly in δ. Therefore, $\partial_t^3 \bar{v}_\delta$ and $\partial_t^2 \partial_{x_i} \bar{v}_\delta$ are bounded in $L^p(\Omega_T)$ for all $p \in (1, +\infty)$ and for a given ε, M, by the maximal regularity and the Gagliardo-Nirenberg inequality, respectively. Finally, it remains to prove that $\partial_{x_i, x_j, x_l} \bar{v}_\delta$ is bounded in $L^p(\Omega_T)$ for all $i, j, l = 1, \ldots, N$. In order to do that, we take the (weak) space derivative in the equation (3.2.12), to get

$$\partial_{x_i} \partial_t \bar{v}_\delta - d_v \partial_{x_i} \Delta \bar{v}_\delta = (\partial_{x_i} \bar{v}_\delta) g_{\varepsilon,M} + \bar{v}_\delta (\check{\widetilde{g_M}} *_{t,x} \partial_{x_i} \varphi_\varepsilon). \tag{3.2.24}$$

Therefore using (3.2.14), (3.2.16), the maximal regularity implies

$$\|\partial_{x_i, x_j, x_l} \bar{v}_\delta\|_{L^p(\Omega_T)} \le C(\varepsilon, M, T),$$

for all $i, j, l = 1, \ldots, N$, so that $\bar{v}_\delta$ bounded in $W^{3,p}(\Omega_T)$. Then, (3.2.19), (3.2.20) with $k = 3$ allows to conclude. □

Remark 3.4.
We can improve *Lemma 3.2.4* by showing

$$\bar{v}_\delta \in C^m(\overline{\Omega_T}), \qquad \forall\, m \in \mathbb{N}, \text{ uniformly in } \delta. \tag{3.2.25}$$

We get the result by induction on k in (3.2.19). Indeed, we take the (weak) time and space derivatives in the equation (3.2.12) to get the suitable $W^{k,p}$ control. Then, we use the maximal regularity and the Gagliardo-Nirenberg inequality to conclude.

- **The equation satisfied by $\bar{a}_\delta$**

Recalling the definition of s_M in (3.1.14) and using (R1), (3.1.5), (3.2.17), then there exists a constant $S(\varepsilon, M, T) > 0$ depending on ε, M, T but not on δ, such that

$$|s_M(t, x)| \le S(\varepsilon, M, T), \qquad \text{a.e. in } \Omega_T. \tag{3.2.26}$$

Therefore thanks to *Proposition 3.2.2* and the uniqueness proven in *Subsection A.4.4*, the problem (3.2.13) admits a unique nonnegative solution so that $\bar{a}_\delta$ is well defined. Moreover by *Proposition 3.2.2* again, $\bar{a}_\delta$ verifies the following estimates (which are uniform in δ)

$$\|\bar{a}_\delta\|_{L^\infty(\Omega_T)} \le e^{TS(\varepsilon,M,T)}\|a_{\text{in},\varepsilon}\|_{L^\infty(\Omega)}, \tag{3.2.27}$$

and

$$\begin{aligned}\|\partial_t \bar{a}_\delta\|^2_{L^2(\Omega_T)} + \|\nabla \bar{a}_\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + a_0\|\Delta \bar{a}_\delta\|^2_{L^2(\Omega_T)} \\ \le \|\nabla a_{in,\varepsilon}\|^2_{L^2(\Omega)} + C\|s_M \bar{a}_\delta\|^2_{L^2(\Omega_T)},\end{aligned} \tag{3.2.28}$$

where the constant $C > 0$ depends on a_0, a_1, ε.

Proof of Lemma 3.2.3.
By (3.2.14), (3.2.27) we get the result with

$$Q := e^{T\max\{C_g, S(\varepsilon,M,T)\}} \max\left\{\|a_{in,\varepsilon}\|^2_{L^\infty(\Omega)}, \|v_{in,\varepsilon}\|^2_{L^\infty(\Omega)}\right\}. \tag{3.2.29}$$

□

3.2.2 Compactness of the map Φ

• The compactness of $\bar{v}_\delta$
The compactness of $\bar{v}_\delta$ in E follows from (3.2.14), (3.2.15) using the Rellich-Kondrakov theorem [11].

• The compactness of $\bar{a}_\delta$
In order to prove the compactness of $\bar{a}_\delta$ in E, we use the Aubin-Lions Lemma [69], stating that

$$W = \left\{w \in L^\infty(0,T;\, H^1(\Omega)) \mid \partial_t w \in L^2(0,T;(H^1)'(\Omega))\right\}$$

is compactly embedded into $L^\infty(0,T;\, L^2(\Omega))$. Then, (3.2.27), (3.2.28) allow to conclude.

3.2.3 Continuity of the map Φ

The aim of this paragraph is to prove the continuity of $\Phi : E^2 \to E^2$. In order to do that, we consider any sequence $(a_n)_n, (v_n)_n \in E$ such that $a_n \to a$ in E and $v_n \to v$ in E, as $n \to +\infty$.

For the sake of simplicity, we neglect in this paragraph the subscript δ in $a_\delta, v_\delta, \bar{a}_\delta, \bar{v}_\delta$ and the notations of the time-space extension of g_M. Hence, we introduce the notations below for all $n \in \mathbb{N}$,

$$\mu_n := \mu(a_n, \bar{v}_n), \qquad \mu := \mu(a, \bar{v}),$$
$$s_{M,n} := s_M(a_n, \bar{v}_n, \partial_t \bar{v}_n), \qquad s_M := s_M(a, \bar{v}, \partial_t \bar{v}), \qquad g_{M,n} := g_M(U(a_n, v_n), v_n),$$

and $\bar{v}_n, \bar{a}_n$ satisfy respectively

$$\partial_t \bar{v}_n = d_v \Delta \bar{v}_n + \bar{v}_n(g_{M,n} *_{t,x} \varphi_\varepsilon), \quad \nabla \bar{v}_{n|_{(0,T)\times\partial\Omega}} \cdot \sigma = 0, \quad \bar{v}_n(0,x) = v_{\text{in},\varepsilon}(x),$$

and

$$\partial_t \bar{a}_n = (\mu_n *_x \varphi_\delta)\Delta \bar{a}_n + \bar{a}_n s_{M,n}, \quad \nabla \bar{a}_{n_{|(0,T)\times\partial\Omega}} \cdot \sigma = 0, \quad \bar{a}_n(0,x) = a_{\text{in},\varepsilon}(x).$$

- **The equation satisfied by** $\bar{v}_n - \bar{v}$

We multiply by $\bar{v}_n - \bar{v}$ the equation satisfied by $\bar{v}_n - \bar{v}$ and we integrate on Ω. Hence, it holds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\bar{v}_n - \bar{v})^2 dx + d_v \int_\Omega |\nabla(\bar{v}_n - \bar{v})|^2 dx = \int_\Omega \Big(\bar{v}_n(g_{M,n} *_{t,x} \varphi_\varepsilon) - \bar{v}(g_M *_{t,x} \varphi_\varepsilon)\Big)(\bar{v}_n - \bar{v}) dx =: I_{rea}.$$

Then, we get

$$\begin{aligned} I_{rea} &= \int_\Omega (g_{M,n} *_{t,x} \varphi_\varepsilon)(\bar{v}_n - \bar{v})^2 dx + \int_\Omega \bar{v}((g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon)(\bar{v}_n - \bar{v}) dx \\ &\leq \Big(\|g_{M,n} *_{t,x} \varphi_\varepsilon\|_{L^\infty(\Omega)} + \frac{1}{2}\|\bar{v}\|^2_{L^\infty(\Omega)}\Big)\|\bar{v}_n - \bar{v}\|^2_{L^2(\Omega)} + \frac{1}{2}\|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore, we conclude for all $t \in (0, T)$

$$\begin{aligned} \frac{d}{dt}\|(\bar{v}_n - \bar{v})(t)\|^2_{L^2(\Omega)} \leq \Big(2\|(g_{M,n} *_{t,x} \varphi_\varepsilon)(t)\|_{L^\infty(\Omega)} + \|\bar{v}(t)\|^2_{L^\infty(\Omega)}\Big)\|(\bar{v}_n - \bar{v})(t)\|^2_{L^2(\Omega)} \\ + \|((g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon)(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.2.30}$$

By denoting for the sake of simplicity

$$\alpha(t) := 2\|(g_{M,n} *_{t,x} \varphi_\varepsilon)(t)\|_{L^\infty(\Omega)} + \|\bar{v}(t)\|^2_{L^\infty(\Omega)} \geq 0,$$

for all $t \in (0, T)$, and using Gronwall's lemma in (3.2.30), we get

$$\begin{aligned} &\|(\bar{v}_n - \bar{v})(t)\|^2_{L^2(\Omega)} \\ &\quad\leq e^{\int_0^t \alpha(s)ds}\|(\bar{v}_n - \bar{v})(0)\|^2_{L^2(\Omega)} \\ &\qquad+ e^{\int_0^t \alpha(s)ds}\int_0^t \|((g_{M,n} - g_M) *_{s,x} \varphi_\varepsilon)(s)\|^2_{L^2(\Omega)}\, e^{-\int_0^s \alpha(z)dz} ds \\ &\quad\leq e^{t\|\alpha\|_{L^\infty(0,t)}}\|(\bar{v}_n - \bar{v})(0)\|^2_{L^2(\Omega)} + e^{\int_0^t \alpha(s)ds}\|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega_t)} \\ &\quad\leq e^{T\|\alpha\|_{L^\infty(0,T)}}\Big(\|(\bar{v}_n - \bar{v})(0)\|^2_{L^2(\Omega)} + \|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega_T)}\Big), \end{aligned} \tag{3.2.31}$$

with

$$\begin{aligned} \|\alpha\|_{L^\infty(0,T)} &\leq 2\|g_{M,n} *_{t,x} \varphi_\varepsilon\|_{L^\infty(\Omega_T)} + \|\bar{v}\|^2_{L^\infty(\Omega_T)} \\ &\leq 2\|g_{M,n}\|_{L^\infty(\Omega_T)}\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^{N+1})} + \|\bar{v}\|^2_{L^\infty(\Omega_T)} \\ &\leq 2\sup_{n\in\mathbb{N}}\|g_{M,n}\|_{L^\infty(\Omega_T)} + \|\bar{v}\|^2_{L^\infty(\Omega_T)} =: C(M, T). \end{aligned}$$

Thus, taking the supremum in time in the l.h.s. of (3.2.31) we obtain

$$\begin{aligned} \|\bar{v}_n - \bar{v}\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq e^{TC(M,T)}\Big(\|(\bar{v}_n - \bar{v})(0)\|^2_{L^2(\Omega)} \\ &\qquad + \|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega_T)}\Big), \\ &= e^{TC(M,T)}\|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega_T)}, \end{aligned} \tag{3.2.32}$$

where the second term in (3.2.32) is estimated as follows

$$
\begin{aligned}
&\|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega_T)} \\
&\qquad \le \|g_{M,n} - g_M\|^2_{L^2(\mathbb{R}^{N+1})} \|\varphi_\varepsilon\|^2_{L^1(\mathbb{R}^{N+1})} \\
&\qquad = \|g_{M,n} - g_M\|^2_{L^2(\Omega_T)} \\
&\qquad \le C_1\Big(\|U(a_n, v_n) - U(a, v)\|^2_{L^2(\Omega_T)} + \|v_n - v\|^2_{L^2(\Omega_T)}\Big), \qquad (3.2.33)
\end{aligned}
$$

with $C_1 = \max\{|\partial_1 g_M|^2, |\partial_2 g_M|^2\}$ by (R1). Recalling that U is of class $C^1(\mathbb{R}^2_+)$, (3.2.33) becomes

$$
\|(g_{M,n} - g_M) *_{t,x} \varphi_\varepsilon\|^2_{L^2(\Omega_T)} \le C_2\Big(\|a_n - a\|^2_{L^2(\Omega_T)} + \|v_n - v\|^2_{L^2(\Omega_T)}\Big),
$$

with

$$
C_2 = C_1 \max\{|\partial_1 U|^2, |\partial_2 U|^2 + 1\}.
$$

Finally, (3.2.32) becomes

$$
\|\bar{v}_n - \bar{v}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C_2 e^{TC(M,T)}\Big(\|a_n - a\|^2_{L^2(\Omega_T)} + \|v_n - v\|^2_{L^2(\Omega_T)}\Big), \qquad (3.2.34)
$$

which converges to zero in $L^\infty(0, T; L^2(\Omega))$, as $n \to \infty$.

- **The equation satisfied by** $\bar{a}_n - \bar{a}$

We multiply by $\bar{a}_n - \bar{a}$ the equation satisfied by $\bar{a}_n - \bar{a}$ and we integrate on Ω. Thus it holds

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (\bar{a}_n - \bar{a})^2 dx &= \int_\Omega ((\mu_n *_x \varphi_\delta)\Delta\bar{a}_n - (\mu *_x \varphi_\delta)\Delta\bar{a})(\bar{a}_n - \bar{a}) dx \\
&\quad + \int_\Omega (\bar{a}_n s_{M,n} - \bar{a} s_M)(\bar{a}_n - \bar{a}) dx \\
&=: I_{diff} + I_{rea}. \qquad (3.2.35)
\end{aligned}
$$

Firstly, we observe that the diffusion coefficient $\mu_n *_x \varphi_\delta$ satisfies (3.1.16), uniformly in n, δ, for all $t \in [0, T]$ and a.e. in Ω

$$
0 < a_0 = a_0\|\varphi_\delta\|_{L^1(\mathbb{R}^N)} \le (\mu_n *_x \varphi_\delta)(t, x) \le a_1\|\varphi_\delta\|_{L^1(\mathbb{R}^N)} = a_1. \qquad (3.2.36)
$$

We additionally remark that (3.2.27) and the L^2- boundedness of $\Delta\bar{a}$ in (3.2.28) imply the $L^4(\Omega_T)$ control of $\nabla\bar{a}$, by the Gagliardo-Nirenberg inequality. Indeed, there exists a constant $C(T) > 0$ such that

$$
\|\nabla\bar{a}\|_{L^4(\Omega_T)} \le C(T)\Big(\|\nabla\nabla\bar{a}\|^{1/2}_{L^2(\Omega_T)}\|\bar{a}\|^{1/2}_{L^\infty(\Omega_T)} + \|\bar{a}\|_{L^\infty(\Omega_T)}\Big). \qquad (3.2.37)
$$

Then we compute

$$
\begin{aligned}
I_{diff} &= \int_\Omega (\mu_n *_x \varphi_\delta)\Delta(\bar{a}_n - \bar{a})(\bar{a}_n - \bar{a})dx + \int_\Omega ((\mu_n - \mu) *_x \varphi_\delta)\Delta\bar{a}(\bar{a}_n - \bar{a})dx \\
&= -\int_\Omega \nabla(\mu_n *_x \varphi_\delta)\cdot\nabla(\bar{a}_n - \bar{a})(\bar{a}_n - \bar{a})dx - \int_\Omega (\mu_n *_x \varphi_\delta)|\nabla(\bar{a}_n - \bar{a})|^2 dx \\
&\quad - \int_\Omega \nabla((\mu_n - \mu) *_x \varphi_\delta)\cdot\nabla\bar{a}\,(\bar{a}_n - \bar{a})dx - \int_\Omega ((\mu_n - \mu) *_x \varphi_\delta)\nabla\bar{a}\cdot\nabla(\bar{a}_n - \bar{a})dx \\
&\le \frac{1}{4}\int_\Omega (\mu_n *_x \varphi_\delta)|\nabla(\bar{a}_n - \bar{a})|^2 dx + \frac{1}{a_0}\int_\Omega |\nabla(\mu_n *_x \varphi_\delta)|^2|\bar{a}_n - \bar{a}|^2 dx \\
&\quad - \int_\Omega (\mu_n *_x \varphi_\delta)|\nabla(\bar{a}_n - \bar{a})|^2 dx + \frac{1}{2}\int_\Omega |(\mu_n - \mu) *_x \nabla(\varphi_\delta)|^2|\nabla\bar{a}|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |\bar{a}_n - \bar{a}|^2 dx + \frac{1}{4}\int_\Omega (\mu_n *_x \varphi_\delta)|\nabla(\bar{a}_n - \bar{a})|^2 dx \\
&\quad + \frac{1}{a_0}\int_\Omega |(\mu_n - \mu) *_x \varphi_\delta|^2\,|\nabla\bar{a}|^2 dx \\
&= -\frac{1}{2}\int_\Omega (\mu_n *_x \varphi_\delta)|\nabla(\bar{a}_n - \bar{a})|^2 dx + \frac{1}{a_0}\|\nabla\varphi_\delta\|^2_{L^1(\Omega)}\|\mu_n\|^2_{L^\infty(\Omega)}\|\bar{a}_n - \bar{a}\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\|\nabla\bar{a}\|^2_{L^4(\Omega)}\|(\mu_n - \mu) *_x \nabla\varphi_\delta\|^2_{L^4(\Omega)} + \frac{1}{2}\|\bar{a}_n - \bar{a}\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{a_0}\|\nabla\bar{a}\|^2_{L^4(\Omega)}\|(\mu_n - \mu) *_x \varphi_\delta\|^2_{L^4(\Omega)} \\
&\le \Big(\frac{1}{a_0}\|\nabla\varphi_\delta\|^2_{L^1(\Omega)}\|\mu_n\|^2_{L^\infty(\Omega)} + \frac{1}{2}\Big)\|\bar{a}_n - \bar{a}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\bar{a}\|^2_{L^4(\Omega)}\|\nabla\varphi_\delta\|^2_{L^{4/3}(\Omega)}\|\mu_n - \mu\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{a_0}\|\nabla\bar{a}\|^2_{L^4(\Omega)}\|\varphi_\delta\|^2_{L^{4/3}(\Omega)}\|\mu_n - \mu\|^2_{L^2(\Omega)},
\end{aligned}
$$

by Young's convolution inequality and Hölder's inequality.

Concerning the reaction term in (3.2.35), we have

$$
\begin{aligned}
I_{rea} &= \int_\Omega s_{M,n}(\bar{a}_n - \bar{a})^2 dx + \int_\Omega \bar{a}(s_{M,n} - s_M)(\bar{a}_n - \bar{a})dx \\
&\le \Big(\|s_{M,n}\|_{L^\infty(\Omega)} + \frac{1}{2}\|\bar{a}\|_{L^\infty(\Omega)}\Big)\|\bar{a}_n - \bar{a}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\bar{a}\|_{L^\infty(\Omega)}\|s_{M,n} - s_M\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Therefore, by gathering the obtained estimates we end up with

$$
\frac{d}{dt}\|\bar{a}_n - \bar{a}\|^2_{L^2(\Omega)} \le C_1\|\mu_n - \mu\|^2_{L^2(\Omega)} + C_2\|\bar{a}_n - \bar{a}\|^2_{L^2(\Omega)} + \|\bar{a}\|_{L^\infty(\Omega)}\|s_{M,n} - s_M\|^2_{L^2(\Omega)}, \quad (3.2.38)
$$

with

$$
\begin{aligned}
C_1 &= \|\nabla\bar{a}\|^2_{L^4(\Omega)}\Big(\|\nabla\varphi_\delta\|^2_{L^{4/3}(\Omega)} + \frac{2}{a_0}\|\varphi_\delta\|^2_{L^{4/3}(\Omega)}\Big), \\
C_2 &= \Big(\frac{2\,a_1^2}{a_0}\|\nabla\varphi_\delta\|^2_{L^1(\Omega)} + 1\Big) + \Big(2S(\varepsilon, M, T) + \|\bar{a}\|_{L^\infty(\Omega)}\Big).
\end{aligned}
$$

Therefore, by Gronwall's Lemma and taking the supremum in time for $t \in (0, T)$ we get

$$
\begin{aligned}
\|\bar{a}_n - \bar{a}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C_T\Big(&\|(\bar{a}_n - \bar{a})(0)\|^2_{L^2(\Omega)} + \|\mu_n - \mu\|^2_{L^2(\Omega_T)} \\
&+ \|s_{M,n} - s_M\|^2_{L^2(\Omega_T)}\Big). \quad (3.2.39)
\end{aligned}
$$

Before taking the limit in the previous estimate, we claim the result below.

Claim 3.1.
By the assumptions of Proposition 3.2.1, taking the limit as $n \to +\infty$ it holds

$$\|\mu_n - \mu\|_{L^2(\Omega_T)} \to 0 \qquad \text{and} \qquad \|s_{M,n} - s_M\|_{L^2(\Omega_T)} \to 0.$$

Thus using the *Claim 3.1* in (3.2.39), we conclude that the map Φ is continuous in E thanks to (3.2.34), (3.2.39). □

Proof of the Claim 3.1.
By assumptions μ is in $C(\mathbb{R}_+^2)$ and $n-$uniformly bounded. Thus by (3.2.32) and up to extraction of subsequences, still denoted a_n, $\bar{v}_n$, we have

$$\mu(a_n, \bar{v}_n) \longrightarrow \mu(a, \bar{v}), \text{ a.e. in } \Omega_T \quad \Longrightarrow \quad \mu(a_n, \bar{v}_n) \to \mu(a, \bar{v}), \text{ in } L^2(\Omega_T),$$

as $n \to +\infty$, by the dominated convergence theorem. Similarly, by the definition of s_M in (3.1.14) we have $s_M \in C(\mathbb{R}_+^2 \times \mathbb{R}, \mathbb{R})$ and $U \in C(\mathbb{R}_+^2)$. Moreover by *Lemma 3.2.4* and *Kolmogorov–M. Riesz–Fréchet Theorem* [11], it holds

$$\partial_t \bar{v}_n \longrightarrow \partial_t \bar{v}, \quad \text{in } L^2(\Omega_T).$$

Finally, by the dominated convergence theorem again we get (up to subsequences) as $n \to +\infty$

$$s_M(a_n, \bar{v}_n, \partial_t \bar{v}_n) \longrightarrow s_M(a, \bar{v}, \partial_t \bar{v}), \quad \text{in } L^2(\Omega_T).$$

□

End of proof of Proposition 3.2.1.
By the *Subsections 3.2.1 - 3.2.3*, the map Φ defined in (3.2.11) verifies the hypothesis of Schauder's fixed point theorem. Therefore, there exists at least one solution $(a_\delta \geq 0, v_\delta \geq 0)$ satisfying (3.1.20), (3.1.21) in the sense of *(iii)* in *Proposition 3.2.1*. Moreover, the estimates (3.2.14) - (3.2.16), (3.2.18), (3.2.27), (3.2.28), (3.2.37) are still verified by the fixed point (a_δ, v_δ), giving (3.2.1) - (3.2.3).

□

3.3 $\delta-$uniform estimates and $\delta-$ limit

The aim of this section is to prove the existence of $(a_{\varepsilon,M}, v_{\varepsilon,M})$, satisfying (3.1.12) - (3.1.15). The idea is to take the limit as $\delta \to 0$ in (3.1.20), (3.1.21), thanks to the $\delta-$uniform a priori estimates (3.2.1) - (3.2.3) shown in *Proposition 3.2.1*. We conclude by verifying that the limit satisfies (3.1.12) - (3.1.15) in the strong sense.

Proposition 3.3.1 (Existence of $(a_{\varepsilon,M}, v_{\varepsilon,M})$)**.**
We assume $(a_{in}, v_{in}) \in (H^1 \times L^\infty)(\Omega)$. Then, for any fixed $\varepsilon, M > 0$, there exists a solution $(a_{\varepsilon,M}, v_{\varepsilon,M})$ of (3.1.12) - (3.1.15) in the sense that

(i) $a_{\varepsilon,M} \in L^\infty(\Omega_T)$, $\partial_t a_{\varepsilon,M}, \partial_{x_i,x_j} a_{\varepsilon,M} \in L^2(\Omega_T), \partial_{x_i} a_{\varepsilon,M} \in L^4(\Omega_T)$, *for all* $i, j = 1, \ldots, N$,

(ii) $v_{\varepsilon,M}$, $\partial_t v_{\varepsilon,M}$, $\partial_{x_i x_j} v_{\varepsilon,M}$, $\partial_{x_i} v_{\varepsilon,M} \in L^\infty(\Omega_T)$, *for all* $i, j = 1, \ldots, N$,

(iii) the unknowns $(a_{\varepsilon,M}, v_{\varepsilon,M})$ *satisfy the first two equations of* (3.1.12)*, a.e. in* Ω_T*. Moreover, the boundary conditions in* (3.1.12) *and the initial conditions* (3.1.15) *hold in the sense of traces.*

Proof of Proposition 3.3.1.
Let the solution (a_δ, v_δ), satisfying the system (3.1.20), (3.1.21) in the sense of *Proposition 3.2.1*. Using the $\delta-$uniform estimates achieved in *Proposition 3.2.1*, we take the weak limit in $L^1(\Omega_T)$ as $\delta \to 0$ in (3.1.20), (3.1.21). Firstly, by (3.2.1) - (3.2.3) we get (up to subsequences) for some $a, v \in L^\infty(\Omega_T)$,

$$a_\delta \to a \quad \text{and} \quad v_\delta \to v \quad \text{a. e. in } \Omega_T, \tag{3.3.1}$$

and

$$\partial_t a_\delta \rightharpoonup \partial_t a,\ \Delta a_\delta \rightharpoonup \Delta a, \nabla a_\delta \rightharpoonup \nabla a, \quad \text{weakly in } L^2(\Omega_T), \tag{3.3.2}$$

and for all $p \in [1, +\infty)$

$$\partial_t v_\delta \rightharpoonup \partial_t v,\ \Delta v_\delta \rightharpoonup \Delta v, \nabla v_\delta \rightharpoonup \nabla v, \quad \text{weakly in } L^p(\Omega_T). \tag{3.3.3}$$

Considering the equation satisfied by a_δ in (3.1.20), we obtain by (3.3.2) that the l.h.s. converges weakly in $L^2(\Omega_T)$ to $\partial_t a$, as $\delta \to 0$. Concerning the diffusion term, we first observe that by the $C^0(\mathbb{R}^2_+)$ character of μ together with (3.3.1), we get

$$\mu(a_\delta, v_\delta) \quad \to \quad \mu(a, v), \qquad \text{a.e. in } \Omega_T, \quad \text{as } \delta \to 0.$$

Then, recalling the boundedness of μ in (3.1.16) and using the result in *Proposition A.6.1*, we end up with

$$\mu(a_\delta, v_\delta) \quad \to \quad \mu(a, v), \qquad \text{strongly in } L^p(\Omega_T),\ \forall p < \infty, \ \text{as } \delta \to 0, \tag{3.3.4}$$

implying the strong L^p convergence below

$$\mu(a_\delta, v_\delta) *_x \varphi_\delta \quad \to \quad \mu(a, v), \qquad \text{strongly in } L^p(\Omega_T),\ \forall p < \infty, \ \text{as } \delta \to 0. \tag{3.3.5}$$

Indeed, from (3.3.4) we have $\mu(a, v) \in L^p(\Omega_T)$ and using the regularization by convolution, it holds (abusing notations, we denote by $\mu(a, v)$ the extended by zero function in the space variable x of the function defined in (3.1.13))

$$\mu(a, v) *_x \varphi_\delta \quad \to \quad \mu(a, v), \qquad \text{strongly in } L^p(\Omega_T),\ \forall p < \infty, \ \text{as } \delta \to 0, \tag{3.3.6}$$

then, we compute

$$\begin{aligned}
&\|\mu(a_\delta, v_\delta) *_x \varphi_\delta - \mu(a, v)\|_{L^p(\Omega_T)} \\
&\qquad \le \|(\mu(a_\delta, v_\delta) - \mu(a, v)) *_x \varphi_\delta\|_{L^p(\Omega_T)} + \|\mu(a, v) *_x \varphi_\delta - \mu(a, v)\|_{L^p(\Omega_T)} \\
&\qquad \le \|\varphi_\delta\|_{L^1(\mathbb{R}^N)} \|\mu(a_\delta, v_\delta) - \mu(a, v)\|_{L^p(\Omega_T)} + \|\mu(a, v) *_x \varphi_\delta - \mu(a, v)\|_{L^p(\Omega_T)} \\
&\qquad = \|\mu(a_\delta, v_\delta) - \mu(a, v)\|_{L^p(\Omega_T)} + \|\mu(a, v) *_x \varphi_\delta - \mu(a, v)\|_{L^p(\Omega_T)},
\end{aligned} \tag{3.3.7}$$

giving (3.3.5) by the obtained convergences in (3.3.4), (3.3.6). Therefore, by combining with the weak $L^2(\Omega_T)$ convergence of Δa_δ in (3.3.2), recalling the L^∞ bound for $\mu(a_\delta, v_\delta) *_x \varphi_\delta$:

$$0 \le \mu(a_\delta, v_\delta) *_x \varphi_\delta \le a_1 \|\varphi_\delta\|_{L^1(\mathbb{R}^N)} \le a_1, \qquad \text{a.e. in } \Omega_T,$$

and taking $p = 2$ in (3.3.5), then *Proposition A.6.2* implies

$$(\mu(a_\delta, v_\delta) *_x \varphi_\delta)\Delta a_\delta \quad \rightharpoonup \quad \mu(a,v)\Delta a, \qquad \text{weakly in } L^1(\Omega_T), \;\; \text{as } \delta \to 0. \tag{3.3.8}$$

Concerning the reaction term, we recall the definition of s_M in (3.1.14)

$$\begin{aligned} s_M(a_\delta, v_\delta, \partial_t v_\delta) = \frac{U(a_\delta, v_\delta)}{a_\delta} f_M(U(a_\delta, v_\delta), v_\delta)\partial_1 A(U(a_\delta, v_\delta), v_\delta) \\ + \frac{U(a_\delta, v_\delta)}{a_\delta}\partial_2 B(U(a_\delta, v_\delta), v_\delta)\partial_t v_\delta =: I_\delta + II_\delta. \end{aligned} \tag{3.3.9}$$

Since $(a,v) \mapsto \frac{U(a,v)}{a}$, $\partial_1 A$ and f_M are $C^0(\mathbb{R}^2_+)$ functions (see *Remark 3.2*), then it holds by (3.3.1)

$$\frac{U(a_\delta, v_\delta)}{a_\delta} \to \frac{U(a,v)}{a}, \text{ a.e. in } \Omega_T, \quad \partial_1 A(U(a_\delta, v_\delta), v_\delta) \to \partial_1 A(U(a,v),v), \text{ a.e. in } \Omega_T,$$

and

$$f_M(U(a_\delta, v_\delta), v_\delta) \to f_M(U(a,v),v), \text{ a.e. in } \Omega_T,$$

and thus

$$I_\delta \quad \to \quad \frac{U(a,v)}{a} f_M(U(a,v),v)\partial_1 A(U(a,v),v), \text{ a.e. in } \Omega_T.$$

In addition, $(a,v) \mapsto \frac{U(a,v)}{a}$, f_M and $\partial_1 A$ are $\delta-$uniformly bounded by (3.1.17), (D2), implying

$$\|I_\delta\|_{L^\infty(\Omega_T)} \le C, \tag{3.3.10}$$

where $C > 0$ does not depend on δ. Therefore, by *Proposition A.6.1* we end up with

$$I_\delta \longrightarrow \frac{U(a,v)}{a} f_M(U(a,v),v)\partial_1 A(U(a,v),v), \text{ strongly in } L^p(\Omega_T), \text{ for all } p < \infty, \tag{3.3.11}$$

in particular, weakly in $L^p(\Omega_T)$, for all $p < \infty$. Concerning the second term in (3.3.9), from the continuity of $(a,v) \mapsto \frac{U}{a}$, U, $\partial_2 B$, we obtain

$$\frac{U(a_\delta, v_\delta)}{a_\delta}\partial_2 B(U(a_\delta, v_\delta), v_\delta) \quad \to \quad \frac{U(a,v)}{a}\partial_2 B(U(a,v),v), \qquad \text{a.e. in } \Omega_T,$$

and using (3.1.5), (3.1.17), we get the $L^\infty(\Omega_T)$ boundedness below

$$\left\| \frac{U(a_\delta, v_\delta)}{a_\delta}\partial_2 B(U(a_\delta, v_\delta), v_\delta) \right\|_{L^\infty(\Omega_T)} \le C.$$

Thus, *Proposition A.6.1* again implies

$$\frac{U(a_\delta, v_\delta)}{a_\delta}\partial_2 B(U(a_\delta, v_\delta), v_\delta) \quad \to \quad \frac{U(a,v)}{a}\partial_2 B(U(a,v),v), \qquad \text{strongly in } L^p(\Omega_T),$$

for all $p \in [1,\infty)$. However, the weak convergence of $\partial_t v_\delta$ in (3.3.3) gives by *Proposition A.6.2*

$$II_\delta \quad \rightharpoonup \quad U(a,v)\partial_2 B(U(a,v),v)\partial_t v, \quad \text{weakly in } L^p(\Omega_T), \text{ for all } p < \infty. \tag{3.3.12}$$

Hence, by gathering (3.3.9), (3.3.11) and (3.3.12), we conclude

$$s_M(a_\delta, v_\delta, \partial_t v_\delta) \quad \rightharpoonup \quad s_M(a,v,\partial_t v), \qquad \text{weakly in } L^p(\Omega_T), \text{ for all } p < \infty. \tag{3.3.13}$$

Finally, the L^∞ boundedness of a_δ in (3.2.1) and the a.e. convergence in (3.3.1) give by Proposition A.6.1,

$$a_\delta \quad \longrightarrow \quad a, \qquad \text{strongly in } L^p(\Omega_T), \ \forall p < \infty,$$

that, together with (3.3.13), implies (by *Proposition A.6.2*)

$$a_\delta s_M(a_\delta, v_\delta, \partial_t v_\delta) \quad \to \quad a s_M(a, v, \partial_t v), \qquad \text{weakly in } L^p(\Omega_T), \ \text{ as } \delta \to 0. \tag{3.3.14}$$

Similarly, we take the limit in the equation satisfied by v_δ. From (3.3.3), the evolution term $\partial_t v_\delta$ and the diffusion term Δv_δ converge weakly in $L^p(\Omega_T)$ for all $p \in [1, \infty)$. Concerning the reaction term $v_\delta g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta)$, from the $C^0(\mathbb{R}^2_+)$ character of U and $g_{\varepsilon,M}$ and using the a.e. convergences in (3.3.1), we get (for any fixed $\varepsilon, M > 0$) as $\delta \to 0$,

$$g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta) \quad \to \quad g_{\varepsilon,M}(U(a, v), v), \qquad \text{a.e. in } \Omega_T.$$

Therefore, by (3.3.1) it holds

$$v_\delta \, g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta) \quad \to \quad v \, g_{\varepsilon,M}(U(a, v), v), \qquad \text{a.e. in } \Omega_T.$$

Moreover, (3.2.1) gives the L^∞ boundedness of $g_{\varepsilon,M}$, i.e.

$$\|g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta)\|_{L^\infty(\Omega_T)} \le C \quad \Longrightarrow \quad \|v_\delta \, g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta)\|_{L^\infty(\Omega_T)} \le C.$$

Therefore, *Proposition A.6.1* implies

$$v_\delta g_{\varepsilon,M}(U(a_\delta, v_\delta), v_\delta) \quad \to \quad v \, g_{\varepsilon,M}(U(a, v), v), \qquad \text{strongly in } L^p(\Omega_T), \ \forall \, p \in [1, \infty), \tag{3.3.15}$$

and, in particular weakly in $L^p(\Omega_T)$, for any $p \in [1, \infty)$. Thus, we take the weak limit in $L^1(\Omega_T)$ as $\delta \to 0$, in the first two equations of (3.1.20), using (3.3.2), (3.3.3) (for the evolution terms), (3.3.8), (3.3.3) (for the diffusion terms) and (3.3.14), (3.3.15) (for the reaction terms). Finally, we pass to the limit in the boundary condition of (3.1.20) using the weak convergence of $\Delta a_\delta, \Delta v_\delta$ in (3.3.2), (3.3.3) and the continuity of the trace operator Tr $: H^2(\Omega) \to H^1(\partial\Omega)$. Hereafter, we restore the $\varepsilon, M-$dependences notations so that we refer to the a.e limit of a_δ, v_δ as $a_{\varepsilon,M}, v_{\varepsilon,M}$, respectively, and the weak limit of $\partial_t a_\delta, \Delta a_\delta, \nabla a_\delta$ and $\partial_t v_\delta, \Delta v_\delta, \nabla a_\delta$ as $\partial_t a_{\varepsilon,M}, \Delta a_{\varepsilon,M}, \nabla a_{\varepsilon,M}$ and $\partial_t v_{\varepsilon,M}, \Delta v_{\varepsilon,M}, \nabla v_{\varepsilon,M}$, respectively. Finally, using the weak lower semicontinuity property of the $L^p(\Omega_T)$ norm for $p \in (1, +\infty]$ and estimates (3.2.1) - (3.2.3), we conclude that $a_{\varepsilon,M}, v_{\varepsilon,M}$ satisfy points *(i), (ii), (iii)* of *Proposition 3.3.1*. □

3.4 Existence for the truncated-regularized system in the variables (u, v)

In this section, we prove the existence of strong solutions to the truncated-regularized system (3.1.7) - (3.1.9) for the unknowns $(u_{\varepsilon,M}, v_{\varepsilon,M})$ (see *Proposition 3.4.1*). The crucial result is the *Lemma 3.4.2* below, proving the equivalence between the system in a *non divergence* form (3.1.12) - (3.1.15), satisfied by $(a_{\varepsilon,M}, v_{\varepsilon,M})$, and the system (3.1.7) - (3.1.9), satisfied by $(u_{\varepsilon,M}, v_{\varepsilon,M})$. In other words, taking a solution $(a_{\varepsilon,M}, v_{\varepsilon,M})$ of (3.1.12) - (3.1.15) in the sense of *Proposition 3.3.1*, there exists a solution to (3.1.7) - (3.1.9) in the sense of *Proposition 3.4.1*, according to the change of variable (3.1.10).

Proposition 3.4.1 (Existence of $(u_{\varepsilon,M}, v_{\varepsilon,M})$).
We assume (D1), (D2), (D3), (R1) and we consider the initial data $u_{in} \in (L^4 \cap H^1)(\Omega)$, $v_{in} \in (L^\infty \cap H^3)(\Omega)$ compatible with Neumann boundary condition. Then, for any fixed $\varepsilon, M > 0$ there exists a nonnegative solution $(u_{\varepsilon,M}, v_{\varepsilon,M})$ for the system (3.1.7) - (3.1.9), in the sense that

(i) $u_{\varepsilon,M} \in L^\infty(\Omega_T)$, $\partial_t u_{\varepsilon,M}, \partial_{x_i,x_j} A(u_{\varepsilon,M}, v_{\varepsilon,M}) \in L^2(\Omega_T)$, $\partial_{x_i} u_{\varepsilon,M} \in L^4(\Omega_T)$,

(ii) $v_{\varepsilon,M}$, $\partial_t v_{\varepsilon,M}$, $\partial_{x_i,x_j} v_{\varepsilon,M}$, $\partial_{x_i} v_{\varepsilon,M} \in L^\infty(\Omega_T)$, $i, j = 1, \dots, N$,

(iii) the unknowns $(u_{\varepsilon,M}, v_{\varepsilon,M})$ satisfy the equations of (3.1.7) a.e. in Ω_T while the trace of $(u_{\varepsilon,M}, v_{\varepsilon,M})$ on $[0, T] \times \partial\Omega$ satisfies a.e. the Neumann boundary conditions. Moreover, the initial conditions (3.1.9) are satisfied by the traces of $(u_{\varepsilon,M}, v_{\varepsilon,M})$ on $\{0\} \times \Omega$.

Lemma 3.4.2.
Let $(a_{\varepsilon,M}, v_{\varepsilon,M})$ be a solution to (3.1.12) - (3.1.15) given by Proposition 3.3.1. Then, $(u_{\varepsilon,M}, v_{\varepsilon,M})$ satisfies the system (3.1.7) - (3.1.9) in the sense of Proposition 3.4.1, with $u_{\varepsilon,M} := U(a_{\varepsilon,M}, v_{\varepsilon,M})$.

Proof of Lemma 3.4.2.
For the sake of simplicity, in this proof we neglect the subscripts ε, M in $a_{\varepsilon,M}, v_{\varepsilon,M}$ since ε and M are fixed.

For any ε, M fixed, let (a, v) be a solution to (3.1.12) - (3.1.15) given by *Proposition 3.3.1*. Then, we can find by density a sequence $(a_n, v_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega_T)$ such that

$$
\begin{aligned}
&a_n \to a, && \partial_t a_n \to \partial_t a, && \partial_{x_i} a_n \to \partial_{x_i} a, && \partial_{x_i x_j} a_n \to \partial_{x_i x_j} a,\\
&v_n \to v, && \partial_t v_n \to \partial_t v, && \partial_{x_i} v_n \to \partial_{x_i} v,
\end{aligned}
$$

strongly in $L^2(\Omega_T)$ for any $i, j = 1, \dots, N$, as $n \to +\infty$.

Then we set $u_n = U(a_n, v_n)$ and $u = U(a, v)$ and recall that (D2), (3.1.10) imply the C^1 character of U. Hence, we find for any $\xi \in C_c^\infty((0, T) \times \Omega)$

$$
\begin{aligned}
\int_{\Omega_T} u\partial_t\xi dxdt &= \lim_{n\to\infty} \int_{\Omega_T} u_n \partial_t \xi dxdt\\
&= \lim_{n\to\infty} -\int_{\Omega_T} (\partial_1 U(a_n, v_n)\partial_t a_n + \partial_2 U(a_n, v_n)\partial_t v_n)\xi dxdt\\
&= -\int_{\Omega_T} (\partial_1 U(a, v)\partial_t a + \partial_2 U(a, v)\partial_t v)\xi dxdt,
\end{aligned}
$$

implying that u has the weak time derivative

$$\partial_t u = \partial_1 U(a, v)\partial_t a + \partial_2 U(a, v)\partial_t v, \tag{3.4.1}$$

which is bounded in $L^2(\Omega_T)$. Likewise, we find that u has the weak ∂_{x_i} derivative bounded in $L^4(\Omega_T)$ and given by

$$\partial_{x_i} u = \partial_1 U(a, v)\partial_{x_i} a + \partial_2 U(a, v)\partial_{x_i} v, \qquad \forall i = 1, \dots N. \tag{3.4.2}$$

Thus the claimed bounds for $\partial_t u$ and $\partial_{x_i} u$ are shown. Note also that $\partial_{x_i x_j} A(u, v) = \partial_{x_i x_j} a$. Then, since (a.e. in Ω_T) we have

$$\partial_t a - \mu\Delta a = U(a, v) f_M(U(a, v), v)\partial_1 A(U(a, v), v) + \partial_2 A(U(a, v), v)\partial_t v,$$

we see that

$$\frac{1}{\mu}\partial_t a - \frac{1}{\mu}\partial_2 A \partial_t v - \Delta a = U(a,v) f_M(U(a,v),v),$$

and using (3.4.1) (and $\partial_1 A(u,v)\,\partial_1 U(A(u,v),v) = 1$), we end up with (a.e. in Ω_T)

$$\partial_t u - \Delta(A(u,v)) = u f_M(u,v).$$

Moreover, the trace of ∇a on $[0,T]\times\partial\Omega$ is the trace of $\nabla(A(u,v))$, so that the Neumann boundary condition in (3.1.7) is satisfied. The same holds for the initial conditions. Finally, the equation, boundary condition and initial condition related to v are identical in the system satisfied by (a,v) and in the system satisfied by (u,v) (when $U(a,v)$ is replaced by u in the equations). □

Proof of Proposition 3.4.1.
It follows as a consequence of *Proposition 3.3.1* and identity (3.4.2). In particular, $(u_{\varepsilon,M}, v_{\varepsilon,M})$ satisfies $(i), (ii)$ thanks to the regularity of $(a_{\varepsilon,M}, v_{\varepsilon,M})$ shown in $(i), (ii)$ of *Proposition 3.3.1* and identity (3.4.2), using (D2), (3.1.10), (3.1.17). □

3.5 $\varepsilon, M-$uniform estimates and $\varepsilon, M-$ limit

In *Lemma 3.5.1* below, we prove some $\varepsilon, M-$uniform estimates satisfied by $(u_{\varepsilon,M}, v_{\varepsilon,M})$, in order to pass to the limit and show *Theorem 3.1.1*. Hereafter, we will denote $C(T)$ as a strictly positive constant which depends on T and may change from line to line in the computations.

Lemma 3.5.1 ($\varepsilon, M-$uniform estimates)**.**
Let $(u_{\varepsilon,M}, v_{\varepsilon,M})$ be given by Proposition 3.4.1 and recall that $\nabla u_{in} \in L^2(\Omega)$. Then, the following estimates hold uniformly in ε and M,

(i) there exists a constant $C_1(T) > 0$ such that for all $\varepsilon, M > 0$,

$$\|v_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \le C_1(T), \tag{3.5.1}$$

(ii) there exists a constant $C_2(T) > 0$ such that for all $\varepsilon, M > 0$,

$$\|u_{\varepsilon,M}\|_{L^\infty(0,T;L^4(\Omega))} \le C_2(T), \tag{3.5.2}$$

(iii) there exists a constant $C_3(T) > 0$ such that for all $i, j = 1, \dots, N$ and $\varepsilon, M > 0$

$$\|\partial_t v_{\varepsilon,M}\|_{L^4(\Omega_T)} + \|\partial_{x_i x_j} v_{\varepsilon,M}\|_{L^4(\Omega_T)} + \|\partial_{x_i} v_{\varepsilon,M}\|_{L^8(\Omega_T)} \le C_3(T), \tag{3.5.3}$$

and

$$\|\partial_{x_i} v_{\varepsilon,M}\|_{L^\infty(0,T;L^2(\Omega))} \le C_3(T), \tag{3.5.4}$$

(iv) there exists a constant $C_4(T) > 0$ such that for all $i, j = 1, \dots, N$ and $\varepsilon, M > 0$

$$\begin{aligned}\|\partial_t u_{\varepsilon,M}\|_{L^2(\Omega_T)} + \|\partial_{x_i} u_{\varepsilon,M}\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_{x_i}(A(u_{\varepsilon,M}, v_{\varepsilon,M}))\|_{L^\infty(0,T;L^2(\Omega))} \\ + \|\partial_{x_i x_j}(A(u_{\varepsilon,M}, v_{\varepsilon,M}))\|_{L^2(\Omega_T)} \le C_4(T),\end{aligned} \tag{3.5.5}$$

(v) there exists a constant $C_5(T) > 0$ *such that*

$$\|\partial_t v_{\varepsilon,M}\|_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq C_5(T), \tag{3.5.6}$$

and if $N \leq 3$

$$\|\partial_t v_{\varepsilon,M}\|_{L^2(0,T;L^\infty(\Omega))} \leq C_5(T), \tag{3.5.7}$$

and

$$\|u_{\varepsilon,M}\|_{L^\infty(\Omega_T)} + \|\partial_{x_i} u_{\varepsilon,M}\|_{L^4(\Omega_T)} \leq C_5(T). \tag{3.5.8}$$

Proof.
(i) Estimate (3.5.1) follows from the first inequality of (3.2.1).

(ii) Firstly, we prove the $L^2(\Omega_T)$ boundedness of $\nabla v_{\varepsilon,M}$, implying the $L^2(\Omega_T)$ boundedness of $u_{\varepsilon,M}$, uniformly in ε, M. Then, we get (3.5.2).

We multiply by $v_{\varepsilon,M}$ the second equation of (3.1.7) and we integrate on Ω to get,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_{\varepsilon,M}^2\, dx + d_v \int_\Omega |\nabla v_{\varepsilon,M}|^2 dx = \int_\Omega v_{\varepsilon,M}^2 g_{\varepsilon,M} dx,$$

thus, by integrating in time for $t \in (0,T)$ and using (3.1.9), (3.5.1), we get

$$\begin{aligned}\frac{1}{2}\int_\Omega v_{\varepsilon,M}(t)dx + d_v \int_{\Omega_t} |\nabla v_{\varepsilon,M}|^2 dxds &\leq \frac{1}{2}\int_\Omega v_{\varepsilon,M}^2(0)dx + C_g\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^{N+1})}\int_{\Omega_T} v_{\varepsilon,M}^2 dxdt\\ &\leq C(T),\end{aligned}$$

where $C(T)$ does not depend on ε. Therefore, taking the supremum in time we end up with the uniform estimate

$$\|\nabla v_{\varepsilon,M}\|_{L^2(\Omega_T)} \leq C(T). \tag{3.5.9}$$

Next, in order to prove the $L^2(\Omega_T)$ boundedness of $u_{\varepsilon,M}$, we multiply by $u_{\varepsilon,M}$ the first equation of (3.1.7) and we integrate on Ω_T. Thus, recalling notation (3.1.11) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (u_{\varepsilon,M})^2 dx &= -\int_\Omega \partial_1 A_{\varepsilon,M}|\nabla u_{\varepsilon,M}|^2 dx - \int_\Omega \partial_2 A_{\varepsilon,M}\nabla u_{\varepsilon,M}\cdot\nabla v_{\varepsilon,M} dx + \int_\Omega u_{\varepsilon,M}^2 f_M dx\\ &\leq -\frac{1}{2}\int_\Omega \partial_1 A_{\varepsilon,M}|\nabla u_{\varepsilon,M}|^2 dx + \frac{1}{2}\int_\Omega \frac{|\partial_2 A_{\varepsilon,M}|^2}{\partial_1 A_{\varepsilon,M}}|\nabla v_{\varepsilon,M}|^2 dx + C_f\int_\Omega u_{\varepsilon,M}^2\, dx\\ &\leq \frac{a_2^2}{2a_0}\int_\Omega |\nabla v_{\varepsilon,M}|^2 dx + C_f\int_\Omega u_{\varepsilon,M}^2\, dx,\end{aligned}$$

by (D2), i.e.

$$\frac{d}{dt}\int_\Omega u_{\varepsilon,M}^2 dx - 2C_f\int_\Omega u_{\varepsilon,M}^2 dx \leq \frac{a_2^2}{a_0}\int_\Omega |\nabla v_{\varepsilon,M}|^2 dx. \tag{3.5.10}$$

Thus, we multiply the inequality (3.5.10) by $e^{-2C_f t}$, for $t \in [0,T]$ to get

$$\frac{d}{dt}\Big(e^{-2C_f t}\int_\Omega u_{\varepsilon,M}^2 dx\Big) \leq \frac{a_2^2}{a_0}e^{-2C_f t}\int_\Omega |\nabla v_{\varepsilon,M}|^2 dx,$$

and integrating over $(0,t)$ for $t \in (0,T)$, we obtain

$$e^{-2C_f t}\|u_{\varepsilon,M}(t)\|_{L^2(\Omega)}^2 \leq \|u_{\varepsilon,M}(0)\|_{L^2(\Omega)}^2 + \frac{a_2^2}{a_0}\int_{\Omega_t} e^{-2C_f s}|\nabla v_{\varepsilon,M}(s)|^2 dxds,$$

so that

$$\|u_{\varepsilon,M}(t)\|^2_{L^2(\Omega)} \le e^{2C_f t}\|u_{\varepsilon,M}(0)\|^2_{L^2(\Omega)} + \frac{a_2^2}{a_0} e^{2C_f t}\|\nabla v_{\varepsilon,M}(t)\|^2_{L^2(\Omega_t)}.$$

Finally, taking the supremum in time we end up with

$$\|u_{\varepsilon,M}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C_T\Big(\|u_{\varepsilon,M}(0)\|^2_{L^2(\Omega)} + \|\nabla v_{\varepsilon,M}\|^2_{L^2(\Omega_T)}\Big) \le C_T, \tag{3.5.11}$$

using (3.5.9) and the regularity of the initial datum $u_{\text{in}} \in L^2(\Omega)$.

Thanks to (3.5.1), (3.5.11), $g_{\varepsilon,M}$ is bounded in $L^2(\Omega_T)$ uniformly in ε, M, so that the r.h.s. of the equation of $v_{\varepsilon,M}$ in (3.1.7) has an $L^2(\Omega_T)$ norm which is uniformly bounded in ε, M. Thus, we get by the maximal regularity

$$\|\partial_t v_{\varepsilon,M}\|_{L^2(\Omega_T)} + \|\partial_{x_i x_j} v_{\varepsilon,M}\|_{L^2(\Omega_T)} \le C_3\,(T)\,, \tag{3.5.12}$$

and by the Gagliardo-Nirenberg inequality

$$\|\nabla v_{\varepsilon,M}\|_{L^4(\Omega_T)} \le C'\|\Delta v_{\varepsilon,M}\|^{1/2}_{L^2(\Omega_T)}\|v_{\varepsilon,M}\|^{1/2}_{L^\infty(\Omega_T)} + C''\|v_{\varepsilon,M}\|^{1/2}_{L^\infty(\Omega_T)} \le C(T), \tag{3.5.13}$$

uniformly in ε, M.

In order to get the uniform bound of $u_{\varepsilon,M}$ in $L^\infty(0,T;L^4(\Omega))$, we multiply the first equation of (3.1.7) by $u^3_{\varepsilon,M}$ and we integrate on Ω

$$\begin{aligned} \frac{1}{4}\int_\Omega \partial_t(u^4_{\varepsilon,M})dx &= -\int_\Omega \nabla(A_{\varepsilon,M})\nabla(u^3_{\varepsilon,M})dx + \int_\Omega u^4_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M})dx \\ &=: I_{diff} + I_{rea}. \end{aligned} \tag{3.5.14}$$

Then, we compute

$$\begin{aligned} I_{diff} &= -3\int_\Omega \Big(\partial_1 A_{\varepsilon,M}\nabla u_{\varepsilon,M} + \partial_2 A_{\varepsilon,M}\nabla v_{\varepsilon,M}\Big)\cdot \nabla u_{\varepsilon,M}(u^2_{\varepsilon,M})dx \\ &= -3\int_\Omega \partial_1 A_{\varepsilon,M}(u^2_{\varepsilon,M})|\nabla u_{\varepsilon,M}|^2 dx - 3\int_\Omega \partial_2 A_{\varepsilon,M}(u^2_{\varepsilon,M})\nabla u_{\varepsilon,M}\cdot\nabla v_{\varepsilon,M}\,dx \\ &\le -3\int_\Omega \partial_1 A_{\varepsilon,M}(u^2_{\varepsilon})|\nabla u_{\varepsilon,M}|^2 dx + \frac{3}{2}\int_\Omega \partial_1 A_{\varepsilon,M}(u^2_{\varepsilon,M})|\nabla u_{\varepsilon,M}|^2 dx \\ &\quad + \frac{3}{2}\int_\Omega \frac{|\partial_2 A_{\varepsilon,M}|^2}{\partial_1 A_{\varepsilon,M}} u^2_{\varepsilon,M}|\nabla v_{\varepsilon,M}|^2 dx \\ &\le \frac{3}{4}\frac{a_2^2}{a_0}\int_\Omega u^4_{\varepsilon,M}\,dx + \frac{3}{4}\frac{a_2^2}{a_0}\int_\Omega |\nabla v_{\varepsilon,M}|^4 dx, \end{aligned}$$

using (D2). Using (R1), the reaction term is estimated as follows

$$I_{rea} = \int_\Omega u^4_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M})dx \le C_f\int_\Omega u^4_{\varepsilon,M}\,dx,$$

so that (3.5.14) becomes

$$\frac{1}{4}\frac{d}{dt}\int_\Omega u^4_{\varepsilon,M}dx \le \Big(\frac{3}{4}\frac{a_2^2}{a_0} + C_f\Big)\int_\Omega u^4_{\varepsilon,M}\,dx + \frac{3}{4}\frac{a_2^2}{a_0}\int_\Omega |\nabla v_{\varepsilon,M}|^4 dx,$$

i.e.

$$\frac{d}{dt}\int_\Omega u_{\varepsilon,M}^4 dx - \alpha \int_\Omega u_{\varepsilon,M}^4 dx \le \frac{3a_2^2}{a_0}\int_\Omega |\nabla v_{\varepsilon,M}|^4 dx, \tag{3.5.15}$$

with

$$\alpha := \frac{3a_2^2}{a_0} + 4C_f > 0.$$

Thus, we multiply the inequality (3.5.15) by $e^{-\alpha t}$, for $t \in [0,T]$ to get

$$\frac{d}{dt}\Big(e^{-\alpha t}\int_\Omega u_{\varepsilon,M}^4 dx\Big) \le \frac{3a_2^2}{a_0} e^{-\alpha t}\int_\Omega |\nabla v_{\varepsilon,M}|^4 dx,$$

Finally, we integrate in time for $t \in (0,T)$ to get

$$e^{-\alpha t}\|u_{\varepsilon,M}(t)\|^4_{L^4(\Omega)} \le \|u_{\varepsilon,M}(0)\|^4_{L^4(\Omega)} + \frac{3a_2^2}{a_0}\int_{\Omega_t} e^{-\alpha s}|\nabla v_{\varepsilon,M}|^4 dx ds,$$

so that

$$\|u_{\varepsilon,M}(t)\|^4_{L^4(\Omega)} \le e^{\alpha t}\|u_{\varepsilon,M}(0)\|^4_{L^4(\Omega)} + \frac{3a_2^2}{a_0} e^{\alpha t}\|\nabla v_{\varepsilon,M}\|^4_{L^4(\Omega_t)},$$

thus, taking the supremum in time we end up with

$$\|u_{\varepsilon,M}\|^4_{L^\infty(0,T;L^4(\Omega))} \le C_T\Big(\|u_{\varepsilon,M}(0)\|^4_{L^4(\Omega)} + \|\nabla v_{\varepsilon,M}\|^4_{L^4(\Omega_T)}\Big) \le C_T, \tag{3.5.16}$$

using (3.5.13).

(iii) Thanks to (3.5.1), (3.5.2), $g_{\varepsilon,M}$ is bounded in $L^4(\Omega_T)$ uniformly in ε, M, so that the r.h.s. of the equation of $v_{\varepsilon,M}$ in (3.1.7) has an $L^4(\Omega_T)$ norm which is uniform in ε, M. Thus, we get (3.5.3) by the maximal regularity and the Gagliardo-Nirenberg inequality. The estimate (3.5.4) follows multiplying by $-\Delta v_{\varepsilon,M}$ the equation satisfied by $v_{\varepsilon,M}$, using the Cauchy-Schwarz inequality and recalling that $v_{\varepsilon,M} g_{\varepsilon,M}$ has an $L^2(\Omega_T)$ norm which is bounded in ε, M.

(iv) Since by *Lemma 3.4.2* the system (3.1.7) is equivalent to (3.1.12), the idea consists in proving (3.5.5) for $a_{\varepsilon,M}$ and then to use the change of variable (3.1.10) to get the suitable control for $u_{\varepsilon,M}$.

The inequalities (3.5.2), (3.1.17) imply that $a_{\varepsilon,M}$ is bounded in $L^\infty(0,T;L^4(\Omega))$ uniformly in ε, M. On the other hand, the estimates (3.5.2), (3.5.3) show that s_M is bounded in $L^4(\Omega_T)$ uniformly in ε, M, so that the r.h.s. of the equation for $a_{\varepsilon,M}$ in (3.1.12) has an $L^2(\Omega_T)$ norm which is bounded in ε, M. Therefore, as in (3.2.10) we define

$$C_4' := 2\frac{a_1^2}{a_0} + 1 \qquad \text{and} \qquad C_4'' := 2\Big(\frac{a_1}{a_0}\Big)^2 + \frac{1}{a_0} + 2,$$

to get

$$\begin{aligned}\|\partial_t a_{\varepsilon,M}\|^2_{L^2(\Omega_T)} + \|\nabla a_{\varepsilon,M}\|^2_{L^\infty(0,T;L^2(\Omega))} + a_0\|\Delta a_{\varepsilon,M}\|^2_{L^2(\Omega_T)} &\\ \le C_4'\|\nabla a_{\text{in}}\|^2_{L^2(\Omega)} + C_4''\|a_{\varepsilon,M} s_M\|^2_{L^2(\Omega_T)} &\\ \le C_4(T), &\end{aligned} \tag{3.5.17}$$

uniformly in ε, M. Recalling that by (3.1.10) we have in the weak sense

$$\partial_1 A_{\varepsilon,M}\partial_t u_{\varepsilon,M} = \partial_t a_{\varepsilon,M} - \partial_2 A_{\varepsilon,M}\partial_t v_{\varepsilon,M},$$

then it holds using (D2)

$$\|\partial_t u_{\varepsilon,M}\|^2_{L^2(\Omega_T)} \le \frac{2}{a_0^2}\|\partial_t a_{\varepsilon,M}\|^2_{L^2(\Omega_T)} + 2\Big(\frac{a_2}{a_0}\Big)^2\|\partial_t v_{\varepsilon,M}\|^2_{L^2(\Omega_T)} \le C(T), \tag{3.5.18}$$

uniformly in ε, M thanks to (3.5.12), (3.5.17). Similarly, it holds

$$\nabla(A(u_{\varepsilon,M}, v_{\varepsilon,M})) = \partial_1 A_{\varepsilon,M}\nabla u_{\varepsilon,M} + \partial_2 A_{\varepsilon,M}\nabla v_{\varepsilon,M}, \tag{3.5.19}$$

so that using (D2) again

$$\begin{aligned}\|\nabla u_{\varepsilon,M}\|^2_{L^\infty(0,T;\,L^2(\Omega))} &\le \frac{2}{a_0^2}\|\nabla a_{\varepsilon,M}\|^2_{L^\infty(0,T;\,L^2(\Omega))} + 2\Big(\frac{a_2}{a_0}\Big)^2\|\nabla v_{\varepsilon,M}\|^2_{L^\infty(0,T;\,L^2(\Omega))}\\ &\le C(T),\end{aligned} \tag{3.5.20}$$

thanks to (3.5.4), (3.5.17). Finally, estimates (3.5.17) - (3.5.20) imply (3.5.5).

(v) In order to prove (3.5.6), we take the weak time derivative in the second equation of (3.1.7)

$$\partial_t(\partial_t v_{\varepsilon,M}) - d_v\Delta(\partial_t v_{\varepsilon,M}) = (\partial_t v_{\varepsilon,M})g_{\varepsilon,M} + v_{\varepsilon,M}\partial_1 g_{\varepsilon,M}\partial_t u_{\varepsilon,M} + v_{\varepsilon,M}\partial_2 g_{\varepsilon,M}\partial_t v_{\varepsilon,M}, \tag{3.5.21}$$

thus the r.h.s. of (3.5.21) is bounded in $L^2(\Omega_T)$, uniformly in ε, M. Indeed, the first term has an $L^2(\Omega_T)$ uniform control because $g_{\varepsilon,M}$ and $\partial_t v_{\varepsilon,M}$ are uniformly bounded in $L^4(\Omega_T)$ by (3.5.1), (3.5.2) and (3.5.3), respectively. The last two terms have an $L^2(\Omega_T)$ norm which is uniformly bounded in ε, M by (R2), (3.5.1), (3.5.2). Therefore, using the assumption $v_{\text{in}} \in H^3(\Omega)$, by the maximal regularity applied to (3.5.21) and by the Gagliardo Nirenberg inequality, we get the estimate below uniformly in ε, M

$$\|\partial_t^2 v_{\varepsilon,M}\|_{L^2(\Omega_T)} + \|\nabla\nabla\,(\partial_t v_{\varepsilon,M})\|_{L^2(\Omega_T)} + \|\nabla(\partial_t v_{\varepsilon,M})\|_{L^2(\Omega_T)} \le C(T),$$

implying the following uniform in ε, M inequality, using (3.5.12)

$$\|\partial_t v_{\varepsilon,M}\|_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))} \le C(T),$$

we get thus (3.5.6). Moreover by the continuous injection $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ if $N \le 3$, we end up with

$$\|\partial_t v_{\varepsilon,M}\|_{L^2(0,T;\,L^\infty(\Omega))} \le C(T), \qquad \text{if } N \le 3, \quad \text{uniformly in } \varepsilon, M, \tag{3.5.22}$$

giving (3.5.7). The inequality above gives $a_{\varepsilon,M}$ bounded in $L^\infty(\Omega_T)$, uniformly in ε, M. Then, using the equivalence of the systems (3.1.7) and (3.1.12) we obtain the $L^\infty-$ boundedness of $u_{\varepsilon,M}$, that is (3.5.8). Indeed, by the definition of s_M in (3.1.14), the estimate (3.5.22) implies

$$\int_0^T \sup_{x\in\Omega}|s_M(t,x)| \le C(T), \qquad \text{if } N \le 3, \quad \text{uniformly in } \varepsilon, M,$$

so that by similar computations as in (3.2.27), using (3.5.22) we get uniformly in ε, M

$$\|a_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \le C(T), \qquad \text{if } N \le 3, \tag{3.5.23}$$

giving the first part of (3.5.8). Concerning the $L^4(\Omega_T)-$boundedness of $\partial_{x_i} u_{\varepsilon,M}$ for all $i = 1,\dots N$, we first prove the $L^4(\Omega_T)-$boundedness of $\partial_{x_i} A(u, v)$ by the Gagliardo-Nirenberg inequality. Indeed, taking the fourth power we get for all $i, j = 1,\dots,N$, with $N \leq 3$

$$\|\partial_{x_i} a_{\varepsilon,M}\|^4_{L^4(\Omega)} \leq C_1 \|\partial_{x_i x_j} a_{\varepsilon,M}\|^2_{L^2(\Omega)} \|a_{\varepsilon,M}\|^2_{L^\infty(\Omega)} + C_2 \|a_{\varepsilon,M}\|^4_{L^\infty(\Omega)}.$$

Then, by integrating in time over $(0, T)$ and using the Hölder's inequality we obtain

$$\begin{aligned}\|\partial_{x_i} a_{\varepsilon,M}\|^4_{L^4(\Omega_T)} &\leq C_1 \|\partial_{x_i x_j} a_{\varepsilon,M}\|^2_{L^2(\Omega_T)} \|a_{\varepsilon,M}\|^2_{L^\infty(\Omega_T)} + C_2 \|a_{\varepsilon,M}\|^4_{L^4(0,T,L^\infty(\Omega))} \\ &\leq C(T),\end{aligned}$$

which is uniform in ε, M, using the estimates (3.5.17), (3.5.23). Finally, using (3.5.3), (3.5.19), (D2) and the above inequality we end up for all $i = 1,\dots,N$ and $N \leq 3$,

$$\|\partial_{x_i} u_{\varepsilon,M}\|^4_{L^4(\Omega_T)} \leq C(a_0, a_2)(\|\partial_{x_i} a_{\varepsilon,M}\|^4_{L^4(\Omega_T)} + \|\partial_{x_i} v_{\varepsilon,M}\|^4_{L^4(\Omega_T)}) \leq C(a_0, a_2, T).$$

□

3.6 Proof of the main result

In this section we prove *Theorem 3.1.1*. Before proceeding, we introduce the notations below for any $\varepsilon, M > 0$ and $u_{\varepsilon,M},\ v_{\varepsilon,M}, u,\ v \geq 0$,

$$A_{\varepsilon,M} := A(u_{\varepsilon,M}, v_{\varepsilon,M}), \quad B_{\varepsilon,M} := B(u_{\varepsilon,M}, v_{\varepsilon,M}), \quad A := A(u, v), \quad B := B(u, v).$$

Proof of Theorem 3.1.1.
By the $\varepsilon, M-$uniform estimates in *Lemma 3.5.1*, we get for some $u \in L^\infty(0, T; L^4(\Omega))$ and $v \in L^\infty(\Omega)$ (up to subsequences)

$$u_{\varepsilon,M} \to u, \qquad v_{\varepsilon,M} \to v, \qquad \text{a.e. in } \Omega_T \quad \text{as } \varepsilon \to 0,\ M \to +\infty, \tag{3.6.1}$$

and

$$\begin{aligned} \partial_t u_{\varepsilon,M} &\rightharpoonup \partial_t u, & & \text{weakly in } L^2(\Omega_T), \\ \partial_t v_{\varepsilon,M} \rightharpoonup \partial_t v, \qquad & \Delta v_{\varepsilon,M} \rightharpoonup \Delta v, & & \text{weakly in } L^4(\Omega_T), \\ \nabla v_{\varepsilon,M} &\rightharpoonup \nabla v, & & \text{weakly in } L^8(\Omega_T). \end{aligned} \tag{3.6.2}$$

In addition, we claim

$$A(u_{\varepsilon,M}, v_{\varepsilon,M}) \to A(u, v), \qquad \text{strongly in } L^1(\Omega_T). \tag{3.6.3}$$

Indeed, (3.6.1) and estimates (3.5.1), (3.5.2) give by *Proposition A.6.1*

$$u_{\varepsilon,M} \to u, \qquad v_{\varepsilon,M} \to v, \qquad \text{strongly in } L^2(\Omega_T). \tag{3.6.4}$$

However, the $C^0(\mathbb{R}^2_+)$ character of A and B (with B defined by (3.1.4)) implies

$$A_{\varepsilon,M} \to A \qquad \text{and} \qquad B_{\varepsilon,M} \to B, \qquad \text{a.e. in } \Omega_T, \tag{3.6.5}$$

and by (3.1.4) we have (see *Proposition A.6.1*)

$$B(u_{\varepsilon,M},\ v_{\varepsilon,M}) \to B(u, v), \qquad \text{strongly in } L^2(\Omega_T).$$

Thus, we compute

$$\begin{aligned}\|A_{\varepsilon,M} - A\|_{L^1(\Omega_T)} &\le \|(B_{\varepsilon,M} - B)u_{\varepsilon,M}\|_{L^1(\Omega_T)} + \|B(u_{\varepsilon,M} - u)\|_{L^1(\Omega_T)}\\ &\le \|u_{\varepsilon,M}\|_{L^2(\Omega_T)}\|B_{\varepsilon,M} - B\|_{L^2(\Omega_T)} + a_1\|u_{\varepsilon,M} - u\|_{L^1(\Omega_T)},\end{aligned}$$

where for the first term we used Cauchy-Schwarz inequality and the $\varepsilon, M-$uniformly boundedness of $u_{\varepsilon,M}$ in $L^2(\Omega_T)$, by (3.5.16). Therefore, we get (3.6.3) which implies

$$\Delta A(u_{\varepsilon,M}, v_{\varepsilon,M}) \to \Delta A(u, v), \qquad \text{in } D'(\Omega_T),$$

where $D'(\Omega_T)$ stands for the space of distributions on Ω_T. On the other hand, the inequality (3.5.5) implies the existence of $\omega \in L^2(\Omega_T)$ s.t. (up to subsequences)

$$\Delta A(u_{\varepsilon,M}, v_{\varepsilon,M}) \rightharpoonup \omega, \qquad \text{weakly in } L^2(\Omega_T), \tag{3.6.6}$$

so that by uniqueness of the limit in $D'(\Omega_T)$, we identify

$$\omega = \Delta A(u, v), \qquad \text{in } L^2(\Omega_T). \tag{3.6.7}$$

Moreover if $N \le 3$, $(u_{\varepsilon,M}, v_{\varepsilon,M})$ does not depend on M for M large enough, by the Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$.

Now, we take the $D'(\Omega_T)$ limit as $\varepsilon \to 0$, $M \to +\infty$, if $N > 3$ and as $\varepsilon \to 0$ if $N \le 3$, in (3.1.7) - (3.1.9). From the obtained convergences (3.6.1) - (3.6.7), it remains to prove the $D'(\Omega_T)$ limit of the reaction terms $u_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M})$ and $v_{\varepsilon,M} g_{\varepsilon,M}(u_{\varepsilon,M}, v_{\varepsilon,M})$. By the a.e. convergence of $u_{\varepsilon,M}$, $v_{\varepsilon,M}$ in (3.6.1) and recalling the $C^0(\mathbb{R}^2_+)$ character of f, we see that for $N \le 3$, $f_M(u_{\varepsilon,M}, v_{\varepsilon,M}) = f(u_\varepsilon, v_\varepsilon)$ converges a.e. towards $f(u, v)$, as ε tends to zero. Otherwise, if $N > 3$, for a given $(t, x) \in \Omega_T$ (outside of a zero measure set), because of (3.6.1), it holds for M large enough

$$f_M(u_{\varepsilon,M}, v_{\varepsilon,M}) = f(u_{\varepsilon,M}, v_{\varepsilon,M}) \quad \to \quad f(u, v), \qquad \text{as } \varepsilon \to 0,\ M \to +\infty,$$

so that (still using (3.6.1))

$$u_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M}) \quad \to \quad u\, f(u, v), \qquad \text{as } \varepsilon \to 0,\ M \to +\infty. \tag{3.6.8}$$

Moreover, by assumption (R1) and estimates (3.5.2), (3.5.1) we end up with

$$\|u_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M})\|_{L^2(\Omega_T)} \le C_T,$$

which gives using (3.6.8) (see *Proposition A.6.1*)

$$u_{\varepsilon,M} f_M(u_{\varepsilon,M}, v_{\varepsilon,M}) \quad \to \quad u f(u, v), \qquad \text{strongly in } L^1(\Omega_T),\ \text{as } \varepsilon \to 0,\ M \to +\infty,$$

and thus in $D'(\Omega_T)$. Finally, it remains to treat the reaction term $v_{\varepsilon,M} g_{\varepsilon,M}(u_{\varepsilon.M}, v_{\varepsilon,M})$, where $g_{\varepsilon,M}$ is defined in (3.1.8). Recalling the $C^0(\mathbb{R}^2_+)$ character of g, it holds for $N \le 3$, $g_M(u_{\varepsilon,M,v_{\varepsilon,M}}) = g(u_\varepsilon, v_\varepsilon)$ which converges a.e. in Ω_T towards $g(u, v)$, as ε goes to zero. Otherwise, for a given $(t, x) \in \Omega_T$ (outside of a zero measure set), because of (3.6.1), it holds for M large enough

$$g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) = g(u_{\varepsilon,M}, v_{\varepsilon,M}) \quad \to \quad g(u, v), \qquad \text{as } \varepsilon \to 0,\ M \to +\infty.$$

Moreover, by assumption (R1) and estimates (3.5.1), (3.5.2), we get for any $N \geq 1$

$$||g_M(u_{\varepsilon,M}, v_{\varepsilon,M})||_{L^4(\Omega_T)} \leq C_T,$$

giving (by Proposition (A.6.1)) as $\varepsilon \to 0,\ M \to +\infty$,

$$g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) \quad \to \quad g(u,v), \qquad \text{strongly in } L^r(\Omega_T),\ r < 4. \tag{3.6.9}$$

By the same argument as in (3.3.7), we have from the regularization with standard mollifiers (abusing notations for the extended by zero function in the variables (t, x)),

$$g(u,v) *_{t,x} \varphi_\varepsilon \quad \to \quad g(u,v), \qquad \text{strongly in } L^r(\Omega_T),\ r < 4, \tag{3.6.10}$$

and

$$\begin{aligned} &||g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) *_{t,x} \varphi_\varepsilon - g(u,v)||_{L^p(\Omega_T)} \\ &\qquad \leq ||g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) - g(u,v)||_{L^p(\Omega_T)} + ||g(u,v) *_{t,x} \varphi_\varepsilon - g(u,v)||_{L^p(\Omega_T)}, \end{aligned}$$

giving by (3.6.9), (3.6.10)

$$g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) *_{t,x} \varphi_\varepsilon \quad \to \quad g(u,v), \qquad \text{strongly in } L^r(\Omega_T),\ r < 4,$$

and thus, by (3.6.1) (up to subsequences)

$$v_{\varepsilon,M}\big(g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) *_{t,x} \varphi_\varepsilon\big) \quad \to \quad v\, g(u,v), \qquad \text{a.e. in } \Omega_T.$$

Moreover, by assumption (R1), estimates (3.5.1), (3.5.2) and Young's inequality for convolution we get

$$||v_{\varepsilon,M} g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) *_{t,x} \varphi_\varepsilon||_{L^4(\Omega_T)} \leq ||v_{\varepsilon,M}||_{L^\infty(\Omega_T)} ||g_M(u_{\varepsilon,M}, v_{\varepsilon,M})||_{L^4(\Omega_T)} ||\varphi_\varepsilon||_{L^1(\mathbb{R}^{N+1})} \leq C_T.$$

Therefore, by *Proposition A.6.1* we end up with

$$v_{\varepsilon,M} g_M(u_{\varepsilon,M}, v_{\varepsilon,M}) *_{t,x} \varphi_\varepsilon \quad \to \quad v\, g(u,v), \qquad \text{strongly in } L^r(\Omega_T),\ r < 4,$$

and thus in $D'(\Omega_T)$. Then, using the obtained convergence above, all the terms in the first two equations of (3.1.7) converge in $D'(\Omega_T)$. We conclude by taking the limit in the boundary conditions of (3.1.7), using the continuity of the trace operator and the weak convergence of $\Delta A(u_{\varepsilon,M}, v_{\varepsilon,M})$, $\Delta v_{\varepsilon,M}$ in (3.6.2), (3.6.6), (3.6.7). Finally, using the lower semicontinuity property of the $L^p(\Omega_T)$ norm for $p \in (1, +\infty]$ we conclude that u, v satisfy (3.1.1) - (3.1.3), in the sense of *Theorem 3.1.1*. □

3.7 Uniqueness

This section is devoted to the uniqueness of the solution to (3.1.1) - (3.1.3) when $N \leq 2$ (see *Theorem 3.7.1*). Before doing that, we show a corollary of *Theorem 3.1.1* concerning the regularity of the solution when $N \leq 2$.

Corollary 3.1.
Let (u, v) be the solution given by Theorem 3.1.1. There exists a constant $C > 0$ such that

$$||\nabla v||_{L^2(0,T;L^\infty(\Omega))} \leq C, \qquad \text{if } N \leq 2, \tag{3.7.1}$$

and

$$||\nabla u||_{L^2(0,T;L^\infty(\Omega))} \leq C, \qquad \text{if } N = 1. \tag{3.7.2}$$

Proof.
We show (3.7.1) using Sobolev inequality (3.2.19), (3.2.20) . Thanks to the boundedness of v in $L^4(0,T;W^{2,4}(\Omega))$ for any $N \geq 1$, shown in *Theorem 3.1.1*, and the Sobolev embedding (3.2.19), (3.2.20), there exists a constant $C > 0$ such that for all $t \in (0,T)$

$$\|v(t,\cdot)\|_{C^{1,\gamma}(\bar{\Omega})} \leq C(\gamma, N, \Omega)\|v(t,\cdot)\|_{W^{2,4}(\Omega)}, \qquad \text{with } \gamma = 1 - \frac{N}{4}, \quad N \leq 2.$$

The inequality above together with the boundednees of v in $L^4(0,T;W^{2,4}(\Omega))$ imply that ∇v is bounded in $L^2(0,T;L^\infty(\Omega))$ when $N \leq 2$.

Similarly, in order to prove (3.7.2) we use the $L^2(0,T;H^2(\Omega))$−boundedness of A, shown in *Theorem 3.1.1*, and the Sobolev inequality (3.2.19), (3.2.20) when $N = 1$. Thus, there exists a constant $C(\Omega) > 0$ such that for any $t \in (0,T)$

$$\|A(t,\cdot)\|_{C^{1,1/2}(\bar{\Omega})} \leq C(\Omega)\|A(t,\cdot)\|_{H^2(\Omega)},$$

implying the $L^2(0,T;W^{1,\infty}(\Omega))$−boundedness of A. Then, by computing ∇A in the weak sense, for any $t \in (0,T)$ we end up with

$$\|\partial_1 A(t,\cdot)\nabla u(t,\cdot)\|_{L^\infty(\Omega)} \leq \|\nabla A(t,\cdot)\|_{L^\infty(\Omega)} + \|\partial_2 A(t,\cdot)\nabla v(t,\cdot)\|_{L^\infty(\Omega)},$$

thus taking the square in the above inequality, integrating in time over $(0,T)$ and using (D2), (3.7.1), we conclude

$$a_0^2\|\nabla u\|^2_{L^2(0,T;L^\infty(\Omega))} \leq \|\nabla A\|^2_{L^2(0,T;L^\infty(\Omega))} + a_2^2\|\nabla v\|^2_{L^2(0,T;L^\infty(\Omega))} \leq C.$$

□

Before stating the uniqueness result, we introduce some notations and definitions. For $w, z \in L^q(0,T;L^p(\Omega))$, $p, q \in [1,+\infty]$, we define

$$M(t)_{(p,w,z)} := \max\{\|w(t)\|_{L^p(\Omega)}, \|z(t)\|_{L^p(\Omega)}\}, \quad \text{a.e. } t \in (0,T)\,, \tag{3.7.3}$$

and for $h = h(u,v)$, $k = k(u,v)$ with $u, v \geq 0$ and s.t.

$$\sup_{u,v\geq 0} h, \ \sup_{u,v\geq 0} k \ < +\infty,$$

we denote

$$S(h,k) := \max\{\sup_{u,v\geq 0} h, \ \sup_{u,v\geq 0} k\}. \tag{3.7.4}$$

Moreover, we denote the Jacobian matrix of the application $(u,v) \mapsto (f(u,v), g(u,v))$, as (for all $u, v \geq 0$)

$$J(u,v) := \begin{bmatrix} \partial_1 f(u,v) & \partial_2 f(u,v) \\ \partial_1 g(u,v) & \partial_2 g(u,v) \end{bmatrix}, \tag{3.7.5}$$

and we consider the following matrix norm

$$\forall\, P \in \mathscr{M}_{m,n}(\mathbb{R}), \qquad \mathcal{N}(P) := \max_{1\leq i\leq m} \sum_{1\leq j\leq n} |P_{ij}|, \qquad m, n \in \mathbb{N}. \tag{3.7.6}$$

We are now ready to prove the uniqueness for system (3.1.1) - (3.1.3) that is obtained as an immediate consequence of the stability result below.

Theorem 3.7.1 (Uniqueness)**.**
Let $N \leq 2$. We assume (D1)*,* (D2)*,* (D3)*,* (R1) *and the following additional hypothesis:*

(i) the diffusivity function A is such that

$$\sup_{u,v\geq 0} \mathcal{N}(Hess(A(u,v))) < +\infty, \tag{U1}$$

where $Hess(A)$ stands for the Hessian matrix of A;

(ii) the Jacobian matrix (3.7.5) *satisfies*

$$\sup_{u,v\geq 0} \mathcal{N}(J(u,v)) < +\infty. \tag{U2}$$

Then, taking two solutions (u_i, v_i), $i = 1,2$ of (3.1.1)*,* (3.1.2)*, corresponding to the nonnegative initial data $(u_{i,\,in}, v_{i,\,in})$, $i = 1,2$, in the sense of Theorem 3.1.1, there exists a constant $C_{uniq} > 0$, depending on Ω, T, a_0, a_2, d_v, C_f, C_g, on*

$$\sup_{u,v\geq 0} \mathcal{N}(Hess(A(u,v))),\ \sup_{u,v\geq 0} \mathcal{N}(J(u,v)),\ \|u_2\|_{L^\infty(\Omega_T)},\ \|v_2\|_{L^\infty(\Omega_T)},$$

and on $\|\nabla v_2\|_{L^2(0,T;L^\infty(\Omega))}$, $\|\nabla u_2\|_{L^p(0,T;L^q(\Omega))}$ with

$$(p,q) = \begin{cases} (2,\infty), & \text{if} \quad N = 1, \\ (4,4), & \text{if} \quad N = 2, \end{cases} \tag{3.7.7}$$

such that

$$\begin{aligned} &\|u_1 - u_2\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \|v_1 - v_2\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \\ &\qquad \leq C_{uniq}(\|u_{1,in} - u_{2,in}\|^2_{L^2(\Omega)} + \|v_{1,in} - v_{2,in}\|^2_{L^2(\Omega)}). \end{aligned} \tag{3.7.8}$$

Finally, if $u_{1,in} = u_{2,in}$ and $v_{1,in} = v_{2,in}$ for a.e. $x \in \Omega$, then

$$u_1(t,x) = u_2(t,x) \qquad \text{and} \qquad v_1(t,x) = v_2(t,x), \qquad \text{a.e. } (t,x) \in \Omega_T,$$

so that the uniqueness holds for (3.1.1) - (3.1.3).

Remark 3.5.
We prove the stability inequality (3.7.8) only for $N \leq 2$, otherwise a further regularity on ∇u_i is needed, with respect to the regularity obtained in *Theorem 3.1.1*. In particular if $N = 3$, by applying the same tools used to get (3.7.8), we need ∇u_i bounded in $L^8(0,T;\, L^4(\Omega))$ (see (3.7.22)).

Proof.
For a better readability, we first introduce the notations

$$\begin{gathered} A_i := A(u_i,v_i), \qquad f_i := f(u_i,v_i), \qquad g_i := g(u_i,v_i), \qquad i = 1,2, \\ \partial_1 A_i := \partial_1 A(u_i,v_i), \qquad \partial_2 A_i := \partial_2 A(u_i,v_i), \qquad i = 1,2, \end{gathered}$$

and

$$\mathcal{N}_J := \sup_{i,j=1,2;\, u_i,v_j\geq 0} \mathcal{N}(J(u_i,v_j)), \qquad \mathcal{N}_H := \sup_{i,j=1,2;\, u_i,v_j\geq 0} \mathcal{N}(Hess(A(u_i,v_j))).$$

Now, we compute the equations satisfied by $u_1 - u_2$ and $v_1 - v_2$ and we multiply by $u_1 - u_2$ and $\lambda(v_1 - v_2)$, respectively, where the parameter $\lambda > 0$ will be chosen later. Then, we integrate over Ω and we add the obtained formulations to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |u_1 - u_2|^2 dx + \lambda \int_\Omega |v_1 - v_2|^2 dx\Big) \\
&= -\int_\Omega (\partial_1 A_1 \nabla u_1 + \partial_2 A_1 \nabla v_1)\cdot \nabla(u_1 - u_2)dx \\
&\quad + \int_\Omega \Big(\partial_1 A_2 \nabla u_2 + \partial_2 A_2 \nabla v_2\Big)\cdot \nabla(u_1 - u_2)dx \\
&\quad - d_v \lambda \int_\Omega |\nabla(v_1 - v_2)|^2 dx \\
&\quad + \int_\Omega (u_1 f(u_1, v_1) - u_2 f(u_2, v_2))(u_1 - u_2)dx, \\
&\quad + \lambda \int_\Omega (v_1 g(u_1, v_1) - v_2 g(u_2, v_2))(v_1 - v_2)dx \\
&=: I_{diff} + I_{rea}. \qquad (3.7.9)
\end{aligned}$$

It is worth noticing that as $N \leq 2$, the solutions are bounded in both components. The reaction part is then estimated as below

$$\begin{aligned}
I_{rea} &= \int_\Omega f_1 |u_1 - u_2|^2 dx + \int_\Omega u_2 (f_1 - f_2)(u_1 - u_2)dx \\
&\quad + \lambda \int_\Omega g_1 |v_1 - v_2|^2 dx + \lambda \int_\Omega v_2 (g_1 - g_2)(v_1 - v_2)dx \\
&\leq S(f_1, g_1)\Big(\int_\Omega |u_1 - u_2|^2 dx + \lambda \int_\Omega |v_1 - v_2|^2 dx\Big) \\
&\quad + M(t)_{(\infty, u_2, v_2)}\Big(\int_\Omega |\partial_1 f(\xi_u, v_1)||u_1 - u_2|^2 dx + \lambda \int_\Omega |\partial_2 g(u_2, \xi_v)||v_1 - v_2|^2 dx\Big) \\
&\quad + M(t)_{(\infty, u_2, v_2)} \int_\Omega \Big(|\partial_2 f(u_2, \xi_v)| + \lambda |\partial_1 g(\xi_u, v_1)|\Big)|u_1 - u_2||v_1 - v_2| dx, \qquad (3.7.10)
\end{aligned}$$

with from now on $\xi_u \in (\min\{u_1, u_2\}, \max\{u_1, u_2\})$, $\xi_v \in (\min\{v_1, v_2\}, \max\{v_1, v_2\})$ and M, S defined in (3.7.3), (3.7.4), respectively. Then, we have

$$\begin{aligned}
\int_\Omega |\partial_1 f(\xi_u, v_1)||u_1 - u_2|^2 dx + \lambda \int_\Omega |\partial_2 g(u_2, \xi_v)||v_1 - v_2|^2 dx \\
\leq \mathcal{N}_J \int_\Omega |u_1 - u_2|^2 dx + \lambda \mathcal{N}_J \int_\Omega |v_1 - v_2|^2 dx,
\end{aligned}$$

and using Young's inequality, we estimate the last integral in (3.7.10) as

$$\begin{aligned}\int_\Omega |\partial_2 f(u_2,\xi_v)||u_1-u_2||v_1-v_2|dx + \lambda\int_\Omega |\partial_1 g(\xi_u,v_1)||u_1-u_2||v_1-v_2|dx\\ \le \frac{1}{2}\int_\Omega |\partial_2 f(u_2,\xi_v)|\Big(\frac{1}{\lambda}|u_1-u_2|^2+\lambda|v_1-v_2|^2\Big)dx\\ +\frac{1}{2}\int_\Omega |\partial_1 g(\xi_u,v_1)|\Big(\lambda|u_1-u_2|^2+\lambda|v_1-v_2|^2\Big)dx\\ \le \mathcal{N}_J\Big(\frac{1}{2}(\lambda+\frac{1}{\lambda})\int_\Omega |u_1-u_2|^2dx+\lambda\int_\Omega |v_1-v_2|^2dx\Big).\end{aligned}$$

Therefore, the estimate (3.7.10) becomes

$$I_{rea} \le C_{rea}(t)\left(\|u_1-u_2\|^2_{L^2(\Omega)}+\lambda\|v_1-v_2\|^2_{L^2(\Omega)}\right), \tag{3.7.11}$$

with, for all $t \in [0,T]$,

$$C_{rea}(t) := 2\max\Big\{f_1, g_1, \mathcal{N}_J\Big\}\Big(1+M(t)_{(\infty,u_2,v_2)}\Big(1+\frac{1}{2}(\lambda+\frac{1}{\lambda})\Big)\Big). \tag{3.7.12}$$

Concerning the diffusion part, it holds, using the Young inequality and (D2),

$$\begin{aligned} I_{diff} =& -\int_\Omega \partial_1 A_1|\nabla(u_1-u_2)|^2dx - d_v\lambda\int_\Omega |\nabla(v_1-v_2)|^2dx\\ &-\int_\Omega \partial_1(A_1-A_2)\nabla u_2\cdot\nabla(u_1-u_2)dx - \int_\Omega \partial_2 A_1\nabla(v_1-v_2)\cdot\nabla(u_1-u_2)dx\\ &-\int_\Omega \partial_2(A_1-A_2)\nabla v_2\cdot\nabla(u_1-u_2)dx\\ \le& -\frac{1}{4}\int_\Omega \partial_1 A_1|\nabla(u_1-u_2)|^2dx - d_v\lambda\int_\Omega |\nabla(v_1-v_2)|^2dx\\ &+\int_\Omega \frac{|\nabla u_2|^2}{\partial_1 A_1}|\partial_1(A_1-A_2)|^2dx + \int_\Omega \frac{|\partial_2 A_1|^2}{\partial_1 A_1}|\nabla(v_1-v_2)|^2dx\\ &+\int_\Omega \frac{|\nabla v_2|^2}{\partial_1 A_1}|\partial_2(A_1-A_2)|^2dx\\ \le& -\frac{a_0}{4}\int_\Omega |\nabla(u_1-u_2)|^2dx - (d_v\lambda-\frac{a_2^2}{a_0})\int_\Omega |\nabla(v_1-v_2)|^2dx\\ &+\int_\Omega \frac{|\nabla u_2|^2}{\partial_1 A_1}|\partial_1(A_1-A_2)|^2dx + \int_\Omega \frac{|\nabla v_2|^2}{\partial_1 A_1}|\partial_2(A_1-A_2)|^2dx. \end{aligned} \tag{3.7.13}$$

Now, we focus on the two integrals in (3.7.13). The second one is estimated, for $N = 1, 2$, using (3.7.1) in *Corollary 3.1*, as follows

$$\begin{aligned}&\int_\Omega \frac{|\nabla v_2|^2}{\partial_1 A_1}|\partial_2(A_1-A_2)|^2dx\\ &\le \frac{2}{a_0}\|\nabla v_2\|^2_{L^\infty(\Omega)}\int_\Omega\Big(|\partial_{21}A(\xi_u,v_1)|^2|u_1-u_2|^2+|\partial_{22}A(u_2,\xi_v)|^2|v_1-v_2|^2\Big)dx\\ &\le \frac{2}{a_0}\max\Big\{1,\frac{1}{\lambda}\Big\}\mathcal{N}_H^2\|\nabla v_2\|^2_{L^\infty(\Omega)}\int_\Omega\Big(|u_1-u_2|^2+\lambda|v_1-v_2|^2\Big)dx.\end{aligned} \tag{3.7.14}$$

As the first integral in (3.7.13) is estimated depending on the value of N, to continue the proof we treat the cases $N = 1$ and $N = 2$ separately, in the following two paragraphs.

• **The case** $N = 1$

Similarly as (3.7.14), the hypothesis (U1) and (3.7.2) imply

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u_2|^2}{\partial_1 A_1}|\partial_1(A_1 - A_2)|^2 dx \\
&\leq \frac{2}{a_0}\|\nabla u_2\|^2_{L^\infty(\Omega)} \int_\Omega \Big(|\partial_{11}A(\xi_u, v_1)|^2|u_1 - u_2|^2 + |\partial_{12}A(u_2, \xi_v)|^2|v_1 - v_2|^2\Big)dx \\
&\leq \frac{2}{a_0}\max\Big\{1, \frac{1}{\lambda}\Big\}\|\nabla u_2\|^2_{L^\infty(\Omega)}\, \mathcal{N}_H^2 \int_\Omega \Big(|u_1 - u_2|^2 + \lambda|v_1 - v_2|^2\Big)dx. \qquad (3.7.15)
\end{aligned}$$

Therefore, using (3.7.14), (3.7.15) the term I_{diff} is estimated as below when $N = 1$

$$\begin{aligned}
I_{diff} \leq &-\frac{a_0}{4}\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} - (d_v\lambda - \frac{a_2^2}{a_0})\|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)} \\
&+ C_1(t)\Big(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}\Big), \qquad (3.7.16)
\end{aligned}$$

with, for all $t \in [0, T]$,

$$C_1(t) := \frac{4}{a_0}\max\Big\{1, \frac{1}{\lambda}\Big\}M^2(t)_{(\infty,|\nabla u_2|,|\nabla v_2|)}\mathcal{N}_H^2, \qquad (3.7.17)$$

with $M(t)$ defined in (3.7.3).

Finally, plugging (3.7.11), (3.7.16) into (3.7.9) we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}\Big) \\
&\qquad \leq -\frac{a_0}{4}\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} - (d_v\lambda - \frac{a_2^2}{a_0})\|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)} \\
&\qquad\quad + (C_1(t) + C_{rea}(t))\Big(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}\Big), \qquad (3.7.18)
\end{aligned}$$

i.e.

$$\begin{aligned}
&\frac{d}{dt}(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}) + C_{diff,1}\Big(\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} + \|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)}\Big) \\
&\qquad \leq 2(C_1(t) + C_{rea}(t))\Big(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}\Big), \qquad (3.7.19)
\end{aligned}$$

where λ is chosen in such a way that $\lambda > \frac{a_2^2}{a_0 d_v}$, and

$$C_{diff,1} := 2\min\Big\{\frac{a_0}{4},\ d_v\lambda - \frac{a_2^2}{a_0}\Big\} > 0. \qquad (3.7.20)$$

• **The case** $N = 2$

In order to estimate the first integral in (3.7.13), we use (D2), the Cauchy-Schwarz inequality and the $L^4(\Omega_T)$ boundedness of ∇u_2, given by *Corollary 3.1*, to get

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u_2|^2}{\partial_1 A_1}|\partial_1(A_1 - A_2)|^2 dx \\
&\qquad \leq \frac{2}{a_0}\int_\Omega |\nabla u_2|^2\Big(|\partial_{11}A(\zeta_u, v_1)|^2|u_1 - u_2|^2 + |\partial_{12}A(u_2, \zeta_v)|^2|v_1 - v_2|^2\Big)dx \\
&\qquad \leq \frac{2}{a_0}\mathcal{N}_H^2\|\nabla u_2\|^2_{L^4(\Omega)}\Big(\|u_1 - u_2\|^2_{L^4(\Omega)} + \|v_1 - v_2\|^2_{L^4(\Omega)}\Big). \qquad (3.7.21)
\end{aligned}$$

Then, the Gagliardo-Nirenberg inequality allows us to estimate the L^4 norm of $(u_1 - u_2)$ (resp. $(v_1 - v_2)$) in terms of the L^2 norm of $(u_1 - u_2)$ (resp. $(v_1 - v_2)$) and $\nabla(u_1 - u_2)$ (resp. $\nabla(v_1 - v_2)$), as follows

$$\begin{aligned}
&\|\nabla u_2\|^2_{L^4(\Omega)}\|u_1 - u_2\|^2_{L^4(\Omega)} \\
&\le 2C_{GN}\|\nabla u_2\|^2_{L^4(\Omega)}(\|\nabla(u_1 - u_2)\|_{L^2(\Omega)}\|u_1 - u_2\|_{L^2(\Omega)} + \|u_1 - u_2\|^2_{L^2(\Omega)}) \\
&\le \delta C_{GN}\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} + \frac{C_{GN}}{\delta}\|\nabla u_2\|^4_{L^4(\Omega)}\|u_1 - u_2\|^2_{L^2(\Omega)} \\
&\quad + 2C_{GN}\|\nabla u_2\|^2_{L^4(\Omega)}\|u_1 - u_2\|^2_{L^2(\Omega)} \\
&\le \delta C_{GN}\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} \\
&\quad + C_{GN}\max\left\{2, \frac{1}{\delta}\right\}\|\nabla u_2\|^2_{L^4(\Omega)}(1 + \|\nabla u_2\|^2_{L^4(\Omega)})\|u_1 - u_2\|^2_{L^2(\Omega)} \\
&\le \delta C_{GN}\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} + C_{GN}\max\left\{2, \frac{1}{\delta}\right\}(1 + \|\nabla u_2\|^2_{L^4(\Omega)})^2\|u_1 - u_2\|^2_{L^2(\Omega)} \\
&\le \delta C_{GN}\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} + C_D(t)\|u_1 - u_2\|^2_{L^2(\Omega)},
\end{aligned} \tag{3.7.22}$$

with, for all $t \in (0, T)$,

$$C_D(t) := 2C_{GN}\max\left\{2, \frac{1}{\delta}\right\}(1 + \|\nabla u_2(t)\|^4_{L^4(\Omega)}), \tag{3.7.23}$$

where we denote C_{GN} the best constant involved in the Gagliardo-Nirenberg inequality and with $\delta > 0$ to be chosen later.

Similarly, it holds

$$\|\nabla u_2\|^2_{L^4(\Omega)}\|v_1 - v_2\|^2_{L^4(\Omega)} \le \delta C_{GN}\|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)} + C_D(t)\|v_1 - v_2\|^2_{L^2(\Omega)},$$

so that (3.7.21) is estimated as

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u_2|^2}{\partial_1 A_1}|\partial_1(A_1 - A_2)|^2 dx \\
&\qquad \le \frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big(\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} + \|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)}\Big) \\
&\qquad\quad + \frac{2}{a_0}\mathcal{N}_H^2 C_D(t)(\|u_1 - u_2\|^2_{L^2(\Omega)} + \|v_1 - v_2\|^2_{L^2(\Omega)}) \\
&\qquad \le \frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big(\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} + \|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)}\Big) \\
&\qquad\quad + \frac{2}{a_0}\max\left\{1, \frac{1}{\lambda}\right\}\mathcal{N}_H^2 C_D(t)(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}).
\end{aligned} \tag{3.7.24}$$

Therefore, by (3.7.14), (3.7.24), the term I_{diff} is estimated as below when $N = 2$,

$$\begin{aligned}
I_{diff} \le &-\Big(\frac{a_0}{4} - \frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big)\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} - \Big(d_v\lambda - \frac{a_2^2}{a_0} - \frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big)\|\nabla(v_1 - v_2)\|^2_{L^2(\Omega)} \\
&+ C_2(t)(\|u_1 - u_2\|^2_{L^2(\Omega)} + \lambda\|v_1 - v_2\|^2_{L^2(\Omega)}),
\end{aligned} \tag{3.7.25}$$

with, for any $t \in [0, T]$,

$$C_2(t) := \frac{2}{a_0}\max\left\{1, \frac{1}{\lambda}\right\}\mathcal{N}_H^2\Big(C_D(t) + \|\nabla v_2(t)\|^2_{L^\infty(\Omega)}\Big), \tag{3.7.26}$$

and $C_D(t)$ defined in (3.7.23).

Finally, plugging (3.7.11), (3.7.25) into (3.7.9) we end up with

$$\frac{1}{2}\frac{d}{dt}\Big(\|u_1-u_2\|^2_{L^2(\Omega)}+\lambda\|v_1-v_2\|^2_{L^2(\Omega)}\Big)$$
$$\le -\Big(\frac{a_0}{4}-\frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big)\|\nabla(u_1-u_2)\|^2_{L^2(\Omega)}-\Big(d_v\lambda-\frac{a_2^2}{a_0}-\frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big)\|\nabla(v_1-v_2)\|^2_{L^2(\Omega)}$$
$$+(C_2(t)+C_{rea}(t))(\|u_1-u_2\|^2_{L^2(\Omega)}+\lambda\|v_1-v_2\|^2_{L^2(\Omega)}). \quad (3.7.27)$$

Now, we choose $\delta>0$ such that

$$\frac{a_0}{4}-\frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2>0 \qquad \Longleftrightarrow \qquad 0<\delta<\frac{a_0^2}{8C_{GN}\mathcal{N}_H^2}, \quad (3.7.28)$$

and we choose $\lambda>0$ such that, for $\delta>0$ given above,

$$d_v\lambda-\frac{a_2^2}{a_0}-\frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2>0 \qquad \Longleftrightarrow \qquad \lambda>\frac{a_2^2+2\delta C_{GN}\mathcal{N}_H^2}{a_0 d_v}.$$

Taking into account the admissible values of δ in (3.7.28), the above inequality is satisfied if

$$\lambda>\frac{a_2^2}{a_0 d_v}+\frac{a_0}{4d_v}. \quad (3.7.29)$$

Therefore we obtain

$$\frac{d}{dt}(\|u_1-u_2\|^2_{L^2(\Omega)}+\lambda\|v_1-v_2\|^2_{L^2(\Omega)})$$
$$+C_{diff,2}\|\nabla(u_1-u_2)\|^2_{L^2(\Omega)}+\|\nabla(v_1-v_2)\|^2_{L^2(\Omega)}$$
$$\le 2(C_2(t)+C_{rea}(t))(\|u_1-u_2\|^2_{L^2(\Omega)}+\lambda\|v_1-v_2\|^2_{L^2(\Omega)}), \quad (3.7.30)$$

with

$$C_{diff,2}\coloneqq 2\min\Big\{\frac{a_0}{4}-\frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2,\ d_v\lambda-\frac{a_2^2}{a_0}-\frac{2\delta}{a_0}C_{GN}\mathcal{N}_H^2\Big\}>0. \quad (3.7.31)$$

To conclude, we define for all $t\in[0,T]$ and $N=1,2$

$$\hat{C}_N(t)\coloneqq 2(C_N(t)+C_{rea}(t)), \quad (3.7.32)$$
$$y(t)\coloneqq \|(u_1-u_2)(t,\cdot)\|^2_{L^2(\Omega)}+\lambda\|(v_1-v_2)(t,\cdot)\|^2_{L^2(\Omega)},$$
$$\omega(t)\coloneqq \|\nabla((u_1-u_2)(t,\cdot))\|^2_{L^2(\Omega)}+\|\nabla((v_1-v_2)(t,\cdot))\|^2_{L^2(\Omega)},$$

with $C_{rea}(t)$ and $C_N(t)$ defined in (3.7.12), (3.7.17), (3.7.26). Then, in both cases $N=1,2$, the inequalities (3.7.19), (3.7.30) rewrite, for all $t\in[0,T]$, as

$$\frac{d}{dt}y(t)+C_{diff,N}\,\omega(t)\le\hat{C}_N(t)\,y(t). \quad (3.7.33)$$

Thus, we multiply (3.7.33) by $e^{-\int_0^t\hat{C}_N(s)ds}$ to get

$$\frac{d}{dt}\Big(y(t)e^{-\int_0^t\hat{C}_N(s)ds}\Big)+C_{diff,N}\,w(t)e^{-\int_0^t\hat{C}_N(s)ds}\le 0.$$

Then by integrating in time, we obtain

$$y(t) + C_{diff,N}\, e^{\int_0^t \hat{C}_N(s)ds} \int_0^t w(s) e^{-\int_0^s \hat{C}_N(\sigma)d\sigma} ds \leq y(0) e^{\int_0^t \hat{C}_N(s)ds}.$$

Finally, taking the supremum in $t \in (0, T)$ we conclude

$$\|y(t)\|_{L^\infty(0,T)} + C_{diff,N} \int_0^T \omega(s)ds \leq e^{\int_0^T \hat{C}_N(s)ds} \|y(0)\|_{L^\infty(0,T)}, \tag{3.7.34}$$

giving (3.7.8) with C_{uniq} that depends on $C_{diff,N}$ and $C_N(t)$ for $t \in (0, T)$ and $N = 1, 2$. □

Remark 3.6.
It is worth noticing that the constant C_{uniq} in (3.7.8) does not depend on the solution (u_1, v_1), since the constants $C_{diff,N}$ and $C_N(t)$, defined in (3.7.20), (3.7.31), (3.7.32), do not depend on (u_1, v_1), for $N \leq 2$. This will be fundamental to obtain the weak-strong stability result in *Theorem 3.3* (see *Section 3.8*).

3.8 Weak-strong stability and uniqueness

The aim of this section is twofold. On the one hand we will prove that, under slightly stricter assumptions, the cross-diffusion system (2.1.7) - (2.1.13), in *Chapter 2*, is included in the class of cross-diffusion systems (3.1.1) - (3.1.3) analysed in this chapter, i.e. it satisfies (D1), (D2), (D3), (R1) and (U1), (U2). Therefore, if the initial data satisfy the hypothesis of *Theorem 3.1.1*, the system (2.1.7) - (2.1.13) admits a strong solution, in the sense of *Theorem 3.1.1*. Moreover, this solution is unique when $N \leq 2$, thanks to the uniqueness result in *Theorem 3.7.1* (see *Theorem 3.2*).

On the other hand, we will show that, if in addition $\alpha \geq 1$ (and thus, $\beta \geq 1$), the stability estimate (3.7.8) turns into a weak-strong stability result, when $N \leq 2$ (see *Theorem* 3.3). A direct consequence of the latter result is the weak-strong uniqueness of the solution to (2.1.7) - (2.1.13) (see *Corollary 3.8.1*). We refer to [6, 48] for further weak-strong stability results applied to cross-diffusion systems.

Before proceeding, for the reader's convenience, we recall here the system (2.1.7) - (2.1.11), i.e.

$$\begin{cases} \partial_t u - \Delta(A(u, v)) = F_u(u, v), & \text{in } (0, +\infty) \times \Omega, \\ \partial_t v - \Delta(d_v v) = F_v(u, v), & \text{in } (0, +\infty) \times \Omega, \end{cases} \tag{3.8.1}$$

with

$$A(u, v) := d_a u_a^*(u, v) + d_b u_b^*(u, v), \tag{3.8.2}$$

and where $(u_a^*(u, v),\ u_b^*(u, v))$ satisfies the nonlinear system

$$\begin{cases} u_a + u_b = u, \\ Q(u_a, u_b, v) := \phi(b u_b + d v)\, u_b - \psi(a u_a + c v)\, u_a = 0. \end{cases} \tag{3.8.3}$$

Moreover, the reaction functions are given by

$$\begin{aligned} F_u(u, v) &:= f_u(u_a^*(u, v), u_b^*(u, v), v), \\ F_v(u, v) &:= f_v(u_a^*(u, v), u_b^*(u, v), v), \end{aligned} \tag{3.8.4}$$

where f_u and f_v are defined in (2.1.2), (2.1.11) and the transition functions ϕ, ψ are the same transition functions considered in *Chapter 2* (see (H1)). However, we assume now that both ϕ and ψ are lower bounded by a strictly positive constant, i.e. we assume,

$$\psi(x) := (A+x)^{\alpha}, \qquad \phi(x) := (B+x)^{\beta}, \qquad \forall\, x \geq 0, \tag{H1}$$

with

$$A > 0,\ B > 0, \tag{H2'}$$

and

$$0 < \alpha \leq \beta \leq \min\left\{\frac{6}{N}, (\sqrt{7}+2)\alpha + \sqrt{7}+1\right\}. \tag{H3}$$

Thanks to (H1), (H2'), as shown in *Section 2.3*, for all $u, v \geq 0$, there exists a unique $(u_a^*(u,v), u_b^*(u,v))$ solution to the nonlinear system (3.8.3). It can be written as

$$u_a^*(u,v) = r_a^*(u,v)u \qquad \text{and} \qquad u_b^*(u,v) = r_b^*(u,v)u, \tag{3.8.5}$$

with

$$r_a^*(u,v),\ r_b^*(u,v) \in (0,1), \qquad r_a^*(u,v) + r_b^*(u,v) = 1. \tag{3.8.6}$$

Moreover, thanks to the strict positivity of A and B in (H2'), we will see in *Subsection 3.8.2* that $u_a^*(u,v)$ and $u_b^*(u,v)$ are two differentiable maps, from $\mathbb{R}_+^2$ to $\mathbb{R}_+$, and that the diffusivity function $A(u,v)$ in (3.8.2) is a $C^2(\mathbb{R}_+^2, \mathbb{R}_+)$ function. This will be fundamental to obtain *Theorem 3.2* (see *Subsection 3.8.2*).

Assuming in addition that $\alpha \geq 1$ (and thus, $\beta \geq 1$), we have that u lies in $L^4(\Omega_T)$ (see point *(i)* of *Theorem 2.5.2*) so that F_u belongs to $L^2(\Omega_T)$. Therefore, thanks to the regularity of $\nabla A(u,v)$, we have also that $\partial_t u$ belongs to $L^2(0,T;(H^1(\Omega))')$ and the very weak solution, given by *Theorem 2.5.2*, is in fact a weak solution, i.e. it satisfies, for all $T > 0$ and for all test functions $\xi_1, \xi_2 \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$,

$$-\int_0^T\int_\Omega (\partial_t \xi_1) u\,dxdt - \int_\Omega \xi_1(0,\cdot)u^{\text{in}}dx + \int_0^T\int_\Omega \nabla\xi_1 \cdot \nabla\left(d_a u_a^*(u,v) + d_b u_b^*(u,v)\right)dxdt$$
$$= \int_0^T\int_\Omega \xi_1 F_u(u,v)\,dxdt,$$

and

$$-\int_0^T\int_\Omega (\partial_t \xi_2)\,v\,dxdt - \int_\Omega \xi_2(0,\cdot)\,v^{\text{in}}\,dx + d_v\int_0^T\int_\Omega \nabla\xi_2 \cdot \nabla v\,dxdt$$
$$= \int_0^T\int_\Omega \xi_2\,F_v(u,v)\,dxdt.$$

This will be fundamental when we apply *Theorem 3.7.1*, in order to obtain the weak-strong stability in *Theorem 3.3*, stated in the following subsection.

We are now ready to state the announced results.

3.8.1 Statements of the main results

Theorem 3.2 (Existence and uniqueness of strong solutions)**.**
Let $N \geq 1$. We assume (H1), (H2'), (H3) and we consider nonnegative initial data satisfying $u_{in} \in (L^4 \cap H^1)(\Omega)$, $v_{in} \in (L^\infty \cap H^3)(\Omega)$. Then, for all $T > 0$ there exists a strong nonnegative (for each component) solution (u, v) of (3.8.1) - (3.8.4), in the sense of Theorem 3.1.1. In addition, if $N \leq 2$, the strong solution is unique.

Theorem 3.3 (Weak-strong stability)**.**
Let $N \leq 2$. We assume (H1), (H2'), (H3), $\alpha \geq 1$ and we consider a pair of initial data $(u_{w,in}, v_{w,in})$ satisfying (2.1.13), (H4), and a pair of nonnegative initial data $(u_{s,in}, v_{s,in})$ satisfying $u_{s,in} \in (L^4 \cap H^1)(\Omega)$, $v_{s,in} \in (L^\infty \cap H^3)(\Omega)$. Let (u_w, v_w) be a weak solution to (3.8.1) - (3.8.4), corresponding to $(u_{w,in}, v_{w,in})$, given by Theorem 2.5.2, and (u_s, v_s) be the unique strong solution, corresponding to $(u_{s,in}, v_{s,in})$, given by Theorem 3.2. Then, there exists a constant $C_{ws} > 0$ depending on Ω, T, a_0, a_2, d_v, C_f, C_g, on

$$\sup_{u,v\geq 0} \mathcal{N}(Hess(A(u, v))),\ \sup_{u,v\geq 0} \mathcal{N}(J(u, v)),\ \|u_s\|_{L^\infty(\Omega_T)},\ \|v_s\|_{L^\infty(\Omega_T)},$$

with $\mathcal{N}$ defined in (3.7.6), and on $\|\nabla v_s\|_{L^2(0,T;L^\infty(\Omega))}$, $\|\nabla u_s\|_{L^p(0,T;L^q(\Omega))}$, with

$$(p, q) = \begin{cases} (2, \infty), & \text{if} \quad N = 1, \\ (4, 4), & \text{if} \quad N = 2, \end{cases} \tag{3.8.7}$$

such that

$$\|u_w - u_s\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \|v_w - v_s\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \\ \leq C_{ws}\left(\|u_{w,in} - u_{s,in}\|^2_{L^2(\Omega)} + \|v_{w,in} - v_{s,in}\|^2_{L^2(\Omega)}\right), \tag{3.8.8}$$

Corollary 3.8.1 (Weak-strong uniqueness)**.**
Under the assumptions of Theorem 3.3, if $u_{w,in} = u_{s,in}$ and $v_{w,in} = v_{s,in}$, for a.e. $x \in \Omega$, we have

$$u_w(t, x) = u_s(t, x) \qquad \text{and} \qquad v_w(t, x) = v_s(t, x), \quad \text{for a.e. } (t, x) \in \Omega_T,$$

so that the weak solution is a strong solution.

As announced, these results follow as soon as we prove that the system (3.8.1) - (3.8.4) satisfies (D1), (D2), (D3), (R1) and (U1), (U2). This is the goal of the next subsection.

3.8.2 Proof of the main results

We start by showing that, assuming (H1), (H2'), (H3), the diffusivity function $A(u, v)$ is a $C^2(\mathbb{R}^2_+, \mathbb{R}_+)$ function. As $A(0, v) = 0$, for all $v \geq 0$, assumption (D1) will follow. Then, we will show that $A(u, v)$ satisfies also (D2) and (U1). Assumption (D3) is then a consequence of (U1). From the definition of $A(u, v)$, this corresponds to analyse the properties of $u^*_a(u, v)$ and $u^*_b(u, v)$.

We first observe that, thanks to the assumption (H2'), the transition functions ϕ, ψ are $C^\infty(\mathbb{R}_+, \mathbb{R}_+)$. Moreover, we know that $u^*_b(u, v)$ is the unique zero of the increasing function

$q(u_b; u, v)$ defined in (2.3.1). Then, the implicit function theorem guarantees the C^1 character of u_b^* with respect to (u, v), and thus of $u_a^*(u, v) = u - u_b^*(u, v)$.

Next, we proceed by computing the gradient of u_a^*, u_b^*. By differentiating (3.8.2), we obtain

$$\partial_i A(u, v) = d_a \partial_i u_a^*(u, v) + d_b \partial_i u_b^*(u, v), \qquad i = 1, 2. \tag{3.8.9}$$

Let us denote

$$\psi^* := \psi(au_a^*(u, v) + cv), \qquad \phi^* := \phi(bu_b^*(u, v) + dv),$$

$$\partial_i Q^* := \partial_i Q(u_a^*(u, v), u_b^*(u, v), v)\,, \qquad \partial_{ij} Q^* := \partial_{ij} Q(u_a^*(u, v), u_b^*(u, v), v)\,, \quad i, j = 1, 2, 3\,,$$

and

$$u_a^* := u_a^*(u, v), \qquad u_b^* := u_b^*(u, v)\,.$$

By differenting the identity $Q(u_a^*, u_b^*, v) = 0$, with respect to u and v and using $u = u_a^* + u_b^*$, we get

$$\partial_1 u_a^* \, \partial_1 Q^* + (1 - \partial_1 u_a^*) \, \partial_2 Q^* = 0,$$
$$\partial_2 u_a^* \, \partial_1 Q^* - \partial_2 u_a^* \, \partial_2 Q^* + \partial_3 Q^* = 0,$$

that implies

$$\partial_1 u_a^* = \frac{\partial_2 Q^*}{\partial_2 Q^* - \partial_1 Q^*} = 1 - \partial_1 u_b^*, \tag{3.8.10}$$

$$\partial_2 u_a^* = \frac{\partial_3 Q^*}{\partial_2 Q^* - \partial_1 Q^*} = -\partial_2 u_b^*. \tag{3.8.11}$$

We compute now the gradient of the conversion function $Q(u_a, u_b, v)$ in (2.1.3) and we obtain, thanks to the positivity of ϕ, ψ, ϕ', ψ', for all $u_a, u_b, v \geq 0$,

$$\begin{aligned} \partial_1 Q(u_a, u_b, v) &= -\psi(au_a + cv) - au_a \psi'(au_a + cv) < 0, \\ \partial_2 Q(u_a, u_b, v) &= \phi(bu_b + dv) + bu_b \phi'(bu_b + dv) > 0, \end{aligned} \tag{3.8.12}$$

and

$$\partial_3 Q(u_a, u_b, v) = du_b \phi'(bu_b + dv) - cu_a \psi'(au_a + cv). \tag{3.8.13}$$

As Q is a $C^\infty(\mathbb{R}_+^3, \mathbb{R})$ function and since it holds $\partial_2 Q(u_a, u_b, v) - \partial_1 Q(u_a, u_b, v) > 0$, for all $u_a, u_b, v \geq 0$, we see by (3.8.10), (3.8.11), that $\partial_i u_a^*, \partial_i u_b^*$, $i = 1, 2$, are differentiable, so that $A(u, v) \in C^2(\mathbb{R}_+^2, \mathbb{R}_+)$. Therefore, the assumption (D1) is satisfied.

In order to prove (D2), we have to estimate the gradient of u_a^*, u_b^*. From (3.8.12), it holds for all $u_a, u_b, v \geq 0$,

$$-1 < \frac{\partial_1 Q}{\partial_2 Q - \partial_1 Q} < 0 \qquad \text{and} \qquad 0 < \frac{\partial_2 Q}{\partial_2 Q - \partial_1 Q} < 1. \tag{3.8.14}$$

Moreover, by (3.8.13) we have

$$-\frac{cu_a \psi'}{\phi + \psi + bu_b \phi' + au_a \psi'} \leq \frac{\partial_3 Q}{\partial_2 Q - \partial_1 Q} \leq \frac{du_b \phi'}{\phi + \psi + bu_b \phi' + au_a \psi'},$$

giving

$$-\frac{c}{a} \leq \frac{\partial_3 Q}{\partial_2 Q - \partial_1 Q} \leq \frac{d}{b}. \tag{3.8.15}$$

Therefore, using (3.8.14) in (3.8.10), we get

$$\partial_1 u_a^* \in (0,1) \qquad \text{and} \qquad \partial_1 u_b^* \in (0,1), \tag{3.8.16}$$

while using (3.8.15) in (3.8.11), we obtain

$$-\frac{c}{a} \leq \partial_2 u_a^* \leq \frac{d}{b} \qquad \text{and} \qquad -\frac{d}{b} \leq \partial_2 u_b^* \leq \frac{c}{a}. \tag{3.8.17}$$

Finally, from (3.8.9), we get on the one hand, using $\partial_1 u_a^* + \partial_1 u_b^* = 1$,

$$0 < \min\{d_a, d_b\} \leq \partial_1 A(u,v) \leq \max\{d_a, d_b\},$$

and on the other hand, using (3.8.17)

$$|\partial_2 A(u,v)| \leq d_a|\partial_2 u_a^*| + d_b|\partial_2 u_b^*| \leq (d_a + d_b)\left(\frac{c}{a} + \frac{d}{b}\right).$$

Therefore, $A(u,v)$ satisfies (D2) by taking

$$a_0 := \min\{d_a, d_b\}, \qquad a_1 := \max\{d_a, d_b\}, \qquad a_2 := (d_a + d_b)\left(\frac{c}{a} + \frac{d}{b}\right).$$

In order to verify (U1), we need to compute the hessian of $A(u,v)$ in (3.8.2), i.e. to compute the hessian of u_a^* and u_b^*. Observing that $\partial_{ij} u_a^* = -\partial_{ij} u_b^*$, for $i,j = 1,2$, and that $\partial_{12} u_a^* = \partial_{21} u_a^*$, by the regularity of u_a^*, we will compute only $\partial_{11} u_a^*, \partial_{12} u_a^*, \partial_{22} u_a^*$.

From (3.8.10), it holds

$$\begin{aligned}
\partial_{11} u_a^* &= \frac{\partial_u (\partial_2 Q^*)}{\partial_2 Q^* - \partial_1 Q^*} - \frac{\partial_2 Q^* \left(\partial_u (\partial_2 Q^*) - \partial_u (\partial_1 Q^*)\right)}{(\partial_2 Q^* - \partial_1 Q^*)^2} \\
&= \frac{-\partial_u (\partial_2 Q^*)\, \partial_1 Q^* + \partial_u (\partial_1 Q^*)\, \partial_2 Q^*}{(\partial_2 Q^* - \partial_1 Q^*)^2}.
\end{aligned}$$

Then, using (3.8.14), (3.8.16) and $\partial_{12} Q = \partial_{21} Q = 0$, we get

$$\begin{aligned}
|\partial_{11} u_a^*| &\leq \frac{|\partial_u (\partial_2 Q^*)| + |\partial_u (\partial_1 Q^*)|}{\partial_2 Q^* - \partial_1 Q^*} \\
&\leq \frac{|\partial_{22} Q^*| \partial_1 u_b^* + |\partial_{11} Q^*| \partial_1 u_a^*}{\partial_2 Q^* - \partial_1 Q^*} \\
&\leq \frac{|\partial_{22} Q^*|}{\partial_2 Q^* - \partial_1 Q^*} + \frac{|\partial_{11} Q^*|}{\partial_2 Q^* - \partial_1 Q^*}.
\end{aligned} \tag{3.8.18}$$

Similarly, from (3.8.10) again, it holds

$$\begin{aligned}
\partial_{12} u_a^* &= \frac{\partial_v (\partial_2 Q^*)}{\partial_2 Q^* - \partial_1 Q^*} - \frac{\partial_2 Q^* \left(\partial_v (\partial_2 Q^*) - \partial_v (\partial_1 Q^*)\right)}{(\partial_2 Q^* - \partial_1 Q^*)^2} \\
&= \frac{-\partial_v (\partial_2 Q^*)\, \partial_1 Q^* + \partial_v (\partial_1 Q^*)\, \partial_2 Q^*}{(\partial_2 Q^* - \partial_1 Q^*)^2},
\end{aligned}$$

and using (3.8.14), (3.8.17), we get

$$
\begin{aligned}
|\partial_{12}u_a^*| &\leq \frac{|\partial_v(\partial_2 Q^*)| + |\partial_v(\partial_1 Q^*)|}{\partial_2 Q^* - \partial_1 Q^*} \\
&\leq \frac{\left|\partial_{22}Q^*\partial_2 u_b^* + \partial_{23}Q^*\right|}{\partial_2 Q^* - \partial_1 Q^*} + \frac{\left|\partial_{11}Q^*\partial_2 u_a^* + \partial_{13}Q^*\right|}{\partial_2 Q^* - \partial_1 Q^*} \\
&\leq \left(\frac{c}{a} + \frac{d}{b}\right)\left(\frac{|\partial_{22}Q^*| + |\partial_{11}Q^*|}{\partial_2 Q^* - \partial_1 Q^*}\right) + \frac{|\partial_{23}Q^*| + |\partial_{13}Q^*|}{\partial_2 Q^* - \partial_1 Q^*}\,. \qquad (3.8.19)
\end{aligned}
$$

Finally, we compute $\partial_{22}u_a^*$ from (3.8.11) and we obtain

$$
\partial_{22}u_a^* = \frac{\partial_v(\partial_3 Q^*)}{\partial_2 Q^* - \partial_1 Q^*} - \frac{\partial_3 Q^*\left(\partial_v(\partial_2 Q^*) - \partial_v(\partial_1 Q^*)\right)}{(\partial_2 Q^* - \partial_1 Q^*)^2}\,.
$$

Using (3.8.15), we have

$$
|\partial_{22}u_a^*| \leq \frac{|\partial_v(\partial_3 Q^*)|}{\partial_2 Q^* - \partial_1 Q^*} + \frac{|\partial_v(\partial_2 Q^*)| + |\partial_v(\partial_1 Q^*)|}{\partial_2 Q^* - \partial_1 Q^*}\left(\frac{d}{b} + \frac{c}{a}\right).
$$

Recalling that $\partial_{12}Q^* = \partial_{21}Q^* = 0$ and using (3.8.17), we end up with

$$
\begin{aligned}
|\partial_{22}u_a^*| \leq\ & \frac{|\partial_{31}Q^*|\,|\partial_2 u_a^*| + |\partial_{32}Q^*|\,|\partial_2 u_b^*| + |\partial_{33}Q^*|}{\partial_2 Q^* - \partial_1 Q^*} \\
&+ \frac{|\partial_{22}Q^*|\,|\partial_2 u_b^*| + |\partial_{23}Q^*| + |\partial_{11}Q^*|\,|\partial_2 u_a^*| + |\partial_{13}Q^*|}{\partial_2 Q^* - \partial_1 Q^*}\left(\frac{d}{b} + \frac{c}{a}\right) \\
\leq\ & \frac{|\partial_{33}Q^*|}{\partial_2 Q^* - \partial_1 Q^*} + \frac{|\partial_{31}Q^*| + |\partial_{32}Q^*|}{\partial_2 Q^* - \partial_1 Q^*}\left(\frac{d}{b} + \frac{c}{a}\right) \\
&+ \frac{|\partial_{23}Q^*| + |\partial_{13}Q^*|}{\partial_2 Q^* - \partial_1 Q^*}\left(\frac{d}{b} + \frac{c}{a}\right) + \frac{|\partial_{22}Q^*| + |\partial_{11}Q^*|}{\partial_2 Q^* - \partial_1 Q^*}\left(\frac{d}{b} + \frac{c}{a}\right)^2. \qquad (3.8.20)
\end{aligned}
$$

Therefore, in order to estimate (3.8.18) - (3.8.20), we need to estimate the ratios $\frac{|\partial_{ij}Q|}{\partial_2 Q - \partial_1 Q}$, $i, j = 1, 2, 3$.

From (3.8.12), the Hessian of Q is given by

$$
Hess(Q) = \begin{pmatrix} -2a\psi' - a^2 u_a\psi'' & 0 & -c\psi' - acu_a\psi'' \\ 0 & 2b\phi' + b^2 u_b\phi'' & d\phi' + bdu_b\phi'' \\ -c\psi' - acu_a\psi'' & d\phi' + bdu_b\phi'' & d^2u_b\phi'' - c^2 u_a\psi'' \end{pmatrix}. \qquad (3.8.21)
$$

Then, for ϕ, ψ in (H1), (H2') and for all $x \geq 0$, we have

$$
\left|\frac{\psi'(x)}{\psi(x)}\right| = \frac{\psi'(x)}{\psi(x)} = \frac{\alpha(A+x)^{\alpha-1}}{(A+x)^\alpha} \leq \frac{\alpha}{A}, \qquad (3.8.22)
$$

$$
\left|\frac{\psi''(x)}{\psi'(x)}\right| = \frac{|\psi''(x)|}{\psi'(x)} = \frac{\alpha|\alpha-1|(A+x)^{\alpha-2}}{\alpha(A+x)^{\alpha-1}} \leq \frac{|\alpha-1|}{A}, \qquad (3.8.23)
$$

$$
\left|\frac{\phi'(x)}{\phi(x)}\right| = \frac{\phi'(x)}{\phi(x)} = \frac{\beta(B+x)^{\beta-1}}{(B+x)^\beta} \leq \frac{\beta}{B}, \qquad (3.8.24)
$$

$$
\left|\frac{\phi''(x)}{\phi'(x)}\right| = \frac{|\phi''(x)|}{\phi'(x)} = \frac{\beta|\beta-1|(B+x)^{\beta-2}}{\beta(B+x)^{\beta-1}} \leq \frac{|\beta-1|}{B}\,. \qquad (3.8.25)
$$

Using (3.8.21) and the strict positivity of $\psi, \phi, \psi'\phi'$, we have

$$\frac{|\partial_{11}Q|}{\partial_2 Q - \partial_1 Q} \le \frac{2a\psi' + a^2 u_a|\psi''|}{\phi + \psi + au_a\psi' + bu_b\phi'} \le 2a\frac{\psi'}{\psi} + a\frac{|\psi''|}{\psi'} \le \frac{2a\alpha}{A} + \frac{a|\alpha-1|}{A}. \tag{3.8.26}$$

Similarly, it holds

$$\frac{|\partial_{22}Q|}{\partial_2 Q - \partial_1 Q} \le \frac{2b\phi' + b^2 u_b|\phi''|}{\phi + \psi + au_a\psi' + bu_b\phi'} \le 2b\frac{\phi'}{\phi} + b\frac{|\phi''|}{\phi'} \le \frac{2b\beta}{B} + \frac{b|\beta-1|}{B}, \tag{3.8.27}$$

and

$$\frac{|\partial_{33}Q|}{\partial_2 Q - \partial_1 Q} \le \frac{d^2 u_b|\phi''| + c^2 u_a|\psi''|}{\phi + \psi + au_a\psi' + bu_b\phi'} \le \frac{d^2}{b}\frac{|\phi''|}{\phi'} + \frac{c^2}{a}\frac{|\psi''|}{\psi'} \le \frac{d^2|\beta-1|}{bB} + \frac{c^2|\alpha-1|}{aA}. \tag{3.8.28}$$

Next, we get

$$\frac{|\partial_{13}Q|}{\partial_2 Q - \partial_1 Q} \le \frac{c\psi' + acu_a|\psi''|}{\phi + \psi + au_a\psi' + bu_b\phi'} \le c\frac{\psi'}{\psi} + c\frac{|\psi''|}{\psi'} \le \frac{c\alpha}{A} + \frac{c|\alpha-1|}{A}, \tag{3.8.29}$$

and

$$\frac{|\partial_{23}Q|}{\partial_2 Q - \partial_1 Q} \le \frac{d\phi' + bdu_b|\phi''|}{\phi + \psi + au_a\psi' + bu_b\phi'} \le d\frac{\phi'}{\phi} + d\frac{|\phi''|}{\phi'} \le \frac{d\beta}{B} + \frac{d|\beta-1|}{B}. \tag{3.8.30}$$

To conclude, thanks to (3.8.26) - (3.8.30) and recalling $Q_{12} = Q_{21} = 0$, the ratios $\frac{|\partial_{ij}Q|}{\partial_2 Q - \partial_1 Q}$, $i, j = 1, 2, 3$, and the derivative $\partial_{ij}u_a^* = -\partial_{ij}u_b^*$ are bounded, i.e. $A(u, v)$ satisfies (U1).

It remains to show that assumptions (R1) and (U2) are satisfied by the reaction functions (3.8.4). In order to do that, we first observe that, using (3.8.5), (3.8.6), the functions F_u and F_v can be written as competition reaction functions with non constant coefficients, i.e.

$$F_u(u,v) = uf(u,v) \qquad \text{and} \qquad F_v(u,v) = ug(u,v),$$

with

$$\begin{aligned} f(u,v) &:= f_1(u,v) - uf_2(u,v) - vf_3(u,v), \\ g(u,v) &:= g_1(u,v) - ug_2(u,v) - vg_3(u,v), \end{aligned} \tag{3.8.31}$$

and f_i, g_i, $i = 1, 2, 3$, such that

$$\begin{aligned} f_1(u,v) &= \eta_a r_a^*(u,v) + \eta_b r_b^*(u,v), \\ f_2(u,v) &= \eta_a a\, r_a^*(u,v)^2 + \eta_b b\, r_b^*(u,v)^2 + (\gamma_a + \gamma_b) r_a^*(u,v)\, r_b^*(u,v), \\ f_3(u,v) &= \eta_a c\, r_a^*(u,v) + \eta_b d\, r_b^*(u,v), \end{aligned} \tag{3.8.32}$$

and

$$
\begin{aligned}
g_1(u,v) &= \eta'_v + \eta''_v,\\
g_2(u,v) &= \eta'_v a\, r_a^*(u,v) + \eta''_v b\, r_b^*(u,v),\\
g_3(u,v) &= \eta'_v c + \eta''_v d.
\end{aligned} \tag{3.8.33}
$$

Then, by (3.8.6), we see that f and g satisfy the growing behaviour in assumption (R1).

Next, by computing the gradient of f, defined in (3.8.31), (3.8.32), for all $u, v \geq 0$, we get

$$
\begin{aligned}
\partial_1 f(u,v) &= \partial_1 f_1(u,v) - f_2(u,v) - u\partial_1 f_2(u,v) - v\partial_1 f_3(u,v),\\
\partial_2 f(u,v) &= \partial_2 f_1(u,v) - u\partial_2 f_2(u,v) - f_3(u,v) - v\partial_2 f_3(u,v),
\end{aligned} \tag{3.8.34}
$$

with, for $i = 1, 2$,

$$
\begin{aligned}
\partial_i f_1 &= \eta_a \partial_i r_a^* + \eta_b \partial_i r_b^*,\\
\partial_i f_2 &= 2\eta_a a r_a^* \partial_i r_a^* + 2\eta_b b r_b^* \partial_i r_b^* + (\gamma_a + \gamma_b)(r_a^* \partial_i r_b^* + r_b^* \partial_i r_a^*),\\
\partial_i f_3 &= \eta_a c \partial_i r_a^* + \eta_b d \partial_i r_b^*.
\end{aligned} \tag{3.8.35}
$$

Thus, we need to estimate $\partial_i r_a^*$ and $\partial_i r_b^*$, $i = 1, 2$. By differentiating r_a^* in (2.3.5) with respect to u, using the identity $r_a^* + r_b^* = 1$ and denoting

$$
(\psi')^* := \psi'(a u_a^*(u,v) + cv)\,, \qquad (\phi')^* := \phi'(b u_b^*(u,v) + dv)\,,
$$

we get

$$
\partial_1 r_a^* = -\partial_1 r_b^* = \frac{b(\phi')^* \partial_1 u_b^* \psi^* - a\phi^* \partial_1 u_a^* (\psi')^*}{(\phi^* + \psi^*)^2}.
$$

Thanks to the positivity of ϕ, ψ, ϕ', ψ' and to the bounds of $\partial_1 u_a^*, \partial_1 u_b^*$ in (3.8.16), we obtain

$$
-\frac{a(\psi')^*}{\psi^*} \leq \partial_1 r_a^* \leq \frac{b(\phi')^*}{\phi^*}.
$$

Thus, (3.8.22) and (3.8.24) give

$$
-\frac{a\alpha}{A} \leq \partial_1 r_a^* \leq \frac{b\beta}{B} \qquad \text{and} \qquad -\frac{b\beta}{B} \leq \partial_1 r_b^* \leq \frac{a\alpha}{A}. \tag{3.8.36}
$$

Similarly, for $\partial_2 r_a^*$ and $\partial_2 r_b^*$ we differentiate r_a^* in (2.3.5) with respect to v, to get

$$
\begin{aligned}
\partial_2 r_a^* = -\partial_2 r_b^* &= \frac{(\phi')^*(b\partial_2 u_b^* + d)}{\phi^* + \psi^*} - \frac{\phi^*\left((\phi')^*(b\partial_2 u_b^* + d) + (\psi')^*(a\partial_2 u_a^* + c)\right)}{(\phi^* + \psi^*)^2}\\
&= \frac{\psi^*(\phi')^*(b\partial_2 u_b^* + d) - \phi^*(\psi')^*(a\partial_2 u_a^* + c)}{(\phi^* + \psi^*)^2}\,,
\end{aligned} \tag{3.8.37}
$$

and

$$
-\frac{(\psi')^*}{\psi^*}\,(a\partial_2 u_a^* + c) \leq \partial_2 r_a^*(u,v) \leq \frac{(\phi')^*}{\phi^*}\left(b\partial_2 u_b^* + d\right).
$$

Thus, using (3.8.17), (3.8.22), (3.8.24), we obtain the following bounds for $\partial_2 r_a^*$, $\partial_2 r_b^*$,

$$
-\frac{\alpha}{A}\left(\frac{ad}{b} + c\right) \leq \partial_2 r_a^* \leq \frac{\beta}{B}\left(\frac{bc}{a} + d\right), \tag{3.8.38}
$$

and

$$-\frac{\beta}{B}\left(\frac{bc}{a}+d\right) \leq \partial_2 r_b^* \leq \frac{\alpha}{A}\left(\frac{ad}{b}+c\right). \tag{3.8.39}$$

Finally, by differentiating the identities in (3.8.5) with respect to u, we obtain

$$\partial_1 u_a^* = r_a^* + u\partial_1 r_a^*, \qquad \partial_1 u_b^* = r_b^* + u\partial_1 r_b^*,$$

so that, thanks to (3.8.6), (3.8.16), the following bound holds

$$|u\partial_1 r_a^*|,\, |u\partial_1 r_b^*| \leq 1. \tag{3.8.40}$$

On the other hand, by differentiating the identities in (3.8.5) with respect to v, it holds

$$\partial_2 u_a^* = u\partial_2 r_a^* \qquad \text{and} \qquad \partial_2 u_b^* = u\partial_2 r_b^*,$$

so that, (3.8.17) implies

$$-\frac{c}{a} \leq u\partial_2 r_a^* \leq \frac{d}{b} \qquad \text{and} \quad -\frac{d}{b} \leq u\partial_2 r_b^* \leq \frac{c}{a}\,. \tag{3.8.41}$$

To conclude, all terms in (3.8.34), (3.8.35) are bounded thanks to the bounds of r_a^*, r_b^* in (3.8.6), the bounds of $\partial_i r_a^*, \partial_i r_b^*$ in (3.8.36), (3.8.38), (3.8.39), the bounds of $u\partial_i r_a^*, u\partial_i r_b^*$ in (3.8.40), (3.8.41), the boundedness of v, shown in *Theorem 2.5.2*, and the bounds of f_i defined in (3.8.32).

The gradient of g, defined in (3.8.31), (3.8.33), is also bounded because of (3.8.6) (3.8.40), (3.8.41). Thus, we conclude that the reaction functions of the cross-diffusion system (3.8.1) - (3.8.4) satisfy the assumptions (R1) and (U2).

Chapter 4

Perspectives

The results of this thesis suggest natural extensions and future directions of research.

As a possible perspective, I'm interested in the linear stability analysis of the spatially homogeneous equilibria of the system (3.8.1) - (3.8.4), in order to yield Turing instability. Moreover, we recall that the system (3.8.1) - (3.8.4) is the natural generalization of the cross-diffusion system (1.1.6) - (1.1.9), introduced in *Chapter 1* and for which we proved that no segregation of species occurs. Therefore, it is natural to investigate how the presence of the cross-diffusion terms and the competitive reaction terms in (3.8.1) - (3.8.4) influence the pattern formation and if Turing instability occurs (see [9, 10]). Moreover, by following the works in [68, 82], one could also explore the bifurcation structure of the system to understand how it changes, depending on the values of the parameters.

A forthcoming work concerns the regularity of the solution to the cross-diffusion system (2.1.7) - (2.1.13), obtained by the mesoscopic approach (see *Chapter 2*). It is based on a priori estimates shown by the analysis of the energy functional below and by a bootstrap argument,

$$\mathcal{E}_p(u_a, u_b, v) := \int_\Omega h_{a,p}(u_a, v)\, dx + \int_\Omega h_{b,p}(u_b, v)\, dx, \qquad p > 2, \tag{4.0.1}$$

(see (2.2.1), (2.2.2)). We recall that in *Chapter 2*, we considered a subfamily of $\{\mathcal{E}_p\}_{p\geq 1}$, in order to get enough compactness and then to pass to the limit. More specifically, in *Lemma 2.7.1* we obtained the energy estimates for the following values of p

$$p = 1, \qquad p = 1 + \frac{1}{\beta + 1}, \qquad p = 1 + \frac{1}{\alpha + 1}, \qquad p = 2.$$

Now, the idea is to estimate uniformly in ε the evolution of $\mathcal{E}_p$ for any $p > 2$. This improves the regularity of the solution $(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon)$ to the mesoscopic system (2.1.1) - (2.1.6) and thus, the regularity of the solution $(u := u_a^* + u_b^*, v)$ to the macroscopic system (2.1.7) - (2.1.13), obtained as $\varepsilon \to 0$. Then, we perform a bootstrap argument, i.e. we estimate the evolution of $\mathcal{E}_p$, using the estimates from the evolution of $\mathcal{E}_{p-1}$.

A third research project consists on studying the existence of non-homogeneous stationary solutions to a class of cross-diffusion systems, including the system (3.1.1) - (3.1.3) considered in *Chapter 3*, and its asymptotic behavior. This is the subject of a current work in collaboration with E. Montefusco and it is detailed below.

Following the work of *Lou et al.* in [62, 63, 65], we study the existence of non-homogeneous stationary solutions to a class of *full cross-diffusion* systems of two equations, i.e. when both equations include cross-diffusion terms. The system we are interested in generalises the system introduced in *Chapter 3* and writes as below

$$\begin{cases} \partial_t u - \Delta(A(u,v)) = uf(u,v), & \text{in } \Omega_T, \\ \partial_t v - \Delta(B(u,v)) = vg(u,v), & \text{in } \Omega_T, \\ \nabla(A(u,v)) \cdot \sigma = \nabla(B(u,v)) \cdot \sigma = 0, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

We want to study the asymptotic behavior of the system above, which entails analysing the existence, uniqueness and regularity of the nonnegative solutions to the following elliptic cross-diffusion system,

$$\begin{cases} -\Delta(A(u,v)) = uf(u,v), & \text{in } \Omega, \\ -\Delta(B(u,v)) = vg(u,v), & \text{in } \Omega, \\ \nabla(A(u,v)) \cdot \sigma = \nabla(B(u,v)) \cdot \sigma = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.0.2}$$

The strategy for studying the existence to (4.0.2) consists in adapting the tools used to prove the existence of strong solutions to the triangular parabolic cross-diffusion system (3.1.1) - (3.1.3), in *Theorem 3.1.1*. The uniqueness and the qualitative properties of the solutions to the system (4.0.2) will be treated thereafter, in order to complement the analysis.

We point out that in *Chapter 3*, we took advantage of the triangular structure of the system (3.1.1), (3.1.2), that allowed us to apply the maximal regularity in the equation satisfied by v. On the contrary, since the system (4.0.2) is a *full cross-diffusion* system, we cannot apply the classical results of elliptic theory. However, here we take advantage of the absence of the evolutionary terms $\partial_t u, \partial_t v$.

As in *Chapter 3*, the main difficulty in studying (4.0.2) comes from the nonlinear diffusion terms $\Delta(A(u,v))$ and $\Delta(B(u,v))$. The key ingredient that we use to handle this difficulty is the introduction of a suitable change of variable. In order to do that, we make the following assumptions on system (4.0.2): the diffusivity functions A, B are such that

$$A, B \in C^2(\mathbb{R}_+^2, \mathbb{R}_+) \quad \text{with} \quad A(0,v) = 0, \quad \text{and} \quad B(0,v) = 0, \qquad \text{for all } v \geq 0. \tag{E1}$$

There exist $a_0, a_1, a_2 > 0$ such that for all $u, v \geq 0$,

$$0 < a_0 \leq \partial_1 A(u,v) \leq a_1 \qquad \text{and} \qquad |\partial_2 A(u,v)| \leq a_2. \tag{E2}$$

There exist $b_0, b_1, b_2 > 0$ such that for all $u, v \geq 0$,

$$0 < b_0 \leq \partial_2 B(u,v) \leq b_1 \qquad \text{and} \qquad |\partial_1 B(u,v)| \leq b_2. \tag{E3}$$

There exist $a_3, b_3 > 0$ such that for all $u, v \geq 0$,

$$|\partial_{12} A(u,v)| \leq a_3 \qquad \text{and} \qquad |\partial_{21} B(u,v)| \leq b_3. \tag{E4}$$

The functions f, g are $C^1(\mathbb{R}_+^2)$ and there exist the constants $C_f, C_g, C_g' > 0$ such that for all $u, v \geq 0$

$$\begin{aligned} -C_f(1+u+v) \leq f(u,v) \leq C_f, \\ -C_g(1+u+v) \leq g(u,v) \leq C_g, \\ |\partial_1 g(u,v)|, |\partial_2 g(u,v)| \leq C_g'. \end{aligned} \tag{R1}$$

The change of variable that we propose is the natural generalization of the one introduced in (3.1.10) in *Chapter 3*. By strongly using the assumptions (E2), (E3), it writes as follows

$$\begin{cases} a = A(u,v), \\ b = B(u,v), \end{cases} \iff \begin{cases} u = U(a,b), \\ v = V(a,b), \end{cases} \tag{4.0.3}$$

so that at a formal level, (a,b) satisfies

$$\begin{cases} -\Delta a = a\, r(a,b), & \text{in } \Omega, \\ -\Delta b = b\, s(a,b), & \text{in } \Omega, \\ \nabla a \cdot \sigma = \nabla b \cdot \sigma = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.0.4}$$

where for all $a, b \geq 0$

$$\begin{aligned} r(a,b) &:= \frac{U(a,b)}{a} f(U(a,b), V(a,b)), \\ s(a,b) &:= \frac{V(a,b)}{b} g(U(a,b), V(a,b)). \end{aligned} \tag{4.0.5}$$

As in *Chapter 3*, we observe that assumptions (E2), (E3) imply for all $a > 0$ and $b \geq 0$

$$0 < \frac{1}{a_1} \leq \frac{U(a,b)}{a} \leq \frac{1}{a_0}, \tag{4.0.6}$$

and for all $a \geq 0$ and $b > 0$

$$0 < \frac{1}{b_1} \leq \frac{V(a,b)}{b} \leq \frac{1}{b_0}. \tag{4.0.7}$$

Thus, (4.0.6), (4.0.7), together with (R1), imply the following upper bounds

$$r(a,b) \leq R \qquad \text{and} \qquad s(a,b) \leq S, \tag{4.0.8}$$

where R, S are strictly positive constants depending on the parameters involved in (E2), (E3), (R1).

Thanks to the change of variable (4.0.3), the existence of strong solutions to the system (4.0.2) follows from the existence of strong solutions to (4.0.4) - (4.0.8). Thus, it remains to show the existence of strong solutions to (4.0.4) - (4.0.8), assuming (E1) - (E4), (R1). We hope to be able to use for that Schauder's fixed point theorem.

We introduce the Hilbert space $H^1(\Omega)$ and its closed bounded convex subset

$$B_Q := B_{H^1(\Omega)}(0, Q) = \{w \in H^1(\Omega) \text{ s.t. } \|w\|^2_{H^1(\Omega)} \leq Q\}, \tag{4.0.9}$$

with the constant $Q > 0$. Then, denoting z^+ the nonnegative part of z, we consider the map

$$\Phi : (a,b) \in \left(H^1(\Omega) \times H^1(\Omega)\right) \to (\bar{a}, \bar{b}), \tag{4.0.10}$$

where $\bar{a}, \bar{b}$ satisfy

$$\begin{cases} -\Delta \bar{a} + k_a \bar{a} = a^+(r(a,b) + k_a)^+, & \text{in } \ \Omega, \\ \nabla \bar{a} \cdot \sigma = 0, & \text{in } \ \partial\Omega, \end{cases} \tag{4.0.11}$$

and

$$\begin{cases} -\Delta \bar{b} + k_b \bar{b} = b^+(s(a,b) + k_b)^+, & \text{in } \Omega, \\ \nabla \bar{b} \cdot \sigma = 0, & \text{in } \partial\Omega, \end{cases} \tag{4.0.12}$$

where the reaction functions r, s are defined in (4.0.5) and k_a, k_b are two nonnegative constants to be determined in such a way that it is possible to apply Schauder's fixed point theorem. Then, by classical results of elliptic equations (see [43]) and using (4.0.6), (4.0.7), we hope to prove that the map Φ satisfies Schauder's fixed point theorem.

Appendix A

Appendix

A.1 The parabolic maximal regularity estimate

In this section we state the following classical result for the heat equation (see [79]).

Proposition A.1.1. *We consider the problem below, with $T > 0$ and $\Omega \subset \mathbb{R}^N$ a smooth bounded domain*

$$\begin{cases} \partial_t v - d\Delta v = g, & \text{on } \Omega_T, \\ \nabla v \cdot \sigma = 0, & \text{on } (0,T) \times \partial\Omega, \\ v(0,x) = v_{in}(x) \geq 0, & \text{on } \Omega, \end{cases} \tag{A.1.1}$$

with the diffusion coefficient $d > 0$. We assume that ∇v_{in} lies in $L^q(\Omega)$ and g in $L^q(\Omega_T)$ with $q \in (1,2]$, then there exists a constant $C > 0$, only depending on d, q, Ω such that the strong solution v of (A.1.1) satisfies

$$\|\partial_t v\|_{L^q(\Omega_T)} + \|\nabla\nabla v\|_{L^q(\Omega_T)} \leq C \left(\|g\|_{L^q(\Omega_T)} + \|\nabla v_{in}\|_{L^q(\Omega)}\right).$$

Moreover, if $\nabla\nabla v_{in}$ belongs to $L^q(\Omega)$ and g to $L^q(\Omega_T)$ with $q \in (2, +\infty)$, then there exists a constant $C > 0$, only depending on d, q, Ω such that v satisfies

$$\|\partial_t v\|_{L^q(\Omega_T)} + \|\nabla\nabla v\|_{L^q(\Omega_T)} \leq C \left(\|g\|_{L^q(\Omega_T)} + \|\nabla\nabla v_{in}\|_{L^q(\Omega)}\right).$$

A.2 Chapter 1 : Proof of Proposition 1.4.4

Proof. The Routh matrix associated to M^ε writes as (see [66])

$$R_{M^\varepsilon} := \begin{bmatrix} 1 & \det_2 M^\varepsilon \\ -\mathrm{tr} M^\varepsilon & -\det M^\varepsilon \\ \dfrac{(\det_2 M^\varepsilon)(\mathrm{tr} M^\varepsilon) - \det M^\varepsilon}{\mathrm{tr} M^\varepsilon} & 0 \\ -\det M^\varepsilon & 0 \end{bmatrix},$$

with

$$\det_2 M^\varepsilon := [M^\varepsilon]_{11} + [M^\varepsilon]_{22} + [M^\varepsilon]_{33},$$

and where $[M^\varepsilon]_{ii}$ are the following minors:

$$[M^\varepsilon]_{11} := \begin{vmatrix} M^\varepsilon_{22} & M^\varepsilon_{23} \\ M^\varepsilon_{32} & M^\varepsilon_{33} \end{vmatrix}, \qquad [M^\varepsilon]_{22} := \begin{vmatrix} M^\varepsilon_{11} & M^\varepsilon_{13} \\ M^\varepsilon_{31} & M^\varepsilon_{33} \end{vmatrix}, \qquad [M^\varepsilon]_{33} := \begin{vmatrix} M^\varepsilon_{11} & M^\varepsilon_{12} \\ M^\varepsilon_{21} & M^\varepsilon_{22} \end{vmatrix}.$$

By the Routh-Hurwitz criterion [66], M^ε is stable if and only if there are no sign variations in the first column entries of R_{M^ε}, i.e., if and only if M^ε satisfies

$$\begin{cases} \mathrm{tr} M^\varepsilon < 0, \\ (\det_2 M^\varepsilon)(\mathrm{tr} M^\varepsilon) - \det M^\varepsilon < 0, \\ \det M^\varepsilon < 0. \end{cases} \tag{A.2.1}$$

From the expression of M^ε, we get

$$\mathrm{tr}\, M^\varepsilon = -\eta_a - \eta_b\alpha - \eta_v(1-\alpha) - \frac{r}{\varepsilon} < 0,$$

and

$$\begin{aligned} [M^\varepsilon]_{11} &= \eta_v \frac{1-\alpha}{\varepsilon}\phi_1 > 0, \\ [M^\varepsilon]_{22} &= \eta_v(1-\alpha)\Big(\eta_a + \frac{\beta}{\varepsilon}\Big) > 0, \\ [M^\varepsilon]_{33} &= \eta_a\eta_b\alpha + \frac{1}{\varepsilon}\eta_a(r-\beta) + \frac{1}{\varepsilon}\eta_b\alpha\beta > 0, \end{aligned}$$

which imply

$$\det_2 M^\varepsilon > 0.$$

Furthermore,

$$\det M^\varepsilon = \Big(-\eta_a + \frac{1}{\varepsilon}\partial_1\bar{Q}\Big)[M^\varepsilon]_{11} - \frac{\eta_v}{\varepsilon}\partial_1\bar{Q}\frac{1-\alpha}{\varepsilon}(\partial_2\bar{Q} - \partial_3\bar{Q}) = -\frac{\eta_a\eta_v\phi_1}{\varepsilon}(1-\alpha) < 0\,.$$

It remains to check the second inequality in (A.2.1), that is a consequence of the previous computations and of the identity

$$\det M^\varepsilon = -\eta_a[M^\varepsilon]_{11}\,.$$

Indeed,

$$\begin{aligned} (\det_2 M^\varepsilon)(\mathrm{tr}\, M^\varepsilon) - \det M^\varepsilon &= ([M^\varepsilon]_{11} + [M^\varepsilon]_{22} + [M^\varepsilon]_{33})\mathrm{tr}\, M^\varepsilon + \eta_a[M^\varepsilon]_{11} \\ &= ([M^\varepsilon]_{22} + [M^\varepsilon]_{33})\mathrm{tr}\, M^\varepsilon - [M^\varepsilon]_{11}\Big(\eta_b\alpha + \eta_v(1-\alpha) + \frac{r}{\varepsilon}\Big) < 0\,. \end{aligned}$$

Thus, M^ε is stable for all $\varepsilon > 0$.

Concerning the matrix N^ε, we define the quantities

$$D_1 := d_a + d_b + d_v > 0\,, \quad D_2 := d_a d_v + d_b d_v + d_a d_b > 0\,, \quad D_3 := d_a d_b d_v\,, \tag{A.2.2}$$

and

$$\begin{aligned} A &:= d_a(M^\varepsilon_{22} + M^\varepsilon_{33}) + d_b(M^\varepsilon_{11} + M^\varepsilon_{33}) + d_v(M^\varepsilon_{11} + M^\varepsilon_{22}) < 0, \\ B &:= d_b d_v M^\varepsilon_{11} + d_a d_v M^\varepsilon_{22} + d_a d_b\, M^\varepsilon_{33} < 0, \\ C &:= d_a\,[M^\varepsilon]_{11} + d_b[M^\varepsilon]_{22} + d_v[M^\varepsilon]_{33} > 0\,. \end{aligned} \tag{A.2.3}$$

Thus, using the previous computations, we obtain

$$\operatorname{tr} N^{\varepsilon} = \operatorname{tr} M^{\varepsilon} - D_1\lambda_n < 0\,,$$

$$\det_2 N^{\varepsilon} = \det_2 M^{\varepsilon} + D_2\,\lambda_n^2 - A\,\lambda_n > 0,$$

and

$$\det N^{\varepsilon} = \det M^{\varepsilon} - D_3\,\lambda_n^3 + B\,\lambda_n^2 - C\,\lambda_n < 0\,.$$

To conclude, it remains to check the sign of the quantity below:

$$\begin{aligned}(\det_2 N^{\varepsilon})(\operatorname{tr} N^{\varepsilon}) - \det N^{\varepsilon} = &(\det_2 M^{\varepsilon})(\operatorname{tr} M^{\varepsilon}) - \det M^{\varepsilon}\\ &+ \lambda_n^3(-D_1D_2 + D_3) + \lambda_n^2(D_2\operatorname{tr} M^{\varepsilon} + AD_1 - B)\\ &+ \lambda_n(-D_1\det_2 M^{\varepsilon} - A\operatorname{tr} M^{\varepsilon} + C)\,.\end{aligned}$$

The latter is indeed strictly negative, using again the negativity of the entries of M^{ε}, the positivity of the minors $[M^{\varepsilon}]_{ii}$, definitions (A.2.2) and (A.2.3) and

$$-D_1D_2 + D_3 < 0\,,\qquad AD_1 - B < 0\,,\qquad -D_1\det_2 M^{\varepsilon} + C < 0\,.$$

Then, by the Routh-Hurwitz criterion again, N^{ε} is stable for all strictly positive ε. □

A.3 Chapter 2 : Proof of *Lemma* (2.7.2)

Proof of Lemma 2.7.2.
In order to show (2.7.22), we observe that it is equivalent to prove that for any $\gamma > 0, \delta \in (0,1)$ and $\lambda > 0$, there exists $C(\gamma,\delta) > 0$

$$\Big(\frac{\eta}{\lambda}\Big)^{\gamma}\Big(1 - \frac{\eta}{\lambda}\Big) \leq C(\gamma,\delta) - \delta\Big(\frac{\eta}{\lambda}\Big)^{\gamma+1}, \qquad \eta \geq 0, \tag{A.3.1}$$

with the optimal constant

$$C(\gamma,\delta) = \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{1-\delta}\right)^{\gamma}. \tag{A.3.2}$$

The idea to prove (A.3.1) is to show the nonnegativity of the following polynomial for any $\eta \geq 0$,

$$\mathcal{P}(\eta) := (1-\delta)\Big(\frac{\eta}{\lambda}\Big)^{\gamma+1} - \Big(\frac{\eta}{\lambda}\Big)^{\gamma} + C(\gamma,\delta),$$

provided (A.3.2). By analysing the monotonicity of $\mathcal{P}$, we get

$$\mathcal{P}'(\eta) = \Big(\frac{1}{\lambda}\Big)^{\gamma}\eta^{\gamma-1}\Big((1-\delta)(1+\gamma)\frac{\eta}{\lambda} - \delta\Big),$$

which admits a unique local minimum

$$\eta_{min} := \frac{\gamma\lambda}{(1-\delta)(1+\gamma)} > 0.$$

Therefore, observing that $\mathcal{P}(0) = C(\gamma,\delta) > 0$ we compute

$$\mathcal{P}(\eta_{min}) = (1-\delta)\Big(\frac{\gamma}{(1-\delta)(1+\gamma)}\Big)^{\gamma+1} - \Big(\frac{\gamma}{(1-\delta)(1+\gamma)}\Big)^{\gamma} + C(\gamma,\delta),$$

which is nonnegative iff

$$C(\gamma,\delta) \geq \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{1}{1-\delta}\right)^{\gamma}.$$

Thus, the regularity of $\mathcal{P}$ allows to conlcude.

□

A.4 Chapter 3 : Existence and estimates for a *non divergence* linear parabolic problem

This section aims to show *Proposition 3.2.2*. The strategy of proof is based on a density argument: firstly, we consider a regularized version of (3.2.4) which admits a pointwise and unique solution by the classical parabolic theory. Then, we prove some uniform a priori estimates for this regularized system in order to taking the limit and finally end up with the estimates for the solutions to equation (3.2.4).

Proof of Proposition 3.2.2.
By density there exist $(\gamma_n)_n$, $(r_n)_n \in C_c^\infty(\mathbb{R}^{N+1})$ satisfying (3.2.5),(3.2.6) and $(b_{\text{in},n})_n \in C_c^\infty(\mathbb{R}^N)$ such that

$$\gamma_{n_{|\Omega_T}} \to \gamma, \qquad r_{n_{|\Omega_T}} \to r, \qquad b_{\text{in},n_{|\Omega}} \to b_{\text{in}}, \qquad \text{a.e.} \quad \text{as } n \to +\infty, \tag{A.4.1}$$

up to subsequences still denoted $\gamma_n, r_n, b_{\text{in},n}$ (e.g. regularization by convolution with standard mollifiers). For any fixed $n \in \mathbb{N}$ we consider the problem

$$\begin{cases} \partial_t b_n - \gamma_n(t,x)\Delta b_n = r_n(t,x) b_n, & \text{on } \Omega_T, \\ \nabla b_n \cdot \sigma = 0, & \text{on } (0,T) \times \partial\Omega, \\ b_n(0,x) = b_{\text{in},n}(x) \geq 0, & \text{on } \Omega, \end{cases} \tag{A.4.2}$$

where the uniform bounds hold

$$0 < \gamma_0 \leq \gamma_n(t,x) \leq \gamma_1, \qquad \text{a.e. in } \Omega_T, \tag{A.4.3}$$

and

$$\|r_n\|_{L^2(\Omega_T)} + \|b_{\text{in},n}\|_{L^\infty(\Omega)} + \|\nabla b_{\text{in},n}\|_{L^2(\Omega)} \leq C, \tag{A.4.4}$$

with C not depending on n. According to [57] there exists a solution b_n to the system (A.4.2) s.t. for any $n \in \mathbb{N}$

$$b_n(t,x) \in C([0,T]; H^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)), \tag{A.4.5}$$

and $b_n(t,x)$ satisfies (A.4.2) pointwise. The following paragraphs are devoted to prove some $n-$uniform estimates and properties of b_n in order to take the limit in (A.4.2) as $n \to +\infty$.

A.4.1 *Nonnegativity of* b_n

In order to prove the nonnegativity of b_n a.e. in $\mathbb{R}_+ \times \Omega$, we multiply the first equation of (A.4.2) by the test function $-(b_n)^- := -\min\{0, b_n\} \geq 0$ and then we integrate on Ω. We remark that all the integrals below are well defined.

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (b_n^-)^2 dx &= \int_\Omega \nabla(b_n)\cdot\nabla(\gamma_n b_n^-)dx + \int_\Omega r_n (b_n^-)^2 dx \\
&= \int_\Omega b_n^- \nabla(b_n)\cdot\nabla\gamma_n dx + \int_\Omega \gamma_n \nabla b_n \cdot \nabla(b_n^-) + \int_\Omega r_n(b_n^-)^2 dx \\
&= -\int_\Omega b_n^-\nabla(b_n^-)\cdot\nabla\gamma_n dx - \int_\Omega \gamma_n |\nabla(b_n^-)|^2 dx + \int_\Omega r_n (b_n^-)^2 dx \\
&\le \frac{1}{2}\int_\Omega \gamma_n |\nabla(b_n^-)|^2 dx + \frac{1}{2}\int_\Omega \frac{1}{\gamma_n}(b_n^-)^2 |\nabla\gamma_n|^2 dx \\
&\quad - \int_\Omega \gamma_n |\nabla(b_n^-)|^2 dx + \int_\Omega r_n(b_n^-)^2 dx \\
&\le \Big(\frac{1}{2\gamma_0}\|\nabla\gamma_n\|^2_{L^\infty(\Omega)} + R\Big)\int_\Omega (b_n^-)^2 dx.
\end{aligned}$$

Thus by Gronwall's Lemma and the nonnegativity of $b_{\text{in},n}$, we conclude that for any $n \in \mathbb{N}$

$$b_n^-(t,x) = 0, \quad \text{a.e in } \mathbb{R}_+ \times \Omega \qquad \Longrightarrow \qquad b_n(t,x) \ge 0, \quad \text{a.e in } \mathbb{R}_+ \times \Omega.$$

A.4.2 $L^\infty(\Omega_T)$ bound of b_n

We define for all $(t,x) \in \Omega_T$ and $n \in \mathbb{N}$,

$$\beta_n(t,x) := b_n(t,x)\, e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} \ge 0, \qquad \text{a.e in } \Omega_T.$$

Thus, we compute

$$\begin{aligned}
\partial_t\beta_n - \gamma_n\Delta\beta_n &= (\partial_t b_n)e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} - b_n\,(\sup_{x\in\Omega} r_n\,)e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} \\
&\quad - \gamma_n\Delta(b_n)e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} \\
&= \gamma_n\Delta(b_n)e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} + r_n b_n e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} \\
&\quad - b_n(\sup_{x\in\Omega} r_n)e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} - \gamma_n\Delta(b_n)e^{-\int_0^t \sup_{x\in\Omega} r_n(s,x)ds} \\
&= \beta_n(r_n - \sup_{x\in\Omega} r_n) \le 0,
\end{aligned}$$

by the nonnegativity of β_n. Therefore β_n satisfies

$$\begin{cases}
\partial_t\beta_n - \gamma_n\Delta\beta_n \le 0, & \text{in } \Omega_T, \\
\nabla\beta_n\cdot\sigma = \nabla b_n\cdot\sigma = 0, & \text{in } (0,T)\times\partial\Omega, \\
\beta_n(0,x) = b_{\text{in},n}(x), & \text{in } \Omega,
\end{cases}$$

so that by the maximum principle (see [11]) we get for all $t \in (0,T)$ and $n \in \mathbb{N}$,

$$\|\beta_n(t,\cdot)\|_{L^\infty(\Omega)} \le \|\beta_n(0,\cdot)\|_{L^\infty(\Omega)},$$

i.e.

$$\|b_n(t,\cdot)\|_{L^\infty(\Omega)} \le e^{RT}\|b_{\text{in},n}\|_{L^\infty(\Omega)}. \tag{A.4.6}$$

A.4.3 *Extra $n-$uniform estimates*

The aim of this paragraph is to prove the inequality (3.2.10). In order to do that, we multiply the first equation of (A.4.2) by $-\Delta b_n$ and we integrate on Ω to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla b_n|^2 dx &= -\int_\Omega \gamma_n(\Delta b_n)^2 dx - \int_\Omega r_n b_n \Delta b_n dx\\ &\leq -\frac{1}{2}\int_\Omega \gamma_n(\Delta b_n)^2 dx + \frac{1}{2}\int_\Omega \frac{1}{\gamma_n}(r_n b_n)^2 dx\\ &\leq -\frac{1}{2}\int_\Omega \gamma_n(\Delta b_n)^2 dx + \frac{1}{2\gamma_0}\int_\Omega (r_n b_n)^2 dx,\end{aligned}$$

using the strictly positive lower bound of γ_n in (3.2.5). Finally integrating in time and taking the supremum for $t \in (0,T)$, we end up with

$$||\nabla b_n||^2_{L^\infty(0,T;L^2(\Omega))} + \gamma_0||\Delta b_n||^2_{L^2(\Omega_T)} \leq ||\nabla b_{\text{in},n}||^2_{L^2(\Omega)} + \frac{1}{\gamma_0}||r_n b_n||^2_{L^2(\Omega_T)}. \tag{A.4.7}$$

Next, in order to get the L^2-boundedness of $\partial_t b_n$ we multiply the first equation of (A.4.2) by $\partial_t b_n$ and we integrate on Ω_T to obtain

$$\begin{aligned}\int_{\Omega_T} (\partial_t b_n)^2 dxdt &= \int_{\Omega_T} \gamma_n \Delta b_n \partial_t b_n \, dxdt + \int_{\Omega_T} r_n b_n \partial_t b_n \, dxdt\\ &\leq \frac{1}{2}||\partial_t b_n||^2_{L^2(\Omega_T)} + \gamma_1^2||\Delta b_n||^2_{L^2(\Omega_T)} + ||r_n b_n||^2_{L^2(\Omega_T)},\end{aligned}$$

i.e.

$$||\partial_t b_n||^2_{L^2(\Omega_T)} \leq 2\gamma_1^2||\Delta b_n||^2_{L^2(\Omega_T)} + 2||r_n b_n||^2_{L^2(\Omega_T)}. \tag{A.4.8}$$

Finally using (A.4.4) in (A.4.6), (A.4.7),(A.4.8), up to subsequences we have for some $b \in L^\infty(\Omega_T)$,

$$b_n \to b \quad \text{a. e. in } \Omega_T, \tag{A.4.9}$$

and

$$\partial_t b_n \rightharpoonup \partial_t b, \quad \Delta b_n \rightharpoonup \Delta b, \quad \nabla b_n \rightharpoonup \nabla b, \quad \text{weakly in } L^2(\Omega_T). \tag{A.4.10}$$

Therefore, it remains to take the weak limit as $n \to +\infty$, in system (A.4.2), using the obtained limits (A.4.9), (A.4.10). The weak $L^2(\Omega_T)$ convergence of the evolutionary term follows directly from (A.4.10). Concerning the diffusion part, we observe that γ_n and Δb_n satisfies the assumptions of *Proposition A.6.2* with $p = q = 2$, by (A.4.1), (A.4.3), (A.4.10). Thus, we get

$$\gamma_{n|_{\Omega_T}} \Delta b_n \quad \rightharpoonup \quad \gamma \Delta b, \qquad \text{weakly in } L^1(\Omega_T).$$

About the reaction term $r_n b_n$, we have by (A.4.10)

$$r_{n|_{\Omega_T}} b_n \quad \to \quad rb, \qquad \text{a.e. in } \Omega_T.$$

Moreover, by the $n-$uniform estimates (A.4.6), (A.4.4) we get

$$||r_n b_n||_{L^2(\Omega_T)} \leq C,$$

giving (up to subsequences) by the uniqueness of the a.e. limit

$$r_{n|_{\Omega_T}} b_n \quad \rightharpoonup \quad rb, \qquad \text{weakly in } L^2(\Omega_T).$$

We conclude by taking the limit in the boundary condition of (A.4.2), using the continuity of the trace operator and the weak convergence of Δb_n in (A.4.10). Finally, b satisfies (3.2.4) in the sense of *(iii)* in *Proposition 3.2.2* and the weak lower semicontinuity of the L^p norm gives (3.2.9), (3.2.10). □

A.4.4 Uniqueness of $\bar{a}_\delta$

In this subsection, we prove the uniqueness of $\bar{a}_\delta$ in $L^\infty(0,T;L^2(\Omega))$, satisfying (3.2.13) in the sense of *Proposition 3.2.2*. Let $\bar{a}_1, \bar{a}_2$ two solutions to (3.2.13) with the initial data $a_{\text{in},1}, a_{\text{in},2}$, respectively. Then, we multiply by $\bar{a}_1 - \bar{a}_2$ the equation satisfied by $\bar{a}_1 - \bar{a}_2$ and we integrate on Ω. We have

$$\begin{aligned}\int_\Omega (\bar{a}_1 - \bar{a}_2)\partial_t(\bar{a}_1 - \bar{a}_2)dx = &\int_\Omega (\mu(t,x) *_x \varphi_\delta)(\bar{a}_1 - \bar{a}_2)\Delta(\bar{a}_1 - \bar{a}_2)dx \\ &+ \int_\Omega s_M(t,x)(\bar{a}_1 - \bar{a}_2)^2 dx,\end{aligned}$$

thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (\bar{a}_1 - \bar{a}_2)^2 dx = &-\int_\Omega (\mu(t,x) *_x \varphi_\delta)|\nabla(\bar{a}_1 - \bar{a}_2)|^2 dx \\ &- \int_\Omega (\bar{a}_1 - \bar{a}_2)\nabla(\mu(t,x) *_x \varphi_\delta)\cdot\nabla(\bar{a}_1 - \bar{a}_2)dx \\ &+ \int_\Omega s_M(t,x)(\bar{a}_1 - \bar{a}_2)^2 dx \\ \le &-\frac{1}{2}\int_\Omega (\mu(t,x) *_x \varphi_\delta)|\nabla(\bar{a}_1 - \bar{a}_2)|^2 dx \\ &+ \frac{1}{2}\|\nabla(\mu(t,x) *_x \varphi_\delta)\|^2_{L^\infty(\Omega)}\int_\Omega \frac{(\bar{a}_1 - \bar{a}_2)^2}{(\mu(t,x) *_x \varphi_\delta)}dx \\ &+ S(\varepsilon, M, T)\int_\Omega (\bar{a}_1 - \bar{a}_2)^2 dx \\ \le &\Big(\frac{a_1^2}{2a_0}\|\nabla\varphi_\delta\|^2_{L^1(\mathbb{R}^N)} + S(\varepsilon, M, T)\Big)\int_\Omega (\bar{a}_1 - \bar{a}_2)^2 dx.\end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt}\|\bar{a}_1 - \bar{a}_2\|^2_{L^2(\Omega)} \le \Big(\frac{a_1^2}{a_0}\|\nabla\varphi_\delta\|^2_{L^1(\mathbb{R}^N)} + 2S(\varepsilon, M, T)\Big)\|\bar{a}_1 - \bar{a}_2\|^2_{L^2(\Omega)},$$

implying by Gronwall's Lemma and taking the supremum for $t \in (0,T)$,

$$\|\bar{a}_1 - \bar{a}_2\|^2_{L^\infty(0,T;L^2(\Omega))} \le C(\delta, \varepsilon, M, T)\|\bar{a}_{\text{in},1} - \bar{a}_{\text{in},2}\|^2_{L^2(\Omega)}, \tag{A.4.11}$$

implying the uniqueness of the solution in $L^\infty(0,T;L^2(\Omega))$.

A.5 Chapter 3 : Extra computations

A.5.1 Nonnegativity of $\bar{v}_\delta$

Let $\bar{v}_\delta$ be the solution to (3.2.12) with the initial datum $v_{\text{in},\varepsilon}$ as in (H4). Then, we prove the nonnegativity of $\bar{v}_\delta$ by multiplying the first equation on (3.2.12) by $-\bar{v}_\delta^- := -\min\{0, \bar{v}_\delta\} \ge 0$ and by integrating on Ω

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\bar{v}_\delta^-)^2 dx = -d_v\int_\Omega |\nabla\bar{v}_\delta^-|^2 dx + \int_\Omega (\bar{v}_\delta^-)^2 g_{\varepsilon,M}\, dx \le C_g\int_\Omega (\bar{v}_\delta^-)^2 dx.$$

Then, we integrate in time and we use the nonnegativity of $v_{\text{in},\varepsilon}$ to get

$$\bar{v}_\delta^-(t,x) = 0, \quad \text{a.e in } \mathbb{R}_+\times\Omega \qquad \Longrightarrow \qquad \bar{v}_\delta(t,x) \ge 0, \quad \text{a.e in } \mathbb{R}_+\times\Omega.$$

A.6 Chapter 3 : Useful functional analysis results

In this section, we state two useful functional analysis results (which can easily be deduced from properties stated in [11]). In particular, *Proposition A.6.1* gives a criterion of strong L^p convergence, while *Proposition* A.6.2 is a result of weak convergence for a product of two sequences.

Proposition A.6.1.
Let $U \subset \mathbb{R}^N, N \in \mathbb{N}$ a smooth bounded open set with $|U| < \infty$. We consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p(U)$ with $1 < p < \infty$ s.t.

$$f_n(x) \quad \to \quad f(x), \qquad \text{a.e. on } U, \tag{A.6.1}$$

$\{f_n\}_{n\in\mathbb{N}}$ is bounded uniformly in $n \in \mathbb{N}$, i.e. there exists a constant $C > 0$ not depending on $n \in \mathbb{N}$ s.t.

$$\|f_n\|_{L^p(U)} \leq C. \tag{A.6.2}$$

Then, it holds

$$f_n \quad \to \quad f, \qquad \text{strongly in } L^q(U), \qquad \forall\ q < p. \tag{A.6.3}$$

Proposition A.6.2.
Let $U \subset \mathbb{R}^N, N \in \mathbb{N}$ a smooth bounded open set with $|U| < \infty$. Let $\{f_n\}_{n\in\mathbb{N}} \in L^p(U)$ with $1 \leq p < \infty$ and $\{g_n\}_{n\in\mathbb{N}} \in L^q(U)$ with $1 \leq q < \infty$ s.t.

$$f_n \quad \to \quad f, \qquad \text{strongly in } L^p(U), \tag{A.6.4}$$

and

$$g_n \quad \rightharpoonup \quad g, \qquad \text{weakly in } L^q(U). \tag{A.6.5}$$

Then, it holds

$$f_n\, g_n \quad \rightharpoonup \quad fg, \qquad \text{weakly in } L^r(U). \tag{A.6.6}$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Bibliography

[1] N. D. Alikakos. Lp bounds of solutions of reaction-diffusion equations. *Communications in Partial Differential Equations*, 4(8):827–868, 1979. (See page xxi.)

[2] H. Amann. Dynamic theory of quasilinear parabolic equations. i. abstract evolution equations. *Nonlinear Analysis: Theory, Methods & Applications*, 12(9):895–919, 1988. (See page xxi.)

[3] H. Amann. Dynamic theory of quasilinear parabolic equations. ii. reaction-diffusion systems. *Differential and Integral Equations*, 3(1):13–75, 1990. (See page xxi.)

[4] J. R. Beddington. Mutual interference between parasites or predators and its effect on searching efficiency. *The Journal of Animal Ecology*, pages 331–340, 1975. (See page xvi.)

[5] J. F. Benson and B. R. Patterson. Inter-specific territoriality in a canis hybrid zone: spatial segregation between wolves, coyotes, and hybrids. *Oecologia*, 173(4):1539–1550, 2013. (See pages viii and xv.)

[6] J. Berendsen, M. Burger, V. Ehrlacher, and J.-F. Pietschmann. Uniqueness of strong solutions and weak–strong stability in a system of cross-diffusion equations. *Journal of Evolution Equations*, 20(2):459–483, 2020. (See page 105.)

[7] D. H. Boucher, S. James, and K. H. Keeler. The ecology of mutualism. *Annual Review of Ecology and Systematics*, 13:315–347, 1982. (See page xiii.)

[8] L. Boudin, B. Grec, and F. Salvarani. A mathematical and numerical analysis of the maxwell-stefan diffusion equations. *Discrete and Continuous Dynamical Systems-Series B*, 17(5):1427–1440, 2012. (See page xx.)

[9] M. Breden, C. Kuehn, and C. Soresina. On the influence of cross-diffusion in pattern formation, 2021. (See pages xvi and 115.)

[10] M. Breden, J.-P. Lessard, and M. Vanicat. Global bifurcation diagrams of steady states of systems of pdes via rigorous numerics: a 3-component reaction-diffusion system. *Acta applicandae mathematicae*, 128(1):113–152, 2013. (See page 115.)

[11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011. (See pages 81, 85, 123, and 126.)

[12] E. Brocchieri, L. Corrias, H. Dietert, and Y.-J. Kim. Evolution of dietary diversity and a starvation driven cross-diffusion system as its singular limit. *Journal of Mathematical Biology*, 83(5):1–40, 2021. (See pages viii and xxii.)

[13] M. Burger, M. Di Francesco, J.-F. Pietschmann, and B. Schlake. Nonlinear cross-diffusion with size exclusion. *SIAM Journal on Mathematical Analysis*, 42(6):2842–2871, 2010. (See page xx.)

[14] J. A. Canizo, L. Desvillettes, and K. Fellner. Improved duality estimates and applications to reaction-diffusion equations. *Communications in Partial Differential Equations*, 39(6):1185–1204, 2014. (See page xxi.)

[15] J. G. Cecere, S. Bondì, S. Podofillini, S. Imperio, M. Griggio, E. Fulco, A. Curcio, D. Ménard, U. Mellone, N. Saino, et al. Spatial segregation of home ranges between neighbouring colonies in a diurnal raptor. *Scientific reports*, 8(1):1–9, 2018. (See pages viii and xv.)

[16] L. Chen and A. Jüngel. Analysis of a multidimensional parabolic population model with strong cross-diffusion. *SIAM journal on mathematical analysis*, 36(1):301–322, 2004. (See page xxi.)

[17] L. Chen and A. Jüngel. Analysis of a parabolic cross-diffusion population model without self-diffusion. *Journal of Differential Equations*, 224:39–59, 2006. (See pages xxi, 3, and 32.)

[18] X. Chen, E. S. Daus, and A. Jüngel. Global existence analysis of cross-diffusion population systems for multiple species. *Archive for Rational Mechanics and Analysis*, 227:715–747, 2018. (See page 3.)

[19] X. Chen and A. Jüngel. When do cross-diffusion systems have an entropy structure? *Journal of Differential Equations*, 278:60–72, 2021. (See page xxi.)

[20] E. Cho and Y.-J. Kim. Starvation driven diffusion as a survival strategy of biological organisms. *Bulletin of Mathematical Biology volume*, 75:845—870, 2013. (See page 3.)

[21] B. Choi and Y.-J. Kim. Diffusion of biological organisms: Fickian and fokker–planck type diffusions. *SIAM Journal on Applied Mathematics*, 79(4):1501–1527, 2019. (See page 2.)

[22] Y. Choi, R. Lui, and Y. Yamada. Existence of global solutions for the shigesada-kawasaki-teramoto model with weak cross-diffusion. *Discrete & Continuous Dynamical Systems*, 9(5):1193, 2003. (See page xxi.)

[23] Y. Choi, R. Lui, and Y. Yamada. Existence of global solutions for the shigesada-kawasaki-teramoto model with strongly coupled cross-diffusion. *Discrete & Continuous Dynamical Systems*, 10(3):719, 2004. (See page xxi.)

[24] J. Chung, Y.-J. Kim, O. Kwon, and C.-W. Yoon. Biological advection and cross-diffusion with parameter regimes. *AIMS Mathematics*, 4(6):1721–1744, 2019. (See page 2.)

[25] F. Conforto and L. Desvillettes. Rigorous passage to the limit in a system of reaction-diffusion equations towards a system including cross diffusions. *Communications in Mathematical Sciences*, 12(3):457–472, 2014. (See page xx.)

[26] F. Conforto, L. Desvillettes, and C. Soresina. About reaction–diffusion systems involving the holling-type ii and the beddington–deangelis functional responses for predator–prey models. *Nonlinear Differential Equations and Applications NoDEA*, 25(3):1–39, 2018. (See page xvi.)

[27] F. Conforto, L. Desvillettes, and C. Soresina. About reaction-diffusion systems involving the holling-type ii and the beddington-de angelis functional responses for predator prey models. *Nonlinear Differ. Equ. Appl.*, 25:1501–1527, 2018. (See page 3.)

[28] C. Cosner and A. C. Lazer. Stable coexistence states in the volterra–lotka competition model with diffusion. *SIAM Journal on Applied Mathematics*, 44(6):1112–1132, 1984. (See page xv.)

[29] E. S. Daus, L. Desvillettes, and H. Dietert. About the entropic structure of detailed balanced multi-species cross-diffusion equations. *Journal of Differential Equations*, 266(7):3861–3882, 2019. (See page xxi.)

[30] E. S. Daus, L. Desvillettes, and A. Jüngel. Cross-diffusion systems and fast-reaction limits. *Bulletin des Sciences Mathématiques*, 159:102824, 2020. (See page 3.)

[31] L. Desvillettes. About entropy methods for reaction-diffusion equations. *Rivista di Matematica dell'Università di Parma*, 7:81–123, 2007. (See page 7.)

[32] L. Desvillettes, K. Fellner, M. Pierre, and J. Vovelle. Global existence for quadratic systems of reaction-diffusion. *Advanced Nonlinear Studies*, 7(3):491–511, 2007. (See page xxi.)

[33] L. Desvillettes and V. Giunta. Existence and regularity for a chemotaxis model involved in the modeling of multiple sclerosis. *Ricerche di Matematica*, 70(1):99–113, 2021. (See pages xx and 67.)

[34] L. Desvillettes, T. Lepoutre, and A. Moussa. Entropy, duality, and cross diffusion. *SIAM Journal on Mathematical Analysis*, 46(1):820–853, 2014. (See page xxi.)

[35] L. Desvillettes, T. Lepoutre, A. Moussa, and A. Trescases. On the entropic structure of reaction-cross diffusion systems. *Communications in Partial Differential Equations*, 40(9):1705–1747, 2015. (See page 3.)

[36] L. Desvillettes and C. Soresina. Non-triangular cross-diffusion systems with predator–prey reaction terms. *Ricerche di Matematica*, 68(1):295–314, 2019. (See pages xvi and xx.)

[37] L. Desvillettes and A. Trescases. New results for triangular reaction cross diffusion system. *Journal of Mathematical Analysis and Applications*, 430(1):32–59, 2015. (See pages xx, xxi, 3, and 32.)

[38] P. Deuring. An initial-boundary-value problem for a certain density-dependent diffusion system. *Mathematische Zeitschrift*, 194(3):375–396, 1987. (See page xxi.)

[39] L. C. Evans. *Partial differential equations*, volume 19. American Mathematical Society, 2022. (See pages 77 and 79.)

[40] J. Fontbona and S. Méléard. Non local lotka-volterra system with cross-diffusion in an heterogeneous medium. *Journal of mathematical biology*, 70(4):829–854, 2015. (See page xvii.)

[41] T. Funaki, H. Izuhara, M. Mimura, and C. Urabe. A link between microscopic and macroscopic models of self-organized aggregation. *Networks & Heterogeneous Media*, 7(4):705–740, 2012. (See page 3.)

[42] G. Gambino, M. C. Lombardo, and M. Sammartino. Turing instability and traveling fronts for a nonlinear reaction–diffusion system with cross-diffusion. *Mathematics and Computers in Simulation*, 82(6):1112–1132, 2012. (See page xvi.)

[43] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer, 1977. (See page 117.)

[44] V. Giunta, M. C. Lombardo, and M. Sammartino. Pattern formation and transition to chaos in a chemotaxis model of acute inflammation. *SIAM Journal on Applied Dynamical Systems*, 20(4):1844–1881, 2021. (See page xvi.)

[45] R. L. Hoffman, G. L. Larson, and B. J. Brokes. Habitat segregation of ambystoma gracile and ambystoma macrodactylum in mountain ponds and lakes, mount rainier national park, washington, usa. *Journal of Herpetology*, 37(1):24–34, 2003. (See page xv.)

[46] H. Hoi, T. Eichler, and J. Dittami. Territorial spacing and interspecific competition in three species of reed warblers. *Oecologia*, 87(3):443–448, 1991. (See pages viii and xv.)

[47] C. S. Holling. Cross-scale morphology, geometry, and dynamics of ecosystems. *Ecological monographs*, 62(4):447–502, 1992. (See page xvii.)

[48] X. Huo, A. Jüngel, and A. E. Tzavaras. Weak-strong uniqueness for maxwell–stefan systems. *SIAM Journal on Mathematical Analysis*, 54(3):3215–3252, 2022. (See page 105.)

[49] T. Ichikawa and Y. Yamada. Some remarks on global solutions to quasilinear parabolic system with cross-diffusion. *FUNKCIALAJ EKVACIOJ SERIO INTERNACIA*, 43(2):285–302, 2000. (See page xxi.)

[50] M. Iida, M. Mimura, and H. Ninomiya. Diffusion, cross-diffusion and competitive interaction. *Journal of Mathematical Biology*, 53(4):617–641, 2006. (See pages xvi, xvii, xviii, xx, 3, and 23.)

[51] T. L. Jackson and H. M. Byrne. A mechanical model of tumor encapsulation and transcapsular spread. *Mathematical biosciences*, 180(1-2):307–328, 2002. (See page xx.)

[52] A. Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity*, 28(6):1963, 2015. (See pages xvii, 3, and 32.)

[53] A. Jüngel and I. V. Stelzer. Entropy structure of a cross-diffusion tumor-growth model. *Mathematical Models and Methods in Applied Sciences*, 22(07):1250009, 2012. (See page xx.)

[54] J. U. Kim. Smooth solutions to a quasi-linear system of diffusion equations for a certain population model. Technical report, WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER, 1983. (See page xxi.)

[55] Y.-J. Kim and O. Kwon. Evolution of dispersal with starvation measure and coexistence. *Bulletin of mathematical biology*, 78(2):254–279, 2016. (See page 2.)

[56] K. Kishimoto and H. F. Weinberger. The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains. 1988. (See page xv.)

[57] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1988. (See page 122.)

[58] D. Lamberton. Equations d'évolution linéaires associées à des semi-groupes de contractions dans les espaces l^p. *Journal of functional analysis*, 72(2):252–262, 1987. (See page 79.)

[59] T. Lepoutre and A. Moussa. Entropic structure and duality for multiple species cross-diffusion systems. *Nonlinear Analysis*, 159:298–315, 2017. (See page xxi.)

[60] T. Lepoutre, M. Pierre, and G. Rolland. Global well-posedness of a conservative relaxed cross diffusion system. *SIAM Journal on Mathematical Analysis*, 44(3):1674–1693, 2012. (See page xxi.)

[61] A. Leung. Equilibria and stabilities for competing-species reaction-diffusion equations with dirichlet boundary data. *Journal of Mathematical Analysis and Applications*, 73(1):204–218, 1980. (See page xv.)

[62] Y. Lou and W.-M. Ni. Diffusion, self-diffusion and cross-diffusion. *Journal of Differential Equations*, 131(1):79–131, 1996. (See page 116.)

[63] Y. Lou and W.-M. Ni. Diffusion vs cross-diffusion: an elliptic approach. *Journal of Differential Equations*, 154(1):157–190, 1999. (See page 116.)

[64] Y. Lou, W.-M. Ni, and Y. Wu. On the global existence of a cross-diffusion system. *Discrete & Continuous Dynamical Systems*, 4(2):193, 1998. (See page xxi.)

[65] Y. Lou, W.-M. Ni, and S. Yotsutani. On a limiting system in the lotka–volterra competition with cross-diffusion. *Discrete & Continuous Dynamical Systems*, 10(1&2):435, 2004. (See page 116.)

[66] M. Marcus. Applications of the theory of matrices (fb gantmacher). *SIAM Review*, 6(2):190, 1964. (See pages 23, 119, and 120.)

[67] B. Maurer. Scale. In *Encyclopedia of Ecology*, pages 3154–3159. Academic Press, Oxford, 2008. (See page xvii.)

[68] M. Mimura and K. Kawasaki. Spatial segregation in competitive interaction-diffusion equations. *Journal of Mathematical Biology*, 9(1):49–64, 1980. (See page 115.)

[69] A. Moussa. Some variants of the classical aubin–lions lemma. *Journal of Evolution Equations*, 16:65–93, 2016. (See pages 12, 68, and 81.)

[70] J. D. Murray. *Mathematical biology: I. An introduction*. Springer, 2002. (See pages xiii and xvi.)

[71] L. Nirenberg. An extended interpolation inequality. *Annali Della Scuola Normale Superiore di Pisa-Classe di Scienze*, 20(4):733–737, 1966. (See pages 8 and 79.)

[72] A. Okubo. Diffusion and ecological problems: mathematical models. 1980. (See page xiii.)

[73] S. Ostrander. Macroscopic cross-diffusion models derived from spatially discrete continuous time microscopic models. *SIAM Undergrad. Res. Online*, 4:51–71, 2011. (See page xvii.)

[74] K. J. Painter. Continuous models for cell migration in tissues and applications to cell sorting via differential chemotaxis. *Bulletin of Mathematical Biology*, 71(5):1117–1147, 2009. (See page xx.)

[75] K. J. Painter and T. Hillen. Volume-filling and quorum-sensing in models for chemosensitive movement. *Can. Appl. Math. Quart*, 10(4):501–543, 2002. (See page xx.)

[76] F. Palomares, N. Fernández, S. Roques, C. Chávez, L. Silveira, C. Keller, and B. Adrados. Fine-scale habitat segregation between two ecologically similar top predators. *PloS one*, 11(5):e0155626, 2016. (See page xv.)

[77] C.-V. Pao. Coexistence and stability of a competition—diffusion system in population dynamics. *Journal of Mathematical Analysis and Applications*, 83(1):54–76, 1981. (See page xv.)

[78] M. Pierre and D. Schmitt. Blowup in reaction-diffusion systems with dissipation of mass. *SIAM review*, 42(1):93–106, 2000. (See page xxi.)

[79] J. Prüss and R. Schnaubelt. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *Journal of mathematical analysis and applications*, 256(2):405–430, 2001. (See page 119.)

[80] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Springer, 2019. (See page 7.)

[81] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *Journal of theoretical biology*, 79(1):83–99, 1979. (See pages vii and xvi.)

[82] C. Soresina. Hopf bifurcations in the full skt model and where to find them. *arXiv preprint arXiv:2202.04168*, 2022. (See page 115.)

[83] A. Trescases. On triangular reaction cross-diffusion systems with possible self-diffusion. *Bulletin des Sciences Mathématiques*, 140(7):796–829, 2016. (See page xxi.)

[84] E. Tulumello, M. C. Lombardo, and M. Sammartino. Cross-diffusion driven instability in a predator-prey system with cross-diffusion. *Acta Applicandae Mathematicae*, 132(1):621–633, 2014. (See page xvi.)

[85] A. Turing. The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, 237(641):37–72, 1952. (See pages viii and xv.)

[86] A. Yagi. Global solution to some quasilinear parabolic system in population dynamics. *Nonlinear Analysis: Theory, Methods & Applications*, 21(8):603–630, 1993. (See page xxi.)

[87] A. Yagi. A priori estimates for some quasilinear parabolic system in population dynamics. *Kobe J. Math.*, 14:91–108, 1997. (See page xxi.)

[88] C. Yoon and Y.-J. Kim. Global existence and aggregation in a keller–segel model with fokker–planck diffusion. *Acta Applicandae Mathematicae*, 149(1):101–123, 2017. (See page xvi.)

ÉCOLE DOCTORALE
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Title: Evolutionary dynamics of populations structured by dietary diversity and starvation driven cross-diffusion systems.

Keywords: Cross-diffusion, reaction-diffusion system, entropy methods, existence, uniqueness, regularity, Turing instability.

Abstract: This thesis aims to present some recent results and advances in the analysis of nonlinear parabolic problems arising in biology and ecology. More precisely, we study the existence, regularity and stability of solutions of partial differential equations (PDEs), describing the evolution of two species that diffuse in a homogeneous environment and interact with each other. The PDEs we consider are strongly coupled and the system they give rise to belongs to a class of nonlinear reaction-diffusion systems, called cross-diffusion systems. More precisely, we study a class of triangular starvation driven cross-diffusion systems arising in population dynamics. The tools employed are entropy methods, a priori estimates, fixed point and compactness arguments. Moreover, we analyze the linear stability of homogeneous equilibria in order to investigate Turing instability and pattern formation. Numerical simulations are performed to complement the theoretical results.

Titre: Dynamique évolutive des populations structurées par la diversité alimentaire: systèmes de diffusion croisée.

Mots clés: Diffusion-croisée, système de réaction-diffusion, méthodes d'entropie, existence, unicité, régularité, instabilité de Turing.

Résumé: Cette thèse porte sur l'analyse des problèmes paraboliques non linéaires survenant dans la biologie et l'écologie. Plus précisément, nous étudions l'existence, la régularité et la stabilité des solutions d'une classe d'équations aux dérivées partielles (EDPs), décrivantes l'évolution de deux espèces qui diffusent dans un environnement homogène et interagissent les unes avec les autres. Les EDPs considérées sont fortement couplées et le système lui même appartient à une classe de systèmes de réaction-diffusion non linéaires, appelés systèmes de diffusion croisée. Plus précisément, nous étudions une classe de systèmes triangulaires avec diffusion croisée induits par la diversité alimentaire qui s'appliquent à la dynamique des populations. Les outils utilisés sont les méthodes d'entropie, les estimations a priori, les arguments de point fixe et de compacité. De plus, nous analysons la stabilité linéaire des équilibres homogènes en espace, afin d'étudier l'instabilité de Turing et la formation de motifs (pattern). Des simulations numériques sont réalisées pour compléter les résultats abstraits.