Approximation to Integrable Functions by Modified Complex Shepard Operators

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Abstract

This is a continuation of our recent work: "O. Duman and B. Della Vecchia, Complex Shepard operators and their summability. Results Math. 76 (2021), no. 4, Paper No. 214, 19 pp." In this paper, we introduce the Kantorovich version of complex Shepard operators in order to approximate functions whose pth powers are integrable on the unit square. We also give an application which explains why we need such operators. Furthermore, we study the effects of some regular summability methods on this L_p-approximation.

Keywords: Approximation in the complex plain, Shepard operators, Kantorovich operators, matrix summabiliy methods, Cesàro summability
2010 MSC: 30E10, 40A25, 40C05

1. Introduction

In our recent study [1], we have introduced the complex Shepard operators in order to approximate complex-valued and continuous functions on the unit

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square as follows:

$$\mathbb{S}_{n,\lambda}(f;z)=\frac{\sum_{k,m=0}^{n}|z-z_{k,m}|^{-\lambda}f(z_{k,m})}{\sum_{k,m=0}^{n}|z-z_{k,m}|^{-\lambda}}, \tag{1}$$

where $\lambda>0$, $n\in\mathbb{N}$, $z=x+iy\in K:=[0,1]\times[0,1]$, $i^2=-1$, $x=\Re(z)$, $y=\Im(z)$ and the sample points $z_{k,m,n}:=z_{k,m}=\frac{k}{n}+i\frac{m}{n}$, $k,m\in\{0,1,...,n\}$. In (1) we write $\sum_{k,m=0}^{n}$ for the double summation $\sum_{k=0}^{n}\sum_{m=0}^{n}$. Now let $C(K,\mathbb{C})$ denote the space of all complex-valued and continuous functions on K. Then, in [1] we have proved the following approximation result.

Theorem A. *(see Theorem 1 in [1]) For every $f\in C(K,C)$ and $\lambda\geq 3$,*

$$\lim_{n\to\infty}\|\mathbb{S}_{n,\lambda}(f)-f\|=0,$$

where the symbol $\|\cdot\|$ denotes the usual supremum norm on K.

Recall that the classical Shepard operators, which are highly effective in scattered data interpolation problems, were first introduced by Donald Shepard in 1968 [2]. Since the classical Shepard operators defined on real-valued functions are of interpolated type, linear and positive, they have many applications in current issues of the approximation theory, such as error estimations, direct and inverse approximation, saturation results, rational approximation, allowing results not possible by polynomials (see, for instance, [3, 4, 5, 6, 7, 8, 9]). The superiority of rational approximation over polynomials is well-known: we recall the very elegant results on rational approximation obtained by Herbert Stahl in [10]. There are also several modifications of these operators in solving specific interpolation problems in Computer Aided Geometric Design, such as scattered data and image compression (see [11, 12, 13, 14, 15]).

So far, many approximation operators known for the real case have also been examined on complex domains (see, e.g., [1, 16, 17, 18, 19, 20, 21, 22, 23]).

In this paper, we study the Kantorovich version of complex Shepard operators given by (1) in order to approximate functions in $L_p(K,\mathbb{C})$ $(p\geq 1)$ which is the space of all measurable complex-valued functions f such that $|f|^p$ is integrable on K. So, we will extend the uniform approximation in Theorem A to the L_p spaces and discuss its significant applications. We should note that the Kantorovich Shepard operators for real-valued functions and their approximation properties in the L_p spaces were examined in [24] by Xiou and Zhou. To the best of our knowledge its complex case seems untouched. More precisely, we consider the following Kantorovich version of complex Shepard operators

$$\mathbb{L}_{n,\lambda}(f;z) = (n+1)^2 \sum_{k,m=0}^{n} r_{k,m,n}(\lambda,z) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} f(s,t)dtds, \qquad (2)$$

where

$$r_{k,m,n}(\lambda,z) = \frac{|z-z_{k,m}|^{-\lambda}}{\sum_{k,m=0}^{n} |z-z_{k,m}|^{-\lambda}}.$$

Then, we first obtain a uniform approximation to a continuous complex-valued function on K by means of the operators (2).

Theorem 1. *For every $g \in C(K,\mathbb{C})$ and $\lambda \geq 3$, we have*

$$\lim_{n\to\infty} \|\mathbb{L}_{n,\lambda}(g) - g\| = 0.$$

Since the uniform convergence on K implies L_p-convergence, the next result is an easy consequence of Theorem 1.

Corollary 2. *For every $g \in C(K,\mathbb{C})$ and $\lambda \geq 3$, we have*

$$\lim_{n\to\infty} \|\mathbb{L}_{n,\lambda}(g) - g\|_p = 0,$$

where the symbol $\|\cdot\|_p$ denotes the L_p-norm on K given by

$$\|g\|_p = \left(\int_0^1 \int_0^1 |g(x,y)|^p \, dydx \right)^{1/p}, \ p \geq 1.$$

The following theorem gives an approximation in the space $L_p(K,\mathbb{C})$.

Theorem 3. *For every $f \in L_p(K,\mathbb{C})$ $(p \geq 1)$ and $\lambda \geq 3$, we have*

$$\lim_{n\to\infty} \|\mathbb{L}_{n,\lambda}(f) - f\|_p = 0.$$

2. Auxiliary Results

To prove Theorems 1 and 3, we need the following lemmas.

Lemma 4. (*see [1]*) *Let $n \in \mathbb{N}$ and $z \in K$ with $z \neq z_{k,m}$. Then, for every $\lambda > 0$,*

$$\left(\sum_{k,m=0}^{n} |z - z_{k,m}|^{-\lambda} \right)^{-1} = O\left(n^{-\lambda}\right)$$

holds.

Lemma 5. *For any $z = x + iy \in K$ and $\lambda > 1$,*

$$r_{k,m,n}(\lambda, z) \leq C\left\{(|[(n+1)x] - k| + 1)^2 + (|[(n+1)y] - m| + 1)^2\right\}^{-\lambda/2}$$

holds for $k, m = 0, 1, ..., n$, where C is a positive constant depending at most on λ and $[x]$ is the greatest integer not exceeding x.

Proof. First assume that $z = z_{\alpha,\beta}$ $(\alpha, \beta = 0, 1, ..., n)$. We immediately get

$$r_{k,m,n}(\lambda, z_{\alpha,\beta}) = \delta_{k,\alpha} \cdot \delta_{m,\beta},$$

where $\delta_{a,b}$ is the Kronecker delta. Hence, in this case, the proof is clear. Assume now that $z = x + iy \in K \backslash \{z_{\alpha,\beta} : \alpha, \beta = 0, 1, ..., n\}$. Let $[(n+1)x] = N$ and $[(n+1)y] = M$. One can check that

$$\frac{N}{n+1} \leq x < \frac{N+1}{n+1} \quad \text{and} \quad \frac{M}{n+1} \leq y < \frac{M+1}{n+1}.$$

Then, we have the following four possible cases.

(a) $k < N - 1$ and $m < M - 1$,

(b) $k < N - 1$ and $m \geq M$,

(c) $k \geq N$ and $m < M - 1$,

(d) $k \geq N$ and $m \geq M$.

In the case of (a) from Lemma 4 there exists a positive constant C_1 such that

$$\begin{aligned} r_{k,m,n}(\lambda, z) &\leq C_1 \left|(nx - k) + i\,(ny - m)\right|^{-\lambda} \\ &= C_1 \left\{(nx - k)^2 + (ny - m)^2\right\}^{-\lambda/2}. \end{aligned}$$

Then, we see that

$$
\begin{aligned}
r_{k,m,n}(\lambda,z) &\le C_1\left\{\left(\frac{nN}{n+1}-k\right)^2+\left(\frac{nM}{n+1}-m\right)^2\right\}^{-\lambda/2}\\
&= C_1\left\{\left(\frac{n(N-k)-k}{n+1}\right)^2+\left(\frac{n(M-m)-m}{n+1}\right)^2\right\}^{-\lambda/2}\\
&\le C_1\left\{(N-k-1)^2+(M-m-1)^2\right\}^{-\lambda/2}\\
&\le C_1\left\{\left(\frac{N-k+1}{4}\right)^2+\left(\frac{M-m+1}{4}\right)^2\right\}^{-\lambda/2}\\
&= C\left\{(|[(n+1)x]-k|+1)^2+(|[(n+1)y]-m|+1)^2\right\}^{-\lambda/2},
\end{aligned}
$$

where $C := 4^{\lambda}C_1$. In the case of (b) using the same constants C_1 and C, we may write that

$$
\begin{aligned}
r_{k,m,n}(\lambda,z) &\le C_1\left\{(nx-k)^2+(m-ny)^2\right\}^{-\lambda/2}\\
&\le C_1\left\{\left(\frac{nN}{n+1}-k\right)^2+\left(m-\frac{n(M+1)}{n+1}\right)^2\right\}^{-\lambda/2}\\
&= C_1\left\{\left(\frac{n(N-k)-k}{n+1}\right)^2+\left(\frac{n(m-M-1)+m}{n+1}\right)^2\right\}^{-\lambda/2}\\
&\le C_1\left\{(N-k-1)^2+(m-M-1)^2\right\}^{-\lambda/2}\\
&\le C_1\left\{\left(\frac{N-k+1}{4}\right)^2+\left(\frac{m-M+1}{4}\right)^2\right\}^{-\lambda/2}\\
&= C\left\{(|[(n+1)x]-k|+1)^2+(|[(n+1)y]-m|+1)^2\right\}^{-\lambda/2}.
\end{aligned}
$$

In a similar manner, one can also get the same inequality in the cases of (c) and (d). Therefore, the proof is completed.

Now for each fixed $z = x + iy \in K$ define the function φ_z on K by

$$\varphi_z(w) := |w-z| = \sqrt{(s-x)^2+(t-y)^2}.$$

Then, we also need the next lemma.

Lemma 6. *For any $z = x + iy \in K$, we have*

$$
\mathbb{L}_{n,\lambda}(\varphi_z;z)=\begin{cases} O(n^{-1}), & \text{if } \lambda>3\\ O(n^{-1}\log n), & \text{if } \lambda=3.\end{cases}
$$

Proof. For a given $n \in \mathbb{N}$ and $z = x + iy \in K$, there exist $u, v \in \{0, 1, ..., n\}$ such that $x \in \left[\frac{u}{n+1}, \frac{u+1}{n+1}\right]$ and $y \in \left[\frac{v}{n+1}, \frac{v+1}{n+1}\right]$. Hence, we get from Lemma 5 that

$$r_{k,m,n}(\lambda, z) \leq C \left\{ (|u-k|+1)^2 + (|v-m|+1)^2 \right\}^{-\lambda/2}.$$

Using this we obtain that

$$\begin{aligned}
& \mathbb{L}_{n,\lambda}(\varphi_z; z) \\
= \quad & (n+1)^2 \sum_{k,m=0}^{n} r_{k,m,n}(\lambda, z) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} \sqrt{(s-x)^2 + (t-y)^2}\, dtds \\
\leq \quad & (n+1)^2 \sum_{k,m=0}^{n} r_{k,m,n}(\lambda, z) \frac{\sqrt{(|u-k|+1)^2 + (|v-m|+1)^2}}{(n+1)^3} \\
\leq \quad & \frac{C}{n+1} \sum_{k,m=0}^{n} \left\{ (|u-k|+1)^2 + (|v-m|+1)^2 \right\}^{(1-\lambda)/2} \\
\leq \quad & \frac{C}{n+1} \sum_{k,m=1}^{n} \frac{1}{(k^2+m^2)^{(\lambda-1)/2}} \\
\leq \quad & \frac{C}{n+1} \left(\frac{1}{2^{(\lambda-1)/2}} + 2 \sum_{k=2}^{n} \frac{1}{(k^2+1)^{(\lambda-1)/2}} + \sum_{k,m=2}^{n} \frac{1}{(k^2+m^2)^{(\lambda-1)/2}} \right),
\end{aligned}$$

which implies, for $\lambda \geq 3$,

$$\begin{aligned}
\mathbb{L}_{n,\lambda}(\varphi_z; z) &= O\left(\frac{1}{n}\right) \left(1 + \int_1^n \int_1^n \frac{dtds}{(s^2+t^2)^{(\lambda-1)/2}} \right) \\
&= O\left(\frac{1}{n}\right) \left(1 + \int_0^{2\pi} \int_1^{\sqrt{2}n} \frac{drd\theta}{r^{\lambda-2}} \right).
\end{aligned}$$

Therefore, we see that

$$\mathbb{L}_{n,\lambda}(\varphi_z; z) = \begin{cases} O\left(\frac{1}{n}\right), & \text{if } \lambda > 3 \\ O\left(\frac{\log n}{n}\right), & \text{if } \lambda = 3, \end{cases}$$

which completes the proof.

Lemma 7. *Let $\lambda \geq 3$ and $p \geq 1$. Then, the sequence of linear operators $\{\mathbb{L}_{n,\lambda}\}$ is uniformly bounded from $L^p(K, \mathbb{C})$ into itself, i.e., for every $f \in L^p(K, \mathbb{C})$,*

$$\|\mathbb{L}_{n,\lambda}(f)\|_p \leq B \|f\|_p$$

holds for some absolute positive constant B.

Proof. We may write from Lemma 5 that, for every $k, m \in \{0, 1, ..., n\}$,

$$\begin{aligned}
& \int_0^1 \int_0^1 r_{k,m,n}(\lambda, z) dy dx \\
= & \sum_{u,v=0}^{n} \int_{\frac{u}{n+1}}^{\frac{u+1}{n+1}} \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} r_{k,m,n}(\lambda, x, y) dy dx \\
\leq & C \sum_{u,v=0}^{n} \int_{\frac{u}{n+1}}^{\frac{u+1}{n+1}} \int_{\frac{v}{n+1}}^{\frac{v+1}{n+1}} \left\{ (|u-k|+1)^2 + (|v-m|+1)^2 \right\}^{-\lambda/2} dy dx \\
= & \frac{C}{(n+1)^2} \sum_{u,v=0}^{n} \left\{ (|u-k|+1)^2 + (|v-m|+1)^2 \right\}^{-\lambda/2} \\
= & O\left(\frac{1}{(n+1)^2} \right)
\end{aligned}$$

holds whenever $\lambda \geq 3$. If $f \in L^1(K, \mathbb{C})$, then we see that

$$\begin{aligned}
& \| \mathbb{L}_{n,\lambda}(f) \|_1 \\
= & \int_0^1 \int_0^1 |\mathbb{L}_{n,\lambda}(f; z)| \, dy dx \\
\leq & (n+1)^2 \sum_{k,m=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} |f(s,t)| \, dt ds \right) \int_0^1 \int_0^1 r_{k,m,n}(\lambda, z) dy dx \\
\leq & C \sum_{k,m=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} |f(s,t)| \, dt ds \right),
\end{aligned}$$

which implies

$$\| \mathbb{L}_{n,\lambda}(f) \|_1 \leq C \, \| f \|_1 . \tag{3}$$

On the other hand, if $f \in C(K, \mathbb{C})$, then one can easily check that

$$\| \mathbb{L}_{n,\lambda}(f) \| \leq \| f \| , \tag{4}$$

where the symbol $\| \cdot \|$ denotes the usual supremum norm on K as stated before. Therefore, using (3) and (4), we obtain from the Riesz-Thorin theorem [25] (see also [24]) that, for some $B > 0$,

$$\| \mathbb{L}_{n,\lambda}(f) \|_p \leq B \, \| f \|_p , \quad p \geq 1,$$

holds for every $f \in L^p(K, \mathbb{C})$.

3. Proof of the Main Results

Proof of Theorem 1. Let $g \in C(K, \mathbb{C})$ and $\lambda \geq 3$. By the uniform continuity of g on K, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|g(w) - g(z)| < \varepsilon$$

for all $w = s + it, z = x + iy \in K$ satisfying

$$|w - z| = \sqrt{(s-x)^2 + (t-y)^2} < \delta.$$

Then from the definition of the operators $\mathbb{L}_{n,\lambda}$ we observe that

$$\begin{aligned}
|\mathbb{L}_{n,\lambda}(g;z) - g(z)| &\leq (n+1)^2 \sum_{k,m=0}^{n} r_{k,m,n}(\lambda, z) \\
&\quad \times \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} |g(s,t) - g(x,y)| \, dtds \\
&\leq (n+1)^2 \sum_{k,m=0}^{n} r_{k,m,n}(\lambda, z) \\
&\quad \times \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \int_{\frac{m}{n+1}}^{\frac{m+1}{n+1}} \left(\varepsilon + \frac{2M}{\delta} \sqrt{(s-x)^2 + (t-y)^2} \right) dtds \\
&= \varepsilon + \frac{2M}{\delta} \mathbb{L}_{n,\lambda}(\varphi_z; z),
\end{aligned}$$

where $M := \|g\|$. Then the proof immediately follows from Lemma 6.

Now we give the proof of Theorem 3

Proof of Theorem 3. Let $f \in L^p(K, \mathbb{C})$ $(p \geq 1)$. Then, we can choose a function $g \in C(K, \mathbb{C})$ such that

$$\|\mathbb{L}_{n,\lambda}(f) - f\|_p \leq \|\mathbb{L}_{n,\lambda}(f-g)\|_p + \|\mathbb{L}_{n,\lambda}(g) - g\|_p + \|f - g\|_p .$$

Hence, Lemma 7 implies that

$$\|\mathbb{L}_{n,\lambda}(f) - f\|_p \leq C \|f - g\|_p + \|\mathbb{L}_{n,\lambda}(g) - g\|_p$$

holds for some $C > 0$. We know from Corollary 2 that

$$\lim_{n \to \infty} \|\mathbb{L}_{n,\lambda}(g) - g\|_p = 0.$$

Using this and also the fact that $C(K, \mathbb{C})$ is dense in $L^p(K, \mathbb{C})$, the proof is completed.

4. Applications and Concluding Remarks

We first give an application of Theorem 3 on the set $K = [0, 1] \times [0, 1]$.

Example 1. Define the function f by

$$f(z) = \lfloor 3x \rfloor + ixy, \quad z = x + iy \in K, \tag{5}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the floor function. Then, by Theorem 3 one can obtain an L_p-approximation. Hence, for every $\lambda \geq 3$ and $p \geq 1$, we get

$$\lim_{n \to \infty} \left\| \mathbb{L}_{n,\lambda}(f) - f \right\|_p = 0. \tag{6}$$

Observe that Theorem A is not valid for this function f in (5) since f does not belong to the space $C(K, \mathbb{C})$. To visualize this L_p-approximation in (6) we may consider the real or imaginary part of the function f. Then, one can also easily check that

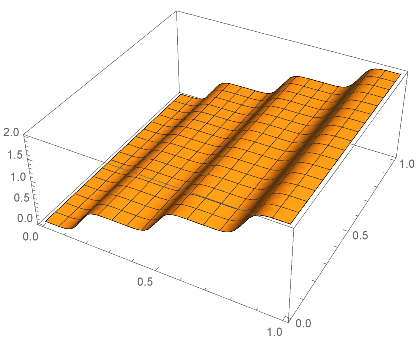
$$\lim_{n \to \infty} \left\| \Re \left(\mathbb{L}_{n,\lambda}(f) \right) - \Re \left(f \right) \right\|_p = 0,$$

which is indicated in Figure 1 for the values $\lambda = 4$ and $n = 3, 8, 12$. Hence, this example explains why it is also needed the Kantorovich version of comlpex Shepard operators.

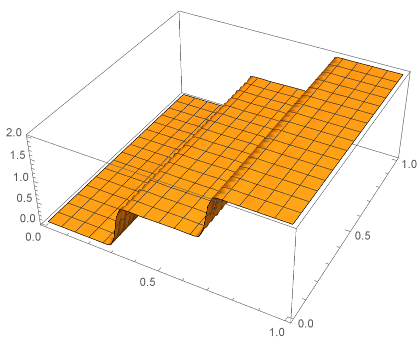
Finally, we can also incorporate regular summability methods in the approximation process of Kantorovich-Shepard operators as in the following way. We should note that a summability method is a common and useful way to overcome the lack of the usual convergence [26, 27]. Recent studies clearly show that regular summability methods are also quite effective in the approximation theory (see, for instance, [28, 29, 30, 31, 32, 33, 34]).

Now we recall some notations and definitions from the summability theory. Let an infinite matrix $A := [a_{jn}]$ $(j, n \in \mathbb{N})$ and a sequence $x := (x_n)$ be given. Then, the A-transformed sequence of (x_n) is defined by

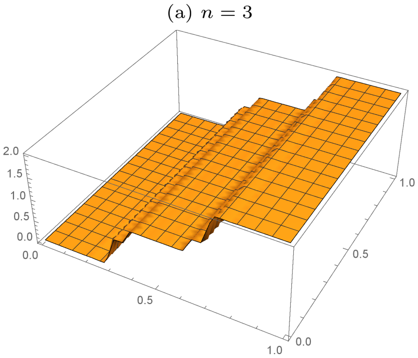
$$Ax := ((Ax)_j) = \sum_{n=1}^{\infty} a_{jn} x_n$$



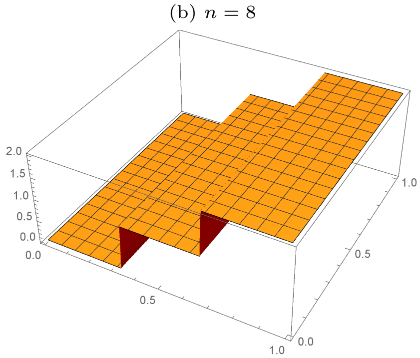
(a) $n = 3$



(b) $n = 8$



(c) $n = 12$



(d) $\Re(f)$

Figure 1: L_p-Approximation to the real part of the function f given by (5) by means of the real part of the operators $\mathbb{L}_{n,\lambda}(f)$ for the values $\lambda = 4$ and $n = 3, 8, 12$

if the above series is convergent for every $j \in \mathbb{N}$. In this case, it is said that A is a summability matrix method. A summability method A is called regular if $\lim Ax = L$ whenever $\lim x = L$ (see [26, 27]). If all entries of a summability matrix method A are nonnegative, then A is called nonnegative. In this section, we will consider $A = [a_{jn}]$ as a nonnegative regular summability matrix. We also say that a sequence (x_n) is A-summable (or A-convergent) to a number L if

$$\lim_{n\to\infty} (Ax)_j = L.$$

It is also possible to give the same definition for a sequence of functions in the space $L_p(K, \mathbb{C})$ $(p \geq 1)$. Let $f_n \in L_p(K, \mathbb{C})$ for every $n \in \mathbb{N}$ and $A = [a_{jn}]$ be a nonnegative regular summability method such that

$$\sum_{n=1}^{\infty} a_{jn} f_n \in L_p(K, \mathbb{C}) \quad \text{for every} \quad j \in \mathbb{N}.$$

Then, we say that (f_n) is A-summable (with respect to the L_p-norm on K) to a function $f \in L_p(K, \mathbb{C})$ if

$$\lim_{j\to\infty} \left\| \sum_{n=1}^{\infty} a_{jn} f_n - f \right\|_p = 0.$$

We note that the concept of A-summability in L_p spaces and its applications to the classical approximation theory (for real-valued functions) were examined by Orhan and Sakaoglu [35] (see also [36]). We now use it in the approximation by complex Kantorovich-Shepard operators given by (2).

With this terminology we can give the following application.

Example 2. Using the complex Kantorovich-Shepard operators in (2), we define the following operators

$$\mathbb{L}^*_{n,\lambda}(f; z) := \begin{cases} 2\mathbb{L}_{n,\lambda}(f; z), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even,} \end{cases} \tag{7}$$

where $f \in L_p(K, \mathbb{C})$ $(p \geq 1)$ and $z \in K$. Then, it is easy to check for any function $f \neq 0$ belonging to $L_p(K, \mathbb{C})$, we cannot get an L_p-approximation to

f by means of the operators $\mathbb{L}^*_{n,\lambda}(f)$, that is, for every $\lambda > 0$,

$$\left\|\mathbb{L}^*_{n,\lambda}(f) - f\right\|_p \nrightarrow 0 \text{ as } n \to \infty.$$

Now consider the Cesàro summability method $C_1 := [c_{jn}]$ given by

$$c_{jn} = \begin{cases} \frac{1}{n}, & \text{if } n = 1, 2, ..., j \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

In this case, we claim that, for every $f \in L_p(K, \mathbb{C})$ $(p \geq 1)$ and $\lambda \geq 3$, the sequence $\left(\mathbb{L}^*_{n,\lambda}(f)\right)$ is C_1-summable (with respect to the L_p-norm on K) to the function f. Indeed, by using the definitions (7) and (8) we may write that

$$\begin{aligned} \sum_{n=1}^{\infty} c_{jn}\mathbb{L}^*_{n,\lambda}(f) &= \frac{1}{j}\sum_{n=1}^{j}\mathbb{L}^*_{n,\lambda}(f) \\ &= \begin{cases} \frac{2\{\mathbb{L}_{1,\lambda}(f)+\mathbb{L}_{3,\lambda}(f)+\cdots+\mathbb{L}_{2m-1,\lambda}(f)\}}{2m-1}, & \text{if } j = 2m-1 \\ \frac{\mathbb{L}_{1,\lambda}(f)+\mathbb{L}_{3,\lambda}(f)+\cdots+\mathbb{L}_{2m-1,\lambda}(f)}{m}, & \text{if } j = 2m, \end{cases} \end{aligned}$$

where $m \in \mathbb{N}$. Then, we observe that

$$\begin{aligned} &\left\|\sum_{n=1}^{\infty} c_{jn}\mathbb{L}^*_{n,\lambda}(f) - f\right\|_p \\ \leq &\begin{cases} \left(\frac{2m}{2m-1}\right)\frac{1}{m}\sum\limits_{k=1}^{m}\left\|\mathbb{L}_{2k-1,\lambda}(f) - f\right\|_p + \left|\frac{2m}{2m-1} - 1\right|\left\|f\right\|_p, & \text{if } j = 2m-1 \\ \frac{1}{m}\sum\limits_{k=1}^{m}\left\|\mathbb{L}_{2k-1,\lambda}(f) - f\right\|_p & \text{if } j = 2m. \end{cases} \end{aligned}$$

Now taking limit as $j \to \infty$ (and hence, as $m \to \infty$) on both sides of the last inequality and also considering Theorem 3 and the regularity of the Cesàro method we obtain that

$$\lim_{j\to\infty}\left\|\sum_{n=1}^{\infty} c_{jn}\mathbb{L}^*_{n,\lambda}(f) - f\right\|_p = 0,$$

which corrects our claim.

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